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Question 1

Part (a)

The constant a is determined by first integrating $m \frac{dN}{dM}$ over the mass range to find an expression involving a for the total mass of the cluster. Using the total mass given in the problem, we solve for a.

In[151]:= ClearAll[a];

$$\text{totalmass} = \int_{0.1}^{20} m\, a\, m^{-2.35}\, dm$$

5.39503 a

Framed[Solve[totalmass == 10^6, a][[1, 1]]]

$\boxed{a \to 185356.}$

Part (b)

Now that we have solved for a, we can find the the total luminosity by integrating $m^4 \frac{dN}{dM}$ over the mass range because L ~ m^4. **Note that the first answer is in units of solar luminosities**. The second answer is found by integrating for stars with mass greater than 5 solar masses and dividing by the luminosity of the entire distribution.

In[152]:= a = 185356;

$$\text{Framed}\left[\int_{0.1}^{20} m^4\, a\, m^{-2.35}\, dm\right]$$

Out[153]= $\boxed{1.96106 \times 10^8}$

$$\text{Framed}\left[\frac{\int_5^{20} m^4\ a\ m^{-2.35}\ dm}{\int_{0.1}^{20} m^4\ a\ m^{-2.35}\ dm}\right]$$

$\boxed{0.974618}$

Part (c)

First, we find a solution for N(m) by integrating dN/dm. Note that the constant of integration is zero. To find the mean mass we integrate $m\,\frac{dN}{dM}$ over the mass range and divide by the difference in the number of stars at both bounds.

$$\int a\ m^{-2.35}\ dm$$

$$-\frac{137301.}{m^{1.35}}$$

$$\text{Num}[m\_] = -\frac{137300.74074074073`}{m^{1.35`}};$$

$$\text{Framed}\left[\frac{1}{\text{Num}[20]-\text{Num}[0.1]}\int_{0.1}^{20} m\ a\ m^{-2.35}\ dm\right]$$

$\boxed{0.325587}$

Part (d)

Using the scaling relation given in the problem, we find the proportionality constant for the scaling relation, i.e. b:

In[155]:= $\text{Solve}\left[b\ (1)^{-2.5} == 10\ ,\ b\right]$

Out[155]= $\{\{b \to 10.\}\}$

Using this constant b, we can solve for the mass for when the age of the cluster is 1 Gyr. Note that **this answer is in units of solar masses.**

$$\text{Framed}\left[\text{Quiet@Solve}\left[10\ (M)^{-2.5} == 1,\ M\right][[1, 1]]\right]$$

$\boxed{M \to 2.51189}$

We can find an expression for L(m) by integrating the same distribution from part (b). Note that again the integration constant is equal to zero. Inputting our solution of M = 2.51 solar masses, we find the luminosity. Note that **the luminosity is in units of solar luminosities**.

$$\int m^4\ a\ m^{-2.35}\ dm$$

$$69\,945.7\ m^{2.65}$$

```
Framed[L → ScientificForm[69945.6603773585` (2.51188643150958`)^2.65`]]
```

$\boxed{L \to 8.03084 \times 10^{5}}$

Question 2

Part (a)

As done in the previous question, find the constant a. However, for this derivation, we are given the total number of stars instead of the total mass, therefore, we can simply integrate dN/dm.

```
ClearAll[a, totalmass];
```

$$\text{totalnumb} = \int_{0.4}^{100} a\, m^{-2.35}\, dm$$

$$2.55055\, a$$

```
Solve[totalnumb == 10^11, a][[1, 1]]
```

$$a \to 3.92072 \times 10^{10}$$

$$a = 3.92072 \times 10^{10};$$

Given this value of a we can find the total number of stars with a mass greater than 8 solar masses by integrating from 8 to the end of the mass range. The fraction of the total number stars that this comprises can be found by dividing by the total number of stars in the mass range.

