財務工程 HW2 學習歷程

Step 1 搞懂 e 在做甚麼

NO: 財務工程 <e>

DATE: 03/07/20

$f(x) = 2^x$

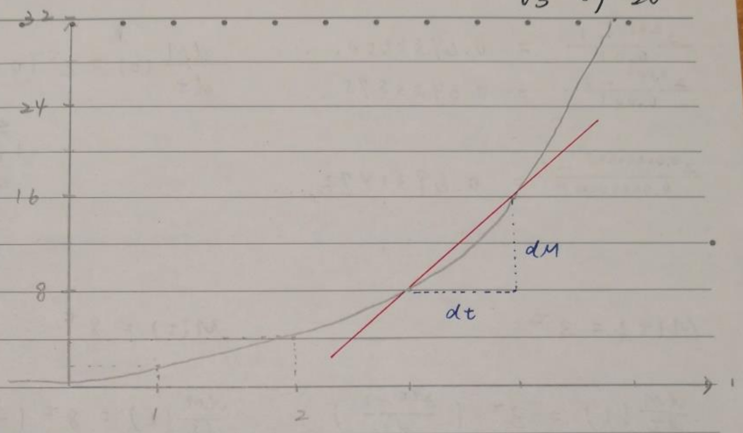
$\Downarrow$

$\underbrace{P(t)}_{\text{Population size}} = 2^t$

t = Time (in days)

$\Downarrow$

$\underbrace{M(t)}_{\text{Population mass}} = 2^t$



$\frac{dM}{dt} = ???$

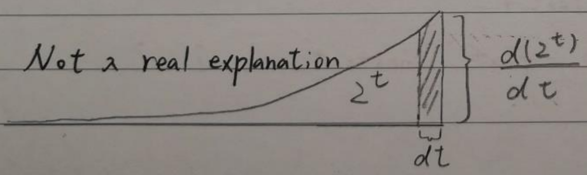
From day 3 to 4: $\frac{\text{"8" creatures}}{\text{1 day}}$

From day 4 to 5: $\frac{\text{"16" creatures}}{\text{1 day}}$

$\underbrace{\frac{d(2^t)}{dt} = 2^t}_{\text{Tempting}} \Rightarrow \underbrace{\frac{2^{t+1} - 2^t}{1} = 2^t}_{\text{Rate of change over one full day.}}$

$\frac{2^{t+dt} - 2^t}{dt} = ???$ Rate of change in a small time

Not a real explanation



More numerical than visual...

$$\frac{dM}{dt}(t) = \frac{2^t 2^{dt} - 2^t}{dt} = 2^t \left(\underbrace{\frac{2^{dt} - 1}{dt}}_{dt \to 0} \right)$$

$$\frac{\overbrace{2^{dt}-1}^{dt \to 0}}{dt}$$

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$$\frac{2^{0.01}-1}{0.01} = 0.6955550\ldots$$

$$\frac{2^{0.001}-1}{0.001} = 0.6933875\ldots$$

$$\vdots$$

$$\frac{2^{0.0000001}-1}{0.0000001} = 0.6931472\ldots$$

$$\frac{dM}{dt}(t) = 2^t\left(\underbrace{0.6931472}_{\underbrace{\frac{2^{dt}-1}{dt}}_{dt \to 0}}\right)$$

$\Rightarrow$ proportional to itself.

$M(t) = 3^t$

$M(t) = 8^t$

$$\frac{dM}{dt}(t) = 3^t\left(\underbrace{\frac{3^{dt}-1}{dt}}_{dt \to 0}\right)$$

$$\frac{dM}{dt}(t) = 8^t\left(\underbrace{\frac{8^{dt}-1}{dt}}_{dt \to 0}\right)$$

$$\frac{3^{0.0000001}-1}{0.0000001} = 1.0986123\ldots$$

$$\frac{8^{0.0000001}-1}{0.0000001} = 2.0794416\ldots$$

$$\Downarrow$$

$$\frac{dM}{dt}(t) = 3^t(1.0986123\ldots)$$

$$\frac{dM}{dt} = 8^t(2.0794416\ldots)$$

$$\frac{d(2^t)}{dt} = 2^t(0.6931\ldots)$$

$$\downarrow \times 3$$

Why ?

$$\frac{d(8^t)}{dt} = 8^t(2.0794\ldots)$$

$$\frac{d(a^t)}{dt} = a^t(1)(\text{Some constant})$$

Is there a base where that constant is 1 ?

