Previously
5.1 controllability

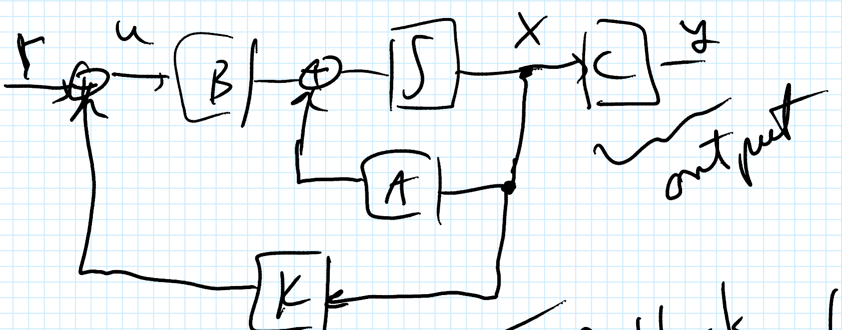
- state feedback & the real meaning of controllability

consider $\dot{x} = Ax + Bu$

one idea about controlling this system is to let

$u = Kx$ or $u = Kx + r$

reference input



output

state feedback (assumes we can measure states)

combine $\boxed{\dot{x} = Ax + Bu}$ original system stability/other depend on eig(A)

$u = Kx + r$

$\Longrightarrow \quad \dot{x} = Ax + B(Kx + r)$

or $\boxed{\dot{x} = (A + BK)x + Br}$ controlled system stability/other depend on eig(A+BK)

since K is "free" to be chosen

Q. Does there exist K,

so that eigenvalues of $(A+BK)$ can be arbitrarily assigned

Theorem Yes, if and only if (A,B) cont.

Ex: $\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$

cont. system ✓

not stable

$u = (K_1 \ \ K_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Kx$

$$(A+BK) = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}(K_1 \ K_2)$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} K_1 & K_2 \\ K_1 & K_2 \end{pmatrix}$$

$$= \begin{pmatrix} -1+K_1 & K_2 \\ K_1 & 2+K_2 \end{pmatrix}$$

eig$(A+BK)$ = solution of

$$\det\begin{pmatrix} \lambda+1-K_1 & -K_2 \\ -K_1 & \lambda-2-K_2 \end{pmatrix} = 0$$

$$\lambda^2 + \left[(1-K_1)+(-2-K_2)\right]\lambda + (1-K_1)(-2-K_2) - K_1K_2 = 0$$

$$\lambda^2 + \left[-1-K_1-K_2\right]\lambda + (-2+2K_1-K_2) = 0$$

suppose I want eig $(\ldots K) = a, b$

suppose want eigenvalues of $(A+BK) = a, b$ $\implies \lambda^2 + (a+b)\lambda + ab = 0$

Ex. $\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$

eig $-1, 2$

$u = Kx$

$A + BK = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} k_1 & k_2 \end{pmatrix}$

$= \begin{bmatrix} -1+k_1 & k_2 \\ 0 & 2 \end{bmatrix}$

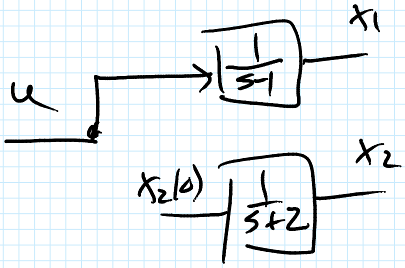
eig $-1+k_1, 2$

$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$
unstable, cont.

$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$
unstable, uncont

$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$
stable, uncontroll

$\dot{x} = \begin{bmatrix} +1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$
unstable, uncont.
stabilizable



Ex: ccf form $G(s) = \dfrac{s^2+s+1}{s^3+s^2+s+1}$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix} \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

let $\dot{x} = Ax + Bu$ } plant

$u = Kx$ } control

$\dot{x} = (A+BK)x$ } closed-loop system

$$A+BK = A + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} k_1 & k_2 & k_3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_1 & k_2 & k_3 \end{pmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k_1-1 & k_2-1 & k_3-1 \end{bmatrix}$$

eig$(A+BK)$ are solutions of

$$s^3 + (k_3-1)s^2 + (k_2-1)s + (k_1-1) = 0$$

suppose I want 3 poles at $s=-1$

$\Rightarrow$ desired characteristic eq is

$$s^3 + 3s^2 + 3s + 1 = 0 = (s+1)^3$$

$$\Rightarrow \boxed{\begin{matrix} k_3 = 4 \\ k_2 = 4 \\ k_1 = 2 \end{matrix}}$$

in Matlab: >> acker

>> place

} Given A, B design K so (A+BK) has desired eigenvalues

5.2 Observability

Define $W_o = \int_0^T e^{A^T t} C^T C e^{At} \, dt$ Observability Gramian

Define observability: for $\dot{x} = Ax + Bu$
$y = Cx + Du$

the system is observable if given measurements $u(t)$ and $y(t)$, we can recover the initial state $x(t_0)$

