COMPUTATIONAL FINANCE & RISK MANAGEMENT

UNIVERSITY of WASHINGTON

Department of Applied Mathematics

MORE(!) Equity Options and RQuantLib Functions

CFRM 425 (019)

R Programming for Quantitative Finance

Lecture References and Topics

- Peter James
  - Option Theory (Wiley, 2003)
    - General background
    - Excellent reference book to have in your arsenal
- Futures Magazine article:
  - http://www.futuresmag.com/2018/05/24/trading-vanilla-exotic-options
- FLEX Options:
  - http://ir.cboe.com/~/media/Files/C/CBOE-IR-V2/press-release/2016/cboe-to-offer-flex-index-options-with-asian-and-cliquet-style-settlement.pdf

- Topics
  - Pricing Methods
    - Closed-Form (No-Arbitrage Theory/Black Scholes Derived)
      - https://campus.datacamp.com/courses/quantitative-risk-management-in-r/estimating-portfolio-value-at-risk-var?ex=8
    - Lattices: Binomial and Trinomial Trees
    - Numerical Solutions of PDE's Monte Carlo
  - Some Basic Exotic Options
    - Binary Options
    - Barrier Options
    - Asian Options

- Ella Fitzgerald and Louis Armstrong

Black-Scholes Closed Form: European Options

- Let
  - S = current equity (stock) price
  - r = risk-free rate of interest (constant)
  - X = exercise price
  - T = time to maturity (year fraction)
  - q = dividend rate
  - $N(z)$ = standard normal cumulative distribution function (CDF)
  - σ = volatility of the underlying asset

- The price of a call option according to the model is

$$c = Se^{-qT}N(d_1) - X\,e^{-rT}N(d_2)$$

- The price of a put option according to the model is

$$p = -Se^{-qT}N(-d_1) + X\,e^{-rT}N(-d_2)$$

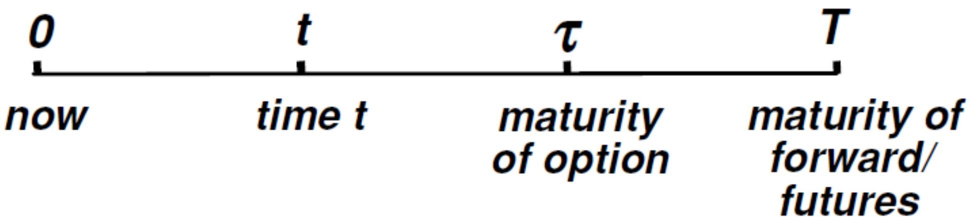
where

$$d_1 = \frac{\left[\log\left(\frac{S}{X}\right) + (r - q + 0.5\sigma^2)T\right]}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\left[\log\left(\frac{S}{X}\right) + (r - q - 0.5\sigma^2)T\right]}{\sigma\sqrt{T}}$$

More Closed Form Pricing Formulae

- Black's Formula (1976) for options on futures (James, (5.7))
- Option expiring at time τ on a futures contract expiring at time *T*:



  - V_{tT} = Value of a futures contract expiring at time *T*, entered into at time *t*
  - All other parameters the same as Black Scholes
  - Also works for forward contracts

$$f_t = e^{-r(\tau - t)} \phi\{V_{tT}\, N[\phi d_1] - X\, N[\phi d_2]\}$$

$$d_1 = \frac{1}{\sigma\sqrt{\tau - t}}\left\{\ln\frac{V_{tT}}{X} + \frac{1}{2}\sigma^2(\tau - t)\right\}; \qquad d_2 = d_1 - \sigma\sqrt{\tau - t};$$

More Closed Form Pricing Formulae

- Garman Kohlhagen for FX options (1983):
  - Replace the dividend rate with the foreign currency risk-free rate
  - S and X are quoted in terms of units of domestic currency per unit of foreign currency

- Closed form pricing formulae for barrier options (coming shortly)

Lattices: Binomial Tree Example (James, Ch 7, p 80)

- Start with simplest case: European equity call option
- Up and down returns u and d are derived from the underlying equity volatility σ (Jarrow-Rudd)
- The probabilities of up and down movements are determined by no-arbitrage arguments (p and 1 – p, respectively)
- Moving forward in time, we can then calculate the underlying prices at each node, using the initial underlying price S_0, u, d, and p
- Moving back, using payoffs and expected values, we get the option values at each node
- The option value determined at t = 0 is the calculated option price
- In the limit, it is the same as the Black Scholes price

