10/13/25 Multiple Linear Regression (Part 2)

Canonical decomposition: $$\underbrace{\sum_{i=1}^{n}(y_i-\bar{y})^2}_{S_{yy}} = \underbrace{\sum_{i=1}^{n}(\hat{y}_i-\bar{y})^2}_{SSreg} + \underbrace{\sum_{i=1}^{n}(y_i-\hat{y}_i)^2}_{RSS}$$

Interpretation: 1) S_{yy} = total variability in y

2) $SSreg$ = variability of y explained by $X_1, \dots, X_p$

3) RSS = variability of y not explained by

F-statistic: $$F = \frac{SSreg/p}{RSS/(n-p-1)}$$ p = # predictor variables in the full model

Generalization: $$F = \frac{\left(RSS(\text{reduced}) - RSS(\text{full})\right)/p_{diff}}{RSS(\text{full})/(n-p-1)}$$

H_0: reduced model is correct

H_1: full model is correct

where p_{diff} = difference in # parameters between full and $\underbrace{\text{reduced}}_{\text{a.k.a "nested"}}$ models

Examples (reduced & full models)

Consider $y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \varepsilon$ ← full model

1) Overall null: $H_0: \beta_1 = \beta_2 = 0 \rightsquigarrow$ reduced model: $y = \beta_0 + \varepsilon$

Q: Difference in df between reduced & full? $3 - 1 = 2$

2) Linear constraint 1: $H_0: \beta_2 = 2\beta_1$

$\rightsquigarrow$ reduced model: $y = \beta_0 + \beta_1 X_1 + 2\beta_1 X_2 + \varepsilon$

One fewer parameter $\quad = \beta_0 + \beta_1 \underbrace{(X_1 + 2X_2)}_{\tilde{X}} + \varepsilon = \beta_0 + \beta_1 \tilde{X} + \varepsilon$

$P_{diff} = 3 - 2 = 1$

3) Linear constraint 2: $H_0: \beta_1 + \beta_2 = 1$. Q: Reduced model?

Df for this reduction?

Reduced model: $Y = \beta_0 + \beta_1 X_1 + (1 - \beta_1) X_2 + \varepsilon$

$$\iff \underbrace{Y - X_2}_{\tilde{Y}} = \beta_0 + \beta_1 \underbrace{(X_1 - X_2)}_{\tilde{X}} + \varepsilon$$

$P_{diff} = 3 - 2 = 1$

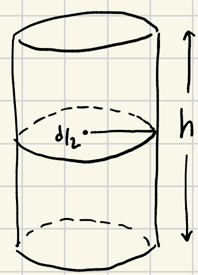
<u>Data example</u> $n = 31$ cherry trees (y = timber volume)

```
> head(trees)
  Diameter Height Volume
1      8.3     70   10.3
2      8.6     65   10.3
3      8.8     63   10.2
4     10.5     72   16.4
5     10.7     81   18.8
6     10.8     83   19.7
```

Volume (ft^3)

Height (ft)

Diameter (inches) $\rightarrow$ measured at 4 ft + 6 in. above ground



Formula: Volume = Area $\times$ height $= \pi r^2 h \qquad r = \text{Const.} \times d$

$\rightsquigarrow \log V = \beta_0 + \log h + 2 \log d \qquad$ (idealized)

MLR model (full): $\log V = \beta_0 + \beta_1 \log h + \beta_2 \log d + \varepsilon$

may be different from 1, 2

Q: Are timber volumes of real cherry trees consistent with the cylinder formula? Consider $H_0: \beta_2 = 2\beta_1$

$\rightsquigarrow$ Reduced model: $\log V = \beta_0 + \beta_1 \log h + 2\beta_1 \log d + \varepsilon$

$$= \beta_0 + \beta_1 (\underbrace{\log h + 2 \log d}_{\tilde{x}}) + \varepsilon$$

(new predictor)

