Project 2: Supersonic Engine Analysis

Aerospace 523: Computational Fluid Dynamics I
Undergraduate Aerospace Engineering
University of Michigan, Ann Arbor

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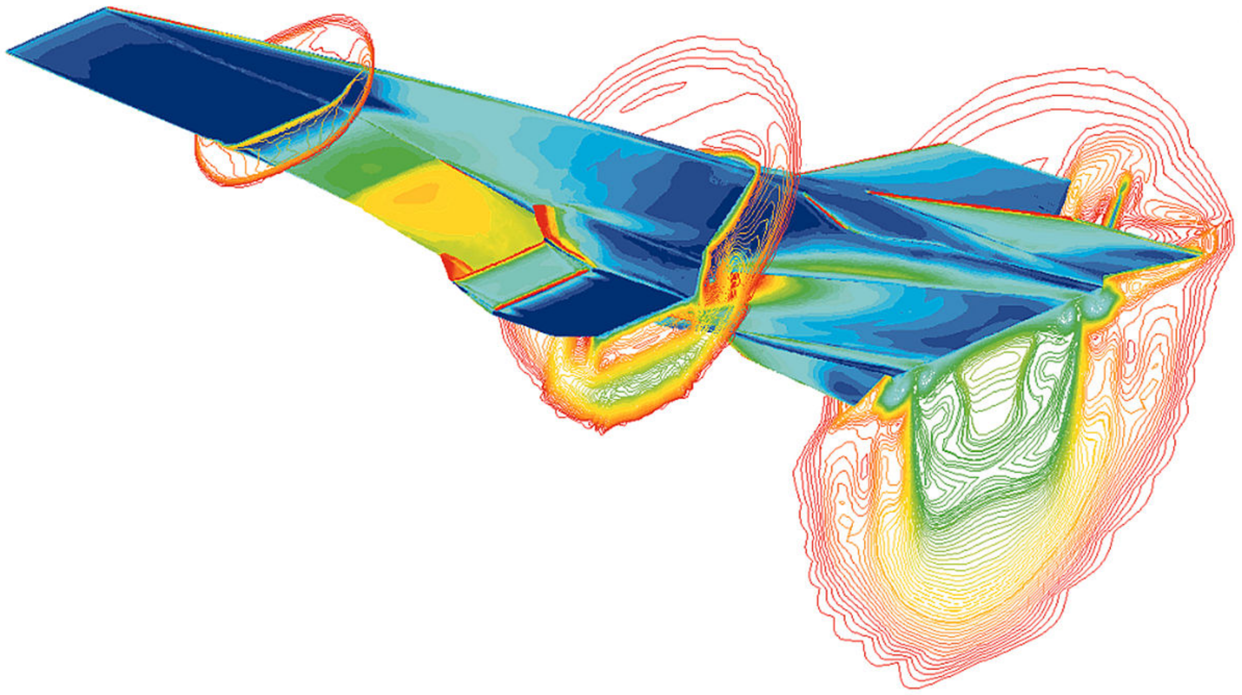

NASA X-43 Hypersonic Airplane

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1 Introduction

In this project you will simulate supersonic flow through a two-dimensional scramjet engine, using a first-order, adaptive, finite-volume method. Combustion will not be included, and your investigation will focus on measuring the total pressure recovery of the engine. The shock structure inside the engine is complex, and accurate simulations will require adapted meshes to resolve the shocks and expansions.

Geometry: Figure 1 shows the geometry of the engine, which consists of two sections: lower and upper. The reference length is the height of the engine channel at the exit, which is d = 1. Note that the units of the measurements are not relevant, as you will be reporting non-dimensional quantities.

