COMPUTATIONAL FINANCE & RISK MANAGEMENT
UNIVERSITY of WASHINGTON
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Value at Risk (VaR) and Expected Shortfall

CFRM 425 (015)

R Programming for Quantitative Finance

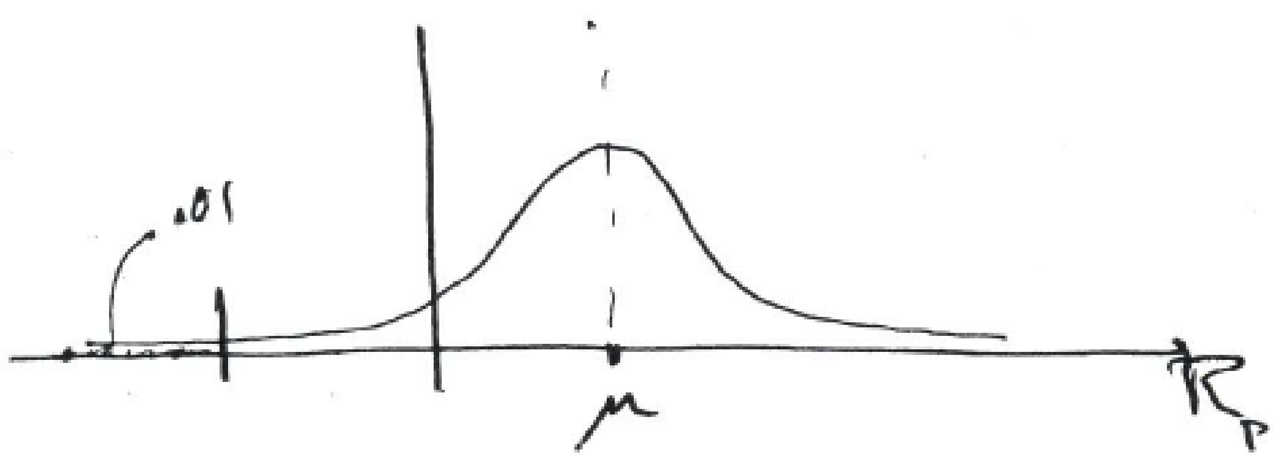
Lecture References/Package Downloads

- Jeet & Vats, Ch 7: VaR and Monte Carlo Simulation (two sections of the chapter)
- Ang, Ch 4: §§4.4-4.5
- Wikipedia: Expected Shortfall (Mathematical Details)
  - https://en.wikipedia.org/wiki/Expected_shortfall

- Topics:
  - Value at Risk (VaR)
    - Introduction
    - Parametric VaR
    - Historical VaR
    - Simulated VaR (brief discussion)
  - Expected Shortfall

Value at Risk (VaR)

- Value at Risk (VaR) is a measure of the worst possible portfolio loss over a given time period (eg one month, one year, two weeks, etc)
- We need to find the distribution of returns for the portfolio, and then determine, say, the 1% percentile
- This says there is a 1% probability of suffering a loss at this percentile level or lower
- So, for example, if R_p represents the portfolio return and has mean μ, our plot might look something like this:



Value at Risk (VaR)

- Remarks:
    1. When we mention VaR in this course, it specifically refers to *Market VaR*; there is also *Credit VaR*, but this is beyond the scope of our class.

    2. We will only consider the case of equity risk; more sophisticated models and systems will also incorporate derivatives pricing models, bonds, and stochastic interest rate and volatility models for measuring VaR.

