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Aerospace 523: Computational Fluid Dynamics
Homework: 3
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1 A-Stable Backwards Difference

In the BDF methods, the time derivative is approximated using one-sided finite differences. The BDF2 method is A-stable, whereas BDF3 is not. Consider a multi-step method in which the time derivative is approximated by the average of the BDF2 and BDF3 time-derivative approximations:

$$\frac{du}{dt} = f \rightarrow \frac{1}{2}\frac{du}{dt}|_{\text{BDF2}} + \frac{1}{2}\frac{du}{dt}|_{\text{BDF3}} = f$$

a. Determine the coefficients α_k and β_k that define this method. What is its order of accuracy?

BDF2:

The expression for the derivative can be expressed as,

$$u_t = \frac{\frac{3}{2}u^{n+1} - 2u^n + \frac{1}{2}u^{n-1}}{\Delta t} = f^{n+1}$$

BDF3:

The expression for the derivative can be expressed as,

$$u_t = \frac{\frac{11}{6}u^{n+1} - 3u^n + \frac{3}{2}u^{n-1} - \frac{1}{3}u^{n-2}}{\Delta t} = f^{n+1}$$

Averaging BDF2 and BDF3 together will give the time-derivative approximation,

$$\Delta t f^{n+1} = \frac{1}{2}\underbrace{\left(\frac{3}{2}u^{n+1} - 2u^n + \frac{1}{2}u^{n-1}\right)}_{\text{BDF2}} + \frac{1}{2}\underbrace{\left(\frac{11}{6}u^{n+1} - 3u^n + \frac{3}{2}u^{n-1} - \frac{1}{3}u^{n-2}\right)}_{\text{BDF3}}$$

Combining like terms results in,

$$\frac{5}{3}u^{n+1} - \frac{5}{2}u^n + u^{n-1} - \frac{1}{6}u^{n-2} = \Delta t f^{n+1}$$

This results in the coefficients α and β to be,

$$\boxed{\alpha_1 = \frac{5}{3}, \quad \alpha_0 = -\frac{5}{2}, \quad \alpha_{-1} = 1, \quad \alpha_{-2} = -\frac{1}{6}, \quad \beta_1 = 1}$$

Order of Accuracy

Firstly, is to start with the Taylor-Series expansion expression,

$$u^{n+k} = u^n + (k\Delta t)u_t^n + \frac{1}{2}(k\Delta t)^2 u_{tt}^n + \frac{1}{6}(k\Delta t)^3 u_{ttt}^n + \frac{1}{24}(k\Delta t)^4 u_{t^{(4)}}^n + \dots \mathcal{O}(\Delta t^5)$$

$$f^{n+k} = u_t^{n+k} = u_t^n + (k\Delta t)u_{tt}^n + \frac{1}{2}(k\Delta t)^2 u_{ttt}^n + \frac{1}{6}(k\Delta t)^3 u_{t^{(4)}} + \frac{1}{24}(k\Delta t)^4 u_{t^{(5)}}^n + \dots \mathcal{O}(\Delta t^5)$$

Conducting Taylor-Expansions:

$$
\begin{aligned}
u^{n+1} &= u^n + \Delta t u_t^n + \frac{1}{2}\Delta t^2 u_{tt}^n + \frac{1}{6}\Delta t^3 u_{ttt}^n + \frac{1}{24}\Delta t^4 u_{t^{(4)}}^n + \dots \mathcal{O}(\Delta t^5) \\
u^n &= u^n \\
u^{n-1} &= u^n - \Delta t u_t^n + \frac{1}{2}\Delta t^2 u_{tt}^n - \frac{1}{6}\Delta t^3 u_{ttt}^n + \frac{1}{24}\Delta t^4 u_{t^{(4)}}^n + \dots \mathcal{O}(\Delta t^5) \\
u^{n-2} &= u^n - 2\Delta t u_t^n + 2\Delta t^2 u_{tt}^n - \frac{4}{3}\Delta t^3 u_{ttt}^n + \frac{2}{3}\Delta t^4 u_{t^{(4)}}^n + \dots \mathcal{O}(\Delta t^5) \\
f^{n+1} &= u_t^n + \Delta t u_{tt}^n + \frac{1}{2}\Delta t^2 u_{ttt}^n + \frac{1}{6}\Delta t^3 u_{t^{(4)}}^n + \frac{1}{24}\Delta t^4 u_{t^{(5)}}^n + \mathcal{O}(\Delta t^5)
\end{aligned}
$$