- From James:

With three steps $\delta t = 0.5/3$: $\quad F_{t\,t+\delta t} = S_t\, e^{(r-q)\delta t} = 1.01005 S_t; \quad e^{-r\,\delta t} = 0.983$

$$u = e^{\sigma\sqrt{\delta t}} = 1.0851; \qquad d = e^{-\sigma\sqrt{\delta t}} = 0.9216; \qquad p = \frac{F_{t\,t+\delta t} - S_t\, e^{-\sigma\sqrt{\delta t}}}{S_t\, e^{\sigma\sqrt{\delta T}} - S_t\, e^{-\sigma\sqrt{\delta t}}} = 0.541$$

S_0 = 100

X = 100

r = 10%

q = 4%

σ = 20%

t = 0.5 year

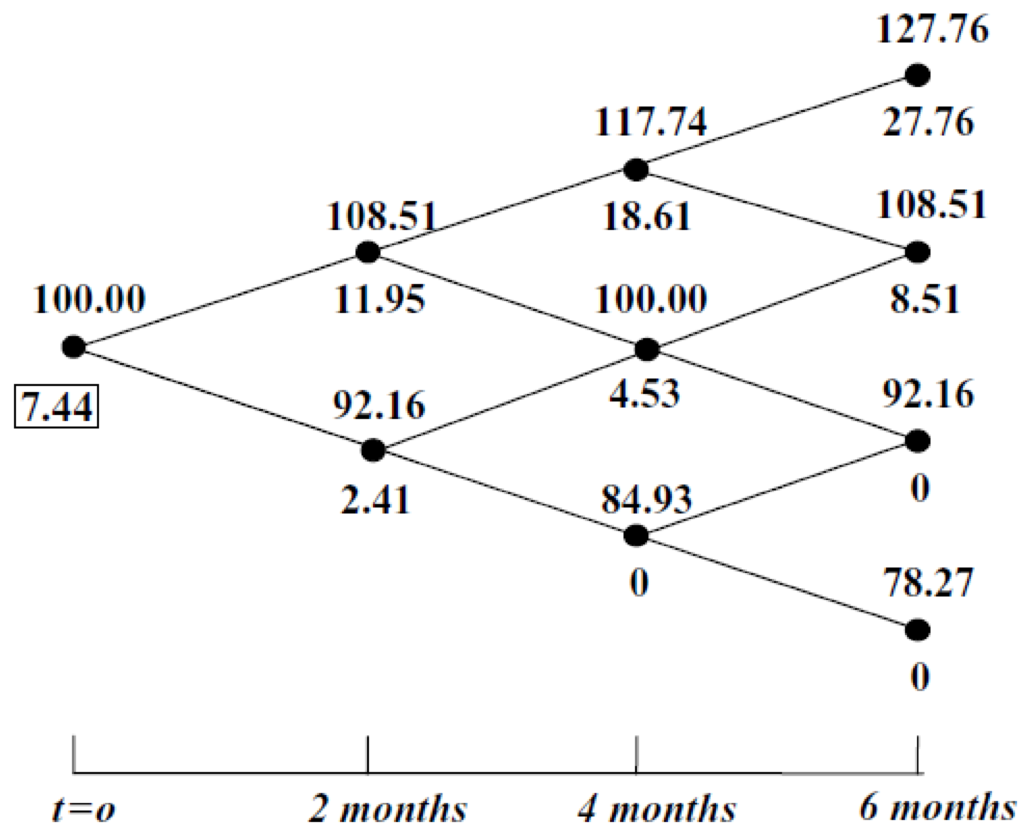


Figure 7.4 European call: Jarrow–Rudd discretization

- We can break this down into three components:
  1. At each node is an object containing the corresponding data values
  2. A generic tree (basically a 2-d array or generic matrix) that holds the type in the 1st component at each node
  3. Calculation logic:
     - ➢ Iterate forward through the tree to calculate the equity prices
     - ➢ Iterate backward through the tree to calculate the expected option values