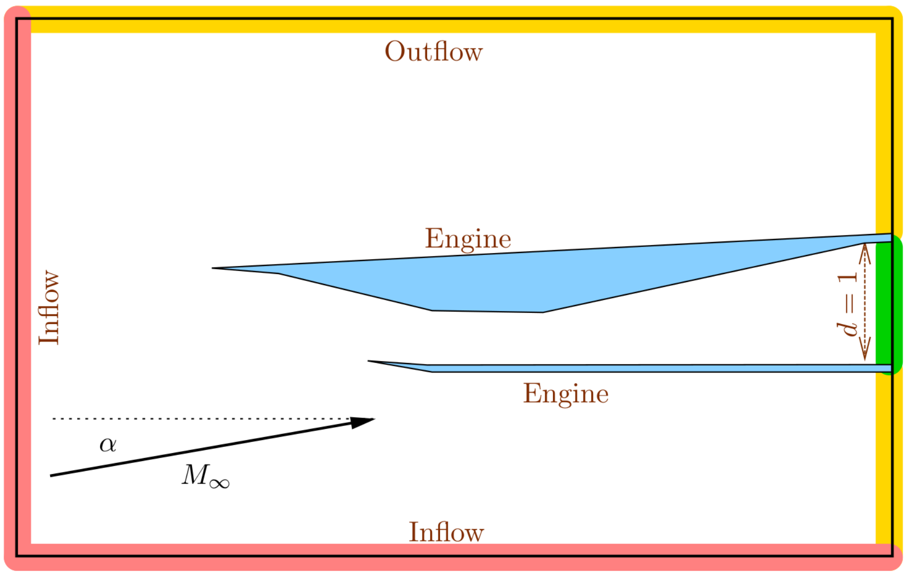

Figure 1: Engine geometry and boundary conditions.

Governing Equations: Use the two-dimensional Euler equations, with a ratio of specific heats of $\gamma = 1.4$.

Units: To avoid ill-conditioning, use "convenient" $\mathcal{O}(1)$ units for this problem, in which the freestream state is

$$\mathbf{u}_\infty = \left[\rho, \quad \rho u, \quad \rho v, \quad \rho E\right]^{\mathbf{T}} = \left[1, \quad M_\infty \cos(\alpha), \quad M_\infty \sin(\alpha), \quad \frac{1}{\gamma(\gamma-1)} + \frac{M_\infty^2}{2}\right]^{\mathbf{T}} \tag{1}$$

where M_∞ is the free-stream Mach number, and α is the angle of attack.

Initial and Boundary Conditions: The computational domain consists of the region around the engine. The inflow portion of the far-field rectangle consists of the left and bottom boundaries. On these boundaries apply free-stream "full-state" conditions, with a free-stream Mach number of $M_\infty = 2.2$. You will investigate angles of attack in the range $\alpha \in [0, 3°]$, with a baseline value of $\alpha = 1°$. On the outflow and engine exit boundaries, assume that the flow is supersonic, which means that no boundary state is needed – the flux is computed from the interior state. On the engine surface, apply the inviscid wall boundary condition.

When initializing the state in a new run, i.e. not when restarting from an existing state, you can set all cells to the same state, based on the free-stream Mach number, M_∞.

Output: Shocks inside the engine are necessary to slow the flow down and compress it for combustion, but they also lead to a loss in total pressure (lost work). A figure of merit is then the *average total pressure recovery* (ATPR), defined by an integral of the engine exit of the ratio of the total pressure to the freestream total pressure,

$$\text{ATPR} = \frac{1}{d}\int_0^d \frac{p_t}{p_{t,\infty}}\, dy, \quad p_t = p\left(1 + \frac{\gamma-1}{2}M^2\right)^{\gamma/(\gamma-1)}, \tag{2}$$

where p is the pressure, p_t is the total pressure, and y measures the vertical distance along the engine exit.

2 Numerical Method

Use the first-order finite volume methods to solve for the flow through the engine. March the solution to steady state using local time stepping, starting from either an initial uniform flow, or from an existing converged or partially-converged state.

Discretization: From the notes, cell i's average, $(\mathbf{u_i})$, evolves in time according to

$$A_i \frac{d\mathbf{u_i}}{dt} + \mathbf{R_i} = \mathbf{0} \rightarrow \frac{d\mathbf{u_i}}{dt} = -\frac{1}{A_i}\mathbf{R_i}. \tag{3}$$

where the flux residual $\mathbf{R_i}$ for a triangular cell is

$$\mathbf{R_i} = \sum_{e=1}^{3} \hat{\mathbf{F}}(\mathbf{u_i}, \mathbf{u_{N(i,e)}}, \vec{n}_{i,e})\Delta l_{i,e} \tag{4}$$

Recall that $N(i,e)$ is the cell adjacent to cell i across edge e, and $\vec{n}_{i,e}, \Delta l_{i,e}$ are the outward normal and length on edge e of cell i. Discretize Equation 3 with forward Euler time integration and use local time stepping to drive the solution to steady state.

