About assignment

$$e^{At} \triangleq \mathcal{L}^{-1}\{(sI-A)^{-1}\}$$

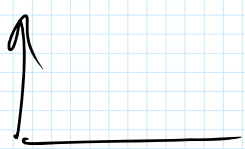
we derived for $\dot{x}(t) = Ax(t)$

the solution is $x(t) = \underbrace{\mathcal{L}^{-1}\{(sI-A)^{-1}\}}_{\text{call this } e^{At}} x(0)$

similarly solution of $\dot{x} = Ax + Bu$, $x(0) = 0$

is

$$x(t) = \int_0^t \underbrace{\mathcal{L}^{-1}\{(sI-A)^{-1}\}\Big|_{t=t-\tau}}_{\text{call this } e^{A(t-\tau)}} u(\tau)\,d\tau$$



Last time 5.0 Controllability / Observability

5.1 Controllability

- Define controllability Grammian

$$W_T = \int_0^T e^{At}BB^Te^{A^Tt}\,dt$$

- Theorem : (A,B) controllable iff $W_T$ invertible (rank n)

- Theorem : PBH criteria
  controllability matrix

- Theorem:

controllability matrix

$$[B \ AB \ A^2B \ \cdots]$$

- Examples

## Proof:

ASIDE ON "Proofs"

(A is true) if and only if (B is true)

$$A \Longleftrightarrow B$$

if (sufficient) A if B

$$B \Rightarrow A$$

A true if B true

B true is sufficient for A true

two ways to show

(1) Assume B true, try to show A true

(2) Assume A false, show B false

$$\left.\begin{array}{c} \text{not } A \Rightarrow \text{not } B \\ B \Rightarrow A \end{array}\right\} \text{equivalent}$$

leads to proof by contradiction

i) Assume B true
ii) Suppose A false
iii) show as a result of ii that B false
iv) $\Rightarrow$ contradicts i
$\Rightarrow B \Rightarrow A$

only if (necessity) $A \Rightarrow B$

B true if A true

B true is necessary for A true

Two ways

(1) Assume A true, show B true

(2) Assume B false, show A false

not B $\Rightarrow$ not A

proof by contradiction

for our theorem, want to show

$$\underbrace{(A,B) \text{ controllable}} \iff \underbrace{W_T^{-1} \text{ exists}}$$

$$A \iff B$$

i) Sufficiency $B \Rightarrow A$

assume $W_T^{-1}$ exists, then show (A,B) cont.

ie. $\exists\, u(t) \ni$ can

reach any $x^*$ from
and $x(t_0)$

let $u(\tau) = B' e^{A'(T-\tau)} W_T^{-1} [x^* - e^{AT} x_0]$

plug $u(\tau)$ into

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$$

*use
prop
as triggers

$$\Rightarrow x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} B B' e^{A'(T-\tau)} W_T^{-1} [x^* - e^{AT} x_0] d\tau$$

$$= e^{At} x_0 + \left[\int_0^t e^{A(t-\tau)} B B' e^{A'(T-\tau)} d\tau\right] W_T^{-1} [x^* - e^{AT} x_0]$$

let $t = T$

$$x(T) = e^{AT} x_0 + \underbrace{\left[\int_0^T e^{A(T-\tau)} B B' e^{A'(T-\tau)} d\tau\right]}_{W_T} W_T^{-1} \{x^* - e^{AT} x_0\}$$

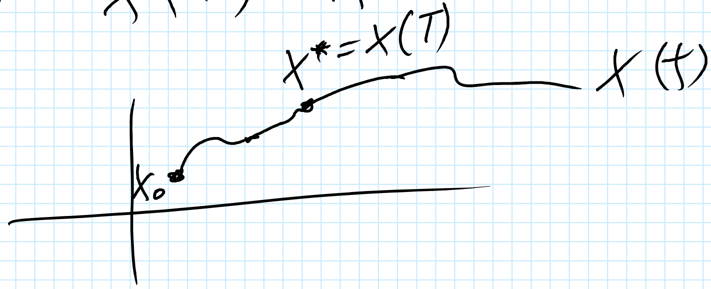
$$= e^{AT} x_0 + \{x^* - e^{AT} x_0\}$$

$x(T) = x^*$ (proof by construction)

