# CFRM 505: Monte Carlo Methods in Finance (Winter 2021)

**CFRM Program, University of Washington**

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**Assignment 3**

1. Provide the simulation algorithm (e.g. R/Python script) that generates sample paths of the exponential OU process. That is,

$$P_t = \exp(X_t),$$

where

$$dX_t = \theta(\mu - X_t)\,dt + \sigma\,dW_t, \qquad t \geq 0. \tag{1}$$

Here, $(W_t)_{t\geq 0}$ is a SBM, and $X_0 = 0$. Show a plot of the sample path. Use at least 100 time-steps. Then, estimate the expectation $E[P_1]$, with $\mu = 1, \theta = 1, \sigma = 0.1$, sample size $n \geq 10,000$. Show the histogram of $P_1$.

2. Suppose a store opens for business between $t = 0$ and $t = 10$ and that arrivals to the store during $[0, 10]$ constitute a non-homogeneous Poisson process with intensity function $\lambda(t) = (2 + t + t^2)/100$. (The fact that the intensity function is increasing might reflect the fact that rush hour occurs at the end of the time period.) Each arrival is equally likely to spend \$100, \$400 or \$900.

   (a) Use the thinning algorithm to simulate arrivals to the store and to estimate the average amount of money that is spent in $[0, 10]$. (You should provide your R/Python script that uses at least $10,000$ samples for your estimate. )

   (b) Estimate the variance of the amount of money that is spent in $[0, 10]$.

3. Let $T$ be the investment horizon. You can invest in a stock $S$ and the riskless money market account. Denote by $S_t$ and $B_t$ the prices at time $t$ of the stock and the money market account, respectively. Assume that $S_t \sim GBM(\mu, \sigma)$ and that $B_t = \exp(rt)$, $t \in [0, T]$. Suppose you can trade at the $m$ equally spaced time points: $\{t_i = iT/m : i = 0, \ldots, m-1\}$.

   Our trading strategy is a *constant proportion* trading strategy. That is, at each trading point $t_i$, we re-balance our portfolio so that $\alpha\%$ of our wealth $W_{t_i}$ is invested in the stock, and $(1-\alpha)\%$ is invested in the money market account.

   You may assume that $T = 1$ years and that $m = 12$ so that we re-balance our portfolio every month. Assume $W_0 = \$100,000$, $S_0 = \$100$, $B_0 = \$1$, $r = 5\%$, $\mu = 15\%$, $\sigma = 20\%$ and $\alpha = 60\%$.

   Write a simulation algorithm to (1) show the distribution of $W_T$ (say, in a histogram) and estimate its mean and variance; (2) estimate the probability that $W_T/W_0 \leq p$ where $p$ is some fixed constant. Run your algorithm for values of $p = .8, .9, 1.0, 1.1, 1.2, 1.3, 1.4$ and 1.5. You should use at least $20,000$ samples and plot $\mathrm{P}(W_T/W_0 \leq p)$ against $p$.

4. Now, suppose you can invest in two stocks, $A$ and $B$, plus the riskless money market account. Let $S_t^a$, $S_t^b$ and $B_t$ denote the prices at time $t$ of stock $A$, stock $B$ and the money market account respectively. Assume that $S_t^a \sim GBM(\mu_a, \sigma_a)$, $S_t^b \sim GBM(\mu_b, \sigma_b)$ and that $B_t = B_0 \exp(rt)$ for all $t \in [0, T]$. Again, there are $m$ equally spaced time points where we can trade.

   Following the *constant proportion* trading strategy, you rebalance the portfolio so that $\alpha_a\%$ of wealth is invested stock $A$, $\alpha_b\%$ of wealth is invested stock $B$, and $(1-\alpha_a-\alpha_b)\%$ is invested in the money market account.

   Take $T = 1$ year, $m = 12$, $S_0^a = \$100$, $S_0^b = \$200$, $B_0 = \$1$, $r = .05$, $\mu_a = .15$, $\sigma_a = .2$, $\mu_b = .2$, $\sigma_b = .25$, $\alpha_a = 40\%$, $\alpha_b = 30\%$, and $W_0 = \$100,000$. Moreover, we assume the Brownian motions driving $S_t^a$ and $S_t^b$ have correlation coefficient $\rho$.

   Write a simulation algorithm to (1) show the distribution of terminal wealth $W_T$ (say, in a histogram) and estimate its mean and variance; (2) estimate the probability that $W_T/W_0 \le p$ where $p$ is some fixed constant. Run your algorithm for values of $p = .9, 1.0, 1.1, 1.2$ and $1.3$, and for values of $\rho = -.5, 0, .5$. You should plot $\mathrm{P}(W_T/W_0 \le p)$ against $p$ for each value of $\rho$.

5. Consider a price process $(S_t)_{t\ge 0}$. The running maximum of $S$ at time $t$ is defined by $\bar{S}_t := \max_{0\le u\le t} S_u$. To measure the distance between the current price and its running maximum, we consider the difference

$$D_t := \frac{\bar{S}_t}{S_t} - 1,$$

   which is called the *relative drawdown* of $S$ at time $t$. This is an essential concept used in evaluating investment managers and their strategies. For many hedge funds, the managers would try to avoid a large drawdown.

   Suppose $S$ is a GBM, with parameters $T = 1, \mu = 5\%, \sigma = 15\%, S_0 = 1$. Simulate a path of $S$ together with the associated path of $\bar{S}$ over the period $[0, T]$.

   Next, plot the path of the relative drawdown $D$ (in percentage) over time. This is typically called the "underwater chart."

   In addition, the maximum relative drawdown over $[0, T]$ is defined by

$$D_T^* := \max_{0\le t\le T} D_t.$$

   For $D_T^*$, plot its (estimated) expected value when $\sigma = 10\%, 12\%, \ldots, 40\%$, using at least 10000 samples each and holding other parameters fixed. How do the expected values vary with the volatility parameter $\sigma$? (Hint: you can choose to construct the paths by using the same set of generated BM increments for different values of $\sigma$, as opposed to starting the simulation afresh and generate a new set of BM increments when you move from $\sigma$ value to another. Both are valid approaches, but the former is much more computationally efficient and will result in a smooth curve.)