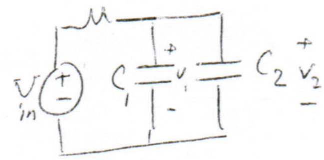
Lecture 8, Feb 16

5.0 Controllability / Obs.

5.1 Controllability

For LTI: $\dot{x} = Ax \Rightarrow x(t) = e^{At} x(0)$ — Con.

$x_{k+1} = A x_k \Rightarrow x_k = A^k x(0)$ — Dis.

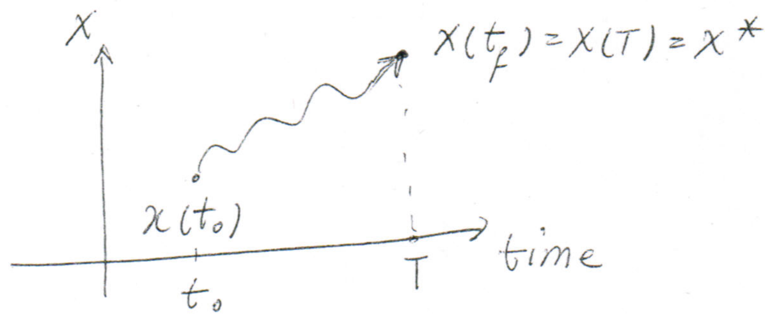


$x_1 = v_1$, $x_2 = v_2$
$u = v_{in}$ → not cont

Only Consider continuous-time:

⇒ **Def:** $\dot{x} = Ax + Bu$ is state-controllable at t_0 if

$\exists\, T > t_0$ s.t. $\forall\, x(t_0)$ and $\forall\, x^*$ $\exists\, u(t)$ for $t \in [t_0, T]$ s.t. $x(T) = x^*$.



Def. The controllability Grammian: $W_T = \int_0^T e^{At} B B^T e^{A^T t}\, dt$

$n \times n \rightarrow [\ \ ][\ \ ][\ \ ]^T[\ \ ]$

Theorem: The system $\dot{x} = Ax + Bu$ is controllable if and only if W_T is invertible (has rank n).

necessary and sufficient condition

Corollary: $u(t) = B^T e^{A^T(T-t)} W_T^{-1} [x^* - e^{AT} x_0]$ takes $\dot{x} = Ax + Bu$ from x_0 to x^* at $t = T$.

Theorem: The following are equivalent: (for $A \in \mathbb{R}^{n \times n}$)

1. rank $W_T = n$
2. $\dot{x} = Ax + Bu$ is state_controllable
3. $\text{rank } [\underbrace{B \; AB \; A^2B \; \cdots \; A^{n-1}B}_{\mathcal{C}(A,B)}] = n$
4. $\text{rank } [A - \lambda I \, \vdots \, B] = n \quad \forall \lambda \in \mathbb{C}$ PBH criteria

$\dim \{B, AB, \ldots, A^{n-1}B\} = \mathbb{R}^n$

$\det(\lambda I - A) = 0$ to find eig

Ex.

$$A=\begin{bmatrix} +1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \qquad B=\begin{array}{cc} B_1 & B_2 \\ \downarrow & \downarrow \end{array} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

B_1 for the first input

$$C=[2 \quad 1 \quad 4]$$

1) $\mathcal{C}(A,B) = [B \quad AB \quad A^2B] = \left[\begin{array}{cc:cc:cc} 1 & 0 & +1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 & 9 \end{array}\right]$

$\text{rank}(\mathcal{C}) = \overset{\text{row}}{\text{rank}} = 3-1 = 2 \rightarrow$ not cont. (2 cont. modes)

2) $\lambda = \overset{\text{unstable}}{+1}, -2, -3$ Diag form → zero row in B (Note 1)

$$\lambda I - A = \begin{bmatrix} \lambda-1 & 0 & 0 \\ 0 & \lambda+2 & 0 \\ 0 & 0 & \lambda+3 \end{bmatrix}$$

$\lambda=+1 \leftarrow \dot{x}_1 = +x_1 + u_1;$

$\lambda=-2 \leftarrow \dot{x}_2 = -2x_2;$ (note 2)

$\lambda=-3 \leftarrow \dot{x}_3 = -3x_3 + u_2;$

associated λ

$$\Rightarrow [\lambda I - A \,\vdots\, B_1] = \left[\begin{array}{ccc:c} \lambda-1 & 0 & 0 & 1 \\ 0 & \lambda+2 & 0 & 0 \\ 0 & 0 & \lambda+3 & 0 \end{array}\right]$$

$$[\lambda I - A \,\vdots\, B_2] = \left[\begin{array}{ccc:c} \lambda-1 & 0 & 0 & 0 \\ 0 & \lambda+2 & 0 & 0 \\ 0 & 0 & \lambda+3 & 1 \end{array}\right]$$

$$[\lambda I - A \,\vdots\, B_1]_{\lambda=+1} = \left[\begin{array}{ccc:c} 0 & 0 & 0 & 1 \\ 0 & +3 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \rightarrow \text{rank} = 3 \quad \text{cont.}$$

$$[\lambda I - A \,\vdots\, B_2]_{\lambda=+1} = \left[\begin{array}{ccc:c} 0 & 0 & 0 & 0 \\ 0 & +3 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right] \rightarrow \text{rank} = 2 \quad \text{not cont.}$$

$$[\lambda I - A \,\vdots\, B_1]_{\lambda=-2} = \left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right] \rightarrow \text{rank} = 2 \quad \text{not cont.}$$

$$[\lambda I - A \,\vdots\, B_2]_{\lambda=2} = \left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 \end{array}\right] \rightarrow \text{rank} = 2 \quad \text{not cont.}$$

↓ ASIDE

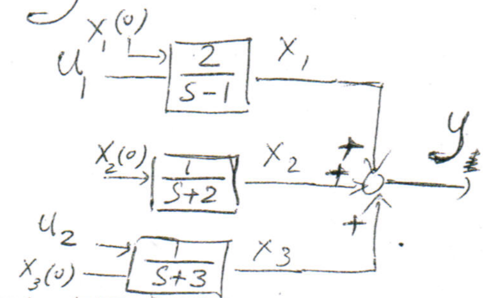
↑ $\text{rank}\left(\left[\begin{array}{cc:cc:cc} 2 & 1 & 1 & 0 & 2 & 1 \\ 2 & 1 & 0 & 2 & 8 & 4 \end{array}\right]\right)$? $\det \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = 2 \neq 0 \rightarrow \text{rank} = 2$

1) write the diag Form. If there is any zero row in B → system is not cont.

ote 2) diag Form: look at B_i: if its j^{th} element is zero you can't control λ_j with B_i.

3) poles of the OL system = eig. values = $+1, -2, -3$

state Feedback → next section



	stable?	Controllable	
$P_1 = +1$	No	Yes! (u_1)	⇒ use u_1 change its place relocate it (Pole Placement)
$P_2 = -2$	Yes	No!	⇒ can't change its place but it doesn't make the CL unstable
$P_3 = -3$	Yes	Yes! (u_2)	⇒ You can relocate it to improve the CL response (control performance)

Cont. canonical form:

ex: $A = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $\Rightarrow \mathcal{C}(A, B) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & -3 & -3 \end{bmatrix}$

$$C = \begin{array}{cc} \begin{bmatrix} 4 & 5 \end{bmatrix} \\ \begin{array}{cc} 1 & 2 \end{array} \end{array}$$

full rank → cont.