Local Time Stepping: To implement local time stepping, a vector of time steps is calculated, one time step for each cell: Δt_i. Defining the CFL number for cell i as,

$$\text{CFL}_i = \frac{\Delta t_i}{2A_i}\sum_{e=1}^{3} |s|_{i,e}\Delta l_{i,e}, \tag{5}$$

where A_i is the area of the cell, the summation is over the three edges of a cell, and $|s|_{i,e}$ is the maximum propagation speed for edge e.

Time stepping requires the value of $\Delta t_i/A_i$ for each cell, and this can be calculated by re-arranging Equation 5,

$$\frac{\Delta t_i}{A_i} = \frac{2\text{CFL}_i}{\sum_{e=1}^{3} |s|_{i,e}\Delta l_{i.e}}. \tag{6}$$

The easiest method to calculate the right-hand-side is to calculate the summation of $|s|_e \Delta l_{i,e}$ during the flux evaluations. Note that the propagation speed $|s|_{i,e}$ should be calculated by the flux function. In local time stepping, the CFL number for each cell is the same: $\text{CFL}_i = \text{CFL} = 1.0$ is a good choice for this project.

Residuals and Convergences: Assess convergence by monitoring the undivided L_1 norm of the residual vector, defined as

$$|\mathbf{R}|_{L_1} = \sum_{\text{cells } i} \sum_{\text{states } k} |R_{i,k}| \tag{7}$$

That is, take the sum of the absolute values of all of the entries in your residual vector (you will be summing the 4 conservation equation residuals in each cell). You should not divide by the number of entries/cells, as the residuals already represent integrated quantities over the cells, so the sum will behave properly with mesh refinement. Deem a solution converged when $|\mathbf{R}|_{L_1} < 10^{-5}$.

Numerical Flux: Use the Roe flux for the interface flux and to impose the full-state far-field boundary condition. This flux is described in the course notes. You will need to verify your flux once implemented, using the following tests:

- Consistency check: $\mathbf{F}(\mathbf{u_L}, \mathbf{u_L}, \vec{n})$ should be the same as $\tilde{\mathbf{F}}(\mathbf{U_L}) \cdot \vec{n}$ (the analytical flux dotted with the normal).
- Flipping the direction: check that $\mathbf{F}(\mathbf{u_L}, \mathbf{u_R}, \vec{n}) = -\mathbf{F}(\mathbf{u_R}, \mathbf{u_L}, -\vec{n})$.
- States with supersonic normal velocity: the flux function should return the analytical flux from the upwind state. The downwind state should not have any effect on flux.

Time Stepping: Use the forward-Euler method to drive the solution to steady state. With local time-stepping, the update on cell i at iteration n can be written as

$$\mathbf{u}_i^{n+1} = \mathbf{u}_i^n - \frac{\Delta t_i^n}{A_i} \mathbf{R_i}(\mathbf{U^n}), \tag{8}$$

where Δt_i^n is the local time step computed from the state at time step n.

Mesh: You are provided with a baseline mesh of 1670 cells, shown in Figure 2. This mesh will not provide very accurate flow solutions, but it will serve as the starting point for adaptation. The included `readme.txt` file describes the structure of the text-based `.gri` mesh file. You are also given python and Matlab codes for reading the `.gri` mesh file and for plotting/processing the mesh.

