# Lecture 1

### Dani Nguyen

#### September 24, 2025

# Contents

	1.1	rtible Matrix Theorem 1   Definition 1   Exercise 2	
1 Invertible Matrix Theorem			
l.	1	Definition	
	et A b	a square n x n matrix. Then the following statements are equivalent	,

That is, for a given A, the statements are either all true or all false.

1. A is invertible.

- 2. A is row equivalent to I.
- 3. A has n pivot positions.
- 4. The equation  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution.
- 5. The columns of A are linearly independent.
- 6. The linear transformation  $\mathbf{x} \mapsto A\mathbf{x}$  is one-to-one.
- 7. The equation  $A\mathbf{x} = \mathbf{b}$  has at least one solution for each  $\mathbf{b}$  in  $\mathbb{R}^n$ .
- 8. The columns of A span  $\mathbb{R}^n$

- 9. The linear transformation  $\mathbf{x} \mapsto A\mathbf{x}$  maps  $\mathbb{R}^n$  onto  $\mathbb{R}^n$ .
- 10. There is an n x n matrix C such that CA = I.
- 11. There is an  $n \times n$  matrix D such that AD = I.
- 12.  $A^T$  is invertible.
- 13. The columns of A form a basis of  $\mathbb{R}^n$ .
- 14. Col A =  $\mathbb{R}^n$ .
- 15. dim Col A = n.
- 16. rank A = n
- 17. Nul A = 0
- 18. dim Nul A = 0
- 19. The number 0 is *not* an eigenvalue of A.
- 20. det  $A \neq 0$ .
- 21.  $(\text{Col A})^{\perp} = \mathbf{0}$
- 22.  $(\text{Nul A})^{\perp} = \mathbb{R}^n$
- 23. Row  $A = \mathbb{R}^n$

#### 1.2 Exercise

**Thm 9.1.16** Let  $A \in M_n$  and let  $\lambda \in \mathbb{C}$ . Then the following statements are equivalent:

- (a)  $\lambda$  is an eigenvalue of A.
- (b)  $A\mathbf{x} = \lambda \mathbf{x}$  for some nonzero  $\mathbf{x} \in \mathbb{C}^n$ .
- (c)  $(A \lambda I)x = 0$  has a nontrivial solution, that is, nullity  $(A \lambda I) > 0$ .
- (d)  $rank(A \lambda I) < n$ .
- (e) A  $\lambda I$  is not invertible.

- (f)  $A^{\top}$   $\lambda I$  is not invertible.
- (g)  $\lambda$  is an eigenvalue of  $A^{\top}$ .

# ${\bf Proof}$

- (a)  $\Leftrightarrow$  (b)
- (b)  $\Leftrightarrow$  (c)
- $(c) \Leftrightarrow (d)$
- $(d) \Leftrightarrow (e)$
- (e)  $\Leftrightarrow$  (f)
- $(f) \Leftrightarrow (g)$