Adel Danandeh HW#1

Exercises page 25

1 a) all the odd natural numbers.

b) Even integer numbers

c) Even numbers

d) Natural numbers divisible by 2 and 3.

e) bi-directional binary numbers (i.e. $10101 \Leftrightarrow 10101$)
left to right is same as right to left.

f) set of all integers.

2 a) $A = \{1, 10, 100\}$

b) $A = \{n \mid n > 5 \text{ for } n \in Z\}$

c) $A = \{n \mid n < 5 \text{ for } n \in N\}$

d) $A = \{a, b\}$

e) $A = \{\}$

f) $\emptyset$

3 a) No b) yes c) $A \cup B = \{x, y, z\}$

d) $A \cap B = \{x, y\}$ e) $A \times B = \{(x,x), (x,y), (y,x), (y,y), (z,x), (z,y)\}$

f) $\{\{x\}, \{y\}, \{x,y\}, \emptyset\}$

(4) $A \times B$ has $a \times b$ elements. Because $A \times B$ is a cartesian product each element of A and B form a set of tuples.

For instance:

$\overset{2 \text{ elements}}{A=\{1,2\}}$ $\quad \overset{2 \text{ element}}{B=\{3,4\}}$ $\Rightarrow A \times B = 4$ elements

$A \times B = \{(1,3),(1,4),(2,3),(2,4)\}$ Therefor there are 4 elements.

(5) C has c elements

$C=\{c_0,c_1,c_2,\ldots\}$

$C=\{c_0,c_1\} \Rightarrow \{\{c_0\},\{c_1\},\{c_0,c_1\},\varnothing\}$ If $n=2 \Rightarrow 4=2^2$ element

$C=\{c_0,c_1,c_2\} \Rightarrow \{\{c_0\},\{c_1\},\{c_2\},\{c_0,c_1\},\{c_0,c_2\},\{c_1,c_2\},\{c_0,c_1,c_2\},\varnothing\}$

If $n=3 \Rightarrow$ power of c is $8=2^3$

The formula is $|P(S)| = 2^n$ where S is the set and n is the number of elements in the set. In this case, we have a set C which it has c elements therefore we have: $|P(C)| = 2^c$ $c \rightarrow$ number of elements in C

6) $X = \{1, 2, 3, 4, 5\}$ $Y = \{6, 7, 8, 9, 10\}$

a) $f(2) = 7$

b) Domain $= [1,5]$ Range $= [6,7]$

c) $g(2, 10) = 6$

d) Domain $= [1,5]$ Range $= [6,10]$

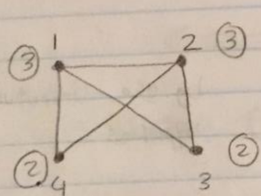
e) $g(4, f(4)) = g(4,7) = 8$

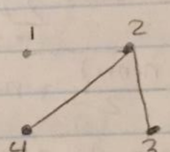
7) a) $A = \{(1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (3,2)\}$ A is reflexive and symmetric
but not transitive. $(1,2) \in A$ and $(2,3) \in A$ but $(1,3) \notin A$.
symmetric because: $(1,2) \in A$ and $(2,1) \in A$ and $(2,3) \in A$ and $(3,2) \in A$.

b) $\leq$ on $\mathbb{Z}$. $\leq$ is reflexive and transitive, but it is not symmetric, since
$0 \leq 1$ but $1 \not\leq 0$.

c) not possible.

8)





This is a path from 3 to 4.

9)

$G = \{\{1,2,3,4,5,6\}, \{(1,4), (1,5), (1,6), (2,4), (2,5), (2,6)$
$(3,4), (3,5), (3,6)\}\}$

(10) We can't divide both sides by $(a-b)$, since $a=b$ therefore we have $a-b=0$ and division by 0 is undefined.

(11) $S(n) = 1+2+ \cdots + n$ a) $S(n) = \frac{1}{2}n(n+1)$

$C(n) = 1^3 + 2^3 + \cdots + n^3$ b) $C(n) = \frac{1}{4}(n^4 + 2n^3 + n^2) = \frac{1}{4}n^2(n+1)^2$

a) Basis step $n=0$ $\frac{0(0+1)}{2} = 0$ $0=0$ ✓

Induction step: Let $n \in \mathbb{N}$ for arbitrary integer $n \geq 0$. We must show that the statement $S(n+1)$ is true. Let's assume: $S(n) = \sum_{i=0}^{n} i = \frac{1}{2}n(n+1)$ for some integer $n \geq 0$. We must prove (using induction hypothesis)

$$S(n) = \sum_{i}^{n+1} i = \frac{1}{2}(n+1)((n+1)+1)$$

$$\sum_{i=0}^{n+1} i = \underbrace{1+2+3+\cdots+n}_{S(n) = \sum_{i=0}^{n} i} + (n+1)$$

$$= \frac{n(n+1)}{2} + n+1$$ by the induction hypothesis

$$= \frac{n(n+1) + 2(n+1)}{2}$$

$$= \frac{(n+1)(n+2)}{2}$$

$$= \frac{(n+1)((n+1)+1)}{2} \quad ✓$$

By the principle of mathematical induction it follows that

$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$ for all natural numbers n. ■

(b) Basis step:

$n=1 \quad \frac{1}{4}(1)^2(1+1)^2 = \frac{4}{4} = 1 \quad 1=1$ ✓

Base case holds.

