HW 5 - ASTR540

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Q1)

a)

Number of electrons

Asking Mathematica:

In[1]:=

number of protons in 50ktons of water

Result

```
1.6714*10^34
```

Out[1]= 1.6714×10^{34}

b)

Number of detections

In[2]:=

```
Flux * DetectionRate * CrossSection * ElectronNumber /.
  {Flux -> 6.7 × 10^10 per centimeter^2 per second, CrossSection → 10^-43 cm^2,
   DetectionRate → 10^-6, ElectronNumber → %} // UnitConvert[#, "1/Days"] &
```

Out[2]= 9.6754 per day

Q2)

a)

Given equations

In[3]:=

```
eqs = {n_e == 8 π / (3 h^3) p_f^3, p_f == Sqrt[2 m_e Ef]}
```

Out[3]= $\left\{n_e == \frac{8\pi p_f^3}{3 h^3},\ p_f == \sqrt{2}\ \sqrt{Ef\, m_e}\right\}$

Solving for fermi energy (E_f)

In[4]:= `sEf = Solve[eqs, {Ef, pf}][[2, 1]]`

Out[4]= $Ef \to \frac{h^2 \left(\frac{3}{\pi}\right)^{2/3} n_e^{2/3}}{8\, m_e}$

b)

Thermal energy of the system

Assuming Z/A ~ 0.5

In[5]:= `sEth = Eth -> 9/8 (2 ne) k T;`

Electron number density

In[6]:= `se = ne → ρ / mp / 2;`

Numerical value of n_e:

In[7]:= `ne -> se[[2]] /. {mp -> mp , ρ → 10^6 g/cm^3}`

Out[7]= $n_e \to 2.9893187 \times 10^{29} / cm^3$

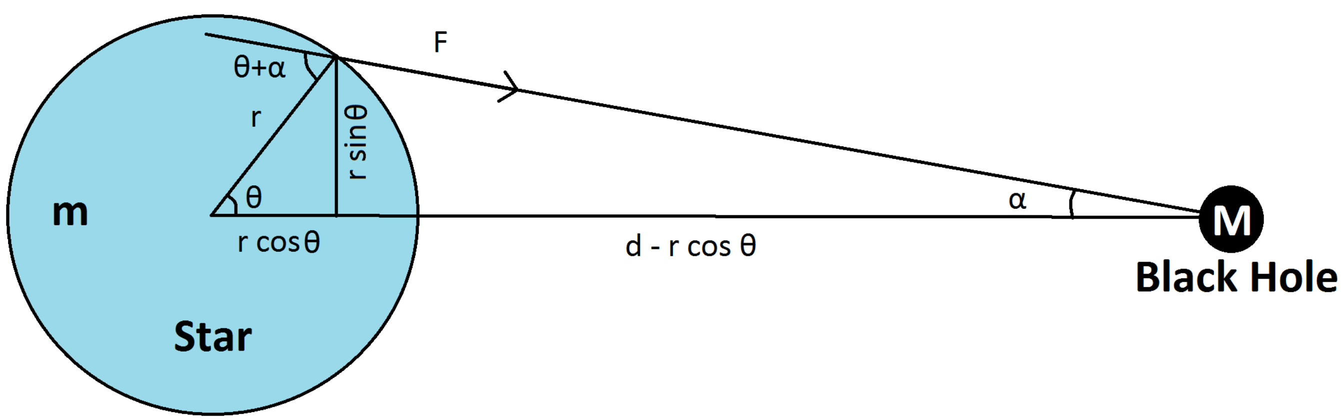
Thermal energy per electron / fermi energy

In[8]:= `(Eth / ne) / Ef //. {sEf, sEth, se} /. {mp -> mp , me -> me , h → h , k -> k , ρ → 1 × 10^6 g/cm^3 , T → 1 × 10^7 K}`

Out[8]= 0.0118933

Q3)

In[300]:=



a)

