HW 6 - ASTR404

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Q1)

Density in terms of radius

In[32]:= `sρ := ρ[r_] → ρc (1 - r/R*)`

$$s\rho := \rho[r\_] \to \rho_c \left(1 - \frac{r}{R_*}\right)$$

Nuclear energy production rate

In[33]:=

$$s\epsilon := \epsilon[r\_] \to \text{If}\left[r < \frac{R_*}{5}, \epsilon_c \left(1 - \frac{r}{\frac{R_*}{5}}\right), 0\right]$$

Total energy produced per second:

$$L = \int_0^{R_*} 4\,\pi\, r^2\, \rho(r)\, \epsilon(r)\, dr$$

Evaluating the integral:

In[34]:=

$$sL = L \to \text{Integrate}\left[4\,\pi\, r^2\, \epsilon[r]\, \rho[r]\ /.\ \{s\rho, s\epsilon\}, \{r, 0., R_*\}, \text{Assumptions} \to R_* > 0\right]$$

Out[34]=

$$L \to 0.007372271\, \epsilon_c\, \rho_c\, R_*^3$$

Q2)

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a)

Degeneracy is important if:

Fermi energy is greater than thermal energy.

In[35]:=

$$eqe = \frac{\left(\frac{3\, n\, h^3}{8\,\pi}\right)^{2/3}}{2\, m_e} > \frac{3}{2}\, k\, Tc;$$

Number density at center given μ

In[36]:=
```
sn[μ_] := n → ρc/(μ (m_p));
```

Scaling relation for central density

In[37]:=
```
sρc = ρc → ρcSun (M/M☉)^(-2/7) /. ρcSun → 160000 kg/m^3;
```

Scaling relation for central temperature

In[38]:=
```
sT = Tc → TcSun (M/M☉)^(4/7) /. TcSun → 1.57×10^7 K;
```

Substituting and finding range of M

In[39]:=
```
Quiet@Reduce[eqe //. {sn[1.17], sρc, sT}, M] /. m_Quantity :> UnitConvert[m, "SolarMass"]
```

Out[39]= $0\,M_\odot < M < 0.2412733\,M_\odot$

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b)

Coulomb interaction energy dominates if:

It is greater than the thermal energy.

In[40]:=
```
eqC = (1.9 e^2 n^(1/3))/(4 π ε0) > 3/2 k Tc;
```

Substituting and finding range of M

In[41]:=
```
Quiet@Reduce[eqC && M > 0 //. {sn[1.29], sρc, sT}, M] /. m_Quantity :> UnitConvert[m, "SolarMass"]
```

Out[41]= $0\,M_\odot < M < 0.01347966\,M_\odot$

Q3)

Luminosity of a shell

The luminosity in a shell of mass dm is the nuclear energy generated - heat absorbed:

In[42]:=
```
eqL := dL → ϵ dm - T S′[t] dm
```

Rate of change in entropy

In[43]:=
$$\text{eqS} = \text{S}'[\text{t}] \to \partial_t \left(\frac{\text{k Log}\left[\frac{\text{P[t]}}{\rho[\text{t}]^{\gamma}}\right]}{\mu\, \text{m}_\text{H} (\gamma - 1)} + \text{const} \right);$$

Substituting this in the first equation

In[44]:= `sL = eqL /. eqS`

Out[44]=
$$\text{dL} \to \text{dm}\,\epsilon - \left(\text{dm k T}\,\rho[\text{t}]^{\gamma} \left(\rho[\text{t}]^{-\gamma}\, \text{P}'[\text{t}] - \gamma\, \text{P}[\text{t}]\, \rho[\text{t}]^{-1-\gamma}\, \rho'[\text{t}]\right)\right) \Big/ \left((-1+\gamma)\, \mu\, \text{P}[\text{t}]\, \text{m}_\text{H}\right)$$

Checking whether this is equivalent to the given equation

In[45]:=
$$\frac{\text{dL}}{\text{dm}} == \epsilon - \frac{\rho[\text{t}]^{-1+\gamma}}{-1+\gamma} \partial_t \left(\frac{\text{P[t]}}{\rho[\text{t}]^{\gamma}}\right) \text{ /. sL /. P} \to \left(\frac{\text{k T}\,\rho[\#]}{\mu\, \text{m}_\text{H}} \&\right) \text{ // Simplify}$$

Out[45]= True

QED.

