5.3 Controllability / Obs. + Minimal Realizations

$$G(s) = \frac{s^2+2s+1}{s^3+5s^2+8s+4} = \frac{s+1}{s^2+4s+4} = \frac{(s+1)^{\cancel{2}}}{\cancel{(s+1)}(s+2)^2}$$

$$\underbrace{A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -8 & -5 \end{bmatrix} \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad C = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}}_{\text{cont., not obs}}$$

$$\underbrace{A = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}}_{\text{cont, obs}}$$

$$\underbrace{A = \begin{bmatrix} 0 & 0 & -4 \\ 1 & 0 & -8 \\ 0 & 1 & -5 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad C = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}}_{\text{not cont; is obs.}}$$

In this example 3 state system is never both cont. and obs

$\Longrightarrow$ call 2 state system <u>minimal</u>

2 state t.f. <u>irreducible</u>

<u>Result / Definitions</u>: ① a minimal system is both cont. & obs.; ② $G(s) = C(sI-A)^{-1}B + D$ is irreducible if and only if (A,B) is cont and (A,C) is obs.; ③ all minimal (A,B,C,D) realization of $G(s)$ have the same order

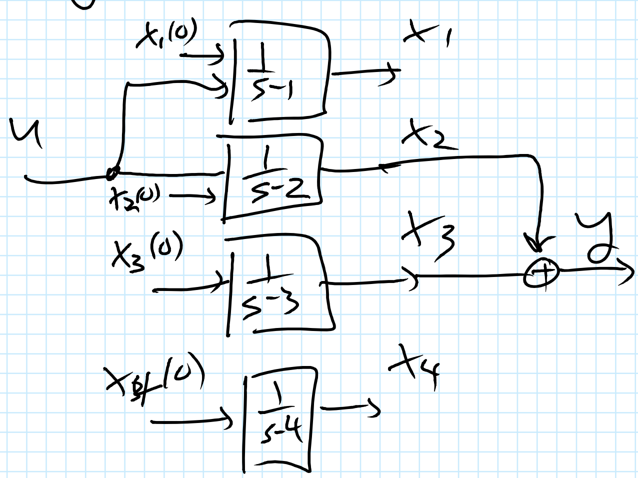
<u>Computational Problem</u>: Given $G(s)$, find minimal (A,B,C,D) realization

5.4 Kalman Decomposition

<u>Ex</u>

$$\dot{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix} x$$



x_1 $C\bar{O}$

x_2 CO

x_3 $\bar{C}O$

x_4 $\bar{C}\bar{O}$

<u>Q</u>: Given (A,B,C,D) which eigenvalues are

cont / obs ?

- <u>controllable decomposition</u> $A \in \mathbb{R}^{n \times n}$

suppose $\mathcal{C}(A,B)$ has rank $q < n$

Define $T = [\underbrace{t_1 \; t_2 \cdots t_q}_{} \; \underbrace{t_{q+1} \cdots t_n}_{}]$

$\mathrm{span}\{t_1, t_2, \cdots t_q\}$ = range space of $\mathcal{C}(A,B)$

ie. $\mathrm{span}\{t_i\}_{i=1}^{q} = \mathbb{R}^q$

chosen to make T^{-1} exist

Then let
$$\bar{A} = T^{-1}AT$$
$$\bar{B} = T^{-1}B$$
$$\bar{C} = CT$$
$$\bar{D} = D$$

This will result in

$$\bar{A} = \left[\begin{array}{c|c} A_1 & A_3 \\ \hline 0 & A_2 \end{array}\right] \quad \bar{B} = \left[\begin{array}{c} B_1 \\ \hline 0 \end{array}\right]$$

$A_1 \in \mathbb{R}^{q \times q}$

(A_1, B_1) cont

Ex: $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\mathcal{C}(A,B) = [B \;\; AB \;\; A^2B]$$

$$= \left[\begin{array}{cc|cc|cc} 0 & 1 & 1 & 1 & \times & \times \\ 1 & 0 & 1 & 0 & \times & \times \\ 0 & 1 & 1 & 1 & \times & \times \end{array}\right]$$

dependent on columns to left

$\times$ means "doesn't matter" because previous set AB is already dependent on B

t_1 t_2

$\Rightarrow$ rank $= 2 < 3 \Rightarrow$ not cont.

