Probability Theory

1. Given $X = \{x_1, .., x_n\}$ $\quad X \sim B(x, p) = \binom{n}{x} p^x (1-p)^{n-x}$

$L(p) = \log P(X|p)$ want $p = \text{argmax} LL(p|X)$

$$\frac{\partial(\log P(X|p))}{\partial p} = \frac{\partial(\log \prod_{n=1}^{n} \binom{n}{x_i} p^{x_i}(1-p)^{n-x_i})}{\partial p} = \frac{\partial}{\partial p} \underbrace{C}_{\text{constant not dep on } p} \log(\prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i})$$

$$= \frac{\partial}{\partial p}\left(\log p^x (1-p)^{n-x}\right) =$$

$X = \sum_{i=1}^{n} x_i$

$$= \frac{\partial(x \log(p))}{\partial p} + \frac{\partial(n-x)(1-p)}{\partial p} = x\frac{1}{p} + \frac{(n-x)(-1)}{1-p}$$

Set $\frac{\partial(LL)}{\partial p} = 0 \Rightarrow \frac{x}{p} = \frac{n-x}{1-p} \Rightarrow \boxed{p = \frac{x}{n}}$

2. $P(pass) = .6$, $P(pass^c) = P(fail) = .4$

(I) $P(S|pass) = .95 = P(S \cap pass)/P(pass) \underset{\text{bayes}}{=} P(Pass|S) \cdot P(S)/P(pass)$

(II) $P(S|fail) = .6 = P(fail|S) \cdot P(S)/P(fail)$

from (I) $P(S \cap pass) = .95 \times .6 = .57$, (II) $P(S \cap fail) = .24$

$P(S) = P(S \cap pass) + P(S \cap fail) = .57 + .24 = .81$

$\Rightarrow P(pass|S) = P(S|p) \cdot P(pass)/P(S) = .95 \times .6/.81 = .70$

$P(pass \cap S^c) + P(pass \cap S) = P(pass) = .6 \Rightarrow P(pass \cap S^c) = .03$

$\Rightarrow P(pass|S^c) = P(S^c \cap pass)/P(S^c) = .03/.19 = .16$

$\boxed{P(pass \mid studied) = 70\%, \ P(pass \mid not\ study) = 16\%}$

4. $X \sim B(.75, 7)$: 1 trail each day for 7 days

a) what is prob of 2 successes in 7 days?

$P(X=2) = \binom{7}{2}(.75)^2(.25)^5$

b) $P(X \geq 4) = 1 - P(X \leq 3) = 1 - P(X=0) - P(X=1) - P(X=2) - P(X=3)$

$= 1 - \binom{7}{0}(.25)^7 - \binom{7}{1}(.75)(.25)^6 - \binom{7}{2}(.75^2)(.25^5) - \binom{7}{3}(.75)^3(.25)^4$

$= .929$

c) Given $X_1, \ldots X_n$ such that $X_i \sim B(.75, 7)\ \forall\ 1 \leq i \leq 100$

X_i are i.i.d so according to the law of large numbers

$\bar{X}_n = \frac{1}{n}(X_1, \ldots X_n) \xrightarrow[n \to \infty]{} \mu \qquad \mu = np = .75 \cdot 7$

so we expect the average result to converge to 5.25

3. $\Sigma_X = \{4,5,6,7\} \quad \Sigma_Y = \{4,5,6,7\} \quad \Sigma_C = \{0,1\}$

$P(C=0) = .3 \quad P(C=1) = .7$

$P(X=4|C=0) = .4 \quad P(X=6|C=0) = .6$ } all other probabilites = 0

$P(X=5|C=1) = .4 \quad P(X=7|C=1) = .6$

$P(1 \leq X \leq 5) = P(X=4) + P(X=5) = P(X=4 \cap C=0) + P(X=4 \cap C=1) + P(X=5 \cap C=0) + P(X=5 \cap C=1)$

$\overset{**}{=} .4 \cdot .3 + 0 + 0 + .4 \cdot .7 = \boxed{.4}$ (ii) verified ✓

$P(X \cap C) = P(X|C) \cdot P(C)$

$P(Y=4|C=0) = .4 \quad P(Y=6|C=0) = .6 \quad P(Y=5|C=1) = .4 \quad P(Y=7|C=1) = .6$

$\Rightarrow$ same as X we get $P(1 \leq Y \leq 5) = .4$

$P(X=4, Y=4, C=0) = (.4)^2(.3) \qquad P(X=6, Y=6, C=0) = (.6)^2(.3)$

$P(X=5, Y=5, C=1) = (.4)^2(.7) \qquad P(X=7, Y=7, C=1) = (.6)^2(.7)$

$P(X=4, Y=6, C=0) = (.4)(.6)(.3) \qquad P(X=6, Y=4, C=0) = (.6)(.4)(.3)$

$P(X=5, Y=7, C=1) = (.4)(.6)(.7) \qquad P(X=7, Y=5, C=1) = (.6)(.4)(.7)$

* easy to see that $X \perp Y | C$ - distribution was built that way.

$P(X \cap Y | C) = P(X \cap Y \cap C)/P(C) = P(X|C) \cdot P(Y|C)$

$\underbrace{P(X=4) = .12 \quad P(Y=4) = .12}_{\text{based on } **} \Rightarrow P(X=4) \cdot P(Y=4) = .24$

$P(X=4 \cap Y=4) = P(X=4, Y=4, C=0) + P(X=4, Y=4, C=1)$

$= .048 + 0 \Rightarrow .24 \neq .048$ so $X \not\perp Y$.