3.3 Transfer Matrices

- Impulse response and transfer function

Last time: - idea of impulse response
- Laplace transforms

compute Laplace transform of the output of a linear system

$$y = H(u)$$

$$y(t) = \int_0^t g(t-\tau)\, u(\tau)\, d\tau$$

system defined by its impulse response $g(t)$

$y(t) = \underbrace{g(t) * u(t)}$ convolution operator

$$Y(s) = \mathcal{L}\{y(t)\}$$

$$= \int_0^\infty y(t) e^{-st} dt$$

$$= \int_0^\infty \left(\int_0^t g(t-\tau) u(\tau) d\tau \right) e^{-st} dt$$

$$= \int_0^\infty \left(\int_0^\infty g(t-\tau) e^{-s(t-\tau)} dt \right) u(\tau) e^{-s\tau} d\tau$$

$$= \underbrace{\left(\int_0^\infty g(v) e^{-sv} dv \right)} \underbrace{\left(\int_0^\infty u(\tau) e^{-s\tau} d\tau \right)}$$

$$Y(s) = G(s) \cdot U(s)$$

$$\mathcal{L}\{y(t)\} \quad \mathcal{L}\{g(t)\} \quad \mathcal{L}\{u(t)\}$$

output impulse response input

define $G(s) = \mathcal{L}\{g(t)\}$ to be

TF transfer function (SISO)

matrix (MIMO)

- <u>Transfer function of LODEs</u>

in general, consider SISO LODE of form

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1\dot{y} + a_0 y = b_n u^{(n)} + b_{n-1}u^{(n-1)} + \cdots + b_1\dot{u} + b_0 u$$

assume all initial conditions are zero and take Laplace of each side

$$y^{(n)} \overset{\mathcal{L}}{\Longleftrightarrow} s^n Y(s)$$

$$s^n Y(s) + a_{n-1}s^{n-1}Y(s) + \cdots + a_1 s Y(s) + a_0 Y(s)$$
$$= b_n s^n U(s) + b_{n-1}s^{n-1}U(s) + \cdots + b_1 s U(s) + b_0 U(s)$$

$$\left[b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0 \right] U(s)$$

$$\Rightarrow \quad Y(s) = \underbrace{\left[\frac{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \right]}_{G(s)\ \text{the SISO transfer function } (1\times 1)} U(s)$$

Summary

$$\begin{tikzcd} & \text{LODE} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \text{TF} \arrow[rr, bend right, "\text{realization theory}"'] & & (A,B,C,D) \arrow[ll, bend right, "?"'] \end{tikzcd}$$

- Transfer function from (A,B,C,D)

suppose system
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

take Laplace

$$sX(s) - X(0) = AX(s) + BU(s)$$
$$sX(s) - AX(s) = BU(s) + X(0)$$
$$(sI - A)X(s) =$$
$$X(s) = (sI - A)^{-1} BU(s) + (sI-A)^{-1} X(0)$$

$$Y(s) = CX(s) + DU(s)$$

$$Y(s) = C X(s) + D U(s)$$

$$Y(s) = C(sI - A)^{-1} x(0) + C(sI-A)^{-1} B U(s) + D U(s)$$

let $x(0) = 0$, then

$$Y(s) = \underbrace{\left[C(sI-A)^{-1} B + D \right]}_{G(s)} U(s)$$

transfer function (SISO)

transfer matrix (MIMO)

Ex

$$\ddot{y} + 3\dot{y} + 2y = u$$

TF

S.S.

$$\frac{Y(s)}{U(s)} = G(s) = \frac{1}{s^2+3s+2}$$

1×1

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = [1 \;\; 0] x$$

$$G(s) \stackrel{?}{=} C(sI-A)^{-1} B$$

$$C(sI-A)^{-1}B = [1 \;\; 0] \left\{ s\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \right\}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$C(sI-A)^{-1}B = \underbrace{\begin{bmatrix}1 & 0\end{bmatrix}}_{(1\times 2)} \underbrace{\left\{ s\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}0 & 1\\ -2 & -3\end{pmatrix}\right\}^{-1}}_{(2\times 2)} \underbrace{\begin{pmatrix}0\\ 1\end{pmatrix}}_{(2\times 1)}$$

$(1\times 2)\,(2\times 2)\,(2\times 1)$

(1×1)

$$= \begin{pmatrix}1 & 0\end{pmatrix}\begin{bmatrix} s & -1\\ +2 & s+3\end{bmatrix}^{-1}\begin{pmatrix}0\\ 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0\end{pmatrix}\left\{\frac{1}{s(s+3)-(-1)(2)}\begin{bmatrix} s+3 & 1\\ -2 & s\end{bmatrix}\right\}\begin{pmatrix}0\\ 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0\end{pmatrix}\left[\frac{1}{s^2+3+2}\begin{bmatrix} s+3 & 1\\ -2 & s\end{bmatrix}\right]\begin{pmatrix}0\\ 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0\end{pmatrix}\left[\frac{1}{s^2+3s+2}\begin{bmatrix}1\\ s\end{bmatrix}\right]$$

$$= \frac{1}{s^2+3s+2}$$

Next time: - Ex of MIMO

- interconnected systems

- "transformations"

→ ASIDE

consider

$$\begin{array}{cccc} A & B & C & D\\ \begin{pmatrix}0 & 1\\ -2 & -3\end{pmatrix}, & \begin{pmatrix}0\\ 1\end{pmatrix}, & \begin{pmatrix}1 & 0\end{pmatrix}, & 0\end{array}$$

$$\frac{1}{s^2+3s+2} \longrightarrow \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \end{pmatrix}, 0$$

note

$$\frac{1}{s^2+3s+2} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$-\frac{1}{s+2} + \frac{1}{s+1}$$

consider $\quad \dot{z} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} z + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$

$$y = [1 \ \ -1] z$$

compute $C(sI-A)^{-1}B$ for this

$$(1 \ \ -1) \cdot \begin{pmatrix} s+1 & 0 \\ 0 & s+2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(1 \ \ -1) \begin{pmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(1 \ \ -1) \begin{pmatrix} \frac{1}{s+1} \\ \frac{1}{s+2} \end{pmatrix} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$= \frac{1}{s^2+3s+2}$$

point: state-space representation for a system is not unique