To test H_0, fit the reduced model & compare RSS values

```
> full.mod <- lm(log(Volume)~log(Height)+log(Diameter),trees)
> reduced.mod <- lm(log(Volume) ~ I(log(Height)+2*log(Diameter)) , trees)
> RSS.reduced <- sum(residuals(reduced.mod)^2); RSS.reduced
[1] 0.1875054
> RSS.full <- sum(residuals(full.mod)^2); RSS.full
[1] 0.1854634
> num <- (RSS.reduced - RSS.full) / 1
> den <- RSS.full / (31 - 3)
> Fstat <- num/den; Fstat
[1] 0.3082937
> qf(0.95,1,31-3)
[1] 4.195972
```

```
> anova(reduced.mod,full.mod)
Analysis of Variance Table

Model 1: log(Volume) ~ I(log(Height) + 2 * log(Diameter))
Model 2: log(Volume) ~ log(Height) + log(Diameter)
  Res.Df     RSS Df Sum of Sq      F Pr(>F)
1     29 0.18751
2     28 0.18546  1  0.002042 0.3083 0.5831
```

$\rightsquigarrow$ don't reject $H_0: \beta_2 = 2\beta_1$

Q: Consider $H_0: \beta_1 = 1$ and $\beta_2 = 2$. Reduced model?

$$\rightsquigarrow \underbrace{\log V - \log h - 2 \log d}_{\tilde{y}} = \beta_0 + \varepsilon$$

(new response)

```
> y.new <- log(trees$Volume) - log(trees$Height) - 2*log(trees$Diameter)
> reduced.mod2 <- lm(y.new~1)
> RSS.reduced2 <- sum(residuals(reduced.mod2)^2); RSS.reduced2
[1] 0.1876858
> RSS.full <- sum(residuals(full.mod)^2); RSS.full
[1] 0.1854634
> num <- (RSS.reduced2 - RSS.full) / 2
> den <- RSS.full / (31 - 3)
> Fstat <- num/den; Fstat
[1] 0.1677617
> qf(0.95,2,31-3)
[1] 3.340386
```

```
> # Equivalent solution using anova() + offset
> reduced.mod2 <- lm(log(Volume) ~ 1, offset = I(log(Height)+2*log(Diameter)), trees)
> anova(reduced.mod2,full.mod)
Analysis of Variance Table

Model 1: log(Volume) ~ 1
Model 2: log(Volume) ~ log(Height) + log(Diameter)
  Res.Df     RSS Df Sum of Sq      F Pr(>F)
1     30 0.18769
2     28 0.18546  2 0.0022224 0.1678 0.8464
```

$\rightsquigarrow$ Don't reject $H_0: \beta_1 = 1$ and $\beta_2 = 2$

```
Coefficients:
               Estimate Std. Error t value Pr(>|t|)
(Intercept)    -6.63162    0.79979  -8.292 5.06e-09 ***
log(Height)     1.11712    0.20444   5.464 7.81e-06 ***
log(Diameter)   1.98265    0.07501  26.432  < 2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.08139 on 28 degrees of freedom
Multiple R-squared:  0.9777,	Adjusted R-squared:  0.9761
F-statistic: 613.2 on 2 and 28 DF,  p-value: < 2.2e-16
```

<u>Categorical predictor variables</u> e.g. treatment / control, disease / no disease

Example: Estimating the effect of cloud seeding on rainfall

i.e. spray silver iodide into cloud

Experiment (Summer 1975, Florida): Measure Suitability for potential cloud seeding ⇝ score S_i for day i.
↳ via weather model

- Select days for which $S_i \geq 1.5$

  ⇝ $n = 24$ "suitable" days (12 seeded)

- For each such day, flip a coin to decide whether to seed
- Measure rainfall following seeding (or not seeding)

```
> cloud.df <- read.csv("cloud.csv")
> head(cloud.df)
  A D    S  Rain
1 0 0 1.75 12.85
2 1 1 2.70  5.52
3 1 3 4.10  6.29
4 0 4 2.35  6.11
5 1 6 4.25  2.45
6 0 9 1.60  3.61
```