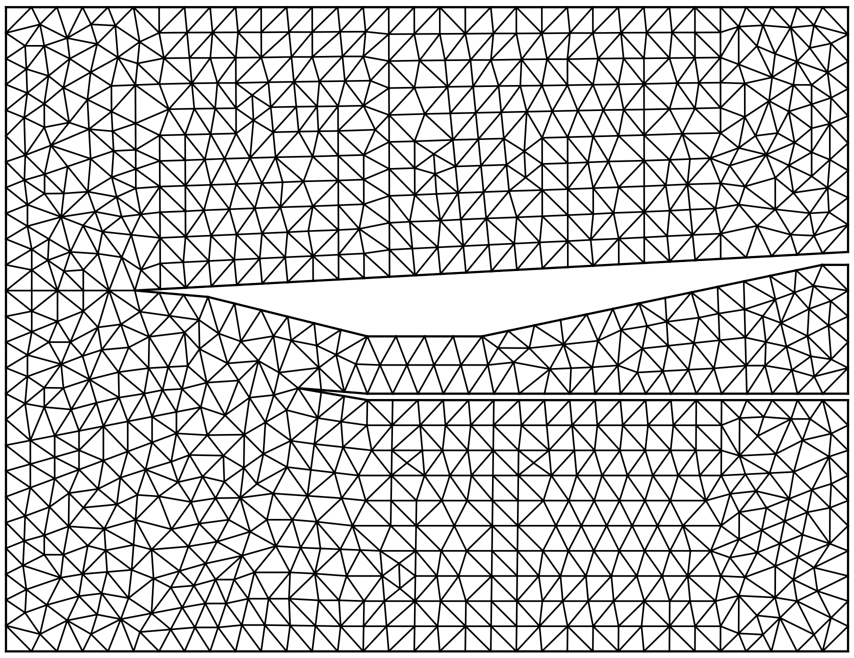

Figure 2: Scramjet baseline mesh.

Output Calculation: The average total pressure recovery output in Equation 2 requires an integral over the engine exit. Approximate this integral by summing over the edges on the exit boundary. For each edge, use the state from the adjacent cell to calculate the total pressure.

3 Adaptation

You will use mesh adaptation to improve solution quality. Adapting a mesh means locally increasing the mesh resolution in regions where errors are likely to be large. This requires a measurement of error and a method for adapting the mesh. A reasonable way to measure error is to look at jumps in the solution between cells. For example, looking at jumps in the Mach number, we can define an error indicator for each interior edge e according to

$$\text{interior: } \epsilon_e = |M_{k^+} - M_{k^-}| h_e.$$

In this formula, M_{k^+} and M_{k^-} are the Mach numbers on the two cells adjacent to edge e, and h_e is the length of edge e.

You can assume that the error indicator on the farfield boundary edges is zero. On the engine boundary (solid wall), define the error indicator by

$$\text{wall: } \epsilon_e = |M_k^{\perp}| h_e,$$

where $M_k^{\perp}$ is the Mach number of the cell's velocity component in the edge normal direction.

After calculating the error indicators ϵ_e over all edges (interior and boundary), sort the indicators in decreasing order and flag a small fraction $f = .03$ of edges with the highest error for refinement. Next, to smooth out the refinement pattern, loop over all cells: if a cell has *any* of its edges flagged for refinement, then flag *all* of its edges for refinement. This will increase the total number of edges for refinement.

Once edges are flagged as described, refine all cells adjacent to flagged edges. These cells will fall into one of three categories, shown in Figure 3, and they should be refined as indicated. At each adaptive iteration, transfer the solution to the new mesh to provide a good initial guess for the next solve.

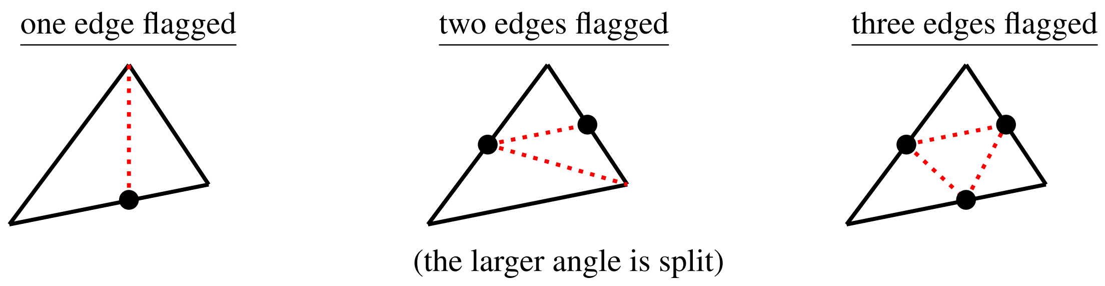

Figure 3: Refinement of triangles given edge splittings.

4 Tasks and Deliverables

In preparation for simulating the scramjet engine inlet performance I will prepare code that will implement Roe Flux to approximate the changing flow state between cells. After verification that the flux is correctly implemented then I will implement a first-order finite volume method to approximate the steady state solution and perform a convergence study on my method. Additionally, I will model Mach number jumps throughout the domain and determine the the averaged total pressure recovery. Finally, I will perform adaptive iterations to then determine the effects of the angle of attack and the averaged total pressure recovery.