Then for the order of accuracy the error gives,

$$
\begin{aligned}
\epsilon^{n+1} &= \frac{5}{3}u^{n+1} - \frac{5}{2}u^n + u^{n-1} - \frac{1}{6}u^{n-2} - \Delta t f^{n+1} \\
&= \frac{5}{3}\left(u^n + \Delta t u_t^n + \frac{1}{2}\Delta t^2 u_{tt}^n + \frac{1}{6}\Delta t^3 u_{ttt}^n + \frac{1}{24}\Delta t^4 u_{t^{(4)}}^n\right) + \dots \\
&- \frac{5}{2}u^n + \dots \\
&+ \left(u^n - \Delta t u_t^n + \frac{1}{2}\Delta t^2 u_{tt}^n - \frac{1}{6}\Delta t^3 u_{ttt}^n + \frac{1}{24}\Delta t^4 u_{t^{(4)}}^n\right) + \dots \\
&- \frac{1}{6}\left(u^n - 2\Delta t u_t^n + 2\Delta t^2 u_{tt}^n - \frac{4}{3}\Delta t^3 u_{ttt}^n + \frac{2}{3}\Delta t^4 u_{t^{(4)}}^n\right) + \dots \\
&- \Delta t\left(u_t^n + \Delta t u_{tt}^n + \frac{1}{2}\Delta t^2 u_{ttt}^n + \frac{1}{6}\Delta t^3 u_{t^{(4)}}^n + \frac{1}{24}\Delta t^4 u_{t^{(5)}}^n\right)
\end{aligned}
$$

Then using Matlab to simplify gives that the error is,

$$
\epsilon^{n+1} = -\frac{1}{6}\Delta t^3 u_{ttt} - \frac{1}{6}\Delta t^4 u_{t^{(4)}} + \frac{1}{40}\Delta t^5 u_{t^{(5)}}
$$

Then from the leading term this gives that the convergence is,

$$
|\epsilon^{n+1}| = \mathcal{O}(\Delta t^{p+1}) = \mathcal{O}(\Delta t^3)
$$

This gives that the order of accuracy is,

$$
\boxed{p = 2}
$$

Since p = 2, the order of accuracy for this scheme is second-order accurate.

b. Perform an eigenvalue-stability analysis and *prove* (analytically) that this method is A-stable. Plot its stability boundary in the $\lambda\Delta t$ complex number plane, and overlay BDF2 and BDF3.

Starting with the expression for this averaged time-derivative,

$$\Delta t f^{n+1} = \frac{5}{3}u^{n+1} - \frac{5}{2}u^{n} + u^{n-1} - \frac{1}{6}u^{n-2}$$

Then from here substituting in $g^{n+k}u_0$ for u^{n+k} and $\lambda g^{n+k}u_0$ for f^{n+k},

$$\lambda\Delta t g^{n+1}u_0 = \frac{5}{3}g^{n+1}u_0 - \frac{5}{2}g^{n}u_0 + g^{n-1}u_0 - \frac{1}{6}g^{n-2}u_0$$

Then taking this expression and dividing by $g^n u_0$ results in,

$$\lambda\Delta t g = \frac{5}{3}g - \frac{5}{2} + g^{-1} - \frac{1}{6}g^{-2}$$

Isolating the $\lambda\Delta t$ term then results in,

$$\lambda\Delta t = \frac{5}{3} - \frac{5}{2}g^{-1} + g^{-2} - \frac{1}{6}g^{-3}$$

Further simplifications without loss of generality gives,

$$\lambda\Delta t = \frac{5}{3} + \frac{1}{6g^3}\left(-15g^2 + 6g - 1\right)$$

Then by definition, this scheme must be stable if the un-stable region (the regions *inside* the marked plots do not extend into the left-hand plan). In this limiting case, this can be re-written as the limit as $\lambda\Delta t \to 0^-$ and solve for the θ value at which this occurs,

$$\lim_{\lambda\Delta t\to 0} = 0 = \frac{5}{3} + \frac{1}{6g^3}\left(-15g^2 + 6g - 1\right)$$