Variable descriptions:

A = action (seed = 1, no seed = 0)

D = days after June 16, 1975

S = suitability score (higher = more suitable)

Rain = amount of rainfall following action (10^7 cubic meters)

Q: What is the effect of cloud seeding on log(rain)?

```
> lmod <- lm(log(Rain)~A+S,cloud.df)
> summary(lmod)
```

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   1.9439     0.6816   2.852  0.00955 **
A             0.5207     0.3793   1.373  0.18429
S            -0.3284     0.2089  -1.572  0.13091
```

What does this number mean?

Interpretation: 0.52 = estimated increase in log(Rain) for seeded days compared to non-seeded days of the same Suitability.

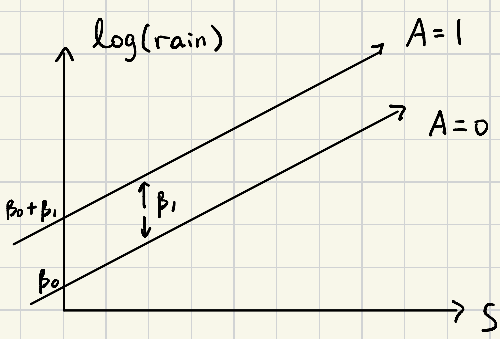
Q: Why?

```
> X <- model.matrix(lmod)
> head(X)
  (Intercept) A    S
1           1 0 1.75
2           1 1 2.70
3           1 1 4.10
4           1 0 2.35
5           1 1 4.25
6           1 0 1.60
```

$$\log(\text{Rain}) = \beta_0 + \beta_1 A + \beta_2 S + \varepsilon = \begin{cases} \beta_0 + \beta_2 S + \varepsilon & \text{if } A = 0 \\ \beta_0 + \beta_1 + \beta_2 S + \varepsilon & \text{if } A = 1 \end{cases}$$

$$\Rightarrow \beta_1 = \underbrace{E[\log(\text{rain}) \mid A=1, S]}_{\beta_0 + \beta_1 + \beta_2 S} - \underbrace{E[\log(\text{rain}) \mid A=0, S]}_{\beta_0 + \beta_2 S}$$

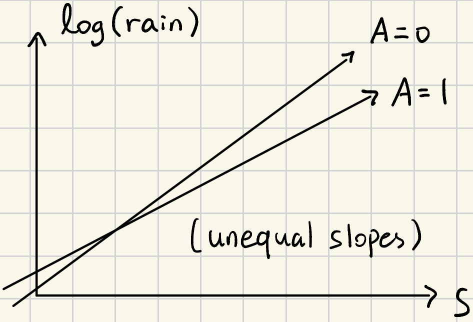
β_2 = effect of unit increase in Suitability score on log(rain), holding A fixed, e.g.



(same slope $\Leftrightarrow$ effect of cloud seeding constant in S)

"additive treatment effect"

More flexible model: $y = \beta_0 + \beta_1 A + \beta_2 S + \beta_3 \overbrace{A \cdot S}^{\text{"interaction"}} + \varepsilon$



$$= \begin{cases} \beta_0 + \beta_2 S + \varepsilon & \text{if } A = 0 \\ (\beta_0 + \beta_1) + (\beta_2 + \beta_3) S + \varepsilon & (A = 1) \end{cases}$$

$\rightsquigarrow$ Effect of A on response

$$= \beta_1 + \beta_3 S$$

Interpretation: effect of seeding depends on suitability of weather conditions

In R:

```
> lmod <- lm(log(Rain)~A*S,cloud.df)
> summary(lmod)
```

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   1.7156     0.8404   2.041   0.0546 .
A             1.2115     1.4832   0.817   0.4236
S            -0.2525     0.2647  -0.954   0.3515
A:S          -0.2148     0.4452  -0.482   0.6347
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.9316 on 20 degrees of freedom
```

