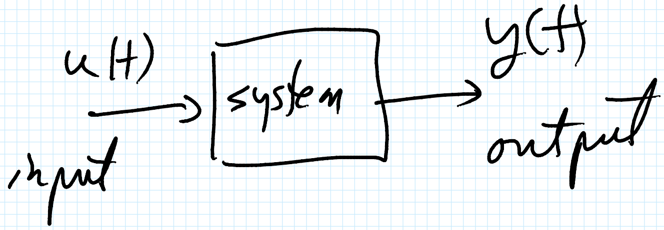
Last time 2.0 State Space Approach (Internal)

3.0 Modeling Dynamic System: Classical Approach (External or Input/Output)

3.1 I/O descriptions



initial time 0

- assume relaxed: output depends only on initial state and input applied after t=0

- assume linear: suppose $y = \underbrace{H}_{\text{operator}}(u)$

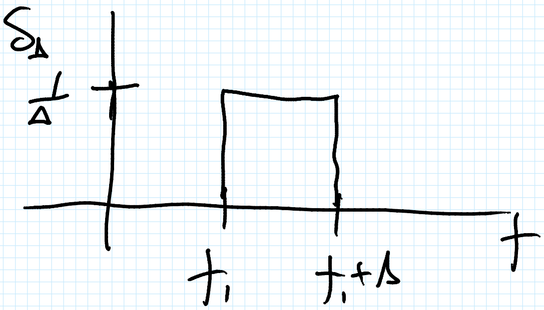
write linear system as

$$y = \underbrace{Hu}_{\text{not multiplication}}$$

$$H(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 H(u_1) + \alpha_2 H(u_2)$$

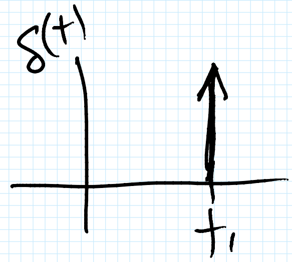
scalar input

Define $\delta_\Delta(t-t_1) = \begin{cases} 0 & t \le t_1 \\ \frac{1}{\Delta} & t_1 < t < t+\Delta \\ 0 & t \ge t_1+\Delta \end{cases}$



Note $\lim_{\Delta \to 0} \delta_\Delta(t-t_1) \triangleq \underbrace{\delta(t-t_1)}_{\text{Dirac delta function}}$

"impulse" function



- infinite magnitude
- infitesimal duration

note: $\displaystyle\int_{-\infty}^{\infty} \delta(t-t_1)\,dt = 1$

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-t_1)\,dt = f(t_1)$$

sifting property

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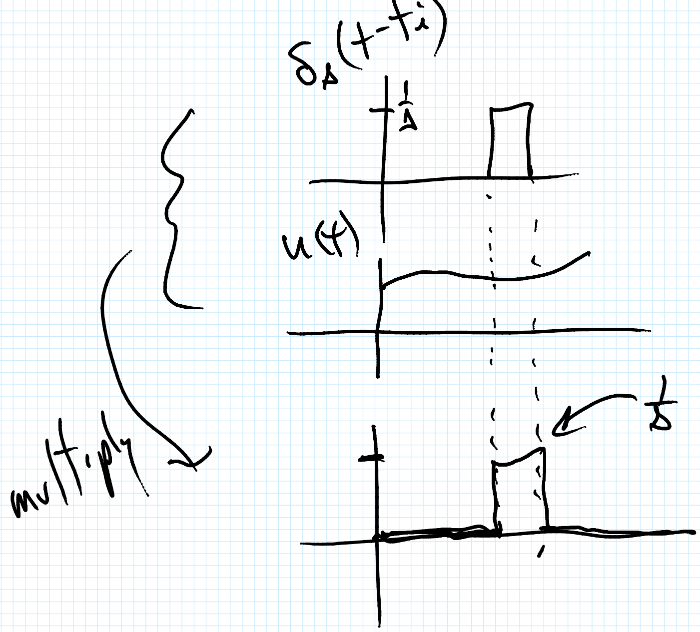
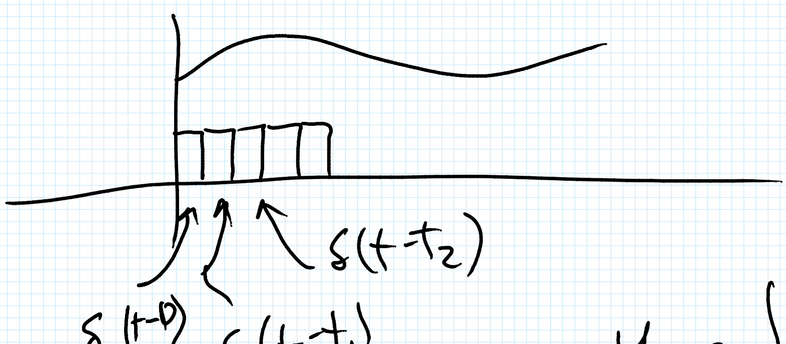
assume $y(t)$, $u(t)$ scalars : $y(t) : \mathbb{R}^+ \to \mathbb{R}^1$

