# HW 1 - Astrophysical Dynamics

## Problem 1

Renaming variables for convinience: $\theta$ -> o, D -> d, $\omega$ -> w

In[1]:=
```
x = l Sin[o[t]]
```
Out[1]=
```
l Sin[o[t]]
```

In[2]:=
```
y = d Cos[w t] - l Cos[o[t]]
```
Out[2]=
```
d Cos[t w] - l Cos[o[t]]
```

---

### a) Hamiltonian mechanics

The potential energy PE is given by:

In[3]:=
```
PE = -m g y
```
Out[3]=
```
-g m (d Cos[t w] - l Cos[o[t]])
```

The kinetic energy KE is given by:

In[4]:=
```
KE = P^2 / (2 m)
```
Out[4]=
$$\frac{P^2}{2\,m}$$

The Hamiltonian H is given by:

In[5]:=
```
H = KE + PE
```
Out[5]=
$$\frac{P^2}{2\,m} - g\,m\,(d\,\text{Cos}[t\,w] - l\,\text{Cos}[o[t]])$$

---

### b)

In[6]:=
```
KE = 1/2 m (D[x, t]^2 + D[y, t]^2)
```
Out[6]=
$$\frac{1}{2}\,m\left(l^2\,\text{Cos}[o[t]]^2\,o'[t]^2 + (-d\,w\,\text{Sin}[t\,w] + l\,\text{Sin}[o[t]]\,o'[t])^2\right)$$

Let L be the lagrangian. Then:

In[7]:=
```
L = KE - PE
```
Out[7]=
$$g\,m\,(d\,\text{Cos}[t\,w] - l\,\text{Cos}[o[t]]) + \frac{1}{2}\,m\left(l^2\,\text{Cos}[o[t]]^2\,o'[t]^2 + (-d\,w\,\text{Sin}[t\,w] + l\,\text{Sin}[o[t]]\,o'[t])^2\right)$$

Momentum p is given by:

In[8]:=
```
p = Simplify[D[L, D[o[t], t]]]
```

Out[8]=
```
l m (-d w Sin[t w] Sin[o[t]] + l o'[t])
```

Hamiltonian H is given by:

In[9]:=
```
H = D[o[t], t] D[L, D[o[t], t]] - L
```

Out[9]=

$$-g\,m\,(d\,\mathrm{Cos}[t\,w] - l\,\mathrm{Cos}[o[t]]) + \frac{1}{2}\,m\,o'[t]\,\left(2\,l^2\,\mathrm{Cos}[o[t]]^2\,o'[t] + 2\,l\,\mathrm{Sin}[o[t]]\,(-d\,w\,\mathrm{Sin}[t\,w] + l\,\mathrm{Sin}[o[t]]\,o'[t])\right) - \frac{1}{2}\,m\,\left(l^2\,\mathrm{Cos}[o[t]]^2\,o'[t]^2 + (-d\,w\,\mathrm{Sin}[t\,w] + l\,\mathrm{Sin}[o[t]]\,o'[t])^2\right)$$

The Euler-Lagrange equation gives:

In[10]:=
```
EQ = Simplify[D[D[L, D[o[t], t]], t] - D[L, o[t]] == 0]
```

Out[10]=

$$l\,m\,\left(-\left(g + d\,w^2\,\mathrm{Cos}[t\,w]\right)\,\mathrm{Sin}[o[t]] + l\,o''[t]\right) == 0$$

Solving the EL equation for o''[t] we get:

In[11]:=
```
S = Solve[EQ, D[o[t], {t, 2}]][[1]]
```

Out[11]=

$$\left\{o''[t] \to \frac{\left(g + d\,w^2\,\mathrm{Cos}[t\,w]\right)\,\mathrm{Sin}[o[t]]}{l}\right\}$$

In[12]:=
```
LHS = o''[t] /. S[[1]]
```

Out[12]=

$$\frac{\left(g + d\,w^2\,\mathrm{Cos}[t\,w]\right)\,\mathrm{Sin}[o[t]]}{l}$$

---

# c) Numerical solution

Substituting values g=1 and l=1

In[13]:=
```
eq1 = o''[t] == LHS /. {g → 1, l → 1}
```

Out[13]=

$$o''[t] == \left(1 + d\,w^2\,\mathrm{Cos}[t\,w]\right)\,\mathrm{Sin}[o[t]]$$

NUmerically solving this system using the inbuilt ODE solver in Mathemtica:

In[14]:=
```
sol = ParametricNDSolveValue[{eq1, o[0] == 0.01, o'[0] == 0}, o, {t, 0, 5}, {d, w}]
```

