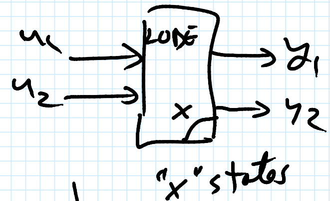
3.4 Transfer Matrix Example

$$\tau_1 \dot{y}_1 + y_1 = K_{11} u_1 + K_{12} u_2(t)$$
$$\tau_{12} \dot{y}_1 + \tau_2 \dot{y}_2 + y_2 = K_{21} u_1$$



$\mathcal{L}$, I.C. $=0$

$$\left[\begin{array}{c|c} \tau_1 s+1 & 0 \\ \hline \tau_{12} s & \tau_2 s+1 \end{array}\right]\begin{pmatrix} Y_1(s) \\ Y_2(s)\end{pmatrix} = \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & 0 \end{array}\right]\begin{pmatrix} U_1(s) \\ U_2(s)\end{pmatrix}$$

$$\Gamma(s)\,\underline{Y}(s) = K\,\underline{U}(s)$$

ASIDE

$$\tau \dot{y} + y = K u$$

$\mathcal{L}$

$$(\tau s+1)\,Y(s) = K\,U(s)$$
$$Y(s) = (\tau s+1)^{-1} K\,U(s) = \frac{K}{\tau s+1}\cdot U(s)$$

$$\underline{Y}(s) = \Gamma^{-1}(s)\,K\,\underline{U}(s)$$

$$\begin{pmatrix} Y_1(s) \\ Y_2(s)\end{pmatrix} = \left[\begin{array}{c|c} \tau_1 s+1 & 0 \\ \hline \tau_{12} s & \tau_2 s+1 \end{array}\right]^{-1}\left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & 0 \end{array}\right]\begin{pmatrix} U_1(s) \\ U_2(s)\end{pmatrix}$$

$$= \underbrace{\underbrace{D(s)^{-1}}_{\text{denominator}}\;\underbrace{N(s)}_{\text{numerator}}\;\underline{U}(s)}_{\text{matrix fraction}}$$

denominator, numerator: both of these are matrices whose elements are polynomials in "s" polynomials

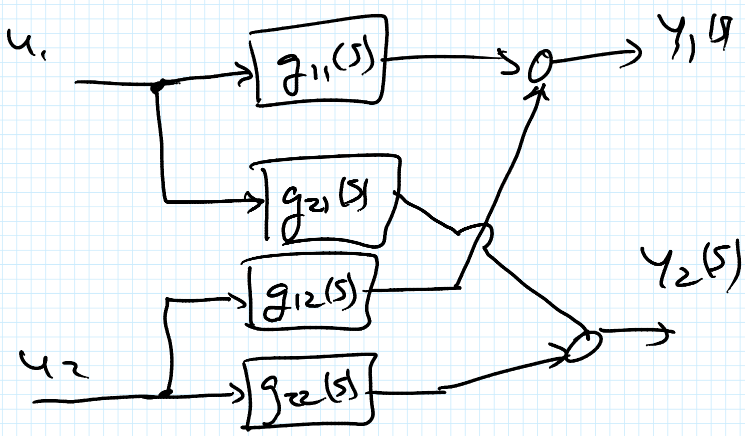
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(s) = \left[\begin{array}{c:c} \dfrac{K_{11}}{\tau_1 s+1} & \dfrac{K_{12}}{\tau_1 s+1} \\ \hdashline \dfrac{(K_{11}\tau_{12}+K_{12}\tau_1)s+1}{(\tau_1 s+1)(\tau_2 s+1)} & \dfrac{K_{12}\tau_{12}s}{(\tau_1 s+1)(\tau_2 s+1)} \end{array}\right] \begin{pmatrix} u_1(s) \\ u_2(s) \end{pmatrix}$$

$$\begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} = \left[\begin{array}{c:c} g_{11}(s) & g_{12}(s) \\ \hdashline g_{21}(s) & g_{22}(s) \end{array}\right] \begin{pmatrix} u_1(s) \\ u_2(s) \end{pmatrix} \qquad g_{ij}(s) \triangleq \left.\frac{Y_i(s)}{U_j(s)}\right|_{\substack{IC=0 \\ u_k=0 \\ k\neq j}}$$

$$Y(s) = D(s)^{-1}N(s)\,U(s)$$
$$= G(s)\,U(s)$$

transfer matrix

block diagram



- <u>state-space</u>

$$\tau_1 \dot{y}_1 + y_1 = K_{11}u_1 + K_{12}u_2$$
$$\tau_{12}\dot{y}_1 + \tau_2\dot{y}_2 + y_2 = K_{21}u_1$$

