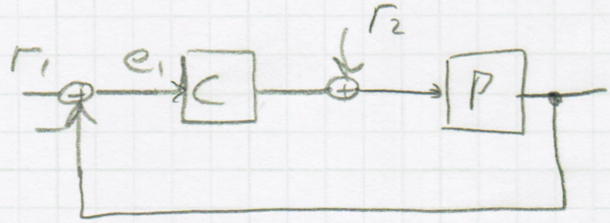
All Stabilizing Controllers

consider



note : $\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \underbrace{\left[\begin{array}{c|c} (I+P(s)C(s))^{-1} & -P(I+CP)^{-1} \\ \hline C(I+PC)^{-1} & (I+CP)^{-1} \end{array}\right]}_{H(s):\ \text{closed-loop system}} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$

we seek all $C(s)$ so that $H(s)$ is BIBO stable

<u>Key result</u>: If $H(s)$ is BIBO stable then closed-loop is also internally stable (assumes $C(s)$, $P(s)$ each irreducible)

small point

- ratio of rationals versus ratio of polynomials
- we will write $P(s) = \frac{n(s)}{d(s)} = \frac{n_p(s)}{d_p(s)}$

  where $n(s), d(s)$ polynomials
  $n_p(s), d_p(s)$ ratios of polynomials

- Ex

$$P(s) = \frac{s}{s+1} = \frac{s/s+2}{s+1/s+2}$$

can use anything here

in general $P(s) = \dfrac{\frac{s}{\alpha(s)}}{\frac{s+1}{\alpha(s)}}$ or $P(s) = \dfrac{n_p}{d_p} = \dfrac{\frac{n(s)}{\alpha(s)}}{\frac{d(s)}{\alpha(s)}}$

Theorem 1: Let $P = \frac{n_p}{d_p}$ where n_p, d_p are stable rationals.

Find rationals $x(s)$, $y(s)$ to solve the Bezout equation:

$$x(s)n_p(s) + y(s)d_p(s) = 1 \quad *$$

Then any stabilizing controller has the form

$$C(s) = \frac{x + q d_p}{y - q n_p}$$

where $q(s)$ is any arbitrary stable rational function with $y - q n_p \neq 0$.

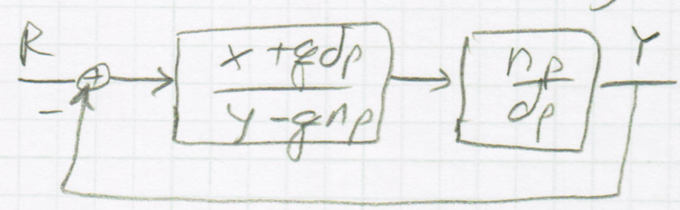
Theorem 2: Using $q(s)$ from Theorem 1,

$$H(s) = \begin{bmatrix} \frac{d_p(y - q n_p)}{d_p(x + q d_p)} & \frac{-n_p(x + q d_p)}{d_p(y - q n_p)} \end{bmatrix}$$

is both BIBO and internally stable

[proof —

compute



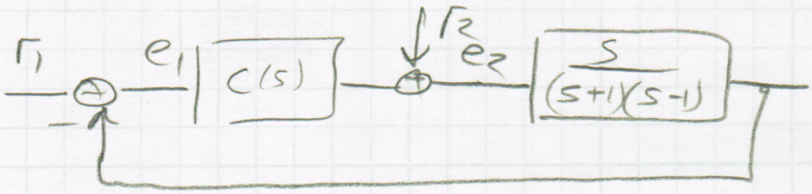
$$\frac{Y}{R} = \frac{CP}{1 + CP} = \frac{\left(\frac{x + q d_p}{y - q n_p}\right)\frac{n_p}{d_p}}{1 + \left(\frac{x + q d_p}{y - q n_p}\right)\frac{n_p}{d_p}}$$

$$= \frac{(x + q d_p) n_p}{(y - q n_p) d_p + (x + q d_p) n_p}$$

$$= \frac{(x + q d_p) n_p}{y d_p - \cancel{q n_p d_p} + x n_p + \cancel{q d_p n_p}} = \frac{(x + q d_p) n_p}{\underbrace{y d_p + x n_p}_{=1 \text{ see * above}}} = (x + q d_p) n_p$$