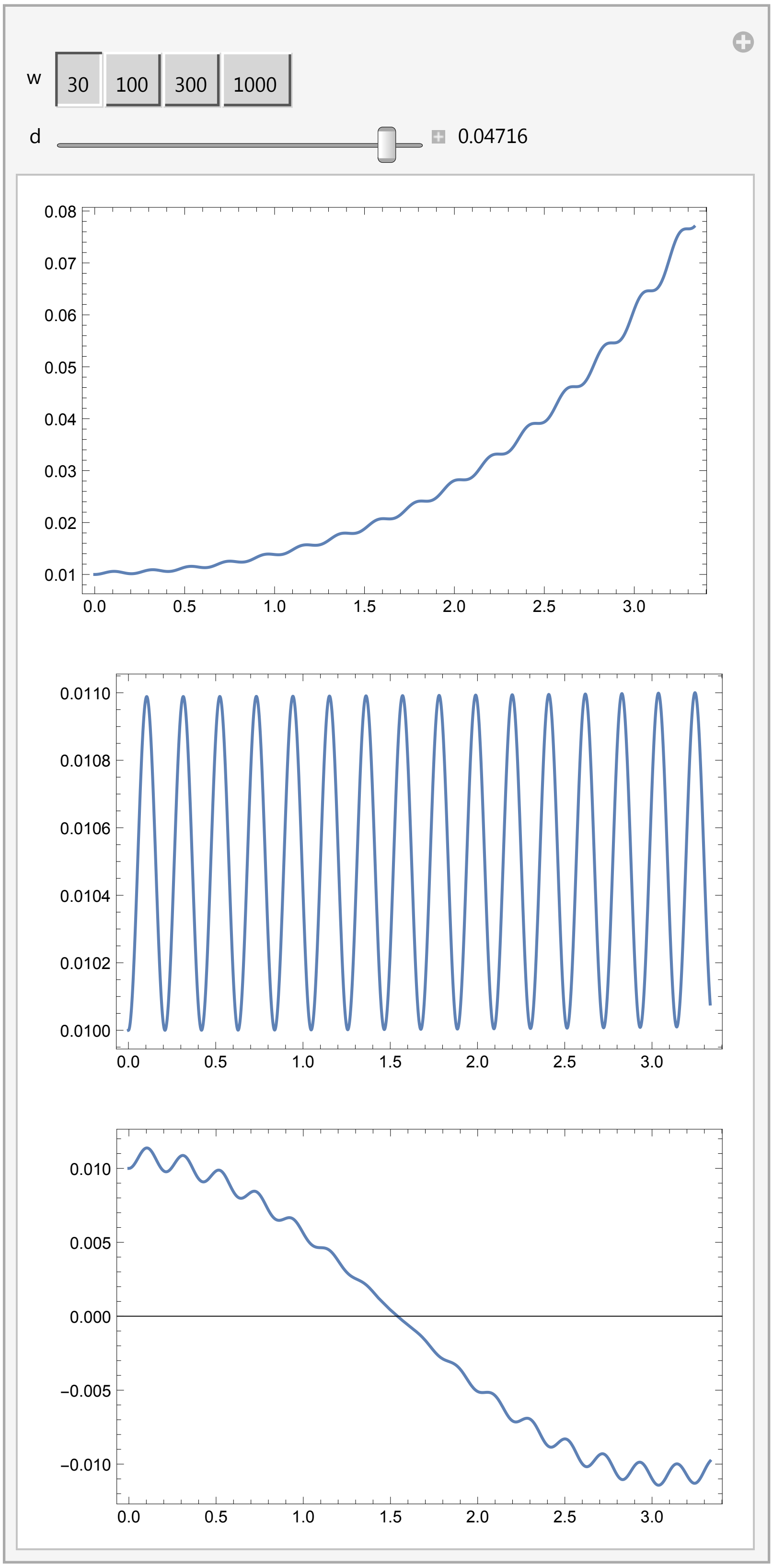
Out[14]=
```
ParametricFunction[ Expression: o
                    Parameters: {d, w} ]
```

# Plots

## w=30, d= (0.02716, 0.04716, 0.06716)

In[15]:=
```
Manipulate[
  Row[Plot[#, {t, 0, 100 / w}, PlotRange → Full, Frame → True, ImageSize → Medium] & /@
    Evaluate[Table[sol[d + i, w][t], {i, {-0.02, 0, 0.02}}]]],
  {w, {30, 100, 300, 1000}}, {{d, 0.04716}, 0, .05, .0001, Appearance → "Labeled"}]
```

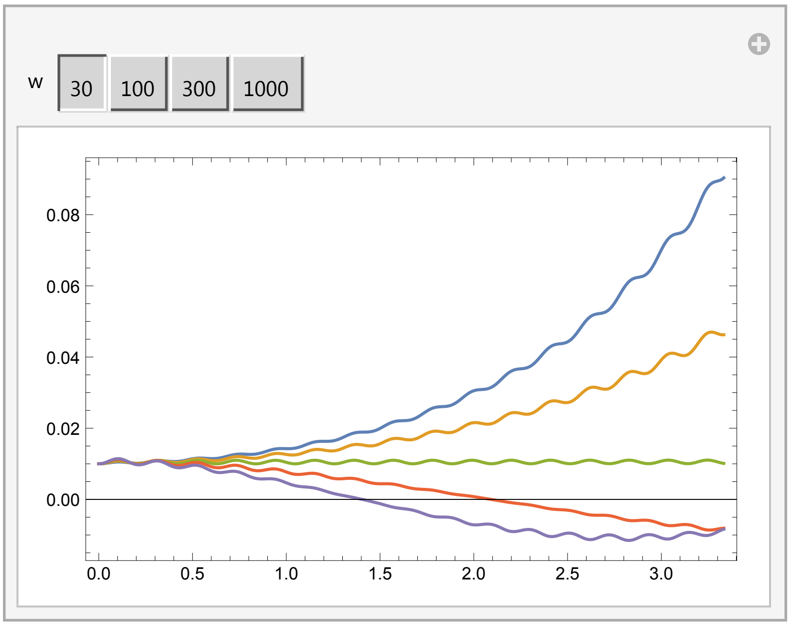
Out[15]=




In[16]:= **w = 30, d = (0.02357, 0.035355, 0.047140, 0.05892, 0.07071)**

In[16]:=
```
Manipulate[Plot[
  Evaluate[Table[sol[d, w][t], {d, .5 Sqrt[2] / w, 1.5 Sqrt[2] / w, .25 Sqrt[2] / w}]],
  {t, 0, 100 / w}, PlotRange → All, Frame → True,
  ImageSize → Medium], {{w, 30}, {30, 100, 300, 1000}}]
```