Radial tidal force exerted at $\theta = 0$

```
In[9]:= FTidal := 2 G dm M r / d^3
```

Gravitational binding force

```
In[10]:= FGrav := G dm m / r^2
```

Equating the two and solving for d

```
In[11]:= sd = Solve[FTidal == FGrav, d][[2, 1]]
```

Out[11]= $d \to \frac{2^{1/3} M^{1/3} r}{m^{1/3}}$

b)

Solving for d = Schwarzschild radius

```
In[12]:= sM = Solve[d == 2 G M / c^2 /. sd, M][[3, 1]]
```

Out[12]= $M \to \frac{c^3 r^{3/2}}{2 G^{3/2} \sqrt{m}}$

Numerical value

```
In[13]:= UnitConvert[
  sM[[2]] /. {r -> Sun (star) ["Radius"], m -> Sun (star) ["Mass"], c -> c, G -> G}, "SolarMass"]
```

Out[13]= $1.617 \times 10^8 M_\odot$

c)

Tangential tidal force at $\theta = \pi / 2$

At $\theta = 90°$, from the figure we get:

```
In[14]:= FTidalT = -G dm M / ((r Sin[θ])^2 + (d - r Cos[θ])^2) Cos[θ + α] /. θ → π / 2
```

Out[14]= $\frac{dm\, G\, M\, \mathrm{Sin}[\alpha]}{d^2 + r^2}$

At r << d, the tangential tidal force per unit mass is:

```
In[15]:= FTidalT / dm /. d^2 + r^2 → d^2 /. Sin[α] → r / d
```

Out[15]= $\frac{G M r}{d^3}$

Q4)

a)

Force (F) on each star due to the other

In[16]:=
```
sF = F → G M^2 / r^2
```

Out[16]= $F \to \frac{G M^2}{r^2}$

Total potential energy (PE)

This is the energy required to bring the system together from infinity:

In[17]:=
```
sPE = PE → Integrate[F /. sF, {r, ∞, a}, Assumptions → a > 0]
```

Out[17]= $PE \to -\frac{G M^2}{a}$

Orbital velocity (v) of each star

Equating gravitational force to centrifugal force:

In[18]:=
```
sv = Solve[M v^2 / (a / 2) == G M^2 / a^2, v][[2, 1]]
```

Out[18]= $v \to \frac{\sqrt{G}\,\sqrt{M}}{\sqrt{2}\,\sqrt{a}}$

Total kinetic energy (KE)

In[19]:=
```
sKE = KE → 2 × 1 / 2 M v^2 /. sv
```

Out[19]= $KE \to \frac{G M^2}{2 a}$

Total energy (TE)

In[20]:=
```
sTE = TE -> KE + PE /. {sKE, sPE}
```

Out[20]= $TE \to -\frac{G M^2}{2 a}$

b)

Rate of change of total energy with time

In[21]:=
```
sEt = TE'[t] -> D[TE /. sTE /. a → a[t], t]
```

Out[21]= $TE'[t] \to \frac{G M^2 a'[t]}{2 a[t]^2}$

Equating this to energy lost due to GWs

In[22]:=
```
eqa = TE'[t] == -2 c^5/(5 G) (2 G M/(c^2 a[t]))^5 /. sEt
```

Out[22]= $\frac{G M^2 a'[t]}{2 a[t]^2} == -\frac{64 G^4 M^5}{5 c^5 a[t]^5}$

Solving the above differential equation

In[23]:=
```
sa = DSolve[{eqa, a[0] == a0}, a[t], t][[4, 1]] // Simplify
```

Out[23]= $a[t] \to \frac{\left(a0^4 c^5 - \frac{512}{5} G^3 M^3 t\right)^{1/4}}{c^{5/4}}$

Here a0 is the current distance between the stars.

c)

Solving a(t) = 0 for a0

In[24]:=
```
sa0 = a0 -> Solve[a[t] == 0 /. sa, a0][[4, 1, 2]]
```

Out[24]= $a0 \to \frac{4 \left(\frac{2}{5}\right)^{1/4} G^{3/4} M^{3/4} t^{1/4}}{c^{5/4}}$

Numerical value of a0

In[25]:=
```
a0 -> UnitConvert[sa0[[2]] /. {G -> G, M -> 1 M☉, t -> 10 Gyr, c -> c}, "AU"]
```

Out[25]= $a0 \to 0.01579$ au

Q5)

a)

Electrostatic energy density

Potential energy of one atom divided by volume of a cube of side r.

