

Lecture 1

Dani Nguyen

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1 Invertible Matrix Theorem

1.1 Definition

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A , the statements are either all true or all false.

1. A is invertible.
2. A is row equivalent to I .
3. A has n pivot positions.
4. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
5. The columns of A are linearly independent.
6. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
7. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each \mathbf{b} in \mathbb{R}^n .
8. The columns of A span \mathbb{R}^n .
9. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps \mathbb{R}^n onto \mathbb{R}^n .
10. There is an $n \times n$ matrix C such that $CA = I$.
11. There is an $n \times n$ matrix D such that $AD = I$.
12. A^T is invertible.

- 13. The columns of A form a basis of \mathbb{R}^n .
- 14. $\text{Col } A = \mathbb{R}^n$.
- 15. $\dim \text{Col } A = n$.
- 16. $\text{rank } A = n$
- 17. $\text{Nul } A = \mathbf{0}$
- 18. $\dim \text{Nul } A = 0$
- 19. The number 0 is *not* an eigenvalue of A .
- 20. $\det A \neq 0$.
- 21. $(\text{Col } A)^\perp = \mathbf{0}$
- 22. $(\text{Nul } A)^\perp = \mathbb{R}^n$
- 23. $\text{Row } A = \mathbb{R}^n$

1.2 Proof