Taking this further, isolating and solving for g gives

$$\lim_{\lambda\Delta t\to 0} = -10g^3 = -15g^2 + 6g - 1$$

Pulling all g terms to one side results in,

$$0 = 10g^3 - 15g^2 + 6g - 1$$

Conducting simple factorization gives,

$$0 = (g-1)\left(10g^2 - 5g + 1\right)$$

Using quadratic formula this gives that g is equivalent to,

$$g = 1, \frac{5 \pm i\sqrt{15}}{20}$$

Solving for the values at which these occurs gives,

$$g = \frac{5 \pm i\sqrt{15}}{20} = \exp[i\theta] = \cos\theta + i\sin\theta$$
$$\theta = 1.1513i \pm 0.6591$$

Again for $g = 1$,

$$g = 1 = \exp[i\theta] = \cos\theta + i\sin\theta$$
$$\theta = 0, \quad \textbf{Physical answer}$$

As shown above, there are three approximated answers in which this averaged time-derivative scheme will cross into the unstable region. However, two of these three are not physical answers as $\theta \in [0, 2\pi] \mid \theta \in \mathbb{R}$ $\therefore$ the limiting case occurs at $\theta = 0$ where $g = 1$ resulting in $\lambda\Delta t = \frac{5}{3} - \frac{5}{3} = 0$ − resulting in an <u>A-stable scheme.</u>

Plotting these eigenvalue stability regions can show and confirm that $\theta = 0$ is the limiting case and for the averaged time-derivative that it is indeed A-stable as the unstable region never crosses into the left-hand plane like in BDF3 scheme. Plotting the un-stable regions gives Figure 1 shown below,

