Euler Bernoulli Beam Bending

$$\frac{1}{\rho} = \frac{M(x)}{EI}$$

At small deflections, curvature may be simplified to

$$\frac{1}{\rho} = \frac{d^2y}{dx^2}$$

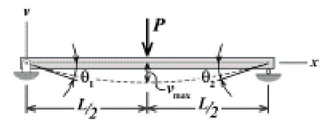
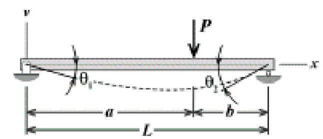
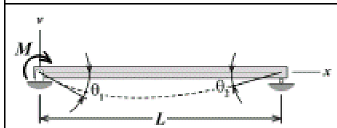
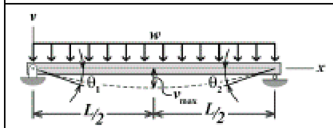
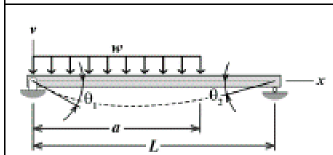
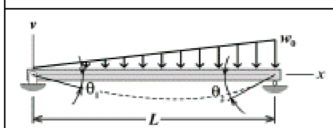
Therefore

$$\frac{d^2y}{dx^2} = -\frac{M}{EI}$$

$$\frac{dy}{dx} = \int_0^x -\frac{M}{EI}dx + A$$

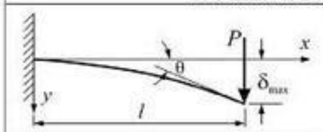
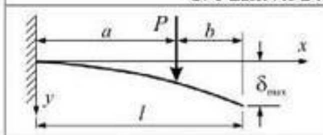
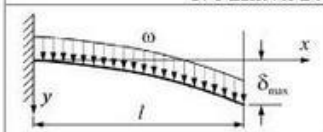
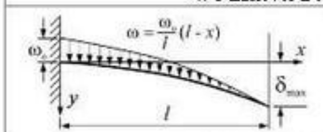
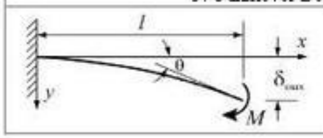
$$y(x) = \int_0^x \int_0^x \left(-\frac{M}{EI}dx + C\right) dx + B$$

Beam Equations

Simply Supported Beams			
Beam	**Slope**	**Deflection**	**Elastic Curve**
	1 $\theta_1 = -\theta_2 = -\dfrac{PL^2}{16EI}$	2 $v_{max} = -\dfrac{PL^3}{48EI}$	3 $v = -\dfrac{Px}{48EI}(3L^2 - 4x^2)$ for $0 \le x \le L/2$
	4 $\theta_1 = -\dfrac{Pb(L^2 - b^2)}{6LEI}$ $\theta_2 = +\dfrac{Pa(L^2 - a^2)}{6LEI}$	5 $v = -\dfrac{Pa^2b^2}{3LEI}$ at $x = a$	6 $v = -\dfrac{Pbx}{6LEI}(L^2 - b^2 - x^2)$ for $0 \le x \le a$
	7 $\theta_1 = -\dfrac{ML}{3EI}$ $\theta_2 = +\dfrac{ML}{6EI}$	8 $v_{max} = -\dfrac{ML^2}{9\sqrt{3}\,EI}$ at $x = L\left(1 - \dfrac{\sqrt{3}}{3}\right)$	9 $v = -\dfrac{Mx}{6LEI}(2L^2 - 3Lx + x^2)$
	10 $\theta_1 = -\theta_2 = -\dfrac{wL^3}{24EI}$	11 $v_{max} = -\dfrac{5wL^4}{384EI}$	12 $v = -\dfrac{wx}{24EI}(L^3 - 2Lx^2 + x^3)$
	13 $\theta_1 = -\dfrac{wa^2}{24LEI}(2L - a)^2$ $\theta_2 = +\dfrac{wa^2}{24LEI}(2L^2 - a^2)$	14 $v = -\dfrac{wa^3}{24LEI}(4L^2 - 7aL + 3a^2)$ at $x = a$	$v = -\dfrac{wx}{24LEI}(Lx^3 - 4aLx^2 + 2a^2x^2 + 4a^2L^2 - 4a^3L + a^4)$ for $0 \le x \le a$ $v = -\dfrac{wa^2}{24LEI}(2x^3 - 6Lx^2 + a^2x + 4L^2x - a^2L)$ 15 for $a \le x \le L$
	16 $\theta_1 = -\dfrac{7w_0L^3}{360EI}$ $\theta_2 = +\dfrac{w_0L^3}{45EI}$	17 $v_{max} = -0.00652\dfrac{w_0L^4}{EI}$ at $x = 0.5193L$	18 $v = -\dfrac{w_0x}{360LEI}(7L^4 - 10L^2x^2 + 3x^4)$

