Hierarchical hyperbolicity of graph products and graph braid groups

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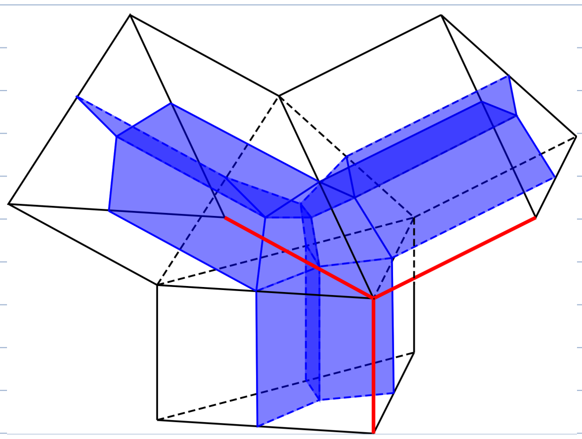
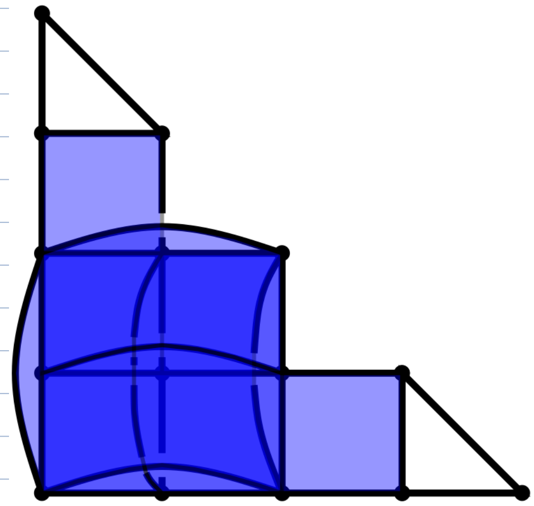
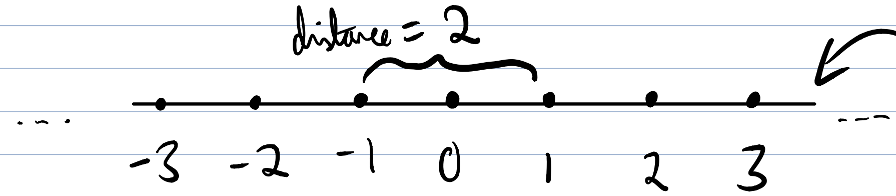




The topic of this thesis is geometric group theory

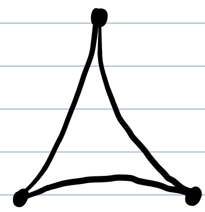
e.g. Integers



Cayley graph

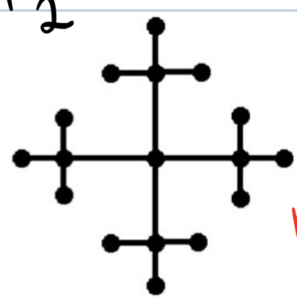
Can use Cayley graph to develop analogues of classical geometric concepts for groups

e.g. Hyperbolicity



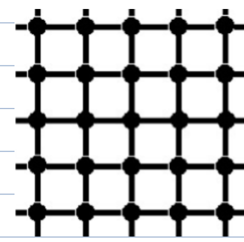
"Triangles in the Cayley graph are thin"

