Proof of "there are infinitely many prime numbers"

We show that if we list the prime $p_1, p_2, p_3, \ldots$ the list continues forever.

Suppose we have reached stage n: $p_1, p_2, p_3, \ldots, p_n$

Can we find another prime to continue the list?

$\boxed{\text{logic}}$

If we can always find another prime, then the list goes on forever and we've shown that there are infinitely many primes...

Can we?

$\boxed{\text{Clever trick from 「Elements」 by Euclid}}$

Look at the number $N = (p_1 \times p_2 \times p_3 \ldots \times p_n) + 1$

Clearly, N is bigger than p_n.

① If N is prime, we have found a prime bigger than p_n, and we can continue the list.

② If N is not prime, it is divisible by a prime, say p. p cannot be any of $p_1, p_2, \cdots p_n$, since dividing N by any ~~of~~ those leaves a remainder ~~of~~ 1.

So $p > p_n$, and that means we found a prime number bigger than p_n.

→ Either way, we can find a prime bigger than p_n and the list can always be continued.