Computational Physics Project

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December 12, 2024

Abstract

The diffraction characteristics of an optical system can be described by the point spread function or PSF. The PSF of an aperture is related to the aperture of an optical system via Fourier Transform. We employ Gaussian random fields to simulate the effects of imperfections in optical systems have on their PSFs. These distorted PSFs are then used to generate an analog of a Hubble telescope image of stars.

1 The Point Spread Function

The point spread function, or PSF describes the diffraction characteristics of light that passes through an aperture in the far field approximation [1]. Mathematically, it is defined in terms of the Fourier transform of said aperture,

$$\mathrm{PSF}(\theta_x, \theta_y) = |\mathrm{FT}[P(x, y)]\left(k_x = \frac{\theta_x}{\lambda}, k_y = \frac{\theta}{\lambda_y}\right)|^2, \tag{1}$$

where FT denotes the Fourier transform, k_x and k_y are the Fourier transform variables, λ wavelength, θ_x and $P(x, y)$ is the complex pupil function of an aperture. The complex pupil function is defined in terms of the aperture function $P_r(x, y)$ and phase function W(x, y),

$$P(x, y) = P_r(x, y)e^{-i2\pi W(x,y)/\lambda}. \tag{2}$$

The aperture function is a binary function which is zero where light cannot pass through and one where it can. For example, we can write the aperture function for a simple circular aperture of radius R_0 as

$$P_r(x, y) = \begin{cases} 1 & x^2 + y^2 < R_0^2 \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The point spread function of a circular aperture is known as the Airy disk and is given by

$$\mathrm{PSF}(\theta_r) = C\left(\frac{J_1(2\pi R_0\theta_r/\lambda)}{2\pi R_0\theta_r/\lambda}\right)^2, \tag{4}$$

where $\theta_r^2 = \theta_x^2 + \theta_y^2$, C is a scaling constant, J_1 is the first order Bessel function of the first kind. The location of the first minimum of the Airy function is given by

$$\theta_{\min} = 1.22\lambda/R_0, \tag{5}$$

in the small angle approximation [1].

The phase function $W(x, y)$ is used to add arbitrary phases to the light at the point in the system. This can be used to simulate atmospheric distortions as well as imperfections in the optical system, both of which will be explored in a later section.