Design matrix X:

```
> X <- model.matrix(lmod); head(X)
  (Intercept) A    S  A:S
1           1 0 1.75 0.00
2           1 1 2.70 2.70
3           1 1 4.10 4.10
4           1 0 2.35 0.00
5           1 1 4.25 4.25
6           1 0 1.60 0.00
```

Q: What is the effect of seeding treatment on log(rain) when $S = 4$?

```
> betas <- coefficients(lmod); betas
(Intercept)           A           S         A:S
  1.7155667   1.2115201  -0.2524720  -0.2147825
```

$\hat{\beta}_1$ A, $\hat{\beta}_2$ S, $\hat{\beta}_3$ A:S

$$\widehat{\Delta \log(\text{rain})} = \hat\beta_1 + \hat\beta_3 S = 1.212 - 0.215\,(4) = 0.352$$

Q: Is this change "significant"? e.g. at 95% confidence level

$\leadsto$ need to compute $se(\hat\beta_1 + \hat\beta_3 S)$

$$Var(\hat\beta_1 + \hat\beta_3 S) = Var(\hat\beta_1) + \underbrace{Var(\hat\beta_3 S)}_{S^2 Var(\hat\beta_3)} + 2\underbrace{Cov(\hat\beta_1, \hat\beta_3 S)}_{S\, Cov(\hat\beta_1, \hat\beta_3)}$$

Recall: $Cov(\hat\beta) = \sigma^2 (X^T X)^{-1}$, $\widehat{Cov}(\hat\beta) = \hat\sigma^2 (X^T X)^{-1}$

$\hat\sigma^2$:

```
> sigma.sq <- sum(lmod$residuals^2)/(nrow(X)-ncol(X)); sigma.sq
[1] 0.8678162
```

$\hat\sigma^2 (X^T X)^{-1}$:

```
> cov.mat <- sigma.sq*solve(t(X) %*% X); cov.mat
            (Intercept)          A           S         A:S
(Intercept)   0.7062426 -0.7062426 -0.21072285  0.21072285
A            -0.7062426  2.1998295  0.21072285 -0.63753033
S            -0.2107229  0.2107229  0.07004638 -0.07004638
A:S           0.2107229 -0.6375303 -0.07004638  0.19821679
```

$\leadsto se(\hat\beta_1 + 4\hat\beta_3) = \sqrt{2.2 + 4^2(0.20) + 2(4)(-0.638)} \approx 0.52$

$\leadsto$ 95% CI for $\beta_1 + \beta_3 S$: $0.352 \pm t_{n-4}\,(0.52) = (-0.73, 1.44)$

$\hookrightarrow$ qt(0.975, 20) = 2.08

Conclusion: Data doesn't show strong evidence of an effect on log(rainfall) due to cloud seeding, after controlling for suitability of weather conditions (point estimate ≥ 0, CI contains 0)

Multi-level categorical predictor

Suppose the predictor has more than 2 levels, e.g. A_1 = no seed, A_2 = silver iodide, A_3 = dry ice. Q: How is this coded in MLR?

Example (Twin data, $n = 27$)

```
> twins.df <- read.csv("twins.csv")
> head(twins.df)
   C IQb IQf
1 C3  94  94
2 C3  73  77
3 C1 129 117
4 C3 103 106
5 C2  78  71
6 C3  85  83
```

- C = Social class of biological parents ("Levels": 1 = high, 2 = middle, 3 = low).
- IQb = IQ of twin raised by biological parents. ← quantitative
- IQf = IQ of twin raised by foster parents.