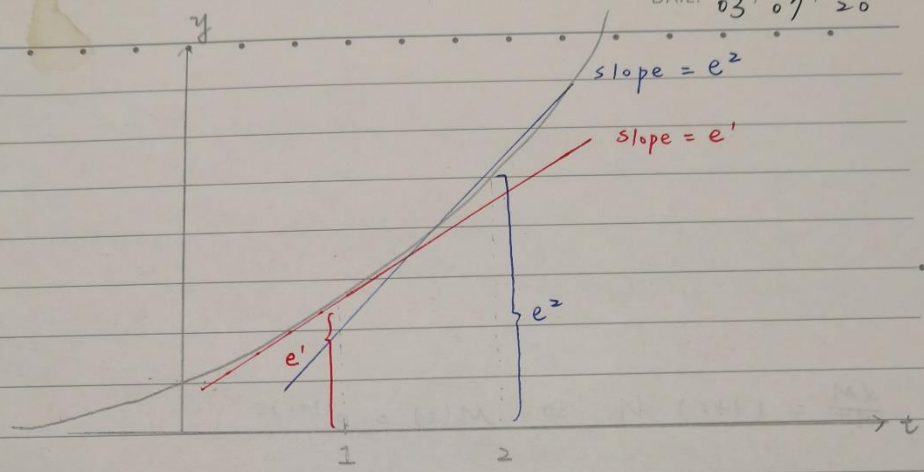
$e = 2.71828\ldots$

$M(t) = e^t$

$$\frac{e^{0.0000001}-1}{0.0000001} = 1.0000000\ldots \qquad \underbrace{\frac{e^{dt}-1}{dt}}_{dt \to 0}$$

$$\Downarrow$$

$$\frac{dM}{dt}(t) = e^t(1.0000000\ldots)$$



$\frac{d(2^t)}{dt} = 2^t(0.6931\ldots)$

$\downarrow \times 3$ Why?

$\frac{d(8^t)}{dt} = 8^t(2.0794\ldots)$ Think of the chain rule

$\frac{d(e^{3t})}{dt} = e^{3t} \cdot 3 \Rightarrow \frac{d(e^{ct})}{dt} = c \cdot e^{ct}$

$2 = e^{\ln(2)} \Rightarrow 2^t = e^{\ln(2)t} \xRightarrow{\text{Derivative}} \ln(2)2^t = \ln(2)e^{\ln(2)t}$

$\ln 2 = 0.6931\ldots$

$\frac{d(2^t)}{dt} = 2^t \overbrace{(0.6931\ldots)}^{\ln 2}$ $(0.6931\ldots) = \ln(2)$

$e^{(0.6931)} = 2$

$\frac{d(3^t)}{dt} = 3^t \overbrace{(1.0986\ldots)}^{\ln 3}$

$\frac{d(7^t)}{dt} = 7^t \overbrace{(1.9459\ldots)}^{\ln 7}$

There are many ways to write down any particular

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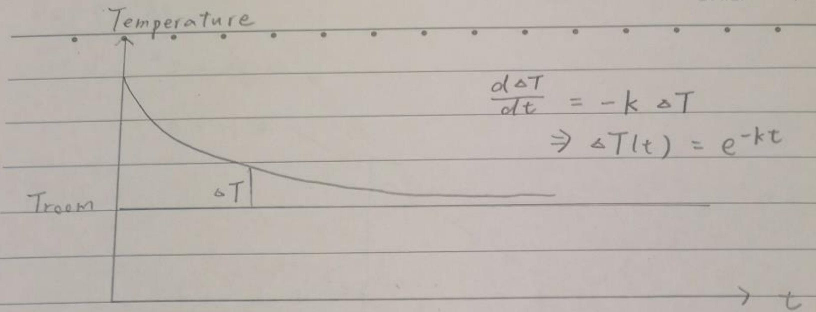

