An Inverse Eigenvalue Problem (IEP) Investigation

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The Forward Problem

Generalized Eigenvalue Problem

Given $A, B \in \mathbb{C}^{n \times n}$, find $(\lambda, \mathbf{x}) \in \mathbb{C} \times (\mathbb{C}^n - \{\mathbf{0}\})$ such that

$$A\mathbf{x} = \lambda B\mathbf{x}.$$

For $B = I$ this becomes the familiar linear eigenvalue problem (LEP). Many algorithms exist to tackle this problem for general A [1]:

$$
\begin{aligned}
&\textbf{for } k = 1, 2, \ldots \\
&\quad z^{(k)} = Aq^{(k-1)} \\
&\quad q^{(k)} = z^{(k)} / \| z^{(k)} \|_2 \\
&\quad \lambda^{(k)} = [q^{(k)}]^H A q^{(k)} \\
&\textbf{end}
\end{aligned}
$$

(a) Power Method

$$
\begin{aligned}
&\textbf{for } k = 1, 2, \ldots \\
&\quad \text{Solve } (A - \mu I) z^{(k)} = q^{(k-1)}. \\
&\quad q^{(k)} = z^{(k)} / \| z^{(k)} \|_2 \\
&\quad \lambda^{(k)} = q^{(k)T} A q^{(k)} \\
&\textbf{end}
\end{aligned}
$$

(b) Shift and Invert

$$
\begin{aligned}
&H = U_0^T A U_0 \qquad \text{(Hessenberg reduction)} \\
&\textbf{for } k = 1, 2, \ldots \\
&\quad \text{Determine a scalar } \mu. \\
&\quad H - \mu I = UR \qquad \text{(QR factorization)} \\
&\quad H = RU + \mu I \\
&\textbf{end}
\end{aligned}
$$

(c) QR Iteration

The Backwards Problem

Why? [2]

- system identification [3]
- principal component analysis [4]
- molecular spectroscopy [5]

Generalized IEP (GIEP)

Given $\lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_k\}$ for $k \leq n$, determine $A, B \in \mathbb{C}^{n \times n}$ so that $\det(A - \lambda_i B) = 0$ for $i = 1, 2, \cdots, k$.

A Novel GIEP Algorithm

Set $A = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_k, 0, \cdots, 0)$ and $B = I$.

...Under-determined! We must impose constraints:

- spectral
- structural

A Structurally Constrained IEP (1)

Consider the following dynamical system:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$
$$\mathbf{y}(t) = C\mathbf{x}(t)$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$.

$\mathbf{u}(t) = K\mathbf{y}(t) = KC\mathbf{x}(t)$ induces the *closed-loop* dynamical system:

$$\dot{\mathbf{x}}(t) = (A + BKC)\mathbf{x}(t)$$

Output Feedback Pole Assignment Problem (OfPAP)

Given $\lambda = \{\lambda_i\}_{i=1}^n \subseteq \mathbb{C}$ where $\overline{\lambda} = \lambda$ find $K \in \mathbb{R}^{m\times p}$ such that:

$$\sigma(A + BKC) = \{\lambda_i\}_{i=1}^n$$

A Structurally Constrained IEP (2)

Linear Parameterized Inverse Eigenvalue Problem (LiPIEP)

For $\{A_j\}_{j=1}^m \subset \mathcal{M} \subseteq \mathbb{C}^{n\times n}$, put $A(\mathbf{c}) = A_0 + c_1 A_1 + \cdots + c_m A_m$.
Given $\lambda = \{\lambda_i\}_{i=1}^n \subseteq \mathbb{C}$ where $\overline{\lambda} = \lambda$, find $\mathbf{c} = \begin{bmatrix} c_1 & \cdots & c_m \end{bmatrix}^T \in \mathbb{C}^m$ such that:

$$\sigma(A(\mathbf{c})) \subseteq \{\lambda_i\}_{i=1}^n$$

Remark:

- When $m = 1, c_1 = 1, A_1 = BKC$ and $\subseteq \to =$, LiPIEP $\Leftrightarrow$ OfPAP.
- When $\mathcal{M}$ is the set of symmetric, real, $n \times n$ matrices and $m = n$, we recover a variant of the LiPIEP, which we denote *LiPIEP2*.

