COLORADO SCHOOL OF MINES
Theory and Design of Advanced Control Systems

Control of a Quadruple-Tank Process

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MAY 7th, 2007

Introduction

The quadruple-tank process consists of an interconnection of four water tanks. Figure 1 illustrates the process. The target is to control the levels y_1 and y_2 in the lower two tanks with two pumps. The flows of the pumps are split up by valves. The flow from pump 1 goes into tanks 1 and 4, pump 2 feeds tanks 2 and 3. The inputs u_1 and u_2 are the voltages applied to the two pumps and the outputs are the voltages representing the levels in the lower tanks. Capacitive electrodes measure the water levels.

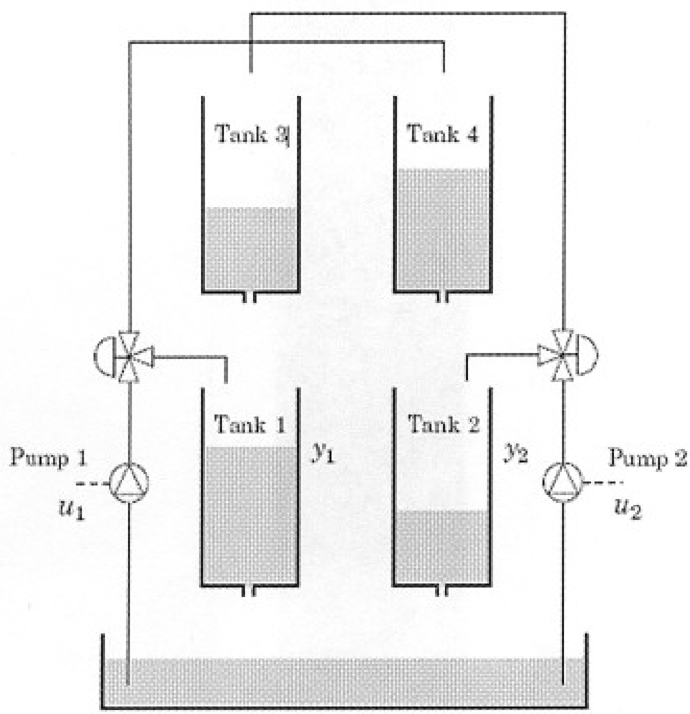

Figure 1. Schematic of the quadruple-tank process

In the following sections, a nonlinear model for the quadruple-tank process will be derived. The model will be linearized around an operation point and various control techniques will be implemented. The relative gain array matrix will be derived in order to define the proper pairing of inputs and outputs for the use of decentralized controllers. Finally, the nonlinear model will be controlled using gain scheduling.

Modeling

In this section a mathematical model of the quadruple-tank process will be derived. For each tank i=1,2,3,4, mass balance and Bernoulli's law give

$$A_i \frac{dh_i}{dt} = -a_i \sqrt{2 \cdot g \cdot h_i} + q_{ini}$$

where:

A_i is the cross-section area of the tank

h_i is the water level

a_i is the outlet cross-sectional area of the tank

g is the acceleration of gravity

q_{ini} is the inflow to the tank

Each pump gives a flow proportional to the control signal

$$q_{pumpi} = k_i \cdot u_i$$

where:

k_i is the pump constant

The flow from the pumps are divided according to the two parameters γ_1, γ_2.

Flow to Tank 1: $\gamma_1 \cdot k_1 \cdot u_1$

Flow to Tank 2: $\gamma_2 \cdot k_2 \cdot u_2$

Flow to Tank 3: $(1-\gamma_2) \cdot k_2 \cdot u_2$

Flow to Tank 4: $(1-\gamma_1) \cdot k_1 \cdot u_1$

The measured level signals are $y_1 = k_c \cdot h_1$ and $y_2 = k_c \cdot h_2$, where k_c is a measurement constant.