$$\frac{d\Delta T}{dt} = -k\,\Delta T$$

$$\Rightarrow \Delta T(t) = e^{-kt}$$

$$\frac{dM}{dt} = (1+r)M \Rightarrow M(t) = e^{(1+r)t}$$

$$FV = P\left(1+\frac{r}{m}\right)^{nm}$$

$$= \left(1+\frac{r}{m}\right)^{nm} \times P$$

令 $1+\frac{r}{m} = R$

$$\frac{dFV}{dn} = \frac{[(1+R)^m]^{n+dn} \times P - [(1+R)^m]^n \times P}{dn}$$

$$= P[(1+R)^m]^n \quad \frac{[(1+R)^m]^{dn} - 1}{dn}$$

2^t $\qquad \frac{d(2^t)}{dt} = ?$

$2 = e^{\ln 2}$ $\qquad \ln 2 \; \underbrace{e^{(\ln 2)t}}_{2^t = A(t)}$

$A(t) = 2^t = e^{(\ln 2)t}$

$\Rightarrow A(t) = e^{(\ln 2)t}$ $\qquad \frac{dM}{dt}(t) = (1+r)M(t)$

$\frac{dA}{dt}(t) = \ln 2 \; A(t)$ $\qquad M(t) = e^{(1+r)t}$

$A(t) = e^{(\ln 2)t}$

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Definition of e

1. $e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$

$e = \lim_{t\to 0} (1+t)^{\frac{1}{t}}$

2. $e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \dots$

3. $\int_1^x \frac{dt}{t} = 1$

4. $\lim_{h\to 0} \frac{x^h - 1}{h} = 1$

Proof of 1 :

$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} n \ln\left(1+\frac{1}{n}\right)$

$= \lim_{n\to\infty} \frac{\ln(1+\overset{\to 0}{\frac{1}{n}}) \to 0}{\frac{1}{n} \to 0}$

By l'Hôpital Rule

$\lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)\cdot\left(-\frac{1}{n^2}\right)}{\left(-\frac{1}{n^2}\right)} = 1$

$\Rightarrow \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n = 1$

$\ln \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = 1$

$\Rightarrow \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$ #

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Proof $[1+(\frac{r}{m})]^m \to e^r$

$\lim_{x\to\infty} (1+\frac{a}{x})^{bx}$, let $\frac{a}{x} = u \Rightarrow x = \frac{a}{u} \Rightarrow \begin{matrix} x\to\infty \\ u\to 0 \end{matrix}$

