3-3-16 Lecture 13

Thursday, March 3, 2016
10:36 AM

- **state-space to matrix fraction**

D_L, N_L, D_R, N_R polynomial matrices

Given
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$G(s) = C(sI-A)^{-1}B + \cancel{D}^{\,D=0} = D_L^{-1}(s)\,N_L(s) = N_R(s)\,D_R^{-1}(s)$$

where
$$\left.\begin{aligned} D_L &= I - C(sI-A+JC)^{-1}J \\ N_L &= C(sI-A+JC)^{-1}B \end{aligned}\right\} \text{ where } A-JC \text{ stable}$$

$$\left.\begin{aligned} D_R &= I - K(sI-A+BK)^{-1}B \\ N_L &= C(sI-A+BK)^{-1}B \end{aligned}\right\} \text{ where } A-BK \text{ stable}$$

Further, using K, J above, define

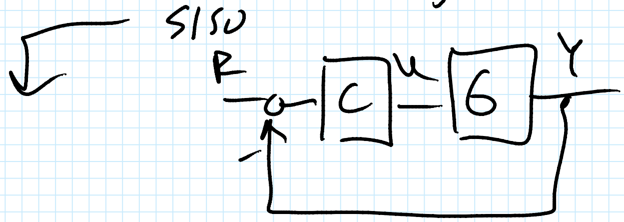
$$Y_L = K(sI-A+BK)^{-1}J$$
$$X_L = I + C(sI-A+BK)^{-1}J$$

$$Y_R = K(sI-A+JC)^{-1}J$$
$$X_R = I + K(sI-A+JC)^{-1}B$$

Then
$$\begin{bmatrix} Y_R & X_R \\ -N_R & D_R \end{bmatrix}\begin{bmatrix} D_L & -X_L \\ N_L & Y_L \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Diagonal terms
$$\left.\begin{aligned} Y_L D_L + X_R N_L &= I \\ D_R Y_R + N_R X_L &= I \end{aligned}\right\} \text{ Bezout equations}$$

• MIMO Block Diagrams



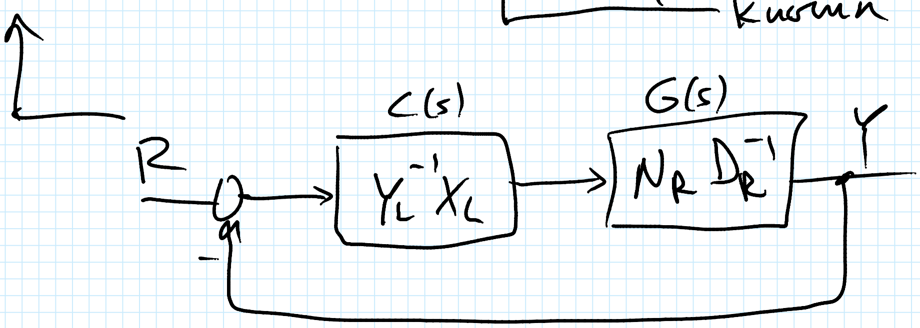
$G = \frac{n(s)}{d(s)}$ → e.g. $C(s) = K_p + \frac{K_I}{s} = \frac{K_p s + K_I}{s}$

$C = \frac{x(s)}{y(s)}$

$$\frac{Y}{R} = \frac{C \cdot G}{1 + C \cdot G} = \frac{n(s)\,x(s)}{d(s)\,y(s) + n(s)\,x(s)}$$