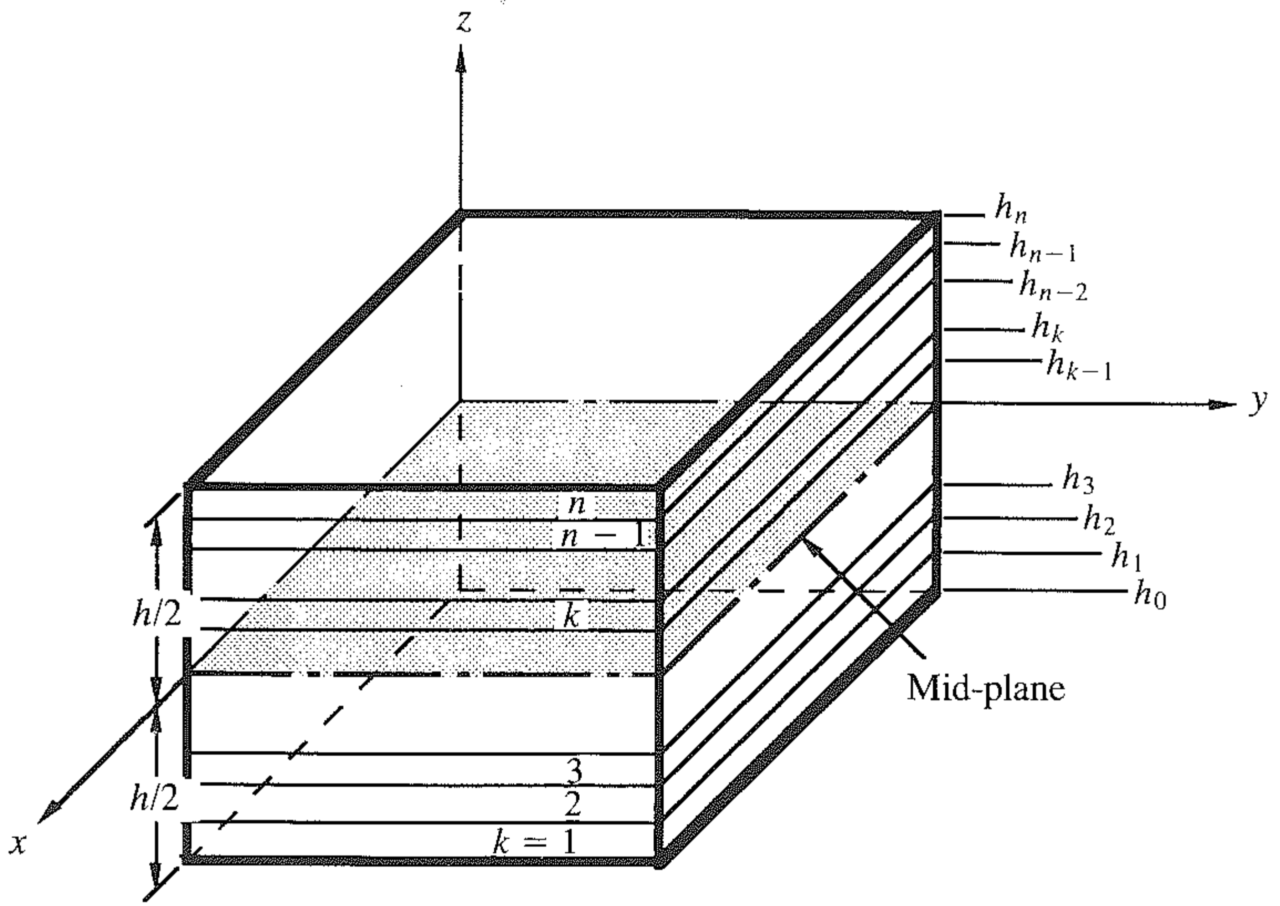
Cantilever Beams

BEAM DEFLECTION FORMULAE

BEAM TYPE	SLOPE AT FREE END	DEFLECTION AT ANY SECTION IN TERMS OF x	MAXIMUM DEFLECTION
1. Cantilever Beam – Concentrated load P at the free end			
	$\theta = \frac{Pl^2}{2EI}$	$y = \frac{Px^2}{6EI}(3l - x)$	$\delta_{max} = \frac{Pl^3}{3EI}$
2. Cantilever Beam – Concentrated load P at any point			
	$\theta = \frac{Pa^2}{2EI}$	$y = \frac{Px^2}{6EI}(3a - x)$ for $0 < x < a$ $y = \frac{Pa^2}{6EI}(3x - a)$ for $a < x < l$	$\delta_{max} = \frac{Pa^2}{6EI}(3l - a)$
3. Cantilever Beam – Uniformly distributed load ω (N/m)			
	$\theta = \frac{\omega l^3}{6EI}$	$y = \frac{\omega x^2}{24EI}(x^2 + 6l^2 - 4lx)$	$\delta_{max} = \frac{\omega l^4}{8EI}$
4. Cantilever Beam – Uniformly varying load: Maximum intensity ω_o (N/m)			
	$\theta = \frac{\omega_o l^3}{24EI}$	$y = \frac{\omega_o x^2}{120lEI}(10l^3 - 10l^2x + 5lx^2 - x^3)$	$\delta_{max} = \frac{\omega_o l^4}{30EI}$
5. Cantilever Beam – Couple moment M at the free end			
	$\theta = \frac{Ml}{EI}$	$y = \frac{Mx^2}{2EI}$	$\delta_{max} = \frac{Ml^2}{2EI}$

Typical Laminate

Fig. 2.5. An n-layered laminate.