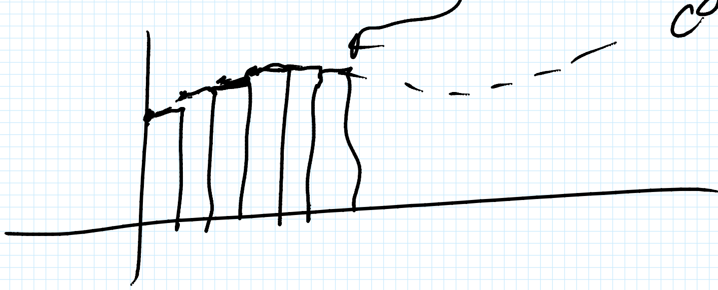
note any $u(t)$ can be approximated as

$$u(t) \approx \sum_i u(t_i)\, \delta_\Delta (t - t_i)\, \Delta$$



$\frac{1}{\Delta}$ smaller $\Longrightarrow$ add Δ in sum above





all these little segments can be added up to $\approx u(t)$

suppose for some system H, we

have $y = H(u)$ where $H(\cdot)$ is linear

then $y \approx H\left(\sum_i u(t_i)\, \delta_\Delta(t-t_i)\, \Delta \right)$

$$\approx \sum H\big(\underbrace{u(t_i)}_{\text{constant}}\, \delta_\Delta(t-t_i)\, \underbrace{\Delta}_{\text{constant}} \big)$$

$$y(t) \approx \sum_i H\left(\delta_\Delta(t-t_i) \right) u(t_i)\, \Delta$$

let $\Delta \to 0$

$$\Longrightarrow y(t) = \int_{-\infty}^{\infty} \underbrace{H(\delta(t-\tau))}\, u(\tau)\, d\tau$$

define $H(\delta(t-\tau))$ as $g(t,\tau)$

called the **impulse response** of the lin...

$y = H(u)$

define $g(t,\tau) = H(\delta(t-\tau))$

$g(t,\tau)$ ↑ the ...

$\delta(t-\tau)$ ↑ impulse input

$g(t, \ldots)$ ... impulse response ... impulse input ... response of the system

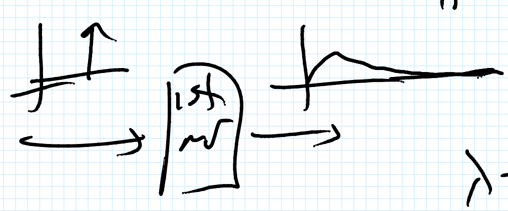
so rewrite

$$y(t) = \int_{-\infty}^{\infty} g(t,\tau)\, u(\tau)\, d\tau$$

ASIDE

$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$

natural freq



$\lambda + p = 0$

ASIDE

for MIMO: $\underline{y} = \int_{-\infty}^{\infty} \underset{\sim}{G(t,\tau)}\, \underline{u}(\tau)\, d\tau$

matrix

$$G = \begin{bmatrix} g_{11}(t,\tau) & \cdots & g_{1p}(t,\tau) \\ \vdots & & \vdots \\ g_{q1}(t,\tau) & & g_{qp}(t,\tau) \end{bmatrix}$$

$\llcorner\!\!\rightarrow$ g_{ij} = impulse response ~~between~~ input j and output i

consider

$$y(t) = \int_{-\infty}^{\infty} g(t,\tau)\, u(\tau)\, d\tau$$

assume causal: output at time t (in the future) does not depend on inputs from the future