Theorem $\dot{x} = Ax + Bu$, $y = Cx + Du$ is obs. iff any (all) of following are true

1) Rank $W_o = n$

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2) rank $\mathcal{O}(A,C) = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$ is n

3) rank $\begin{bmatrix} \lambda I - A \\ \hline C \end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$

- Initial Conditions & Meaning of Obs.

consider $\dot{x} = Ax + Bu$

$\boxed{y = Cx + Du}$

differentiate

$\dot{y} = C\dot{x} + D\dot{u}$

$= C(Ax + Bu) + D\dot{u}$

$\boxed{\dot{y} = CAx + CBu + D\dot{u}}$

again

$\ddot{y} = CA\dot{x} + CB\dot{u} + D\ddot{u}$

$= CA(Ax + Bu) + CB\dot{u} + D\ddot{u}$

$\boxed{\ddot{y} = CA^2x + CABu + CB\dot{u} + D\ddot{u}}$

$\vdots$

$\begin{bmatrix} y(t) \\ \dot{y}(t) \\ \vdots \end{bmatrix} \quad \begin{bmatrix} C \\ CA \\ \vdots \end{bmatrix} \quad \begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & 0 & \cdots & 0 \\ \vdots \end{bmatrix} \begin{pmatrix} u(t) \\ \dot{u}(t) \\ \vdots \end{pmatrix}$

$$\begin{pmatrix} y(t) \\ \dot{y}(t) \\ \ddot{y}(t) \\ \vdots \\ y^{(n-1)}(t) \end{pmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} x + \begin{bmatrix} D & 0 & \cdots & \cdots & 0 \\ CB & D & 0 & \cdots & 0 \\ CAB & CB & D & 0 \cdots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ CA^{n-2}B & \cdots & \cdots & CB & D \end{bmatrix} \begin{pmatrix} u(t) \\ \dot{u}(t) \\ \ddot{u}(t) \\ \vdots \\ u^{(n-1)}(t) \end{pmatrix}$$

true for all $t \Rightarrow$ true at $t=0$

so, given $y(0), \dot{y}(0), \cdots y^{(n-1)}(0), u(0), \cdots u^{(n-1)}(0)$

$\Rightarrow$ can get $x(0)$ if $\mathcal{O}(A,B)$ is invertible

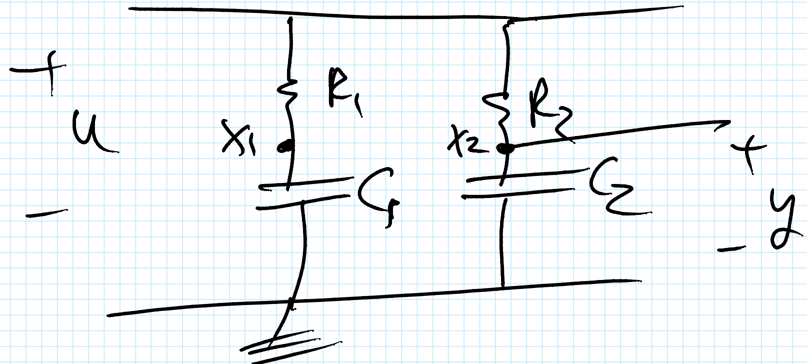
ASIDE

$$G(s) = \frac{s^2+2s+1}{s^3+5s^2+8s+4} = \frac{(s+1)^2}{(s+1)(s+2)^2} = \frac{(s+1)}{s^2+4s+4}$$

$$\underbrace{A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -8 & -5 \end{bmatrix} \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad C = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}}_{\text{cont, not obs}}$$

$$\underbrace{A = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}}_{\text{cont, obs,}}$$

This can happen



$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1/R_1C_1 & 0 \\ 0 & -\frac{1}{R_2C_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{R_1C_1} \\ \frac{1}{R_2C_2} \end{pmatrix} u$$

cont.,
not
obs.

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$