MATHEMATICAL MODELS

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 \dot{y} + a_0 y = b_n u^{(n)} + b_{n-1} u^{(n-1)} + \cdots + b_1 \dot{u} + b_0 u$$

LODE

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_n s^n + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

$$= b_n + \frac{c_{n-1} s^{n-1} + \cdots + c_1 s + c_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

$$c_i = b_i - b_n a_i$$

$$= b_n + \frac{\alpha_1 \gamma_1}{s + p_1} + \cdots + \frac{\alpha_n \gamma_n}{s + p_n}$$

TRANSFER FUNCTION

inspection

$$H(s) = C(sI - A)^{-1} B + D$$

$$\dot{\underline{x}} = A\underline{x} + Bu$$
$$y = C\underline{x} + Du$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & & -a_{n-1} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad \text{phase-variable form}$$

$$C = [c_0 \; c_1 \; \cdots \; c_{n-1}] \quad D = b_n$$

$$A = \begin{bmatrix} -p_1 & & & 0 \\ & -p_2 & & \\ & & \ddots & \\ 0 & & & -p_n \end{bmatrix} \quad B = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \quad \text{diagonal form}$$

$$C = [\gamma_1 \; \gamma_2 \; \cdots \; \gamma_n] \quad D = b_n$$

STATE SPACE

- **EXAMPLE**

LODE

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 3\dddot{u} + 2\ddot{u} + \dot{u} + u$$

TRANSFER FUNCTION

$$G(s) = \frac{3s^3 + 2s^2 + s + 1}{s^3 + 6s^2 + 11s + 6} = 3 + \frac{-16s^2 - 32s - 17}{s^3 + 6s^2 + 11s + 6} = 3 + \frac{-\frac{1}{2}}{s+1} + \frac{17}{s+2} + \frac{-65/2}{s+3}$$

STATE SPACE

$$\left\{ \begin{aligned} \dot{\underline{x}} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y &= [-17 \; -32 \; -16] \, \underline{x} + 3u \end{aligned} \right\} \quad \text{or} \quad \left\{ \begin{aligned} \dot{\underline{x}} &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u \\ y &= [-\tfrac{1}{2} \; 17 \; -\tfrac{65}{2}] \, \underline{x} + 3u \end{aligned} \right\}$$

- **TRANSFORMATIONS** : state space representations are not unique

Given $\left\{ \begin{aligned} \dot{x} &= A_1 x + B_1 u \\ y &= C_1 x + D_1 u \end{aligned} \right\}$, let $x = Tz$ (T invertible), then $\left\{ \begin{aligned} \dot{z} &= A_2 z + B_2 u \\ y &= C_2 z + D_2 u \end{aligned} \right\}$ is a valid state space representation.

where $A_2 = T^{-1} A_1 T$, $B_2 = T^{-1} B_1$, $C_2 = T C_1$, $D_2 = D_1$

Trans. to phase variable form

(1) Find $\mathcal{C}(A,B)^{-1} = [B \; AB \; A^2B \cdots A^{n-1}B]^{-1}$

(2) Let $q_1^T = [0 \cdots 0 \; 1] \, \mathcal{C}(A,B)^{-1}$

(3) Form $Q^{-1} = \begin{bmatrix} q_1^T \\ q_1^T A \\ \vdots \\ q_1^T A^{n-1} \end{bmatrix}$

(4) Then $\left. \begin{aligned} \bar{A} &= Q^{-1} A Q \\ \bar{B} &= Q^{-1} B \end{aligned} \right\}$ are in phase-variable form

Diagonalizing Transformation

(1) Find the eigenvector $\underline{v}_i$ associated with each eigenvalue λ_i (assume distinct λ_i)

(2) Form $P = [\underline{v}_1 \; \underline{v}_2 \; \cdots \; \underline{v}_n]$

(3) Then $\Lambda = P^{-1} A P = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$