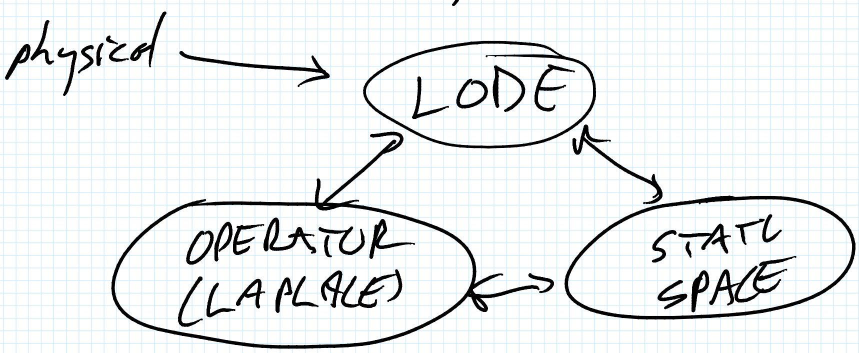
Last time

1.0 Introduction

2.0 Modeling Dynamic Systems: State Space Approach



2.1 Review: Linear, Ordinary, Differential Equations (LODE)

- First-Order

$$\dot{y} + a_0 y = b_0 u$$

$$\tau \dot{y} + y = K u$$

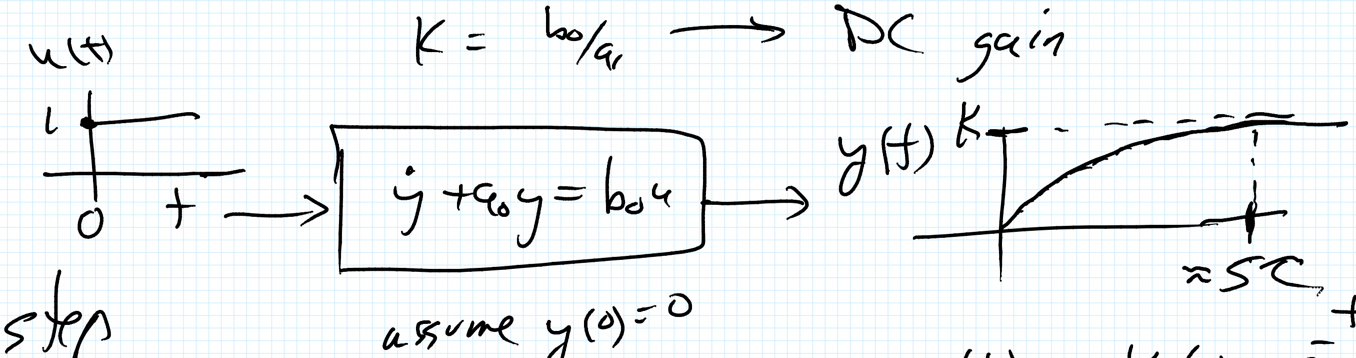
$$\dot{y} \triangleq \frac{d}{dt}\left(y(t)\right)$$

$$\ddot{y} \triangleq \frac{d^2}{dt^2}\left(y(t)\right)$$

etc

$\tau = \frac{1}{a_0} \longrightarrow$ time constant

$K = b_0/a_0 \longrightarrow$ DC gain



$$y(t) = K\left(1 - e^{-t/\tau}\right)$$

- **2nd-order**

$$\ddot{y} + a_1\dot{y} + a_0 y = b_0 u$$

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = K\omega_n^2 u(t)$$

(zeta) ζ = damping factor, natural frequency, K gain

Define characteristic equation

$$\lambda^2 + a_1\lambda + a_0 = 0$$

or

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$$

$$(\lambda + \lambda_1)(\lambda + \lambda_2) = 0$$

poles (or eigenvalues)

solution $y_h(t)$ has 3 possible forms depending on λ_1, λ_2

ASIDE

$$\left.\begin{aligned} &\ddot{y} + a_1\dot{y} + a_0 y = b_0 u \\ &\text{or} \\ &\dot{y} + a_0 y = b_0 u \end{aligned}\right\}$$

$\dot y + a_0 y = b u$

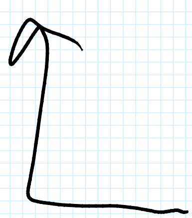
$$y(t) = \underbrace{y_h(t)}_{\substack{\text{response} \\ \text{when } u(t)=0 \\ \text{and } y(0)\neq 0}} + y_f(t)$$

