General homework note: *Please upload your solutions as one* .pdf *document. Scans of hand-written work are acceptable, but please use an app that produces legible documents if using your phone or tablet. Supporting codes should be uploaded separately as a* .zip *archive of source code files.*

1 A-Stable Backwards Difference [30%]

In the BDF methods, the time derivative is approximated using one-sided finite differences. The BDF2 method is A-stable, whereas BDF3 is not. Consider a multi-step method in which the time derivative is approximated by the average of the BDF2 and BDF3 time-derivative approximations:

$$\frac{du}{dt} = f \quad \Rightarrow \quad \frac{1}{2}\frac{du}{dt}\bigg|_{\text{BDF2}} + \frac{1}{2}\frac{du}{dt}\bigg|_{\text{BDF3}} = f$$

a) Determine the coefficients α_k and β_k that define this method. What is its order of accuracy?

b) Perform an eigenvalue-stability analysis and *prove* (analytically) that this method is A-stable. Plot its stability boundary in the $\lambda\Delta t$ complex number plane, and overlay BDF2 and BDF3.

c) Calculate the temporal truncation error of this method, $\tau = \text{LHS}-\text{RHS}$ of the multistep formula, and show that the leading term is half the magnitude of that of BDF2.

2 The Beam-Warming Method [35%]

Consider the Beam-Warming (BW) method applied to the one-dimensional advection equation, $u_t + au_x = 0$, $a > 0$, with initial condition $u(x, 0) = u_0(x)$, $x \in [0, L]$ and periodic boundaries.

a) Derive the modified equation for the BW method and express it in the form

$$u_t + au_x = \alpha u_{xx} - \beta u_{xxx}.$$

Use this equation to determine the order of accuracy of the BW method, and discuss the dispersion relation.

b) Perform a von-Neumann stability analysis of the Beam-Warming method. What is the stability limit for the CFL number σ?

c) Implement the BW method in a computer program using $L = 2$, $a = 0.5$, $u_0(x) = \exp[-100(x/L-0.5)^2]$ and a final time of $T = L/a$ (1 period). Perform spatial and temporal convergence studies of the L_2 solution error to demonstrate the order of accuracy in space and time.

3 Manufactured Solutions [35%]

Discretize the one-dimensional advection-diffusion equation, $u_t + au_x - \nu u_{xx} = 0$, using the trapezoidal method in time and second-order central differences in space. Assume a grid of length L, periodic boundaries, and N spatial intervals.

a) Write a computer program that implements the given method, and run a simulation using the parameters given in problem 2c, $\nu = 0.1$, $N = 64$, and a CFL number of $\sigma = 0.5$. Plot the state at the final time, $u(x, T)$.

b) Apply the method of manufactured solutions to your discretization. The PDE will now need a source term: $u_t + au_x - \nu u_{xx} = s(x, t)$. Derive the form of $s(x, t)$ for the manufactured solution $u^{MS} = \sin(kx - \omega t)$, where k and ω are known constants.

c) Implement the method of manufactured solutions in your discretization, and present the solution at $t = T = L/a$ for $N = 64, \sigma = 0.5$. Use $k = 4\pi/L$ and $\omega = 5a/L$.

d) Using the manufactured solution, perform spatial and temporal convergence studies of your discretization, using the L_2 solution error, and verify that the orders of accuracy match your expectations.