Astrodynamics Final Exam

Daniel George

Q1)

---

a)

Mach number:

In a giant molecular cloud the typical sound speed (C_s) is about 0.1 km/s and bulk velocities (V) are about 1km/s. Therefore the Mach number (V/C_s) is about 10.

The physical significance of having such a high value of Mach number is that the molecular cloud flows are highly supersonic and therefore the pressure forces are negligible compared to advection.

Reynolds number:

The viscosity (ν) of a diffuse gas is approximately equal to RMS speed times (u) mean free path (λ):

```
In[1]:= ν == u λ
```

The RMS speed of a gas is of the same order of magnitude as the sound speed which is about 0.1 km/s. The mean free path is given by the equation:

```
λ == 1 / (n σ)
```

Where the typical number density (n) is of the order 100 cm^{-3} and molecular size (σ) is about 1 nm^2. Plugging in these values gives the viscosity (ν):

```
In[99]:= ν = 0.1 km/s / (100/cm^3 1 nm ^2)
Out[99]= 1. * 10^6 km^2/s
```

Now the Reynold's number (R) is given by the formula:

```
R == L V / ν
```

Typical value of L is about 30 pc and bulk velocity (V) is about 1 km/s. This gives

```
In[100]:= R = 30 pc 1 km/s / ν
Out[100]= 9.257032744 * 10^8
```

Physically, such a high value of the Reynolds number $(\sim 10^9)$ means that the flow is highly non-viscous and turbulent. Also the turbulence travels at supersonic speeds since bulk velocity is greater than the sound speed.

b)

Molecular viscosity:

The viscosity (v) of the accretion disk is approximately equal to speed of sound times mean free path (λ):

$\nu == Cs\,\lambda$

The typical sound speed is about $5*10^4$ cm/s . The mean free path is given by the equation:

$\lambda == 1/(n\,\sigma)$

Here the typical number density is about $10^{12}/\text{cm}^3$ and molecular size (σ) is about $2*10^{-15}$ cm^2. (source: http://www.astro.washington.edu/courses/astro557/accretion_disks.pdf)
Plugging in these values gives the molecular viscosity (v):

```
In[101]:= ν = 5 * 10^4 / (10^12 * 2 * 10^-15) cm^2/s
Out[101]= 25 000 000 cm^2/s
```

Thus the molecular viscosity is about $2.5*10^7$ cm$^2/s$.

Turbulent viscosity:

The viscous time scale at radius r = 10 AU is observed to be about a million years (source: http://www.astro.washington.edu/courses/astro557/accretion_disks.pdf).
This implies:

$t == r^2/\nu == 10^6\ \text{years}$

We can calculate the viscosity from this equation as:

```
In[102]:= ν = 10 au ^2 / 1 * 10^6 yr ;
```

Converting to cm$^2/s$:

```
In[103]:= UnitConvert[ν, cm^2/s] // N
Out[103]= 7.096500164 * 10^14 cm^2/s
```

Thus the turbulent viscosity required to match observations is about 10^{15} cm$^2/s$.

c)

The plasma beta is given by the following formula (where $n\,k_B$T is the plasma pressure):

```
In[104]:= β = n k_B T / (B^2 / (2 μ0)) ;
```

For a typical fluorescent lamp the temperature T = 5600 Kelvin. The magnetic field is about 10 mT. (source: http://michaelbluejay.com/electricity/emf.html)
Typical density is about 10^{10} m^{-3} (source: https://en.wikipedia.org/wiki/Plasma_(physics))

Plugging in the values we get β equals:

In[105]:=
```
β /. {n -> 1 * 10^10 per meter^3 , T -> 5600 K , kB -> k , μ0 -> μ0 , B -> 10 mT }
```

Out[105]= $1.94317 * 10^{-11}$

---

d)

The total magnetic flux at the surface of the Sun would be equal to the total magnetic flux at the surface of the White Dwarf. This means that:

$B_{WD}\, 4\, \pi\, R_{WD}{}^2 == B_{Sun}\, 4\, \pi\, R_{Sun}{}^2$

The radius of a typical white dwarf is around the same as that of Earth which is approximately 1% the radius of Sun.