In[26]:=
```
EED = 1/(4 π ε0) 1 e ^2 /r * 6/r^3
```

Out[26]= $\frac{\frac{3}{2\pi} e^2/\varepsilon_0}{r^4}$

b)

Radius of the planet

Equating the number of atoms:

In[27]:=
```
sR = NSolve[M/ m_p == 4/3 π R^3/r^3, R][[-1, 1]]
```

Out[27]= $R \to 5.226022 \times 10^{8}\, r \left(M \left(1/\mathrm{kg}\right)\right)^{1/3}$

Gravitational energy density

In[28]:=
```
GED = G M^2/R/(4/3 π R^3) /. sR
```

Out[28]= $$\frac{M^2 \left(3.200564 \times 10^{-36}\, G\right)}{r^4 \left(M \left(1/\mathrm{kg}\right)\right)^{4/3}}$$

c)

Equating the two and solving for M

In[29]:=
```
M -> Solve[EED == GED, M][[1, 1, 2]]/ 0.001 M_⊙ "M_Jupiter"
```

Out[29]= $M \to 8.29625\, M_{\mathrm{Jupiter}}$

Q6)

a)

Moment of inertia (mI) of the neutron star

In[30]:=
```
smI = mI → 2/5 M R^2
```

Out[30]= $mI \to \frac{2\, M\, R^2}{5}$

Angular momentum (J)

In[31]:=
```
sJ = Solve[(J / m/R^2)^2 R == G M/R^2, J][[2, 1]]
```

Out[31]= $J \to \sqrt{G}\, m \sqrt{M}\, \sqrt{R}$

Equating rate of change of angular momentum

In[32]:=
```
eqP = D[mI 2 π / P[t], t] == Ṁ J / m /. smI
```

Out[32]= $-\frac{4 M \pi R^2 P'[t]}{5 P[t]^2} == \frac{J \dot{M}}{m}$

---

b)

Solving the above equation for P(t)

In[33]:=
```
sP = DSolve[{eqP, P[0] == P0}, P[t], t][[1, 1]] /. sJ // Simplify
```

Out[33]= $P[t] \to \frac{4 \sqrt{M}\, P0\, \pi R^{3/2}}{4 \sqrt{M}\, \pi R^{3/2} + 5 \sqrt{G}\, P0\, t\, \dot{M}}$

Finding time at which P = 1ms

In[34]:=
```
t -> UnitConvert[
  NSolve[1 ms == sP[[2]] //. {M -> 1.4 M☉, R -> 10 km, Ṁ -> 10^-9 M☉/yr, P0 -> 1 s, G -> G}, t][[1, 1, 2]],
  "Years"]
```

Out[34]= $t \to 2.5788 \times 10^8\ \text{yr}$

Q7)

Equation for Eddington luminosity

In[35]:=
```
eqEL = 4 π c G M m / (2 A) == 1.3 × 10^38 ergs/s M / 1 M☉
```

Out[35]= $\frac{2 c G m M \pi}{A} == M \left(1.3 \times 10^{38}\ \text{ergs}/(M_\odot\ \text{s})\right)$

Solving the equation to find m

In[36]:=
```
Solve[eqEL, m][[1, 1, 2]] /. {A -> 1.5 m^2, G -> G, c -> c} // UnitSimplify
```

Out[36]= 78.0066 kg