

### 3.3 Transfer Matrices

- Impulse response and transfer function

Last time :- idea of impulse response  
 - Laplace transforms

compute Laplace transform of the output  
 of a linear system

$$y = H(u)$$

$$y(t) = \int_0^t g(t-\tau) u(\tau) d\tau$$

defined by its impulse response  $g(t)$

$$Y(s) = \{y(t)\}$$

$$= \int_0^\infty y(t) e^{-st} dt$$

$$= \int_0^\infty \left( \int_0^t g(t-\tau) u(\tau) d\tau \right) e^{-st} dt$$

$$= \int_0^\infty \left( \int_0^\infty g(t-\tau) e^{-s(t-\tau)} dt \right) u(\tau) e^{-s\tau} d\tau$$

$$= \left( \int_0^\infty g(v) e^{-sv} dv \right) \left( \int_0^\infty u(\tau) e^{-s\tau} d\tau \right)$$

convolution operator

... . . . . .

$$Y(s) = \bar{G}(s) \cdot U(s) \Rightarrow$$

$$\mathcal{L}\{y(t)\} \quad \mathcal{L}\{g(t)\} \quad \mathcal{L}\{u(t)\}$$

↑  
output                   ↑  
                            impulse response                   ↑  
  input

define  $G(s) = \mathcal{L}\{g(t)\}$  to be  
 transfer function ( $SISO$ )  
 matrix ( $M/MO$ )

### Transfer function of LODE

In general, consider LODE of form

$$y^{(n)} + q_{n-1}y^{(n-1)} + \dots + q_1y + q_0y = b_n u^{(n)} + b_{n-1}u^{(n-1)} + \dots + b_1u + b_0u$$

assume all initial conditions are zero  
 and take Laplace of each side

$$y^{(n)} \xrightarrow{\mathcal{L}} s^n Y(s)$$

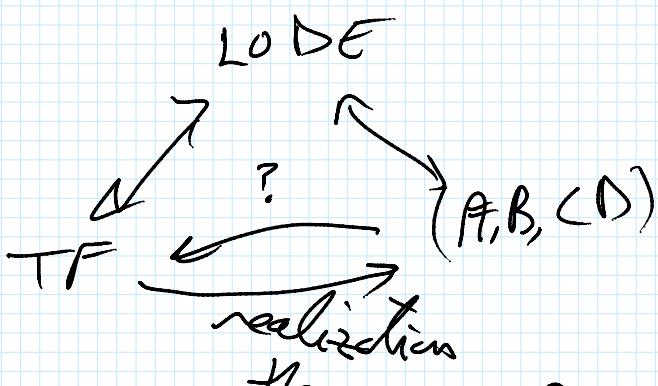
$$\begin{aligned} s^n Y(s) + q_{n-1}s^{n-1}Y(s) + \dots + q_1sY(s) + q_0Y(s) \\ = b_n s^n U(s) + b_{n-1}s^{n-1}U(s) + \dots + b_1sU(s) \\ + b_0U(s) \end{aligned}$$

$$\therefore h_n s^n + h_{n-1}s^{n-1} + \dots + h_1s + h_0 \Rightarrow h(s)$$

$$\Rightarrow Y(s) = \left[ \frac{b_n s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_0}{s^n + q_{n-1} s^{n-1} + \dots + q_1 s + q_0} \right] u(s)$$

$G(s)$ 
 $\rightarrow$ 
 $s \mid s \in$ 
transfer  
function  
( $1 \times 1$ )

Summary



- Transfer function from  $(A, B, C, D)$

Suppose system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

take Laplace

$$sX(s) - X(0) = AX(s) + BU(s)$$

$$sX(s) - AX(s) = BU(s) + X(0)$$

$$(sI - A)X(s) =$$

$$X(s) = (sI - A)^{-1}BU(s) + (sI - A)^{-1}X(0)$$

$\therefore X(s) = (sI - A)^{-1}BU(s) + (sI - A)^{-1}X(0)$

$$Y(s) = C X(s) + D U(s)$$

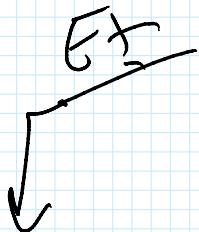
$$Y(s) = C(sI - A)^{-1} X(0) + C(sI - A)^{-1} B U(s) + D U(s)$$

let  $X(0) = 0$ , then

$$Y(s) = \underbrace{[C(sI - A)^{-1} B + D]}_{G(s)} U(s)$$

$G(s)$  transfer function  
(SISO)

transfer matrix  
(MIMO)



$$\ddot{y} + 3\dot{y} + 2y = u$$

TF

$$\frac{Y(s)}{U(s)} = G(s) = \frac{1}{s^2 + 3s + 2}$$

1x1

$s, s_1$

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x\end{aligned}$$

$$G(s) = C(sI - A)^{-1} B$$

$$C(sI - A)^{-1} B = \begin{bmatrix} 1 & 0 \end{bmatrix} \left\{ s \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \right\}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$( (s^2 - \tau) D = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} s(0) + (-2-3) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1)$$

$$\overbrace{(1+2)}^{(1+1)} \underbrace{(2+2)}_{(1+1)} \underbrace{(2+1)}$$

$$= (1 \ 0) \begin{bmatrix} s & -1 \\ -2 & s+3 \end{bmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= (1 \ 0) \left\{ \frac{1}{s(s+3) - (-1)(2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \right\} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= (1 \ 0) \left[ \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \right] \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= (1 \ 0) \left[ \frac{1}{s^2 + 3s + 2} \begin{bmatrix} 1 \\ s \end{bmatrix} \right]$$

$$= \frac{1}{s^2 + 3s + 2}$$

Next time : - Ext of MIMO

- interconnected systems
- "transformations"

→ 45108  
consider

$$A \quad B \quad C \quad D$$

$$- \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, (1 \ 0), 0$$

$$I \text{ (constant)} \xrightarrow{s^2 + 3s + 2} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \end{pmatrix}, 0$$

$$\frac{-1}{s+2} s + \frac{1}{s+1}$$

note

$$\frac{1}{s^2 + 3s + 2} = \frac{1}{s+1} - \frac{1}{s+2}$$

consider  $\dot{z} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} z + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$

$$y = [1 \ -1] z$$

compute  $C(sI - A)^{-1}B$  for this

$$(1 \ -1) \cdot \begin{pmatrix} s+1 & 0 \\ 0 & s+2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(1 \ -1) \begin{pmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(1 \ -1) \begin{pmatrix} \frac{1}{s+1} \\ \frac{1}{s+2} \end{pmatrix} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$= \frac{1}{s^2 + 3s + 2}$$

point: state-space representation for a system is not unique