Examples: ① F_2



✓

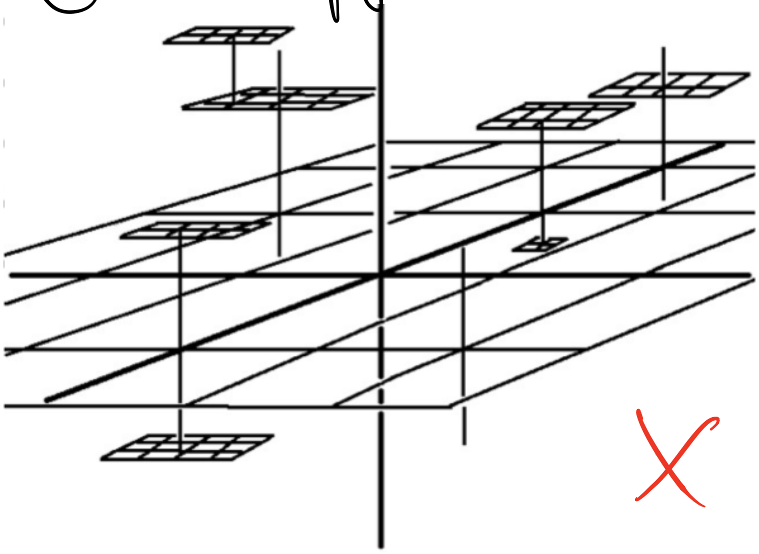
② $\mathbb{Z}^2$



✗

"Flat"

③ "Tree of flats"



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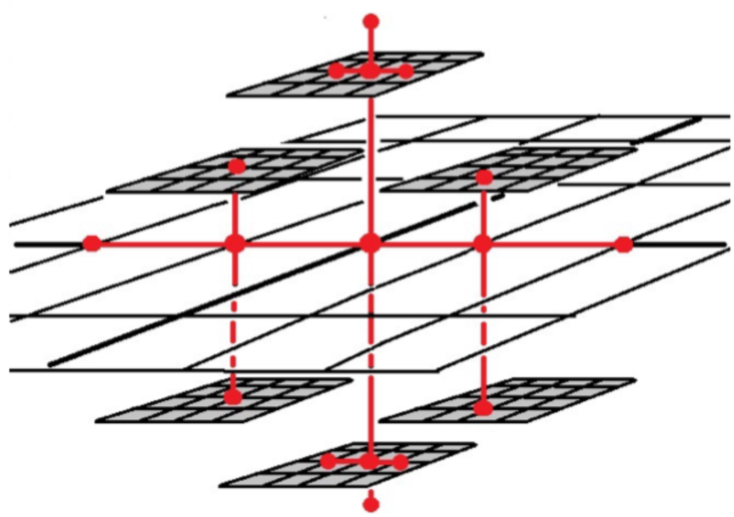


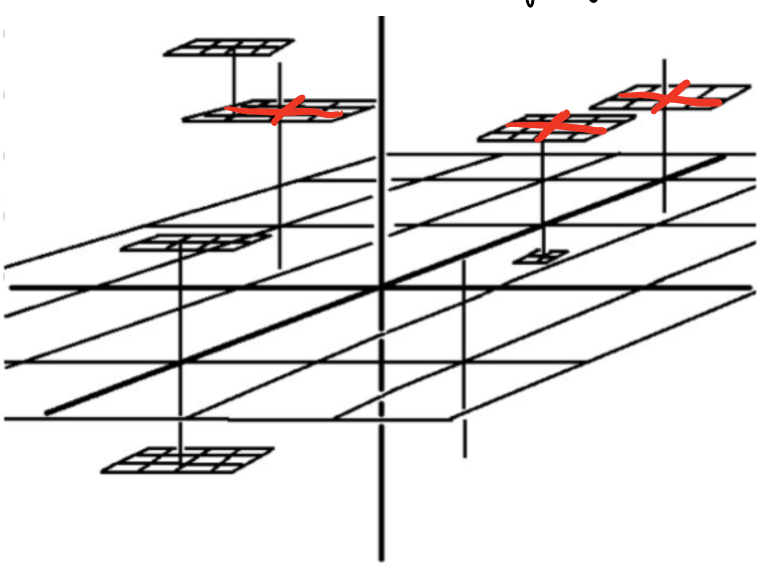
<u>Generalisations</u>:
- Relative hyperbolicity (Gromov, Bowditch, Farb, et al.)
- Acylindrical hyperbolicity (Sela, Bowditch, Osin)
- <u>Hierarchical hyperbolicity</u> (Behrstock-Hagen-Sisto)