In[106]:=
```
RWD = .01 RSun ;
```

The average magnetic field of the Sun is around 1 Gauss.
(source: http://www.windows2universe.org/sun/sun_magnetic_field.html)

In[107]:=
```
BSun = 1 ;
```

Therefore the magnetic field of the White Dwarf will be:

In[108]:=
```
BWD == BSun RSun^2 / RWD^2 G
```

Out[108]= $B_{WD} == 10\,000.\ G$

Thus the magnetic field at the surface of the White Dwarf would be about 10^4 Gauss.

---

e)

Finding the combinations of density, time and radius which can give units of energy:

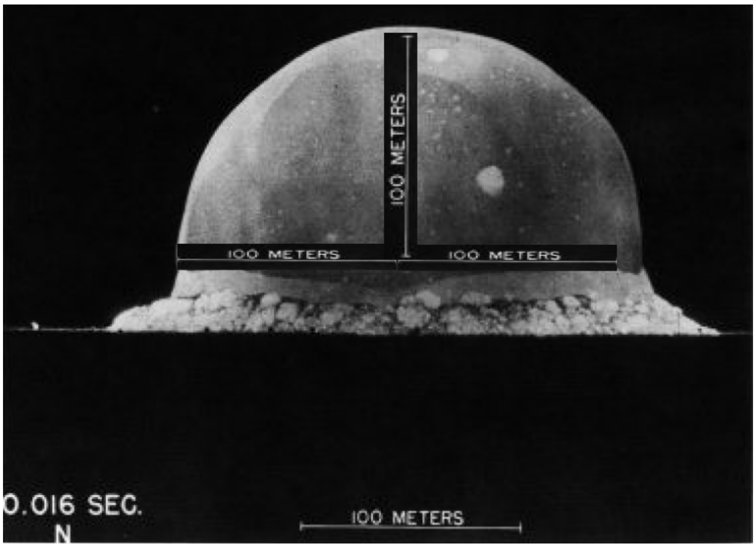
In[109]:=
```
DimensionalCombinations[{"MassDensity", "Radius", "Time"}, "Energy"]
```

Out[109]= $\left\{\frac{\text{MassDensity Radius}^5}{\text{Time}^2}\right\}$

Therefore the only allowed combination is:

$$\text{Energy} = \rho\, \frac{R^5}{t^2}$$

In[127]:=



From the figure above we can see that the radius is approximately 100 Meters. Time is given as 0.016 seconds. The density of air is around 1.2 kg$/m^3$. Plugging in the values we get:

In[110]:=
```
Energy = 1.2 kg/m^3 (100 m)^5/(.016 s)^2
```

Out[110]= $4.6875 * 10^{13}$ kg m^2/s^2

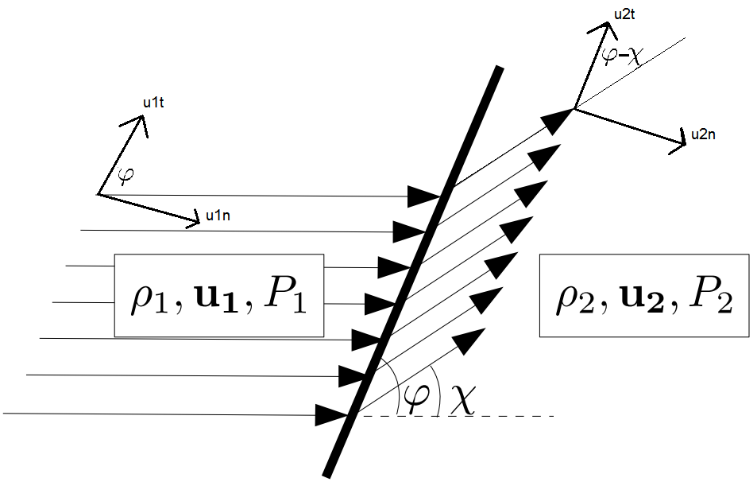
Thus the energy of the explosion is about **$5*10^{13}$ Joules** . In kilo tons of TNT it is:

In[111]:=
```
UnitConvert[Energy, "KilotonsOfTNT"]
```

Out[111]= 11.20339388 kilotons of TNT

Q2)

In[130]:=



From the figure we have:

In[112]:=
```
u1t = u1 Cos[φ]; (*Tangential component of u1*)
u1n = u1 Sin[φ]; (*Normal component of u1*)
M1n = M1 Sin[φ]; (*Normal component of Mach number of u1*)
u2t = u2 Cos[φ - χ]; (*Tangential component of u1*)
u2n = u2 Sin[φ - χ]; (*Normal component of u1*)
M2n = M2 Sin[φ - χ]; (*Normal component of Mach number of u2*)
```

The tangential components remain the same across the shock:

In[114]:=
```
eq1 = u1t == u2t
```

Out[114]= $u1\,\mathrm{Cos}[\varphi] == u2\,\mathrm{Cos}[\varphi - \chi]$

Now we can solve for u2 in terms of u1, φ, χ:

In[115]:=
```
eq2 = u2 → (u2 /. Solve[eq1, u2][[1]])
```

Out[115]= $u2 \to u1\,\mathrm{Cos}[\varphi]\,\mathrm{Sec}[\varphi - \chi]$

Ratio of the normal components of velocity in terms of normal Mach numbers:

The continuity equation and momentum equations can be applied to the normal component of the velocities. Therefore we can use the same equations as in the lecture notes except with velocities and Mach numbers replaced by their normal components:

In[116]:=
```
eq3 = u1n / u2n == (γ + 1) M1n^2 / (2 + (γ - 1) M1n^2)
```

Out[116]= $$\frac{u1\,\mathrm{Csc}[\varphi - \chi]\,\mathrm{Sin}[\varphi]}{u2} == \frac{M1^2\,(1 + \gamma)\,\mathrm{Sin}[\varphi]^2}{2 + M1^2\,(-1 + \gamma)\,\mathrm{Sin}[\varphi]^2}$$

Substituting for u2 from eq2 in eq3:

In[117]:=
```
eq4 = eq3 /. eq2
```

Out[117]= $$\mathrm{Cot}[\varphi - \chi]\,\mathrm{Tan}[\varphi] == \frac{M1^2\,(1 + \gamma)\,\mathrm{Sin}[\varphi]^2}{2 + M1^2\,(-1 + \gamma)\,\mathrm{Sin}[\varphi]^2}$$

---

a)

Finding minimum value of φ

The minimum value of φ occurs when χ becomes zero. Solving the above equation:

In[118]:=
```
Solve[eq4 /. {χ → 0}, Sin[φ]]
```

Out[118]= $$\left\{\left\{\mathrm{Sin}[\varphi] \to -\frac{1}{M1}\right\}, \left\{\mathrm{Sin}[\varphi] \to \frac{1}{M1}\right\}\right\}$$

Since the angles are defined to be positive the only feasible solution is:

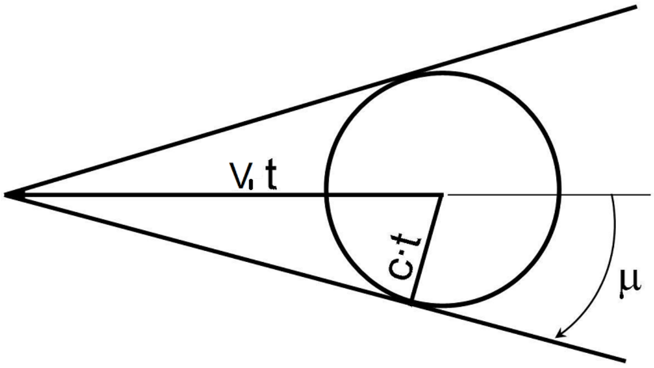
$$\text{Sin}[\varphi] == \frac{1}{\text{M1}}$$

Therefore the minimum value of φ is $\sin^{-1}(1/\text{M1})$

Range of φ:

Physically this makes sense since shocks propagate faster than the Mach wave (which travels at the speed of sound). The sin of Mach angle (μ) can be calculated as the ratio of distance traveled by sound to the distance traveled by the particles:

In[119]:=




```
Sin[μ] = C t / (V1 t) = C / V1 = 1 / M1
```

Thus the minimum value of φ is equal to the Mach angle. The maximum value of φ is evidently 90° and it occurs in the case of a normal shock (for angles greater than 90° we define φ as 180° - φ)

Therefore $\sin^{-1}(1/\text{M1}) < \varphi < \pi/2$

b)

Solving eq4 for cot(χ);

In[120]:=
```
eq5 = Cot[χ] == (Cot[χ] /.
    Solve[Tan[φ] (1 + Cot[χ] Cot[φ]) / (Cot[χ] - Cot[φ]) == eq4[[2]], Cot[χ]][[1]])
```

Out[120]=
$$\text{Cot}[\chi] == -\frac{\left(2 + \text{M1}^2\,\gamma + \text{M1}^2\,\text{Cos}[2\,\varphi]\right)\,\text{Tan}[\varphi]}{2 - \text{M1}^2 + \text{M1}^2\,\text{Cos}[2\,\varphi]}$$

This is identical to the given equation since $\cos(2\,\varphi) = 1 - 2\sin^2(\varphi)$. Checking:

In[121]:=
$$-\frac{\left(2 + \text{M1}^2\,\gamma + \text{M1}^2\,\text{Cos}[2\,\varphi]\right)\,\text{Tan}[\varphi]}{2 - \text{M1}^2 + \text{M1}^2\,\text{Cos}[2\,\varphi]} == \left(\frac{\text{M1}^2\,(1+\gamma)}{2\,\left(\text{M1}^2\,\text{Sin}[\varphi]\text{^}2 - 1\right)} - 1\right)\,\text{Tan}[\varphi] \text{ // Simplify}$$

Out[121]= True

QED.

c)

In the limit of large M_l:

Taking limit of LHS as M1 → ∞:

In[122]:=
```
eq6 = eq5[[1]] == Limit[eq5[[2]], {M1 → ∞}][[1]]
```

Out[122]= $\text{Cot}[\chi] == \frac{1}{2}(\gamma + \text{Cos}[2\varphi])\,\text{Csc}[\varphi]\,\text{Sec}[\varphi]$

The LHS is an extremum when its derivative is zero

Equating the derivative of the LHS to zero:

In[123]:=
```
eq7 = D[eq6[[2]], φ] == 0 // Simplify
```

Out[123]= $(1 + \gamma\,\text{Cos}[2\varphi])\,\text{Csc}[\varphi]\,\text{Sec}[\varphi] == 0$

Solving this equation for φ:

In[124]:=
```
eq8 = Assuming[γ > 1, Simplify[Solve[eq7 && 0 ≤ φ ≤ Pi/2, φ]]][[1]]
```

Out[124]= $\{\varphi \to \frac{1}{2}\text{ArcCos}[-\frac{1}{\gamma}]\}$

We can check if this is a minimum or maximum by calculating the second derivative and checking if it is positive:

In[125]:=
```
Assuming[γ > 1, (D[eq6[[2]], {φ, 2}] /. eq8) ≥ 0 // Simplify]
```

Out[125]= True

Therefore it is a minimum since the second derivative is positive.

Solving for χ

The maximum value of χ occurs when cot(χ) is minimum i.e. when the LHS is minimum. Substituting the minimum value of LHS:

In[126]:=
```
Assuming[γ > 1, Simplify[eq6 /. eq8]]
```

Out[126]= $\sqrt{-1 + \gamma^2} == \text{Cot}[\chi]$

Therefore the maximum value of $\chi = \cot^{-1}\left(\sqrt{-1 + \gamma^2}\right)$. This is can be shown to be identical to $\sin^{-1}(1/\gamma)$:

In[127]:=
```
Assuming[γ > 1, FullSimplify[ArcCot[Sqrt[-1 + γ^2]] == ArcSin[1/γ]]]
```

Out[127]= True

Thus the maximum value of χ is $\sin^{-1}(1/\gamma)$.