EGGN 517
Lecture 19
3-27

will be around 11-3 today

8.2 State FB for MIMO

Given A, B, find K so that $A+BK$ has desired eigenvalues

SISO
• **one way**: - Transform to ccf
- get $\bar{K}$ by inspection

- transform back to get K

5150

another way: - automate above

(1) Let $\alpha_d(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$

be desired characteristic polynomial for $A+BK$

(2) Form $\mathcal{C}(A,B) = [B \; AB \cdots A^{n-1}B]$

(3) Compute $\alpha_d(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I$

(4) $K = -[0 \cdots 0 \; 1]\,\mathcal{C}^{-1}(A,B)\,\alpha_d(A)$

called "Ackerman's formula"

Ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

(1) want poles at $s = -1$

i.e., $\alpha_d(s) = (s+1)^2 = s^2+2s+1$

(i.e. $\delta = [1\ 2\ 1]$)

(2) $\mathcal{C}(A,B) = [B\ \ AB] = \begin{bmatrix}1 & 1\\ 2 & 4\end{bmatrix}$

(3) $\alpha_d(A) = A^2 + 2\cdot A + I = \begin{bmatrix}4 & 0\\ 0 & 9\end{bmatrix}$

(4) $$K = \underbrace{-\underset{1\times 2}{[0\ \ 1]}\underbrace{\underset{2\times 2}{\begin{pmatrix}1 & 1\\ 2 & 4\end{pmatrix}^{-1}}\underset{2\times 2}{\begin{pmatrix}4 & 0\\ 0 & 9\end{pmatrix}}}_{2\times 2}}_{1\times 2} = [4\ \ -9/2]$$

check $A_{cl} = A + \overbrace{BK}^{\begin{bmatrix}4 & -9/2\\ 8 & -9\end{bmatrix}}$

$$= \underset{2\times 2}{\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}} + \underbrace{\underset{2\times 1}{\begin{pmatrix}1\\ 2\end{pmatrix}}\underset{1\times 2}{(4\ \ -9/2)}}_{2\times 2}$$

$$= \begin{bmatrix}5 & -9/2\\ 8 & -7\end{bmatrix} \Rightarrow \lambda_{1,2} = -1$$

Matlab acker, place

- MIMO case

aside: A matrix is cyclic if its characteristic polynomial is also its minimal polynomial

smallest-order polynomial $\psi(s)$ such that $\psi(A)=0$

note: if $\Delta(s)$ is characteristic polynomial $\Rightarrow \Delta(A)=0$

↳ means each eigenvalue has only one Jordan block

Ex: $A = \left[\begin{array}{c|cc|ccc} \lambda_1 & & & & 0 & \\ \hline & \lambda & 1 & & & \\ & 0 & \lambda & & & \\ \hline & & & \lambda & 1 & 0 \\ 0 & & & 0 & \lambda & 1 \\ & & & 0 & 0 & \lambda \end{array}\right]$

char polynomial
$\Delta(s) = (s-\lambda_1)(s-\lambda)^5$

minimal polynomial
$\psi(s) = (s-\lambda_1)(s-\lambda)^3$

so $\Delta(s) \neq \psi(s)$

$\Longrightarrow$ <u>not cyclic</u>

order of biggest Jordan block

Ex:

$$A = \left[\begin{array}{c|ccccc} \lambda_1 & & & & & \\ \hline & \lambda & 1 & 0 & 0 & 0 \\ & 0 & \lambda & 1 & 0 & 0 \\ & 0 & 0 & \lambda & 1 & 0 \\ & 0 & 0 & 0 & \lambda & 1 \\ & 0 & 0 & 0 & 0 & \lambda \end{array}\right]$$

char poly

$\Delta(s) = (s-\lambda_1)(s-\lambda)^5$

min poly

$\psi(s) = (s-\lambda_1)(s-\lambda)^5$

so $\psi(s) = \Delta(s) \Longrightarrow$ cyclic

<u>Fact</u>: If (A, B) contr. B is MIMO and A cyclic then A can be controlled from one input

<u>Ex</u>: $A = \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

note (A,B) is cont. $\quad \begin{array}{cc} | & | \\ b_1 & b_2 \end{array}$

note (A, b_1) is not cont.

(A, b_2) not cont.

$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ but $(A, b_1 + b_2)$ is cont

$$b_1 + b_2 = B\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Bigg\} \qquad A \begin{pmatrix} -3 & 0 \\ 0 & -4 \end{pmatrix} \quad b_1 + b_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$= Bg$

<u>Fact</u> If A is not cyclic then it can be made cyclic with feedback

$\swarrow$

<u>Theorem</u> If (A,B) cont. then $\exists K_0$ so that $(A + BK_0)$ is cyclic

Procedure

(1) Find K_0 so that $(A+BK_0)$ is cyclic

(2) Find g so that $(A+BK_0, Bg)$ is cont.