Ex. (1) $\dot y + 2y = u$

$\boxed{u=0} \Rightarrow \dot y + 2y = 0$

$\Rightarrow y_h(t) = y(0)e^{-2t}$

(2) $\boxed{\begin{matrix} u = 1 \\ y(0) = 0 \end{matrix}}$ $t \geq 0$ $\Bigg\} \Rightarrow y_f(t) = \frac{1}{2}(1-e^{-2t})$ $t \geq 0$



(3) $\dot y + 2y = u$

$\boxed{\begin{matrix} u = 1 \\ y(0) = 1 \end{matrix}} \Rightarrow y(t) = y_h(t) + y_f(t)$

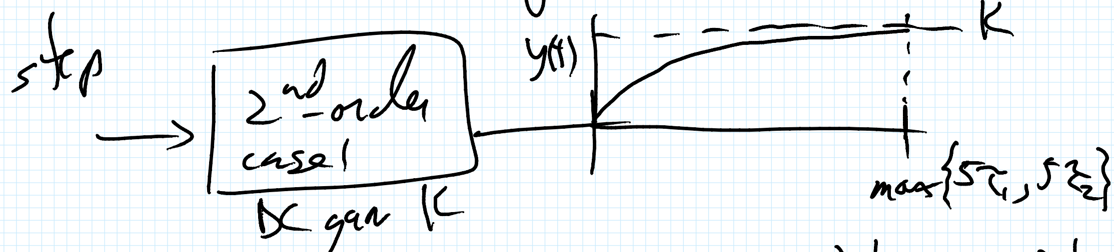
$= \frac{1}{2}(1-e^{-2t}) + e^{-2t}$

3 cases

1) λ_1, λ_2 real distinct (overdamped $\Rightarrow \zeta > 1$)

$y_h(t) = ae^{-\lambda_1 t} + be^{-\lambda_2 t}$

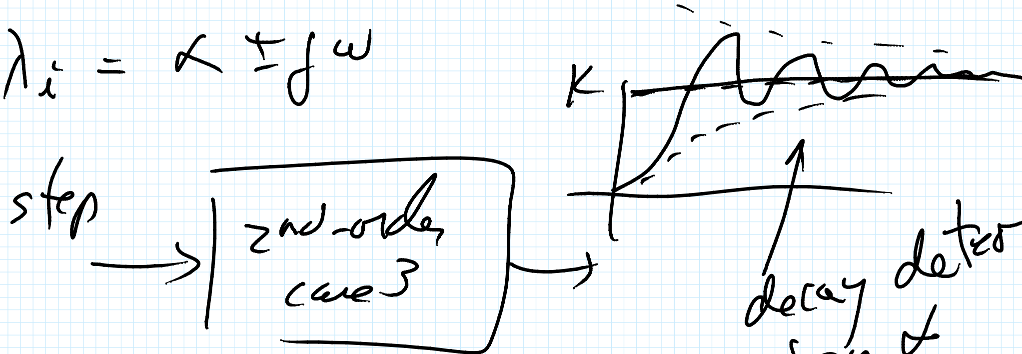
$$y_n(t) = ae^{-\lambda_1 t} + be^{-\lambda_2 t}$$



2) λ_1, λ_2 same $(\lambda_1 = \lambda, \lambda_2 = \lambda)$ $y_n(t) = cte^{-\lambda t} + de^{-\lambda t}$
(critically damped $\Rightarrow \zeta = 1$)

3) $\lambda_{1,2}$ are complex (underdamped $\zeta < 1$)