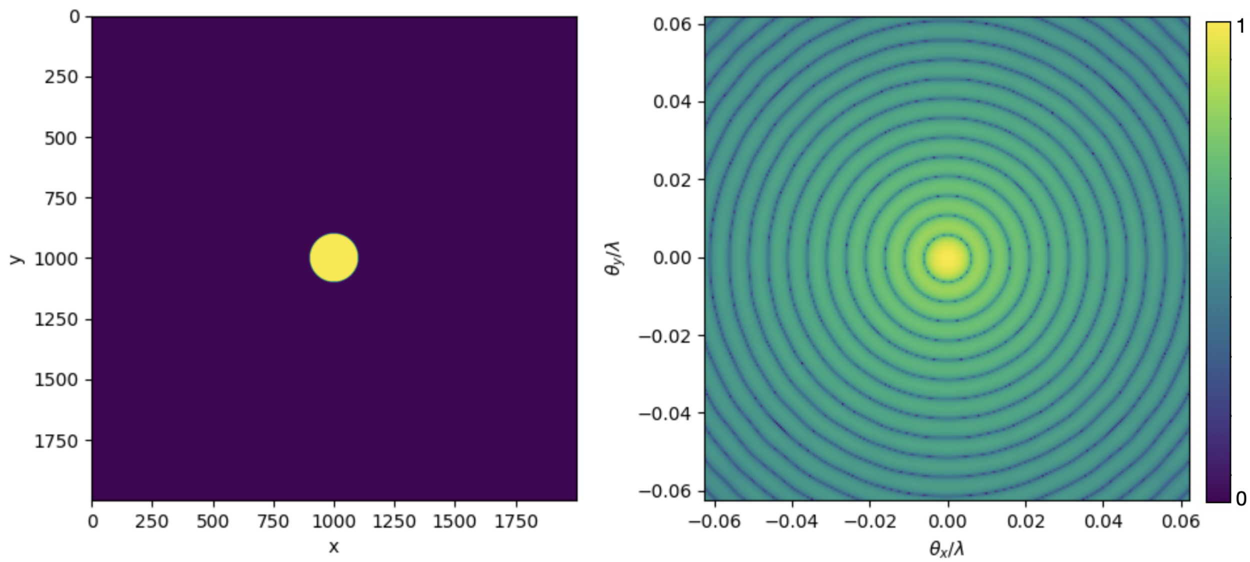

Figure 1: Left: circular aperture of radius 100 generated on a 2000x2000 grid. Right: PSF generated from the FFT of the aperture on the left with a 8x zoom applied and using a log scale.

2 Simulation of a Circular Aperture

To test our general method, we will first examine the well known case of the circular aperture. Figure 1 shows a circular aperture as well as the PSF generated from said aperture using numpy's fft.fft2 method. Qualitatively, the PSF appears to be the expected Airy disk and this agreement is shown quantitatively through the squared residuals and the locations of the first minima shown in figure 2. The Airy function used to generate the residuals is implemented using scipy's built it method. As the Bessel function has no analytic solution, it is defined in therms of the integral

$$J_1 = \frac{1}{\pi}\int_0^{\pi} d\theta \cos\left(\theta - x\sin\theta\right). \tag{6}$$

Thus, to validate scipy's built in Bessel function, we compared the residuals between it and the numeric integration of the above equation using Simpson's rule. The results, shown in figure 3, show excellent agreement between the two methods.

3 Cassegrain Apertures

While the case of a simple circular aperture is a nice example of a point spread function, it is not a very realistic model of a telescope. Many real world telescopes use a Cassegrain design, which has a smaller secondary mirror that sits on top of the main mirror. This can be represented by a donut shaped aperture with inner radius R_1 and outer radius R_0. As the Airy disk represents an intensity, which is proportional to the electric field strength squared, and electric fields obey superposition, we create a Cassegrain PSF simply by subtracting the square root of an Airy disk of radius R_1 from the root of an Airy disk of radius R_0 and squaring the result:

$$PSF(\theta_r) = C\left(\frac{J_1(2\pi R_0\theta_r/\lambda)}{2\pi R_0\theta_r/\lambda} - \frac{J_1(2\pi R_1\theta_r/\lambda)}{2\pi R_1\theta_r/\lambda}\right)^2. \tag{7}$$

