HW 12 - ASTR404

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Q1)

a)

Finding dimensionless combinations of the moment of inertia, radius and other constants:

```
const = First@DimensionalCombinations[{"MomentOfInertia", "Radius"},
   IncludeQuantities -> {"GravitationalConstant", "SpeedOfLight"}]
```

$$\frac{\left(1\, c^{2/3}/\sqrt[3]{G}\right)\ \text{Radius}}{\text{MomentOfInertia}^{1/3}}$$

We can obtain units of ω by cubing the reciprocal of the above and multiplying with ω.

```
ω / const^3
```

$$\frac{\omega\left(1\, G/c^2\right)\ \text{MomentOfInertia}}{\text{Radius}^3}$$

Since J is equal to the moment of inertia times ω, we get the desired result.

b)

Angular rotation rate for earth

Assuming earth is a uniform sphere of constant density.

```
fE = UnitConvert[G 2 / 5 [mass of earth] [radius of earth]^2
   [angular velocity of earth] / [radius of earth]^3 / c^2, "per year"]
```

0.1321" per year

Time taken for one rotation

```
Solve[t fE == 360°, t][[1, 1]]
```

t → 3.094×10^{14} s

---

c)

Angular rotation rate for fastest pulsar

The fastest pulsar is PSR J1748-2446ad according to Google. Assuming it is a uniform sphere of 2 solar mass and 16km radius.

```
fP =
 UnitConvert[G 2/5 2 M☉ 16 km ^2 angular velocity of PSR J1748−2446ad / 16 km ^3/ c ^2, rev / s]
```

52.89 rev/s

Time taken for one rotation

```
Solve[t fP == 360°, t][[1, 1]]
```

t → 0.01891 s

Q2)

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a)

Finding mass from constants

Finding dimensions of each constant, raising them to the powers x, y, z respectively and then solving for values giving units of mass.

```
mS = G ^x c ^y ℏ ^z /.
  First@Solve[Exponent[Times @@ MapThread[Power, {Times @@@ Apply[Power, UnitDimensions /@
            {G, c, ℏ}, {2}], {x, y, z}}] // PowerExpand, #] & /@
     {"LengthUnit", "MassUnit", "TimeUnit"} == {0, 1, 0}, {x, y, z}]
```

$1 \sqrt{\hbar} \sqrt{c} / \sqrt{G}$

(Need to report bug in the built-in DimensionalCombinations function...)

Evaluating numerically

```
mS // UnitConvert
```

2.176×10^{-8} kg

---

b)

```
rS = FormulaData[{"BlackHoleEventHorizonRadius", "Standard"}, {"M" → mS}] // Last
```

$3.232456632536755\times10^{-35}$ m

---

c)

```
rS / c // UnitConvert
```

$1.078231471899388\times10^{-43}$ s

Q3)

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a)

Solving for separation in terms of period

Let M be the total mass of the system.

In[21]:=
```
sa = Solve[P[t]^2 / a^3 == 4 π^2 / (G M), a][[1, 1]]
```

Out[21]= $a \to \frac{G^{1/3} M^{1/3} P[t]^{2/3}}{(2\pi)^{2/3}}$

Equation of mass transfer

From the lecture notes we have:

In[19]:=
```
eqM = D[a[t], t] / a == 2 M1dot (M1 - M2) / (M1 M2)
```

Out[19]= $\frac{a'[t]}{a} == \frac{2\,\text{M1dot}\,(\text{M1}-\text{M2})}{\text{M1 M2}}$

Substituting for a from the first equation

In[27]:=
```
eqM /. a'[t] → D[a /. sa, t] /. sa // FullSimplify
```

Out[27]= $3\,\text{M1dot}\left(-\frac{1}{\text{M1}}+\frac{1}{\text{M2}}\right) == \frac{P'[t]}{P[t]}$

This is identical to the given equation.

---

b)

Approximating derivatives up to first order:

```
Solve[1/P ΔP/Δt == 3 M1dot (M1 - M2)/(M1 M2) /.
  {P -> 2.49 days , ΔP → 20 s , Δt -> 100 yr , M1 -> 2.9 M☉ , M2 -> 1.4 M☉ }, M1dot][[1, 1]]
```

M1dot → 5.28853×10^{16} kg/s

This is positive. Therefore, M1 is gaining mass.

Q4)

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a)

Kepler's law

```
sa = a → (2 π / Ω)^2 G (M1 + M2) / (4 π^2)
```

$a \to \frac{G\,(\text{M1}+\text{M2})}{\Omega^2}$

Substituting this in the equation

```
sJ = Jorb -> M1 M2 / (M1 + M2) a^2 Ω /. sa
```

$\text{Jorb} \to \frac{G^2\,\text{M1}\,\text{M2}\,(\text{M1}+\text{M2})}{\Omega^3}$

This is the required relation. The constant of proportionality is: $G^2\,\text{M1}\,\text{M2}\,(\text{M1}+\text{M2})$

b)

Equation for I_1

```
sI = Solve[I1 Ω + Jorb == Jtot /. sJ, I1][[1, 1]]
```

$$I1 \to \frac{-G^2\,M1^2\,M2 - G^2\,M1\,M2^2 + Jtot\,\Omega^3}{\Omega^4}$$

Maximizing over Ω

Finding the point where derivative is zero.

```
sΩ = Solve[D[I1 /. sI, Ω] == 0, Ω][[2, 1]]
```

$$\Omega \to \frac{2^{2/3}\,G^{2/3}\,M2^{1/3}\,\left(M1^2 + M1\,M2\right)^{1/3}}{Jtot^{1/3}}$$

J_1

Substituting this value in the above equation.

```
J1 == I1 Ω /. sI /. sΩ /. sJ /. Jtot → J1 + Jorb // Simplify
```

J1 == 3 Jorb

Q5)

a)

Energy released per kg of iron

Assuming mass of iron produced is approximately same as mass burned.

In[85]:= `EperKg = 7.3 × 10^13 J / 1 kg`

Out[85]= 7.3×10^{13} J/kg

Potential energy of star

In[87]:= `PE = -5.1 × 10^43 J;`

Initial thermal energy of the star

Assuming virial equilibrium.

In[90]:=
```
thermalE = -PE / 2
```
Out[90]= 2.55×10^{43} J

For gravitationally unbound system

The net energy must be zero. This implies:

In[96]:=
```
sE = Solve[thermalE + PE + releasedE == 0, releasedE][[1, 1]]
```
Out[96]= releasedE → 2.55×10^{43} J

Mass of iron that must burn

In[106]:=
```
UnitConvert[releasedE / EperKg /. sE, "SolarMass"]
```
Out[106]= 0.175673 $M_\odot$

---

b)

Energy of ejecta

In[103]:=
```
ejectaE = UnitConvert[1 / 2 1.38 M☉ 5000 km/s ^2, "Joules"]
```
Out[103]= 3.43005×10^{43} J

Mass of additional iron burned

In[105]:=
```
UnitConvert[ejectaE / EperKg, "SolarMass"]
```
Out[105]= 0.236301 $M_\odot$