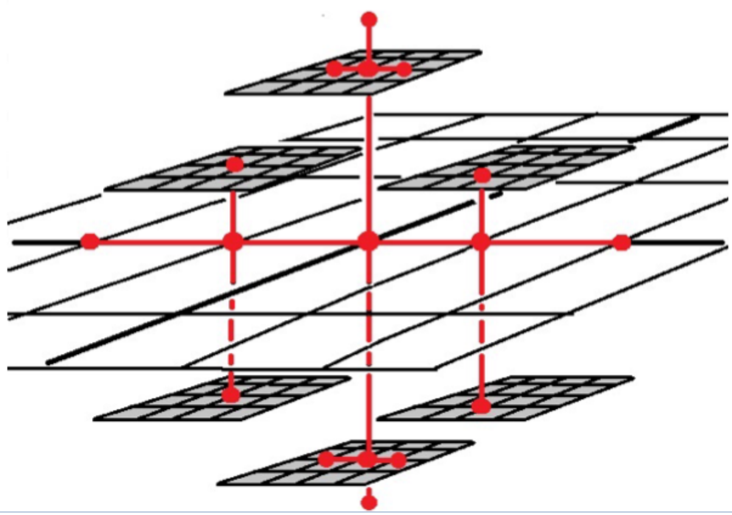
- The geometric information of an HHS X is encoded in a collection of projections
$$\pi_U : X \to \mathcal{C}(U), \quad U \in \mathfrak{S} \quad \text{"domains"}$$
onto hyperbolic spaces $\mathcal{C}(U)$.

- These domains are arranged via a partial order on $\mathfrak{S}$ called nesting ($\sqsubseteq$), and flats are encoded via a relation on $\mathfrak{S}$ called orthogonality ($\perp$).

+ some extra axioms

<u>Example</u>: "Tree of flats"





Can use hierarchical hyperbolicity to detect other forms of hyperbolicity!

$\nexists\, U \perp V$ with $\mathcal{C}(U)$ & $\mathcal{C}(V)$ unbounded $\Leftrightarrow$

- Hyperbolicity & acylindrical hyperbolicity (Behrstock – Hagen – Sisto) $\Leftrightarrow$ Not virtually cyclic & $\sqsubseteq$-maximal $\mathcal{C}(S)$ is unbounded
- Relative hyperbolicity (Russell) "Orthogonality is isolated"

HHS variants

Definition: (Relative HHS) $(X, \mathfrak{S})$ is a relative HHS if it satisfies all HHS axioms except $\mathcal{C}(U)$ need not be hyperbolic when U is $\sqsubseteq$-minimal.

Container axiom: $\forall\, V \sqsubsetneq W\ \exists\, U \sqsubsetneq W$ such that whenever $Q \perp V$ and $Q \sqsubseteq W$, then $Q \sqsubseteq U$.

Definition: (Almost HHS) $(X, \mathfrak{S})$ is an almost HHS if it satisfies all HHS axioms except the container axiom, which is replaced with a uniform bound on the number of pairwise orthogonal domains.

Theorem 1: (B.–Russell) Any almost HHS admits an HHS structure.

Proof: (Sketch)

- Start with an almost HHS $(X, \mathfrak{S})$
- If $V \sqsubseteq W$ and $\exists\, \mathcal{U} \sqsubseteq W$ with $\mathcal{U} \perp V$, then add a dummy domain D_W^V to serve as the container for V in W, where $\mathcal{C}(D_W^V)$ is a single point.
- **Problem**: What is the container for $\overset{\perp \cup \sqsubseteq}{\mathcal{U}}$ in D_W^V?

  Solution: Add domains $D_W^{\mathcal{V}}$ where $\mathcal{V}$ is a pairwise orthogonal set of domains nested in W i.e. $D_W^{\mathcal{V}}$ contains all domains that are nested in W and orthogonal to all $V \in \mathcal{V}$.

  Answer: $D_W^{\mathcal{V} \cup \mathcal{U}}$

- Now define projections & relations for the $D_W^{\mathcal{V}}$ and show the HHS axioms hold for these.

