HW 6 - ASTR404

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Q1)

Density in terms of radius

In[1]:= $s\rho := \rho[r\_] \to \rho_c \left(1 - \frac{r}{R_*}\right)$

Nuclear energy production rate

In[2]:= $s\epsilon := \epsilon[r\_] \to \text{If}\left[r < \frac{R_*}{5}, \epsilon_c \left(1 - \frac{r}{\frac{R_*}{5}}\right), 0\right]$

Total energy produced per second:

$$L = \int_0^{R_*} 4\,\pi\, r^2\, \rho(r)\, \epsilon(r)\, dr$$

Evaluating the integral:

In[3]:= $sL = L \to \text{Integrate}\left[4\,\pi\, r^2\, \epsilon[r]\, \rho[r] \,/.\, \{s\rho, s\epsilon\}, \{r, 0., R_*\}, \text{Assumptions} \to R_* > 0\right]$

Out[3]= $L \to 0.007372271\, \epsilon_c\, \rho_c\, R_*^3$

Q2)

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a)

Degeneracy is important if:

Fermi energy is greater than thermal energy.

In[4]:= $eqe = \frac{\left(\frac{3\, n\, h^3}{8\, \pi}\right)^{2/3}}{2\, m_e} > \frac{3}{2}\, k\, Tc;$

Number density at center given μ

In[5]:= $sn[\mu\_] := n \to \frac{\rho c}{\mu\, (m_p)};$

Scaling relation for central density

In[6]:=
```
sρc = ρc → ρcSun (M/M⊙)^(-2/7) /. ρcSun → 160000 kg/m^3;
```

Scaling relation for central temperature

In[7]:=
```
sT = Tc → TcSun (M/M⊙)^(4/7) /. TcSun → 1.57×10^7 K;
```

Substituting and finding range of M

In[8]:=
```
Quiet@Reduce[eqe //. {sn[1.17], sρc, sT}, M] /. m_Quantity :> UnitConvert[m, "SolarMass"]
```

Out[8]= $0\,M_\odot < M < 0.2412733\,M_\odot$

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b)

Coulomb interaction energy dominates if:

It is greater than the thermal energy.

In[9]:= $eqC = \frac{1.9\, e^2\, n^{1/3}}{4\pi\, \epsilon_0} > \frac{3}{2} k\, Tc;$

Substituting and finding range of M

In[10]:=
```
Quiet@Reduce[eqC && M > 0 //. {sn[1.29], sρc, sT}, M] /. m_Quantity :> UnitConvert[m, "SolarMass"]
```

Out[10]= $0\,M_\odot < M < 0.01347966\,M_\odot$

Q3)

Luminosity of a shell

The luminosity in a shell of mass dm is the nuclear energy generated - heat absorbed:

In[11]:=
```
eqL := dL → ϵ dm - T S′[t] dm
```

Rate of change in entropy

In[12]:= $eqS = S'[t] \to \partial_t\left(\frac{k\,\mathrm{Log}\left[\frac{P[t]}{\rho[t]^{\gamma}}\right]}{\mu\, m_H\,(\gamma-1)} + const\right);$

Substituting this in the first equation

In[13]:= `sL = eqL /. eqS`

Out[13]= $dL \to dm\,\epsilon - \left(dm\,k\,T\,\rho[t]^{\gamma}\left(\rho[t]^{-\gamma}P'[t] - \gamma P[t]\,\rho[t]^{-1-\gamma}\rho'[t]\right)\right) \Big/ \left(\left(-1+\gamma\right)\mu\,P[t]\,m_H\right)$

Checking whether this is equivalent to the given equation

In[14]:= $\frac{dL}{dm} == \epsilon - \frac{\rho[t]^{-1+\gamma}}{-1+\gamma}\partial_t\left(\frac{P[t]}{\rho[t]^{\gamma}}\right)$ /. sL /. $P \to \left(\frac{k\,T\,\rho[\#]}{\mu\,m_H}\,\&\right)$ // Simplify

Out[14]= True

QED.

