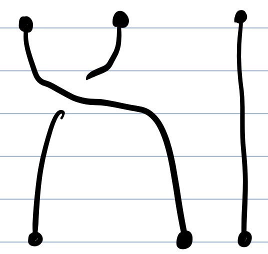
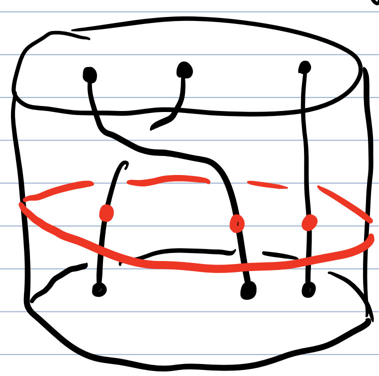
Graph braid groups

- Think of the classical pic of a braid



n strands intertwine with each other via a sequence of half-twists

- Embed this in a cylinder $D^2 \times I$



Each horizontal cross-section gives a disc with n marked points.
As you move down the cylinder, these marked pts ("particles") move around the disc w/o colliding, eventually returning to the same n start locations (up to permutation).

$$C_n(D^2) = \{(x_1, \ldots, x_n) \in (D^2)^n \mid x_i \neq x_j \ \forall i \neq j\}$$

Config space

don't want to distinguish different strands

A braid is a loop in $C_n(D^2)/S_n = UC_n(D^2)$

Braid gp $B_n(D^2) = \pi_1(C_n(D^2)/S_n)$ (Fox '62)

Q: What happens if you replace D^2 with another space X?

Thm: (Birman '69)

$B_n(M) \cong \prod_{i=1}^n \pi_1(M)$ for mfds M w/ $\dim(M) \geq 3$

Two obvious directions we can go:

① Consider $\dim(M) \leq 2$, i.e.

$B_n(\mathbb{R}) = 1$

$B_n(S^1) = \mathbb{Z}$

$B_n(\Sigma)$, Σ surface $\xrightarrow{\text{Birman exact sequence}}$ MCG w/ n marked pts

Theorem: (Birman '69)

$$\underbrace{\pi_1(\mathrm{Homeo}^{+,\partial}(\Sigma))}_{=1 \text{ if } \chi(\Sigma)<0} \to B_n(\Sigma) \to MCG(\Sigma; \{p_1, \dots, p_n\}) \to MCG(\Sigma) \to 1$$

$\Sigma = D^2$: $1 \to B_n(D^2) \xrightarrow{\cong} MCG(D; \{p_1, \dots, p_n\}) \to 1$

$n=1$: $1 \to \pi_1(\Sigma) \to MCG(\Sigma; p) \to MCG(\Sigma) \to 1$

② Weaken the mfd condition, e.g. $B_n(\Gamma)$ for Γ a finite graph

Theorem: (Abrams '00, Crisp–Wiest '04) $UC_n(\Gamma)$ deformation retracts onto a compact special cube complex.

<u>Corollary</u>: (Behrstock–Hagen–Sisto '17) Graph braid groups are hierarchically hyperbolic groups (HHGs).

An HHG structure is roughly:

coset of subgroups

- An index set $\mathfrak{S}$ corresponding to a natural collection of subspaces
- A hyperbolic space $\mathcal{C}(U)$ for each $U \in \mathfrak{S}$
- Projections π_U onto the $\mathcal{C}(U)$ spaces
- A nesting relation "$\sqsubseteq$" giving a partial order on $\mathfrak{S}$
- An orthogonality relation "$\perp$" encoding direct products

<u>The HHG structure on $B_n(\Gamma)$</u>

Should really use $B_n(\Gamma, S) = \pi_1(UC_n(\Gamma), S)$

initial configuration

<u>Proposition</u>: (Genevois) Suppose $\Gamma = \Gamma_1 \sqcup \Gamma_2$.

$$UC_n(\Gamma) \cong \bigsqcup_{k=0}^{n} UC_k(\Gamma_1) \times UC_{n-k}(\Gamma_2)$$

Suppose $S \in UC_n(\Gamma)$ has k particles in Γ_1 and $n-k$ particles in Γ_2.

$$B_n(\Gamma, S) \cong B_k(\Gamma_1, S \cap \Gamma_1) \times B_{n-k}(\Gamma_2, S \cap \Gamma_2)$$

$\rightsquigarrow$ Assume Γ is connected and drop the S

<u>Note</u>: If $\Lambda \subseteq \Gamma$, $k \leq n$, and $S \in UC_k(\Lambda)$, then $B_k(\Lambda, S)$ embeds as a subgroup of

$B_n(\Gamma)$.

