Hierarchical hyperbolicity of graph products

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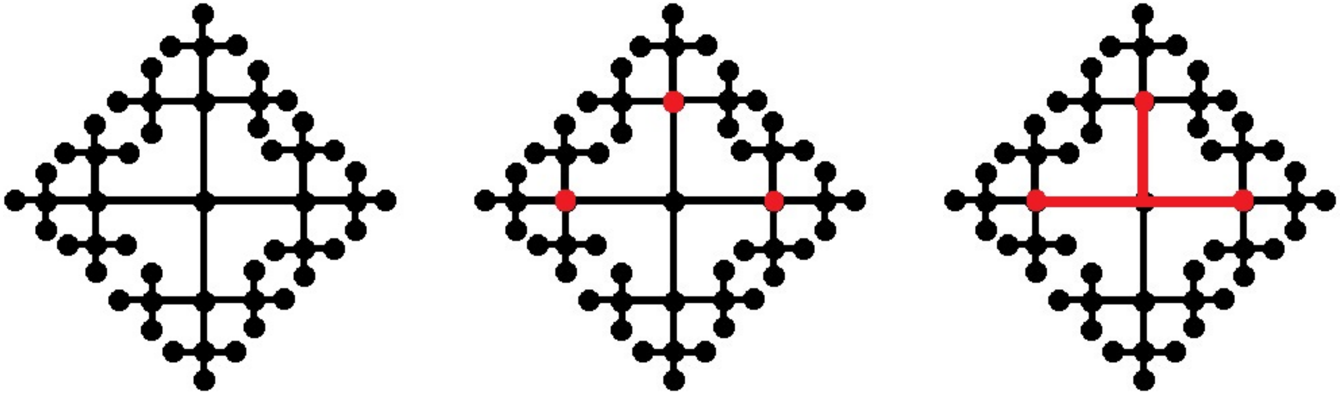
Hyperbolicity

Definition (Hyperbolicity)

A finitely generated group is δ–*hyperbolic* if all geodesic triangles in its Cayley graph are δ–thin.
i.e. Any side of a triangle is contained in the δ–neighbourhood of the other two sides.

e.g. $\mathbb{Z} * \mathbb{Z}$ is 0–hyperbolic.



Hyperbolicity

Generalisations of hyperbolicity:

- Relative hyperbolicity (Gromov, Farb, et al.)
- Hierarchical hyperbolicity (Behrstock–Hagen–Sisto)
  - Axiomatises Masur and Minsky's treatment of mapping class groups using subsurface projections and curve graphs.

e.g. (1) Mapping class groups (groups of homeomorphisms of surfaces up to isotopy)

(2) Most fundamental groups of 3–manifolds

(3) Groups acting properly and cocompactly on CAT(0) cube complexes, e.g. right-angled Artin groups (RAAGs)

(4) **Graph products** (B.–Russell)

Graph products

Definition (Graph product)

Let Γ be a finite simplicial graph with each vertex v labelled by a group G_v. Then the graph product G_Γ is the group

$$G_\Gamma = \left(\mathop{\ast}_{v \in V(\Gamma)} G_v \right) \Big/ \langle\langle [g_v, g_w] \,|\, g_v \in G_v, g_w \in G_w, \; \{v, w\} \; edge \rangle\rangle$$

If $G_v = \mathbb{Z}$ for all v, then this is called a *right-angled Artin group* (RAAG).

Graph products

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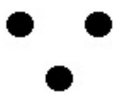
Γ				
G_Γ	F_3	$\mathbb{Z}^3$	$\mathbb{Z} * \mathbb{Z}^2$	$F_2 \times F_2$

Definition of HHG

encode hyperbolicity

"nice subgroups"

Let $X = Cay(G, S)$ for some finite generating set S. A *proto-hierarchy structure* on G consists of:

(1) A collection of geodesic spaces $\{CW\}_{W \in \mathfrak{S}}$ where $\mathfrak{S}$ is some index set.

partial order

encodes direct product subgroups

(2) Projections $\pi_W : X \to CW$.