<u>Consequences</u>: • Streamlines proofs of Abbott–Behrstock–Durham
• Allows us to show graph products of HHGs are HHGs

<u>Definition</u>: (Graph product) Let Γ be a finite simplicial graph and label each $v \in V(\Gamma)$ with a finitely generated group G_v.

The graph product is

$$G_\Gamma = \left(\mathop{\ast}_{v \in V(\Gamma)} G_v \right) \Big/ \langle\langle [g_v, g_w] \mid g_v \in G_v, g_w \in G_w, \{v,w\} \in E(\Gamma) \rangle\rangle$$

<u>Remark</u>: If $G_v = \mathbb{Z}$ $\forall v$ then this defines a RAAG

<u>Theorem</u>: (BHS) RAAGs are HHGs.

<u>Theorem 2</u>: (B.–Russell) Graph products of HHGs are HHGs.

<u>Theorem 3</u>: (B.–Russell) <u>All</u> graph products are <u>relative</u> HHGs.

Sketch of proof:

① Replace the word metric on G_Γ with the

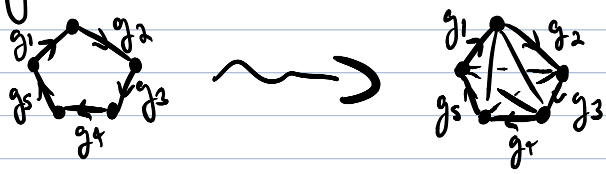
syllable metric, i.e. study $\mathrm{Cay}(G_\Gamma, \underbrace{\bigcup_{v\in V(\Gamma)} G_v}_{\text{infinite generating set given by all elements of vertex groups}})$

↑

Kim-Koberda

$g_1 g_2 g_3 g_4 g_5 = 1$ in G_v

5-cycle



5-simplex

$\mathrm{Cay}(G_\Gamma, \bigcup_{v\in V(\Gamma)} G_v)$ is shown by Genevois to have cubical-like geometry.

② Use this to develop analogues of B-H-S's cubical techniques for RAAGs, showing that G_Γ is an HHS with the syllable metric.

HHS structure

- Index set $\mathfrak{S}$ is given by full subgraphs $\Lambda \subseteq \Gamma$. + cosets

  Λ induces a subgroup $G_\Lambda \leq G_\Gamma$.

Nesting: $\Lambda_1 \sqsubseteq \Lambda_2$ if $\Lambda_1 \subseteq \Lambda_2$

<u>Orthogonality</u>: $G_{\Lambda_1} \times G_{\Lambda_2}$ embeds as a subgroup of G_Γ if every vertex of Λ_1 is connected to every vertex of Λ_2

$\rightsquigarrow \Lambda_1 \perp \Lambda_2$ if $\Lambda_1 \subseteq \mathrm{link}(\Lambda_2)$

Not hyperbolic in general

<u>Hyperbolic spaces</u>:

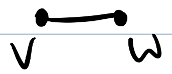
<u>$\Lambda = \{v\}$</u>: $\mathcal{C}(\Lambda) = \begin{cases} \mathrm{Cay}(G_v, S_v) & \text{if using word metric} \\ \mathrm{Cay}(G_v, G_v) & \text{if using syllable metric} \end{cases}$

<u>$|V(\Lambda)| > 1$</u>: $\mathcal{C}(\Lambda) = \mathrm{Cay}\Big(G_\Lambda, \underbrace{\bigcup_{\Omega \subsetneq \Lambda} G_\Omega}_{\text{Includes } \bigcup_v G_v}\Big)$

③ Conclude that G_Γ is a relative HHG with the word metric.

④ Replace the $\sqsubseteq$-minimal domains with the HHG structures of the vertex groups (must show the structures are compatible with the main structure).

