HW 3 - ASTR501

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Q1)

Definition of optical depth τ

In[160]:=
```
sI := I → I0 Exp[-τ]
```

Substituting the above in the formula for magnitude difference

In[166]:=
```
Δm -> -2.5 Log10[I / I0] /. sI // PowerExpand
```

Out[166]= $\Delta m \to 1.08574\,\tau$

Q2)

a)

Substituting quantities in formula for Einstein radius θE

In[170]:=
```
θE → UnitConvert[(Sqrt[(DAB (4 M☉ G/c^2))/(DA DB)]) 1 rad //. {DA -> 8 kpc , DAB → DA - DB, DB -> 4 kpc }, 1" ]
```

$$\theta E \to \text{UnitConvert}\left[\left(\sqrt{\frac{DAB\left(4 M_{\odot} G/c^2\right)}{DA\,DB}}\right) 1\,\text{rad} \;//.\; \{DA \to 8\,\text{kpc},\ DAB \to DA - DB,\ DB \to 4\,\text{kpc}\},\ 1''\right]$$

Out[170]= $\theta E \to 0.0010089''$

b)

Since surface brightness (which is flux times solid angle) is conserved

In[173]:=
```
Solve[Sobs == Ssource /. {Sobs → Fobs / ΔΩobs, Ssource -> Fsource / ΔΩsource}, ΔΩobs]
```

$$\text{Solve}[S_{obs} == S_{source} \;/.\; \{S_{obs} \to F_{obs}/\Delta\Omega_{obs},\ S_{source} \to F_{source}/\Delta\Omega_{source}\},\ \Delta\Omega_{obs}]$$

Out[173]= $\left\{\left\{\Delta\Omega_{obs} \to \frac{F_{obs}\,\Delta\Omega_{source}}{F_{source}}\right\}\right\}$

Since observed flux is higher than that of source, the subtended solid angle increases.

Q3)

a)

Planck function

In[17]:=
```
sB = Bν → 2 h ν^3 / c^2 / (Exp[h ν / (k T)] - 1)
```

Out[17]= $B\nu \to \dfrac{2\,h\,\nu^3}{c^2\left(-1+e^{\frac{h\nu}{k\,T}}\right)}$

Derivative of B_ν with respect to T

In[90]:=
```
D[Bν /. sB, T]
```

Out[90]= $\dfrac{2\,e^{\frac{h\nu}{k\,T}}\,h^2\,\nu^4}{c^2\left(-1+e^{\frac{h\nu}{k\,T}}\right)^2 k\,T^2}$

Rosseland mean opacity (k_R) definition, using $k_\nu = j_\nu / B_\nu$

$$\frac{1}{k_R} = \frac{\int_0^\infty \frac{B\nu\,\frac{\partial(B\nu)}{\partial T}}{j\nu}\,d\nu}{\int_0^\infty \frac{\partial(B\nu)}{\partial T}\,d\nu}$$

b)

kν = ke = constant (electron scattering opacity)

Therefore, k_R = k_e from part a)

Q4)

a)

Length of the chord traversed by the ray

In[96]:=
```
sd := d → 2 (√(-b^2 + R^2))
```

Substituting above in the formula for optical depth

In[97]:=
```
sτ = τ → Integrate[α[ν], {x, 0, d /. sd}]
```
Out[97]= $\tau \to 2\sqrt{-b^2+R^2}\,\alpha[\nu]$

---

b)

Using the formula for I and setting I_0 to zero and the source function to be the planck function

In[98]:=
```
sI = I -> I0 Exp[-τ] + Bν (1 - Exp[-τ]) /. I0 → 0
```
Out[98]= $I \to B\nu\,(1-e^{-\tau})$

Integrating I cos(θ) to get flux and setting exp(-τ) to zero in this limit

In[99]:=
```
sF = F -> Integrate[I Cos[θ] Sin[θ] /. sI /. {sB, Exp[-τ] → 0},
   {θ, 0, π/2}, {ϕ, 0, 2π}, {ν, 0, ∞}, Assumptions → {h > 0, k > 0, T > 0, c > 0}]
```
Out[99]= $F \to \frac{2\,k^4\,\pi^5\,T^4}{15\,c^2\,h^3}$

