HW 3 - ASTR540

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Q1)

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a)

Equations for binary system

oP is the orbital period, M is the total mass, mP and mS are the masses of the primary and secondary, x is the ratio mP / M, dS and dP are the distances to the center of mass, vS and vP are the actual velocities, whereas vSr and vPr are the observed radial velocities.

In[134]:=
```
eqs = {2 Pi dS / oP == vS (*Time period of secondary*),
    2 Pi dP / oP == vP (*Time period of primary*),
    vS mS == mP vP (*Conservation of momentum*),
    mP * vP^2 / dP == G mS mP / (dP + dS)^2 (*Centripetal force equals gravity*)
   } /. {vS -> vSr / Sin@i, vP -> vPr / Sin@i, mP → x M, mS → (1 - x) M};
```

Solving the equations numerically

In[182]:=
```
sol = Quiet@Solve[eqs /. {oP → 2 days , vSr → 100 km/s , vPr → 200 km/s }, {M, x, dP, dS}][[2]];
```

Mass ratio of the stars

In[136]:=
```
x / (1 - x) /. sol
```

Out[136]= 0.500

---

b)

m $\sin^3$(i) for each star

In[137]:=
```
ms3 = {x, 1 - x} UnitConvert[M Sin[i]^3, "SolarMass"] /. sol
```

Out[137]= { 1.865 $M_\odot$, 3.73 $M_\odot$ }

c)

Mean value of $\sin^3(i)$

For azimuthal symmetry the ϕ components of the integral cancel. Therefore the mean of $\sin^3(i)$ is:

```
In[138]:= si = Sin[i]^3 -> Integrate[Sin[i]^3 Sin[i] , {i, 0, Pi}] / Integrate[Sin[i] , {i, 0, Pi}]
```

Out[138]= $\mathrm{Sin}[i]^3 \to \frac{3\pi}{16}$

Substituting this value in b)

```
In[139]:= ms3 / (Sin[i]^3 /. si)
```

Out[139]= $\{3.17\,M_\odot,\ 6.33\,M_\odot\}$

Q2)

a)

Solving equations in Q1 with different parameters

```
In[165]:= sol2 = Quiet@Solve[eqs /. {oP → 5 yr , vSr → 20 km/s , vPr → 5 km/s }, {M, x, dP, dS}][[2]];
```

Why i = 90° is a good approximation

Taylor series of 90° + δ is approximately 1 as shown below:

```
In[162]:= Series[Sin[Pi / 2 + δ]^3, {δ, 0, 1}]
```

Out[162]= $1 + O[\delta]^2$

Masses of the stars at i = 90°

```
In[166]:= m90 = UnitConvert[{x, 1 - x} M /. sol2 /. i → Pi / 2, "SolarMass"]
```

Out[166]= $\{2.364\,M_\odot,\ 0.591\,M_\odot\}$

b)

Relative velocities assuming i = 90°

The relative velocity is the sum of the observed radial velocities.

```
vrel = 25 km/s ;
```

Radius of secondary

Time taken to till minima is the time taken to travel through the diameter of secondary.

```
In[180]:= "rS" -> UnitConvert[Solve[vrel == 2 rS / 0.3 days , rS][[1, 1, 2]], "SolarRadius"]
Out[180]= rS → 0.46585190510424 R⊙
```

Radius of primary

Time taken to complete minima is the time taken to travel through the diameter of primary minus twice the diameter of secondary.

```
In[181]:= "rP" -> UnitConvert[Solve[vrel == (2 rP - 2 rS) / 1 days /. srS, rP][[1, 1, 2]], "SolarRadius"]
Out[181]= rP → 2.018691588785 R⊙
```

Sensitivity to inclination

These values we obtained for radii are sensitive to changes in inclination because the crossing will not happen across the equatorial planes of the stars for i ≠ 90° hence we would not be able to use the diameter in the above equations.

Q3)

Components of pressure

```
In[143]:= {Pg, Prad} = {ρ k T / m, 1/3 a T^4};
```

Constants at core and surface (from the Wikipedia)

```
In[144]:= core = {ρ -> 150 g/cm^3 , T → 1.5 × 10^7 K , m → .5 hydrogen (element) [ atomic mass ]};
          surface = {ρ -> 1/10^9 g/cm^3 , T → 5777 K , m → 1.3 hydrogen (element) [ atomic mass ]};
```

Substituting values at core

```
In[146]:= UnitConvert[{Pg, Prad} /. core, "Pascals"]
Out[146]= { 3.7120333934702 × 10^16 Pa , 1.2767158155126 × 10^13 Pa }
```

Therefore we can see that the kinetic pressure dominates.