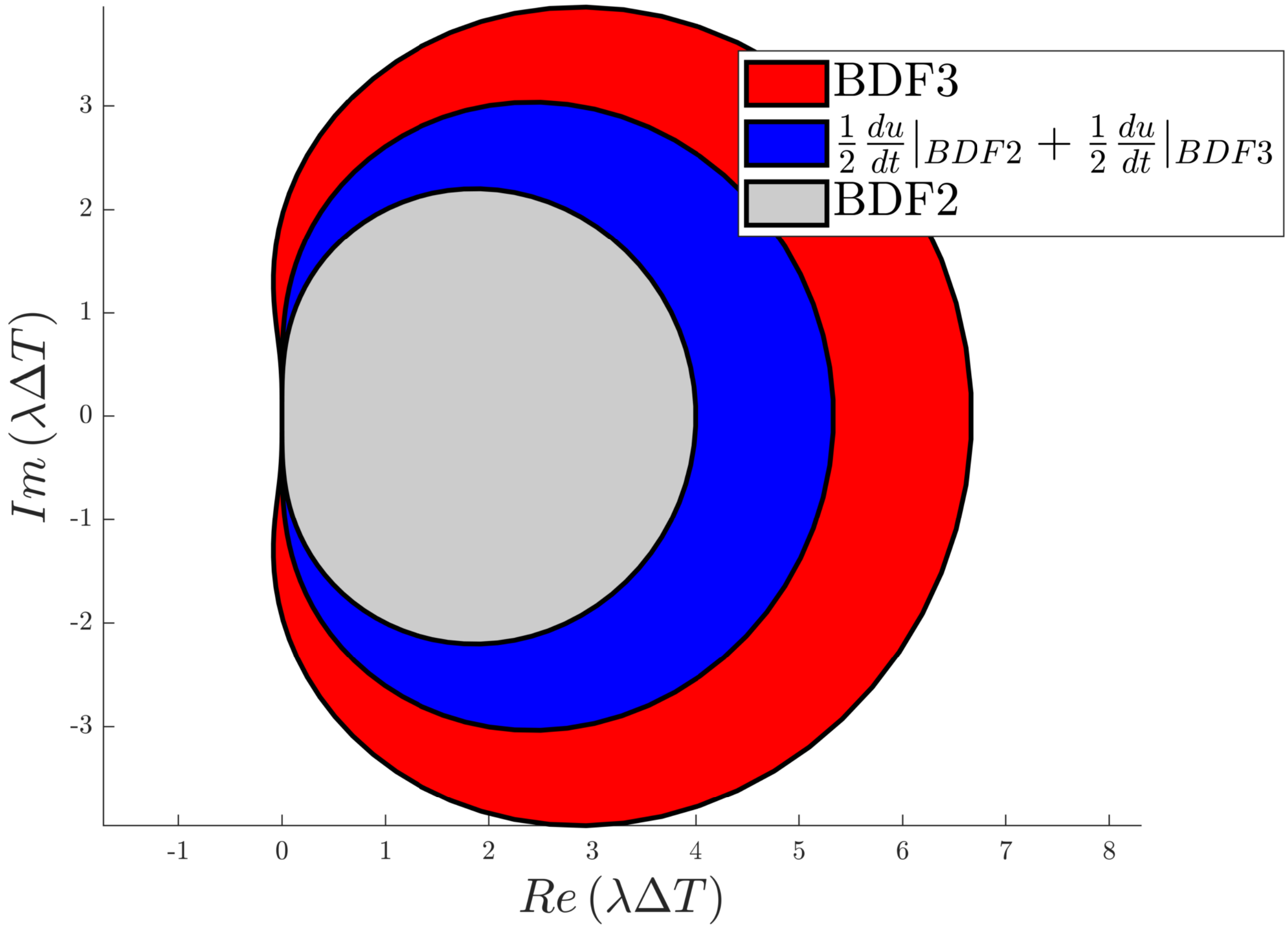

Figure 1: Eigenvalue stability region for BDF2, BDF3, and the averaged time-derivative approximation.

Shown above in Figure 1 are the un-stable regions for BDF2, BDF3, and the averaged time-derivative of the two. Outside of these regions the schemes remain stable where the left-hand plane where Re($\lambda\mathbf{\Delta T}$) $<$ 0 is the A-stable region of the schemes.

c. Calculate the temporal truncation error of this method, $\tau = \text{LHS} - \text{RHS}$ of the multistep formula, and show that the leading term is half the magnitude of that of BDF2.

From part a. of this question, I found that the local error was,

$$\epsilon^{n+1} = -\frac{1}{6}\Delta t^3 u_{ttt} - \frac{1}{6}\Delta t^4 u_{t(4)} + \frac{1}{40}\Delta t^5 u_{t(5)}$$

Thus, the truncation error is

$$\epsilon^{n+1} = \underbrace{-\frac{1}{6}\Delta t^3 u_{ttt} - \frac{1}{6}\Delta t^4 u_{t(4)} + \frac{1}{40}\Delta t^5 u_{t(5)} + \ldots \mathcal{O}(\Delta t^6)}_{\text{truncation error: } \mathcal{O}(\Delta t^3)}$$

However, proving that the leading term is half the magnitude of that of BDF2, I will use the LHS and RHS definitions from BDF2 and use the Taylor-Series expansions from part a. and simplify as,

$$\begin{aligned}
\text{LHS} &= \frac{3}{2}u^{n+1} - 2u^n + \frac{1}{2}u^{n-1} \\
&= \frac{3}{2}\left(u^n + \Delta t u_t^n + \frac{1}{2}\Delta t^2 u_{tt}^n + \frac{1}{6}\Delta t^3 u_{ttt}^n + \frac{1}{24}\Delta t^4 u_{t(4)}^n + \frac{1}{120}\Delta t^5 u_{t(5)}^n\right) + \ldots \\
&- 2u^n + \ldots \\
&+ \frac{1}{2}\left(u^n - \Delta t u_t^n + \frac{1}{2}\Delta t^2 u_{tt}^n - \frac{1}{6}\Delta t^3 u_{ttt}^n + \frac{1}{24}\Delta t^4 u_{t(4)}^n - \frac{1}{120}\Delta t^5 u_{t(5)}^n\right) \\
&= \Delta t u_t^n + \Delta t^2 u_{tt}^n + \frac{1}{6}\Delta t^3 u_{ttt}^n + \frac{1}{12}\Delta t^4 u_{t(4)}^n + \frac{1}{120}\Delta t^5 u_{t(5)}^n \\
\text{RHS} &= \Delta t f^{n+1} \\
&= \Delta t\left(u_t^n + \Delta t u_{tt}^n + \Delta t^2 u_{ttt}^n + \Delta t^3 u_{t(4)}^n + \Delta t^4 u_{t(5)}^n\right)
\end{aligned}$$

Taking the difference between the two gives,

$$\text{LHS} - \text{RHS} = \left(\frac{1}{6} - \frac{1}{2}\right)\Delta t^3 u_{ttt}^n + \left(\frac{1}{12} - \frac{1}{6}\right)\Delta t^4 u_{t(4)}^n + \left(\frac{1}{120} - \frac{1}{24}\right)\Delta t^5 u_{t(5)}^n$$

$$\tau = \text{LHS} - \text{RHS} = -\frac{1}{3}\Delta t^3 u_{ttt}^n - \frac{1}{12}u_{t(4)}^n - \frac{1}{30}\Delta t^5 u_{t(5)}^n$$