4.1 Roe Flux Overview

Roe flux, is an alternative flux that carefully upwinds waves one by one and is given by Equation 9 below. [1]

$$\hat{\mathbf{F}} = \frac{1}{2}\left(\mathbf{F_L} + \mathbf{F_R}\right) - \frac{1}{2}\left|\frac{\partial \mathbf{F}}{\partial \mathbf{u}}(\mathbf{u}^*)\right|\left(\mathbf{u_R} - \mathbf{u_L}\right) \tag{9}$$

In this expression $\left|\frac{\partial \mathbf{F}}{\partial \mathbf{u}}(\mathbf{u}^*)\right|$ refers to the absolute values of the eigenvalues, i.e. $\mathbf{R}|\mathbf{\Lambda}|\mathbf{L}$, in the eigenvalue decomposition. $\mathbf{u}^*$ is an intermediate state that is based on $\mathbf{u_L}$ and $\mathbf{u_R}$. This intermediate choice is important for nonlinear problems, and the Roe flux uses the Roe-average state, a choice that yields exact single-wave solutions to the Riemann problem. However, for Euler equations Roe flux is given by Equation 10 below.

$$\hat{\mathbf{F}} = \frac{1}{2}(\mathbf{F_L} + \mathbf{F_R}) - \frac{1}{2}\begin{bmatrix} |\lambda|_3\Delta\rho + C_1 \\ |\lambda|_3\Delta(\rho\vec{v}) + C_1\vec{v} + C_2\hat{n} \\ |\lambda|_3\Delta(\rho E) + C_1 H + C_2(\vec{v}\cdot\hat{n}) \end{bmatrix} \tag{10}$$

Where further expansions of the constants above give,

$$C_1 = \frac{G_1}{c^2}(s_1 - |\lambda|_3) + \frac{G_2}{c}s_2, \qquad C_2 = \frac{G_1}{c}s_2 + (s_1 - |\lambda|_3)G_2$$

$$G_1 = (\gamma - 1)\left(\frac{q^2}{2}\Delta\rho - \vec{v}\cdot\Delta(\rho\vec{v}) + \Delta(\rho E)\right), \qquad G_2 = -(\vec{v}\cdot\hat{n})\Delta\rho + \Delta(\rho\vec{v})\cdot\hat{n}$$

$$s_1 = \frac{1}{2}\left(|\lambda|_1 + |\lambda|_2\right), \qquad s_2 = \frac{1}{2}(|\lambda|_1 - |\lambda|_2)$$

$$\Delta\mathbf{u} = \mathbf{u_R} - \mathbf{u_L}$$

$$\mathbf{F_L} = \tilde{\mathbf{F}}(\mathbf{u_L})\cdot\hat{n}, \qquad \mathbf{F_R} = \tilde{\mathbf{F}}(\mathbf{u_R})\cdot\hat{n}$$

However, to prevent expantion shocks, an entropy fix is required. The simple solution to this is to keep all eigenvalues away from zero such that,

$$\text{if } |\lambda|_i < \epsilon \text{ then } \lambda_i = \frac{\epsilon^2 + \lambda_i^2}{2\epsilon}, \quad \forall\ i\ \in\ [1,\ 4]$$

Where ϵ is a small fraction of the Roe-averaged speed of sound, e.g. $\epsilon = 0.1c$

4.1.1 Roe Flux Function

In this project I will implement Roe Flux into Python3 that will be further implemented when writing the finite-volume method to determine the flow through the scramjet. Essentially this function is as follows:

Inputs This function inputs the left state and the right state of a given edge. This will allow the finite-volume method solver to simply call this function when determining the fluxes in and out of a given cell. Furthermore, this function will also input the normal vector and a logical on whether to return test cases or not.

Generating Arguments Going further, this code then will determine the states of the left and right side such as $\rho,\ u,\ v,\ P,\ H$ to determine the flux and approximate the Roe-average state. With the left and right hand fluxes determined what's left is the Roe-averages.

Roe-Average Determining the Roe-average is done by passing all the calculated values into a separate subfunction that will determine the Roe-averages from a weighted averaged of the densities to the state properties. Additionally in this function it will calculate the wave propagating eigenvalues to remove discontinuities from the calculation.

Final Calculation Then with the Roe-Average and the fluxes determined, simply conducted the average of the fluxes subtracted by half the sum of the running waves.[2]

4.1.2 Subsonic and Supersonic Implementation Tests

Consistency Check: First and foremost is a simple check to see if the Roe flux at steady state is equal to the flux of a single state vector acting in the same direction of the normal. In this I simply returned the values in Python3 and tabulated the results in order to check the consistency. In this test I assumed $\alpha = 0°$, $M_\infty = 2.2$, $\vec{n} = \begin{bmatrix}1, & 0\end{bmatrix}$ and used this initial state for u_l. Performing the consistency check I get Table 1 below aligning with theory.