"Dummy variable": a (binary) predictor variable indicating whether each case belongs to a particular level of the categorical predictor. (K categories require K−1 dummy variables.)

e.g. X matrix for twin data:

```
> lmod <- lm(IQf~IQb+C,twins.df)
> X <- model.matrix(lmod); head(X)
  (Intercept) IQb CC2 CC3
1           1  94   0   1
2           1  73   0   1
3           1 129   0   0
4           1 103   0   1
5           1  78   1   0
6           1  85   0   1
```

$$CC2_i = \begin{cases} 1 & \text{if pair } i \text{ born middle class} \\ 0 & \text{otherwise} \end{cases}$$

$$CC3_i = \begin{cases} 1 & \text{if pair } i \text{ born lower class} \\ 0 & \text{otherwise} \end{cases}$$

Q: What if $CC2_i = CC3_i = 0$? Pair i born upper class ($CC1_i = 1 - CC2_i - CC3_i$)

MLR model: $\mathrm{IQf} = \underbrace{\beta_0 + \beta_1 \mathrm{IQb} + \beta_2 \mathrm{CC2} + \beta_3 \mathrm{CC3}}_{\text{"mean function" } f(\mathrm{IQb}, \mathrm{CC2}, \mathrm{CC3})} + \varepsilon$

Interpretation:

β_1 : A unit increase in IQb is associated with a β_1 increase in mean IQf among twins born in the same social class.

β_2 = Contrast in mean IQ between foster twins originally born middle class versus those born upper class (CC2 − CC1)

$$= \underbrace{f(\mathrm{IQb}, 1, 0)}_{\beta_0 + \beta_1 \mathrm{IQb} + \beta_2} - \underbrace{f(\mathrm{IQb}, 0, 0)}_{\beta_0 + \beta_1 \mathrm{IQb}}$$ holding IQb fixed

e.g.

```
> lmod <- lm(IQf~IQb+C,twins.df)
> summary(lmod)
```

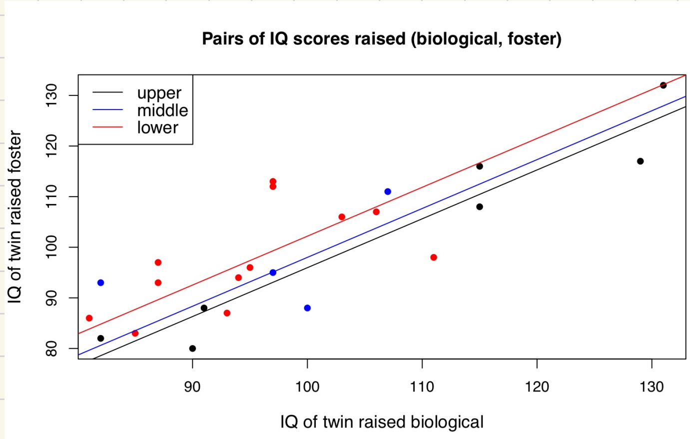
⇝

Coefficients:

		Estimate	Std. Error	t value	Pr(>|t|)	
	(Intercept)	-0.6076	11.8551	-0.051	0.960	
	IQb	0.9658	0.1069	9.031	5.05e-09	***
	CC2	2.0353	4.5908	0.443	0.662	
$\hat{\beta}_3$	CC3	6.2264	3.9171	1.590	0.126	

Interp: Mean IQ among foster twins whose biological twin has, e.g. IQ = 100, is estimated to be 6.23 points higher among those born to a low social class, compared to those born to high social class.

```
> c1 <- which(twins.df$C=="C1")
> c2 <- which(twins.df$C=="C2")
> c3 <- which(twins.df$C=="C3")
> plot(twins.df$IQb[c1],twins.df$IQf[c1],pch=16,
+      xlab="IQ of twin raised biological",
+      ylab="IQ of twin raised foster",cex.lab=1.2,
+      main="Pairs of IQ scores raised (biological, foster)")
> points(twins.df$IQb[c2],twins.df$IQf[c2],pch=16,col="blue")
> points(twins.df$IQb[c3],twins.df$IQf[c3],pch=16,col="red")
>
> beta <- coefficients(lmod); beta
(Intercept)        IQb        CC2        CC3
 -0.6076000  0.9658077  2.0353362  6.2264260
> abline(a=beta[1],b=beta[2])
> abline(a=beta[1]+beta[3],b=beta[2],col="blue")
> abline(a=beta[1]+beta[4],b=beta[2],col="red")
>
> legend("topleft",c("upper","middle","lower"),
+        col=c("black","blue","red"),lty=1,cex=1.2)
```