(3) Nesting ($\sqsubseteq$), orthogonality ($\perp$) and ~~transversality ($\pitchfork$)~~ relations between elements of $\mathfrak{S}$.

We say X is (relatively) *hierarchically hyperbolic* if each CW is ($\sqsubseteq$-minimal or) hyperbolic and $\{CW\}_{W \in \mathfrak{S}}$ "captures precisely the geometry of X" via the projections and relations.

$d_X(x,y) \asymp \sum d_{CW}(\pi_W(x), \pi_W(y))$

Theorem (B.–Russell)

Graph products are relatively hierarchically hyperbolic groups.

Proto-hierarchy structure of G_Γ: index set and nesting

Each subgraph $\Lambda \subseteq \Gamma$ generates a subgroup $\langle \Lambda \rangle \leqslant G_\Gamma$ isomorphic to G_Λ. We call these the **graphical subgroups**.

- Index set: $\mathfrak{S} = \{g\langle \Lambda \rangle\}/\sim$ where $g\langle \Lambda \rangle \sim h\langle \Lambda \rangle$ iff $g\langle \Lambda \rangle$ and $h\langle \Lambda \rangle$ are parallel.
- Nesting: $g\langle \Lambda \rangle \sqsubseteq h\langle \Omega \rangle$ if $\Lambda \subseteq \Omega$ and there is some $k \in G_\Gamma$ such that $g\langle \Lambda \rangle$ is parallel to $k\langle \Lambda \rangle$ and $h\langle \Omega \rangle$ is parallel to $k\langle \Omega \rangle$.

Proto-hierarchy structure of G_Γ: orthogonality

Definition (Link, star, and join)

- The *link* of Λ, denoted $\mathrm{lk}(\Lambda)$, is the set of vertices of $\Gamma \setminus \Lambda$ which are connected to every vertex of Λ.
- The *star* of Λ, denoted $\mathrm{st}(\Lambda)$, is $\Lambda \cup \mathrm{lk}(\Lambda)$.
- We say Λ is a *join* if it can be written as $\Lambda = \Lambda_1 \sqcup \Lambda_2$ where every vertex of Λ_1 is connected to every vertex of Λ_2.

- A join subgraph of Γ generates a subgroup of G_Γ which splits as a direct product.
- $\langle \mathrm{st}(\Lambda) \rangle$ is the largest subgroup of G_Γ which splits as a direct product with $\langle \Lambda \rangle$ as one of the factors:
  $\langle \mathrm{st}(\Lambda) \rangle \cong \langle \Lambda \rangle \times \langle \mathrm{lk}(\Lambda) \rangle$. $\langle \Lambda \rangle \times \langle \Omega \rangle \leq G_\Gamma$
- Orthogonality: $g\langle \Lambda \rangle \perp h\langle \Omega \rangle$ if $\Omega \subseteq \mathrm{lk}(\Lambda)$ and there is $k \in G_\Gamma$ such that $g\langle \Lambda \rangle$ is parallel to $k\langle \Lambda \rangle$ and $h\langle \Omega \rangle$ is parallel to $k\langle \Omega \rangle$.

Proto-hierarchy structure: $C(g\langle\Lambda\rangle)$

Want to kill any geometry appearing in vertex groups.

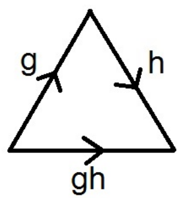
Definition (Syllable metric) [Kim-Koberda]

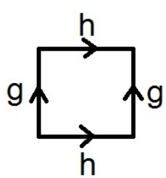
$S(\Gamma)$ is the graph where:

$S(\Gamma) = \mathrm{Cay}(G_\Gamma, \cup_v G_v)$

- Vertices are elements of G_Γ
- g, h are joined by an edge if $g^{-1}h \in G_v$ for some v.

