FinalExam - ASTR404

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Q1

a)

In[276]:=
```
5. / 100 × 4 × 10^-9 10 M☉ per year
```

Out[276]= $2.\times 10^{-9}\ M_\odot/\mathrm{yr}$

b)

The cool way:

In[497]:=
```
FormulaData["LorentzFactor", {"γ" -> 10.}]
```

Out[497]= $v == -2.9829\times 10^{8}\ \mathrm{m/s}\ ||\ v == 2.9829\times 10^{8}\ \mathrm{m/s}$

The boring way

In[540]:=
```
sv = NSolve[10 == 1 / Sqrt[1 - v^2 / c^2], v][[1, 1]]
```

Out[540]= $v \to 2.9829\times 10^{8}\ \mathrm{m/s}$

Fraction of light speed

In[541]:=
```
v / c /. sv
```

Out[541]= 0.994987

c)

Assuming a photon at 90°,

In[369]:=
```
Quiet@Solve[Tan[θ] == c / (γ v) /. γ → 10 /. sv, θ][[1, 1, 2]] / Degree "°"
```

Out[369]= 5.73917 °

d)

Combining Wien’s law and gravitational redshift formula

In[303]:= $\frac{\lambda\infty}{\lambda} == \frac{T}{T\infty} == \frac{1}{\sqrt{1-\frac{2GM}{c^2 R}}}$ /. $R \to \frac{6GM}{c^2}$

Out[303]= $\frac{\lambda\infty}{\lambda} == \frac{T}{T\infty} == \sqrt{\frac{3}{2}}$

Solving for T∞ at given T

In[307]:= $T\infty \to \sqrt{\frac{2}{3}}\ 3. \times 10^7\ \mathrm{K}$

Out[307]= $T\infty \to 2.44949 \times 10^7\ \mathrm{K}$

---

e)

Total KE

In[290]:= sKE = KE -> electronKE + protonKE /. {electronKE → (γ - 1) me c^2, protonKE → (γ - 1) mp c^2}

Out[290]= $KE \to c^2\ me\ (-1+\gamma) + c^2\ mp\ (-1+\gamma)$

Fraction of electron's KE

In[293]:= electronKE / KE /. electronKE → (γ - 1) me c^2 /. sKE // Simplify

Out[293]= $\frac{me}{me + mp}$

Q2

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a)

In[330]:= sR = Last@Solve[0.68 $M_\odot$ / 1 $M_\odot$ (R / sun radius)^3 == 1.98 × 10^-6, R]

Out[330]= $\{R \to 9.93438 \times 10^6\ \mathrm{m}\}$

b)

In[332]:=
```
sL = First@Solve[6220 K == 4. × 10^7 K (L / 1 L_N^⊙ / (0.68 M_⊙ / 1 M_⊙))^(2/7), L]
```
Out[332]= $\{L \to 1.22688 \times 10^{13}\ \mathrm{W}\}$

c)

Finding cooling age using Mestel model:

In[444]:=
```
st = t → QuantityMagnitude@UnitConvert[8.6 × 10^10 0.68 M_⊙ ^(5/7) L^(-5/7) /. sL] s
```
Out[444]= $t \to 1.28089 \times 10^{23}\ \mathrm{s}$

Time in years:

In[445]:=
```
UnitConvert[t /. st, "Years"]
```
Out[445]= $4.06168 \times 10^{15}\ \mathrm{yr}$

The Mestel model assumes that the gas goes from fully degenerate, isothermal, conductive interior to fully nondegenerate, ideal gas at some radius. In reality this transition is more gradual, therefore our cooling age estimate is wrong.

Q3

a)

In[362]:=
```
UnitConvert[Solve[3 × 10^39 J == .007 m c^2, m][[1, 1, 2]], "SolarMass"]
```
Out[362]= $2.39812 \times 10^{-6}\ M_\odot$

b)

Radius estimate

In[363]:=
```
sR = R -> UnitConvert[
   Solve[1.37 M_⊙ / 1 M_⊙ (R / sun radius)^3 == 1.98 × 10^-6, R][[-1, 1, 2]], "SolarRadius"]
```
Out[363]= $R \to 0.0113094\ R_\odot$

Correcting for relativistic effects

In[364]:=
```
sR = R → sR[[2]] / 4
```
Out[364]= R → 0.00282735 $R_\odot$

---

c)

Net force on atmosphere

In[365]:=
```
sF = F -> G Mwd Matm / R^2
```
Out[365]= $F \to \frac{\text{Matm Mwd } (1\, G)}{R^2}$

Pressure at base

In[366]:=
```
sP = P -> F / A /. sF /. A → 4 π R^2
```
Out[366]= $P \to \frac{\text{Matm Mwd } \left(\frac{1}{4\pi} G\right)}{R^4}$

Mass of atmosphere

In[446]:=
```
sM = Matm ->
   UnitConvert[Solve[10^19 N/m^2 == P /. sP /. sR /. Mwd -> 1.37 M☉ , Matm][[1, 1, 2]], "SolarMass"]
```
Out[446]= Matm → 5.19736×10^{-6} $M_\odot$

---

d)

Taking ratio of accreted mass to time between outbursts:

In[452]:=
```
sMd = MwdDot -> Matm / 10 yr /. sM
```
Out[452]= MwdDot → 5.19736×10^{-7} $M_\odot$/yr

Using formula from HW 12:

In[455]:=
```
Solve[Pdot / P == 3 MwdDot (Mwd - Ms) / (Mwd Ms) /. sMd /.
   {Mwd -> 1.37 M☉ , Ms -> 1.5 M☉ , P -> 1.2306 days }, Pdot]
```
Out[455]= {{Pdot → -3.32552×10^{-10}}}

The period is decreasing, therefore the orbit is contracting (from Kepler's 3rd law).