To find

known



from earlier $Y = G \cdot C (I + GC)^{-1} R$

$= G (I + CG)^{-1} C \, R$

$$\boxed{Y = N_R (Y_L D_R + X_L N_R)^{-1} X_L R}$$

compare to SISO

$y = n(dy + nx)^{-1} x \cdot R$

call $Y_L D_R + X_L N_R = \Delta(s)$

let $\Delta_d(s)$ = desired $\Delta(s)$

$\Rightarrow$ Given $D_R, N_R, \Delta_d(s)$

Find X_L, Y_L

so that $\Delta(s) = \Delta_d(s)$

solvable if (A, B, C) is cont/obs

i.e. $G(s)$ is minimal

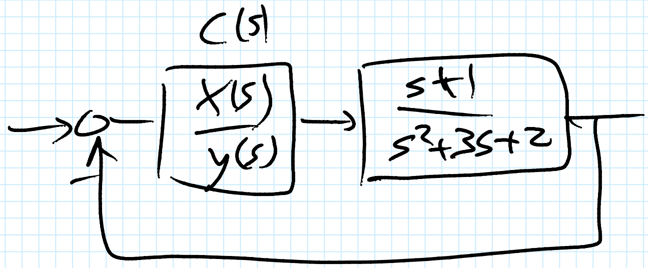
In MIMO case $G(s)$ minimal means no common factors between $\underbrace{D_R(s) \text{ and } N_R(s)}_{\text{matrices}}$

common factors means there is a matrix $U(s)$ so that

$D_R = \bar{D}_R(s) U(s) \Rightarrow D_R^{-1} = U^{-1} \bar{D}_R^{-1}$

$N_R = \bar{N}_R(s) U(s)$ $\quad G(s) = N_R D_R^{-1}$

Ex: 5150



$G = \dfrac{\cancel{(s+1)}}{\cancel{(s+1)}(s+2)} = \dfrac{1}{s+2}$

$$G(s) = \frac{n(s)}{d(s)} \quad \text{where} \quad n(s) = (s+1)$$
$$d(s) = s^2 + 3s + 2$$
$$= (s+1)(s+2)$$

common factor $(s+1)$

for this feedback system

characteristic polynomial $\delta(s) = y(s) \cdot d(s) + x(s)\, n(s)$

$$= y(s)(s+1)(s+2) + x(s)(s+1)$$
$$= (s+1)\left[\, y(s)(s+2) + x(s) \right)$$

common factor equivalent to uncontrollable eigenvalue in A matrix

$\Rightarrow$ have to be working with a minimal, irreducible system to design controllers

when N, D have no common factors they are called <u>co-prime</u>

6.4 Poles & Zeros of MIMO Systems

for (A, B, C, D) it's easy: poles = eigenvalues

- SISO case

$$G(s) = \frac{n(s)}{d(s)}$$

zeros = roots of $n(s)$ (zeros)

poles = roots of $d(s)$ (zeros)

- poles determine stability, modes of system
- zeros? define signals that can't pass through system

$$u(s) \longrightarrow \boxed{\frac{s-1}{s^2+5s+6}} \longrightarrow y = G \cdot u$$

$u(t) = e^{t} + 20$

$u(s) = \frac{1}{s-1}$

$$\Rightarrow y = \frac{\cancel{s-1}}{(\quad)} \cdot \frac{1}{\cancel{s-1}} = \frac{1}{s^2+5s+6}$$

so $y(t) \to 0$

but $u(t) \to \infty$

input is blocked

- MIMO case

Given matrix $G(s)$, $s = c$ is a zero of $G(s)$ if there exists constant vectors

v, w so that $G(c)v = 0$ and $w\,G(c) = 0$

i.e., $G(s)\big|_{s=c}$ loses rank

Ex: $G(s) = \begin{bmatrix} \frac{4}{(s+1)(s+4)} & \frac{-1}{s+1} \\ \frac{2}{s+1} & \frac{-1}{2(s+1)(s+2)} \end{bmatrix}$

Note: $G(-3) = \begin{bmatrix} 2 & 1/2 \\ -1 & -1/4 \end{bmatrix}$ has rank 1
$s=-3$

$\Longrightarrow -3$ is a zero

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can also do this with matrix matrix fractions

$$G = N_R D_R^{-1} = D_L^{-1} N_L$$

zeros are values of s where N_R (and N_L) lose rank

- Poles of MIMO systems

with matrix fraction, poles are where D_R (and D_L) lose rank

in terms of $G(s)$, define poles as the roots of pole polynomial roots of $\underbrace{\text{pole polynomial}}_{}$

least common denominator of all $\underbrace{\text{minors}}_{}$ of $G(s)$

determinants of all $m \times m$ submatrices of a matrix