• Example $\ddot{y} + 3\dot{y} + 2y = 2\ddot{u} + 7\dot{u} + 7$ $\quad y(0) = \dot{y}(0) = -1$, $u(0) = \dot{u}(0) = 1$, $u(t) = 1 + 2\sigma$

$$G(s) = \frac{2s^2+7s+7}{s^2+3s+2} = 2 + \frac{s+3}{s^2+3s+2} = 2 + \frac{2}{s+1} + \frac{-1}{s+2}$$

$$\dot{\underline{x}} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \underline{x} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \underline{x} + 2u$$

To set $\underline{x}(0)$: $\quad \begin{pmatrix} y(0) \\ \dot{y}(0) \end{pmatrix} = \begin{bmatrix} C \\ CA \end{bmatrix} \underline{x}(0) + \begin{bmatrix} D & 0 \\ CB & D \end{bmatrix} \begin{pmatrix} u(0) \\ \dot{u}(0) \end{pmatrix}$

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \underline{x}(0) + \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \underline{x}(0) + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \underline{x}(0)$$

$$\underline{x}(0) = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ -2 \end{pmatrix} = -1 \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

$$\underline{x}(0) = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

Then

$$y(t) = Ce^{At}\underline{x}_0 + C\int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau + D u(t)$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{pmatrix} -4 \\ 3 \end{pmatrix} + \begin{bmatrix} 1 & 1 \end{bmatrix} \int_0^t \begin{bmatrix} e^{-(t-\tau)} & 0 \\ 0 & e^{-2(t-\tau)} \end{bmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} d\tau + 2$$

$$= -4e^{-t} + 3e^{-2t} + \int_0^t \left(2e^{-(t-\tau)} - e^{-2(t-\tau)}\right) d\tau + 2$$

$$= -4e^{-t} + 3e^{-2t} + 2e^{-t}(e^{t}-1) - \tfrac{1}{2}e^{-2t}\left[e^{2t}-1\right] + 2$$

$$= -4e^{-t} + 3e^{-2t} + 2 - 2e^{-t} + \tfrac{1}{2} - \tfrac{1}{2}e^{-2t} + 2$$

$$= \underbrace{-4e^{-t} + 3e^{-2t}}_{\text{I.C.}_1} \quad \underbrace{7/2 - 2e^{-t} + \tfrac{1}{2}e^{-2t}}_{\text{Forced response}}$$

$$y(t) = 7/2 - 6e^{-t} + \tfrac{7}{2}e^{-2t}$$

check: Do same exercise using Laplace:

$$\ddot{y} + 3\dot{y} + 2y = 2\ddot{u} + 7\dot{u} + 7u \qquad y(0) = \dot{y}(0) = u(0) = \dot{u}(0) = 1$$

$$\Downarrow$$

$$Y(s)[s^2 + 3s + 2] - s\,y(0) - \dot{y}(0) - 3y(0) = U(s)[2s^2 + 7s + 7] - s\,2u(0) - 2\dot{u}(0) - 7u(0)$$

$$Y(s) = \frac{2s^2 + 7s + 7}{s(s^2 + 3s + 2)} - \frac{s + 5}{s^2 + 3s + 2}$$

$$= \frac{7}{2} \cdot \frac{1}{s} - \frac{2}{s+1} + \frac{1/2}{s+2} - \left[\frac{4}{s+1} - \frac{3}{s+2}\right]$$

$$y(t) = \underbrace{\frac{7}{2} - 2e^{-t} + \frac{1}{2}e^{-2t}}_{\text{forced}} - \underbrace{4e^{-t} + 3e^{-2t}}_{\text{I.C.}}$$

$$= 7/2 - 6e^{-t} + 7/2\, e^{-2t} \qquad \checkmark$$