Then re-writing both truncation errors gives,

$$\begin{aligned}
\tau_{BDF2} &= -\frac{1}{3}\Delta t^3 u_{ttt}^n - \frac{1}{12}u_{t(4)}^n - \frac{1}{30}\Delta t^5 u_{t(5)}^n \\
\tau_{\text{Avg}} &= -\frac{1}{6}\Delta t^3 u_{ttt} - \frac{1}{6}\Delta t^4 u_{t(4)} + \frac{1}{40}\Delta t^5 u_{t(5)}
\end{aligned}$$

By inspection of the truncation errors above, we see that the leading term for the averaged time-derivative is indeed half that of the BDF2 scheme.

2 The Beam-Warming Method

Consider the Beam-Warming (BW) method applied to the one-dimensional advecton equation, $u_t + au_x = 0,\ a > 0$, with initial condition $u(x,0) = u_0(x),\ x \in [0, L]$ and periodic boundaries.

a. Derive the modified equation for the BW method and express it in the form

$$u_t + au_x = \alpha u_{xx} - \beta u_{xxx}$$

Use this equation to determine the order of accuracy of the BW method, and discuss the dispersion relation.

Starting with the modified equation for Beam-Warming method,

$$u_j^{n+1} = u_j^n - \frac{\sigma}{2}\left(3u_j^n - u_{j-1}^n + u_{j-2}^n\right) + \frac{\sigma^2}{2}\left(u_{j-2}^n - 2u_{j-1}^n + u_j^n\right)$$

Conducting the Taylor series expansions for these nodes gives,

$$\begin{aligned}
u_{j-2}^n &= u_j^n - 2\Delta x u_x + 2\Delta x^2 u_{xx} - \frac{4}{3}\Delta x^3 u_{xxx} + \frac{2}{3}\Delta x^4 u_{x^{(4)}} + \ldots \mathcal{O}(\Delta x^5) \\
u_{j-1}^n &= u_j^n - \Delta x u_x + \frac{1}{2}\Delta x^2 u_{xx} - \frac{1}{6}\Delta x^3 u_{xxx} + \frac{1}{24}\Delta x^4 u_{x^{(4)}} + \ldots \mathcal{O}(\Delta x^5) \\
u_j^{n+1} &= u_j^n + \Delta t u_t + \frac{1}{2}\Delta t^2 u_{tt} + \frac{1}{6}\Delta t^3 u_{ttt} + \frac{1}{24}\Delta t^4 u_{t^{(4)}} + \ldots \mathcal{O}(\Delta t^5)
\end{aligned}$$

Expanding the right-hand side of the expression I get,

$$\text{RHS} = u_j^n - \frac{\sigma}{2}\left(3u_j^n - 4u_{j-1}^n + u_{j-2}^n\right) + \frac{\sigma^2}{2}\left(u_{j-2}^n - 2u_{j-1}^n + u_j^n\right)$$

Expressing each quantity I get,

$$\begin{aligned}
3u_j^n - 4u_{j-1}^n + u_{j-2}^n &= 3u_j^n + \ldots \\
&\quad - 4\left(u_j^n - \Delta x u_x + \frac{1}{2}\Delta x^2 u_{xx} - \frac{1}{6}\Delta x^3 u_{xxx} + \frac{1}{24}\Delta x^4 u_{x^{(4)}} + \ldots \mathcal{O}(\Delta x^5)\right) + \ldots \\
&\quad + u_j^n - 2\Delta x u_x + 2\Delta x^2 u_{xx} - \frac{4}{3}\Delta x^3 u_{xxx} + \frac{2}{3}\Delta x^4 u_{x^{(4)}} + \ldots \mathcal{O}(\Delta x^5) \\
&= 2\Delta x u_x - \frac{2}{3}\Delta x^3 u_{xxx} + \frac{1}{2}\Delta x^4 u_{x^{(4)}} + \mathcal{O}(\Delta x^5) \\
u_{j-2}^n - 2u_{j-1}^n + u_j^n &= u_j^n - 2\Delta x u_x + 2\Delta x^2 u_{xx} - \frac{4}{3}\Delta x^3 u_{xxx} + \frac{2}{3}\Delta x^4 u_{x^{(4)}} + \ldots \mathcal{O}(\Delta x^5) + \ldots \\
&\quad - 2\left(u_j^n - \Delta x u_x + \frac{1}{2}\Delta x^2 u_{xx} - \frac{1}{6}\Delta x^3 u_{xxx} + \frac{1}{24}\Delta x^4 u_{x^{(4)}} + \ldots \mathcal{O}(\Delta x^5)\right) + \ldots \\
&\quad + u_j^n \\
&= \Delta x^2 u_{xx} - \Delta x^3 u_{xxx} + \frac{7}{12}\Delta x^4 u_{x^{(4)}} - \frac{1}{4}\Delta x^5 u_{x^{(5)}} + \mathcal{O}(\Delta x^6)
\end{aligned}$$