Observation: The contrast in mean IQf scores between, e.g. upper versus lower class, doesn't depend on IQb in this model

Consider a "full model" where slope may vary by social class:

$$IQf = \beta_0 + \beta_1 IQb + \beta_2 CC2 + \beta_3 CC3 + \beta_4 CC2 \cdot IQb + \beta_5 CC3 \cdot IQb + \varepsilon \qquad (\star)$$

$$= \begin{cases} \beta_0 + \beta_1 IQb + \varepsilon & \text{if } CC2 = CC3 = 0 \\ (\beta_0 + \beta_2) + (\beta_1 + \beta_4) IQb + \varepsilon & \text{if } CC2 = 1,\ CC3 = 0 \\ (\beta_0 + \beta_3) + (\beta_1 + \beta_5) IQb + \varepsilon & \text{if } CC2 = 0,\ CC3 = 1 \end{cases}$$

(equivalent to 3 separate simple LR)

In R:

```
> lmod2 <- lm(IQf~C+IQb+IQb*C,twins.df)
> summary(lmod2)
```

lm(IQf ~ IQb*C) also works

```
Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
(Intercept) -1.872044  17.808264  -0.105    0.917    
CC2          2.688068  31.604178   0.085    0.933    
CC3          9.076654  24.448704   0.371    0.714    
IQb          0.977562   0.163192   5.990 6.04e-06 ***
CC2:IQb     -0.004995   0.329525  -0.015    0.988    
CC3:IQb     -0.029140   0.244580  -0.119    0.906    
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 7.921 on 21 degrees of freedom
Multiple R-squared:  0.8041,	Adjusted R-squared:  0.7574 
F-statistic: 17.24 on 5 and 21 DF,  p-value: 8.31e-07
```

```
> round(coefficients(lmod2),2)
(Intercept)         CC2         CC3         IQb     CC2:IQb     CC3:IQb 
      -1.87        2.69        9.08        0.98        0.00       -0.03 
```

Fitted model:

$$\widehat{IQf} = \begin{cases} -1.87 + 0.98\, IQb & \text{if } CC2 = CC3 = 0 \\ (-1.87 + 2.69) + 0.98\, IQb & \text{if } CC2 = 1 \\ (-1.87 + 9.08) + (0.98 - 0.03)\, IQ_b & \text{if } CC3 = 1 \end{cases}$$

Q: Based on this ↑, what is the estimated IQ_f for twins born into lower class (CC3), holding $IQ_b = 100$ fixed?

Point estimate: $\hat{\beta}_0 + \hat{\beta}_3 + (\hat{\beta}_1 + \hat{\beta}_5) \cdot (100) \approx 102$

Confidence interval:

```
> round(predict(lmod2,data.frame(C="C3",IQb=100),interval="confidence",level=0.95),2)
     fit   lwr    upr
1 102.05 96.53 107.57
```

Another (equivalent) way to fit (☆)

```
> lmod2.1 <- lm(IQf~C+IQb:C-1,twins.df)
> summary(lmod2.1)
```

means: no intercept, but now 3 dummy variables (instead of 3-1=2)

Same fit as

lm(IQf ~ C * IQb)

```
Coefficients:
          Estimate Std. Error t value Pr(>|t|)
CC1       -1.8720    17.8083  -0.105  0.91728
CC2        0.8160    26.1092   0.031  0.97536
CC3        7.2046    16.7513   0.430  0.67151
CC1:IQb    0.9776     0.1632   5.990 6.04e-06 ***
CC2:IQb    0.9726     0.2863   3.397  0.00272 **
CC3:IQb    0.9484     0.1822   5.206 3.69e-05 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 7.921 on 21 degrees of freedom
Multiple R-squared:  0.9948,	Adjusted R-squared:  0.9933
F-statistic: 663.2 on 6 and 21 DF,  p-value: < 2.2e-16
```

