# EGGN 517 Lecture 16

Mar 6

continuing: Arbitrary Pole Placement

Last: $\delta(s) = n_c(s)\, n_p(s) + d_c(s)\, d_p(s)$

polynomials

$n_c = \beta_p s^p + \cdots + \beta_0$
$d_c = \alpha_p s^p + \cdots + \alpha_0$
} $p$-th order

$n_p = b_n s^n + \cdots + b_0$
$d_p = s^n + \cdots + a_0$
} $n$-th order

$\delta(s) = s^q + \cdots + \delta_1 s + \delta_0$ } $\underbrace{(n+p)}_{q}$th order

$$x = (\alpha_p\ \alpha_{p-1} \cdots \alpha_1\ \alpha_0\ \beta_p\ \beta_{p-1} \cdots \beta_1\ \beta_0)$$

$$\delta = (1\ \delta_{q-1} \cdots \delta_1\ \delta_0)$$

$$\Rightarrow \quad \delta = A\, x$$

sylvester matrix

$\delta \in \mathbb{R}^{q+1} = \mathbb{R}^{n+p+1}$ $\qquad A \in \mathbb{R}^{(q+1)\times(2p+2)}$

$x \in \mathbb{R}^{2p+2}$

Q, when is $A$ square?

A, when $q+1 = n+p+1 = 2p+2$

$$p = n-1$$

$\Rightarrow$ if controller has order $n-1$ (and $n_p, d_p$ have no common zeros)

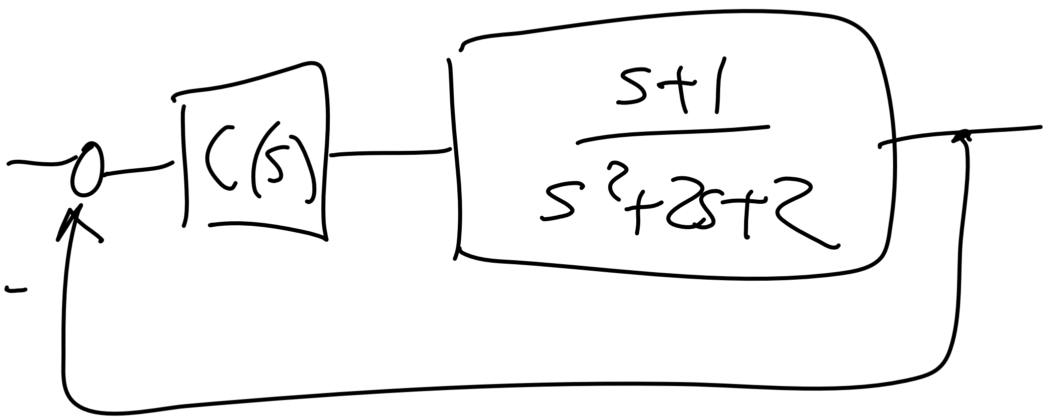
$\Rightarrow$ can invert $A$

$\Longrightarrow$ given desired $\delta$, we can find $x$

Q: when ~~is~~ does $\text{range}(A) = \mathbb{R}^{q+1}$

A: when $p \geq n-1$

Ex:




$n = 2 \Longrightarrow$ pick $p = n-1 = 1$

$$\Longrightarrow \quad C(s) = \frac{\beta_1 s + \beta_0}{s + \alpha_0}$$

$$\delta(s) = n_c n_p + d_c d_p$$

$$= n_c(s+1) + d_c(s^2+2s+2)$$

$$= (\beta_1 s + \beta_0)(s+1) + (s+\alpha_0)(s^2+2s+2)$$

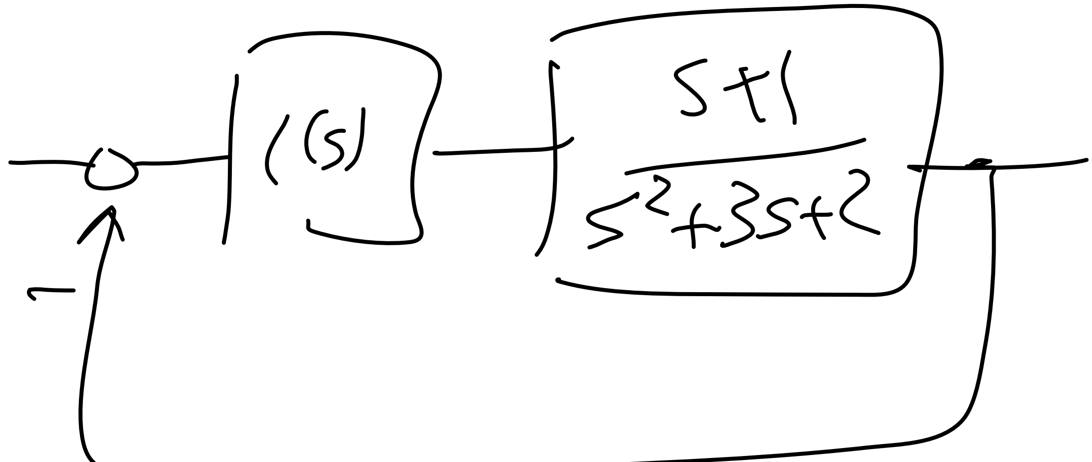
$$= 1s^3 + \delta_2 s^2 + \delta_1 s + \delta_0$$