	A	B	C	D	E	F	G	H	I	J	K
19	**Share**										
20		0	1	2	3	4	5	6	7	8	9
21	9										155.16
22	8									147.77	141.20
23	7								140.73	134.47	128.49
24	6							134.02	128.07	122.37	116.93
25	5						127.64	121.96	116.54	111.36	106.41
26	4					121.56	116.15	110.99	106.06	101.34	96.84
27	3				115.77	110.62	105.70	101.01	96.51	92.22	88.12
28	2			110.25	105.35	100.67	96.19	91.92	87.83	83.93	80.20
29	1		105.00	100.33	95.87	91.61	87.54	83.65	79.93	76.38	72.98
30	0	**100.00**	**95.55**	91.31	87.25	83.37	79.66	76.12	72.74	69.50	66.41

Figure 10.4 Nine-step share price tree using JR parameters (sheet JREuro)

- Lattice methods are often used for American options and determining optimal exercise

More Generalized Lattice (3-D)

- We could also have a 3-D lattice for a two-asset binomial tree option pricing model
- It now becomes even more important to incorporate abstractions to separate the lattice (tree) from the pricing information (figure also from the James text):

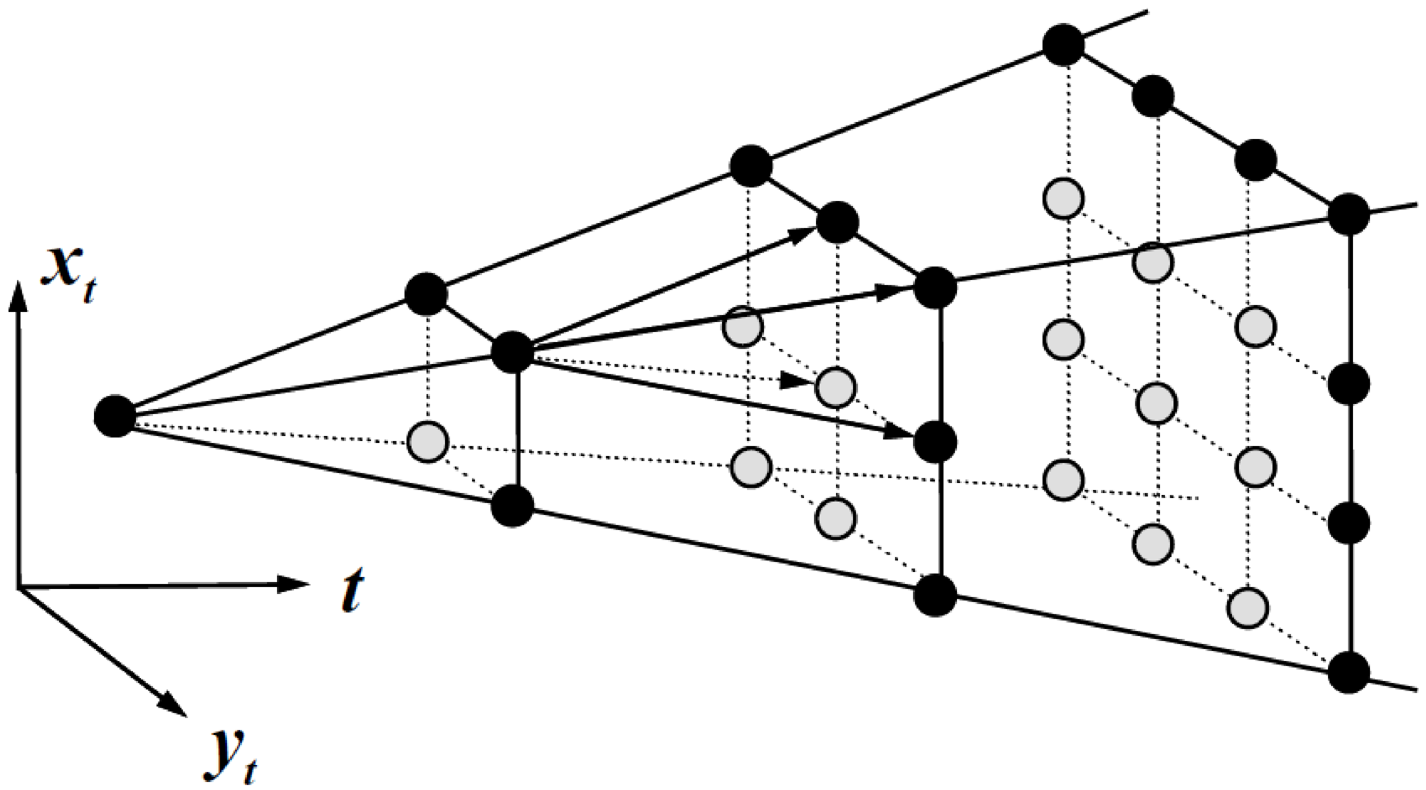


Figure 12.1 Binomial tree for two assets

- Wes Montgomery, guitar

Numerical Solutions of PDE's

- The Black Scholes PDE, and almost any other PDE coming out of option pricing is a parabolic PDE
- James, p 265, § 23.5:

$$\frac{\partial f_t}{\partial t} + rS_t \frac{\partial f_t}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f_t}{\partial S_t^2} = rf_t$$

- Numerical methods:
  - Finite Difference Method (FDM)
    - ➢ Explicit FDM
    - ➢ Implicit FDM
    - ➢ Crank-Nicolson
    - ➢ Barone-Adesi and Whaley
  - Finite Element Methods

- Numerical PDE methods are also well-suited for American options

Monte Carlo

- Simplest case: pricing options where interest rates and volatilities are constants
- For options, we need random scenarios for the price of the underlying security; these are generated by an equity price generator (next slide)
- At the last time step in the simulation, calculate each payoff, take the sum (including out-of-the money zero payoffs), and then compute the mean: This gives us the price of a European option
- We could relax these assumptions by bringing in the market term structure in place of a constant interest rate
- One further addition would be a term structure of volatilities (similar to an interest rate term structure these are also time dependent)

Monte Carlo: Equity Price Generator

- We will use the standard lognormal result:

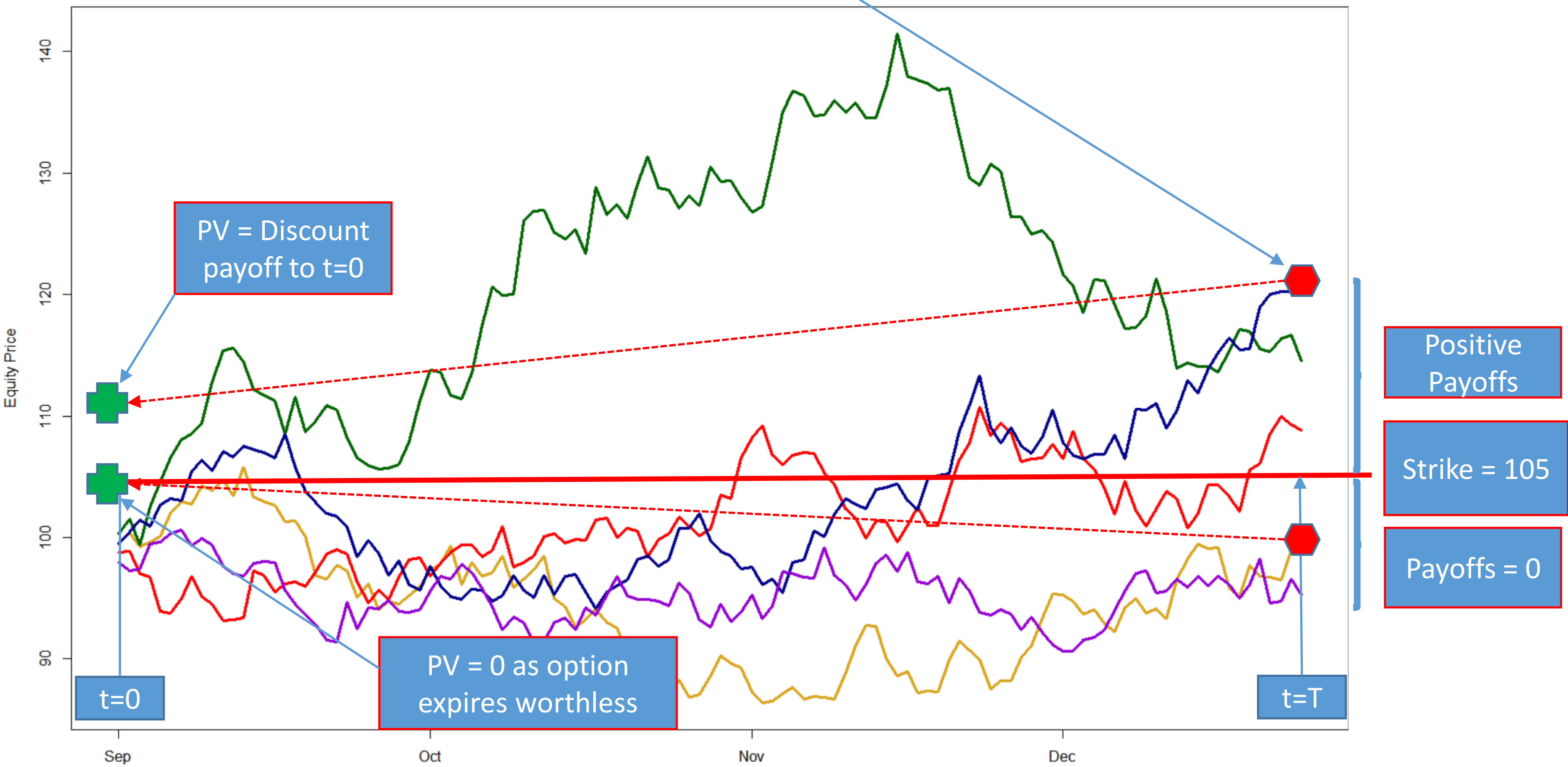
$$S_t = S_{t-1} e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma \varepsilon_t \sqrt{\Delta t}}$$

