CFRM505: Monte Carlo Methods in Finance (Winter 2021)

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Final Exam (take-home, open-book, work alone)

Exam Questions

Instructions: Attempt all questions. Points are for reference only and may be subject to change by the instructor after the exam.

1. (15pt) Imagine that we want to obtain samples of $X \sim N(0,1)$ using the acceptance-rejection method by generating samples of $Y \sim N(1, \sigma^2)$.

   (a) As simple as this approach may sound, it doesn't always work. Specifically, show analytically that this acceptance-rejection method yields samples of the target $N(0,1)$ random variable *if and only if* $\sigma^2 > 1$.

   (b) For any fixed σ^2, such that $\sigma^2 > 1$, derive analytically the minimum value of the constant a in the acceptance-rejection method.

2. A Poisson random variable with parameter $\lambda > 0$, denoted by $L \sim Poi(\lambda)$, has the probability mass function:

$$\mathbb{P}\{L = k\} = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

   Define a new probability measure $\mathbb{P}_t$ by exponential tilting/twisting:

$$\mathbb{P}_t\{L = k\} = \frac{e^{tk}}{\mathbb{E}\{e^{tL}\}}\mathbb{P}\{L = k\}, \qquad t \in \mathbb{R}.$$

   (8pt) What is the distribution of L under measure $\mathbb{P}_t$? Derive it analytically and give the parameter of the distribution.

   (2pt) Also, write down the range of values of t in which L has a *larger variance* under the new probability measure $\mathbb{P}_t$ (no proof needed).

3. (10pt) Let $Z_1, \ldots, Z_4$ be IID $N(0,1)$ random variables. Suppose we want to estimate the probability

$$\mathbb{P}\left\{\sum_{i=1}^{4} Z_i^2 \geq 20\right\}.$$

   Describe an *importance sampling* method that involves simulating 4 IID $N(0, \sigma^2)$ random variables with $\sigma^2 > 1$. State what you simulate and give the estimator (what's inside your expectation). (Note: take σ as a given fixed constant. You do not need to find the optimal σ. No coding needed.)

4. (25pt) Suppose the random terminal stock price under the historical measure $\mathbb{P}$ is modeled by

$$S_T = S_0 \exp\left((\mu - \frac{V^2}{2})T + V\sqrt{T}Z \right) \tag{1}$$

where $Z \sim N(0,1)$, V is a volatility random variable such that $V \sim U(0.1, 0.5)$. Assume Z and V are independent.

We wish to estimate the probability that the call option will be in the money:

$$\theta = \mathbb{P}\{S_T > K\},$$

where S_T is given in (1) above. Other parameters: $S_0 = 10, K = 11, \mu = 0.08, T = 1$.

(a) Implement in R/Python a simple MC simulation procedure, which involves simulating both Z and V, to estimate the probability θ. Also show the histogram of your estimator.

(b) Can V be used as a control variate? Yes/No. If not, explain in one sentence why not. If yes, simply state the estimator. No need to write down the algorithm for this part

(c) Describe a MC simulation procedure that simulates only one random variable (per sample) in estimating the probability θ. Then, implement in R/Python and return the estimate of the probability θ. Also show the histogram of your estimator.

5. (20pt) Let us construct two default indicators, denoted by $Y_i = 1_{\{X_i > 2\}}$, where

$$X_i = \sqrt{\rho}Z + \sqrt{1-\rho}\epsilon_i, \quad 0 < \rho < 1,$$

for $i = 1, 2$, and Z and ϵ_i's are IID $N(0,1)$.

(i) First, compute analytically the conditional expectation $E[Y_i \,|\, Z]$ and the conditional variance $Var(Y_i \,|\, Z)$. (note: $Var(Y_i \,|\, Z) = E[Y_i^2|Z] - (E[Y_i|Z])^2$.)

(ii) We want to estimate the *variance* of

$$\sum_{i=1}^{2} Y_i.$$

Use the *conditional MC* method: relate $Var(Y_1 + Y_2)$ to the conditional expectation and conditional variance that you've computed, and use this observation to derive the MC algorithm. Implement this in R/Python using sample size of at least 10000. You can set $\rho = 0.5$ in your implementation.

6. Consider the Black-Scholes model where the underlying stock S is $GBM(r, \sigma)$. For a given $p > 0$, the *power put* option pays $(K - S_T^p)^+$ at time T. The corresponding no-arbitrage option price at time 0 with initial stock price S_0 is given by the risk-neutral expectation:

$$p = \mathbb{E}^Q\{e^{-rT}(K - S_T^p)^+\}.$$

(a) (10pt) Describe the pathwise (PW) method to estimate the sensitivity of the option price with respect to interest rate r. In particular, make sure to state what random variable(s) you're simulating and write down the resulting estimator. Implement this in R/Python and return the mean and variance of the price sensitivity to r.

(b) (10pt) Describe the likelihood ratio (LR) method to estimate the sensitivity of the option price with respect to interest rate r. In particular, make sure to state what you're simulating and give the resulting estimator. Implement this in R/Python and return the mean and variance of the price sensitivity to r.

For both (a) and (b): Parameters: $S_0 = 10$, $\sigma = 20\%, r = 3\%, T = 1, p = 2$, $K = 10$. Use at least 100,000 samples (use up to 1 million if feasible).