Methods for Measuring Value at Risk (VaR)

- Methods for Measuring Value at Risk (VaR)
  - Parametric Methods (typically with the assumption of normality)
  - Historical VaR (no distribution assumption)
  - Monte Carlo Simulation (no distribution assumption)
- For this section of the course, we will consider the parametric case with assumption of normality, and the historical case
- In higher level CFRM courses, more sophisticated methods, such as fat-tailed distributions, and Monte Carlo simulation, are presented

Speaking of Fat Tails

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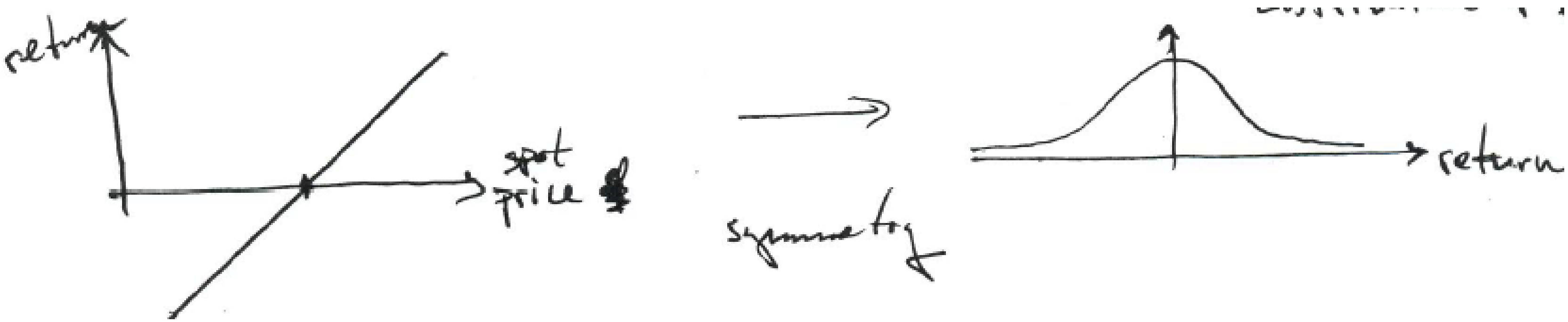
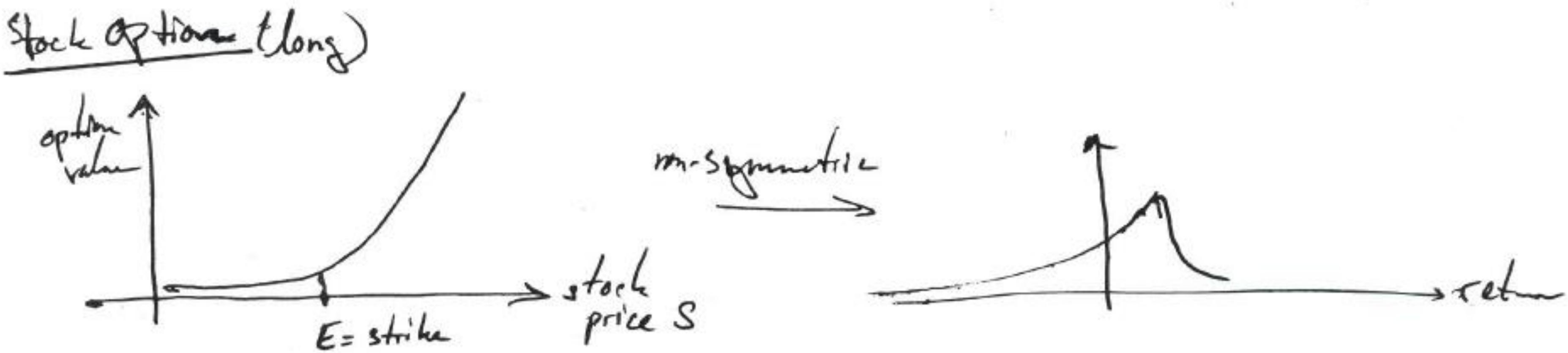
Value at Risk (VaR)

- Parametric VaR with normality assumption for returns is the simplest case, but coding it up is illustrative of some of the tasks you may have in practice
- It is used in practice, but with caveats:
  - The symmetry of the normal distribution is appropriate for investments such as equities, FX, and futures, but will not capture the asymmetry in returns found in options portfolios
  - It will not capture the fat tails that are present in reality