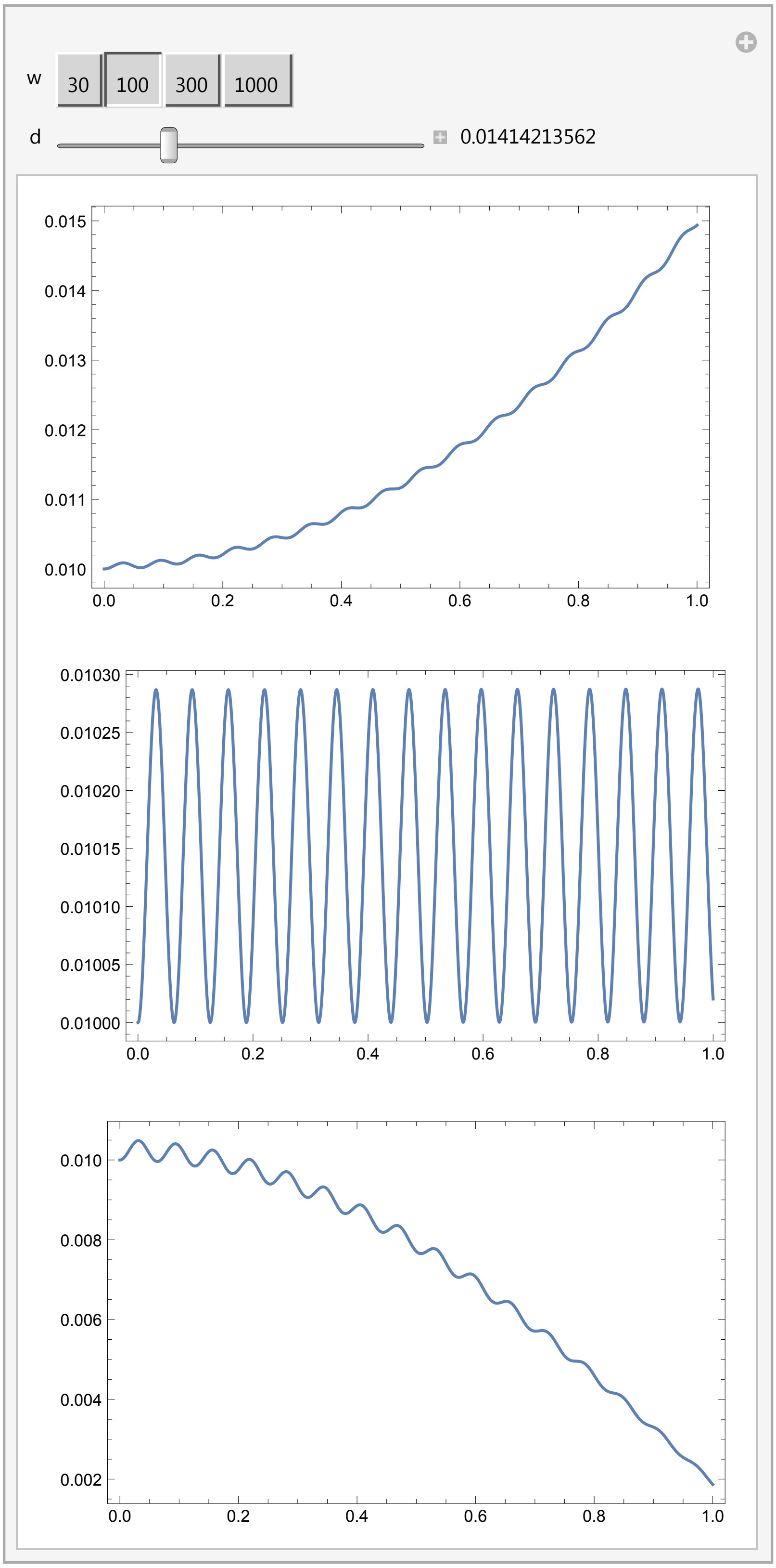
Out[16]=




w=100, d= 0.00414, 0.01414, 0.02414

In[17]:=
```
Manipulate[
  Row[Plot[#, {t, 0, 100 / w}, PlotRange → Full, Frame → True, ImageSize → Medium] & /@
    Evaluate[{sol[d - 0.01, w][t], sol[d, w][t], sol[d + 0.01, w][t]}]],
  {{w, 100}, {30, 100, 300, 1000}}, {{d, Sqrt[2.0] / w}, 0,
   .05, .0001, Appearance → "Labeled"}]
```

