Lecture 1

Dani Nguyen

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1 Invertible Matrix Theorem

1.1 Definition

Let A be a square n x n matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

1. A is invertible.
2. A is row equivalent to I.
3. A has n pivot positions.
4. The equation A$\mathbf{x} = \mathbf{0}$ has only the trivial solution.
5. The columns of A are linearly independent.
6. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
7. The equation A$\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
8. The columns of A span $\mathbb{R}^n$
9. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
10. There is an n x n matrix C such that CA = I.
11. There is an n x n matrix D such that AD = I.
12. A^T is invertible.

13. The columns of A form a basis of $\mathbb{R}^n$.
14. Col A = $\mathbb{R}^n$.
15. dim Col A = n.
16. rank A = n
17. Nul A = 0
18. dim Nul A = 0
19. The number 0 is *not* an eigenvalue of A.
20. det A $\neq$ 0.
21. (Col A)$^\perp$ = $\mathbf{0}$
22. (Nul A)$^\perp$ = $\mathbb{R}^n$
23. Row A = $\mathbb{R}^n$

Exercise

Thm 9.1.16 Let A $\in M_n$ and let $\lambda \in \mathbb{C}$. Then the following statements are equivalent:

1. λ is an eigenvalue of A.
2. A$\mathbf{x} = \lambda\mathbf{x}$ for some nonzero $\mathbf{x} \in \mathbb{C}^n$.
3. (A - λI)$\mathbf{x} = \mathbf{0}$ has a nontrivial solution, that is, nullity(A-λI) > 0.
4. rank(A - λI) < n.
5. A - λI is not invertible.
6. A$^\top$ - λI is not invertible.
7. λ is an eigenvalue of A$^\top$.

Proof