I. WKB VELOCITY DIVERGENCE RESULTS

To calculate the CDM or neutrino velocity divergence power, we use the following integral.

$$
\begin{aligned}
P_{\Theta_i}(k,\tau) = &\Bigg| \int_{-\infty}^{\tau_0} d\tau' G_{\Theta_i}(\tau-\tau') \frac{3}{2} H_M^2(\tau') w_M^0(k,\tau') + \\
&\int_{\tau_0}^{\tau} d\tau' G_{\Theta_i}(\tau-\tau') \frac{3}{2} H^2(\tau') w^0(k,\tau') \Bigg|^2 + \\
&\sum_{m=1}^{m_{max}} \left| \int_{\tau_0}^{\tau} d\tau' G_{\Theta_i}(\tau-\tau') \frac{3}{2} H^2(\tau') w^m(k,\tau') \right|^2 .
\end{aligned}
\tag{1}
$$

In most cases, the integrals in Equation 1 were calculated using a trapezoidal integration method. However, when calculating the neutrino velocity divergence power, we found that the integrand varied rapidly at low neutrino masses and high wavenumbers, likely due to the dependence of the time τ on these quantities. In this case, the trapezoidal method was inaccurate. To increase the accuracy of our integration, we use the following approximation.

We begin by writing the Green's function integral as:

$$
\int_{\tau_0}^{\tau} d\tau' F(\tau') G(\tau-\tau') = \sum_{\tau''=\tau_0}^{\tau-d\tau''} \int_{\tau''}^{\tau''+d\tau''} d\tau' \frac{F(\tau')}{F_{\text{lin}}(\tau')} F_{\text{lin}}(\tau') G_i(\tau''-\tau) \tag{2}
$$

where F_{lin} is the quantity F computed in linear theory. If we choose a small enough $d\tau''$, then $\frac{F(\tau')}{F_{\text{lin}}(\tau')}$ may be approximated as constant over the range τ'' to $\tau''+d\tau''$ and may be removed from the integral, so that we have:

$$
\int_{\tau_0}^{\tau} d\tau' F(\tau') G(\tau-\tau') = \sum_{\tau''=\tau_0}^{\tau-d\tau''} \frac{F(\tau')}{F_{\text{lin}}(\tau')} \int_{\tau''}^{\tau''+d\tau''} d\tau' F_{\text{lin}}(\tau') G(\tau-\tau') \tag{3}
$$

If we make the assumption of matter domination, we can calculate the integral of F_{lin} analytically, eliminating any spurious features from the numerical integral. We find that the assumption of matter domination introduces a <15% error into the integral at redshift $z = 0$. However the correction to the integral is a very well-behaved function of the scalefactor and the limits of the integral, and can be well-approximated by a third order polynomial ($r^2 \approx 0.96$). We use these functions to correct the linear integrals. For most of this analysis, the trapezoidal method is used; we will make note whenever we use this secondary method of integration instead. Due to the similarity between the concept behind this method and that of the Wentzel-Kramers-Brillouin integration used to find approximate solutions to the Schrödinger equation, we will refer to this method as a WKB integration method.

In Figure 1, we compare the results of the trapezoidal and WKB integration method for the reconstruction of the CDM velocity divergence power, a case when the integrand is slowly varying and WKB integration is not required. Figure 1 shows that while this WKB integration appears to be overly sensitive to some variations in the integrand, causing the spike at $k = 0.07$ h Mpc^{-1}, overall the method reproduces the integral with less than 25% error up until $k = 0.5$ h Mpc^{-1}. This isn't great, but things get worse when we try using this method to integration the density power spectrum.

In Figure 2, we compare the results of the trapezoidal and WKB integration method for the reconstruction of the CDM density power at z=0. As you can see, the WKB integration method overestimates the integral by 30% for $k > 0.1$ h Mpc^{-1}. Decreasing $d\tau''$ in Equation (3) does not improve the results.

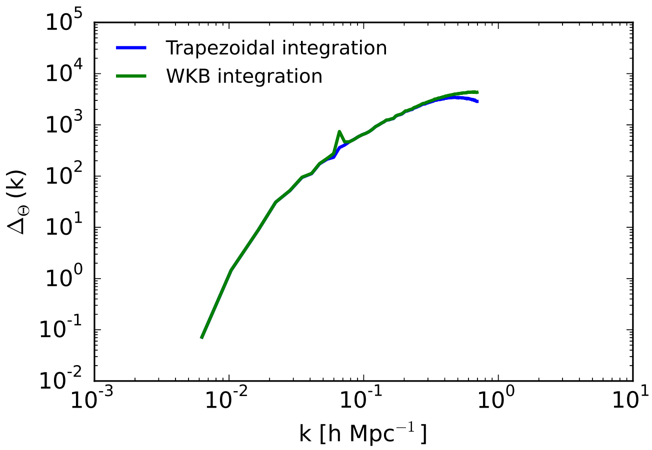
II. VARIABLE INTERPOLATION SCHEME

Next step is to try a variable interpolation scheme. We have a varying function that we are integrating over that contains $\sin(kc_s(\tau-\tau'))$, where we are integrating over τ from $\tau(z_0)$ to τ'. We can calculate the number of oscillations that are in this function from $\tau(z_0)$ to $\tau(z=0)$ or today. This is given by:

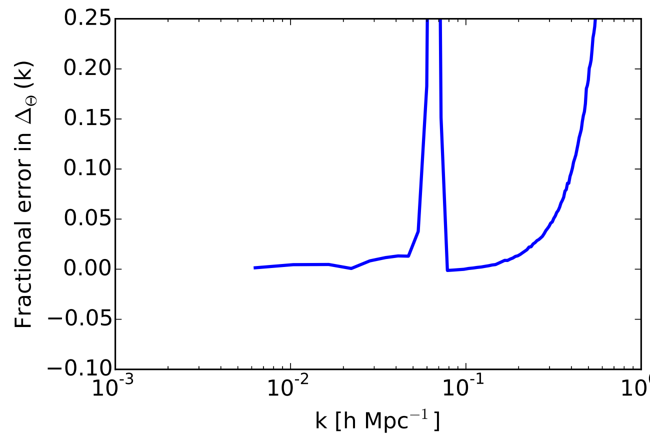
$$
\#\text{ of oscillations} = \frac{kc_s}{2\pi}(\tau(z=0) - \tau(z_0)) \tag{4}
$$

Say we want N sample points per oscillation, where N should be about 8-10. Then we need a total number of points:

$$
\#\text{ of sample points} = N\frac{kc_s}{2\pi}(\tau(z=0) - \tau(z_0)) \tag{5}
$$

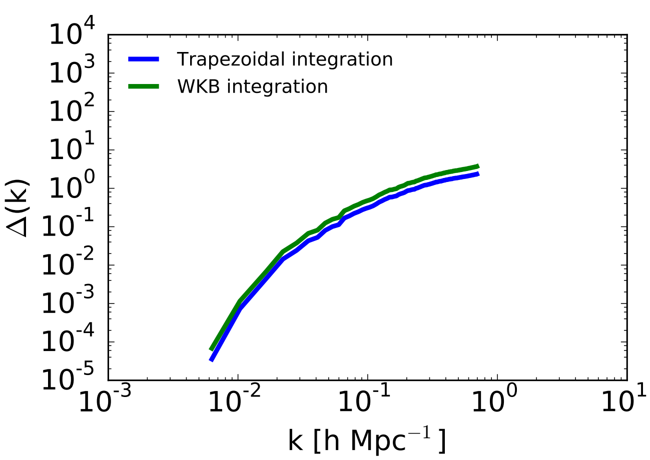


(a) Trapezoidal and WKB integrations

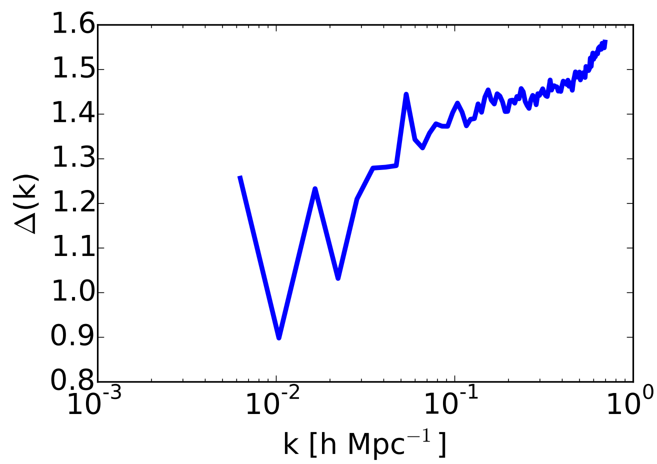


(b) Error in WKB integration

Figure 1: A comparison of the trapezoidal and WKB integration methods, specifically for the velocity divergence power of cold dark matter. In figure 1a, the integration results are compared, while in figure 1b the fractional error of the WKB method compared to the trapezoidal method is shown.



(a) Trapezoidal and WKB integrations



(b) Error in WKB integration

Figure 2: A comparison of the trapezoidal and WKB integration methods, specifically for the velocity divergence power of cold dark matter. In figure 2a, the integration results are compared, while in figure 2b the fractional error of the WKB method compared to the trapezoidal method is shown.