Table 1: Roe Flux consistency check

Flux	ρ	ρu	ρv	ρE
$F(u_l, u_l, \vec{n})$	2.200	5.554	0.000	10.824
$\vec{F}(\vec{U}_l) \cdot \vec{n}$	2.200	5.554	0.000	10.824
ΔF	0.00e+00	0.00e+00	0.00e+00	0.00e+00

Direction Flipping Next is to check that there is agreement with flipping the states and the norm vector and returning the same results without error. In this test case I will assume that the left state will be $\alpha = 0°$, $M_\infty = 2.2$, $\vec{n} = \begin{bmatrix}1, & 0\end{bmatrix}$ initially and for the right state the same but with $M_\infty = 2.4$ initially. Tabulating the results gives Table 2 below.

Table 2: Roe Flux flipped direction check

Flux	ρ	ρu	ρv	ρE
$F(u_l, u_r, \vec{n})$	2.250	5.964	-0.230	11.346
$-F(u_r, u_l, -\vec{n})$	2.250	5.964	-0.230	11.346
ΔF	0.00e+00	0.00e+00	0.00e+00	0.00e+00

Supersonic Normal Velocity Shown and confirmed above, my Roe Flux implementation passes this test. This function returns the analytical flux from the upwind state and the downwind state does not have any effect on the flux.

4.2 Implementing Finite Volume Method

[25%] Implement a first-order finite volume method to obtain a steady-state flow solution on a given mesh. The code should:

- read and process the mesh files,
- iterate until the residual norm is less than the specified tolerance,
- calculate the average total pressure recovery output,
- monitor and log the residual norm and output convergence

You may also wish to equip your code with the ability to write restart files at the end of the simulation and periodically during a run. In your report, describe your code structure, algorithms, and data structures.

4.3 Convergences

[15%] Run your code on the baseline mesh at $\alpha = 1^\circ$, starting with a uniform freestream state at this α. Perform the following post-processing:

- Plot the convergence of the given residual L_1 norm versus time step iterations.
- Plot the convergence of the ATPR output versus time step iterations.
- For the converged solution, make two field plots: one of the Mach number, and one of the total pressure.

4.4 Implementing Mach Number Jumps

[25%] Implement the mesh adaptation algorithm based on the Mach number jumps. Run your algorithm for $\alpha = 1^\circ$ with at least 5 adaptive iterations, and perform the following post-processing:

- Plot the sequence of your adapted meshes.
- Plot the Mach number and total pressure fields on your finest mesh.
- Plot the ATPR output versus number of cells in the mesh – one data point per adaptive iteration.

Discuss the results, including areas targeted for adaption and the convergence of the output.

4.5 Adaptive Iterations

[15%] Perform adaptive iterations for $\alpha = [0.5, 1, 1.5, 2, 2.5, 3]$ degrees. Run the same number of adaptive iterations for each α at least 5. Plot the ATPR output from your finest mesh versus alpha, and discuss the trend. Include flowfield plots to augment your discussion.

Appendices

A Additional Supporting Code

Algorithm 1: Python Edge Hash

```
import numpy as np
from scipy import sparse


#-----------------------------------------------------------
# Identifies interior and boundary edges given element-to-node
# IE contains (n1, n2, elem1, elem2) for each interior edge
# BE contains (n1, n2, elem) for each boundary edge
def edgehash(E, B):
    Ne = E.shape[0]; Nn = np.amax(E)+1
    H = sparse.lil_matrix((Nn, Nn), dtype=np.int)
    IE = np.zeros([int(np.ceil(Ne*1.5)),4], dtype=np.int)
    ni = 0
    for e in range(Ne):
        for i in range(3):
            n1, n2 = E[e,i], E[e,(i+1)%3]
            if (H[n2,n1] == 0):
                H[n1,n2] = e+1
            else:
                eR = H[n2,n1]-1
                IE[ni,:] = n1, n2, e, eR
                H[n2,n1] = 0
                ni += 1
    IE = IE[0:ni,:]
    # boundaries
    nb0 = nb = 0
    for g in range(len(B)): nb0 += B[g].shape[0]
    BE = np.zeros([nb0,4], dtype=np.int)
    for g in range(len(B)):
        Bi = B[g]
        for b in range(Bi.shape[0]):
            n1, n2 = Bi[b,0], Bi[b,1]
            if (H[n1,n2] == 0): n1,n2 = n2,n1
            BE[nb,:] = n1, n2, H[n1,n2]-1, g
            nb += 1
    return IE, BE
```