(3) find k^T so that eig of
$$(A+BK_0)+(Bg)k^T$$
are at desired locations

(4) Claim is $K = K_0 + gk^T$ makes eig of $A+BK$ what we want

check
$$A+BK$$
$$A+B(K_0+gk^T)$$
$$\underset{n\times n}{A} + \underset{n\times q}{B}\underset{q\times n}{K_0} + \underset{n\times q}{B}\underset{q\times 1}{g}\underset{1\times n}{k^T}$$

Ex) $\dot{x} = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u$

$A = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$ not cyclic $\quad \Delta(s) = (s+3)^2$

$\psi(s) = (s+3)$

$\psi(A) = (A + 3 \cdot I) = 0$

(1) K_0 so that $A + BK_0$ cyclic

almost anything will work

Pick K_0

$A + BK_0 = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix} + \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \overset{K_0}{\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}}}_{\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}}$

$\boxed{K_0 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}}$ $\quad = \begin{pmatrix} -3 & 0 \\ 0 & -4 \end{pmatrix} \Longrightarrow$ cyclic

(2) Find g so that $[(A + BK_0), (Bg)]$ cont.

use $g = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Longrightarrow Bg = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(3) Solve $(A+BK_0)+(Bg)k^T$ to give poles we want. suppose want poles at $-1, -2 \Rightarrow \alpha_d(s) = s^2+3s+2$

$$\Rightarrow \begin{pmatrix} -3 & 0 \\ 0 & -4 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} k_1 & k_2 \end{pmatrix}$$

find $\boxed{k^T = (-2 \quad 6)}$

ASIDE

$$\begin{pmatrix} -3 & 0 \\ 0 & -4 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} k_1 & k_2 \end{pmatrix}$$

$$\det\Big(sI - \underbrace{\begin{pmatrix} -3+k_1 & k_2 \\ k_1 & -4+k_2 \end{pmatrix}}\Big) = 0$$

$$(s+3-k_1)(s+4-k_2) - k_1k_2 = 0$$

$$= s^2+3s+2$$

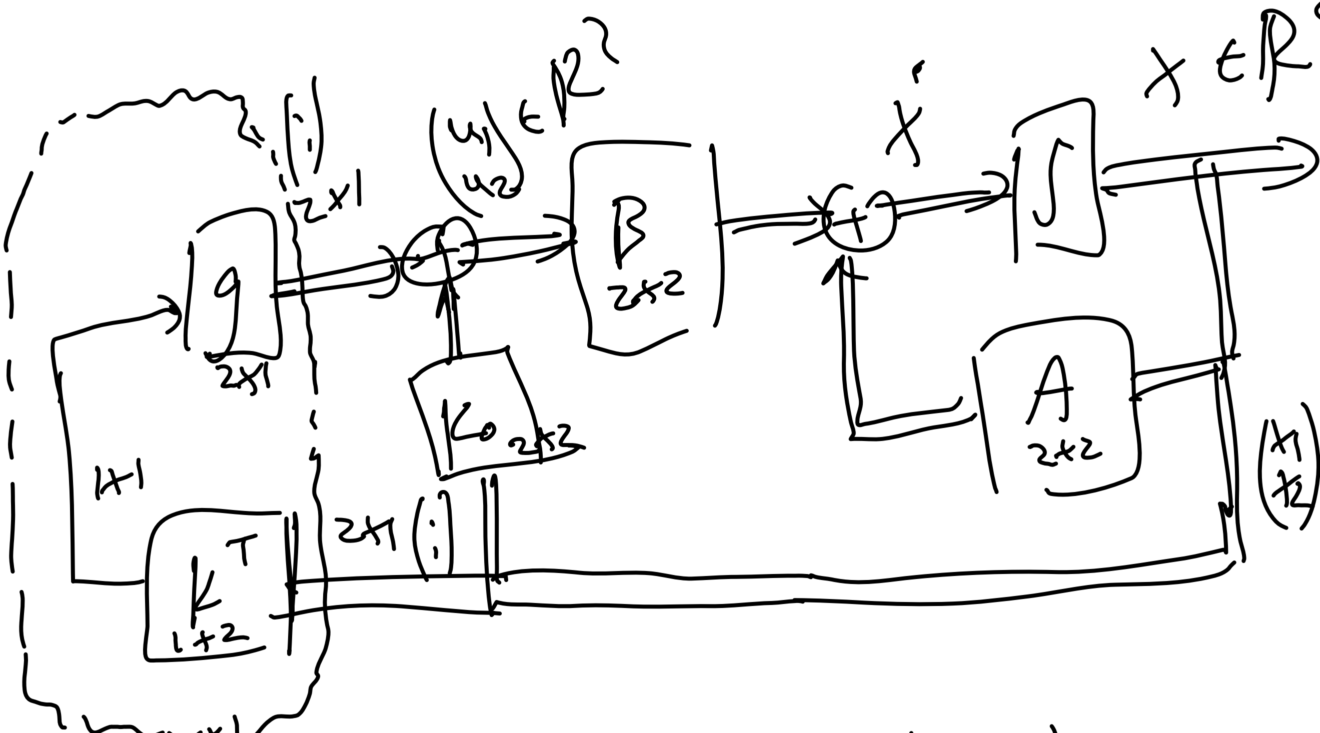
$(4) \Rightarrow K = K_0 + g k^T$

$$= \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} -2 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 6 \\ -2 & 5 \end{pmatrix} \qquad K$$

check: $A + BK = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & 6 \\ -2 & 5 \end{pmatrix}$

$$= \begin{pmatrix} -5 & 6 \\ -2 & 2 \end{pmatrix} \text{ eig } -1, -2$$



$$\underset{2\times 2}{K} = \underset{2\times 1}{g} \; \underset{1\times 2}{k^T} = \begin{pmatrix} | \end{pmatrix} \begin{pmatrix} - \end{pmatrix} = \begin{pmatrix} \# \end{pmatrix}$$