COMPUTATIONAL FINANCE & RISK MANAGEMENT
UNIVERSITY of WASHINGTON
Department of Applied Mathematics

Equity Options
Black-Scholes Formula

CFRM 425 (018)

R Programming for Quantitative Finance

Lecture References

- Ang Ch 9: §§9.2 - 9.4 (Old CFRM 425 text)
- M Jackson & M Staunton (Old CFRM 506 text)
  - Advanced Modelling in Finance Using Excel and VBA (Wiley, 2001)
    - Ch 9: §§9.1, 9.2, 9.6-9.10 (Introduction)
    - Ch 11: §§11.1-11.4
    - Sample spreadsheets and VBA code for equity options (CD with book)
- Peter James
  - Option Theory (Wiley, 2003)
    - General background
    - Excellent reference book to have in your arsenal
- Wilmott, Howison, Dewynne
  - The Mathematics of Financial Derivatives, A Student Introduction (Cambridge University Press, 2002)
    - Ch 2: Asset Price Random Walks (and Ito's Lemma)
    - Ch 3: The Black-Scholes Model
    - Ch's 2 and 3 provide a very nice mathematical derivation of Black-Scholes

Derivatives and Options

- A derivative is a financial instrument dependent on an underlying asset (eg equity, bond) or rate (eg interest rate, foreign exchange rate)
- An equity option is a derivative that gives the holder the right to buy or sell the equity at a given price (the strike price, also called the exercise price) on or before an expiration date
  - Call option: option to buy
  - Put option: option to sell
- Basic option exercise types
  - European: may only be exercised on the expiration date
  - American: may be exercised any time before or on the expiration date
  - Bermudan: may be exercised on specific dates before expiration, or on the expiration date (Bermuda lies between Europe and the US)
- We will look at the Black-Scholes option pricing formula for European options, and use the results for exercises in R
- A brief mention of American options is included at the end (a popular interview question…)

- Published in 1973

European Options and the Black-Scholes Formula

- Let
  - S = current equity (stock) price
  - r = risk-free rate of interest (constant)
  - X = exercise price
  - T = time to maturity (year fraction)
  - q = dividend rate
  - $N(z)$ = standard normal cumulative distribution function (CDF)
- The price of a call option according to the model is

$$c = Se^{-qT}N(d_1) - X\,e^{-rT}N(d_2)$$

- The price of a put option according to the model is

$$p = -Se^{-qT}N(-d_1) + X\,e^{-rT}N(-d_2)$$

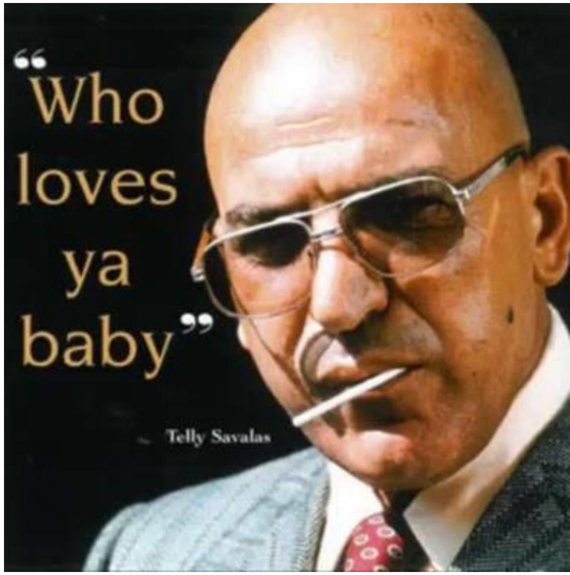
where

$$d_1 = \frac{\left[\log\left(\frac{S}{X}\right) + (r - q + 0.5\sigma^2)T\right]}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\left[\log\left(\frac{S}{X}\right) + (r - q - 0.5\sigma^2)T\right]}{\sigma\sqrt{T}}$$

- For hedging purposes, risk values – also called "option Greeks" – also typically need to be calculated. These are mathematical derivatives with respect to the parameters in option pricing models.
- By differentiating through the Black-Scholes formula, we conveniently arrive at closed-form formulae for these values as well
- First, rewrite the Black-Scholes formula as follows, with $\phi = 1$ for a call option, and $\phi = -1$ for a put (James, § 5.4):

$$f = \phi S e^{-qT} N(\phi d_1) - \phi X e^{-rT} N(\phi d_2)$$



Famous Greek: Telly Savalas (Kojak)