Algorithm 2: Python Plot Mesh

```
import numpy as np
import matplotlib.pyplot as plt
from readgri import readgri

#-----------------------------------------------------------
def plotmesh(Mesh, fname):
    V = Mesh['V']; E = Mesh['E']; BE = Mesh['BE']

    f = plt.figure(figsize=(12,12))
    #plt.tripcolor(V[:,0], V[:,1], triangles=E)
    plt.triplot(V[:,0], V[:,1], E, 'k-')
    for i in range(BE.shape[0]):
        plt.plot(V[BE[i,0:2],0],V[BE[i,0:2],1], '-', linewidth=2, color='black')
    dosave = not not fname
    plt.axis('equal')
    plt.axis('off')
    plt.tick_params(axis='both', labelsize=12)
    f.tight_layout(); plt.show(block=(not dosave))
    if (dosave): plt.savefig(fname, bbox_inches='tight')
    plt.close(f)

#-----------------------------------------------------------
def main():
    Mesh = readgri('mesh0.gri')
    plotmesh(Mesh, [])

if __name__ == "__main__":
    main()
```

Algorithm 3: Python Read Grid

```python
import numpy as np
from scipy import sparse

#-----------------------------------------------------------
# Identifies interior and boundary edges given element-to-node
# IE contains (n1, n2, elem1, elem2) for each interior edge
# BE contains (n1, n2, elem, bgroup) for each boundary edge
def edgehash(E, B):
    Ne = E.shape[0]; Nn = np.amax(E)+1
    H = sparse.lil_matrix((Nn, Nn), dtype=np.int)
    IE = np.zeros([int(np.ceil(Ne*1.5)),4], dtype=np.int)
    ni = 0
    for e in range(Ne):
        for i in range(3):
            n1, n2 = E[e,i], E[e,(i+1)%3]
            if (H[n2,n1] == 0):
                H[n1,n2] = e+1
            else:
                eR = H[n2,n1]-1
                IE[ni,:] = n1, n2, e, eR
                H[n2,n1] = 0
                ni += 1
    IE = IE[0:ni,:]
    # boundaries
    nb0 = nb = 0
    for g in range(len(B)): nb0 += B[g].shape[0]
    BE = np.zeros([nb0,4], dtype=np.int)
    for g in range(len(B)):
        Bi = B[g]
        for b in range(Bi.shape[0]):
            n1, n2 = Bi[b,0], Bi[b,1]
            if (H[n1,n2] == 0): n1,n2 = n2,n1
            BE[nb,:] = n1, n2, H[n1,n2]-1, g
            nb += 1
    return IE, BE

#-----------------------------------------------------------
def readgri(fname):
    f = open(fname, 'r')
    Nn, Ne, dim = [int(s) for s in f.readline().split()]
    # read vertices
    V = np.array([[float(s) for s in f.readline().split()] for n in range(Nn)])
    # read boundaries
    NB = int(f.readline())
    B = []; Bname = []
    for i in range(NB):
        s = f.readline().split(); Nb = int(s[0]); Bname.append(s[2])
        Bi = np.array([[int(s)-1 for s in f.readline().split()] for n in range(Nb)])
        B.append(Bi)
    # read elements
    Ne0 = 0; E = []
    while (Ne0 < Ne):
        s = f.readline().split(); ne = int(s[0])
        Ei = np.array([[int(s)-1 for s in f.readline().split()] for n in range(ne)])
        E = Ei if (Ne0==0) else np.concatenate((E,Ei), axis=0)
        Ne0 += ne
    f.close()
    # make IE, BE structures
    IE, BE = edgehash(E, B)
    Mesh = {'V':V, 'E':E, 'IE':IE, 'BE':BE, 'Bname':Bname }
    return Mesh

#-----------------------------------------------------------
def writegri(Mesh, fname):
    V = Mesh['V']; E = Mesh['E']; BE = Mesh['BE']; Bname = Mesh['Bname'];
    Nv, Ne, Nb = V.shape[0], E.shape[0], BE.shape[0]
    f = open(fname, 'w')
    f.write('%d␣%d␣2\n'%(Nv, Ne))
    for i in range(Nv):
        f.write('%.15e␣%.15e\n'%(V[i,0], V[i,1]));
    nbg = 0
    for i in range(Nb): nbg = max(nbg, BE[i,3])
    nbg += 1
    f.write('%d\n'%(nbg))
    for g in range(nbg):
        nb = 0
        for i in range(Nb): nb += (BE[i,3] == g)
        f.write('%d 2 %s\n'%(nb, Bname[g]))
        for i in range(Nb):
            if (BE[i,3]==g): f.write('%d %d\n'%(BE[i,0]+1, BE[i,1]+1))
    f.write('%d 1 TriLagrange\n'%(Ne))
    for i in range(Ne):
        f.write('%d %d %d\n'%(E[i,0]+1, E[i,1]+1, E[i,2]+1))
    f.close()


#-----------------------------------------------------------
def main():
    Mesh = readgri('xflow/v2/capsule.gri')
    writegri(Mesh, 'xflow/v2/test.gri')

if __name__ == "__main__":
    main()
```