$\lim_{u\to 0} [(1+u)^{\frac{1}{u}}]^{ab}$

$= [\underbrace{\lim_{u\to 0} (1+u)^{\frac{1}{u}}}_{= e}]^{ab}$

$= e^{ab}$ #

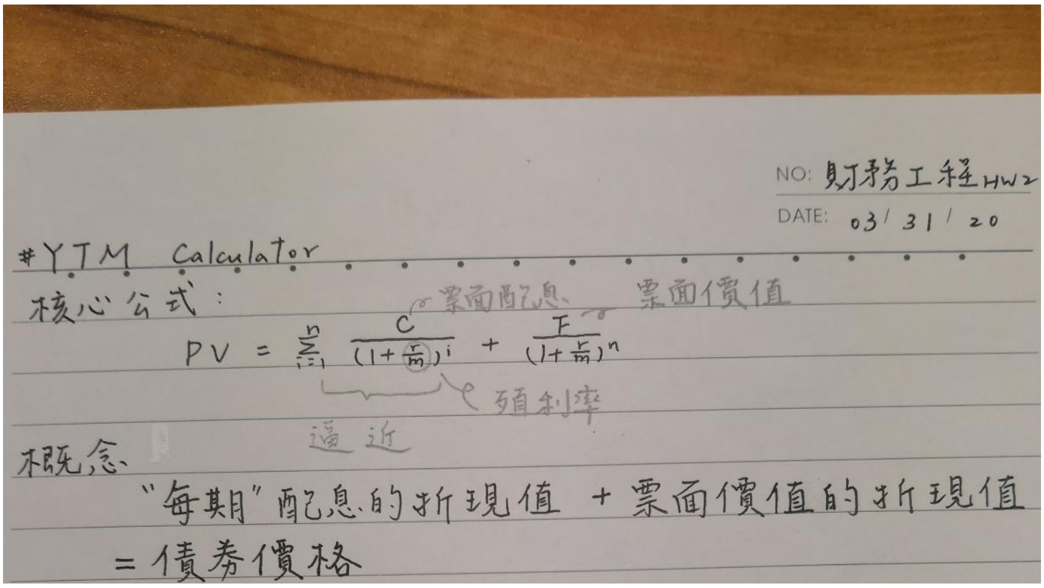
Step 2 寫 YTM Calculator 時所遇的瓶頸

#程式

```
In [135]: #計算YTM
          #輸入基本資料
          print("請輸入當期債券價格(Current Bond Price)")
          CBP = float(input())
          print("請輸入債券票面價格(Bond Par Value)")
          BPV = float(input())
          print("請輸入債券票面利率%(Coupon Rate%)")
          CR = float(input())/100
          print("請輸入債券到期年限(Years to Maturity)")
          YM = float(input())
          print("請輸入債券年付息次數(Numbers of Payments per Year)")
          NP = int(input())

          #計算殖利率(YTM)
          #先求年化報酬率
          C = BPV * CR

          #想法: 用當期債券價格減去估計Present value 取最近的值

          int_YM = int(YM)
          for r in range(1, 100001):
              iPV = 0
              for i in range(int_YM):
                  iPV += C/((1+r/100000)**(i+1))
              d_BPV = BPV / (1+r/100000)**(int_YM)
              PV = iPV + d_BPV
              if(PV - CBP) <= 0:
                  print("債券殖利率:" ,end =" ")
                  YTM = 100*round(r/100000/NP, 4)
                  strYTM = str(YTM)
                  print(strYTM + "%")
                  break

          請輸入當期債券價格(Current Bond Price)
          101
          請輸入債券票面價格(Bond Par Value)
          100
          請輸入債券票面利率%(Coupon Rate%)
          5
          請輸入債券到期年限(Years to Maturity)
          3
          請輸入債券年付息次數(Numbers of Payments per Year)
          2
          債券殖利率: 2.32%
```

撰寫程式的過程中發現 YTM 應該是要用逼近的，一開始想要偷吃步先算出總價值(Future Value)再除以債券價格得到債券殖利率，但幾經程式計算比對便明白每期配息的折現幅度不相同，故無法使用此方式。 之後參考老師提供的網站以及網路上的資料後決定使用最簡單粗暴的方式—暴力展開，將利率部分設定為 r/100000，方便我們將其逼近至小數點後第四位。

#數值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#YTM Calculator

核心公式：

$$PV = \sum_{i=1}^{n} \frac{\overset{\text{票面配息}}{C}}{(1+\underbrace{r}_{\text{殖利率}})^i} + \frac{\overset{\text{票面價值}}{F}}{(1+r)^n}$$

逼近

概念

"每期"配息的折現值 + 票面價值的折現值

= 債券價格

Step 3 零息債券、Spot rate

#程式

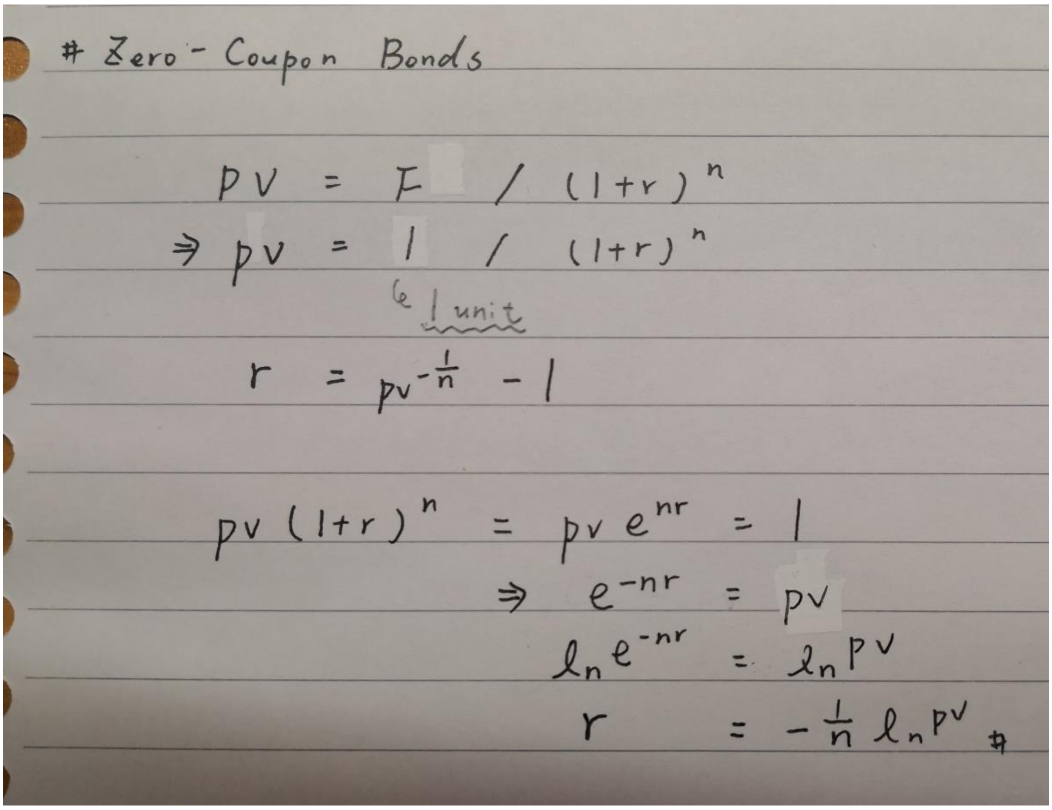
```
In [4]: #計算spot rate
        #輸入基本資料
        import math
        print("請輸入零息債券到期年數(Duration of spot rate)")
        Y = float(input())
        print("請輸入零息債券價格(Price of unit zero-coupon bond)")
        P = float(input())

        Yield = round( ((1/P)**(1/Y) -1), 4 ) * 100
        Yield = round(Yield, 2)
        strYield = str(Yield)
        print("spot rate of interest:" ,end =" ")
        print(strYield + "%")

        ForceOfInterest = round( (-1/Y)*(math.log(P)), 4 ) * 100
        strF = str(ForceOfInterest)
        print("spot force of interest:" ,end =" ")
        print(strF + "%")

        請輸入零息債券到期年數(Duration of spot rate)
        8
        請輸入零息債券價格(Price of unit zero-coupon bond)
        0.5
        spot rate of interest: 9.05%
        spot force of interest: 8.66%
```