Problem: This only gives an almost HHG structure



$$\begin{array}{ccccc} & & G_{\{v,w\}} & & \\ & \sqcup\!| & & \cup\!| & \\ & G_v & \perp & G_w & \\ \sqcup\!| & & \cup\!| & & \\ U \perp & & Q & & \end{array}$$

No container for U in $G_{\{v,w\}}$

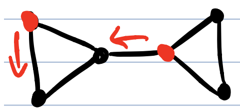
Solution: Use Theorem 1

Graph braid groups

<u>Definition</u>: Let $n \geq 1$ and let Γ be a finite graph such that:

n particles

Γ



① Every path between vertices of Γ of valence ≥ 3 has length $\geq n-1$

② Every homotopically essential loop in Γ has length $\geq n+1$

The carrier $C(x)$ of $x \in \Gamma$ is the lowest dimensional simplex of Γ containing x.

vertex or edge

- <u>Combinatorial configuration space</u>: $C_n(\Gamma) = \Gamma^n \setminus D$

where $D = \{(x_1, \dots, x_n) \in \Gamma^n \mid C(x_i) \cap C(x_j) \neq \emptyset \text{ for some } i \neq j\}$

No two particles can occupy the same edge of Γ
Can consider the particles to be jumping between adjacent vertices of Γ

- <u>Unordered combinatorial configuration space</u>:

$$UC_n(\Gamma) = C_n(\Gamma)/S_n$$

Symmetric group acts by permutation of coordinates

- <u>Graph braid group</u>: $B_n(\Gamma, S) = \pi_1(UC_n(\Gamma), S)$

Initial configuration of particles on Γ

<u>Proposition</u>: (Genevois) Suppose $\Gamma = \Gamma_1 \sqcup \Gamma_2$.

$$UC_n(\Gamma) \simeq \bigsqcup_{k=0}^{n} UC_k(\Gamma_1) \times UC_{n-k}(\Gamma_2)$$

If $S \in UC_n(\Gamma)$ has k particles in Γ_1 and $n-k$ particles in Γ_2, then

$$B_n(\Gamma, S) \cong B_k(\Gamma_1, S \cap \Gamma_1) \times B_{n-k}(\Gamma_2, S \cap \Gamma_2)$$

$\rightsquigarrow$ Assume Γ is connected and drop the basepoint S

<u>Theorem</u>: (Abrams, Genevois) $UC_n(\Gamma)$ is a compact special cube complex.

<u>Theorem</u>: (B-H-S) Let X be a special cube complex with finitely many hyperplanes. Then its universal cover is an HHS and $\pi_1(X)$ is an HHG.

<u>Corollary</u>: $B_n(\Gamma)$ is an HHG.

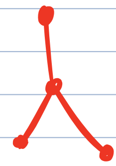
HHG structure

Index set $\mathfrak{S}$ is given by triples $\langle \Lambda, k, S \rangle$ where $\Lambda \subseteq \Gamma$, $k \leq n$, $S \in UC_k(\Lambda)$. These determine graph braid subgroups $B_k(\Lambda, S) \leq B_n(\Gamma)$.

Lemma: (Genevois) $B_k(\Lambda, S)$ is unbounded iff Λ contains a cycle/star subgraph in one of its components. Otherwise, $B_k(\Lambda, S)$ is trivial.

≥ 1 particle in this component

≥ 2 particles in this component



Nesting: $\langle \Lambda, k, S \rangle \sqsubseteq \langle \Omega, l, T \rangle$ if $\Lambda \subseteq \Omega$, $k \leq l$, and S and $T \cap \Lambda$ are in the same component of $UC_k(\Lambda)$.

Orthogonality: $\langle \Lambda \sqcup \Omega, k+l, S \sqcup T \rangle \cong \langle \Lambda, k, S \rangle \times \langle \Omega, l, T \rangle$

$\langle \Lambda, k, S \rangle \perp \langle \Omega, l, T \rangle$ if $\Lambda \cap \Omega = \emptyset$ and $k + l \leq n$.

<u>Theorem</u>: (B., Genevois) $B_n(\Gamma)$ is hyperbolic iff either:

i) $n=1$.

ii) $n=2$ and Γ does not contain two disjoint cycles

iii) $n=3$ and Γ does not contain two disjoint cycles, nor a disjoint star and cycle.

iv) $n \geq 4$ and Γ does not contain two disjoint cycles/stars.