Label edge between g and h by $g^{-1}h$.

- Triangles: 

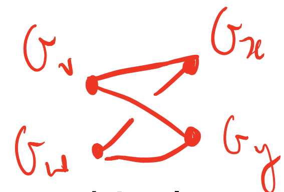
g, h in common vertex group
- Squares: 

g, h in adjacent vertex groups

Proto-hierarchy structure: $C(g\langle\Lambda\rangle)$

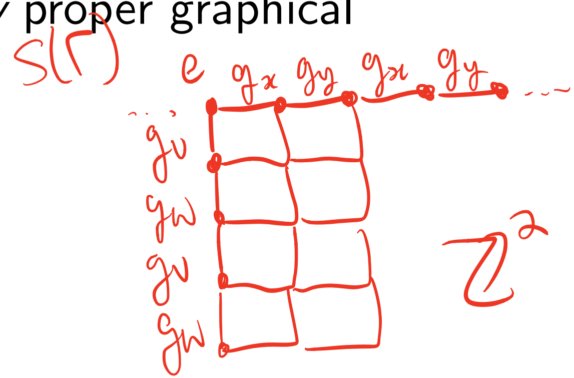
We denote distance in $S(\Gamma)$ by d_{syl} and call $\mathrm{d}_{syl}(g, h)$ the *syllable distance* between g and h.

Good news: $S(\Gamma)$ has a rich geometry similar to that of cube complexes, developed extensively by Genevois.



Bad news: $S(\Gamma)$ is still not hyperbolic.

Solution: Kill all geometry occurring in *any* proper graphical subgroup.



Proto-hierarchy structure: $C(g\langle\Lambda\rangle)$

word metric

Definition

Let $\Lambda \subseteq \Gamma$. If $\Lambda = v$, define $C(\langle\Lambda\rangle) = G_v$. Otherwise, $C(\langle\Lambda\rangle)$ is the graph where:

- Vertices are elements of $\langle\Lambda\rangle$
- g, h are joined by an edge if $g^{-1}h \in \langle\Omega\rangle$ for some strict subgraph $\Omega \subsetneq \Lambda$.

For each coset $g\langle\Lambda\rangle$ we can define $C(g\langle\Lambda\rangle)$ similarly.

Proto-hierarchy structure: $C(g\langle\Lambda\rangle)$

Theorem (B.-Russell)

For each $g\Lambda \in \mathfrak{S}$, either $g\Lambda$ is $\sqsubseteq$-minimal or $C(g\langle\Lambda\rangle)$ is $\frac{7}{2}$-hyperbolic.

Proof: (Sketch)

Projections

Genevois and Martin construct a *gate map* $\mathfrak{g}_\Lambda \colon S(\Gamma) \to \langle \Lambda \rangle$:

Definition

$\mathfrak{g}_\Lambda(x)$ is the longest initial subword of x contained in $\langle \Lambda \rangle$ (with respect to syllable length).

- $\mathfrak{g}_\Lambda(x)$ is the unique element of $\langle \Lambda \rangle$ such that $\mathrm{d}_{syl}(x, \mathfrak{g}_\Lambda(x)) = \mathrm{d}_{syl}(x, \langle \Lambda \rangle)$.
- This defines a nearest point projection.
- <u>Projections</u>: $G_\Gamma \hookrightarrow S(\Gamma) \xrightarrow{gate} \langle \Lambda \rangle \hookrightarrow C(\langle \Lambda \rangle)$

Consequences

Theorem (B.-Russell)

Graph products are hierarchically hyperbolic relative to their vertex groups.

Corollary

Any graph product of hierarchically hyperbolic groups is again a hierarchically hyperbolic group.

This is a strengthening of a result of Berlai and Robbio.

Corollary

Any graph product endowed with the syllable metric is hierarchically hyperbolic.