Example : Kalman Decomposition

$$A = \begin{bmatrix} 0 & -6 & 7 & -15 & -2 \\ 1 & 6 & -6 & 8 & 4 \\ 1 & 4 & -4 & 1 & 2 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -2 \\ 1 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 3 & -2 & 2 & 2 \\ 1 & 2 & -1 & 0 & 1 \end{bmatrix}$$

$$\mathcal{C}(A,B) = \underbrace{\begin{bmatrix} 0 & -2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{rank} = 3} \begin{matrix} -6 & 0 \\ 4 & 1 \\ 2 & 1 \\ 0 & 0 \\ 0 & 0 \\ \times & \times \end{matrix} \quad \text{———}$$

$$\mathcal{O}(C,A) = \begin{bmatrix} 1 & 3 & -2 & 2 & 2 \\ 1 & 2 & -1 & 0 & 1 \\ \hline 1 & 4 & -3 & 3 & 2 \\ 1 & 2 & -1 & 1 & 2 \\ 1 & 6 & -5 & 7 & 4 \end{bmatrix} \begin{matrix} \\ \\ \\ \times \\ \times \\ \times \end{matrix} \quad \Big\} \text{ rank} = 3$$

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Then $v_1, v_2, v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ span $\mathcal{C}(A,B)$

also, need to find $\hat{v}_1, \hat{v}_2$ so that

$\mathcal{O}(C,A)[v_1 \; v_2] = 0$

let $b=1$, $e=0$ $\Rightarrow \begin{bmatrix} 1 & -2 & 2 \\ 1 & -1 & 0 \\ 1 & 3 & 3 \end{bmatrix} \begin{pmatrix} a \\ c \\ d \end{pmatrix} = \begin{matrix} -3 \\ -2 \\ -4 \end{matrix} \Rightarrow \hat{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{bmatrix} 1 & 3 & -2 & 2 & 2 \\ 1 & 2 & -1 & 0 & 1 \\ 1 & 4 & -3 & 3 & 2 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} =$$

let $b=0$, $e=1$ $\begin{bmatrix} 1 & -2 & 2 \\ 1 & -1 & 0 \\ 1 & -3 & 3 \end{bmatrix} \begin{pmatrix} a \\ c \\ d \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ -2 \end{pmatrix} \Rightarrow \hat{v}_1 = \begin{pmatrix} -2 \\ 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}$

can check $\mathcal{O}(A,C)\,[v_1\ v_2] = 0$ ✓

Now, trick is to find which one of the $\hat{v}_1, \hat{v}_2$ spans the unobservable subspace of the controllable subspace

But, it must be $\hat{v}_2$, because of the form of v_1, v_2, v_3, so, we form

$$T = v_1\ v_2\ \hat{v}_2\ Q_1\ \hat{v}_1$$

where Q_1 is picked to make the whole thing invertible

$$T = \begin{bmatrix} 0 & -2 & -1 & 0 & -2 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then we compute

$$\bar{A} = T^{-1} A T = \left[\begin{array}{ccc:c:c} 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 6 & 0 \\ -1 & 2 & -1 & 1 & 0 \\ \hdashline 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right] \qquad \bar{B} = T^{-1} B = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right]$$

$$\bar{C} = CT = \left[\begin{array}{cc:c:c:c} 1 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

cont part $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & 2 & -1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\qquad co \quad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

obs part $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 6 \\ 0 & 0 & -2 \end{bmatrix}$ $\qquad c\bar{o} \quad [-1]$

$\qquad \bar{c}o \quad [-2]$

$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$ $\qquad \bar{c}\bar{o} \quad [-3]$