- Symmetric returns (eg, equity holding):



- Non-symmetric returns:



(Parametric) Value at Risk (VaR)

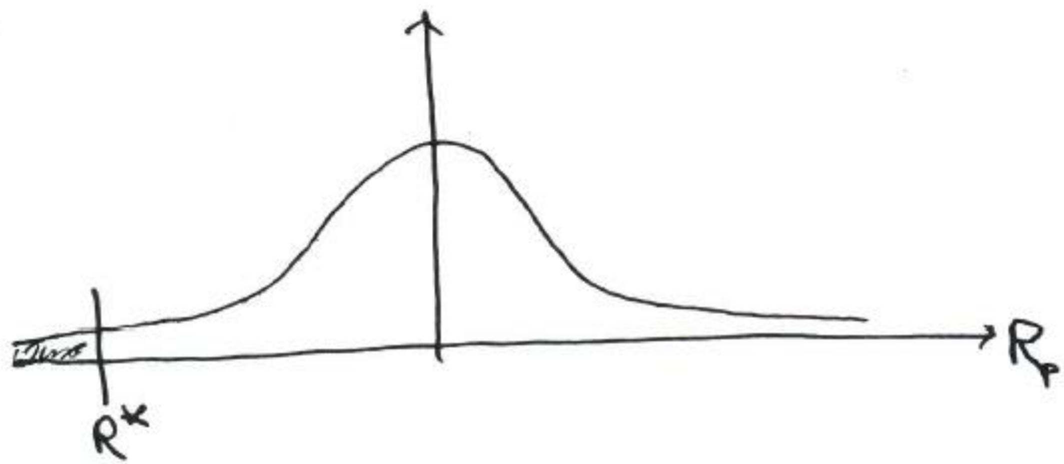
- Consider a portfolio of n assets (eg equities) with annual returns R_1, $R_2, \dots, R_n$, and the weights of each in the portfolio are $w_1,\ w_2, \dots, w_n$. Assume that $R_i \sim N\left(0, \sigma_i^2\right), i = 1, \dots, n$.
- Remark: This assumption will give us the *Relative VaR*
- We need to find the distribution of

$$R_p = \sum_{i=1}^{n} w_i R_i = \boldsymbol{w}'\boldsymbol{R} \in \mathbb{R}$$

where the p subscript stands for *portfolio.*

- As R_p is the sum of normal random variables, it itself is normal
- $E\left(R_p\right) = 0$ as it is the weighted sum of zero expectations
- It remains to find the variance of the portfolio, σ_P^2
  - As in the case of our earlier portfolio optimization examples, we have a covariance matrix $\boldsymbol{\Sigma}$
  - We then have $\sigma_P^2 = \boldsymbol{w}'\boldsymbol{\Sigma}\,\boldsymbol{w}$, and hence $R_p \sim N(0, \sigma_P^2)$

- Now, let R^* be the 1%-tile value for the distribution of R_p



- Apply the usual transformation to $N(0,1)$: $Z = \dfrac{R_p}{\sigma_P}$ $(\mu_p = 0)$
- From the standard normal table we get $\dfrac{R^*}{\sigma_P} = -2.33$, and hence

$$R^* = -2.33\sigma_P$$

- Now, if our initial portfolio value is, say, V_0, then the one-year VaR is

$$-2.33\sigma_p V_0$$

Value at Risk (VaR)

- Two more remarks
  - When making the normality assumption, one typically uses log returns from asset price data, viz

    $R_{i,t} = \ln(\frac{S_{i,t}}{S_{i,t-1}})$, where $S_{i,t}$ = price of asset i at time t.
  - We have two ways of constructing the covariance matrix
    - Brute force
    - Single Index Model (less computation; follows from the CAPM model, which we will cover later)

- Question: How would we calculate annual VaR if working with daily returns?