i.e.,

$$y(t) = H\left(u_{[-\infty,t]}^{(t)}\right)$$

$$\Rightarrow y(t) = \int_{-\infty}^{t} g(t,\tau)\, u(\tau)\, d\tau$$

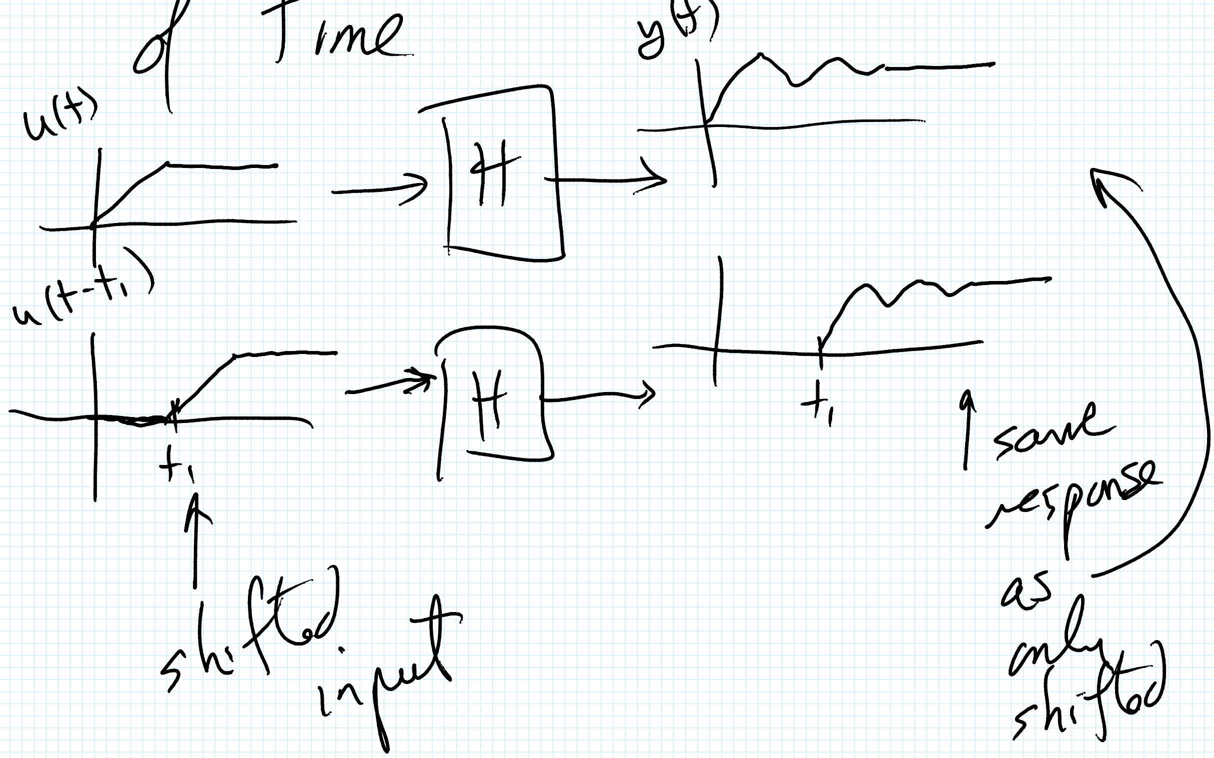
relaxed

$$\Rightarrow y(t) = \int_{t_0}^{t} g(t,\tau)\, u(\tau)\, d\tau$$

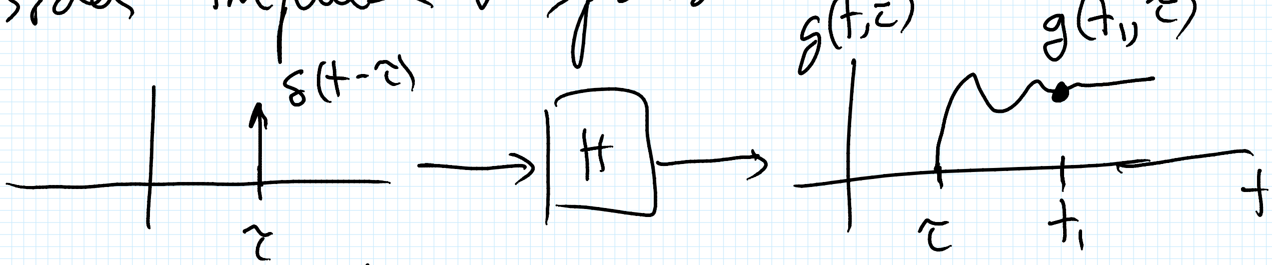
Time-invariance (TI)