  where S_t = underlying (eg equity) price at time t, Δt = time step year fraction, r = risk-free rate, σ = volatility, and $\varepsilon_t \sim \mathrm{N}(0, 1)$ iid.

- Our scenarios are then random realizations of price paths chopped up into time steps each of length Δt .
- In the limit, the Monte Carlo price will converge to the Black-Scholes price; same for lattice and PDE methods for pricing European options under these assumptions
- This Monte Carlo method, however, is not suitable for pricing American options
- Monte Carlo simulations can be used, however, in conjunction with evaluating optimal payoffs over the time horizon to price American options; eg, the Longstaff Schwartz Least Squares Monte Carlo model

Monte Carlo Option Pricing

- European Option: A tradeable contract that gives the holder the right to buy or sell a share of stock at a predetermined strike price on its expiration date
- A simple graphical example of a call option is shown below (right to purchase at strike price)
  - Assume the vertical axis represents changes from an underlying asset currently valued at $100/share
  - The red line represents a strike price of $105
  - The positive payoffs at expiration will be the blue (~$15), green (~$10) and red (~$5) scenarios



Monte Carlo Option Pricing

- In our five-scenario example, to determine the option price:
  - If the risk-free interest rate is, say 1.2%,
  - and the time to expiration is four months (1/3 of a year),
  - the option value would be

$$\frac{e^{-(0.012)(0.333)}\{\boxed{(120\ -105)\ +\ \ (115\ -105)\ +\ (110\ -105)+0+0}\}}{5}=5.98$$

