Lecture 1

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Contents

1 Invertible Matrix Theorem

1.1 Definition

Let A be a square n x n matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

1. A is invertible.

2. A is row equivalent to I.

3. A has n pivot positions.

4. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

5. The columns of A are linearly independent.

6. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

7. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

8. The columns of A span $\mathbb{R}^n$

9. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

10. There is an n x n matrix C such that $CA = I$.

11. There is an n x n matrix D such that $AD = I$.

12. A^T is invertible.

13. The columns of A form a basis of $\mathbb{R}^n$.

14. Col A $= \mathbb{R}^n$.

15. dim Col A $=$ n.

16. rank A $=$ n

17. Nul A $= 0$

18. dim Nul A $= 0$

19. The number 0 is *not* an eigenvalue of A.

20. det A $\neq 0$.

21. $(\text{Col A})^{\perp} = \mathbf{0}$

22. $(\text{Nul A})^{\perp} = \mathbb{R}^n$

23. Row A $= \mathbb{R}^n$

1.2 Exercise

Thm 9.1.16 Let A $\in M_n$ and let $\lambda \in \mathbb{C}$. Then the following statements are equivalent:

(a) λ is an eigenvalue of A.

(b) $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero $\mathbf{x} \in \mathbb{C}^n$.

(c) $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution, that is, nullity$(A - \lambda I) > 0$.

(d) rank$(A - \lambda I) <$ n.

(e) $A - \lambda I$ is not invertible.

(f) $A^\top - \lambda I$ is not invertible.

(g) λ is an eigenvalue of $A^\top$.

Proof

(a) $\Leftrightarrow$ (b)

By definition 9.1.1, if $A\mathbf{x} = \lambda\mathbf{x}$ and $\mathbf{x} \neq \mathbf{0}$ then $(\lambda, \mathbf{x})$ is an eigenpair of A, meaning λ is an eigenvalue of A.

(b) $\Leftrightarrow$ (c)

These are restatements of one another.

(c) $\Leftrightarrow$ (d)

By rank nullity theorem, if nullity$(A - \lambda I) > 0$, then rank$(A - \lambda I) < $ n.

(d) $\Leftrightarrow$ (e)

By property 1 and 16 of IMT, if rank$(A - \lambda I) < $ n then $(A - \lambda I)$ is not invertible.

(e) $\Leftrightarrow$ (f)

(f) $\Leftrightarrow$ (g)