Q4)

a)

Final density of bubble

In[15]:= `ρf = Series[ρ[r + dr], {dr, 0, 1}]`

Out[15]= $\rho[r] + \rho'[r]\,dr + O[dr]^2$

Gravitational force per unit volume on the bubble

In[16]:= `Fg = ρf g`

Out[16]= $g\,\rho[r] + g\,\rho'[r]\,dr + O[dr]^2$

b)

Pressure of gas at final position of bubble

In[17]:= `ρgf = Series[ρg[r + dr], {dr, 0, 1}]`

Out[17]= $\rho g[r] + \rho g'[r]\,dr + O[dr]^2$

Entropy is conserved for adiabatic process

In[18]:= `eqEn = D[P[r] / ρg[r]^γ, r] == 0`

Out[18]= $\rho g[r]^{-\gamma}P'[r] - \gamma P[r]\,\rho g[r]^{-1-\gamma}\rho g'[r] == 0$

Solving above for ρ_g'(r)

In[19]:= `sρg = Solve[eqEn, ρg'[r]][[1, 1]]`

Out[19]= $\rho g'[r] \to \frac{\rho g[r]\,P'[r]}{\gamma P[r]}$

Buoyant force per unit volume on the bubble

In[20]:=
```
Fb = ρgf g /. sρg
```

Out[20]= $g\,\rho g[r] + \frac{g\,\rho g[r]\,P'[r]\,dr}{\gamma\,P[r]} + O[dr]^2$

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c)

Net force up to $O(dr^2)$

Subtracting buoyant force from gravitational force and using Newton's second law:

In[21]:=
```
eqF = ρf r''[t] == Fb - Fg /. ρg[r] → ρ[r] // Normal
```

Out[21]= $\rho[r]\,r''[t] + dr\,\rho'[r]\,r''[t] == dr\left(\frac{g\,\rho[r]\,P'[r]}{\gamma\,P[r]} - g\,\rho'[r]\right)$

Solving for r''(t)

In[22]:=
```
sr = Solve[eqF, r''[t]][[1, 1]]
```

Out[22]= $r''[t] \to \frac{dr\,(g\,\rho[r]\,P'[r] - g\,\gamma\,P[r]\,\rho'[r])}{\gamma\,P[r]\,(\rho[r] + dr\,\rho'[r])}$

Expanding up to $O(dr^2)$

In[23]:=
```
sr[[1]] -> Series[sr[[2]], {dr, 0, 1}]
```

Out[23]= $r''[t] \to \left(\frac{g\,P'[r]}{\gamma\,P[r]} - \frac{g\,\rho'[r]}{\rho[r]}\right) dr + O[dr]^2$

This is equivalent to the given equation, where N^2 is:

$$N^2 \to -g\left(\frac{P'(r)}{\gamma\,P(r)} - \frac{\rho'(r)}{\rho(r)}\right)$$

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d)

Entropy equation

In[24]:=
```
sS := S → P[r] ρ[r]^-γ
```

Rate of change in entropy with radius

In[25]:= $\frac{\rho[r]^\gamma\,S'[r]}{\gamma\,P[r]} == \left(\frac{\rho[r]^\gamma\,\partial_r\,(S\ /.\ sS)}{\gamma\,P[r]}\ //\ \text{Simplify}\right)$

Out[25]= $\frac{\rho[r]^\gamma\,S'[r]}{\gamma\,P[r]} == \frac{P'[r]}{\gamma\,P[r]} - \frac{\rho'[r]}{\rho[r]}$

Since g, P, ρ, and γ are positive, we have that N^2 is positive only if S'(r) is negative.

Q5)

Formula for Eddington luminosity

In[26]:= fEdd[M_, κ_] := UnitConvert[$\frac{4\pi M\ c\ G}{\kappa}$, "SolarLuminosity"]

a) Star with mass 0.072 $M_\odot$

Eddington luminosity

In[27]:= fEdd[0.072 $M_\odot$, 0.001 m^2/kg]

Out[27]= 94 061.28 $L_\odot$

Actual luminosity

In[28]:= 10^-4.3 $L_\odot$

Out[28]= 0.00005011872 $L_\odot$

This is much lesser than the Eddington luminosity; therefore the radiation pressure is not significant for such stars.

b) Star with mass 120 $M_\odot$

Eddington luminosity

In[29]:= fEdd[120 $M_\odot$, 0.04 m^2/kg]

Out[29]= $3.91922 \times 10^6\ L_\odot$

Actual luminosity

In[30]:= 10^6.252 $L_\odot$

Out[30]= $1.786488 \times 10^6\ L_\odot$

This is comparable to the Eddington luminosity. Therefore the radiation pressure is significant for such stars.