Simple Existence Results for the (Li)PIEP [2]

Theorem 1 (Xu, 1998) [6]

Given a set of n complex numbers $\{\lambda_k\}_{k=1}^n$, then for almost all $\{A_i\}_{i=0}^n \subset \mathbb{C}^{n\times n}$, there exists $\mathbf{c} \in \mathbb{C}^n$ such that $\sigma(A(\mathbf{c})) = \{\lambda_k\}_{k=1}^n$. Furthermore, there are at most $n!$ distinct solutions.

Theorem 2 (Helton et al., 1997) [7]

For almost all $A_0 \in \mathbb{C}^{n\times n}$ and almost all $\{\lambda_k\}_{k=1}^n$, there is a $\mathbf{c} \in \mathbb{C}^n$ such that $\sigma(A(\mathbf{c})) = \{\lambda_k\}_{k=1}^n$ **if and only if** the following two conditions hold:

1. The matrices $A_1, \cdots, A_n$ are linearly independent; and,
2. $\mathrm{trace}(A_i) \neq 0$ *for some* $i = 1, 2, \cdots, n$.

Newton's Method for LiPIEP2

Recall: For differentiable $f : \mathbb{R} \to \mathbb{R}$, one Newton step is given as:

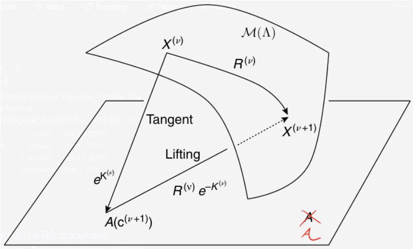
$$x^{(\nu+1)} = x^{(\nu)} - (f'(x^{(\nu)}))^{-1} f(x^{(\nu)}).$$

Idea: $f(x^{(\nu+1)})$ is a "lift" of the x-intercept of $f'(x^{(\nu)})$.

Given $\lambda = \{\lambda_k\}_{k=1}^{n}$, put

$$\mathcal{A} := \{A(\mathbf{c}) \,:\, \mathbf{c} \in \mathbb{R}^n\}$$
$$\mathcal{M}(\Lambda) := \left\{ Q\Lambda Q^T \,:\, Q \in \mathcal{O}(n) \right\}$$

where $\Lambda := \mathrm{diag}(\lambda)$.



Geometry of Newton's Method for PIEP [2]

Code Aside

Definitions & StIEP

Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$.

We say A is *non-negative* if $a_{ij} \geq 0$ for $1 \leq i, j \leq n$.

We say A is *(row) stochastic* if $\sum_{k=1}^{n} a_{ik} = 1$ for $i = 1, 2, \cdots, n$.

StIEP

Given $\lambda = \{\lambda_k\}_{k=1}^{n} \subseteq \mathbb{C}$ where $\overline{\lambda} = \lambda$, construct a (row) stochastic matrix $C \in \mathbb{R}^{n \times n}$ so that

$$\sigma(C) = \{\lambda_k\}_{k=1}^{n}$$

Restriction of the StIEP to Real Values

Real Stochastic IEP (RStIEP)

Given a set $\mathcal{S} = \{1\} \cup \{\lambda_i \in \mathbb{R} : -1 \leq \lambda_i \leq 1\}_{i=2}^{n}$, construct a stochastic matrix C such that $\sigma(C) = \mathcal{S}$

If $\lambda_1 = 1, \Lambda = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$, then we may formulate the constrained optimization problem as:

$$\begin{gathered} \text{minimize} \\ [2 \times \mathcal{J}(P,R)]^{1/2} := \left\| P\Lambda P^{-1} - R \odot R \right\| = \|\Gamma(P) - \Xi(R)\| = \|\Delta(P,R)\| \\ \text{subject to } P \in GL(\mathbb{R}, n) \end{gathered}$$

where $\odot$ denotes the *Hadamard* product.
With $[M, N] = MN - NM$ denoting the *Lie bracket*, the gradient is given as [2]:

$$\nabla \mathcal{J}(P,R) = \left(\left[\Delta(P,R), \Gamma(P)^T \right] P^{-T}, -2\Delta(P,R) \odot R \right)$$

Code Aside

Now for the good part . . .