Inductive step: Let $n \in N$ for any arbitrary integer $n \geq 0$. We must show that $p(n+1)$ is true. We have:

$$C(n) = \sum_{i=1}^{n} i^3 = \frac{1}{4} n^2 (n+1)^2$$ for some integer $n \geq 0$. We must show that:

$$C(n+1) = \sum_{i=1}^{n+1} i^3 = \frac{1}{4}(n+1)^2((n+1)+1)^2$$

$$= \underbrace{1^3 + 2^3 + \cdots + n^3}_{C(n)} + (n+1)^3$$

$$= \frac{1}{4} n^2 (n+1)^2 + (n+1)^3$$ By the induction hypothesis

$$= (n+1)^2 \left[\frac{1}{4} n^2 + (n+1) \right] \quad \times 4/4$$

$$= \frac{1}{4}(n+1)^2 \left[n^2 + 4 + 4 \right]$$

$$= \frac{1}{4}(n+1)^2 (n+2)^2 = \frac{1}{4}(n+1)^2((n+1)+1)^2$$ which was to be shown.

By the principle of mathematical induction it follows that

$$\sum_{i=1}^{n} i^3 = \frac{1}{4} n^2 (n+1)^2$$ for all natural numbers n. ∎

From previous proofs, we have:

$$C_n = \frac{1}{4}\left(n^4 + 2n^3 + n^2\right) = \frac{1}{4} n^2 (n+1)^2 = \left[\frac{n(n+1)}{2}\right]^2 = (S_n)^2$$

$$C_n = (S_n)^2$$

(12) If we have $n=2$ two hourses of different colors, the proof works if this case were true.

- Prove $\overline{(A \cap B)} = (\bar{A} \cup \bar{B})$

$$\forall x\, [x \in \overline{A \cap B} \Leftrightarrow x \in \bar{A} \cap \bar{B}]$$

$\forall x,\ x \in \overline{A \cap B} \Leftrightarrow x \notin [A \cap B]$ Definition of complement

$\Leftrightarrow \neg [x \in A \cap B]$ Definition of $\notin$

$\Leftrightarrow \neg [(x \in A) \wedge (x \in B)]$ Def of intersection

$\Leftrightarrow \neg (x \in A) \vee \neg (x \in B)$ De Morgan's Law

$\Leftrightarrow (x \in \bar{A}) \vee (x \in \bar{B})$ Def of complement

$\Leftrightarrow x \in \bar{A} \cup \bar{B}$ Def of Union

Hence (*) $\forall x [x \in \overline{A \cap B} \Leftrightarrow x \in \bar{A} \cup \bar{B}]$

For the other direction; $x \in \bar{A} \cup \bar{B}$ Then x is in $\bar{A}$ or $\bar{B}$. Therefore, x is not in A or x is not in B, and thus not in the intersection of these two sets. Hence x is in $\overline{A \cap B}$, which completes the proof. (*) ■

- Show the set of odd numbers is countable.

__proof__: Let E be the set of odd numbers and

$f(x) = 2x+1$ be a function from N to E. Then we have:

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 1 & 3 & 5 & 7 & 9 & 11 \end{array}$$

Then f is one-to-one and onto. To show that it is

injective;

$f(n) = f(m)$, then $2n + \cancel{1} = 2m + \cancel{1}$

$n = m$.

since the function is a one-to-one mapping onto the

natural numbers, we've shown that odd numbers are

Countable. $\boxtimes$

Prove: For all $n \in N$, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Basis: Show it when $n=1$ we have

$$1^2 = \frac{1(1+1)(2+1)}{6} = \frac{6}{6} = 1 \checkmark$$

so both sides are equal to 1

Induction Hypothesis:

Let $k \in \mathbb{N}$ $(k \geq 1)$, suppose $n = k$ is true, then we have:

$$\circledast \quad \sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{by induction hypothesis}$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)\left[k(2k+1) + 6(k+1)\right]}{6}$$

$$= \frac{(k+1)\left[2k^2 + 7k + 1\right]}{6}$$

$$= \frac{(k+1)\left[(2k+3)(k+1)\right]}{6} = \frac{(k+1)(2(k+1)+1)}{6}$$

This holds for $n = k+1$, and the proof of the induction step is complete.

By principle of induction, it follows that $\circledast$ is true for all $n \in \mathbb{N}$. ■