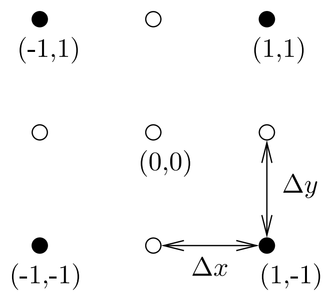
<u>General homework note</u>: *Please upload your solutions as one* .pdf *document. Scans of hand-written work are acceptable, but please use an app that produces legible documents if using your phone or tablet. Supporting codes should be uploaded separately as a* .zip *archive of source code files.*

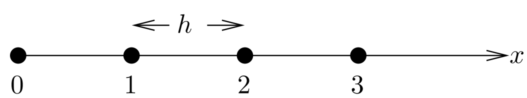
1 Lagrange interpolation [20%]

Use the method of *Lagrange interpolation* to derive an expression for the mixed derivative, u_{xy}, at the center of the stencil pictured below, i.e. point (0,0), using the values of u only at the four corner nodes in a uniform grid. Note, this problem is solved using the method of undetermined coefficients in the notes.



2 Truncation error analysis [20%]

Use the method of undetermined coefficients to derive a finite-difference approximation to the second derivative of u at point 0, $u_{xx}|_0$, using u values at points 0,1,2 on a uniform grid. Repeat this with the addition of u at point 3. In both cases, determine the order of accuracy of your formula.



3 Domain mapping [20%]

A reference-to-global coordinate mapping $\vec{x}(\vec{\xi})$ is given by

$$\begin{aligned} x &= \xi + \frac{1}{2}\eta^2, \\ y &= 1 + \eta - \frac{1}{4}\xi\eta, \end{aligned}$$

where $\vec{\xi} = (\xi, \eta)$ and $\vec{x} = (x, y)$. Consider a unit square domain in reference space, $\vec{\xi} \in [0, 1]^2$.

a) Show what this domain looks like in global space by providing a computer-generated plot of the domain edges.

b) Determine an analytical expression for the 2×2 mapping Jacobian matrix, $\mathbf{J} \equiv \frac{\partial \vec{x}}{\partial \vec{\xi}}$, and its determinant, $J \equiv \det(\mathbf{J})$.

c) Calculate the normal vectors at the four domain edge midpoints (where the midpoint is identified in reference space).

d) Calculate the area of the domain in global space by an analytical integration over the reference unit square.

4 Area calculation [20%]

In the last problem of homework 1, you identified loops of nodes using edges. You should have found that these loops enclosed a computational domain surrounding a three-element airfoil. Moreover, the edges were constructed such that the computational domain is always on the left as an edge is traversed from the first node to the second. This corresponds to a clockwise ordering for the inner loops, and a counter-clockwise ordering for the outer loop. In this problem, your task is to compute the area of that computational domain. Do this by a numerical (midpoint) integration along the edges, after applying the divergence theorem for a vector function $\vec{F} = x\hat{\imath}$. Repeat with $\vec{F} = y\hat{\jmath}$ and show that you obtain the same answer.

5 Advection-diffusion [20%]

Apply the finite-difference method to solve the steady advection-diffusion equation with a source term,

$$u_x - \nu u_{xx} = \sin(\pi x), \qquad u(0) = u(1) = 0.$$

Write a code to solve this problem on a uniform grid of N intervals. Use central difference formulas for both the first and second derivatives. Using $N = 16$, plot the solution for $\nu = 0.1$ and $\nu = 1$ on one figure. For $\nu = 0.1$, run simulations using $N = 4, 8, 16, 32$, and plot these on a single figure. Comment on the effect of ν and N on the solution. Also state whether and why the solution makes sense physically, based on your understanding of advection-diffusion.