CFRM 505: Monte Carlo Methods in Finance (Winter 2021)

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Assignment 5

1. (*Black-Scholes Deltas & Gammas*) Let's estimate the Delta and Gamma of Black-Scholes **Puts**. For Deltas, use the pathwise (PW) and likelihood ratio (LR) methods. For Gammas, use the pure LR method, the combined LR-PW method, as well as the combined PW-LR method. In addition,

   (i) Report the Deltas and Gammas using the Black-Scholes formula. These values allow you to check your estimates.

   (ii) Report the sample variances of your estimators.

   Implement the corresponding MC simulation algorithm in R/Python. Provide your answers in tables like the ones below.

   Parameters: $S_0 = 100$, $\sigma = 25\%, r = 3\%, T = 0.5$, at strikes $K = 90, 100, 110$. Use at least 100,000 samples (use up to 1 million if feasible).

Table 1: Estimating the Delta of European Puts under the B-S Model

Delta	$K = 90$	$K = 100$	$K = 110$
Formula			
PW			
LR			
Variance of PW Estimator			
Variance of LR Estimator			

Table 2: Estimating the Gamma of European Puts under the B-S Model

Gamma	$K = 90$	$K = 100$	$K = 110$
Formula			
LR			
LR-PW			
PW-LR			
Variance of LR Estimator			
Variance of LR-PW Estimator			
Variance of PW-LR Estimator			

2. Write an R/Python script to implement the loss probability formula:

$$P(Y \leq y) \approx \Phi\left(\frac{\Phi^{-1}(1-p) - \sqrt{1-\rho}\,\Phi^{-1}(1-y)}{\sqrt{\rho}}\right).$$

Fix $p = 1\%$. Plot $P(Y \leq y)$ as a function of y, for $\rho = 0.2, 0.5, 0.8$.

Next, numerically compute the PMF:

$$P(L = \ell) = \int_{\mathbb{R}} \binom{m}{\ell} \tilde{p}(z)^{\ell}(1-\tilde{p}(z))^{m-\ell} f_Z(z)dz,$$

where f_Z is the pdf of a $N(0,1)$ rv and $\tilde{p}(z)$ is the conditional default probability (same for all firms, but conditioned on the realization $Z = z$). Take $m = 10$, 25, 50. It will become more computationally expensive as m increases. Again, take $p = 1\%$. Plot the PMF $P(L = \ell)$ as a function of ℓ, for $\rho = 0.2, 0.5, 0.8$.

Define $(X_i)_{i=1,\ldots,m}$ so that they're dependent via a common factor Z:

$$X_i = \sqrt{\rho}Z + \sqrt{1-\rho}\epsilon_i,$$

where Z and ϵ_i's are IID $N(0,1)$. Let $Y_i = 1_{\{X_i > x\}}$, and $L = \sum_{i=1}^{m} Y_i$ (number of defaults from m firms).

In R/Python simulate the distribution of $L = \sum_{i=1}^{m} Y_i$. Take $\rho = 0.5$, and default probability $p = 1\%$ and $m = 50$. Show the histogram of L (note: L takes value from 0 to m). Also, estimate the loss probability $P(L > 10)$. You should use a sample size of at least $20,000$ for L (and apply importance sampling if your estimate is 0).