

Lecture 1

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1 Invertible Matrix Theorem

1.1 Definition

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A , the statements are either all true or all false.

1. A is invertible.
2. A is row equivalent to I .
3. A has n pivot positions.
4. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
5. The columns of A are linearly independent.
6. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
7. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each \mathbf{b} in \mathbb{R}^n .
8. The columns of A span \mathbb{R}^n .

9. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps \mathbb{R}^n onto \mathbb{R}^n .
10. There is an $n \times n$ matrix C such that $CA = I$.
11. There is an $n \times n$ matrix D such that $AD = I$.
12. A^T is invertible.
13. The columns of A form a basis of \mathbb{R}^n .
14. $\text{Col } A = \mathbb{R}^n$.
15. $\dim \text{Col } A = n$.
16. $\text{rank } A = n$
17. $\text{Nul } A = \mathbf{0}$
18. $\dim \text{Nul } A = 0$
19. The number 0 is *not* an eigenvalue of A .
20. $\det A \neq 0$.
21. $(\text{Col } A)^\perp = \mathbf{0}$
22. $(\text{Nul } A)^\perp = \mathbb{R}^n$
23. $\text{Row } A = \mathbb{R}^n$

1.2 Exercise

Thm 9.1.16 Let $A \in M_n$ and let $\lambda \in \mathbb{C}$. Then the following statements are equivalent:

- (a) λ is an eigenvalue of A .
- (b) $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero $\mathbf{x} \in \mathbb{C}^n$.
- (c) $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution, that is, $\text{nullity}(A - \lambda I) > 0$.
- (d) $\text{rank}(A - \lambda I) < n$.
- (e) $A - \lambda I$ is not invertible.

(f) $A^\top - \lambda I$ is not invertible.

(g) λ is an eigenvalue of A^\top .

Proof

(a) \Leftrightarrow (b)

By definition 9.1.1, if $A\mathbf{x} = \lambda\mathbf{x}$ and $\mathbf{x} \neq \mathbf{0}$ then (λ, \mathbf{x}) is an eigenpair of A , meaning λ is an eigenvalue of A .

(b) \Leftrightarrow (c)

These are restatements of one another.

(c) \Leftrightarrow (d)

By rank nullity theorem, if $\text{nullity}(A - \lambda I) > 0$, then $\text{rank}(A - \lambda I) < n$.

(d) \Leftrightarrow (e)

By property 1 and 16 of IMT, if $\text{rank}(A - \lambda I) < n$ then $(A - \lambda I)$ is not invertible.

(e) \Leftrightarrow (f)

(f) \Leftrightarrow (g)

Since (a) is equivalent to (e), the same is true for A^\top .