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let $x_1 = y_1 \Rightarrow \dot{x}_1 = \dot{y}_1$

$x_2 = y_2 \Rightarrow \dot{x}_2 = \dot{y}_2$

$$\Rightarrow \boxed{\dot{x}_1 = \frac{K_{11}}{\tau_1}u_1 + \frac{K_{12}}{\tau_1}u_2 - \frac{1}{\tau_1}x_1}$$

$$\Rightarrow \boxed{\dot{x}_1 = \frac{K_{11}}{\tau_1}u_1 + \frac{K_{12}}{\tau_1}u_2 - \frac{1}{\tau}x_1}$$

similarly $\boxed{\dot{x}_2 = \frac{K_{21}}{\tau_2}u_1 - \frac{x_2}{\tau_1} - \frac{\tau_{12}}{\tau_2}\left(\frac{K_{11}}{\tau_1}u_1 + \frac{K_{12}}{\tau_1}u_2 - \frac{1}{\tau_1}x_1\right)}$

or
$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{bmatrix}-\frac{1}{\tau_1} & 0\\ \frac{\tau_{12}}{\tau_1\tau_{21}} & -\frac{1}{\tau_2}\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}$$

$$\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$b_{11} = K_{11}/\tau_1$

$b_{12} = K_{12}/\tau_1$

$b_{21} = \frac{K_{21}}{\tau_2} - \frac{K_{11}\tau_{12}}{\tau_1\tau_2}$

$b_{22} = -\frac{K_{12}}{\tau_{2}}$

i.e.
$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Exercise 1: show for this example
$$G(s) = C(sI-A)^{-1}B$$

2.0 Modelling using State-Space (cont.)

2.4 Transformations

Example: $\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = -16\ddot{u} - 32\dot{u} - 17u$

$s^3+6s^2+11s+6 = (s+1)(s+2)(s+3)$

residue

$$G(s) = \frac{Y(s)}{U(s)} = \frac{-16s^2-32s-17}{s^3+6s^2+11s+6} = \frac{-\frac{1}{2}}{s+1} + \frac{17}{s+2} + \frac{-65/2}{s+3}$$

$$\Rightarrow \begin{cases}\dot{x} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -6 & -11 & -6\end{bmatrix}x + \begin{bmatrix}0\\0\\1\end{bmatrix}u\\ y = [-17\ -32\ -16]x\end{cases}$$

claim, can also write

$$\left(\dot{z} = \begin{bmatrix}-1 & 0 & 0\\ 0 & -2 & 0\\ & & -3\end{bmatrix}z + \begin{bmatrix}1\\1\\1\end{bmatrix}u\right.$$

$$\left\{ y = [-17\ \ -2\ \ -16]x \right.$$

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

$$\left\{ \begin{aligned} \dot{z} &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} z + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} u \\ y &= [-\tfrac{1}{2}\ \ 17\ \ -\tfrac{65}{2}] z \end{aligned} \right.$$

$$\dot{z} = \bar{A}z + \bar{B}u$$
$$y = \bar{C}z$$

<u>Claim</u>

$$C(sI - A)^{-1}B = \bar{C}(sI - \bar{A})^{-1}\bar{B}$$

each state-space description gives same transfer function

- <u>General</u>

given $\dot{x} = Ax + Bu$
$y = Cx + Du$

let $z = P^{-1}x$ or $x = Pz$ where P^{-1} exists

then $\dot{x} = Ax + Bu$

becomes $P\dot{z} = APz + Bu$

or $\dot{z} = (P^{-1}AP)z + (P^{-1}B)u$

$y = (CP)z + Du$

$$(A, B, C, D) \xrightarrow{z = P^{-1}x} (P^{-1}AP,\ P^{-1}B,\ CP,\ D)$$

or $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$

Key points:

- eigenvalues of A are same as

"eigenvalues" of A

- $C(sI-A)^{-1}B+D = \bar{C}(sI-\bar{A})^{-1}\bar{B}+\bar{D}$

Exercise 2: show this fact

- both (A,B,C,D) and $(\bar{A},\bar{B},\bar{C},\bar{D})$ are valid state-space descriptions and are equivalent to each other

- Canonical Forms

(A,B,C,D) can appear in different forms:

- physical form
- controller form (phase variable form)
- observer form
- diagonal form

- Each form corresponds to a different set of state variables
- Do this because different canonical forms are easier to compute with for different purpose
  - controller form for state

feedback design

- diagonal form for computing output of a system

• Initial States

Given $y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 \dot{y} + a_0 y = b_{n-1} u^{(n)} + \dots + b_0 u$

if I know $u(t)$, I need $y(0), \dot{y}(0)$ --- $y^{(n-1)}(0)$ and $u(0), \dot{u}(0)$ --- $u^{(n-1)}(0)$ to compute $y(t)$

Given
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$
if I know $u(t)$, I need $x(0)$ to compute $y(t)$

2.5 Solution of State Equations

• matrix exponential

consider

$$\dot{x} = Ax \quad \text{vector-matrix}$$

$\mathcal{L}$

$$sX(s) - X(0) = AX(s)$$

$$(sI - A)X(s) = X(0)$$

$$X(s) = (sI - A)^{-1} X(0)$$

$\mathcal{L}^{-1}$

$$x(t) = \underbrace{\mathcal{L}^{-1}\{(sI-A)^{-1}\}}_{\triangleq\, e^{At}}\, x(0)$$

$$\boxed{x(t) = e^{At}x(0)}$$

matrix exponential

ASIDE

scalar: $\dot{x} = -3x$

$\Rightarrow x(t) = e^{-3t}x(0)$

Ex. suppose $\dot{z} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} z(t)$ $\quad z(0) = \begin{pmatrix} z_1(0) \\ z_2(0) \end{pmatrix}$

$$\Rightarrow z(t) = e^{At} z(0)$$

$$e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\}$$

$$= \mathcal{L}^{-1}\left\{\left[s\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\right]^{-1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+3s+2}\left[\begin{array}{c|c} s+3 & 1 \\ \hline -2 & s \end{array}\right]\right\}$$

$$= \mathcal{L}^{-1}\left\{ \begin{bmatrix} \frac{s+3}{s^2+3s+2} & \frac{1}{s^2+3s+2} \\ \frac{-2}{s^2+3s+2} & \frac{s}{s^2+3s+2} \end{bmatrix} \right\}$$

$$s^2+3s+2 = (s+1)(s+2)$$

$$= \begin{bmatrix} a_1 e^{-t} + a_2 e^{-2t} & a_5 e^{-t} + a_6 e^{-2t} \\ a_3 e^{-t} + a_4 e^{-2t} & a_7 e^{-t} + a_8 e^{-2t} \end{bmatrix}$$

e^{-t}, e^{-2t} modes of system

$$z(t) = \begin{bmatrix} \quad \end{bmatrix} \begin{pmatrix} z_1(0) \\ z_2(0) \end{pmatrix}$$

Example

$$\dot{x} = \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix} x \qquad x(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

$$x(t) = e^{+At} x(0)$$

$$e^{+At} = \mathcal{L}^{-1}\left\{ (sI - A)^{-1} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \begin{pmatrix} s+3 & 0 \\ 0 & s+4 \end{pmatrix}^{-1} \right\}$$

clearly

$$x_1(t) = e^{-3t} x_1(0)$$
$$x_2(t) = e^{-4t} x_2(0)$$

$$= \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{1}{s+3} & 0 \\ 0 & \frac{1}{s+4} \end{bmatrix}\right\}$$

$$= \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{-4t} \end{bmatrix}$$

$$\Rightarrow x(t) = \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{-4t} \end{bmatrix}\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

e^{At} easy to compute on diagonal