HW 9 - ASTR540

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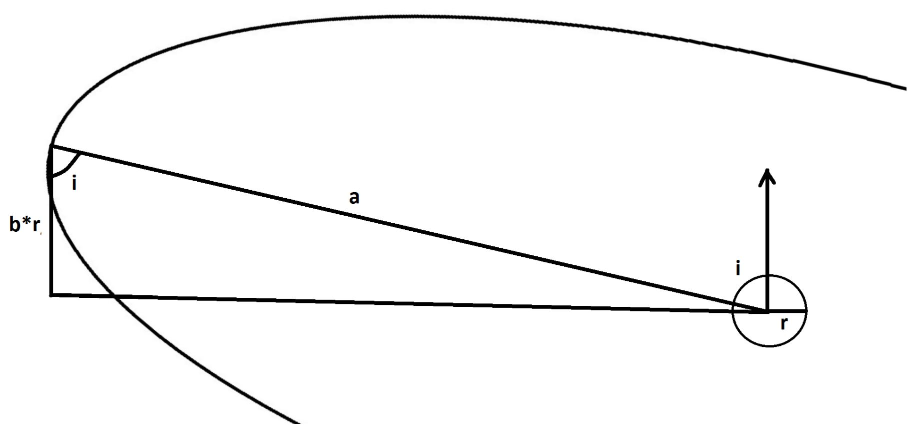
Q1)

In[61]:=



a)

Impact parameter

From the figure cos(i) = b r /a

Time taken

Dividing width of chord by velocity:

In[1]:= `2 Sqrt[r^2 - b^2 r^2] / (2 π a / τ)`

Out[1]= $$\frac{\sqrt{r^2 - b^2 r^2}\ \tau}{a\ \pi}$$

This is equivalent to the given equation.

b)

Transit criterion

Distance to projection on plane of sky is less than sum of radii: a cos(i) < rS+rP

Probability for eclipse

Taking ratio of solid angles:

In[2]:=
```
2 π Integrate[Sin[θ], {θ, ArcCos[(rS + rP) / a], π - ArcCos[(rS + rP) / a]}] / (4 π)
```

Out[2]= $\frac{rP + rS}{a}$

---

c)

Transition depth

Approximately .00025 from figure.

Radius of star relative to Earth’s radius

In[3]:=
```
rP1 = UnitConvert[Solve[.00025 == (rP / rS) ^2 /. rS → 0.91 R☉, rP][[2, 1, 2]], "EarthMeanRadius"]
```

Out[3]= $1.571177\ R_1$

Orbital period

From the figure the orbital period is approximately 4.5 days.

Separation

Using Kepler’s 3rd law:

In[4]:=
```
a1 = UnitConvert[(4.5 days ^2 G 0.91 M☉ / (4 π^2))^(1/3), "AU"]
```

Out[4]= 0.05169163 au

Transit time

From the figure, the duration of the transit is .03 days

Duration of flat

From the figure, the duration of the flat part is .02 days

Impact parameter

Using the formula in Q2 b):

In[5]:=
$$b1 = \frac{1}{(t_{Flat} - t_{Total})^2} (t_{Flat}^2 - t_{Flat}^2\, tDepth - 2\, t_{Flat}\, t_{Total} - 2\, t_{Flat}\, tDepth\, t_{Total} + t_{Total}^2 - tDepth\, t_{Total}^2) \;/.\; \{tDepth \to .00025,\ t_{Flat} \to 0.02\ \text{days},\ t_{Total} \to 0.03\ \text{days}\}$$

Out[5]= 0.99375

Inclination angle

```
In[6]:= ArcCos[b1 0.92 R☉N / a1] / Degree
Out[6]= 85.28203
```

Mass of planet

```
In[7]:= mP1 = UnitConvert[
          Solve[mP vP == mS vS /. {vP → 2 π a1 / 4.5 days , vS →  1.6 m/s , mS -> 0.91 M☉ }, mP][[1, 1, 2]],
          "EarthMass"]
Out[7]= 3.879174 M⊕
```

Density of the planet

```
In[8]:= UnitConvert[mP1 / (4 / 3 π rP1^3), g/cm³ ]
Out[8]= 5.514228 g/cm³
```

Temperature of planet

Assuming the planet is a perfect black-body:

```
In[9]:= NSolve[4 π r^2 σ T^4 / (4 π a^2) π rP^2 (1 - A) == 4 π rP^2 σ Tp^4 /.
          {rP → rP1, T -> 5700 K , a → a1, r -> 0.92 R☉N , A → .3}, Tp][[-1, 1]]
Out[9]= Tp → 1060.636 K
```

Q2)

a)

Solving for upper limit of ρ

In[10]:=
$$\text{Solve}\left[t_{dur} == \frac{r\,\tau}{\pi\, a} \;/.\; a \to (\tau^2\, G\, \rho\, 4/3\, \pi\, r^3 / (4 \pi^2))^{(1/3)},\ \rho\right][[1, 1]]$$

Out[10]=
$$\rho \to \frac{3\,\tau}{G\,\pi^2\, t_{dur}^3}$$

b)

t_{Total}

Let v and rP be the velocity and radius of the planet, then the total time of contact is the time taken to travel a distance of the chord plus twice the radius of the planet. Also let b2 = b^2, then:

In[11]:=
```
eqT := tTotal == v (2 Sqrt[r^2 - r^2 b2] + 2 rP)
```

t_{Flat}

The time of the flat part is the time taken to travel the distance of the chord minus twice the radius of the planet.