$$\lambda_i = \alpha \pm j\omega$$



decay determined by α
oscillation by ω

- <u>higher-order</u>

ex. $\dddot{y} + 2\ddot{y} + 3\dot{y} + y = 2\ddot{u} + 7\dot{u} + u$

3rd-order

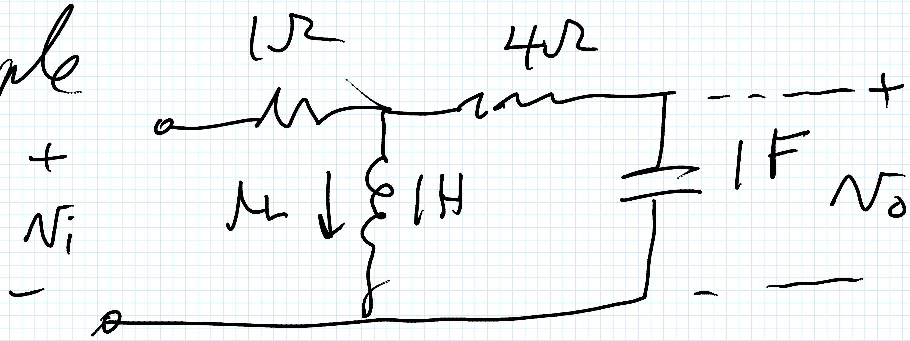
3 1st orders or 1 1st order, 1 second order

all higher-order decompose into

1st & 2nd order

2.2 State Space Concepts

• Example



$$v_i = \frac{dv_o}{dt} + i_L + \frac{di_L}{dt}$$

$$v_o = -4\frac{dv_o}{dt} + \frac{di_L}{dt}$$

combine

$$\ddot{v}_o + \dot{v}_o + \frac{1}{5}v_o = \frac{1}{5}\dot{v}_i$$

let $x_1 = v_o$

$x_2 = \dot{v}_o = \dot{x}_1$

also note $\ddot{v}_o = \dot{x}_2$

plug into

$$\dot{x}_2 + x_2 + \frac{1}{5}x_1 = \frac{1}{5}\dot{v}_i$$

$$\boxed{\begin{aligned}\dot{x}_2 &= -\tfrac{1}{5}x_1 - x_2 + \tfrac{1}{5}\dot{v}_i \\ \dot{x}_1 &= x_2\end{aligned}}$$

↑ won't work

different analysis to get here

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ \cdots & \cdots \end{bmatrix} x + \begin{pmatrix} 0 \\ \cdots \end{pmatrix} v_i$$

define $z_1 = v_o$, $z_2 = i_L$ } variables associated with energy storage

plug into equation and rearrange

$$\dot{z}_1 = -\frac{1}{5}z_1 - \frac{1}{5}z_2 + \frac{1}{5}v_i$$

$$\dot{z}_2 = \frac{1}{5}z_1 - \frac{4}{5}z_2 + \frac{4}{5}v_i$$

$$\begin{pmatrix} \dot{z} \\ \cdots \end{pmatrix} = \begin{bmatrix} -1/5 & -1/5 \\ \cdots & \cdots \end{bmatrix}\begin{pmatrix} z_1 \\ \cdots \end{pmatrix} + \begin{pmatrix} \frac{1}{5} \\ 4 \end{pmatrix} v_i$$

distinct analysis

$$\begin{pmatrix}\dot z_1\\ \dot z_2\end{pmatrix} = \begin{bmatrix}-1/5 & -1/5\\ 1/5 & -4/5\end{bmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix} + \begin{pmatrix}1/5\\ 4/5\end{pmatrix} v_i$$

$$v_0 = \begin{bmatrix}1 & 0\end{bmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix}$$

$$\dot x = \begin{bmatrix}0 & 1\\ -1/5 & -1\end{bmatrix} x + \begin{pmatrix}0\\ 1\end{pmatrix} v_i$$

$$v_0 = \begin{bmatrix}0 & 1/5\end{bmatrix} x$$

also a valid state space description, with different states

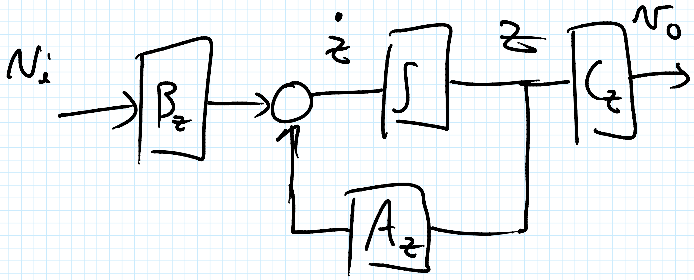
$$\dot z = A_z \cdot z + B_z v_i$$

$$v_0 = C_z z$$

output

state variable

input



state variables: the smallest set of variables such that knowledge of the value of those variables at time t_0 and knowledge of the input at time $t \ge t_0$ for all t is sufficient to calculate the output for all time $t \ge t_0$