Let $\lambda \in \mathbb{R}, a \in \mathbb{R}^+ - \{0\}$ such that $|\lambda| < a$.

Put

$$h_{\min} = \lambda/(a+\lambda) \text{ and } h_{\max} = a/(a+\lambda),$$

and pick $r \in \mathbb{R}$ satisfying $\max(h_{\min}, 0) \leq r \leq \min(h_{\max}, 1)$.

The *scalar splitting operator* [8] is defined to be

$$\widehat{S}(a, \lambda, r) := \frac{a}{h_{\max}} \begin{pmatrix} r & h_{\max} - r \\ r - h_{\min} & 1 - r \end{pmatrix}.$$

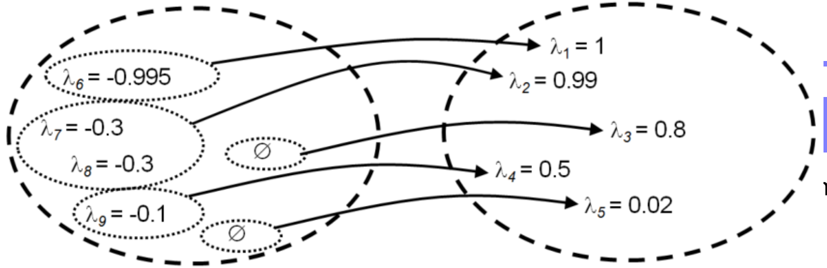
Note: Eigenvalues of $\widehat{S}(a, \lambda, r)$ are a and λ.
Denote the *state-splitting operator* by:

$$\widehat{S}_M(A, \lambda, r, k) := \left(\begin{array}{c|c|c} \mathbf{A}_{11} & \underline{c}_{1k}^T & \mathbf{A}_{12} \\ \hline \underline{r}_{k1} & a_{kk} & \underline{r}_{k2} \\ \hline \mathbf{A}_{21} & \underline{c}_{2k}^T & \mathbf{A}_{22} \end{array}\right) \xrightarrow[k]{\text{split at}} \left(\begin{array}{c|cc|c} \mathbf{A}_{11} & r\underline{c}_{1k}^T & (1-r)\underline{c}_{1k}^T & \mathbf{A}_{12} \\ \hline \underline{r}_{k1} & \multicolumn{2}{c|}{\multirow{2}{*}{$\widehat{S}(a_{kk}, \lambda, r)$}} & \underline{r}_{k2} \\ \underline{r}_{k1} & & & \underline{r}_{k2} \\ \hline \mathbf{A}_{21} & r\underline{c}_{2k}^T & (1-r)\underline{c}_{2k}^T & \mathbf{A}_{22} \end{array}\right)$$

RStIEP Existence Results

Theorem 3 (Suleĭmanova, 1949) [2]

Any n given real numbers $1, \lambda_2, \cdots, \lambda_n$ with $|\lambda_j| < 1$ are the spectrum of some $n \times n$ positive stochastic matrix if the sum of all $|\lambda_j|$ over those $\lambda_j < 0$ is less than 1. If the λ_j's are all negative the condition is al



Th

If

of

2. $\sum_{i=1}^{n} \lambda_i \geq 0$; and,
3. the $(n-p)$ negative values of λ can be grouped into p clusters $\{\mathcal{C}_\ell\}_{\ell=1}^{p}$ corresponding to a positive λ_ℓ such that $\left|\sum_{\gamma \in \mathcal{C}_\ell} \gamma\right| < \lambda_\ell$.

Code Aside

Questions?