In[12]:=
```
eqF := tFlat == v (2 Sqrt[r^2 - r^2 b2] - 2 rP)
```

Transit depth (tDepth)

In[13]:=
```
srP = Solve[tDepth == (rP / r) ^2, rP] [[2, 1]]
```

Out[13]= $rP \to r\sqrt{tDepth}$

Solving for b^2

In[14]:=
```
sb2 = Solve[{eqF, eqT} /. srP, {b2, v}] [[1, 1]]
```

Out[14]= $b2 \to \left(t_{Flat}^2 - tDepth\, t_{Flat}^2 - 2\, t_{Flat}\, t_{Total} - 2\, tDepth\, t_{Flat}\, t_{Total} + t_{Total}^2 - tDepth\, t_{Total}^2\right) / \left(t_{Flat} - t_{Total}\right)^2$

Solving for ρ and substituting above

In[15]:=
```
Solve[tdur == r Sqrt[1 - b2] τ / (π a) /. a → (τ^2 G ρ 4 / 3 π r^3 / (4 π^2)) ^ (1 / 3), ρ] [[1, 1]] /. sb2 //
  FullSimplify // PowerExpand // FullSimplify
```

Out[15]= $$\rho \to \frac{3\, tDepth^{3/2}\, \tau \left(t_{Flat} + t_{Total}\right)^3}{G\, \pi^2\, t_{dur}^3 \left(t_{Flat} - t_{Total}\right)^3}$$

Thus ρ can be found exactly.

Q3)

Einstein radius formula

In[16]:=
```
dE = Sqrt[4 G M / c^2 d / (d 2 d)]
```

Out[16]= $\sqrt{\frac{M\left(2\,G/c^2\right)}{d}}$

Equating to solar radius and solving for d

In[17]:=
```
UnitConvert[NSolve[sun radius / d == dE /. M -> sun mass , d][[1, 1, 2]], "Parsecs"]
```

Out[17]= 0.005312 pc

Q4)

a)

Electrostatic energy density

PE of an atom (due to 6 neighbors) divided by volume of a cube of side r:

In[18]:=
```
EED = 6 / (4 π ε0) 1 e ^2 / r / r^3
```

Out[18]= $\frac{\frac{3}{2\pi}\,e^2/\varepsilon_0}{r^4}$

This is true because on average there is one atom per cube.

b)

Radius of the planet

Equating the number of atoms and solving for R:

In[19]:=
```
sR = NSolve[M / mp == 4 / 3 π R^3 / r^3, R][[-1, 1]]
```

Out[19]= $R \to 5.226022\times10^{8}\ r\left(M\left(1/\mathrm{kg}\right)\right)^{1/3}$

Gravitational energy density

Substituting for R from above:

In[20]:=
```
GED = G M^2 / R / (4 / 3 π R^3) /. sR
```

Out[20]=

$$\frac{M^2 \left(3.200564 \times 10^{-36}\, G\right)}{r^4 \left(M \left(1/\text{kg}\right)\right)^{4/3}}$$

---

c)

Equating the two and solving for M

In[21]:=
```
M -> Solve[EED == GED, M][[1, 1, 2]] / 0.001 M☉ "MJupiter"
```

Out[21]= $M \to 8.296249\, M_{\text{Jupiter}}$

The dependence on r cancels out because both forces follow the same power law.

Q5)

---

a)

Kinetic energy

Equal to negative of PE due to Sun and Earth

In[22]:=
```
kE = G 4 / 3 π 4 km ^3 [ice density]
  ([sun mass] / [sun distance] + [earth mass] / [earth radius]) // UnitConvert
```

Out[22]= $2.360364 \times 10^{23}\ \text{kg}\,\text{m}^2/\text{s}^2$

---

b)

Radius of crater

Equating thermal energy to KE and solving for radius assuming ideal gas law:

In[23]:=
```
NSolve[2 g/cm^3 π r^2 10 km / 30 m_p 3 / 2 k 3500 K == kE, r][[2, 1]]
```

Out[23]= $r \to 50\,996.2\ \text{m}$

c)

Number of impacts on Earth

Taking ratio of cross-sections:

```
In[24]:= nI = 10 * [earth radius]^2 / [moon radius]^2
Out[24]= 134.45
```

Mean time interval between impacts

```
In[25]:= 3000 Myr / nI
Out[25]= 22.313 Myr
```

d)

Probability density function

Assuming uncertainty of 0.1 meter with dN/dm = const*m^-3:

```
In[26]:= sc = NSolve[const (4/3 π 4 km^3 [ice density])^-3 (4 π 4 km^2 [ice density] [.1m]) == nI, const][[1, 1]]
Out[26]= const → 1.057759 × 10^32 kg^2
```

Total mass of comets

```
In[27]:= Integrate[4 π r^2 [ice density] const (4/3 π r^3 [ice density])^-2 /. sc, {r, 0.2 km, 4 km}]
Out[27]= 3.442166 × 10^21 kg
```

Total mass of water on Earth

```
In[28]:= [surface area earth] * .71 * 3.7 km * [water density] // UnitConvert
Out[28]= 1.339942 × 10^21 kg
```

Therefore the total mass of comets landing on earth is comparable to the amount of water.