<u>Index set</u>: $\mathfrak{S}$ = Cosets of $B_k(\Lambda, S)$ (up to an equivalence relation)

<u>Nesting</u>: $B_k(\Lambda, S) \sqsubseteq B_l(\Omega, T)$ if $\Lambda \subseteq \Omega$, $k \leq l$, S & $T \cap \Lambda$ are in the same component of $UC_k(\Lambda)$

<u>Orthogonality</u>: $B_k(\Lambda, S) \perp B_l(\Omega, T)$ if $\Lambda \cap \Omega = \emptyset$ and $k + l \leq n$

$$B_{k+l}(\Lambda \sqcup \Omega, S \sqcup T) \cong B_k(\Lambda, S) \times B_l(\Omega, T)$$

<u>Curve graph</u>: $\mathcal{C}(B_k(\Lambda, S))$ = Factored Contact graph of univ. cover of component of $UC_k(\Lambda)$ containing S

Can use hierarchical hyperbolicity to detect other forms of hyperbolicity! $\nexists\, U \perp V$ with $\mathcal{C}(U)$ & $\mathcal{C}(V)$ unbounded

① Hyperbolicity & acylindrical hyperbolicity (Behrstock–Hagen–Sisto)

Not virtually cyclic and $\sqsubseteq$-maximal $\mathcal{C}(S)$ is unbounded

Lemma : (B.) $\mathcal{C}(B_n(\Lambda,S))$ is unbdd iff Λ is connected & $B_n(\Lambda,S)$ is unbdd.

② Relative hyperbolicity (Russell)

"Orthogonality is isolated"

$\exists\, \mathcal{I} \subseteq \mathfrak{G}$ s.t.

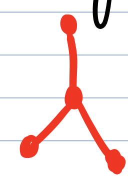
① $U \perp V \Rightarrow U, V \subseteq I \in \mathcal{I}$

② $W \subseteq I, I' \in \mathcal{I} \Rightarrow I = I'$

Lemma : (Genevois) $B_n(\Lambda,S)$ has infinite diameter if and only if Λ has a component containing a cycle/star subgraph

Otherwise, $B_n(\Lambda,S)$ is trivial.

≥ 1 particle in this component



≥ 2 particles

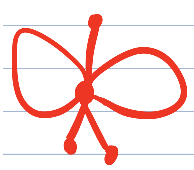
Theorem 1 : (Genevois) $B_n(\Gamma)$ is hyperbolic iff either:

New proof (B.)

i) $n=1$.

ii) $n=2$ and Γ does not contain two disjoint cycles

iii) $n=3$ and Γ does not contain two disjoint cycles nor a disjoint star and cycle.



iv) $n \geq 4$ and Γ does not contain two disjoint stars/cycles

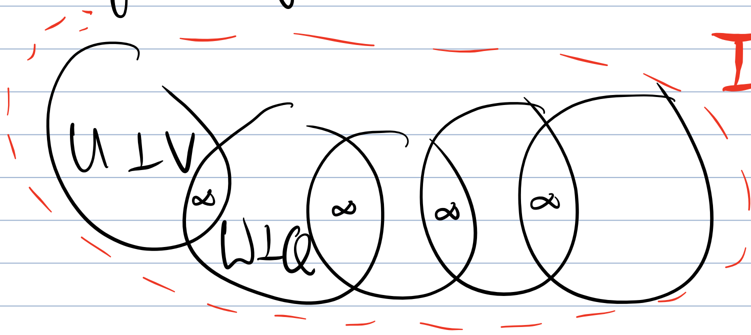
Theorem 2: (Genevois) $B_n(\Gamma)$ is either cyclic or acylindrically hyperbolic.

↑ New proof (B.)

Proof: Use B-H-S + $B_n(\Gamma)$ torsion-free
+ $B_n(\Gamma)$ trivial or unbounded

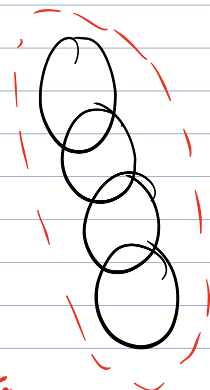
Conjecture 3: There exists an HHG structure $\Upsilon = \mathfrak{G} \cup \mathfrak{G}'$ s.t. the subgroup generated by a maximal "chain" of orthogonal domains is in Υ

Take a seq. of pairs of disjoint subgraphs of Γ s.t. two consecutive pairs intersect in a cycle/star



$I \in \Upsilon$

These will be used to isolate orthogonality



NB In general these subgps are not in $\mathfrak{G}$

Theorem 4: (B.) Suppose Conjecture 3 is true. → $\exists$ maximal $\overbrace{\text{chain of chain of} \ldots \text{of chains}}^{k \text{ times}}$ I s.t. $I = B_n(\Gamma)$

① If $B_n(\Gamma)$ has "hypergraph index" k, then $B_n(\Gamma)$ is strongly thick of order at most k. In particular, $B_n(\Gamma)$ is not rel. hyp.

② If $B_n(\Gamma)$ has "hypergraph index" ∞, then $B_n(\Gamma)$ is rel. hyp.

↳ i.e. chains eventually all have finite intersection