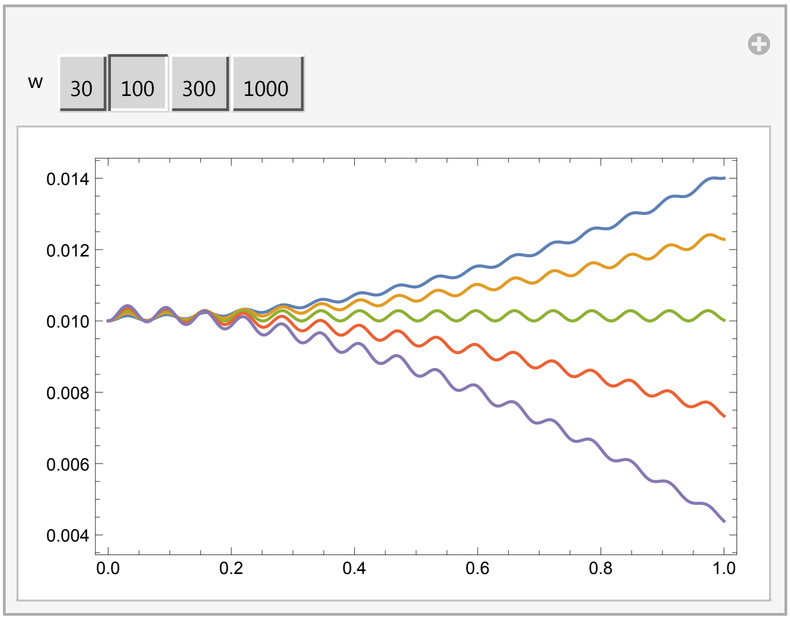
Out[17]=




In[18]:= **w = 100, d = (0.0070710, 0.010606, 0.014142, 0.017677, 0.021213)**

In[18]:=
```
Manipulate[Plot[
  Evaluate[Table[sol[d, w][t], {d, .5 Sqrt[2] / w, 1.5 Sqrt[2] / w, .25 Sqrt[2] / w}]],
  {t, 0, 100 / w}, PlotRange → All, Frame → True, ImageSize → Medium],
 {{w, 100}, {30, 100, 300, 1000}}]
```

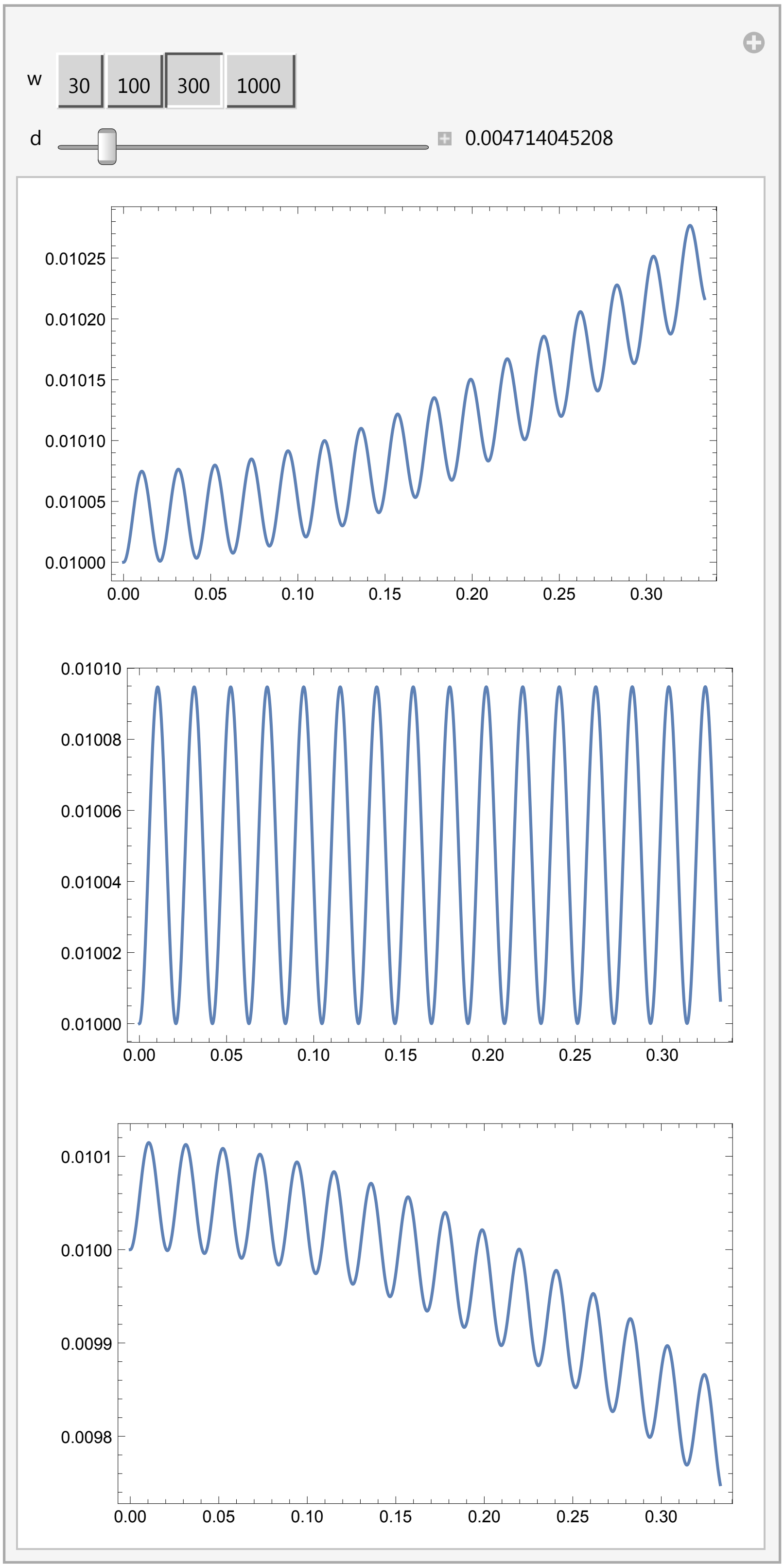
Out[18]=



## w=300, d=0.00371, 0.00471, 0.00571

In[19]:=
```
Manipulate[
  Row[Plot[#, {t, 0, 100 / w}, PlotRange → Full, Frame → True, ImageSize → Medium] & /@
    Evaluate[{sol[d - 0.001, w][t], sol[d, w][t], sol[d + 0.001, w][t]}]],
  {{w, 300}, {30, 100, 300, 1000}}, {{d, Sqrt[2.0] / w}, 0,
   .05, .0001, Appearance → "Labeled"}]
```

