COMPUTATIONAL FINANCE & RISK MANAGEMENT
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Capital Asset Pricing Model (CAPM) Rolling Windows Regression

CFRM 425 (016)

R Programming for Quantitative Finance

Lecture References/Package Downloads

- Jeet & Vats, Ch 5, CAPM
- Ang, Ch 5: §§5.1 and 5.3
- Bodie, Kane, Marcus
  - ➢ Investments 10th Edition (McGraw Hill, 2014), selections from:
    - Ch 6: Capital Allocation to Risky Assets
    - Ch 7: Optimal Risky Portfolios
    - Ch 8: Index Models
    - Ch 9: The Capital Asset Pricing Model (CAPM)
- M Jackson & M Staunton (Old CFRM 506 textbook)
  - ➢ Advanced Modelling in Finance Using Excel and VBA (Wiley, 2001)
    - Ch 6: Portfolio Optimisation, §§6.1-6.3 (revisited)
    - Ch 7: Asset Pricing, §§7.1-7.5
    - Sample spreadsheets and VBA code for portfolio optimization and asset pricing
- Francois-Serge Lhabitant
  - ➢ Hedge Funds: Quantitative Insights (Wiley Finance, 2004)
    - Ch 4, §§4.1 & 4.2 (Sharpe Ratio and CAPM)
- Wikipedia, Harry Markowitz
  - ➢ https://en.wikipedia.org/wiki/Harry_Markowitz

- Topics:
  - The Capital Asset Pricing Model (CAPM)
  - Rolling Windows Regression

The CAPM

Pan Am advert, 1965

Harry Markowitz: Portfolio Theory

- Seminal paper in 1952: "Portfolio Selection"
- As described in Jackson & Staunton:
  - A mean–variance approach to examining the trade-off between return (as measured by an asset's mean return) and risk (measured by an asset's variance of return), with investor preferences based on utility theory from economics
  - Subsequently led to the development by Sharpe, Lintner, and Treynor, of the Capital Asset Pricing Model (CAPM), an equilibrium model describing expected returns on equities. The CAPM introduced beta (regression coefficient) as a measure of diversifiable risk, arguing that the creation of portfolios served to minimise the specific risk element of total risk (variance).
  - Reduced the optimisation problem to that of finding mean–variance efficient portfolios, with efficient points having the highest expected return for a given level of risk.
- Note: The Ang book puts CAPM before Markowitz Theory (Ch 7)

Portfolio Optimization Example

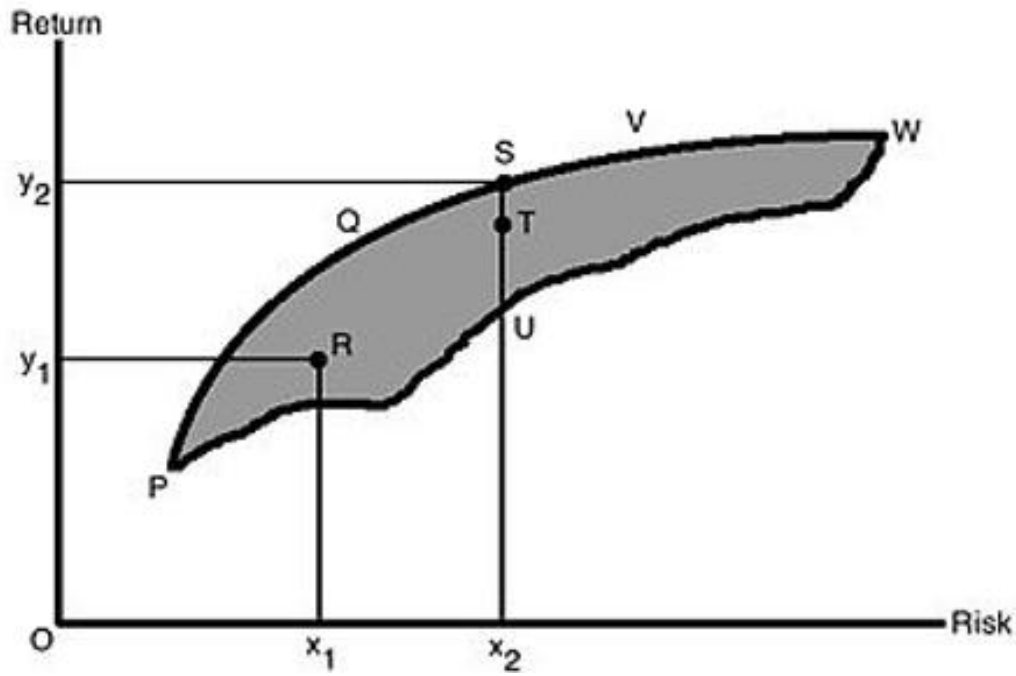
- The classic Markowitz application is a *quadratic optimization problem*, to find the optimal weights among several assets that would minimize the risk (variance) subject to a target return (based on the means of historical returns) and correlations between the (risky) assets
- The definition of a *risky asset* here is that the variance of each is greater than zero
- We will see later what happens when a risk-free asset enters into the model
- The optimization problem:
- Where:
  - $\{\omega_i\}$ is the set of fund weights,
  - $\boldsymbol{\mu}$ is the empirical fund mean vector,
  - and $\boldsymbol{\Sigma}$ is the covariance matrix