Q4)

a)

Final density of bubble

In[46]:= `ρbf = Series[ρb[r + dr], {dr, 0, 1}]`

Out[46]= $\rho\text{b}[\text{r}] + \rho\text{b}'[\text{r}]\, \text{dr} + \text{O}[\text{dr}]^2$

Gravitational force per unit volume on the bubble

In[47]:= `Fg = ρbf g`

Out[47]= $\text{g}\,\rho\text{b}[\text{r}] + \text{g}\,\rho\text{b}'[\text{r}]\, \text{dr} + \text{O}[\text{dr}]^2$

b)

Pressure of gas at final position of bubble

In[48]:= `ρgf = Series[ρg[r + dr], {dr, 0, 1}]`

Out[48]= $\rho\text{g}[\text{r}] + \rho\text{g}'[\text{r}]\, \text{dr} + \text{O}[\text{dr}]^2$

Entropy is conserved for adiabatic process

In[49]:= `eqEn[ρ_] := D[P[r] / ρ[r]^γ, r] == 0`

Solving above for ρ'(r) assuming pressure equilibrium

```
In[50]:= sρg = Solve[eqEn[ρg], ρg'[r]][[1, 1]]
```

Out[50]= $\rho g'[r] \to \frac{\rho g[r]\, P'[r]}{\gamma P[r]}$

Buoyant force per unit volume on the bubble

```
In[51]:= Fb = ρgf g /. sρg
```

Out[51]= $g\, \rho g[r] + \frac{g\, \rho g[r]\, P'[r]\, dr}{\gamma P[r]} + O[dr]^2$

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c)

Net force

Subtracting buoyant force from gravitational force and using Newton’s second law:

```
In[52]:= eqf = ρ[r] r''[t] == Fg - Fb /. ρb | ρg → ρ // Normal
```

Out[52]= $\rho[r]\, r''[t] == dr \left(-\frac{g\, \rho[r]\, P'[r]}{\gamma P[r]} + g\, \rho'[r]\right)$

Solving for r’’(t)

```
In[53]:= Solve[eqf, r''[t]][[1, 1]] // Simplify
```

Out[53]= $r''[t] \to dr\, g \left(-\frac{P'[r]}{\gamma P[r]} + \frac{\rho'[r]}{\rho[r]}\right)$

This is equivalent to the given equation.

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d)

Entropy equation

```
In[54]:= sS = S → P[r] / ρ[r]^γ
```

Out[54]= $S \to P[r]\, \rho[r]^{-\gamma}$

Rate of change in entropy with radius

```
In[75]:= D[S[r], r] / (P[r] ρ[r]^-γ) == (D[S /. sS, r] / (P[r] ρ[r]^-γ) // Simplify)
```

Out[75]= $\frac{\rho[r]^{\gamma}\, S'[r]}{P[r]} == \frac{P'[r]}{P[r]} - \frac{\gamma\, \rho'[r]}{\rho[r]}$

Since g is positive we have that N is positive only if S’(r) is negative.

Q5)

Formula for Eddington luminosity

In[56]:=
```
fEdd[M_, κ_] := UnitConvert[4 π M c G / κ, "SolarLuminosity"]
```

$$\text{fEdd}[M\_, \kappa\_] := \text{UnitConvert}\left[\frac{4\pi M\, c\, G}{\kappa}, \text{"SolarLuminosity"}\right]$$

a) Star with mass 0.072 $M_\odot$

Eddington luminosity

In[57]:= fEdd[0.072 $M_\odot$, 0.001 m^2/kg]

Out[57]= 94 061.28 $L_\odot$

Actual luminosity

In[58]:= 10^-4.3 $L_\odot$

Out[58]= 0.00005011872 $L_\odot$

This is much lesser than the Eddington luminosity; therefore the radiation pressure is not significant for such stars.

b) Star with mass 120 $M_\odot$

Eddington luminosity

In[59]:= fEdd[120 $M_\odot$, 0.04 m^2/kg]

Out[59]= 3.91922×10^6 $L_\odot$

Actual luminosity

In[60]:= 10^6.252 $L_\odot$

Out[60]= 1.786488×10^6 $L_\odot$

This is comparable to the Eddington luminosity. Therefore the radiation pressure is significant for such stars.