此程式遇到的難點在如何簡潔的表達程式，老師提供的範例網站有非常詳細的推導過程，我在執行的過程中，使用round()四捨五入至小數點後第四位，但可能是程式或我個人的問題，在輸入某些數值會導致 round()函數不能四

捨五入，因此在此程式中使用兩次 round()，並在下一個計算 forward rate 程式中改進。

#數值

首先有關 e 部分在 Step 1 有詳盡說明


Zero-Coupon Bonds

$$pv = F \;/\; (1+r)^n$$

$$\Rightarrow pv = 1 \;/\; (1+r)^n$$

← 1 unit

$$r = pv^{-\frac{1}{n}} - 1$$

$$pv\,(1+r)^n = pv\, e^{nr} = 1$$

$$\Rightarrow e^{-nr} = pv$$

$$\ln e^{-nr} = \ln pv$$

$$r = -\frac{1}{n} \ln pv$$ #

Step 4 forward rate

#程式

In [2]:
```python
#計算forward rate
#輸入基本資料
import math
print("請輸入Time due for the beginning of forward rate:")
Y1 = float(input())
strY1 = str(Y1)
print("請輸入Duration of forward rate:")
Y2 = float(input())
strY2 = str(Y2)
strY12 = str(Y1+Y2)

print("請輸入Price of " + strY1 + " year unit zero coupon bond(0~1之間):")
P1 = float(input())
print("請輸入Price of " + strY12 + " year unit zero coupon bond(0~1之間):")
while True:
    P2 = float(input())
    if P2 >= P1:
        print("價格錯誤 請重新輸入")
    else :
        break

FV = (P1/P2) ** (1/Y2) - 1
FV = round(100*FV , 2)
strFV = str(FV)
print(strY2 + " year forward rate of interest beginning " + strY1 + " years from now:" )
print(strFV + "%")

FFV = (1/Y2) * math.log(P1/P2)
FFV = round(FFV * 100 , 2)
strFFV = str(FFV)
print(strY2 + " year forward force of interest beginning " + strY1 + " years from now:" )
print(strFFV + "%")
```

```
請輸入Time due for the beginning of forward rate:
4
請輸入Duration of forward rate:
3
請輸入Price of 4.0 year unit zero coupon bond(0~1之間):
0.8
請輸入Price of 7.0 year unit zero coupon bond(0~1之間):
0.5
3.0 year forward rate of interest beginning 4.0 years from now:
16.96%
3.0 year forward force of interest beginning 4.0 years from now:
15.67%
```

此程式在設計時參考上一題作法，將 round()分成兩步驟進行，一方面讓程式看起來較為乾淨、增加易讀性，另一方面也避免先前的程式錯誤。另外此題最麻煩的是 Robustness，若輸入一些奇怪的數容易導致程式錯誤，因為時間關係這裡只在輸入 P2 時加入錯誤偵測，其他較不易出錯的地方則尚需時間加強。

#數值

\# forward rate

$$(1+y_t)^t (1+f_{t,r})^r = (1+y_{t+r})^{t+r}$$

$$(1+f_{t,r})^r = (1+y_{t+r})^{t+r} / (1+y_t)^t$$

$$= P_t / P_{t+r}$$