$q = 2 = \#$ ind. colums of $\mathcal{C}(A,B)$

let $T = [t_1 \; t_2 \; t_3]$

arbitrary chosen to make T^{-1} exist

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

compute $\bar{A} = T^{-1}AT = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$

$$\bar{B} = T^{-1}B = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \end{array}\right]$$

> eigenvalues at $s = 1$ cont.

one eigenmode at $s=1$ not cont.

remember

$$X(s) = (sI - A)^{-1}B = (sI - \bar{A})^{-1}\bar{B} \quad (3\times 2)$$

- **Observable decomposition**

2 ways:

(1) use duality: if $\dot{x} = Ax + Bu$, $y = Cx$ is obs or not obs then

$$\dot{z} = A^T z + C^T u$$
$$y = B^T z$$

is cont or not cont

(2) Directly: let $\{v_1^T\ v_2^T \cdots v_q^T\}$ be the linearly ind. rows of $\mathcal{O}(A, C)$ then define

$$V^{-1} = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_q^T \\ v_{q+1}^T \\ \vdots \\ v_n^T \end{bmatrix}$$

arbitrary so that V exists

$$\begin{bmatrix} \vdots \\ V_n^T \end{bmatrix}\Big) \quad V \text{ exists}$$

then $\bar{A} = V^{-1}AV = \begin{bmatrix} \bar{A}_1 & 0 \\ \bar{A}_3 & \bar{A}_2 \end{bmatrix}$

$\bar{C} = CV = [\bar{C}_1 \; 0]$

$\Longrightarrow$ pair $(\bar{A}_1, \bar{C}_1)$ obs

- <u>Kalman Decomposition</u>

$\dot{x} = Ax + Bu$

$y = Cx$

there is a Q, with $\bar{A} = Q^{-1}AQ$, $\bar{B} = Q^{-1}B$, $\bar{C} = CQ$

so that $\bar{A}, \bar{B}, \bar{C}$ have the following form

$$\bar{A} = \begin{bmatrix} A_{11} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix} \quad \bar{B} = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C} = [C_1 \; 0 \; C_3 \; 0]$$

$A_{11} \quad (0 \quad \begin{bmatrix} A_{11} & 0 \end{bmatrix}$ cont, not obs where A_{11} CO $\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$ cont, not obs

A_{22} $C\bar{O}$

A_{33} $\bar{C}O$ $\begin{bmatrix} A_{11} & A_{13} \\ 0 & A_{33} \end{bmatrix}$ obs, not cont.

A_{44} $\bar{C}\bar{O}$

6.0 Modeling in the MIMO case

6.1 Transfer Matrix to State-Space (Realization theory)

we know $(A, B, C, D) \longrightarrow G(s)$

$G = C(sI-A)^{-1}B+D$

diagonal, ccf, obs. canonical form, ~~form~~ arbitrary form

Q. $G(s) \xrightarrow{?} (A, B, C, D)$

$G(s) \longrightarrow$ ccf uses common denominator of rows

$G(s) \longrightarrow$ ocf uses common denominator of columns

$G(s) \longrightarrow$ diagonal: "partial fractions" with matrices

(Gilbert's method)

Ex, $\boxed{\text{Diagonal ccf}}$

$$G(s) = \begin{bmatrix} \frac{3}{s-1} & \frac{2}{s-2} \\ \frac{1}{s-3} & \frac{1}{s-1} \end{bmatrix} = \left[\begin{array}{c|c} \frac{3s-9}{s^2-4s+3} & \frac{2s-2}{s^2-3s+2} \\ \hline \frac{s-1}{s^2-4s+3} & \frac{s-2}{s^2-3s+2} \end{array}\right]$$

common denominators $(s-1)(s-3)$

common denominator $(s-1)(s-2)$

$$\dot{x} = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -3 & 4 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 3 \end{array}\right] x + \left[\begin{array}{c|c} 0 & 0 \\ 1 & 0 \\ \hline 0 & 0 \\ 0 & 1 \end{array}\right] u$$

$$y = \left[\begin{array}{cc|cc} -9 & 3 & -2 & 2 \\ \hline -1 & 1 & -2 & 1 \end{array}\right] x$$