characteristics of system don't change as a function of time

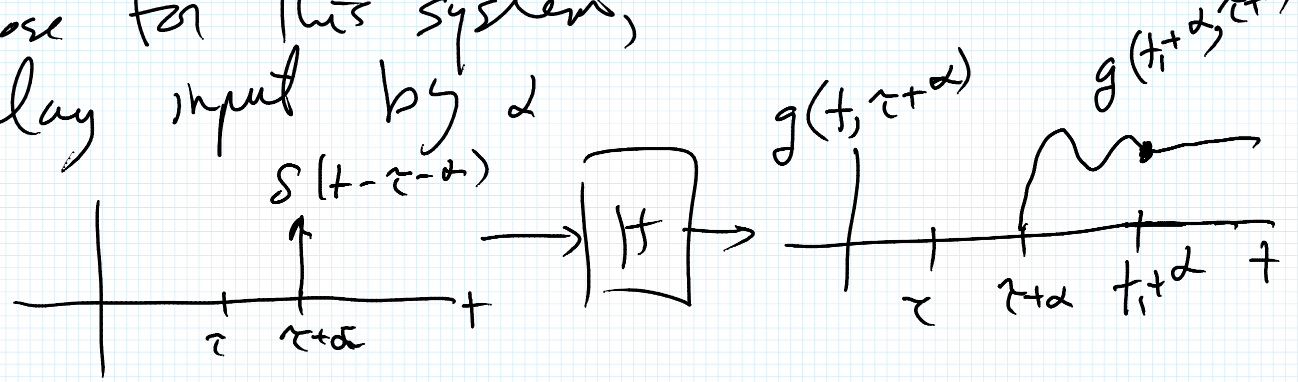
Ex:



consider impulse response



suppose for this system, delay input by α



if TI system we need

$$g(t_1,\tau) = g(t_1+\alpha, \tau+\alpha)$$

this is true for any t_1

$$\Rightarrow \quad g(t,\tau) = g(t+\alpha, \tau+\alpha)$$

also be true for any α

so pick $\alpha = -\tau$

$$\Rightarrow g(t,\tau) = g(t-\tau, 0)$$

$$\triangleq g(t-\tau)$$

so finally

$$\boxed{y(t) = \int_{t_0}^{t} g(t-\tau)u(\tau)\,d\tau}$$

classic convolution integral

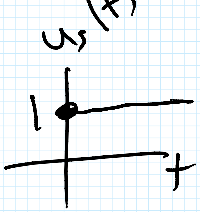
for **L**inear, causal, relaxed, time-**I**nvariant systems

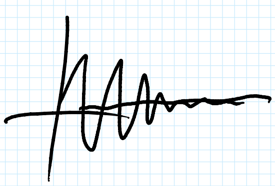
$\boxed{\text{LTI systems}}$

3.2 Laplace Transforms in 15 mins or less

Def: $$F(s) = \mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} f(t)e^{-st}\,dt$$

use tables





$f(t)$	$F(s)$
e^{-at}	$\frac{1}{s+a}$
$u_s(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$	$\frac{1}{s}$ } signal
$\delta(t)$	1
$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

etc.

properties $y(t) = \int f(t)\,dt$ $f(t) \overset{\mathcal{L}}{\Longleftrightarrow} F(s)$

$$\Rightarrow \quad Y(s) = \underbrace{\frac{1}{s}}_{\text{system}} \cdot F(s)$$

$$\dot{f}(t) \overset{\mathcal{L}}{\Longleftrightarrow} sF(s) - f(0)$$

$$\underbrace{\left[\int_0^t f_1(t-\tau) f_2(\tau)\,d\tau\right]}_{\underbrace{f_1 * f_2}_{\text{convolution}}} \Longleftrightarrow F_1(s) \cdot F_2(s)$$

use some of this to solve diff. eq.

Ex. $\ddot{y} + 3\dot{y} + 2y = 5\,u(t)$

$u(t) = u_s(t)$ (step)

$y(0) = 1$

$\dot{y}(0) = 2$

$\mathcal{L}$ (

$$\left[s^2Y(s) - sy(0) - \dot{y}(0)\right] + 3\left[sY(s) - y(0)\right] + 2Y(s) = \frac{5}{s}$$

$$Y(s) = \underbrace{\frac{5}{s(s^2+3s+2)}}_{\text{related to input}} - \underbrace{\frac{s+1}{s^2+3s+2}}_{\text{from initial conditions}}$$

partial fraction expansion

$$Y(s) = \frac{5/2}{s} - \frac{5}{s+1} + \frac{5/2}{s+2} - \frac{1}{s+2}$$

table

$$y(t) = \frac{5}{2} - 5e^{-t} + \frac{5}{2}e^{-2t} - e^{-2t}$$

$$= \frac{5}{2} \text{ as } t \to \infty$$