mean-variance portfolio optimization

$$
\begin{aligned}
&min_b && \omega^T \boldsymbol{\Sigma} \omega \\
&s.t. && \omega^T \boldsymbol{\mu} = \mu_p \\
& && \omega^T \mathbf{1} = 1 \\
& && \omega_{min} \geq \omega_i \geq \omega_{max}
\end{aligned}
$$

Portfolio Optimization Example

- If we were to plot out the feasible set of mean/variance combinations of all of the asset allocations, it would resemble the following (taken from the Wikipedia link cited at the outset):



- Return is measured in terms of historical expected returns
- Risk is measured by the variance of the portfolio

Portfolio Optimization Example

- Essential Summary: The theory states that the optimal return and variance must reside on the curve that bounds the feasible set above; this is called *the efficient frontier*

- We use this property in the following discussion, where a risk-free asset is introduced into the model

The Capital Allocation Line (CAL)

- Recall: An asset is *risky* ⇔ its variance is positive; ie, $\sigma > 0$.
- The *Capital Allocation Line* (CAL) represents a continuum between investment in a risk-free asset (eg a bank account, 1-month US Treasury Bills) and a risky asset or portfolio
- The desired allocation depends on the risk profile of the investor, and will be on a point along this line
- It lies in the same Risk (σ) and Expected Return (μ, or $E(r)$) plane as the efficient frontier, starting at the point (r_f , 0), and ending

  at (σ, $E(r)$)
- If we call the rate of return on the risk-free asset r_f, then the excess return is the difference between the $E(r)$ and r_f
- The slope of the CAL is the Sharpe Ratio, which is a measure of risk-adjusted return, or excess return per unit of risk for an asset A:

$$\frac{E(r_A) - r_f}{\sigma_A}$$

- For example, from BKM 10 E, Ch 6, p 179:

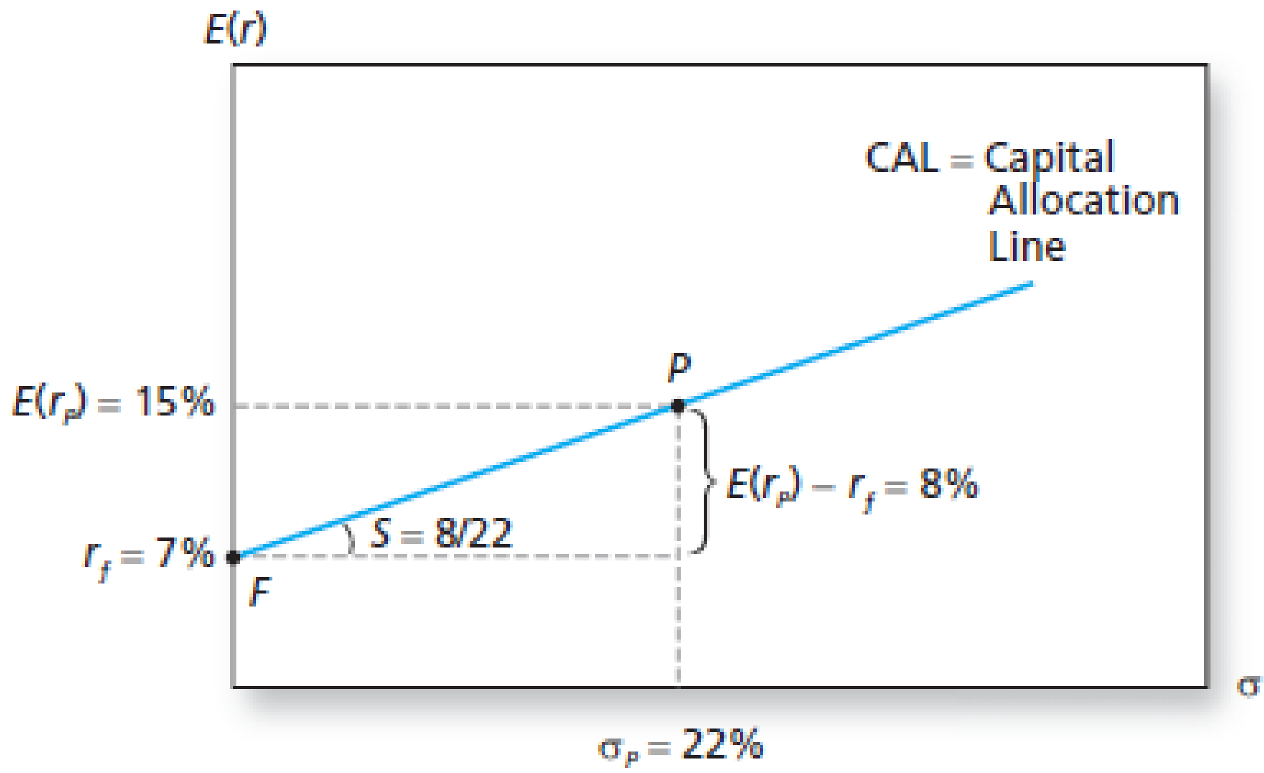


Figure 6.4 The investment opportunity set with a risky asset and a risk-free asset in the expected return–standard deviation plane

- So why do we care about the CAL?
  - Relationship with the Efficient Frontier
  - Relates to the CAPM model
- The upper end point of the CAL – ie, at the point where the portfolio is fully invested in risky assets – must lie at a point tangent to the efficient frontier, to be optimally an efficient risky portfolio
  - Suppose not (BKM, 10 E, p 215)
  - Point B dominates point A
  - More optimal points will dominate point B until tangency is reached on the efficient frontier

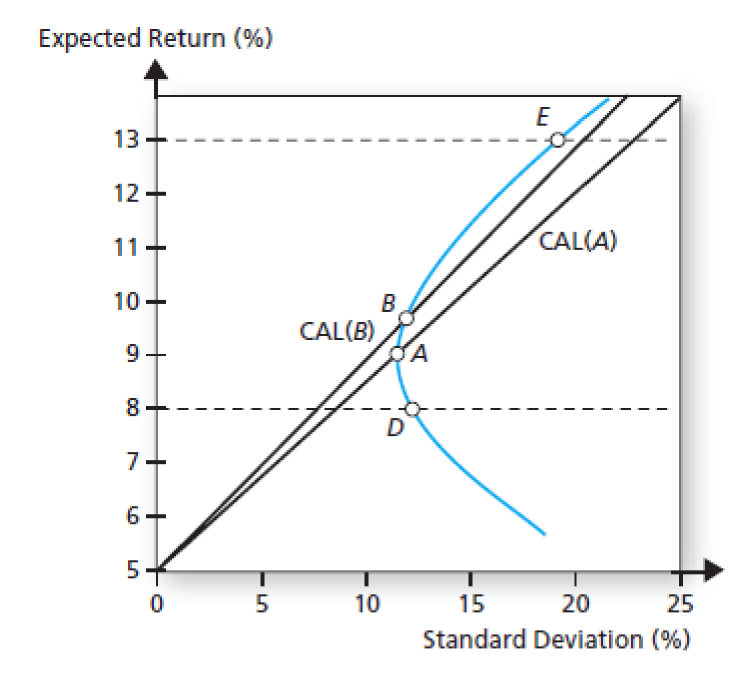


Figure 7.6 The opportunity set of the debt and equity funds and two feasible CALs

- Putting it all together, then, the CAL will lie tangent to the efficient frontier
- The allocation between risk-free and the risky asset portfolio will depend on the investor's risk tolerance, along the CAL
- An example is found on p 217 of BKM (10 E):