Out[19]=

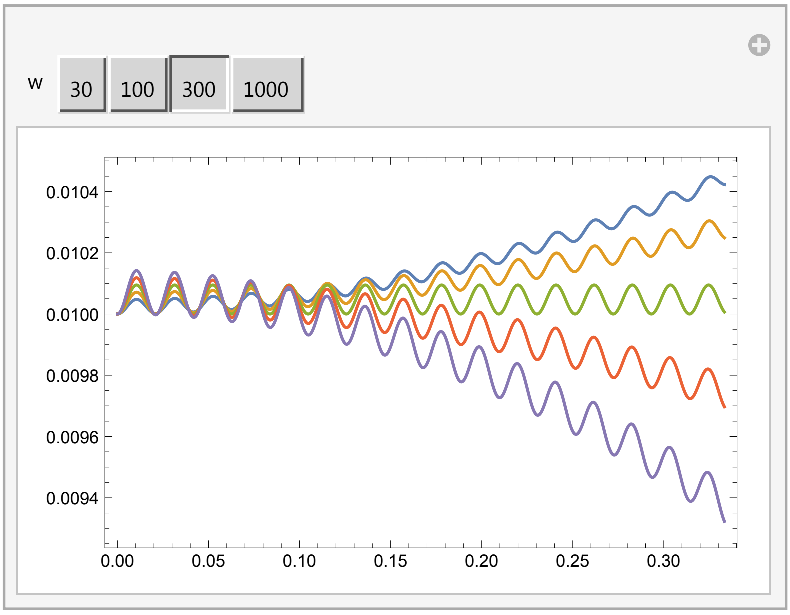



In[20]:= **w = 300,**

**d = (0.002357022, 0.0035355, 0.0047140, 0.0058925, 0.007071067)**

In[20]:=
```
Manipulate[Plot[
  Evaluate[Table[sol[d, w][t], {d, .5 Sqrt[2] / w, 1.5 Sqrt[2] / w, .25 Sqrt[2] / w}]],
  {t, 0, 100 / w}, PlotRange → All, Frame → True, ImageSize → Medium],
 {{w, 300}, {30, 100, 300, 1000}}]
```