• General form

$$\dot{x} = Ax + Bu$$
$$y = Cx + \underbrace{Du}_{\text{feedthrough term}}$$

no $\dot{u}$ on right

$x \in \mathbb{R}^n$

u input vector

y output vector

2.3 Deriving the Controllable Canonical Form

structure in A, B, C
- controller
- observer
- diagonal (Jordan)

less structure
- physical

• Easiest

$$\left(y^{(4)} \triangleq \ddddot{y} \triangleq \frac{d^4}{dt}(y) \right)$$

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = u$$

to solve for $y(t)$, need $u(t)$ and

$y(0), \dot{y}(0), \ddot{y}(0) \cdots\cdots y^{(n-1)}(0)$

$\Rightarrow$ define $x_1 = y$

$x_2 = \dot{y} = \dot{x}_1$

$\boxed{\dot{x}_1 = x_2}$

$$x_2 = \dot{y} = \dot{x}_1$$

$$\vdots$$

$$x_n = y^{(n-1)} = \dot{x}_{n-1}$$

note $\quad y^{(n)} = \dot{x}_n$

$$\left\{ \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \end{aligned} \right.$$

so, diff eq looks like

$$\dot{x}_n + a_{n-1}x_n + a_{n-2}x_{n-1} + \cdots + a_1 x_2 + a_0 x_1 = u$$

$$\dot{x}_n = -a_0 x_1 - a_1 x_2 - \cdots - a_{n-1} x_n + u$$

combine boxes to get

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & \cdots & -a_{n-1} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \underline{x}$$

Ex,

$$y^{(3)} + 2\ddot{y} + \dot{y} + 3y = u$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -1 & -2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

- harder case

Ex: $\ddot{y} + a_1 \dot{y} + a_0 y = b_1 \dot{u} + b_0 u$

$$x_1 = y$$

$$x_2 = \dot{y} = \boxed{\dot{x}_1 = x_2}$$

$$\dot{x}_2 = \ddot{y}$$

$$\boxed{\dot{x}_2 + a_1 x_2 + a_0 x_1 = b_1 \dot{u} + b_0 u}$$

$$\Rightarrow \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ b_0 \end{pmatrix} u + \underbrace{\begin{pmatrix} 0 \\ b_1 \end{pmatrix} \dot{u}}$$

bad; don't want $\dot{u}$ on Right

$\Rightarrow$ ~~trick~~ technique - introduce auxilary variable $\bar{x}$, so that $\ddot{x} + a_1\dot{x} + a_0 x = u$ $\quad *$

$\Longrightarrow \quad \dddot{x} + a_1\ddot{x} + a_0\dot{x} = \dot{u}$ $\quad **$

plug $*$, $**$ into LODE!

$$\ddot{y} + a_1\dot{y} + a_0 y = b_0 u + b_1\dot{u}$$

$$\ddot{y} + a_1\dot{y} + a_0 y = b_0(\ddot{x} + a_1\dot{x} + a_0 x) + b_1(\dddot{x} + a_1\ddot{x} + a_0\dot{x})$$

$$= (b_1\dddot{x} + b_0\ddot{x}) + a_1(b_1\ddot{x} + b_0\dot{x}) + a_0(b_1\dot{x} + b_0 x)$$

$$y = b_1\dot{x} + b_0 x$$

$\Longrightarrow$ suggest, let $x_1 = x$

$x_2 = \dot{x}$

$\Longrightarrow \quad y = b_0 x_1 + b_1 x_2$

$$\dot{x}_2 = \ddot{x} = u - a_1 x_2 - a_0 x_1$$

finally, get

$$\begin{pmatrix}\dot{x}_1 \\ \dot{x}_2\end{pmatrix} = \begin{bmatrix} 0 & 1 \\ & \end{bmatrix}\begin{pmatrix}x_1 \\ x_2\end{pmatrix} + \begin{pmatrix}0 \\ 1\end{pmatrix}u$$

$$\begin{cases} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y = \begin{bmatrix} b_0 & b_1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{cases}$$

$$\left\{ \ddot{y} + a_1 \dot{y} + a_0 y = b_0 u + b_1 \dot{u} \right.$$

<u>Exercise</u> Derive a state-space controller form for

$$\boxed{\ddot{y} + a_1 \dot{y} + a_0 y = b_2 \ddot{u} + b_1 \dot{u} + b_0 u}$$