$$\text{Framed}\left[\frac{\int_{8}^{100} a\, m^{-2.35}\, dm}{\int_{0.4}^{100} a\, m^{-2.35}\, dm}\right]$$

$$\boxed{0.0169537}$$

The total number of stars with mass greater than 8 solar masses

$$\text{Framed}\left[\int_{8}^{100} a\, m^{-2.35}\, dm\right]$$

$$\boxed{1.69537 \times 10^{9}}$$

The total mass of remnants assuming a typical mass of 1.4 solar masses. **Note that this answer is in units of solar masses.**

$$\text{Framed}\left[1.69537 \times 10^{9} * 1.4\right]$$

$$\boxed{2.37352 \times 10^{9}}$$

Part (b)

Using the total number of stars that eventually collapse and multiplying by the typical iron ejecta and dividing by the mass of gas in the Milky Way we obtain a fraction for Z_{Fe}

In[159]:= $Z_{Fe} \to \frac{1.6953703915395846`*^9 * 0.05}{5 * 10^{10}}$

Out[159]= $Z_{Fe} \to 0.00169537$

This is incredibly close to the value for Z_{Fe} of the sun thus demonstrating that the sun is a second generation star, i.e. it was formed using interstellar gas with a Z_{Fe} around Z = 0.00177

Part (c)

We must divide the total number of stars by 4 so that we demonstrate that we are only counting pairs. If we assue a random draw, than we must multiply by the probability for selecing each star with mass m greater than 8 solar masses, i.e. multiplying the total number of binary pairs by the probability of m>8 solar masses squared.

$\frac{10^{11}}{4} * 0.0169537^2$

7.1857×10^6

Part (d)

We equate the binding energy between the binary pairs (each 8 solar masses) to the kinetic energy of two 1.4 solar mass neutron stars with kick velocity of 500 km/s. Solving for the maximal initial separation gives:

```
Framed[UnitConvert[Quiet@
  Solve[G (8 M⊙) (8 M⊙) / r == 1/2 (1.4 M⊙ + 1.4 M⊙) (500 km/s)^2, r][[1, 1, 2]], "pc"]]
```

$\boxed{7.86437 \times 10^{-7}\ \text{pc}}$

Part (e)

Multiplying the total number of stars forming binary pairs (each >8 solar masses) from part (c) by the uniform distribution of separations, i.e. $7.19 \times 10^{-7} * \frac{7.86 \times 10^{-7}}{\Delta d}$ where Δd = 0.01 pc - 0 pc = 0.01 pc we obtain:

```
Framed[7.185698592249999`*^6 * 7.864374591716565`*^-7 pc / 0.01 pc]
```

$\boxed{565.11}$

Question 3

The relaxation time is given roughly as ~ $\frac{v^3}{G^2 M \rho} * \frac{1}{\ln\Lambda}$. To find approximations to the time, we employ the equivalent expression below which gives the relaxation time in years which we subsequently convert to more natural units. For part (a), we assume 1 M⊙ stars with velocity dispersion 1 km/s and a half-mass radius of 2 pc. Assuming a total mass of 300 M⊙, we find a mean density by employing the half-mass radius by saying that $\rho \sim 150\ M_\odot / \left(\frac{4}{3}\pi\ (2\ \text{pc})^3\right)$. See below for the full approximate calculation. For this estimate, order of magnitudes are sufficient for the answer.

Part (a)

$$\text{UnitConvert}\left[0.95 * 10^{10} \left(\frac{1\ \text{km/s}}{200\ \text{km/s}}\right)^3 \left(\frac{150\ M_\odot \Big/ \left(\frac{4}{3}\pi\left(2\ \text{pc}\right)^3\right)}{10^6\ 1\ M_\odot\ *\ 1/\text{pc}^3}\right)^{-1} \left(\frac{1\ M_\odot}{1\ M_\odot}\right)^{-1} \text{yr}, \text{"Myr"}\right]$$

265.29 Myr

For part (b), we use several sources (i.e. https://en.wikipedia.org/wiki/Velocity_dispersion, https://en.wikipedia.org/wiki/Elliptical_galaxy, https://en.wikipedia.org/wiki/Galaxy_cluster, Velocity dispersions in galaxy clusters (Girardi, M.; Biviano, A.; Giuricin, G.; Mardirossian, F.; Mezzetti, M.)), to find an approximate velocity dispersion for a galaxy cluster as 200 km/s, a typical galaxy mass of 10^{10} solar masses, a typical cluster containing 1000 galaxies, and a cluster radius of roughly 1.5 Mpc. Using this information, we deduce the relaxation approximation below:

Part (b)

$$\text{UnitConvert}\left[0.95 * 10^{10} \left(\frac{200\ \text{km/s}}{200\ \text{km/s}}\right)^3 \left(\frac{1000 * 10^{10}\ M_\odot \Big/ \left(\frac{4}{3}\pi\left(1.5\ \text{Mpc}\right)^3\right)}{10^6\ 1\ M_\odot\ *\ 1/\text{pc}^3}\right)^{-1} \left(\frac{10^{10}\ M_\odot}{1\ M_\odot}\right)^{-1} \text{yr}, \text{"Gyr"}\right]$$

1343.03 Gyr

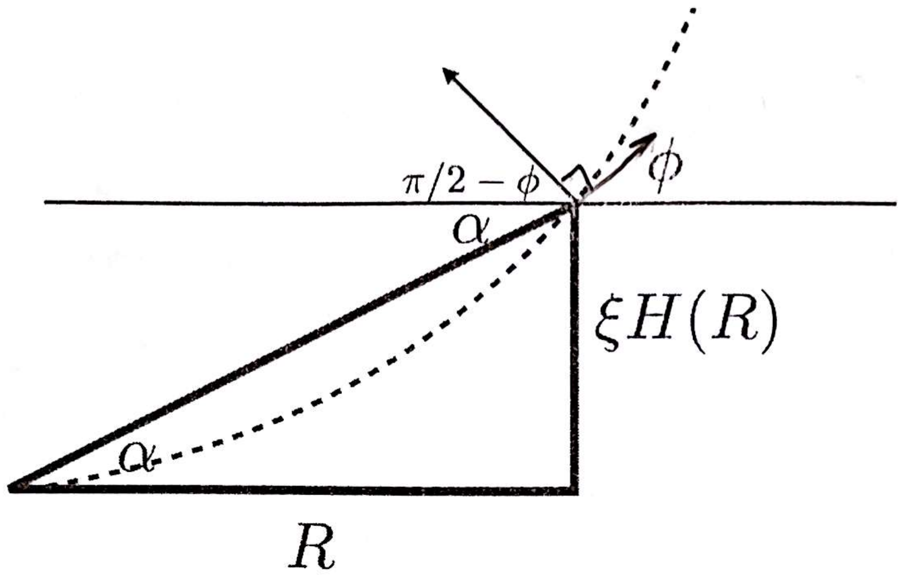
Part (c)

The mass of dark matter in the galaxy cluster constitutes the majority of the mass in the system. Therefore, the effective mass density of galaxies is significantly reduced. This, in turn, largely increases the relaxation time. For instance, if we reduce the density to 1% of its estimated value in Part (b), we obtain a relaxation time of ~10^5 Gyr, significantly larger (three orders of magnitude) than the relaxation time of the galaxy cluster without dark matter:

$$\text{UnitConvert}\left[0.95 * 10^{10} \left(\frac{200\ \text{km/s}}{200\ \text{km/s}}\right)^3 \left(\frac{0.01 * 1000 * 10^{10}\ M_\odot \Big/ \left(\frac{4}{3}\pi\left(1.5\ \text{Mpc}\right)^3\right)}{10^6\ 1\ M_\odot\ *\ 1/\text{pc}^3}\right)^{-1} \left(\frac{10^{10}\ M_\odot}{1\ M_\odot}\right)^{-1} \text{yr}, \text{"Gyr"}\right]$$

134 303. Gyr

Question 4



$$\theta = \pi/2 - \phi + \alpha$$

Part (a)

The angle corresponding to the angle between the normal line and the ray connecting the central source and the height ξ H(R) is given by $\theta = \pi/2 - \phi + \alpha$ where we have defined ϕ and α below:

In[66]:=
```
ClearAll[θ];
```

In[67]:=
```
ϕ = ArcTan[ξ H'[R]];
α = ArcTan[ξ H[R]/R];
```

```
Framed[θ → π/2 - ϕ + α]
```