Setting up the relationships I get that,

$$\begin{aligned}
u_j^n + \Delta t u_t + \frac{1}{2}\Delta t^2 u_{tt} + \frac{1}{6}\Delta t^3 u_{ttt} + \frac{1}{24}\Delta t^4 u_{t^{(4)}} + \ldots \mathcal{O}(\Delta t^5) &= u_j^n - \ldots \\
&\quad \frac{\sigma}{2}\left(2\Delta x u_x - \frac{2}{3}\Delta x^3 u_{xxx} + \frac{1}{2}\Delta x^4 u_{x^{(4)}} + \mathcal{O}(\Delta x^5)\right) + \ldots \\
&\quad \frac{\sigma^2}{2}\left(\Delta x^2 u_{xx} - \Delta x^3 u_{xxx} + \frac{7}{12}\Delta x^4 u_{x^{(4)}} - \frac{1}{4}\Delta x^5 u_{x^{(5)}} + \mathcal{O}(\Delta x^6)\right) \\
&= u_j^n - \sigma\Delta x u_x + \frac{\sigma^2}{2}\Delta x^2 u_{xx} - \frac{1}{6}(3\sigma^2 - 2\sigma)\Delta x^3 u_{xxx} + \frac{1}{24}\left(7\sigma^2 - 6\sigma\right)\Delta x^4 u_{x^{(4)}}
\end{aligned}$$

Starting with subtracting the u_j^n terms and expanding σ gives,

$$\begin{aligned}\Delta t u_t + \frac{1}{2}\Delta t^2 u_{tt} + \frac{1}{6}\Delta t^3 u_{ttt} + \frac{1}{24}\Delta t^4 u_{t^{(4)}} + \dots \mathcal{O}(\Delta t^5) \\ = \frac{a\Delta t}{\Delta x}\left(-\Delta x u_x + \frac{\sigma}{2}\Delta x^2 u_{xx} - \frac{1}{6}(3\sigma - 2)\Delta x^3 u_{xxx} + \frac{1}{24}\left(7\sigma - 6\right)\Delta x^4 u_{x^{(4)}}\right)\end{aligned}$$

From here I will simplify by dividing through by Δt and distributing Δx,

$$\begin{aligned}u_t + \frac{1}{2}\Delta t u_{tt} + \frac{1}{6}\Delta t^2 u_{ttt} + \frac{1}{24}\Delta t^3 u_{t^{(4)}} + \dots \mathcal{O}(\Delta t^4) \\ = a\left(-u_x + \frac{\sigma}{2}\Delta x u_{xx} + \frac{a}{6}(3\sigma - 2)\Delta x^2 u_{xxx} + \frac{1}{24}\left(7\sigma - 6\right)\Delta x^4 u_{x^{(4)}}\right)\end{aligned}$$

Collecting the one-dimensional advection term to the same side,

$$\begin{aligned}u_t + a u_x &= -\frac{1}{2}\Delta t u_{tt} - \frac{1}{6}\Delta t^2 u_{ttt} - \frac{1}{24}\Delta t^3 u_{t^{(4)}} + \frac{\sigma a}{2}\Delta x u_{xx} + \dots \\ &+ \frac{a}{6}(3\sigma - 2)\Delta x^2 u_{xxx} + \frac{1}{24}\left(7\sigma - 6\right)\Delta x^4 u_{x^{(4)}}\end{aligned}$$