B Roe Flux Python Implementation

Algorithm 4: Roe Flux Implementation

```
import numpy as np

def RoeFlux(Ul, Ur, n, return_test):
    gam = 1.4

    # Left-Side arguments
    rhol = Ul[0]; ul = Ul[1]/rhol; vl = Ul[2]/rhol
    pl = (gam - 1)*(Ul[3] - 0.5*rhol*(ul**2 + vl**2))
    Hl = (Ul[3] + pl)/rhol

    # Right-Side arguments
    rhor = Ur[0]; ur = Ur[1]/rhor; vr = Ur[2]/rhor;
    pr = (gam - 1)*(Ur[3] - 0.5*rhor*(ur**2 + vr**2))
    Hr = (Ur[3] + pl)/rhor

    # Left and Right side fluxes
    FL = np.array([np.dot([rhol*ul, rhol*vl], n),np.dot([rhol*ul**2 + pl, rhol*ul*vl
        ], n),np.dot([rhol*ul*vl, rhol*vl**2 + pl], n),np.dot([rhol*ul*Hl, rhol*vl*
        Hl], n)])
    FR = np.array([np.dot([rhor*ur, rhor*vr], n),np.dot([rhor*ur**2 + pr, rhor*ur*vr
        ], n),np.dot([rhor*ur*vr, rhor*vr**2 + pr], n),np.dot([rhor*ur*Hr, rhor*vr*
        Hr], n)])

    RHS = ROE_Avg(ul,vl,ur,vr, rhol, rhor, Hl, Hr, pr, pl)
    F = 0.5*(FL + FR) - 0.5*RHS

    if return_test:
        return F, FL, FR
    else:
        return F

def ROE_Avg(ul,vl,ur,vr, rhol, rhor, Hl, Hr, Pr, Pl):
    vell = np.sqrt(ul**2 + vl**2); velr = np.sqrt(ur**2 + vr**2)
    url = (np.sqrt(rhol)*vell + np.sqrt(rhor)*velr)/(np.sqrt(rhol) + np.sqrt(rhor))
    hrl = (np.sqrt(rhol)*Hl + np.sqrt(rhor)*Hr)/(np.sqrt(rhol) + np.sqrt(rhor))
    rhorl = np.sqrt(rhor*rhol)
    arl = np.sqrt((1.4 - 1.0)*(hrl - 0.5*url**2))

    ls = np.array([url, url+arl, url-arl])
    ws = np.array([rhor-rhol - (Pr-Pl)/arl**2, ur-ul + (Pr-Pl)/(rhorl*arl), ur-ul -
        (Pr-Pl)/(rhorl*arl)])
    es = np.array([[1, 1, 1], [1, 1, 1], [url, url + arl, url-arl], [1/2*url**2, hrl
        +url*arl, hrl - url*arl]])
    es[:,1] *= rhorl/(2*arl); es[:,2] *= -rhorl/(2*arl)

    RHS = 0
    for i in range(3):
        if abs(ls[i]) < 0.1*arl:
            phil = (ls[i]**2 + (0.1*arl)**2)/(2*0.1*arl)
        else:
            phil = abs(ls[i])
        RHS += 0.5*phil*es[:,i]*ws[i]

    return RHS
```

References

[1] K. Fidkowski, “Computational fluid dynamics,” September 2020.

[2] Gryphon, “Roe flux differencing scheme: The approximate riemann problem.”