EGGN417: Modern Control Design: Solutions to State Space Equations

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Lecture 21

Contents

1 Introduction

The goal of this lecture is to develop solutions to the state space equations

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and all matrices with compatible dimensions. By solution, we mean, given initial condition $x(0)$ and input $u(t)$, $t \geq 0$, find an explicit expression for $y(t)$, $t \geq 0$.

However, lets start with the simpler problem: Given $x(0)$, solve

$$\dot{x}(t) = Ax(t). \tag{1}$$

This is the homogenous (zero input) part of the state dynamics.

2 The State Transition Matrix

To find the solutions, let's try to get some insight from the scalar case

$$\dot{x}(t) = ax(t).$$

This equations says that the solution should have a velocity ($\dot{x}(t)$) that is proportional to the value of x at t, with proportionality constant a. We already know from differential equations that the solution is $x(t) = e^{at}x(0)$. This depends on the key property of exponentials

$$\frac{d}{dt}e^{at} = ae^{at}$$

which has the proper relationship between time derivative and value. To get the solution in the matrix case, we will need the matrix equivalent to the exponential.

Definition 1. The following is the matrix exponential (MATLAB command `expm`)

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

This series is convergent for all finite A.

Example 2. Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Because A^2 is a matrix of all zeros, we can see that

$$e^A = I + A + 0 + 0 + \cdots = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

It is important to notice that e^A is not just the exponential of each element of A, that is

$$e^A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} e^0 & e^1 \\ e^0 & e^0 \end{bmatrix}$$

The matrix exponential can be come a function of time by multiplying A by the scalar t to get e^{At}.

Lemma 3. *The following are properties of the matrix exponential, where $A \in \mathbb{R}^{n\times n}$ and t, t_1 and t_2 are scalars.*

1. $e^{At}\big|_{t=0} = I$
2. $e^{A(t_1+t_2)} = e^{At_1}e^{At_2}$
3. $e^{A+B} \neq e^A e^B$ *unless* $AB = BA$.
4. $(e^{At})^{-1} = e^{-At}$
5. $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$

Proof:

1. evaluate definition
2. definition and careful algebra
3. definition and careful algebra
4. use $t_1 = t$, $t_2 = -t$ in $e^{A(t_1+t_2)} = e^{At_1}e^{At_2}$.
5. Using the definition,

$$\begin{aligned}
\frac{d}{dt}e^{At} &= \frac{d}{dt}\left(I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \cdots\right) \\
&= A + \frac{2}{2!}A^2t + \frac{3}{3!}A^3t^2 + \cdots \qquad \text{[element-wise differentiation of uniformly convergent series]} \\
&= A\left(I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \cdots\right) \\
&= Ae^{At}
\end{aligned}$$

$\square$

It is property 5 that indicates that the matrix exponential is going to be the right answer for the solution to (1).

Theorem 4. *For $A \in \mathbb{R}^{n\times n}$, the solution to*

$$\dot{x}(t) = Ax(t)$$

with initial condition $x(0)$ is given by

$$x(t) = e^{At}x(0)$$

Proof: To prove it, we simply evaluate $\dot{x}(t)$ for our potential solution:

$$\dot{x}(t) = \frac{d}{dt}e^{At}x(0) = Ae^{At}x(0)$$

Since $e^{At}x(0) = x(t)$, we have verified that

$$\dot{x}(t) = Ax(t)$$

□

In the context of solutions of state space systems, the matrix e^{At} is called the *state transition matrix*

3 State Space Solution

To develop the solution to the complete state space system, we start by finding the state trajectory with a non-zero input.

Theorem 5. *The state space system*

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with initial condition $x(0)$ has solution

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Proof: We need to show that $x(t)$ satisfies the state space equations. Take the derivative of both sides with respect to time:

$$\begin{aligned}
\dot{x}(t) &= \frac{d}{dt}e^{At}x(0) + \frac{d}{dt}\left(e^{At}\int_0^t e^{-A\tau}Bu(\tau)d\tau\right) \\
&= \frac{d}{dt}e^{At}x(0) + \frac{d}{dt}\left(e^{At}\right)\int_0^t e^{-A\tau}Bu(\tau)d\tau + e^{At}e^{-At}Bu(t)
\end{aligned}$$

Using the properties of the state transition matrix and simplifying

$$\begin{aligned}
\dot{x}(t) &= Ae^{At}x(0) + Ae^{At}\int_0^t e^{-A\tau}Bu(\tau)d\tau + Bu(t) \\
\dot{x}(t) &= A\left(e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\right) + Bu(t) \\
\dot{x}(t) &= Ax(t) + Bu(t)
\end{aligned}$$

The solution is verified. □

Example 6. Find the solution for the (double integrator) state space system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

when

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solution: Note that

$$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, the series definition for the matrix exponential can be exactly calculated as

$$e^{At} = I + At = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{aligned} x(t) &= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \int_0^t \begin{bmatrix} 1 & t-\tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau \\ &= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \int_0^t \begin{bmatrix} t-\tau \\ 1 \end{bmatrix} d\tau \\ &= \begin{bmatrix} t \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{t^2}{2} \\ t \end{bmatrix} \end{aligned}$$

Once the state trajectory is available, the complete input/output solution can be found by simply plugging it into the output equation.

