HOMEWORK 2, Due October 21, 2020

Problem statements that seem clear to one person may not be clear to another and it is difficult to fully specify all details and notes that could prevent you from interpreting these problem differently than intended. If you need clarification, please post to the Canvas Discussion Forum or contact the instructor or TA.

1. (2 points each) Indicate whether each statement is True or False.

   (a) For the Furniture Maker problem, a change in the price of chairs will cause a different production schedule to be optimal.

   (b) Eliminating non-binding constraints can cause the optimal solution of an LP to become unbounded.

   (c) A linear program with multiple inequality constraints must have some of them as non-binding.

   (d) The constraint $5x_1 + 3x_2 \leq 6$ becomes more restrictive as the righthand side value is decreased.

   (e) A linear program with a closed and bounded feasible region can never have an unbounded optimal solution.

2. An investor can invest in exchange-traded equity with an expected return of 6% per year. Alternatively the investor can invest in private equity in the secondary market and expect to earn a return of 12% per year, but this investment requires a 3 year holding period. Finally, the investor also has the opportunity to invest in a primary private equity offering with an expected return of 16% per year, but this investment has a 5 year holding period. Each of these investment opportunities is available in the marketplace every year. The investor has $10,000,000 to invest.
   Assume that investment decisions are decided at the beginning of each year (therefore exchange-traded equity should be treated as though it has a 1 year holding period) and that all investments must be liquidated at the end of year 12. In addition to the budget relationships that relate purchases to available funds, the investor requires that at every point in the investment horizon, the average remaining holding period across all investments is no greater than two years.
   What strategy (i.e., choice of investments each year) should the investor follow over the next 12 years to maximize the expected portfolio value at the end of year 12?

   (a) (10 points) Formulate this problem as a linear program. Indicate and define all decision variables and model constraints. Be sure to differentiate between variables and parameters.

   (b) Solve using R and the glpk solver and submit your script as part of your homework solution.

      i. (4 points) Please identify the optimal values of all variables.

ii. (4 points) Which variables are basic in the optimal solution?

iii. (4 points) Which constraints are active at the optimal solution?

iv. (4 points) Which constraints are inactive at the optimal solution?

v. (4 points) How much could the available investment amount (i.e., the $10,000,000) decrease before a different solution is optimal?

3. Joe and Donald are running against each other for the office of County Dog Catcher. As election day approaches polls have Donald trailing by 6% (Joe has 53% with Donald at 47%), there are still a few strategies they can try to gain votes. Of course, the payoff for each depends on the strategy selected by their opponent. Donald's remaining campaign funds are $4000 while Joe holds $1000 in his account. Each must decide on a strategy without knowing what the opponent will do.
The table below shows the benefit to Joe's campaign if each campaign selects the associated pure strategies and spends all remaining funds on it. Ignore the fact that it's difficult to get a partial endorsement from the Mayor (maybe it just means it's just a lukewarm endorsement). Assume candidates are able to split funds across strategies with proportionately lower benefit.

Donald \ Joe	30 second radio	Newspaper ad	Send 10,000 emails
30 second radio ad	2%	-5%	5%
200 windshield flyers	3%	-15%	10%
Mayor's endorsement	-8%	5%	-6%

(a) (10 points) Formulate this two-player, zero-sum game as a linear programming problem.

(b) (15 points) Solve the linear program using your favorite method and tools. Report on the optimal strategy for each company and the expected outcome (gain or loss of votes). Who's expected to win?

(c) (10 points) What is the dual of Joe's linear programming problem?

Extra Credit

1. Solve the Cash Flow Matching problem using Excel. Highlight cells associated with decision variables in blue, cells associated with RHS in yellow, cells associated with objective function coefficients in green, and the cell holding the objective function value in red.

(a) (1 points) Please identify the optimal values of all variables.

(b) (1 points) Which variables are basic in the optimal solution?

(c) (1 points) Which constraints are active at the optimal solution?

(d) (1 points) Which constraints are inactive at the optimal solution?

(e) (1 points) What is the impact of a decrease of \$1 in the size of the line of credit? Support your answer using sensitivity information from the optimal solution rather than simply resolving.

2. (3 points) Refer to Homework 1, Extra Credit problem 1. Following the hint, we can replace each unrestricted variable by 2 non-negative variables. n unrestricted variables become $2n$ non-negative variables. Can you instead develop an equivalent linear program that replaces n unrestricted variables by $n+1$ non-negative variables?

3. (3 points) On slide 15 of the LinearProgramming3.pdf slide deck, the formulation for the "Standard dual" is not included. Derive the formulation of the Standard Dual using the definition embedded in the Von Neuman Primal and Dual. Show your work.

4. (2 points) Can the primal and dual both be infeasible? (A simple "yes" or "no" answer is unlikely to get you many points)

5. **Finding a Center** (7 points) The center can be defined in different ways, but the idea is to find a point that is equally away from the boundaries of the feasible region. Identifying a center is an important topic in interior-point solution methods for linear programming, because the algorithm often requires fewer iterations and has greater numerical stability if a series of solutions is found that stays away from the constraints/boundaries of the feasible region.
Let $Ax \leq b$, $x \geq 0$ define the feasible region. Each constraint defines a hyperplane in R^n and the distance from a point $\hat{x}$ to a hyperplane is

$$d_i(\hat{x}) = \frac{b_i - A_{i,*}\hat{x}}{\|A_{i,*}\|} \text{ with } \|A_{i,*}\| = \left(\sum_{j=1}^{n} A_{i,j}^2\right)^{\frac{1}{2}}$$

$\|A_{i,*}\|$ is also known as the L-2 norm.
Let $\bar{b}_i = \frac{b_i}{\|A_{i,*}\|}$ and $\bar{A}_{i,*} = \frac{A_{i,*}\hat{x}}{\|A_{i,*}\|}$ and $\bar{d}_i(\hat{x}) = \bar{b}_i - \bar{A}_{i,*}$. $\bar{d}_i(\hat{x})$ is both the distance from $\hat{x}$ to the i^{th} hyperplane as well as the slack in the newly scaled i^{th} constraint.
The center can be determined from the optimization problem

$$\begin{aligned} \text{Maximize(Minimum}_{i=1,\ldots,m} \quad & d_i) \\ \text{subject to:} \quad & \bar{A}x + Id = \bar{b} \\ & x \geq 0 \\ & d \geq 0 \end{aligned}$$

(a) Reformulate this optimization problem as a linear program.

(b) Find the center of the constraints (including non-negativity) of the Furniture Maker example.

(c) Plot the feasible region and the identified center.