<u>Proof</u>: Use B-H-S's characterisation of hyperbolicity together with Genevois' characterisation of diameter of $B_k(\Lambda, S)$.

<u>Theorem</u>: (B., Genevois) $B_n(\Gamma)$ is either cyclic or acylindrically hyperbolic.

<u>Proof</u>: $B_n(\Gamma)$ is either trivial or unbounded

$B_n(\Gamma)$ is torsion-free

Apply B-H-S.

<u>Conjecture</u>: We can use Russell's isolated orthogonality criterion to provide a similar characterisation of relative hyperbolicity.

Definition: (Orthogonality graphs) The orthogonality graphs $\mathcal{O}_i$ of $B_n(\Gamma)$ are hypergraphs, defined inductively:

① $\mathcal{O}_0$: • Vertices are triples $\langle \Lambda, k, S \rangle$ s.t. $B_k(\Lambda, S)$ has ∞-diameter

• Hyperedges are maximal collections $E = \{\langle \Lambda_1, k_1, S_1 \rangle, \dots, \langle \Lambda_m, k_m, S_m \rangle\}$ of pairwise orthogonal domains

• Define $\langle E \rangle$ to be the subgroup generated by the graphical subgroups in $\langle E \rangle$.

② Define equivalence relation $\equiv_i$ on $\mathcal{E}(\mathcal{O}_i)$ by setting

hyperedges

$E \equiv_i E'$ if $\exists$ sequence $E = E_1, \dots, E_m = E'$ of hyperedges s.t. $\langle E_j \rangle \cap \langle E_{j+1} \rangle$ has ∞-diameter $\forall j$

③ Define $V(\mathcal{O}_{i+1}) = V(\mathcal{O}_0)$ and $E \in \mathcal{E}(\mathcal{O}_{i+1})$ if and only if $E = E_1 \cup \dots \cup E_m$ for some maximal collection of $\equiv_i$-equivalent hyperedges of $\mathcal{O}_i$.

Define the hypergraph index of $B_n(\Gamma)$ to be the smallest i such that $\exists\, E \in \mathcal{E}(\mathcal{O}_i)$ with $\langle E \rangle = B_n(\Gamma)$

If no such i exists, define the hypergraph index to be ∞.

<u>Conjecture</u>: (B.) ① If $B_n(\Gamma)$ has hypergraph index k, then $B_n(\Gamma)$ is thick of order at most k
② If $B_n(\Gamma)$ has hypergraph index ∞, then $B_n(\Gamma)$ is relatively hyperbolic.

<u>Proof</u>: (Sketch) ① True by construction.

② <u>Claim</u>: $\mathcal{I}_i = \{\langle E\rangle \in \mathfrak{S} \mid E \in \mathcal{E}(\Theta_i)\}$ isolates orthogonality for some i

<u>Proof</u>: • By definition, every pair of orthogonal domains is nested into some $\langle E\rangle$ for $E \in \mathcal{E}(\Theta_0)$.
$\Rightarrow$ every pair of orthogonal domain is nested into some $\langle E\rangle$ for $E \in \mathcal{E}(\Theta_i)$
$\Rightarrow \mathcal{I}_i$ contains all orthogonality

• If necessary, remove any finite-diameter domain from $\mathfrak{S}$; this still gives an HHS structure.

• If all $\equiv_i$-equivalence classes are trivial, then $\langle E\rangle \cap \langle E'\rangle$ has finite diameter for all $E, E' \in \mathcal{E}(\Theta_i)$, thus no domain is nested into two hyperedges of Θ_i
$\Rightarrow \mathcal{I}_i$ isolates orthogonality

- If there is a non-trivial $\equiv_i$-equivalence class, then we get $E \in \mathcal{E}(\Theta_{i+1})$ which properly contains some hyperedge of Θ_i. But $|V(\Theta_0)| < \infty$. Thus either we eventually get isolated orthogonality, or we get $\langle E \rangle = B_n(\Gamma)$ (contradiction)

<u>Problem</u>: We do not know if $\langle E \rangle$ is a graphical subgroup.