Now with the expression for the one-dimensional advection solved for, I will relate temporal derivatives to spatial indices by conducting expansions,

$$\begin{aligned}u_{tt} &= -\frac{1}{2}\Delta t u_{ttt} - a u_{xt} + \frac{\sigma a}{2}\Delta x u_{xxt} + \mathcal{O}(\Delta x^2, \Delta t^2) \\ u_{tx} &= -\frac{1}{2}\Delta t u_{ttx} - a u_{xx} + \frac{\sigma a}{2}\Delta x u_{xxx} + \mathcal{O}(\Delta x^2, \Delta t^2) \\ u_{ttt} &= -a u_{xtt} + \mathcal{O}(\Delta x, \Delta t) \\ u_{txx} &= -a u_{xxx} + \mathcal{O}(\Delta x, \Delta t) \\ u_{ttx} &= -a u_{xxt}\end{aligned}$$

With the higher mixed-derivatives solved for, backtracking will find the α and β coefficients,

$$\begin{aligned}u_{txx} &= -a u_{xxx} \\ u_{ttx} &= a^2 u_{xxx} \\ u_{ttt} &= -a^3 u_{xxx} \\ u_{tx} &= -\frac{1}{2}\Delta t a^2 u_{xxx} - a u_{xx} + \frac{\sigma a}{2}\Delta x u_{xxx} \\ &= -a u_{xx} + \cancelto{0}{\left(\frac{\sigma a}{2}\Delta x - \frac{a^2\Delta t}{2}\right)} u_{xxx} = -a u_{xx} \\ u_{tt} &= \frac{1}{2}\Delta t(a^3 u_{xxx}) + a^2 u_{xx} - \frac{\sigma a^2}{2}\Delta x u_{xxx} = a^2 u_{xx} \\ u_t + a u_x &= -\frac{1}{2}\Delta t a^2 u_{xx} + \frac{1}{6}\Delta t^2 a^3 u_{xxx} + \frac{\sigma a}{2}\Delta x u_{xx} + \frac{a}{6}(3\sigma - 2)\Delta x^2 u_{xxx} + \mathcal{O}(\Delta x^3, \Delta t^3) \\ &= \underbrace{\cancelto{0}{\left(-\frac{1}{2}\Delta t a^2 + \frac{\sigma a}{2}\Delta x\right)}}_{\alpha} u_{xx} + \underbrace{\left(\frac{1}{6}\Delta t^2 a^3 + \frac{a}{6}(3\sigma - 2)\Delta x^2\right)}_{-\beta} u_{xxx} \\ &= 0 \cdot u_{xx} + a\left(\frac{1}{6}\frac{\Delta x^2}{\Delta x^2}\Delta t^2 a^2 + \frac{a}{6}(3\sigma - 2)\Delta x^2\right) \\ &= 0 \cdot u_{xx} + a\left(\frac{\Delta x^2}{6}\sigma^2 + \frac{a}{6}(3\sigma - 2)\Delta x^2\right) u_{xxx}\end{aligned}$$

After further simplifications,

$$u_t + au_x = 0 \cdot u_{xx} + \frac{a\Delta x^2}{6}\left(\sigma^2 - 3\sigma + 2\right) u_{xxx}$$

This gives that the α and β expressions are,

$$\boxed{\alpha = 0, \quad \beta = -\frac{a\Delta x^2}{6}(\sigma^2 - 3\sigma + 2)}$$

Looking above to the dispersion (the coefficient β) will denote how waves of different frequencies will move at different speeds. This dispersion term will be the cause of oscillations where they were not present before.

b. Perform a von-Neumann stability analysis of the Beam-Warming method. What is the stability limit for the CFL number σ?

In order to complete the von-Neumann stability analysis, first look to the β coefficient to be greater than or equal to zero as the limiting case,

$$\sigma^2 - 3\sigma + 2 \geq 0$$

Performing simple factorization,

$$(\sigma - 2)(\sigma - 1) \geq 0$$

This gives the roots to be,

$$\sigma = 1,\ 2$$

Since these are the roots, and this is a concave-up parabolic function then from $\sigma \in [1, 2] < 0$ thus the actual limits for the CFL number σ as,

$$\boxed{\sigma \in [0, 1] \cup [2, \infty)}$$

c. Implement the BW method in a computer program using $L = 2,\ a = 0.5,\ u_0(x) = exp[-100(x/L - 0.5)^2]$ and a final time of $T = L/a$ (1 period). Perform spatial and temporal convergence studies to demonstrate the order of accuracy in space and time.

