HW 7 - ASTR540

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Q1

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a)

Integrating and equating to given mass:

```
sa = First@Solve[Integrate[m * a m^-2.35, {m, .1, 20}] == 10^6]
{a → 185 356.}
```

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b)

Luminosity

```
Integrate[m^4 * a m^-2.35, {m, .1, 20}] 1 L⊙N /. sa
```

$1.96105 \times 10^{8} L^{\odot}_{N}$

Fraction

Integrating with different limits and dividing:

```
Divide @@ (Integrate[m^4 * a m^-2.35, {m, #, 20}] & /@ {5, .1})
0.974618
```

---

c)

Total mass divided by number of stars:

```
10^6 / Integrate[a m^-2.35, {m, .1, 20}] /. sa
0.325587
```

---

d)

Max mass

Solving for age equal to one:

```
mMax = m /. First@NSolve[10 m^-2.5 == 1, m]
2.51189
```

Luminosity

Substituting the above into luminosity equation:

```
Integrate[m^4 * a m^-2.35, {m, .1, mMax}] 1 L_N^⊙ /. sa
```

```
802926. L_N^⊙
```

Q2

---

a)

Finding constant

Integrating and equating to total number:

```
sa2 = First@Solve[Integrate[a m^-2.35, {m, .4, 100}] == 10^11]
{a → 3.92072 × 10^10}
```

Ratio

Integrating with different limits and dividing:

```
ratio = Divide @@ (Integrate[a m^-2.35, {m, #, 100}] & /@ {8, .4})
0.0169537
```

Total number

```
nT = Integrate[a m^-2.35, {m, 8, 100}] /. sa2
1.69537 × 10^9
```

Mass of remnants

Assuming mass of black-hole/neutron star = 1.4 solar mass

```
nT 1.4 M⊙
```

```
2.37352 × 10^9 M⊙
```

b)

Total mass of stars times mass of ejecta divided by mass of galaxy (assuming mass of milky way is 5*10^10 solar mass)

```
nT .05 / (5 × 10^10)
```

```
0.00169537
```

c)

Probability of a star being in a binary times the probability of it having mass greater than 8 solar mass

```
p = 1 / 2 ratio
```

```
0.00847686
```

Probability of both stars satisfying the above conditions times total number of stars

```
nS = p^2 10^11
```

```
7.18572 × 10^6
```

d)

```
sr = First@Quiet@
   Solve[PE == KE /. {PE → G M M / r, KE → 1 / 2 (2 m) v^2} /. {M -> 8 M⊙ , m -> 1.4 M⊙ , v -> 500 km/s }, r]
```

```
{r → 2.42669 × 10^10 m }
```

e)

Multiply the number in c) with the probability of distance being lesser than above:

```
nS r / 0.01 pc /. sr
```

```
565.112
```

Q3

Equation of relaxation

```
eqR = relaxationTime == k v^3 / (M ρ)
```

```
relaxationTime == k v^3/(M ρ)
```

Finding constant

```
sk = First@
  Solve[eqR /. {relaxationTime → .95 × 10^10 yr , v -> 200 km/s , ρ -> 1 × 10^6 M⊙ / 1 pc^3 , M -> 1 M⊙ }, k]
```

```
{k → 5.09548 × 10^-12 kg^2yr^4/m^6 }
```

---

a)

```
eqR /. {v -> 1 km/s , M -> 1 M⊙ , ρ -> 150 M⊙ / 2 pc ^3} /. sk
```

```
relaxationTime == 6.33333 × 10^7 yr
```

---

b)

Using values from Wikipedia:

```
eqR /. {v -> 200 km/s , M -> 10^10 M⊙ , ρ -> 10^13 M⊙ / 1.5 Mpc ^3} /. sk
```

```
relaxationTime == 3.20625 × 10^11 yr
```

---

c)

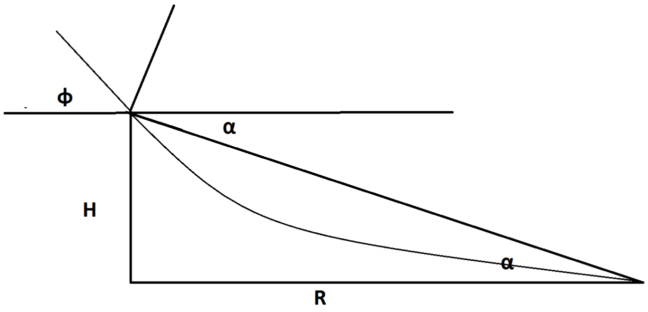
Assuming only 1% total mass of galaxy is matter:

```
eqR /. {v -> 200 km/s , M -> 10^10 M⊙ , ρ -> 10^11 M⊙ / 1.5 Mpc ^3} /. sk
```

```
relaxationTime == 3.20625 × 10^13 yr
```

Q4

We are defining $\xi H \to H$



a)

From the figure:

```
sθ = θ → π / 2 - ϕ + α /. {α → ArcTan[H / R], ϕ → ArcTan[H'[R]]}
```

$$\theta \to \frac{\pi}{2} + \text{ArcTan}\left[\frac{H}{R}\right] - \text{ArcTan}[H'[R]]$$

$$\frac{\partial H}{\partial R}$$

0

Substituting $p = \frac{\partial \log(H)}{\partial \log(R)}$, with $\frac{\partial H}{\partial R} = \frac{H}{R} \frac{\partial \log(H)}{\partial \log(R)}$ and expanding a Taylor series up to first order in H.

```
sθ2 = θ → (Series[sθ[[2]] /. {H'[R] → p H / R}, {H, 0, 1}] // Normal)
```

$$\theta \to \frac{\pi}{2} + \frac{H(1-p)}{R}$$

b)

The heating is given by:

```
L / (4 π R^2) Cos[θ] /. sθ
```

$$-\frac{1}{4\pi R^2} L \,\text{Sin}\left[\text{ArcTan}\left[\frac{H}{R}\right] - \text{ArcTan}[H'[R]]\right]$$

This is zero when R approaches zero with H/R a constant since the derivative is approximately H/R for small H and R.

Using the approximation gives:

```
L / (4 π R^2) Cos[θ] /. sθ2
```

$$-\frac{L \,\text{Sin}\left[\frac{H(1-p)}{R}\right]}{4\pi R^2}$$

This is clearly zero when p approaches 1 with H/R a constant.

c)

Cooling rate is given by

```
σ T^4
```

Where T is the effective temperature.

d)

Distance from the center to the point at height z is

$$\sqrt{R^2 + z^2}$$

This implies the gravitational potential is

$$-\frac{G M}{\sqrt{R^2 + z^2}}$$

Performing a Taylor expansion around z

$$\text{Series}\left[-\frac{G M}{\sqrt{R^2 + z^2}}, \{z, 0, 2\}\right] \text{ // Normal // PowerExpand}$$

$$-\frac{G M}{R} + \frac{G M z^2}{2 R^3}$$

This is equivalent to the required expression, since $\Omega = \sqrt{G M / R^3}$

e)

Since P is ρ cs^2:

```
sρ = DSolve[{1/ρ[z] D[cs^2 ρ[z], z] == -D[1/2 Ω^2 z^2, z], ρ[0] == ρ0}, ρ[z], z][[1, 1]]
```

$$\rho[z] \to e^{-\frac{z^2 \Omega^2}{2 cs^2}} \rho 0$$

Using H = cs/Ω

```
sρ /. cs → H Ω
```

$$\rho[z] \to e^{-\frac{z^2}{2 H^2}} \rho 0$$