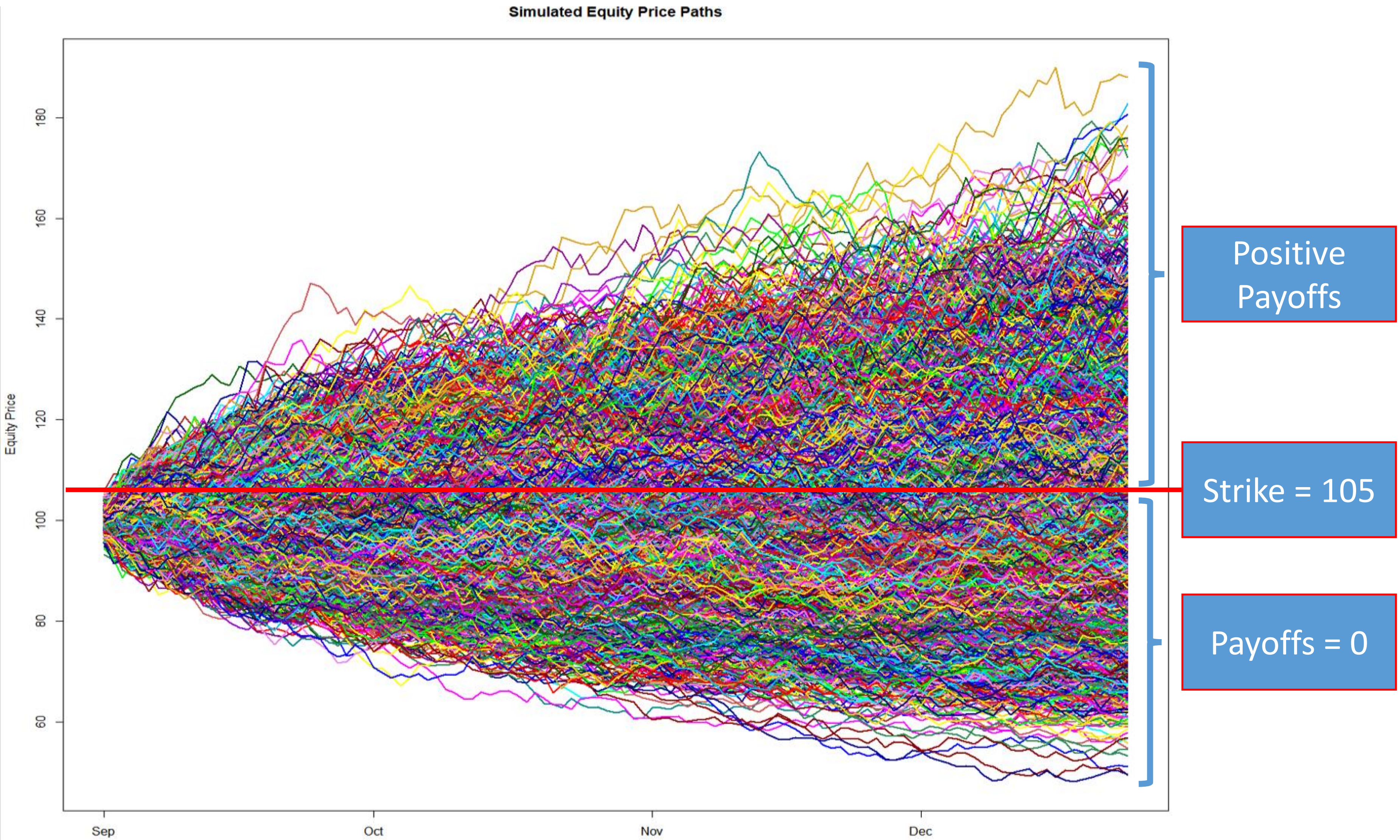
Payoffs at expiration

Discount factor

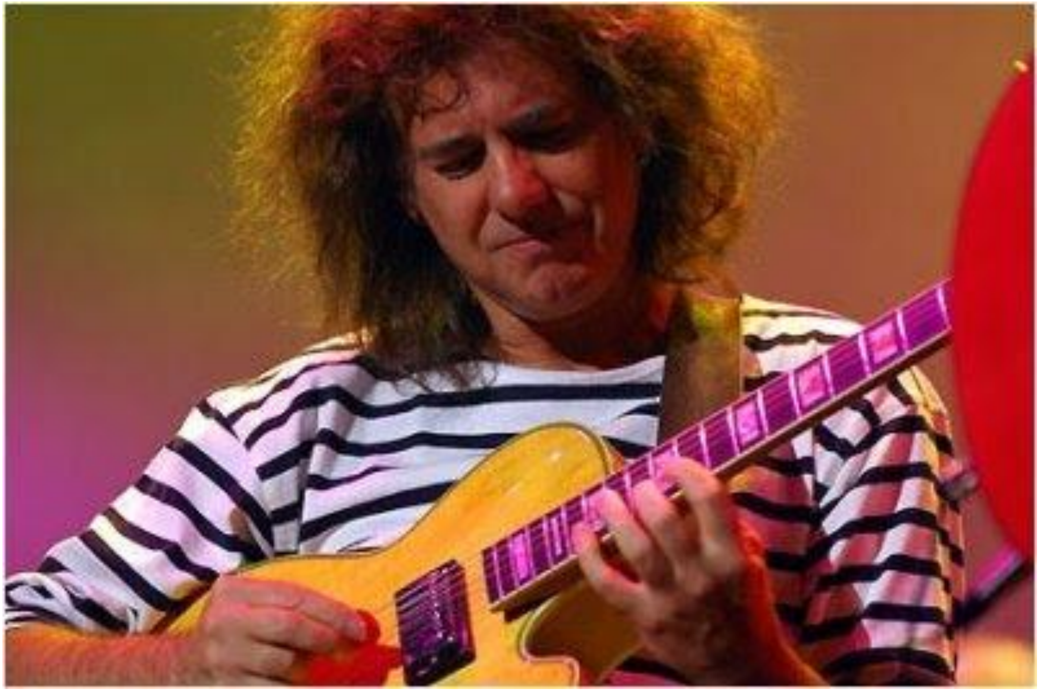
Calculate average to get option price

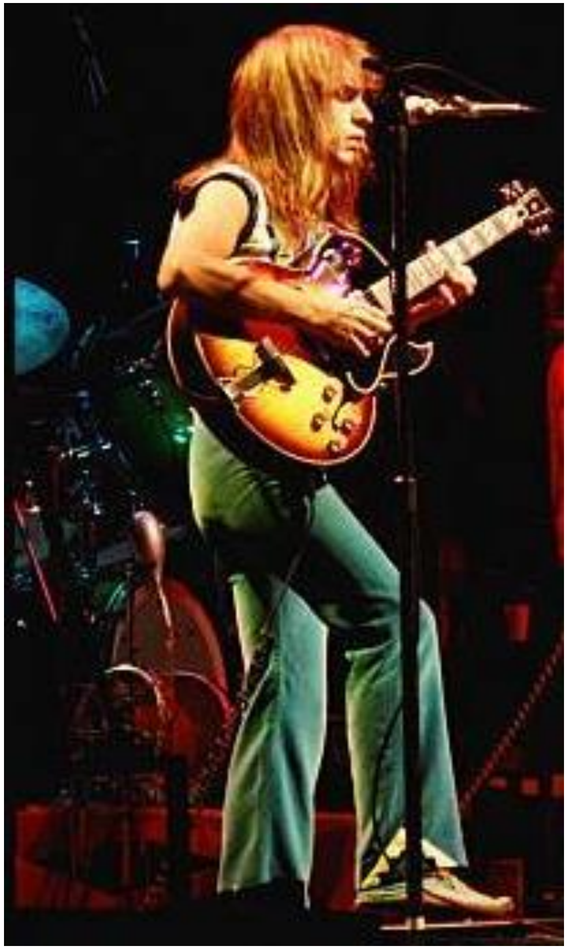
Monte Carlo Option Pricing

- In reality, the number of scenarios required can be on the order of 10,000 – 100,000
- This can lead to computationally intensive operations
- First, however, we need to generate *a single equity price path* (next slide)



- Wes Montgomery Family Tree
  - Pat Metheny and Steve Howe
  - Both credit Wes Montgomery as a primary influence





- Payoff is a fixed amount if the underlying asset is greater than (or less than) the strike price S_T when the option expires at time *T*, and zero otherwise
- For (a simple) example:
  - Payoff = \$1 if $S_T > X$
  - Payoff = 0 otherwise
  - The price of such an option at time *t* < *T* is then

$$e^{-rT} N(d_2)$$

  - This is the present value of the (risk-neutral) probability that $X < S_T$
  - Set the argument above to $-d_2$ for the put case, $X > S_T$

Single Barrier European Options

- Knock-in barriers: Option becomes live if the underlying crosses a barrier level, and then runs to expiration
- Knock-out barriers: Option is live until (if) the underlying crosses a barrier level, at which point is expires worthless
- May be applied to call or put European options
- What is interesting is that closed form solutions exist for every possible combination
- See James, Ch 15