$\Rightarrow$ $W_T^{-1}$ exists $\Rightarrow$ $|A, B|$ cont

ASIDE
not $x(t) = x^*$ for all $t \geq T$

only $x(T) = x^*$




ii) necessity $A \Rightarrow B$

controllable $\Rightarrow W_T^{-1}$ exists

do it by contradiction

not $B \Rightarrow$ not $A$

suppose $W_T$ not invertible, show $\exists$ an $x^*$ you can't reach

If $W_T$ is singular (not invertible)

then $x' W_T x = 0$ for some $x \neq 0$

so $$x' \left[ \int_0^T e^{At} B B' e^{A't} \, dt \right] x = 0$$

$$\int_0^T (x' e^{At} B)(B' e^{A't} x) \, dt = 0$$

$$\int_0^T v^T(t) \, v(t) \, dt = 0$$

$\Rightarrow v(t) = 0$ for all $t$

so if $W_T$ is not invertible it means $\exists\ x \neq 0$ so that

$$x' e^{At} B = 0 \text{ for all } t$$

call this $x$ to be $x^*$, then let's try to go from $x_0 = 0$ to $x^*$

if possible then $\exists$ input $u(\tau)$ so that

$$x(T) = x^* = \int_0^T e^{A(T-\tau)} B u(\tau) d\tau$$

so

$$x^{*'} x^* = x^{*'} \int_0^T e^{A(T-\tau)} B u(\tau) d\tau$$

$$\underbrace{x^{*'} x^*}_{\text{non-zero}} = \int_0^T \underbrace{x^{*'} e^{A(T-\tau)} B}_{=0} u(\tau) d\tau = 0$$

$\Rightarrow x^* = 0$

$\Rightarrow$ contradiction.

$\Rightarrow$ not controllable

i.e. $W_T$ not invertible $\Rightarrow$ not controllable

or controllable $\Rightarrow$ $W_T$ invertible

$\blacksquare$

QED

Ex, consider $\dot{x} = \begin{bmatrix}-1 & 0\\ 0 & 2\end{bmatrix}x + \begin{pmatrix}1\\0\end{pmatrix}u$

$y = (1\ \ 1)x$

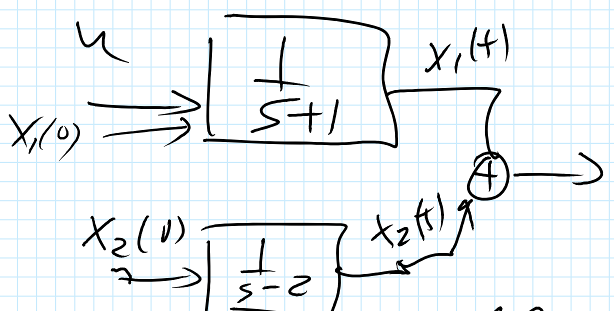
$\dot{x}_1 = -x_1 + u$

$\mathcal{L}\ \ sX_1 + X_1 = u + x_1(0)$

$X_1 = \frac{u(s)}{s+1} + \frac{x_1(0)}{s+1}$

$[B\ \ AB] = \begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix} \Rightarrow \text{rank } 1 < 2$

$\Rightarrow$ not cont




with $u(t)=0$

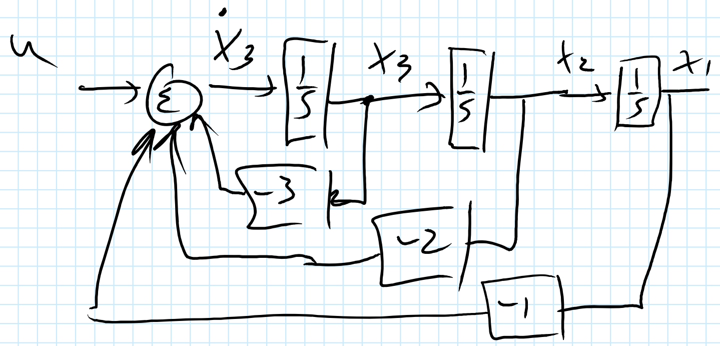
$y(t) = x_1(0)e^{-t} + x_2(0)e^{2t}$

In an uncontrollable system the input cannot influence some of the states

Ex $\dot{x} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -2 & -3\end{bmatrix}x + \begin{pmatrix}0\\0\\1\end{pmatrix}u$

Ex: $\dot{x} = \begin{bmatrix} \cdots \\ -1 & -2 & -3 \end{bmatrix} x + \begin{pmatrix} \; \end{pmatrix}$

$$\mathcal{C}(A,B) = [B \;\; AB \;\; A^2B] = \begin{array}{ccc} \left[\begin{array}{c|c|c} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 7 \end{array}\right] \\ B \quad AB \quad A^2B \end{array} \Rightarrow \text{rank } 3$$




simulation diagram