Substituting the above flux to calculate luminosity

In[100]:=
```
L → 4 π R^2 F /. sF
```
Out[100]= $L \to \frac{8\,k^4\,\pi^6\,R^2\,T^4}{15\,c^2\,h^3}$

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c)

Series expansion of the integrand for small τ

In[101]:=
```
int = Normal@Series[I Cos[θ] Sin[θ] /. sI /. sB, {τ, 0, 1}] /. sτ /. b → R Sin[θ]
```
Out[101]= $\frac{4\,h\,\nu^3\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]\,\sqrt{R^2-R^2\,\mathrm{Sin}[\theta]^2}\,\alpha[\nu]}{c^2\left(-1+e^{\frac{h\nu}{kT}}\right)}$

Integrating the above

In[102]:=
```
sF2 = F -> Integrate[int, {θ, 0, π/2},
   {ϕ, 0, 2π}, {ν, 0, ∞}, Assumptions → {h > 0, k > 0, T > 0, c > 0}]
```
Out[102]= $F \to \mathrm{Integrate}\left[\frac{8\,h\,\pi\,\sqrt{R^2}\,\nu^3\,\alpha[\nu]}{3\,c^2\left(-1+e^{\frac{h\nu}{kT}}\right)}, \{\nu, 0, \infty\}, \mathrm{Assumptions} \to \{h>0, k>0, T>0, c>0\}\right]$

Luminosity at the above flux

In[103]:= `L → 4 π R^2 F /. sF2`

Out[103]= $L \to 4\pi R^2\,\text{Integrate}\left[\frac{8\,h\,\pi\sqrt{R^2}\,\nu^3\,\alpha[\nu]}{3\,c^2\left(-1+e^{\frac{h\nu}{kT}}\right)},\ \{\nu, 0, \infty\},\ \text{Assumptions} \to \{h > 0, k > 0, T > 0, c > 0\}\right]$

Assuming $\alpha(\nu)$ is a constant independent of ν, then flux is

In[104]:= `sF3 = F → (F //. {sF2, α[ν] → α})`

Out[104]= $F \to \frac{8\,k^4\,\pi^5\sqrt{R^2}\,T^4\,\alpha}{45\,c^2\,h^3}$

Luminosity at the above flux when α is a constant

In[105]:= `L → 4 π R^2 F /. sF3`

Out[105]= $L \to \frac{32\,k^4\,\pi^6\left(R^2\right)^{3/2}\,T^4\,\alpha}{45\,c^2\,h^3}$

---

d)

Substituting the expressions for I, Bν, and τ

In[106]:= `Iν -> I /. sI /. sτ /. sB`

Out[106]= $I\nu \to \frac{2\left(1-e^{-2\sqrt{-b^2+R^2}\,\alpha[\nu]}\right)h\,\nu^3}{c^2\left(-1+e^{\frac{h\nu}{kT}}\right)}$

---

e)

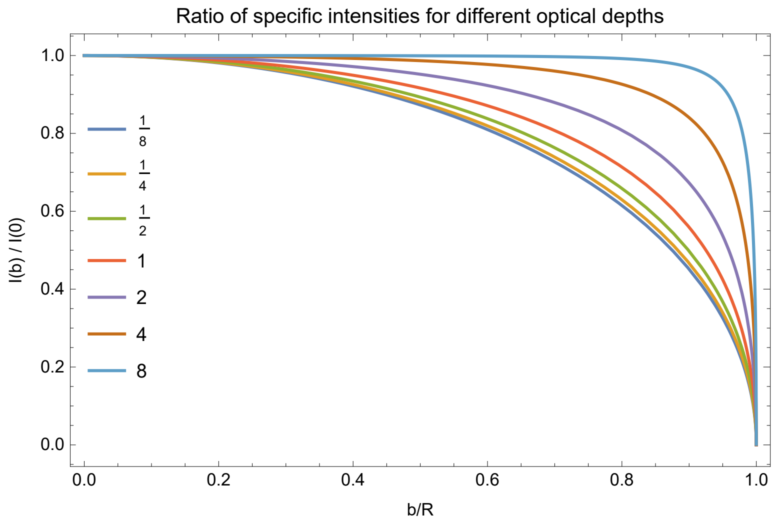
Defining function to find ratio of specific intensities:

In[107]:= `IRatio[τ_, b_] := (1 - Exp[-τ Sqrt[1 - b^2]]) / (1 - Exp[-τ])`

Plotting ratio vs b for different τ:

In[108]:=
```
With[{list = {1/8, 1/4, 1/2, 1, 2, 4, 8}},
 Plot[Evaluate@Table[IRatio[τ, b], {τ, list}], {b, 0, 1}, Frame → True, PlotRange → All,
  PlotLegends → Placed[list, {Left, Center}], FrameLabel → {"b/R", "I(b) / I(0)"},
  PlotLabel → "Ratio of specific intensities for different optical depths",
  ImageSize → 400]]
```

Out[108]=



f)

Integrating the general expression

In[114]:=
```
sF4 = F -> Integrate[I Cos[θ] Sin[θ] //. {sI, sB, sτ, b → R Sin[θ]},
   {θ, 0, π/2}, {ϕ, 0, 2π}, {ν, 0, ∞}, Assumptions → {h > 0, k > 0, T > 0, c > 0}]
```

Out[114]= $F \to \text{Integrate}\left[\left(e^{-2\sqrt{R^2}\,\alpha[\nu]}\, h\,\pi\,\nu^3 \left(1 + 2\sqrt{R^2}\,\alpha[\nu] + e^{2\sqrt{R^2}\,\alpha[\nu]}\left(-1 + 2R^2\alpha[\nu]^2\right)\right)\right) \Big/ \left(c^2\left(-1 + e^{\frac{h\nu}{kT}}\right) R^2\alpha[\nu]^2\right), \{\nu, 0, \infty\}, \text{Assumptions} \to \{h > 0, k > 0, T > 0, c > 0\}\right]$

This cannot be simplified further since we do not know $\alpha(\nu)$ as a function of ν. If it is a constant then flux is

In[117]:=
```
sF5 = sF4 /. α[ν] → α // PowerExpand
```

Out[117]= $F \to \left(e^{-2R\alpha}\, k^4\, \pi^5\, T^4 \left(1 + 2R\alpha + e^{2R\alpha}\left(-1 + 2R^2\alpha^2\right)\right)\right) \Big/ \left(15\, c^2\, h^3\, R^2\, \alpha^2\right)$

The luminosity is

In[125]:=
```
4 π R^2 F /. sF5
```

Out[125]=
$$\frac{4\, e^{-2R\alpha}\, k^4\, \pi^6\, T^4 \left(1 + 2R\alpha + e^{2R\alpha}\left(-1 + 2R^2\alpha^2\right)\right)}{15\, c^2\, h^3\, \alpha^2}$$

In the limit $\alpha \to 0$, the flux is

In[120]:=
```
Series[F /. sF5, {α, 0, 1}]
```

Out[120]=
$$\frac{8\,k^4\,\pi^5\,R\,T^4\,\alpha}{45\,c^2\,h^3} + O[\alpha]^2$$

This is the same flux as part c)

In the limit $\alpha \to \infty$, the flux is

In[124]:=
```
Normal@Series[F /. sF5 /. α → 1/x, {x, 0, 1}] /. x → 1/α
```

Out[124]=
$$\frac{2\,k^4\,\pi^5\,T^4}{15\,c^2\,h^3} + \frac{2\,e^{-2\,R\,\alpha}\,k^4\,\pi^5\,T^4}{15\,c^2\,h^3\,R\,\alpha}$$

The second term is zero in the limit as exp(-α) → 0 as $\alpha \to \infty$. Therefore this is the same flux as part b)

Therefore the limits match.