a
$$\begin{pmatrix} 1 \\ \delta_2 \\ \delta_1 \\ \delta_0 \end{pmatrix} = \underbrace{\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 2 & 1 & 1 \\ 0 & 2 & 0 & 1 \end{array}\right]}_{\text{sylvester matrix}} \begin{pmatrix} 1 \\ \alpha_0 \\ \beta_1 \\ \beta_0 \end{pmatrix}$$

suppose want $\delta(s) = s^3 + 5s^2 + 8s + 5$

$\Longrightarrow \quad \alpha_0 = 2, \ \beta_1 = \beta_2 = 1 \quad C(s) = \dfrac{s+1}{s+2}$

Ex1

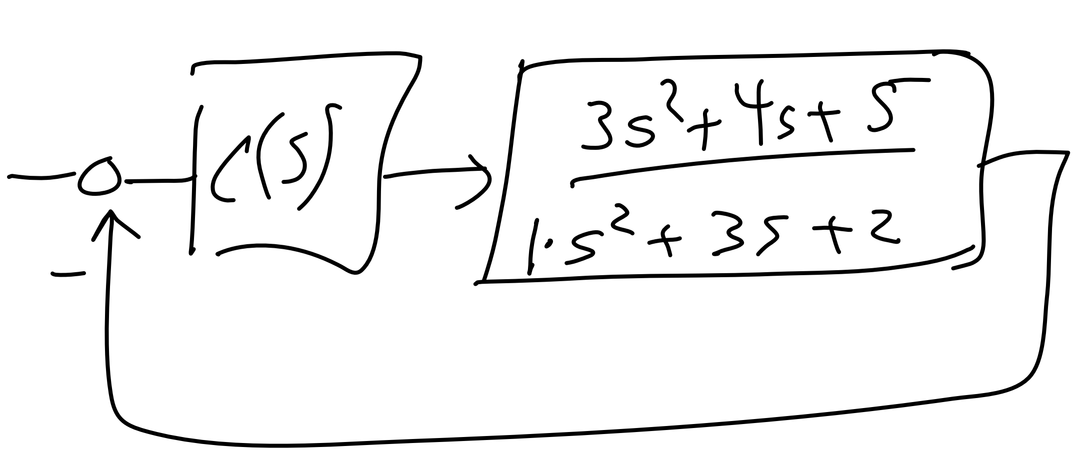



$$\delta = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 3 & 3 & 1 & 1 \\ 0 & 2 & 0 & 1 \end{array}\right] x$$

won't be invertible

because $\left.\begin{aligned} n_p &= s+1 \\ d_p &= (s+1)(s+2) \end{aligned}\right\}$ common zero

Ex1




$n = 2$

$p = 1$

$$C(s) = \frac{\beta_1 s + \beta_0}{\alpha_1 s + \alpha_0}$$

$$n_c n_p = 3\beta_1 s^3 + \cdots$$

$$\begin{pmatrix} \delta_3 \\ \delta_2 \\ \delta_1 \\ \delta_0 \end{pmatrix} = \left[\begin{array}{cc|cc} 1 & 0 & 3 & 0 \\ 3 & 1 & 4 & 3 \\ 2 & 3 & 5 & 4 \\ 0 & 2 & 0 & 5 \end{array}\right] \begin{pmatrix} \alpha_1 \\ \alpha_0 \\ \beta_1 \\ \beta_0 \end{pmatrix}$$