The following definitions are used:

$$
\begin{aligned}
[\mathrm{BS}] &= \mathrm{e}^{-rT}\,\phi\{S_0\,\mathrm{e}^{(m+\frac{1}{2}\sigma^2)T}\,\mathrm{N}[\phi(\sigma\sqrt{T} - Z_X)] - X\,\mathrm{N}[-\phi Z_X]\} \\
[\mathrm{G}] &= \mathrm{e}^{-rT}\,\phi\{S_0\,\mathrm{e}^{(m+\frac{1}{2}\sigma^2)T}\,\mathrm{N}[\phi(\sigma\sqrt{T} - Z_K)] - X\,\mathrm{N}[-\phi Z_K]\} \\
[\mathrm{H}] &= A\,\mathrm{e}^{-rT}\,\phi\{S_0\,\mathrm{e}^{2b+(m+\frac{1}{2}\sigma^2)T}\,\mathrm{N}[\psi(Z'_X - \sigma\sqrt{T})] - X\,\mathrm{N}[\psi Z'_X]\} \\
[\mathrm{J}] &= A\,\mathrm{e}^{-rT}\,\phi\{S_0\,\mathrm{e}^{2b+(m+\frac{1}{2}\sigma^2)T}\,\mathrm{N}[\psi(Z'_K - \sigma\sqrt{T})] - X\,\mathrm{N}[\psi Z'_K]\}
\end{aligned}
$$

$$
\psi = \begin{cases} +1 & \text{up to barrier} \\ -1 & \text{down to barrier} \end{cases} \qquad \phi = \begin{cases} +1 & \text{call} \\ -1 & \text{put} \end{cases}
$$

$$
m = r - q - \tfrac{1}{2}\sigma^2; \qquad b = \ln(K/S_0); \qquad A = \exp(2mb/\sigma^2) = (K/S_0)^{2m/\sigma^2}
$$

$$
Z_X = \frac{\ln(X/S_0) - mT}{\sigma\sqrt{T}}; \qquad Z'_X = \frac{\ln(X/S_0) - mT - 2b}{\sigma\sqrt{T}}
$$

$$
Z_K = \frac{\ln(K/S_0) - mT}{\sigma\sqrt{T}}; \qquad Z'_K = \frac{\ln(K/S_0) - mT - 2b}{\sigma\sqrt{T}}
$$

Single Barrier European Options

The formulas for all the single barrier options are given in Tables 15.1 and 15.2.

Table 15.1 Single barrier knock-in options

Calls	Puts	Formula
$C_{d-i}(X < K)$	$P_{u-i}(K < X)$	[BS] – [G] + [J]
$C_{d-i}(K < X)$	$P_{u-i}(X < K)$	[H]
$C_{u-i}(X < K)$	$P_{d-i}(K < X)$	[G] + [J] – [H]
$C_{u-i}(K < X)$	$P_{d-i}(X < K)$	[BS]

Table 15.2 Single barrier knock-out options

Calls	Puts	Formula
$C_{d-o}(X < K)$	$P_{u-o}(K < X)$	[G] – [J]
$C_{d-o}(K < X)$	$P_{u-o}(X < K)$	[BS] – [H]
$C_{u-o}(X < K)$	$P_{d-o}(K < X)$	[BS] – [G] – [J] + [H]
$C_{u-o}(K < X)$	$P_{d-o}(X < K)$	0

Asian Options

- Options in which
  - the underlying asset is an average price, or
  - the strike price is an average of the underlying asset

- Average price options with payoffs for call and put of

$$\max[0, Av_T - X] \quad \text{and} \quad \max[0, X - Av_T]$$

- Average strike options with payoffs

$$\max[0, S_T - Av_T] \quad \text{and} \quad \max[0, Av_T - S_T]$$

- Two types of averages:
  - Arithmetic
  - Geometric

Asian Options

- Arithmetic: The average of a set of prices is most simply defined as

$$A_N = (N+1)^{-1} \sum_{n=0}^{N} S_n$$

- Geometric: An alternative type of average, defined by

$$G_N = \{S_0 \times S_1 \times \cdots \times S_N\}^{1/(N+1)}$$

is called the ***geometric average***. Taking logarithms of both sides gives

$$g_N = \ln \frac{G_N}{S_0} = \frac{1}{N+1} \sum_{n=0}^{N} \ln S_n = \frac{1}{N+1} \sum_{n=0}^{N} x_n$$

The x_n are normally distributed (see Section 3.1) and we know that the sum of normal random variables is itself normally distributed; the distribution of g_N is therefore normal.

- ***Use of Black Scholes Model for Geometric Average Price Options***: The A closed form pricing model can be constructed by substituting in averaged values in the Black Scholes formula for European options

- For ***Arithmetic Average Strike Options***, the result is more complicated

- Both are discussed in detail in James, Ch 17

[END]