Lecture 1

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Contents

1 Invertible Matrix Theorem

1.1 Definition

Let A be a square n x n matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

1. A is invertible.
2. A is row equivalent to I.
3. A has n pivot positions.
4. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
5. The columns of A are linearly independent.
6. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
7. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
8. The columns of A span $\mathbb{R}^n$
9. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
10. There is an n x n matrix C such that CA = I.
11. There is an n x n matrix D such that AD = I.

12. A^T is invertible.
13. The columns of A form a basis of $\mathbb{R}^n$.
14. Col A = $\mathbb{R}^n$.
15. dim Col A = n.
16. rank A = n
17. Nul A = 0
18. dim Nul A = 0
19. The number 0 is *not* an eigenvalue of A.
20. det A $\neq$ 0.
21. $(\text{Col A})^{\perp} = \mathbf{0}$
22. $(\text{Nul A})^{\perp} = \mathbb{R}^n$
23. Row A = $\mathbb{R}^n$

1.2 Exercise

Thm 9.1.16 Let A $\in M_n$ and let $\lambda \in \mathbb{C}$. Then the following statements are equivalent:

(a) λ is an eigenvalue of A.

(b) A$\mathbf{x} = \lambda\mathbf{x}$ for some nonzero $\mathbf{x} \in \mathbb{C}^n$.

(c) (A - λI)$\mathbf{x} = \mathbf{0}$ has a nontrivial solution, that is, nullity(A - λI) > 0.

(d) rank(A - λI) < n.

(e) A - λI is not invertible.

(f) $A^{\top}$ - λI is not invertible.

(g) λ is an eigenvalue of $A^{\top}$.

Proof

(a) $\Leftrightarrow$ (b)

By definition 9.1.1, if $A\mathbf{x} = \lambda\mathbf{x}$ and $\mathbf{x} \neq \mathbf{0}$ then $(\lambda, \mathbf{x})$ is an eigenpair of A, meaning λ is an eigenvalue of A.

(b) $\Leftrightarrow$ (c)

These are restatements of one another.

(c) $\Leftrightarrow$ (d)

By rank nullity theorem, if nullity(A - λI) > 0, then rank(A - λI) $<$ n.

(d) $\Leftrightarrow$ (e)

By property 1 and 16 of IMT, if rank(A - λI) $<$ n then (A - λI) is not invertible.

(e) $\Leftrightarrow$ (f)

Suppose we have $A - \lambda I = \begin{bmatrix} a - \lambda I & b \\ c & d - \lambda I \end{bmatrix}$, and suppose it is not invertible, its determinant is $(a - \lambda I)(d - \lambda I) - bc = 0$. Consider the case of $A^\top - \lambda I$, we have $\begin{bmatrix} a - \lambda I & c \\ b & d - \lambda I \end{bmatrix}$, whose determinant is $(a - \lambda I)(d - \lambda I) - cb$. And by commutativity of multiplication, $bc = cb$, so the determinant is also 0, meaning $A^\top - \lambda I$ is also not invertible.

(f) $\Leftrightarrow$ (g)

Since (a) is equivalent to (e), the same is true for $A^\top$.