if $\delta_3 = 1 \Rightarrow \beta_1 = 1/3$

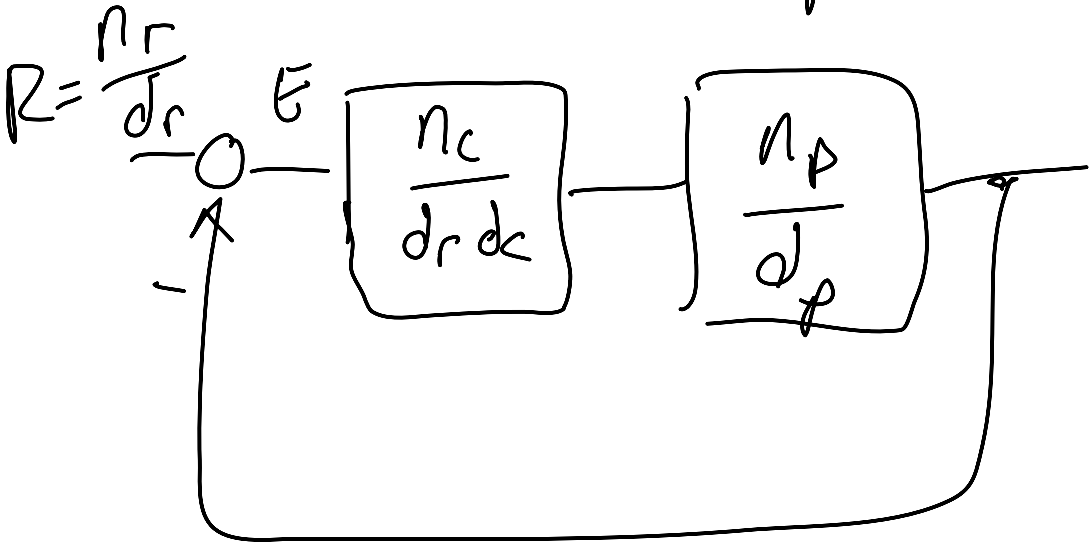
$\alpha_1 = 1$

## • Servo with Pole Placement

Above: Regulator – interested in stability

Now: Servo – tracking

Trick: incorporate internal model principle:




(Aside: really only need to care about unstable parts of $d_r$)

if stable

$\Rightarrow e_{ss} \to 0$

because $d_r$ is in $C(s)$

$$\delta(s) = \underbrace{n_p(s)}_{\text{order } n}\ \underbrace{n_c(s)}_{r+p} + \underbrace{d_p(s)}_{\text{order } n}\ \underbrace{d_r(s)}_{\text{order } r}\ \underbrace{d_c(s)}_{\text{order } p}$$

$\delta(s)$: order $n+r+p$

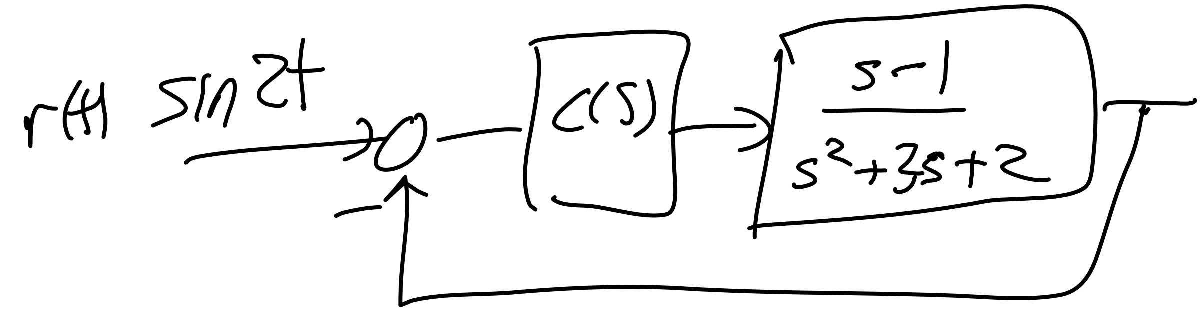
for $\delta = A x$

$$\underbrace{\delta \in \mathbb{R}^{n+r+p+1}}_{\#\text{ of constraints}} \qquad \underbrace{x \in \mathbb{R}^{2p+r+2}}_{\text{free parameters}}$$

$$\Rightarrow 2p+r+2 \geq n+r+p+1$$

or $\boxed{p \geq n-1}$

Ex,



$$\left.\begin{array}{l} R(s) = \dfrac{1}{s^2+4} \\[2ex] n = 2 \Rightarrow p = n-1 = 1 \end{array}\right\} \Rightarrow C(s) = \frac{\beta_3 s^3 + \beta_2 s^2 + \beta_1 s + \beta_0}{(s^2+4)(\alpha_1 s + \alpha_0)}$$