ASIDE

$$g_{11} = \frac{3s-9}{s^2-4s+3} \quad A_1 = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix} \quad B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$C_{11} = [-9 \;\; 3]$$

$$G = C(sI-A)^{-1}B$$

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \quad \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$$

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

$$C(sI-A)^{-1}B$$

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} (sI-A_1)^{-1} & 0 \\ 0 & (sI-A_2)^{-1} \end{pmatrix} \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$$

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} (sI-A_1)^{-1}B_1 & 0 \\ 0 & (sI-A_2)^{-1}B_2 \end{pmatrix}$$

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} (sI-A_1)^{-1}B_1 & \\ 0 & (sI-A_2)^{-1}B_2 \end{pmatrix}$$

$$\left[\begin{array}{c|c} C_{11}(sI-A_1)^{-1}B_1 & C_{12}(sI-A_2)^{-1}B_2 \\ \hline C_{21}(sI-A_1)^{-1}B_1 & C_{22}(sI-A_2)^{-1}B_2 \end{array}\right]$$

Ex: $\boxed{\text{Block ccf}}$

$$G = \left[\begin{array}{c|c} \frac{3}{s-1} & \frac{2}{s-2} \\ \hline \frac{1}{s-3} & \frac{1}{s-1} \end{array}\right] = \frac{1}{\underbrace{s^3-6s^2+11s-6}_{\substack{(s-1)(s-2)(s-3) \\ \text{common denominator}}}} \left[\begin{array}{c|c} 3(s^2-5s+6) & 2(s^2-4s+3) \\ \hline 1(s^2-3s+2) & 1(s^2-5s+6) \end{array}\right]$$

$$= \frac{1}{s^3 \underset{\alpha_1}{-6}s^2 \underset{\alpha_2}{+11}s \underset{\alpha_3}{-6}} \left\{ s^2 \underbrace{\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}}_{N_1} + s \underbrace{\begin{bmatrix} -15 & -8 \\ -3 & -5 \end{bmatrix}}_{N_2} + \underbrace{\begin{bmatrix} 18 & 6 \\ 2 & 6 \end{bmatrix}}_{N_3} \right\}$$

$$\Longrightarrow \quad \dot{x} = Ax + Bu, \quad y = Cx \quad \text{so that } G = C(sI-A)^{-1}B$$

where

$$A = \begin{array}{c} \begin{array}{ccc} \quad -\alpha_1 & \quad -\alpha_2 & \quad -\alpha_3 \end{array} \\ \left[\begin{array}{c|c|c} 6\begin{pmatrix}1&0\\0&1\end{pmatrix} & -11\begin{pmatrix}1&0\\0&1\end{pmatrix} & 6\begin{pmatrix}1&0\\0&1\end{pmatrix} \\ \hline \begin{pmatrix}1&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} \\ \hline \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}1&0\\0&1\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} \end{array}\right] \end{array} \qquad B = \left[\begin{array}{c} \begin{pmatrix}1&0\\0&1\end{pmatrix} \\ \hline \begin{pmatrix}0&0\\0&0\end{pmatrix} \\ \hline \begin{pmatrix}0&0\\0&0\end{pmatrix} \end{array}\right]$$

$C = [N_1 \,\vdots\, N_2 \,\vdots\, N_3]$

Ex: Diagonal (Gilberts)

$$G(s) = \begin{bmatrix} \frac{3}{s-1} & \frac{2}{s-2} \\ \frac{1}{s-3} & \frac{1}{s-1} \end{bmatrix} = \frac{1}{s-1}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{s-2}\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} + \frac{1}{s-3}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$\dot{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Typically minimal