- Then:

$$\Delta \coloneqq \frac{\partial f}{\partial S} = \phi e^{-qT} N(\phi d_1)$$

$$\Gamma \coloneqq \frac{\partial^2 f}{\partial S^2} = N(d_1)\,\frac{e^{-qT}}{S\sigma\sqrt{T}}$$

$$\Theta \coloneqq \frac{\partial f}{\partial T} = \phi q S e^{-qT} N(\phi d_1) - \phi r X e^{-rT} N(\phi d_2) - \frac{S e^{-qT} N(d_1)\sigma}{2\sqrt{T}}$$

$$v\ (vega) \coloneqq \frac{\partial f}{\partial \sigma} = X e^{-rT} N(d_2)\sqrt{T}$$

$$\rho \coloneqq \frac{\partial f}{\partial r} = \phi T X e^{-rT} N(\phi d_2)$$

European Options and the Black-Scholes Formula

- Given these closed form formulae, you can now implement them them in R
- Their actual utility is in hedging portfolios with options and risk management
- The topic of hedging is in other CFRM courses on derivatives, so we will not pursue it in detail in this course
- One interesting result, however, is to write the Black-Scholes PDE in terms of the hedge values (Wilmott et al, p 43):
  - The PDE for the value of an option $f(S,t)$ is

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

  - Substituting in the option Greeks (risk values), we get

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = rf$$

  - Because r and σ are assumed constant, ρ and $vega$ are not present; however, sensitivities to these parameters are certainly considered in more advanced option modeling

Put-Call Parity (a typical interview question)

- Put-Call Parity for European options (with no dividend):
  - Let c and p be the respective prices of a call and a put option on the same underlying equity at time $t = 0$, with expiration at time T, with no dividend paid on the underlying. S is the price of the underlying at $t = 0$.
  - Then, by the no-arbitrage property:

$$c - p = S - Xe^{-rT}$$

  - Using Black Scholes, and the identity $N(d_i) + N(-d_i) = 1,\ i = 1, 2$:

$$p = -SN(-d_1) + Xe^{-rT}N(d_2)$$

- This will be covered in more detail in courses on options and derivatives

- Mentioned here as you may also be asked about it during an interview

Should You Exercise an American Option Early?

- Another typical interview question; again assumes there is no dividend paid on the underlying equity.
- For simplicity, assume time we are at time $t = 0$, with expiration at time T. Let V_c^{am} be the current price of the American option, and V_c^{eur} that of the European option.
- Because an American option carries the right to early exercise, we should expect it to be worth more than an otherwise equivalent European option:

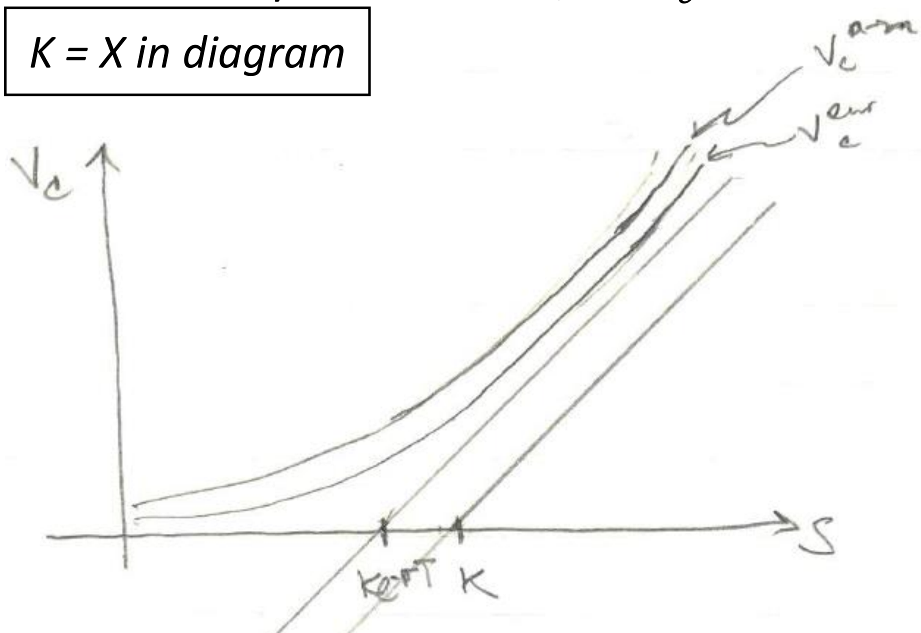
$$V_c^{am} > V_c^{eur} \geq S - Xe^{-rT} \text{ (discounted value of a positive payoff at expiration)}$$

- But,

$$S - Xe^{-rT} \geq S - X = \text{payoff at expiration}$$

- Since we would only get $S - X$ at an early exercise date, but $V_c^{am} > S - X$, there would be no reason to exercise early.

K = X in diagram



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