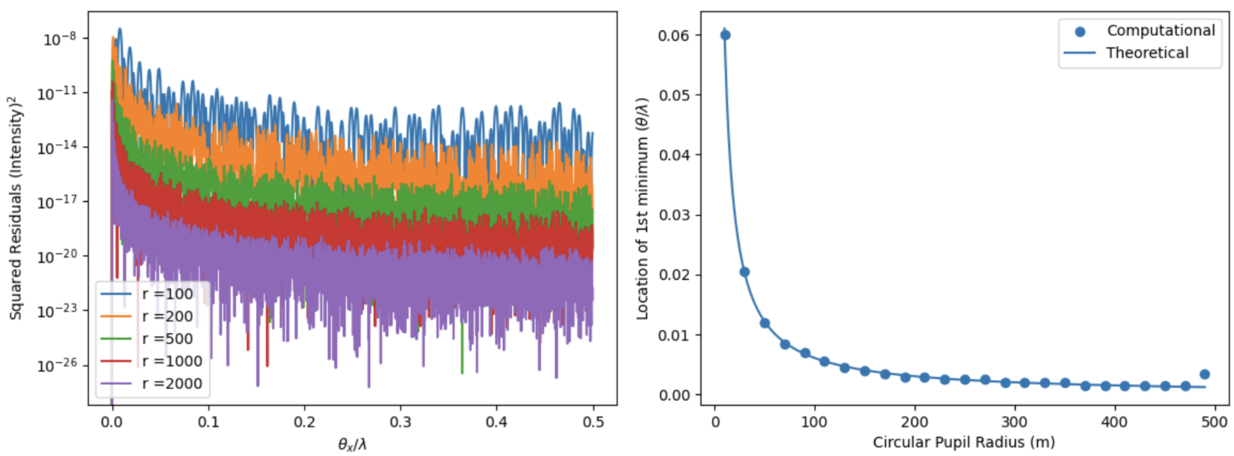

Figure 2: Left: Squared residuals of normalized computational PSF and Airy function for various pupil radii r on a 2000x2000 grid. Right: Theoretical and computational location of 1st minimum of the airy disk as a function of pupil radius.

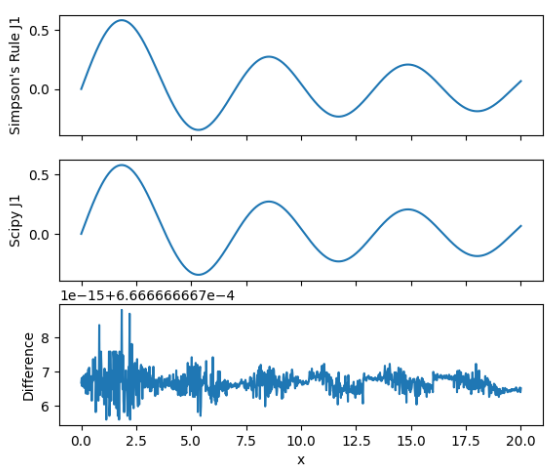

Figure 3: Comparison of Scipy's $J_1(x)$ with the numeric solution to eq 6 found using Simpson's rule.

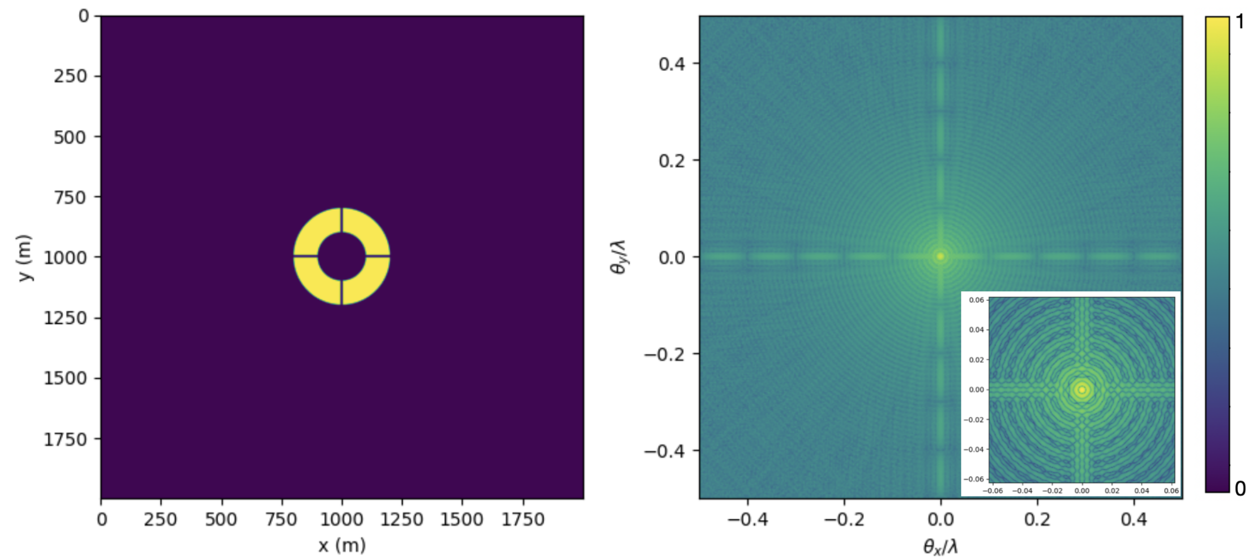

Figure 4: Left: Cassegrain aperture with mirror supporting struts. Right: PSF of the Cassegrain aperture on the left using a log scale. Inset is 8x zoom of (0, 0).