$\longleftarrow$

Example



$$P = \frac{s}{(s+1)(s-1)} = \frac{s/(s+1)^2}{s-1/s+1} \quad (\text{using } \alpha(s) = (s+1)^2$$

$$\Longrightarrow \quad n_p = s/(s+1)^2$$

$$d_p = (s-1)/(s+1)$$

now solve $n_p x + d_p y = 1$ using Sylvester matrix approach

one solution: let $x = \frac{n_x}{s+2}$ $\quad y = \frac{n_y}{s+2}$

so $n_p x + n_d y = 1$ is $\frac{s}{(s+1)^2} \cdot \frac{n_x}{s+2} + \frac{s-1}{s+1} \cdot \frac{n_y}{s+2} = 1$

or

$$n_x s + n_y (s-1)(s+1) = (s+1)^2 (s+2)$$

Note: This becomes part of $\delta(s)$

or

$$s(\beta_1 s + \beta_0) + (s^2 - 1)(\alpha_1 s + \alpha_0) = s^3 + 4s^2 + 5s + 2$$

$$\Rightarrow \begin{pmatrix} 1 \\ 4 \\ 5 \\ 2 \end{pmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right] \begin{pmatrix} \beta_1 \\ \beta_2 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} \Longrightarrow \begin{array}{l} \beta_1 = 6 = \beta_0 \\ \alpha_1 = 1 \\ \alpha_0 = -2 \end{array}$$

or $\quad x(s) = \frac{6s+6}{s+2}, \quad y(s) = \frac{s-2}{s+2}$

Finally

$$C(s) = \frac{x + q d_p}{y - q n_p} = \frac{\frac{6s+6}{s+2} + q(s)\frac{s-1}{s+1}}{\frac{s-1}{s+2} - q(s)\frac{s}{(s+1)^2}}$$

$$\boxed{C(s) = \frac{6(s^3+3s^2+3s+1) + q(s)(s^3+2s^2-s-2)}{(s^3-3s-2) - q(s)(s^2+2s)}}$$

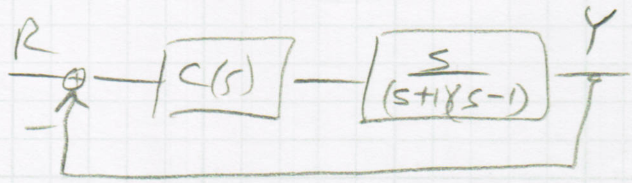
⇒ Claim: Pick any $q(s)$, then the resulting $C(s)$ makes $H(s)$ stable

check: Pick $q(s) = -6$

$$\Rightarrow C(s) = \frac{6(s^3+3s^2+3s+1) - 6(s^3+2s^2-s-2)}{(s^3-3s-2) + 6(s^2+2s)}$$

$$C(s) = \frac{6(s^2+4s+3)}{s^3+6s^2+9s-2}$$

now plug in to



$$\text{so } \frac{Y}{R} = \frac{CP}{1+CP} = \frac{n_c n_p}{n_c n + d_c d}$$

$$= \frac{6(s^2+4s+3)\cdot s}{6(s^2+4s+3)\cdot s + (s^3+6s^2+9s-2)(s+1)(s-1)}$$

$$= \frac{6\cancel{(s+1)}(s+3)\cdot s}{6\cancel{(s+1)}(s+3)\cdot s + \cancel{(s+1)}(s^3+6s^2+9s-2)(s-1)}$$

$$= \frac{6s(s+3)}{s^4+5s^3+9s^2+7s+2}$$

$$\boxed{\frac{Y}{R} = \frac{6s(s+3)}{(s+1)^3(s+2)}}$$

note: similar for other entries of $H(s)$