Value at Risk (VaR)

- Historical VaR
  - No distributional assumption
  - Can capture fat tails in actual data
- Procedure
  - Calculate a set of portfolio returns based on a “large” set of individual portfolio asset returns; assume static asset weights
  - Locate the α quantile
    - ➢ Sort returns in ascending order into a vector, and find the quantile by dividing by the total number of data points; or,
    - ➢ Use the **`quantile(.)`** function in R; or,
    - ➢ Approximate a density using Kernel Density Estimation, using the R **`density(.)`** function, and then locate the quantile value
      - Described briefly in the Ang book
      - However, is a more advanced topic that really requires more explanation
  - Multiply the total portfolio value by the quantile value to get the historical VaR value
- The methods described above are different (and simpler) than that described in Ang, but they use the same underlying mathematics

Value at Risk (VaR)

- Answer to scaling question
- On p 145 of the Ang book:

Using the Square Root of T rule to Scale 1-Day VaR To a T-Day VaR
Assuming the returns are independent and identically distributed (i.i.d), we can calculate a T-Day VaR by multiplying the 1-Day VaR by $\sqrt{T}$. For example, our 1 % 1-Day Historical VaR was \$ 39,744. Assuming returns are i.i.d., the 1 % 10-Day Historical VaR is equal to \$ 125,682.

```
> VaR01.10day<-quantile(-sim.portPnL$Port.PnL,0.99)*sqrt(10)
> VaR01.10day
     99%
125681.3
```

Value at Risk (VaR)

- No accompanying sample code; will be an exercise
- With guidance as you work through it (your instructor does not recommend the Ang book's examples and sample code)
  - Parametric VaR
  - Historical VaR
- Discussion of Simulated VaR
  - Again, no distributional assumptions
  - Can capture fat tails
  - Simulation process, however, itself may use a normality assumption
  - See board for more information
  - Uses the standard lognormal result for generating equity prices:

$$S_t = S_{t-1} e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma \varepsilon_t \sqrt{\Delta t}}$$

    where S_t = underlying (eg equity) price at time t, Δt = time step year fraction, r = risk-free rate, σ = volatility, and $\varepsilon_t \sim \mathrm{N}(0, 1)$ iid.
  - Multiple asset prices require correlated generation
- The PerformanceAnalytics package has VaR (and ES) functionality
- However, it is reasonably straightforward to code by yourself

Expected Shortfall

Expected Shortfall

- VaR is that it does not capture the shape of the tail of the distribution. That is, VaR does not allow us to answer the question, “if losses exceed VaR, how bad should we expect that loss to be?”
- Wikipedia says “Informally, and non rigorously, (expected shortfall) amounts to saying "in case of losses so severe that they occur only alpha percent of the time, what is our average loss“”.
- Ang: ES tells us how much we should expect to lose if our losses exceed VaR (p 138)
- The answer to this is called expected shortfall (ES). ES is known by many other names, such as tail VaR and tail loss
- Actuaries call this Conditional Tail Expectation (CTE); actual calculations may differ to some extent based on regulations
- We will look at
  - Gaussian ES
  - Historical ES

Expected Shortfall

- Historical ES: Compute the mean portfolio return over the lower tail
- Do this one first (before parametric version) – better conceptual view
- Simple calculation:
  - Extract the subset of all portfolio returns at or less than VaR for returns
  - Calculate the mean portfolio return from this set

- Gaussian ES

$$\mu + \sigma f(F^{-1}(\alpha))/\alpha$$

where **`f(.)`** the standard normal pdf, and **`F(.)`** is its cdf

- In R parlance:

$$\mu + \sigma * \mathrm{dnorm}(\mathrm{qnorm}(\alpha))/\alpha$$

- These are derived from an averaged integral of VaR over $[0, \alpha]$ (see the Wikipedia article for mathematical details; accept as given here)

[END]