Out[69]=

$$\boxed{\theta \to \frac{\pi}{2} + \text{ArcTan}\left[\frac{\xi H[R]}{R}\right] - \text{ArcTan}[\xi H'[R]]}$$

If we recall that $\frac{dH(R)}{dR} = \frac{H}{R}\frac{d\ln H}{d\ln R}$, then we can express the above result in terms of p:

```
Clear
```

In[71]:=
```
Framed[θ → π/2 + ArcTan[ξ H/R] - ArcTan[ξ H/R p]]
```

Out[71]=

$$\boxed{\theta \to \frac{\pi}{2} + \text{ArcTan}\left[\frac{H\,\xi}{R}\right] - \text{ArcTan}\left[\frac{H\,p\,\xi}{R}\right]}$$

Because H/R << 1, we can employ a Taylor expansion to approximate to first order θ:

In[74]:= `Framed[Normal@Series[π/2 + ArcTan[H ξ/R] - ArcTan[H p ξ/R], {H, 0, 1}]]`

Out[74]= $\boxed{\frac{\pi}{2}+\frac{H\,(\xi-p\,\xi)}{R}}$

Therefore, θ is ~ $\frac{\pi}{2}+\frac{H\,(\xi-p\,\xi)}{R}$

Part (b)

The heating from the central source (assuming an unobstructed view) is equal to:

In[76]:= $\frac{L}{4\,\pi\,R^2}\,\text{Cos}[\theta]$;

where L is luminosity, θ is the angle from part (a), and R is roughly distance from the central source to the unit disk surface area where we have assumed again that H/R << 1. An exact expression for distance would employ the Pythagorean theorem with side lengths R and z = ξ H(R). Evaluating the above expression using our θ gives:

In[79]:= `Framed[FullSimplify[L/(4 π R^2) Cos[π/2 + ArcTan[H ξ/R] - ArcTan[H p ξ/R]]]]`

Out[79]= $\boxed{-\frac{L\,\text{Sin}\left[\text{ArcTan}\left[\frac{H\,\xi}{R}\right]-\text{ArcTan}\left[\frac{H\,p\,\xi}{R}\right]\right]}{4\,\pi\,R^2}}$

or using our Taylor expansion approximation...

In[80]:= `Framed[FullSimplify[L/(4 π R^2) Cos[π/2 + H (ξ - p ξ)/R]]]`

Out[80]= $\boxed{\frac{L\,\text{Sin}\left[\frac{H\,(-1+p)\,\xi}{R}\right]}{4\,\pi\,R^2}}$

Both of these clearly go to zero as H/R becomes a constant as p → 1 and the Sin[0] is 0 thus making the entire expression equal to zero. We can also employ the small angle approximation and say that

$$\frac{L\,\text{Sin}\left[\frac{H\,(-1+p)\,\xi}{R}\right]}{4\,\pi\,R^2} \sim \frac{L}{4\,\pi\,R^2}\,\frac{H\,(-1+p)\,\xi}{R}$$

Part (c)

If we assume that the emitting area is a perfect blackbody, than the cooling rate (i.e. $\frac{dQ}{dt}$) is equivalent to

In[81]:= `Framed[σ Teff^4]`

Out[81]= $\boxed{\sigma\,T_{eff}^4}$

Part (d)

The distance between the central source and the unit area at height z is given by

In[82]:= Framed[$\sqrt{R^2 + z^2}$]

Out[82]= $\boxed{\sqrt{R^2 + z^2}}$

Therefore, the gravitational potential is given as

In[84]:= ClearAll[ϕ];

In[86]:= Framed[$\phi \to -\frac{G M}{\sqrt{R^2 + z^2}}$]

Out[86]= $\boxed{\phi \to -\frac{G M}{\sqrt{R^2 + z^2}}}$

We can approximate this potential by another Taylor Expansion to show that ϕ is roughly

In[97]:= Normal@Series[$-\frac{G M}{\sqrt{R^2 + z^2}}$, {z, 0, 2}]

Out[97]= $-\frac{G M}{\sqrt{R^2}} + \frac{G M \sqrt{R^2}\, z^2}{2 R^4}$

In[99]:= Framed[$\phi \to -\frac{G M}{R} + \frac{G M R z^2}{2 R^4}$]

Out[99]= $\boxed{\phi \to -\frac{G M}{R} + \frac{G M z^2}{2 R^3}}$

Therefore, we recover that $\phi \sim (1/2)\, \Omega^2 z^2$ where $\Omega = \sqrt{GM/R^3}$.