Q4

a)

```
In[456]:= sv = First@NSolve[ 4 × 10^10 m / v - 4 × 10^10 m / c == 3 h , v]

Out[456]= {v → 3.65851 × 10^6 m/s }
```

b)

Using virial theorem and equating initial energy of envelope to final energy

```
In[467]:= UnitConvert[
  Solve[1 / 100 5. × 10^46 J + M PE / 2 == 1 / 2 M v^2 /. PE → -10^13 J/kg /. sv, M][[1, 1, 2]], "SolarMass"]

Out[467]= 21.5059 M☉
```

c)

```
In[474]:= UnitConvert[NSolve[{2 R Sin[42 °] / c == 360 days , 1.66" == 2 R / d}, {d, R}][[1, 2, 2]], "Kpc"]

Out[474]= 56.1168 kpc
```

Q5

a)

Someday all of science will be inside FormulaData:

```
In[477]:= FormulaData["EscapeVelocity", {"m" -> 19.2 M☉ , "r" -> 16.2 R☉ }]

Out[477]= v == 672 529. m/s
```

Actually doing it:

```
In[480]:= sv = v -> Sqrt[((2 G) 19.2 M☉) / (16.2 R☉)] // UnitConvert

Out[480]= v → 672 529. m/s
```

b)

Distance at which wind velocity is escape velocity

In[482]:= $sd = \text{Solve}\left[v == \sqrt{\frac{(2\,G)\ 14.8\,M_{\odot}}{d}}\ /.\ sv,\ d\right][[1, 1]]$

Out[482]= $d \to 8.68506 \times 10^{9}\ \text{m}$

Orbital separation

Using Kepler's 3rd law:

In[494]:= $sa = \text{Solve}\left[5.599829\ \text{days}\ ^{\wedge}2 == 4\,\pi^{\wedge}2 \big/ \left(G\left(14.8\,M_{\odot} + 19.2\,M_{\odot}\right)\right) a^{\wedge}3,\ a\right][[-1, 1]]$

Out[494]= $a \to 2.99088 \times 10^{10}\ \text{m}$

Roche lobe

Using Eggleton approximation:

In[493]:= $srL = \text{Solve}\left[rL / a == .49\,q^{\wedge}(2/3) \big/ \left(.6\,q^{\wedge}(2/3) + \text{Log}\left[1 + q^{\wedge}(1/3)\right]\right)\ /.\ q \to 14.8\,M_{\odot} \big/ 19.2\,M_{\odot}\ /.\ sa,\ rL\right][[1, 1]]$

Out[493]= $rL \to 1.06661 \times 10^{10}\ \text{m}$

Ratio

In[495]:= $d / rL\ /.\ sd\ /.\ srL$

Out[495]= 0.814266

c)

Angle θ subtended to tangent of sphere with radius d centered at the black-hole:

In[509]:= $s\theta = \text{Quiet@Solve}\left[\text{Sin}[\theta] == d/a\ /.\ sd\ /.\ sa,\ \theta\right][[1, 1]]$

Out[509]= $\theta \to 0.294628$

Solid angle for falling into sphere:

In[511]:= $\Omega = 2\,\pi\left(1 - \text{Cos}[\theta]\right)\ /.\ s\theta$

Out[511]= 0.270742

Accretion rate of black-hole is probability of falling into sphere times wind rate

In[526]:=
```
accRate = Ω / (4 π) 2.5/10^6 M☉/yr
```
Out[526]= $5.38623 \times 10^{-8}\ M_\odot/\mathrm{yr}$

---

d)

Why even bother...

In[519]:=
```
FormulaData[{"BlackHoleEventHorizonRadius", "Standard"}, {"M" -> 14.8 M☉ }]
```
Out[519]= r == 43 707.2 m

Fine I'll do it.

In[520]:=
```
rSch = UnitConvert[(2 G/c^2) 14.8 M☉ ]
```
Out[520]= 43 707.2 m

This is much smaller than the wind capture radius d.

Angle subtended at Schwarzschild radius:

In[521]:=
```
sθ2 = Quiet@Solve[Sin[θ] == rSch / a /. sd /. sa, θ][[1, 1]]
```
Out[521]= $\theta \to 1.46135 \times 10^{-6}$

Accretion rate at Schwarzschild radius:

In[523]:=
```
2.5/10^6 M☉/yr 2 π (1 - Cos[θ]) / (4 π) /. sθ2
```
Out[523]= $1.33477 \times 10^{-18}\ M_\odot/\mathrm{yr}$

---

e)

In[531]:=
```
L → G M Mdot / R /. {R → 3 rSch, Mdot → accRate, M -> 14.8 M☉ } // UnitConvert
```
Out[531]= $L \to 5.08721 \times 10^{31}\ \mathrm{kg\,m^2/s^3}$

In[535]:=
```
LEdd → 4 π G 14.8 M☉ c / κ /. κ → .2 (1 + .5) cm^2/g // UnitConvert
```
Out[535]= $\mathrm{LEdd} \to 2.46646 \times 10^{32}\ \mathrm{kg\,m^2/s^3}$

The Eddington luminosity is about 4 times larger.

Yeah, it's a a feasible explanation since the value we derived is close to the actual value shown below and still less than Eddington luminosity.

In[537]:= Cygnus X–1 luminosity

Out[537]= $1.3 \times 10^{32}\ \mathrm{W}$

Whoops. Ran out of time! Don’t want to lose 40% again.