3 Manufactured Solutions

Discretize the one-dimensional advection-diffusion equation, $u_t + au_x - \nu u_{xx} = 0$, using the trapezoidal method in time and second-order central differences in space. Assume a grid of length L, periodic boundaries, and N spatial intervals.

a. Write a computer program that implements the given method, and run a simulation using the parameters given in problem 2c, $\nu = 0.1,\ N = 64$, and a CFL number of $\sigma = 0.5$. Plot the state at the final time, $u(x, T)$.

b. Apply the method of manufactured solutions to your discretization. The PDE will now need a source term: $u_t + au_x - \nu u_{xx} = s(x, t)$. Derive the form of $s(x, t)$ for the manufactured solution $u^{MS} = \sin(kx - \omega t)$, with $k = 4\pi/L$ and $\omega = 0.5a/L$.

c. Implement the method of manufactured solutions in your discretization, and present the solution at $t = T = L/a$ for $N = 64,\ \sigma = 0.5$.

d. Using the manufactured solution, perform spatial and temporal convergence studies of your discretization and verify that the orders of accuracy match your expectations.

Matlab Code for A-Stable Backwards Difference

Algorithm 1: Matlab Code for determining A-Stable backwards differences.

```
%~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
clear all; clc; close all
set(groot,'defaulttextinterpreter','latex');
set(groot, 'defaultAxesTickLabelInterpreter','latex');
set(groot, 'defaultLegendInterpreter','latex');
%~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% a
syms u ut utt uttt utttt uttttt dt

unp1 = u + dt*ut + 1/2*dt^2*utt + 1/6*dt^3*uttt + 1/24*dt^4*utttt + 1/120*dt^5*uttttt;
un = u;
unm1 = u - dt*ut + 1/2*dt^2*utt - 1/6*dt^3*uttt + 1/24*dt^4*utttt + 1/120*dt^5*uttttt;
unm2 = u - 2*dt*ut + 2*dt^2*utt - 4/3*dt^3*uttt + 2/3*dt^4*utttt + (-2)^5/120*dt^5*uttttt;
fnp1 = ut + dt*utt + 1/2*dt^2*uttt + 1/6*dt^3*utttt + 1/24*dt^4*uttttt;

epsi = 5/3*unp1 - 5/2*un + unm1 - 1/6*unm2 - dt*fnp1;
pretty(simplify(epsi))
%~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% b
num = 100;
bdf2 = zeros(num, 1);
bdf3 = zeros(num, 1);
avg_der = zeros(num, 1);
thetlin = linspace(0, 2*pi, num);
g = exp(thetlin.*1i);
syms ldt
for i = 1:num
    eqn = g(i) == 4/3 - 1/(3*g(i)) + 2/3*ldt*g(i);
    sol = double(solve(eqn, ldt));
    bdf2(i) = sol;

    eqn = g(i) == 18/11 -9/(11*g(i)) + 2/(11*g(i)^2) + 6/11*ldt*g(i);
    sol = double(solve(eqn, ldt));
    bdf3(i) = sol;

    eqn = 5/3*g(i) - 5/2 + g(i)^(-1) - 1/6*g(i)^(-2) == ldt*g(i);
    sol = double(solve(eqn, ldt));
    avg_der(i) = sol;
end

figure()
hold on
fill(real(bdf3), imag(bdf3),[1,1,1],'facealpha', 1, 'FaceColor',[1,0,0],'EdgeColor','k','
    linewidth',1.8)
fill(real(avg_der), imag(avg_der),[1,1,1],'facealpha', 1, 'FaceColor',[0,0,1],'EdgeColor','k','
    linewidth',1.8)
fill(real(bdf2), imag(bdf2),[1,1,1],'facealpha', 1, 'FaceColor',[0.8,0.8,0.8],'EdgeColor','k','
    linewidth',1.8)
xlim([-2.5, 0])
axis equal
xlabel('$Re \left( \lambda \Delta T \right)$','fontsize', 18)
ylabel('$Im \left( \lambda \Delta T \right)$','fontsize', 18)
legend({'BDF3','$\frac{1}{2}\frac{du}{dt}|_{BDF2} + \frac{1}{2}\frac{du}{dt}|_{BDF3}$','BDF2'}, '
    fontsize', 18, 'location', 'best', 'interpreter', 'latex')
set(gcf, 'Color', 'w', 'Position', [100 100 1000 500]);
export_fig('eigs.eps')

syms ldt theta
g = exp(1i*theta);
eqn = 0 == 5/3 - 5/2*g^-1 + g^-2 - 1/6*g^-3;
sol = double(solve(eqn, theta));
```