The following differential equations represent the mass balances in this quadruple-tank process:

$$\frac{dh_1}{dt} = -\frac{a_1}{A_1}\sqrt{2gh_1} + \frac{a_3}{A_1}\sqrt{2gh_3} + \frac{\gamma_1 k_1}{A_1}u_1$$
$$\frac{dh_2}{dt} = -\frac{a_2}{A_2}\sqrt{2gh_2} + \frac{a_4}{A_2}\sqrt{2gh_4} + \frac{\gamma_2 k_2}{A_2}u_2$$
$$\frac{dh_3}{dt} = -\frac{a_3}{A_3}\sqrt{2gh_3} + \frac{(1-\gamma_2)k_2}{A_3}u_2$$
$$\frac{dh_4}{dt} = -\frac{a_4}{A_4}\sqrt{2gh_4} + \frac{(1-\gamma_1)k_1}{A_4}u_1$$

Linearizing around a stationary operating point, the state-space representation becomes:

$$\dot{x} = \begin{bmatrix} -\frac{1}{T_1} & 0 & \frac{A_3}{A_1 T_3} & 0 \\ 0 & -\frac{1}{T_2} & 0 & \frac{A_4}{A_2 T_4} \\ 0 & 0 & -\frac{1}{T_3} & 0 \\ 0 & 0 & 0 & -\frac{1}{T_4} \end{bmatrix} \cdot x + \begin{bmatrix} \frac{\gamma_1 k_1}{A_1} & 0 \\ 0 & \frac{\gamma_2 k_2}{A_2} \\ 0 & \frac{(1-\gamma_2)k_2}{A_3} \\ \frac{(1-\gamma_1)k_1}{A_4} & 0 \end{bmatrix} \cdot u$$

$$y = \begin{bmatrix} k_c & 0 & 0 & 0 \\ 0 & k_c & 0 & 0 \end{bmatrix} \cdot x$$

where the time constants are: $T = \frac{A_i}{a_i}\sqrt{\frac{2h_i}{g}}$

The transfer matrix from u to y is obtained through: $G(s) = C(sI - A)^{-1}B + D$

$$G(s) = \begin{bmatrix} \frac{\gamma_1 k_1 k_c}{A_1(s+1/T_1)} & \frac{(1-\gamma_2)k_2 k_c}{A_1(s+1/T_1)(s+1/T_3)T_3} \\ \frac{(1-\gamma_1)k_1 k_c}{A_2(s+1/T_2)(s+1/T_4)T_4} & \frac{\gamma_2 k_2 k_c}{A_2(s+1/T_2)} \end{bmatrix}$$

For:

$A_i = 28\ cm^2$	$k_c = 0.5$ V/cm	$h_1 = h_2 = 12$ cm
$a_i = 0.06\ cm^2$	$g = 981\ cm/s^2$	$h_3 = h_4 = 1$ cm
$k_i = 2.9\ cm^3/V$	$\gamma_1 = \gamma_2 = 0.7$	

The linearized state-space representation of the system becomes:

$$\dot{x} = \begin{bmatrix} -0.0137 & 0 & 0.0475 & 0 \\ 0 & -0.0137 & 0 & 0.0475 \\ 0 & 0 & -0.0475 & 0 \\ 0 & 0 & 0 & -0.0475 \end{bmatrix} \cdot x + \begin{bmatrix} 0.0725 & 0 \\ 0 & 0.0725 \\ 0 & 0.0311 \\ 0.0311 & 0 \end{bmatrix} \cdot u$$

$$y = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \end{bmatrix} \cdot x$$

And the matrix transfer function becomes:

$$G(s) = \begin{bmatrix} \dfrac{1.015}{28(s+0.0137)} & \dfrac{0.4350}{28(s+0.0137)(s+0.0475)21.071} \\ \dfrac{0.4350}{28(s+0.0137)(s+0.0475)21.071} & \dfrac{1.015}{28(s+0.0137)} \end{bmatrix}$$