Point estimate: $7.02 + 0.95(100) \approx 102$ (same model ⇒ same fitted $\hat{y}$)

Summary so far:

1) Different intercepts & same slope (parallel lines)

   IQf ~ C + IQb

2) Different intercepts & different slopes:

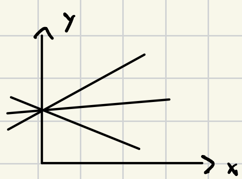
   IQf ~ C * IQb or IQf ~ C - 1 + C:IQb

3) Same intercept & same slope (SLR, ignore C variable)

   IQf ~ IQb

4) Same intercept & different slopes (next)

e.g.



Model 4: $IQf = \beta_0 + \beta_1 IQb + \beta_2 CC2 \cdot IQb + \beta_3 CC3 \cdot IQb + \varepsilon$

In R:

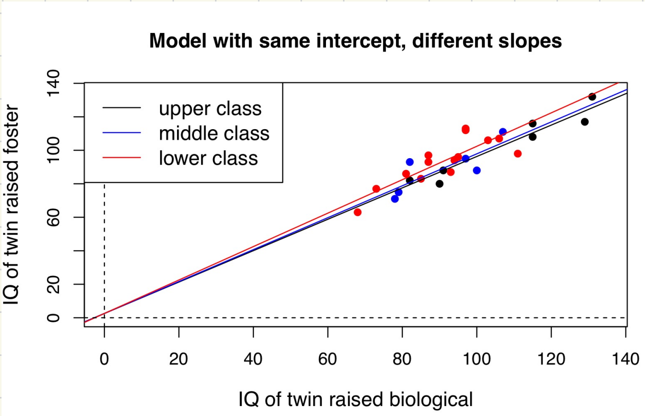
```
> lmod4 <- lm(IQf~IQb+IQb:C,twins.df)
> summary(lmod4)
```

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  2.56226   10.59844   0.242    0.811    
IQb          0.93751    0.09929   9.442 2.23e-09 ***
IQb:CC2      0.01606    0.04723   0.340    0.737    
IQb:CC3      0.06100    0.03905   1.562    0.132    
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 7.594 on 23 degrees of freedom
Multiple R-squared:  0.8027,	Adjusted R-squared:  0.777 
F-statistic:  31.2 on 3 and 23 DF,  p-value: 2.791e-08
```

Q: What's the least squares fit?

$$\widehat{IQf} = \begin{cases} 2.56 + 0.938\, IQ_b & \text{if } C = C1 \\ 2.56 + (0.938 + 0.016)\, IQ_b & \text{if } C = C2 \\ 2.56 + (0.938 + 0.061)\, IQ_b & \text{if } C = C3 \end{cases}$$



All 3 slopes are roughly the same.

Q: How does the fit compare to Model 3 (SLR)?

Task: Test H_0: slopes are all equal $(\beta_2 = \beta_3 = 0)$

reduced model: $IQf = \beta_0 + \beta_1 IQb + \varepsilon$

F-statistic:

$$\frac{\left(RSS(IQf \sim IQb:C) - RSS(IQf \sim IQb) \right)/2}{RSS(IQf \sim IQb:C)/(27-4)} \approx 1.45$$

on 2, 23 df

```
> lmod3 <- lm(IQf~IQb,twins.df)
> anova(lmod3,lmod4)
Analysis of Variance Table

Model 1: IQf ~ IQb
Model 2: IQf ~ IQb + IQb:C
  Res.Df    RSS Df Sum of Sq      F Pr(>F)
1     25 1493.5
2     23 1326.5  2    167.07 1.4485 0.2556
```

$\rightsquigarrow$ Don't reject H_0

Conclusion: SLR model is preferred

Exercise: Can 4) $IQf \sim IQb : C$ and

1) $IQf \sim C + IQb$ be compared

via F-test? Why or why not?

How about 2) $IQf \sim C * IQb$ and 3) $IQf \sim IQb$?