

# Lecture 1

Dani Nguyen

September 30, 2025

## Contents

|          |                                  |          |
|----------|----------------------------------|----------|
| <b>1</b> | <b>Invertible Matrix Theorem</b> | <b>1</b> |
| 1.1      | Definition . . . . .             | 1        |
| 1.2      | Exercise . . . . .               | 2        |

## 1 Invertible Matrix Theorem

### 1.1 Definition

Let  $A$  be a square  $n \times n$  matrix. Then the following statements are equivalent. That is, for a given  $A$ , the statements are either all true or all false.

1.  $A$  is invertible.
2.  $A$  is row equivalent to  $I$ .
3.  $A$  has  $n$  pivot positions.
4. The equation  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution.
5. The columns of  $A$  are linearly independent.
6. The linear transformation  $\mathbf{x} \mapsto A\mathbf{x}$  is one-to-one.
7. The equation  $A\mathbf{x} = \mathbf{b}$  has at least one solution for each  $\mathbf{b}$  in  $\mathbb{R}^n$ .
8. The columns of  $A$  span  $\mathbb{R}^n$ .

9. The linear transformation  $\mathbf{x} \mapsto A\mathbf{x}$  maps  $\mathbb{R}^n$  onto  $\mathbb{R}^n$ .
10. There is an  $n \times n$  matrix  $C$  such that  $CA = I$ .
11. There is an  $n \times n$  matrix  $D$  such that  $AD = I$ .
12.  $A^T$  is invertible.
13. The columns of  $A$  form a basis of  $\mathbb{R}^n$ .
14.  $\text{Col } A = \mathbb{R}^n$ .
15.  $\dim \text{Col } A = n$ .
16.  $\text{rank } A = n$
17.  $\text{Nul } A = \mathbf{0}$
18.  $\dim \text{Nul } A = 0$
19. The number 0 is *not* an eigenvalue of  $A$ .
20.  $\det A \neq 0$ .
21.  $(\text{Col } A)^\perp = \mathbf{0}$
22.  $(\text{Nul } A)^\perp = \mathbb{R}^n$
23.  $\text{Row } A = \mathbb{R}^n$

## 1.2 Exercise

**Thm 9.1.16** Let  $A \in M_n$  and let  $\lambda \in \mathbb{C}$ . Then the following statements are equivalent:

- (a)  $\lambda$  is an eigenvalue of  $A$ .
- (b)  $A\mathbf{x} = \lambda\mathbf{x}$  for some nonzero  $\mathbf{x} \in \mathbb{C}^n$ .
- (c)  $(A - \lambda I)\mathbf{x} = \mathbf{0}$  has a nontrivial solution, that is,  $\text{nullity}(A - \lambda I) > 0$ .
- (d)  $\text{rank}(A - \lambda I) < n$ .
- (e)  $A - \lambda I$  is not invertible.

(f)  $A^\top - \lambda I$  is not invertible.

(g)  $\lambda$  is an eigenvalue of  $A^\top$ .

**Proof**

(a)  $\Leftrightarrow$  (b)

By definition 9.1.1, if  $A\mathbf{x} = \lambda\mathbf{x}$  and  $\mathbf{x} \neq \mathbf{0}$  then  $(\lambda, \mathbf{x})$  is an eigenpair of  $A$ , meaning  $\lambda$  is an eigenvalue of  $A$ .

(b)  $\Leftrightarrow$  (c)

These are restatements of one another.

(c)  $\Leftrightarrow$  (d)

By rank nullity theorem, if  $\text{nullity}(A - \lambda I) > 0$ , then  $\text{rank}(A - \lambda I) < n$ .

(d)  $\Leftrightarrow$  (e)

By property 1 and 16 of IMT, if  $\text{rank}(A - \lambda I) < n$  then  $(A - \lambda I)$  is not invertible.

(e)  $\Leftrightarrow$  (f)

Suppose we have  $A - \lambda I = \begin{bmatrix} a - \lambda I & b \\ c & d - \lambda I \end{bmatrix}$ , and suppose it is not invertible, its determinant is  $(a - \lambda I)(d - \lambda I) - bc = 0$ . Consider the case of  $A^\top - \lambda I$ , we have  $\begin{bmatrix} a - \lambda I & c \\ b & d - \lambda I \end{bmatrix}$ , whose determinant is  $(a - \lambda I)(d - \lambda I) - cb$ . And by commutativity of multiplication,  $bc = cb$ , so the determinant is also 0, meaning  $A^\top - \lambda I$  is also not invertible.

(f)  $\Leftrightarrow$  (g)

Since (a) is equivalent to (e), the same is true for  $A^\top$ .