Theorem 7. *The state space system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

with initial condition $x(0)$ has solution

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \tag{2}$$

4 A view from Laplace Transforms

In order to find the state transition matrix by hand, in most cases it is most useful to utilize Laplace Transforms. Using the definition of matrix exponential, let's find the one-sided Laplace Transform of the state transition matrix. Note that this is really the Laplace Transform of each element of e^{At}, or a total of n^2 functions when A is n by n.

$$\begin{aligned} \mathcal{L}\left\{e^{At}\right\} &= \int_0^\infty e^{At}e^{-st}dt \\ &= \int_0^\infty e^{(A-sI)t}dt \end{aligned}$$

Using the properts of matrix exponentials, we note that $\frac{d}{dt}(A-sI)^{-1}e^{(A-sI)t} = e^{(A-sI)t}$ (when the inverse exists), so

$$\begin{aligned} \int_0^\infty e^{(A-sI)t}dt &= (A-sI)^{-1}e^{(A-sI)t}\Big|_0^\infty \\ &= \lim_{t\to\infty}(A-sI)^{-1}e^{(A-sI)t} - (A-sI)^{-1}e^{(A-sI)0} \end{aligned}$$

if we pick s such that $A - sI$ has all negative eigenvalues (which can always be done) then

$$\lim_{t\to\infty}(A - sI)^{-1}e^{(A-sI)t} = 0$$

and since

$$e^{(A-sI)0} = I$$

we have

$$\mathcal{L}\left\{e^{At}\right\} = -(A - sI)^{-1} = (sI - A)^{-1}$$

Example 8. Find the solution for the state space system

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix}-3 & 1\\ -2 & 0\end{bmatrix}x(t) + \begin{bmatrix}0\\ 1\end{bmatrix}u(t)\\ y(t) &= \begin{bmatrix}1 & 0\end{bmatrix}x(t)\end{aligned}$$

if the input is a unit step.

Solution: First, find the Laplace Transform of the state transition matrix:

$$\begin{aligned}\mathcal{L}\left\{e^{At}\right\} &= \left(\begin{bmatrix}s & 0\\ 0 & s\end{bmatrix} - \begin{bmatrix}-3 & 1\\ -2 & 0\end{bmatrix}\right)^{-1}\\ &= \begin{bmatrix}s+3 & -1\\ 2 & s\end{bmatrix}^{-1}\\ &= \frac{1}{s^2+3s+2}\begin{bmatrix}s & 1\\ -2 & s+3\end{bmatrix}\\ &= \begin{bmatrix}\frac{s}{s^2+3s+2} & \frac{1}{s^2+3s+2}\\ \frac{-2}{s^2+3s+2} & \frac{s+3}{s^2+3s+2}\end{bmatrix}\\ &= \begin{bmatrix}\frac{-1}{s+1} + \frac{2}{s+2} & \frac{1}{s+1} + \frac{-1}{s+2}\\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{2}{s+1} + \frac{-1}{s+2}\end{bmatrix}\end{aligned}$$

The state transition matrix is then (for $t > 0$),

$$e^{At} = \begin{bmatrix}-e^{-t} + 2e^{-2t} & e^{-t} - e^{-2t}\\ -2e^{-t} + 2e^{-2t} & 2e^{-t} - e^{-2t}\end{bmatrix}$$

Now we can plug this into the state space solution formula (2)

$$\begin{aligned}y(t) &= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}-e^{-t} + 2e^{-2t} & e^{-t} - e^{-2t}\\ -2e^{-t} + 2e^{-2t} & 2e^{-t} - e^{-2t}\end{bmatrix}x(0) + \int_0^t \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}-e^{-t} + 2e^{-2t} & e^{-t} - e^{-2t}\\ -2e^{-t} + 2e^{-2t} & 2e^{-t} - e^{-2t}\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}u(t)dt\\ &= \begin{bmatrix}-e^{-t} + 2e^{-2t} & e^{-t} - e^{-2t}\end{bmatrix}x(0) + \int_0^t (e^{-t} - e^{-2t})u(t)dt\end{aligned}$$

Since $u(t) = 1$ in the region of integration, the integral becomes

$$\int_0^t (e^{-t} - e^{-2t})dt = -e^{-t} + \frac{1}{2}e^{-2t}\Big|_0^t = -e^{-t} + \frac{1}{2}e^{-2t} + \frac{1}{2}$$

Thus, the solution is

$$y(t) = (-e^{-t} + 2e^{-2t})x_1(0) + (e^{-t} - e^{-2t})x_2(0) - e^{-t} + \frac{1}{2}e^{-2t} + \frac{1}{2}$$