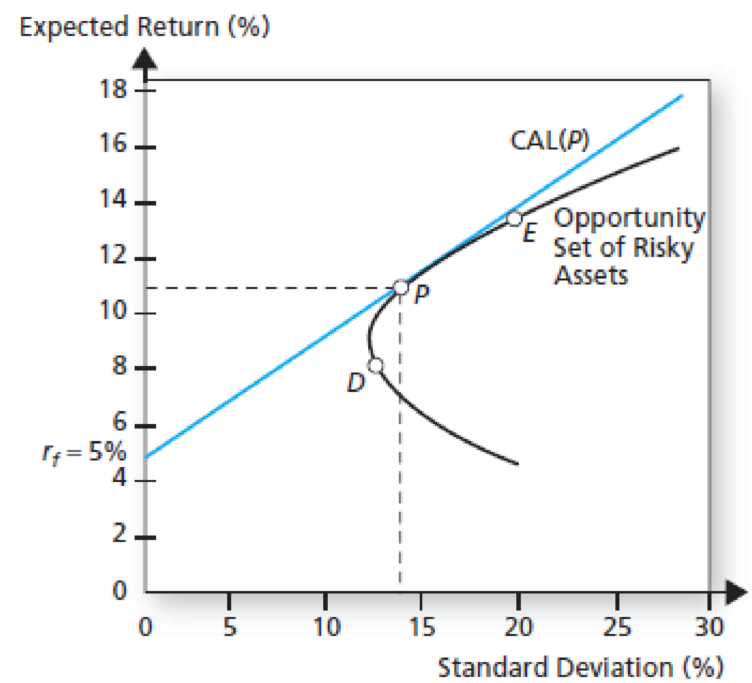


Figure 7.7 The opportunity set of the debt and equity funds with the optimal CAL and the optimal risky portfolio

The Capital Asset Pricing Model (CAPM)

- An extension of the Markowitz paper, due to William Sharpe (of Sharpe Ratio fame) in 1964
- At a high level, CAPM basically says:
  - The *excess return* should be linearly related to the overall ***market*** excess return; eg, that of the S&P 500, or Wilshire 5000 (total market) as a proxy representing the entire US stock market
  - Any risky portfolio or asset must lie on the efficient frontier as defined by "the market"
  - If not, the efficiency of the market will drive the returns back to the equilibrium of the market efficient frontier
  - The CAL is replaced by the *Capital Market Line* (CML): the capital allocation line provided by a risk-free asset (eg 1-month T-bills) and a broad index of common stocks (eg S&P 500) [BKM, 10 E, p 188]
  - The market portfolio minimizes idiosyncratic (specialized) risk due to individual securities comprising the market

The Capital Asset Pricing Model (CAPM)

- BKM, 10 E, p 292:

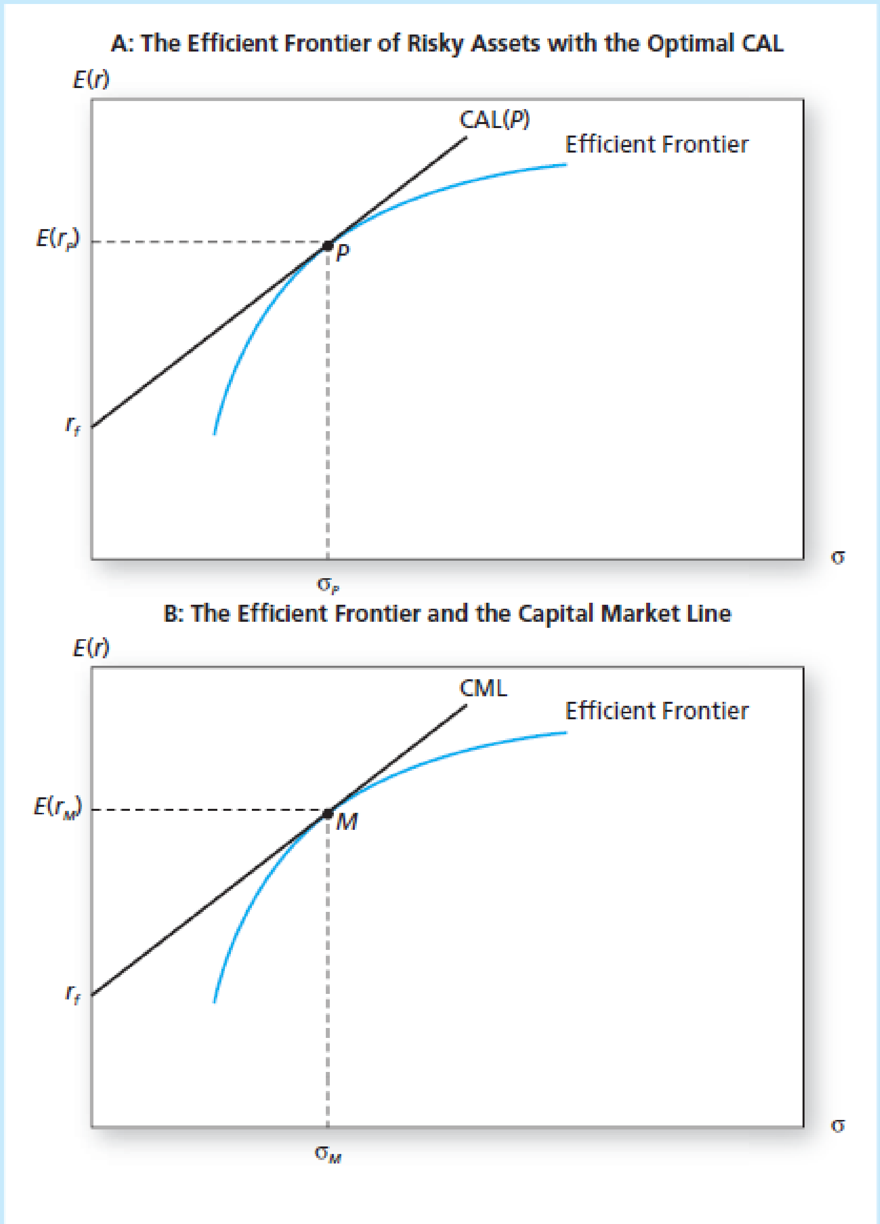


Figure 9.1 Capital allocation line and the capital market line

Estimating Beta Coefficients and the Single Index Model

- The goal here is to estimate the regression slope coefficient (β) of a risky asset with respect to the entire market
- We will also discuss the intercept term (α), its meaning, and the implications under the assumptions of the CAPM

Estimating Beta Coefficients

- *The Single Index Model* (§ 7.1 in J & S), :
  - The model posits a linear relationship between returns on share i and returns on an index, I say. Using R_i and R_I to denote returns in *excess of the risk-free rate* for the share *i* and the index *I* respectively, the equation is written:

$$R_i = \alpha_i + \beta_i R_I + \epsilon_i$$

  - The return on the share is made up of a *systematic part* $\alpha_i + \beta_i R_I$ and a *residual part* ϵ_i
  - β_i is the share's responsiveness to movements in the index.
  - The residual return, ϵ_i, is not related to the index and is specific to share i (represents *non-systematic* or *specific* risk).
  - It is typically assumed that $\epsilon_i \sim N(0, \sigma(\epsilon_i)^2), iid$.
- As a result, we have

$$E(R_i) = \alpha_i + \beta_i E(R_I)$$

$$Var(R_i) = {\beta_i}^2 \sigma_I^2 + \sigma(\epsilon_i)^2$$

- Note that from the 2nd result, the risk is broken down into the systematic part due to the relationship with the market index, and the risk specific to the security.

Estimating Beta Coefficients with CAPM

- We have already covered methods for running simple linear regression models in R
- What becomes interesting, however, is that under the hypothesis that the CAPM is true, the α_i term in the regression is zero. Why?
  - If $\alpha_i > 0$, demand for the stock will bid up in price until its mean return descends to the Efficient Frontier
  - If $\alpha_i < 0$, lack of demand for the stock will cause its price to fall, and thus mean return rise, to the level of the Efficient Frontier
- Now, under CAPM, because $\alpha_i = 0$, we have $E(\alpha_i) = 0$. Also, it is customary to use the subscript M for "Market" rather than I for "index" (as introduced in earlier in Ch 7). As a result, we have

$$E(R_i) = \beta_i E(R_M)$$

- Also, an equilibrium argument (see (9.6), BKM 10E) gives the result

$$\beta_i = \frac{Cov(R_i, R_M)}{\sigma_M^2}$$

Rolling Windows Regression

Rolling Windows Regression

- In Ch 5 of the Ang book, this is limited to the simple regression case of an individual asset (eg AMZN) on an index (eg SPY)
- We also introduce a new and handy tool in the zoo package: the **`rollapply(.)`** function
- Time permitting, we will look at the more general fund tracking case later

Rolling Windows Regression

- The essentials can be found in this code example
- Caution, however, in some of the parameter settings:
  - **`width`** refers to the number of ***rows*** to include in the function input
  - **`align="left"`** refers to positioning the results at the *top* of the time series

```
coeffs<-rollapply(as.zoo(rtns), width=252,
                  FUN=function(X)
                  {
                    roll.reg=lm(AMZN ~ SPY,
                                data = as.data.frame(X))
                    return(roll.reg$coef)
                  },
                  by.column=FALSE, align = "left")
```

Rolling Windows Regression

- See code example for more details and discussion

[END]