To simulate a real system we must add in a number of struts to support the secondary mirror. These appear as lines in the aperture function that divide the aperture into four quadrants (see figure 4 left). This modification has no simple analytic representation and its PSF must be found purely computationally. Using the same PSF generation method used for the circular aperture, we find that the PSF of the Cassegrain aperture with struts has four spikes in intensity matching the symmetry of the struts (see figure 4).

4 PSFs of Imperfect Systems

4.1 Gaussian Random Fields

To simulate an imperfect system we will focus on the previously neglected phase function $W(x, y)$ in the complex pupil function (eq: 2). We want to generate a series of Gaussian random fields to use for $W(x, y)$ as phase distortions to simulate both atmospheric effects and imperfections in the telescope itself.

To generate a Gaussian random field, we must first generate a complex random field $f(x, y)$ whose Fourier transform will be real. For this to be the case, $f(x, y) = f(-x, -y)^*$, ie the real component of the field must be even and the imaginary part must be odd. This means that if we wish generate a $N \times N$ random field, half of the values in said field are determined by the other half. If we have a 2D python array indexed by x and y, we treat $x = -1$ as being equivalent to $x = N - 1$. Luckily, python implements this by default. Thus, all we must do is carefully arrange our $\frac{N^2}{2}$ random numbers and their duplicates in $N \times N$ grid such that the result is even. The correct arrangement is shown in figure 5. An array of random numbers generated using this method is shown in figure 6. Taking the Fourier transform of the random array gives a real result, implying that the random array is indeed obey $f(x, y) = f(-x, -y)^*$.

We generate a Gaussian random field by taking the Fourier transform of the complex random array. This gets multiplied by a power spectrum $P(k)$ of the form:

$$P(k) = k^{\alpha} e^{(-k^2/k_c^2)} \tag{8}$$

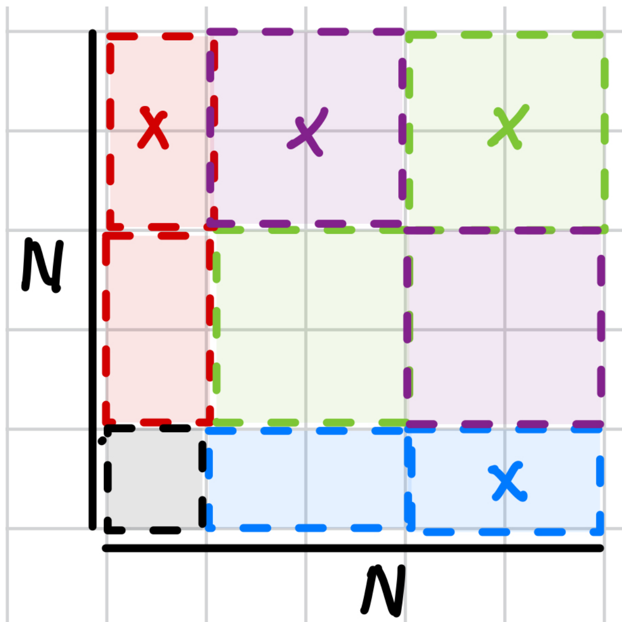

Figure 5: Arrangement of random values in $N \times N$ array to create an even function. Each square on the grid represents a single element in the array. Every square in a colored region without an x is uniquely determined. Those with an x are duplicates of the other region of the same color that have been rotated 180 degrees. The origin is located in the lower left and is highlighted in black.

