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Aerospace 523: Computational Fluid Dynamics
Homework: 4
Due: November 13^{th}, 2020

1 Burger's Equation

In this problem you will solve Burgers equation, $u_t + f_x = 0,\ f = \frac{1}{2}u^2,\ x \in [0, 4)$, periodic boundaries, with the initial condition shown below.

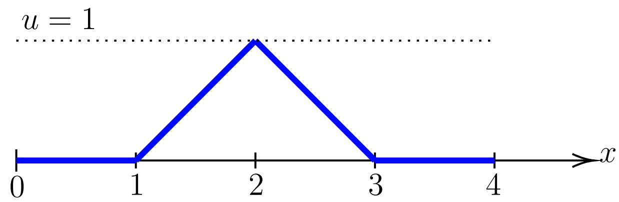

Figure 1: Initial condition to Burgers equation.

For the numerical method, use the finite volume method (FVM) with N_x uniform cells, forward Euler time stepping, a uniform time step, CFL=0.8 (based on the initial condition), and the upwind flux,

$$\hat{F}_{j+\frac{1}{2}} = \frac{1}{2}\left(f_j + f_{j+1}\right) - \frac{1}{2}|\hat{a}_{j+\frac{1}{2}}|\left(u_{j+1} - u_j\right)$$

a. Prior to implementing the FVM, determine the analytical solution using the method of characteristics. Plot the state, $u(x, t)$, at times $t = 0.5,\ 1.0,\ 1.5$ in one figure. In a separate figure, make a space-time diagram of the characteristics, up to $t = 1.5$, and indicate any shock speeds/paths.

b. Implement the FVM and using N_x= 128 and N_x= 512, show the states at the same times as requested in the previous part. Make three plots, one for each time, and overlay the two N_x results and the analytical solution on each plot. Comment on the differences.

c. Perform a convergence study of the FVM, using the L_2 solution error norm at $t = 0.5$, for $N_x = 128,\ 256,\ 512,\ 1024$. Include an error convergence plot and compute/discuss the rate.

2 Shock Tube

A shock tube consists of two chambers containing air ($\gamma = 1.4$) at different states, as shown below. The domain length is $L = 1$, and the diaphragm is in the middle, at $x = 0.5$.

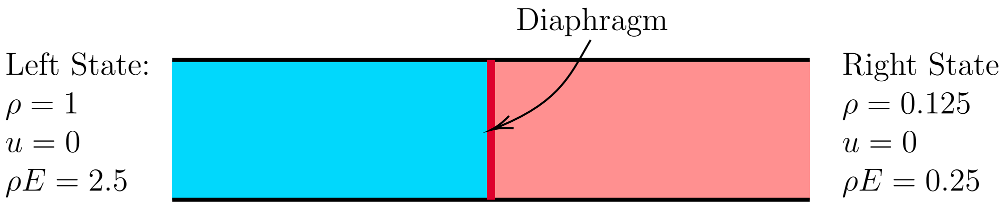

Figure 2: Shock tube chamber configuration.

At $t = 0$, the diaphragm between the two chambers is broken, sending a shock wave and a contact wave into one chamber and an expansion into the other, as shown schematically below for the density.

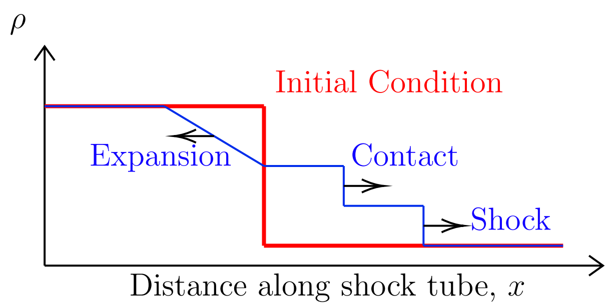

Figure 3: Evolution of unsteady evolution.

In this problem you will use a finite-volume method to simulate the unsteady evolution of the gas state in the shock tube (both chambers). The Euler equations govern the flow, and the units are conveniently chosen to give $\mathcal{O}(1)$ quantities.

a. Write a code that implements a first-order FVM on a uniform grid of N cells, with Dirichlet boundary conditions enforced using the flux function and a constant exterior state, and a constant time step estimated using the CFL condition. Implement both the Rusanov and the HLLE fluxes.Describe your code and ensure that you pass the free-stream preservation test.

b. Run your code to a final time of $T = 0.2$ with both fluxes, on grids of $N = 50,\ 100,\ 200$. Make figures showing the density, momentum, and velocity for each grid, overlaying the two flux function results on each figure (9 figures total). Discuss the behavior of the solution and the differences between the fluxes.