HW 8 - ASTR540

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Q1)

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a)

Number density of photons

In[1]:= $\texttt{snP} := \texttt{nP} \to \frac{Q_*}{4\,\pi\,r^2\,c}$

Mean free path

In[2]:= $\texttt{sl} := \lambda \to \frac{1}{\texttt{nP}\,\sigma_{ion}}$

Ionization timescale

In[3]:= $\tau_{ion} \to \frac{\lambda}{c}$ /. sl /. snP

Out[3]= $\tau_{ion} \to \frac{4\,\pi\,r^2}{\sigma_{ion}\,Q_*}$

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b)

Recombination rate

In[4]:= sR := R_{rec} -> α n^2

Here n is the number density of atoms.

Recombination timescale

Reciprocal of recombination rate.

In[5]:= $\tau_{rec} \to 1 / R_{rec}$ /. sR

Out[5]= $\tau_{rec} \to \frac{1}{n^2\,\alpha}$

c)

Volume ionized in time dt

In[6]:=
```
svol := volume → 4 π r^2 v dt
```

Number of ionized atoms in this volume

In[7]:=
```
snI := nIonized → n volume
```

Number of photons in this volume

In[8]:=
```
snP := nPhotons → Q* dt
```

Equating the above & solving for v

In[9]:=
```
sv = Solve[snI[[2]] == snP[[2]] /. svol, v][[1, 1]]
```

Out[9]= $v \to \frac{Q_*}{4\, n\, \pi\, r^2}$

d)

Numerical values

In[10]:=
```
sN := {Q* → 3 × 10^49 per second , n → 10.^4 /cm^3 }
```

Plugging them in to find v given r

In[11]:=
```
{#, sv /. sN /. List /@ #} &@Thread[r → UnitConvert[{.01 pc , .05 pc , .1 pc }, km ]] // Dataset
```

Out[11]=

$r \to 3.085678 \times 10^{11}$ km	$r \to 1.542839 \times 10^{12}$ km	$r \to 3.085678 \times 10^{12}$ km
$v \to 2.507323 \times 10^{6}$ km/s	$v \to 100\,292.9$ km/s	$v \to 25\,073.23$ km/s

Solving the differential equation for v(r)

In[12]:=
```
sr = DSolve[{r'[t] == (v /. sv /. r → r[t]), r[0] == 0}, r[t], t][[2, 1]]
```

Out[12]= $r[t] \to \frac{\left(\frac{3}{\pi}\right)^{1/3} (t\, Q_*)^{1/3}}{2^{2/3}\, n^{1/3}}$

Time taken to reach 0.2pc

In[13]:=
```
UnitConvert[Solve[sr[[2]] == 0.2 pc /. sN, t][[1, 1, 2]], "Years"]
```

Out[13]= 10.40644 yr

Q2)

a)

Mass flux ($\dot{M}$)

In[14]:=
```
sM := fM → -4 π r^2 v ρ
```

Energy flux

In[15]:=
```
sE := fE → -4 Pi r^2 v (p + u + v^2 ρ/2 - G M/r ρ)
```

b)

Bernoulli parameter

Taking ratio of the fluxes:

In[16]:=
```
sB = Be -> fE / fM /. {sM, sE}
```

Out[16]= $Be \to \frac{p + u - \frac{G M \rho}{r} + \frac{v^2 \rho}{2}}{\rho}$

c)

Substituting equations for u and p

In[17]:=
```
sB2 = sB //. {p → κ ρ^γ, u → p/(-1 + γ)} // Simplify
```

Out[17]= $Be \to -\frac{G M}{r} + \frac{v^2}{2} + \frac{\gamma \kappa \rho^{-1+\gamma}}{-1+\gamma}$

Replacing ρ in terms of $\dot{M}$

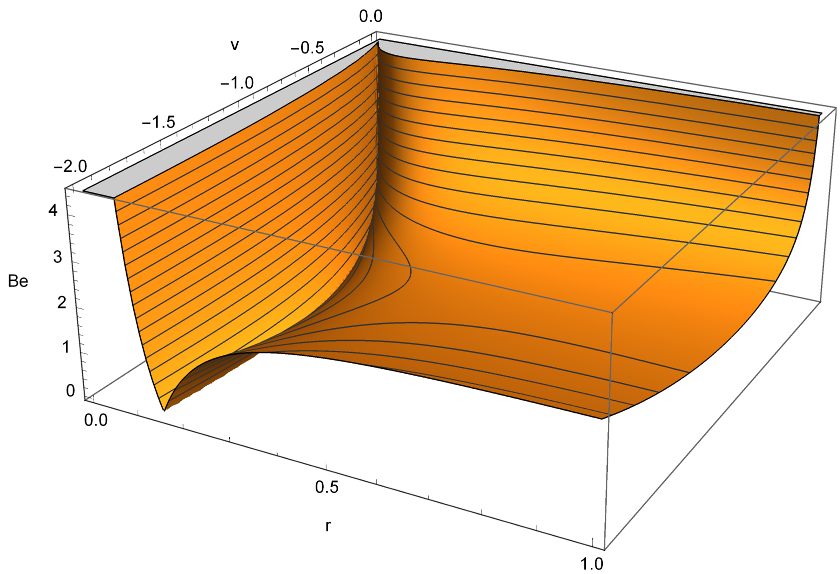
In[18]:=
```
sB3 = sB2 /. Solve[Ṁ == -4 Pi ρ r^2 v, ρ][[1, 1]]
```

Out[18]= $Be \to -\frac{G M}{r} + \frac{v^2}{2} + \frac{(4\pi)^{1-\gamma} \gamma \kappa \left(-\frac{\dot{M}}{r^2 v}\right)^{-1+\gamma}}{-1+\gamma}$

d)

Plot after setting constants = 1 and γ = 4/3

Out[19]=



d)

Saddle point of Be (zero gradient)

In[20]:=
```
srv = Solve[Grad[sB3[[2]], {r, v}] == {0, 0} /. γ → 4/3, {r, v}][[2]]
```

Out[20]=
$$\left\{r \to \frac{9\, G^{7/3}\, M^{7/3}\, \pi^{2/3}}{32\, \kappa^2\, \dot{M}^{2/3}},\ v \to -\frac{4\, \kappa\, \dot{M}^{1/3}}{3\, G^{2/3}\, M^{2/3}\, \pi^{1/3}}\right\}$$

Sound speed at this point

In[21]:=
$$cs \to \sqrt{\frac{p\,\gamma}{\rho}}\ //.\ \left\{p \to \kappa\,\rho^{\gamma},\ \gamma \to 4/3,\ \mathrm{Solve}\left[\dot{M} == -4\,\pi\,\rho\, r^2\, v,\ \rho\right][[1, 1]]\right\}\ /.\ srv\ //\ \mathrm{PowerExpand}$$

Out[21]=
$$cs \to \frac{4\, \kappa\, \dot{M}^{1/3}}{3\, G^{2/3}\, M^{2/3}\, \pi^{1/3}}$$

Therefore, velocity is the negative of the sound speed at the saddle point.

e)

Setting Be = Be(r→∞) and solving for $\dot{M}$

In[22]:=
```
Solve[Assuming[γ > 1, sB3[[2]] == Limit[sB3[[2]], r → ∞]], Ṁ][[1, 1]] // Quiet
```

Out[22]=
$$\dot{M} \to -r^2\, v \left(-\frac{G\, M\, (4\,\pi)^{-1+\gamma}\, (1-\gamma)}{r\,\gamma\,\kappa}\right)^{\frac{1}{-1+\gamma}}$$