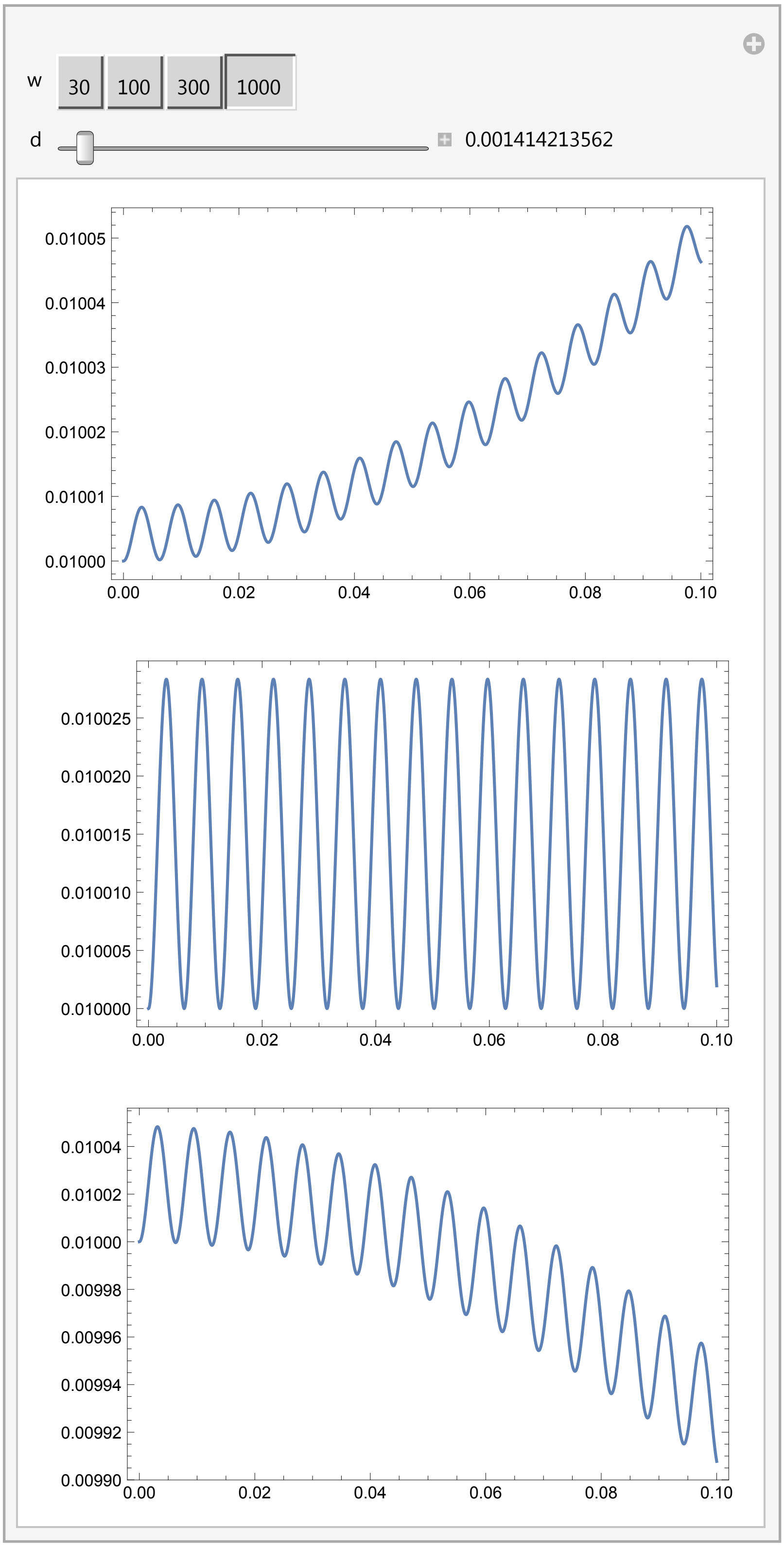
Out[20]=



### w=1000, d=0.0004142, 0.0014142, 0.0024142

In[21]:=
```
Manipulate[
  Row[Plot[#, {t, 0, 100/w}, PlotRange → Full, Frame → True, ImageSize → Medium] & /@
    Evaluate[{sol[d - 0.001, w][t], sol[d, w][t], sol[d + 0.001, w][t]}]],
  {{w, 1000}, {30, 100, 300, 1000}}, {{d, Sqrt[2.0]/w}, 0,
   .05, .0001, Appearance → "Labeled"}]
```

Out[21]=

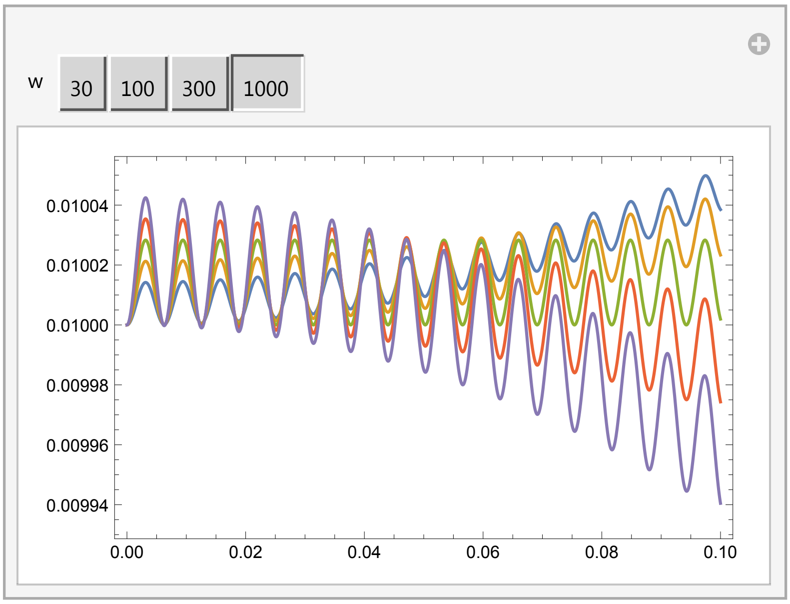



In[22]:= **w = 1000,**

**d = (0.0007071, 0.0010606, 0.0014142, 0.00176776, 0.00212132)**

In[22]:=
```
Manipulate[Plot[
  Evaluate[Table[sol[d, w][t], {d, .5 Sqrt[2] / w, 1.5 Sqrt[2] / w, .25 Sqrt[2] / w}]],
  {t, 0, 100 / w}, PlotRange → All, Frame → True, ImageSize → Medium],
 {{w, 1000}, {30, 100, 300, 1000}}]
```

Out[22]=




For each value of w the transition from exponential growth to periodicity occurs at approximately the value $\sqrt{2}/w$ (as seen in the plots)

# Problem 2

## a)

Postion of first mass = x1, Postion of second mass = x2, Vector connecting the two masses = r, Position of center of mass = X. (Note that x1, x2, r, X are all vectors)

Equation of position of center of mass:

In[1]:= 
```
e1 = X[t] == (m1 x1[t] + m2 x2[t]) / (m1 + m2)
```

Out[1]= $X[t] == \frac{m1\ x1[t] + m2\ x2[t]}{m1 + m2}$

Equation of vector from one body to the other:

In[2]:= 
```
e2 = r[t] == x2[t] - x1[t]
```

Out[2]= $r[t] == -x1[t] + x2[t]$

Solving for x1 and x2 in terms of X and r:

In[3]:= 
```
s = Solve[{e1, e2}, {x1[t], x2[t]}]
```

Out[3]= $\left\{\left\{x1[t] \to -\frac{m2\ r[t] - m1\ X[t] - m2\ X[t]}{m1 + m2},\ x2[t] \to -\frac{-m1\ r[t] - m1\ X[t] - m2\ X[t]}{m1 + m2}\right\}\right\}$

Position of first body in terms of r and X is given by:

In[4]:= 
```
x1[t] = x1[t] /. s[[1]]
```

Out[4]= $-\frac{m2\ r[t] - m1\ X[t] - m2\ X[t]}{m1 + m2}$

Position of second body in terms of r and X is given by:

In[5]:= 
```
x2[t] = x2[t] /. s[[1]]
```

Out[5]= $-\frac{-m1\ r[t] - m1\ X[t] - m2\ X[t]}{m1 + m2}$

The kinetic energy KE is then given by:

In[6]:= 
```
KE = Simplify[1/2 (m1 D[x1[t], t]^2 + m2 D[x2[t], t]^2)]
```

Out[6]= $\frac{m1\ m2\ r'[t]^2 + (m1 + m2)^2\ X'[t]^2}{2\ (m1 + m2)}$

Since X'[t] = Vcm, we can write KE as:

In[7]:= 
```
KE = KE /. {X'[t] → Vcm[t]}
```

Out[7]= $\frac{(m1 + m2)^2\ Vcm[t]^2 + m1\ m2\ r'[t]^2}{2\ (m1 + m2)}$

Now in spherical coordinates r is can be written as: (where we have renamed for convinience the angles $\theta$ -> A and $\phi$ -> B).

In[8]:= 
```
r[t_] = R[t] {Sin[A[t]] Cos[B[t]], Sin[A[t]] Sin[B[t]], Cos[A[t]]}
```

Out[8]= {Cos[B[t]] R[t] Sin[A[t]], R[t] Sin[A[t]] Sin[B[t]], Cos[A[t]] R[t]}

This gives KE in terms of R, A and B as:

In[9]:= `KE = Simplify[((m1 + m2)^2 Vcm[t]^2 + m1 m2 r'[t].r'[t])/(2 (m1 + m2))]`

Out[9]= $$\frac{1}{2\ (m1+m2)}\left((m1+m2)^2\,Vcm[t]^2+m1\ m2\left(R[t]^2\left(A'[t]^2+Sin[A[t]]^2\,B'[t]^2\right)+R'[t]^2\right)\right)$$

Potential energy (PE) is given by:

In[10]:= `PE = -G m1 m2 / R[t]`

Out[10]= $$-\frac{G\ m1\ m2}{R[t]}$$

Thus the total hamiltonian in terms of R, A and B is:

In[11]:= `H = KE + PE`

Out[11]= $$-\frac{G\ m1\ m2}{R[t]}+\frac{1}{2\ (m1+m2)}\left((m1+m2)^2\,Vcm[t]^2+m1\ m2\left(R[t]^2\left(A'[t]^2+Sin[A[t]]^2\,B'[t]^2\right)+R'[t]^2\right)\right)$$

This is independent of the position of the center of mass (X). It can be seen the the orbital motion and the center of mass motion is entirely decoupled. Thus H can be written as the sum of two parts corresponding to each motion.

Moving to center of mass frame we can take Vcm=0. The KE in this frame is then dependent only on the particle seperation. Thus the orbital motion is independant of motion of center of mass. The Hamiltonian in center of mass frame is given by:

In[12]:= `H = H /. {Vcm[t] → 0}`

Out[12]= $$-\frac{G\ m1\ m2}{R[t]}+\frac{m1\ m2\left(R[t]^2\left(A'[t]^2+Sin[A[t]]^2\,B'[t]^2\right)+R'[t]^2\right)}{2\ (m1+m2)}$$

## b)

### i.

We can see that the Hamiltonian is also independent of B[t]. Therefore the corresponding angular momentum is conserved. Hence the motion is confined to a plane.

### ii

Let the the components of momentum be given by $P_R \to$ PR, $P_\theta \to$ PA and $P_\phi \to$ PB

Let L be the lagrangian. In the center of mass frame L is given by:

In[13]:= `L = KE - PE /. {Vcm[t] → 0}`

Out[13]= $$\frac{G\,m1\,m2}{R[t]}+\frac{m1\,m2\left(R[t]^2\left(A'[t]^2+\text{Sin}[A[t]]^2\,B'[t]^2\right)+R'[t]^2\right)}{2\,(m1+m2)}$$

Calculating the momentum from L gives:

In[14]:= `PA[t] == D[L, D[A[t], t]]`

Out[14]= $$PA[t]==\frac{m1\,m2\,R[t]^2\,A'[t]}{m1+m2}$$

In[15]:= `PB[t] == D[L, D[B[t], t]]`

Out[15]= $$PB[t]==\frac{m1\,m2\,R[t]^2\,\text{Sin}[A[t]]^2\,B'[t]}{m1+m2}$$

In[16]:= `PR[t] == D[L, D[R[t], t]]`

Out[16]= $$PR[t]==\frac{m1\,m2\,R'[t]}{m1+m2}$$

Thus we can write the hamiltonian in terms of PA, PB, PR, A, B, and R as

In[17]:= $$H=-\frac{G\,m1\,m2}{R[t]}+\frac{1}{2\,m1\,m2}\left(PR[t]^{\wedge}2+PA[t]^{\wedge}2\big/R[t]^{\wedge}2+PB[t]^{\wedge}2\big/\left(R[t]^{\wedge}2\,\text{Sin}[A[t]]^{\wedge}2\right)\right)(m1+m2)$$

Out[17]= $$\frac{(m1+m2)\left(PR[t]^2+\frac{PA[t]^2}{R[t]^2}+\frac{\text{Csc}[A[t]]^2\,PB[t]^2}{R[t]^2}\right)}{2\,m1\,m2}-\frac{G\,m1\,m2}{R[t]}$$

From hamilton's equations we get:

In[18]:= `D[PB[t], t] == -D[H, B[t]]`

Out[18]= $PB'[t]==0$

Thus PB is conserved and the motion is confined to a plane. Therefore we can reorient our axis in a way such that PB[t] = 0.