Lamina Stress/Strain Relationship

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{pmatrix}$$

Where

$$Q_{11} = \frac{E_{11}}{1 - \nu_{12}\nu_{21}}$$

$$Q_{12} = \frac{\nu_{12}E_{22}}{1 - \nu_{12}\nu_{21}} = \frac{\nu_{21}E_{11}}{1 - \nu_{12}\nu_{21}}$$

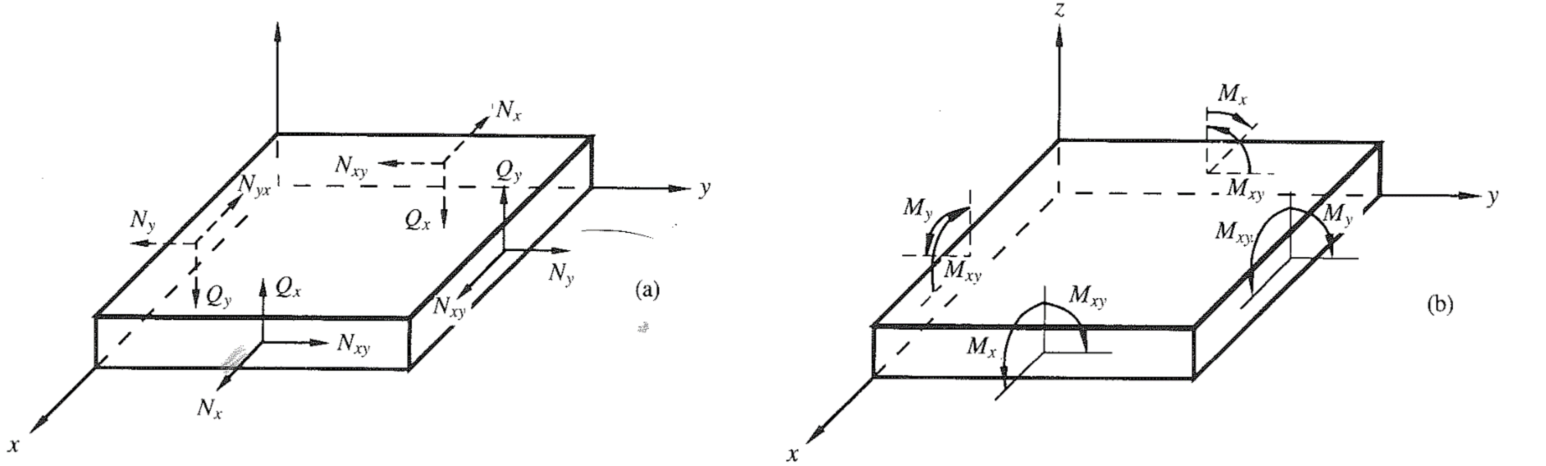
$$Q_{22} = \frac{E_{22}}{1 - \nu_{12}\nu_{21}}$$

$$Q_{66} = G_{12}$$

Rotated

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{pmatrix}$$

Laminate Plates



Stiffness Matrices

$$\left(\frac{N}{M}\right) = \left(\begin{array}{c|c} A & B \\ \hline B & D \end{array}\right)\left(\frac{\varepsilon^{o}}{\kappa}\right)$$

$$\begin{pmatrix} N_x \\ N_y \\ N_{xy} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{pmatrix}\begin{pmatrix} \varepsilon^{o}_{xx} \\ \varepsilon^{o}_{yy} \\ \gamma^{o}_{xy} \end{pmatrix}$$

$$+ \begin{pmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{pmatrix}\begin{pmatrix} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{pmatrix}$$

Stiffness Matrices

$$A_{ij} = \sum_{k=1}^{n} (\bar{Q}_{ij})_k (h_k - h_{k-1})$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} (\bar{Q}_{ij})_k (h_k^2 - h_{k-1}^2)$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} (\bar{Q}_{ij})_k (h_k^3 - h_{k-1}^3)$$

References

- Beer And Johnston, Mechanics of Materials
- Wood, R. J., Avadhanula, S., Sahai, R., Steltz, E. & Fearing, R. S. Microrobot Design Using Fiber Reinforced Composites. *J. Mech. Des.* **130,** 52304 (2008).
- Wood, R. J., Steltz, E. & Fearing, R. S. Optimal energy density piezoelectric bending actuators. *Sensors Actuators A Phys.* **119,** 476–488 (2005).
- Chou, Tsu-Wei. *Microstructural design of fiber composites*. Cambridge University Press, 2005.