Part (e)

Assuming that the disk is isothermal and in equilibrium we can equate:

$$\frac{1}{\rho}\frac{dP}{d\rho} = -\frac{d\phi}{dz}$$

Taking P to be $c_s^2\, \rho$, we obtain:

In[103]:= Framed[DSolve[{$\frac{1}{\rho[z]}$ D[$c_s^2\, \rho[z]$, z] == - D[$\frac{1}{2}\Omega^2 z^2$, z], $\rho[0] == \rho0$}, $\rho[z]$, z][[1, 1]]]

Out[103]= $\boxed{\rho[z] \to e^{-\frac{z^2 \Omega^2}{2 c_s^2}}\, \rho0}$

We can re-express this equation in terms of z and H given the relationship that H = $\frac{c_s}{\Omega}$.

In[104]:= Framed[$\rho[z] \to e^{-\frac{z^2}{2 H^2}}\, \rho0$]

Out[104]= $\boxed{\rho[z] \to e^{-\frac{z^2}{2 H^2}}\, \rho0}$

Part (f)

Again, assuming that the disk is isothermal, we can equate the unit area cooling to the unit area heating from our solutions from previous parts so that:

$$\frac{L}{4\pi R^2}\,\frac{H(-1+p)\,\xi}{R} == \sigma T_{eff}^4$$

But from part (e) we know that

In[109]:= $H = \frac{c_s}{\Omega};$

$c_s = \left(\frac{k\,T}{\mu}\right)^{\frac{1}{2}};$

$\Omega = \left(\frac{G\,M}{R^3}\right)^{\frac{1}{2}};$

In[119]:= $$\frac{L}{4\pi R^2}\,\frac{H(-1+p)\cdot\xi}{R} == \sigma T^4$$

Out[119]= $$\frac{L(-1+p)\sqrt{\frac{k\,T}{\mu}}\;\xi}{4\pi\sqrt{\frac{G\,M}{R^3}}\;R^3} == T^4\,\sigma$$

Solving this for T gives that:

$$T \to \left(\left(\frac{1}{R^3}\right)\left(\frac{k}{\mu G M}\right)\left(\frac{L(p-1)\xi}{4\pi\sigma}\right)^2\right)^{\frac{1}{7}} \sim \frac{1}{R^{3/7}}$$

H // PowerExpand // Simplify

Out[131]= $$\frac{\sqrt{k}\,R^{3/2}\sqrt{T}}{\sqrt{G}\,\sqrt{M}\,\sqrt{\mu}}$$

Therefore,

$$H \to \frac{\sqrt{k}\,R^{3/2}\sqrt{T}}{\sqrt{G}\,\sqrt{M}\,\sqrt{\mu}}$$

But, again, we know that $T \sim 1/R^{3/7}$. Therefore,

In[133]:= ClearAll[H]

In[134]:= $$H \to \frac{\sqrt{k}\,R^{3/2}\sqrt{1/R^{3/7}}}{\sqrt{G}\,\sqrt{M}\,\sqrt{\mu}}$$

Out[134]= $$H \to \frac{\sqrt{k}\,R^{9/7}}{\sqrt{G}\,\sqrt{M}\,\sqrt{\mu}}$$

Therefore, we have shown that H ~ $R^{9/7}$. Thus, p = $\frac{d\ln H}{d\ln R} = \frac{9}{7}\frac{d\ln R}{d\ln R} = \frac{9}{7}$.

And because H ~ $R^{9/7}$, H/R ~ $R^{9/7} / R^{7/7}$ ~ $R^{2/7}$.