In[19]:= `D[PA[t], t] == -D[H, A[t]] == 0`

Out[19]= $$PA'[t]==\frac{(m1+m2)\,\text{Cot}[A[t]]\,\text{Csc}[A[t]]^2\,PB[t]^2}{m1\,m2\,R[t]^2}==0$$

Thus in the new coordinate system with PB[t]=0, PA'[t] is also equal to zero. Hence total angular momentum is conserved.

---

# iii)

From lagrangian dynamics we get:

In[20]:= `PA[t] == D[L, D[A[t], t]]`

Out[20]= $$PA[t]==\frac{m1\,m2\,R[t]^2\,A'[t]}{m1+m2}$$

We have shown that PA'[t] = 0. Therefore PA[t] = constant. This implies:

In[21]:= $PA[t] == \frac{m1\ m2\ R[t]^2\ A'[t]}{m1 + m2} == C$

Out[21]= $PA[t] == \frac{m1\ m2\ R[t]^2\ A'[t]}{m1 + m2} == C$

Let rate of change of area swept per time = Y. Then

In[22]:= `Y = 1/2 R[t]^2 A'[t]`

Out[22]= $\frac{1}{2} R[t]^2 A'[t]$

From the previous equation we get:

In[23]:= `Y == 1/2 C (m1 + m2) / (m1 m2)`

Out[23]= $\frac{1}{2} R[t]^2 A'[t] == \frac{C\ (m1 + m2)}{2\ m1\ m2}$

**Thus Y is a constant. Kepler's second law is obeyed.**

## c)

Since the motion is confined to a plane, the lagrangian in terms of R and B (angle in the plane) is given by:

In[24]:= `L = L /. {A'[t] → 0, A[t] → Pi/2}`

Out[24]= $\frac{G\ m1\ m2}{R[t]} + \frac{m1\ m2\ (R[t]^2\ B'[t]^2 + R'[t]^2)}{2\ (m1 + m2)}$

The angular momentum J is given by:

In[25]:= `J == D[L, D[B[t], t]]`

Out[25]= $J == \frac{m1\ m2\ R[t]^2\ B'[t]}{m1 + m2}$

Let u be the reduced mass of the system:

In[26]:= `u == m1 m2 / (m1 + m2)`

Out[26]= $u == \frac{m1\ m2}{m1 + m2}$

Let us define the constant k as:

In[27]:= `k == G m1 m2`

Out[27]= $k == G\ m1\ m2$

This gives the total energy En as:

In[28]:= `En == 1/2 u R'[t]^2 + J^2/(2 u R[t]^2) - k / R[t]`

Out[28]= $En == \frac{J^2}{2\ u\ R[t]^2} - \frac{k}{R[t]} + \frac{1}{2} u\ R'[t]^2$

This gives the change in angle B as R goes from $R_{min}$ to $R_B$ (R at angle B) as:

In[29]:= `B == J/(Sqrt[2] u) Integrate[1/(R^2 Sqrt[En + k / R - J^2 / (2 u R^2)]), {R, Rmin, RB}]`

Calculating the integral gives:

In[30]:= `1/R == u k^2/J^2 + (1/R_Min - u k/J^2) Cos[B]`

Out[30]= $\frac{1}{R} == \frac{k^2 u}{J^2} + \text{Cos}[B] \left(-\frac{k u}{J^2} + \frac{1}{R_{Min}}\right)$

This is the equation of a conic section which is an ellipse for e < 1

In[31]:= `R[B] == p/(1 + e Cos[B])`

Out[31]= $R[B] == \frac{p}{1 + e\,\text{Cos}[B]}$

where p is the latus rectum and e is the eccentricity which have the values:

In[32]:= `p = J^2/(u k)`

Out[32]= $\frac{J^2}{k u}$

In[33]:= `e = Sqrt[1 + 2 En J^2/(m k^2)]`

Out[33]= $\sqrt{1 + \frac{2\,En\,J^2}{k^2 m}}$

The semi major axis is then given by

In[34]:= `a = p/(1 - e^2)`

Out[34]= $-\frac{k m}{2\,En\,u}$

Area of the ellipse is given by:

In[35]:= `area = Pi a^2 Sqrt[(1 - e^2)] // Simplify`

Out[35]= $\frac{k^2 \sqrt{-\frac{En\,J^2}{k^2 m}}\; m^2 \pi}{2\sqrt{2}\; En^2 u^2}$

Time taken to complete one full orbit (T) is equal to area divided by rate of change of area swept per unit time (Y)

Y is given by:

In[36]:= `Y`

Out[36]= $\frac{1}{2} R[t]^2 A'[t]$

Angular momentum J was shown to be:

In[37]:= `J == u R[t]^2 A'[t]`

Out[37]= $J == u\,R[t]^2 A'[t]$

Therefore

In[38]:= `Y = J/(2 m)`

Out[38]= $\frac{J}{2 m}$

In[39]:=
```
T = area / Y // Simplify
```

Out[39]=
$$\frac{J^3\, m\, \pi}{\sqrt{2}\, k^2 \left(-\frac{En\, J^2}{k^2\, m}\right)^{3/2} u^2}$$

Thus we get T is proportional to En^(-3/2). We have shown previously that the semi major axis a is inversely proportional to En. Therefore we get Kepler's third law i.e T^2 proportional to a^3.