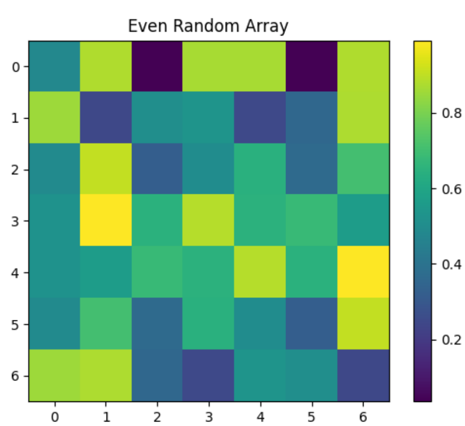

Figure 6: Graphical representation of a random array generated using the method shown in figure 5. The origin is located at the top right corner and indices labeled 6, 5, 4 also correspond to indices -1, -2, -3.

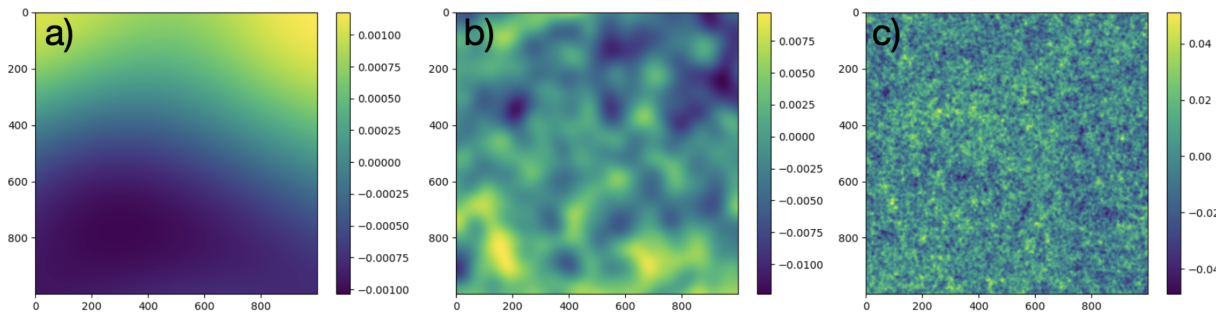

Figure 7: Gaussian random fields generated with $\alpha = -1$ and various values of k_c. a) $k_c = 1$, b) $k_c = 10$, c) $k_c = 100$.

where α and k_c are parameters. Finally, the inverse Fourier transform is taken leaving us with a Gaussian random field with variance specified by $P(k)$. Figure 7 shows Gaussian random fields generated with various values of k_c.

4.2 Small Imperfections

We can simulate small imperfections in the optical system by using a Gaussian field with with power at large k, such as the one shown in figure 7c. The PSF which results from these distortions, which is shown in figure 8, has a much broader central lobe than the undistorted PSF (fig: 4). This destroys much of the Airy-like pattern visible at the center of the undistorted PSF.

To simulate a realistic exposure, we averaged the PSFs generated with 10 different random fields. The results, shown in figure 8 right, show that the time average increases the contrast between the central lobe and the background.

4.3 Atmospheric Distortions

To simulate the effects of atmospheric distortion on the PSF, we will use a Gaussian field which is smooth at small scales and varies at scales much larger than the wavelength of light, such as the one in figure 7a. Distortions at this scale shift the intensity peak of the PSF off center (see figure 9). Taking a long exposure in this case broadens the width of the diffraction peaks caused by the struts and makes them more visible against the low contrast background.