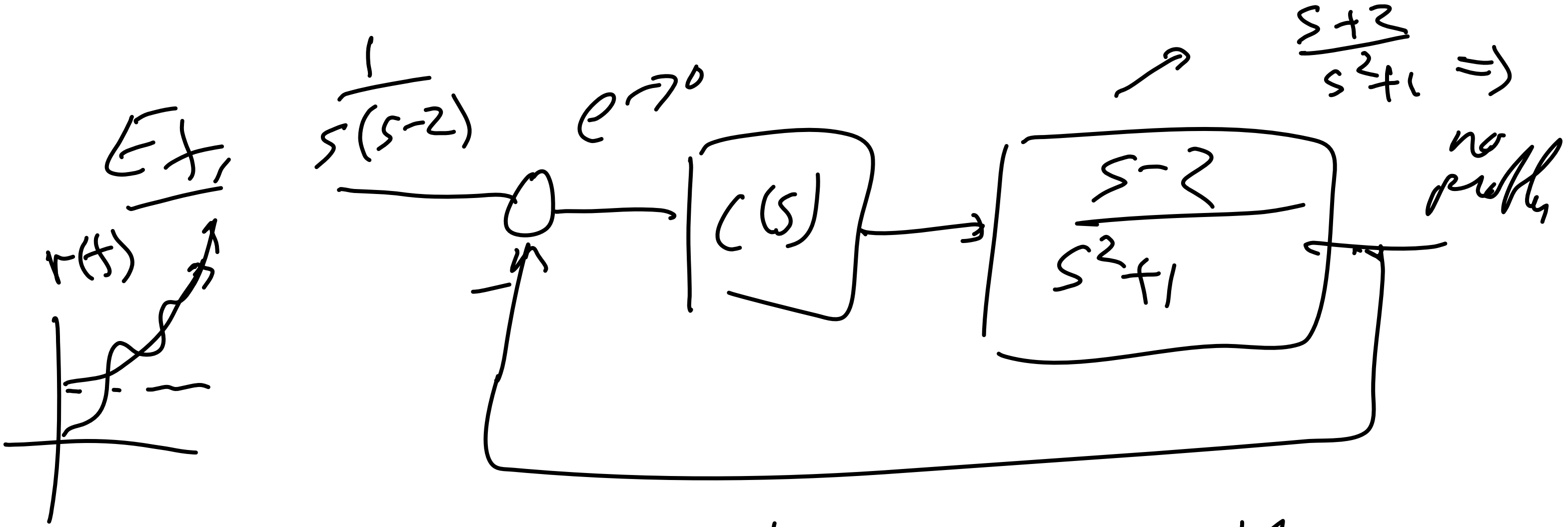
$$\delta(s) = n_p n_c + d_p d_r d_c$$

$$= \underset{\substack{\uparrow \\ 0s^2+s-1}}{(s-1)}(\beta_3 s^3 + \beta_2 s^2 + \beta_1 s + \beta_0) + \underbrace{(s^2+3s+2)(s^2+4)}_{\substack{\text{conv} \\ s^4+3s^3+6s^2+12s+8}}(\alpha_1 s + \alpha_0)$$

$$\begin{pmatrix} \delta_5 \\ \delta_4 \\ \delta_3 \\ \delta_2 \\ \delta_1 \\ \delta_0 \end{pmatrix} = \underbrace{\left[\begin{array}{cc:cccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 0 \\ 6 & 3 & -1 & 1 & 0 & 0 \\ 12 & 6 & 0 & -1 & 1 & 0 \\ 8 & 12 & 0 & 0 & -1 & 1 \\ 0 & 8 & 0 & 0 & 0 & -1 \end{array}\right]}_{\text{modified Sylvester matrix}} \begin{pmatrix} \alpha_1 \\ \alpha_0 \\ \beta_3 \\ \beta_2 \\ \beta_1 \\ \beta_0 \end{pmatrix}$$

<u>Note</u>: invertible if $d_p$ and $d_r$ have no common zeros with $n_p$

Ex.




claim: can't track this signal

naively: $C(s) = \dfrac{\beta_3 s^3 + \beta_2 s^2 + \beta_1 s + \beta_0}{s(s-2)(\alpha_1 s + \alpha_0)}$

$$\delta(s) = (\beta_3 s^3 + \dots + \beta_0)\underbrace{(s-2)} + (\alpha_1 s + \alpha_0)(s)\underbrace{(s-2)}(s^2+1)$$

$$= (s-2)[\text{ stuff }]$$

$\Rightarrow$ can't be stable if applied internal model principle

i.e. can't track zeros of plant