CFRM 504: Final Exam

Directions: You are to work alone on this exam. You may use course notes, homework solutions and textbooks as references. You may *not* discuss the exam with others. Those who are found to have worked with others may face expulsion from the University of Washington. Good luck!

Due date: Thursday, December 17, 2020 at 11:59 PM Pacific Standard time. Please submit your exam through Canvas as a PDF. Late exams will not be accepted under any circumstance.

Please write (clearly) the following statement in the space below, print your name, sign and date this page, and include this sheet with your answers.

"I have worked alone on this exam and did not discuss it with others."

Statement: ______________________________

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Name (print): ______________________________

Signature: ______________________________

Date: ______________________________

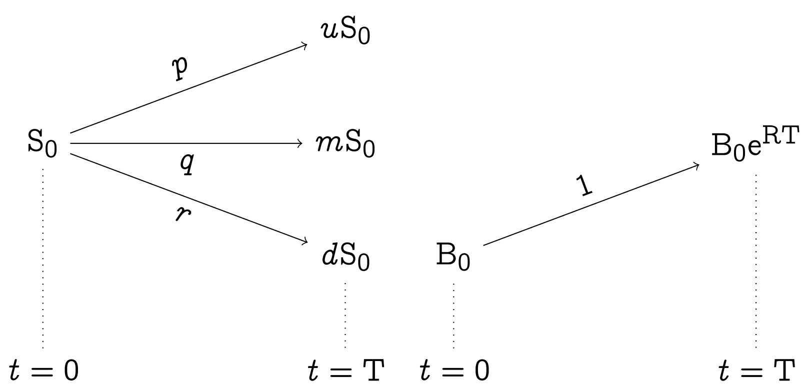

Figure 1: One-period trinomial model

Question 1

Assume $0 < d < m < u < \infty$ and $0 < d < e^{RT} < u < \infty$ and that there is no arbitrage. Consider the one-period trinomial model as depicted in Figure 1. The market is selling an option with payoff $h(S_T)$ for a price V_0. Assume you can trade the stock S, the bond B and the option with payoff $h(S_T)$.

(a) Write a system of equations that you would solve in order to find a portfolio X such that $X_T = g(S_T)$, where g is some given function. Clearly state which variables you would solve for. You do not need to solve the system of equations.

(b) Suppose h is is given by $h(S_T) = \alpha S_T$ where α is a real constant. Is it possible to determine V_0 uniquely? If so, what is V_0? For this question, ignore part (a). If you can determine V_0 uniquely, then please give your answer in terms of parameters of the problem set up (e.g., α, p, q, r, u, m, d, B_0, R, S_0 and T).

(c) In general, can the system of equations you derived in part (a) be solved for any given function g when $h(S_T) = \alpha S_T$? Explain your answer.

Question 2

Consider the Black-Scholes model for a stock S and a bond B, written under the real world measure $\mathbb{P}$

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \qquad dB_t = rB_t dt,$$

where W is a Brownian motion under $\mathbb{P}$. Let $u(t, S_t)$ be the value at time t of a European contract with payoff $\varphi(S_T) = (\log S_T)^2$. And let X be a portfolio composed of a stock and a bond:

$$dX_t = \Delta_t dS_t + (X_t - \Delta_t S_t)\frac{1}{B_t} dB_t.$$

(a) Suppose $\widetilde{\mathbb{P}}$ is a risk-neutral measure with the bond B as numeraire (i.e., X/B is a $\widetilde{\mathbb{P}}$ martingale). What should the dynamics of $\log S$ be under $\widetilde{\mathbb{P}}$? That is, compute $d(\log S_t) = (\ldots)dt + (\ldots)d\widetilde{W}_t$, where $\widetilde{W}$ is a Brownian motion under $\widetilde{\mathbb{P}}$. Explain how you obtain your answer.
(b) Derive an expression for $u(t, S_t)$.
(c) What should X_0 and Δ_t be in order for $X_T = \varphi(S_T)$?

Question 3

In this exercise, you may find the following fact helpful: for any $p \in \mathbb{R}$ we have

$$\mathbb{E}e^{pZ} = e^{bp + \frac{1}{2}p^2 a^2}, \qquad Z \sim \mathcal{N}(b, a^2).$$

Consider the Black-Scholes model for a stock, written under the physical probability measure $\mathbb{P}$. We have

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

Assume the risk-free rate is zero so that the value of a bond is constant $B_t = 1$.
(a) Denote by $V_t^{(p)}$ the value at time t of a *power option*, which pays S_T^p at time T. Compute $V_t^{(p)}$.
(b) Denote by $\Delta_t^{(p)}$ the Delta of the the power option at time t. Compute $\Delta_t^{(p)}$.
(c) Suppose a portfolio consists of one share of $V_t^{(p)}$, α shares of $V_t^{(q)}$ and β shares of $V_t^{(r)}$, where $p, q, r \in \mathbb{R}$. Write a system of equations you would solve in order to find α and β so that the portfolio has zero value and is Delta-neutral. Please give your answer in matrix form

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix}.$$