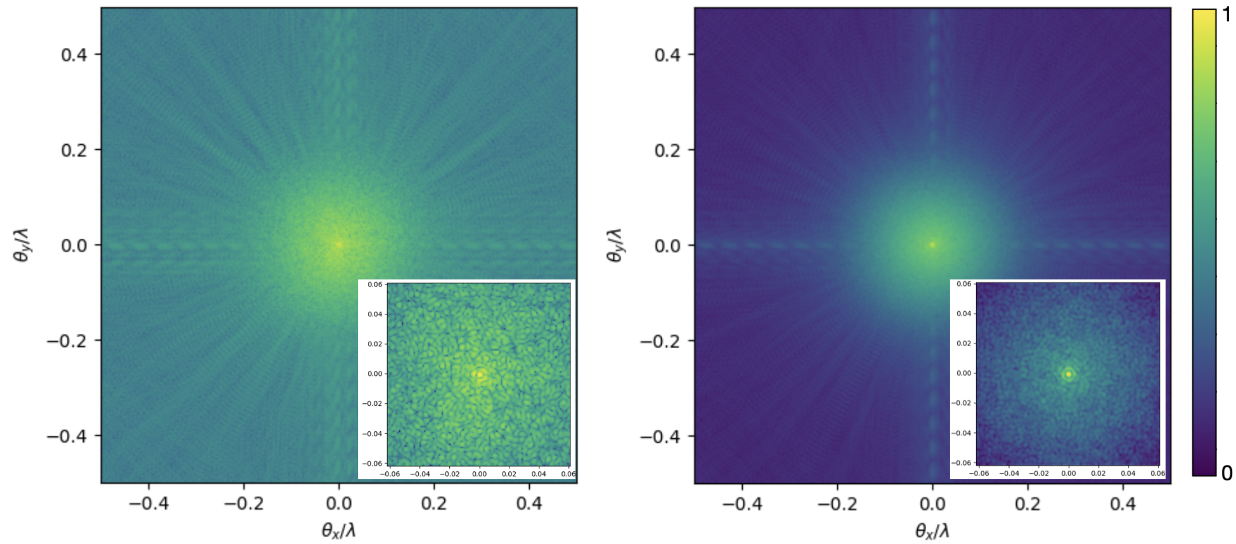

Figure 8: Left: PSF of a Cassegrain aperture with small imperfections applied. Right: Average of 10 Cassegrain PSFs with small imperfections. Insets are 8x zoom.

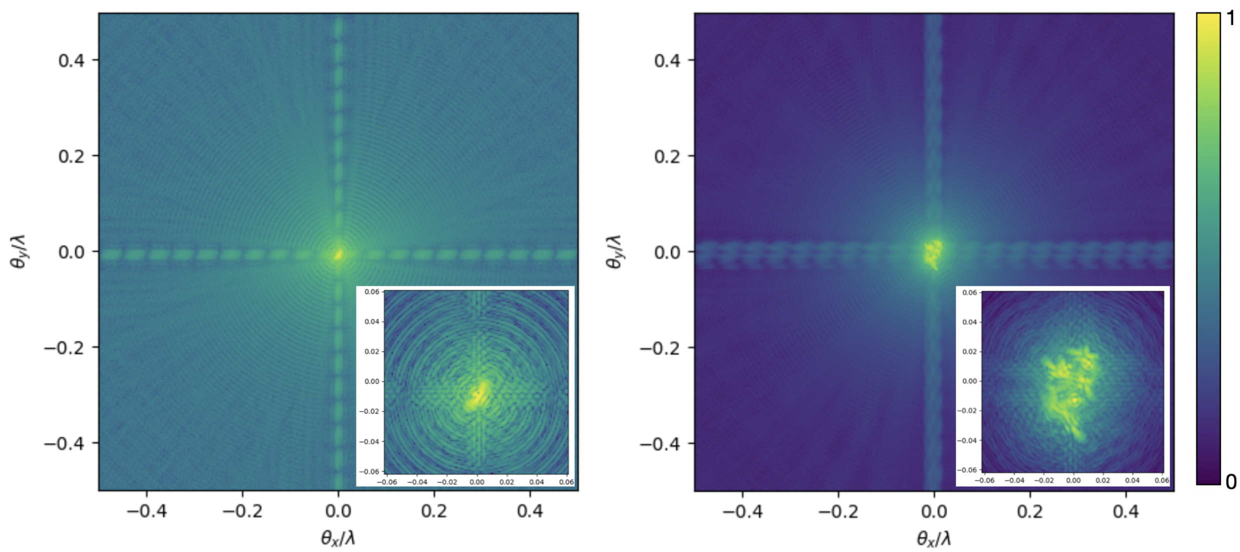

Figure 9: Left: PSF of a Cassegrain aperture with large scale distortions. Right: Average of 10 Cassegrain PSFs with large scale distortions. Insets are 8x zoom.