Substituting values at surface

In[147]:=
```
UnitConvert[{Pg, Prad} /. surface, "Pascals"]
```

Out[147]= {36.657122930047 Pa, 0.28089 Pa}

Therefore we can see that the kinetic pressure dominates again. Hence kinetic pressure must be dominant throughout the sun.

Q4)

a) Q / k_2

Defining the equation

In[148]:=
```
eq = D[a[t], t] == 3 a[t] Ω m / M R^5 / a[t]^5 k / Q
```

Out[148]= $a'[t] == \frac{3\,k\,m\,R^5\,\Omega}{M\,Q\,a[t]^4}$

Getting Ω from Kepler's third law

In[149]:=
```
sΩ = Ω -> Sqrt[G M / a[t]^3]
```

Out[149]= $\Omega \to \sqrt{\frac{G\,M}{a[t]^3}}$

Defining constants

In[150]:=
```
consts = {m → Moon (planetary moon) ["Mass"], M → Earth (planet) ["Mass"],
   R → Earth (planet) ["Radius"], aNow -> N@ Moon (planetary moon) ["SemimajorAxis"]};
```

Substituting numbers and solving for Q/k in SI units

In[151]:=
```
sQ = NSolve[eq /. sΩ /. {a'[t] → 3.8 cm/yr , a[t] → aNow, G -> G} /. consts, Q][[1, 1]]
```

Out[151]= Q → 38.889755084982 k

b) Time to collision

Solving the differential equation analytically

In[152]:=
```
sa = DSolve[{eq /. sΩ, a[0] == a0}, a[t], t][[1, 1]] // FullSimplify // PowerExpand
```

Out[152]= $a[t] \to \frac{G^{1/13}\left(\frac{2\,a0^{13/2}\sqrt{M}\,Q}{\sqrt{G}} + 39\,k\,m\,R^5\,t\right)^{2/13}}{2^{2/13}\,M^{1/13}\,Q^{2/13}}$

Plugging numbers and finding time when a = 0

In[153]:=
```
UnitConvert[
 Solve[a[t] == 0 m /. sa /. {a0 → aNow, G -> G} /. consts /. sΩ /. sQ /. k → 1, t][[1, 1, 2]], "Years"]
```

Out[153]= $-1.5562753036437 \times 10^{9}$ yr

c) Critical distance

Solving the equation for a(T) = 0 for a(0)

In[154]:=
```
sa0 = a0 → ToRadicals@
  Assuming[Q > 0 && M > 0 && G > 0, Solve[a[t] == 0 /. sa, a0, Reals] /. t → T // Simplify][[1, 1, 2, 1]]
```

Out[154]= $a0 \to \dfrac{(-39)^{2/13}\, G^{1/13}\, k^{2/13}\, m^{2/13}\, R^{10/13}\, T^{2/13}}{2^{2/13}\, M^{1/13}\, Q^{2/13}}$

d) Angular velocity

Finding angular velocity at critical distance

In[155]:=
```
Ωc = Sqrt[G M / a0^3] /. sa0
```

Out[155]= $\left(\dfrac{2}{39}\right)^{3/13} \sqrt{-\dfrac{(-1)^{7/13}\, G^{10/13}\, M^{16/13}\, Q^{6/13}}{k^{6/13}\, m^{6/13}\, R^{30/13}\, T^{6/13}}}$

Plugging in the numerical values for orbital period

In[159]:=
```
"Orbital period " ==
 UnitConvert[2 Pi / Abs@Ωc /. {m -> Jupiter (planet) ["Mass"], M → Sun (star) ["Mass"], k → 10^-5 Q,
   G -> G, R → Sun (star) ["Radius"], T -> Sun (star) ["MainSequenceLifetime"]}, "Days"]
```

Out[159]= Orbital period == 6.5 days

e)

The timescale for evolution for orbital eccentricity is usually shorter than the decay timescale.

f)

The tides raised by the sun should lead to evolution of both the moon’s and earth’s orbits.