5 Hubble Images

The Hubble Space Telescope is a Cassegrain telescope which uses four struts to suspend it's secondary mirror. The light reflecting off these struts creates a diffraction pattern giving the image of the stars

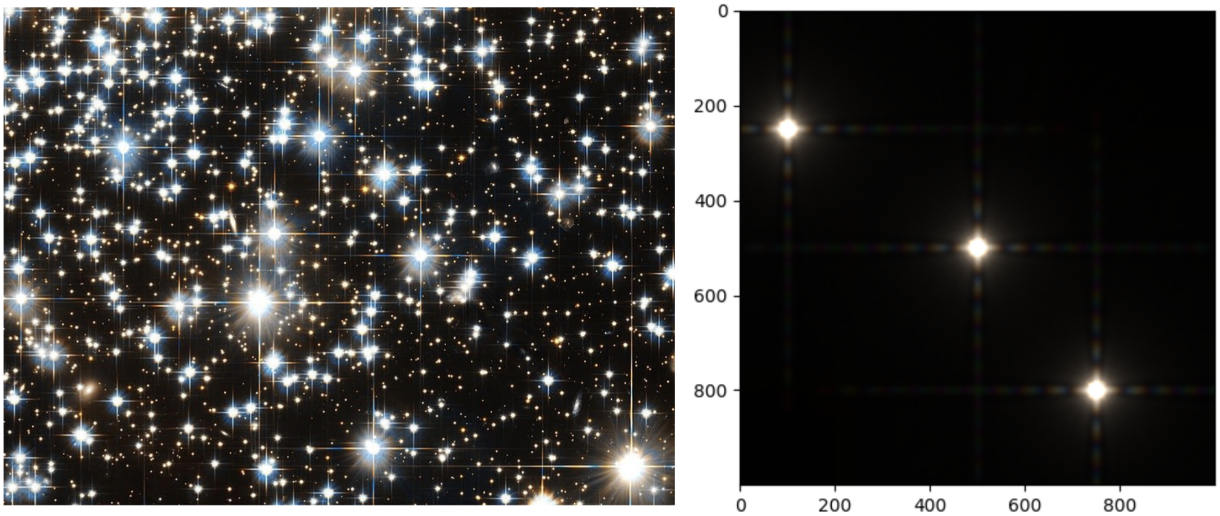

Figure 10: Left: Hubble telescope image of stars where diffraction spikes are clearly visible [2]. Right: Hubble like image generated through the convolution of the point spread function.

distinctive spikes and affects the perceived intensity. The intensity one would see at the focal plane of an optical system is given by the convolution of the PSF and intensity input into the system

$$I_i = I_0 \otimes \mathrm{PSF}, \tag{9}$$

where I_0 is the input intensity and I_i is the intensity at the focal plane [1]. We can thus use the methods we setup in the previous sections to generate realistic Hubble-like images. First, we setup an array of zeros with delta functions a few locations to act as stars. We then convolve this with the PSF using scipy's signal.fftconvolve method. This gives the intensity at the focal plane for a single color. Since the PSF scales according to the wavelength, we can generate RGB color channels by using this process three times with PSFs at different scales. Combining these three color channels produces the image shown on the right side of figure 10.

As can be seen from figure 10, the image generated by our method, like the Hubble image, has a broad intensity peak for each star with four diffraction spikes of alternating color spreading out from it. Thus, we have been able to qualitatively recreate the diffraction effects seen in Hubble telescope images.

References

[1] Frank L. Pedrotti, Leno Matthew Pedrotti, and Leno S. Pedrotti. *Introduction to optics*. eng. 3rd ed. Cambridge: Cambridge university press, 2018. ISBN: 9781108428262.

[2] *Why are stars pointy? — (Diffraction spikes explained)*. en-US. Nov. 2020. URL: `https://wonderdome.co.uk/pointy-stars-diffraction-spikes-explained/`.