Rosenfeld Counting: Proper Conflict-free Coloring of Graphs with Large Maximum Degree

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Joint with Chun-Hung Liu

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Nonrepetitive List-coloring of Paths

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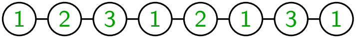

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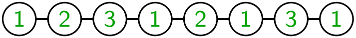

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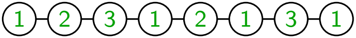

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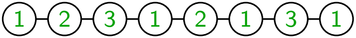

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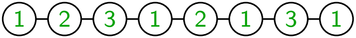

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Since $|\mathcal{C}_1| = 4$, path P_n has more than 2^n nonrepetitive L-colorings.

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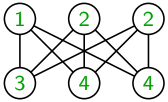
Defn: A proper coloring of G is conflict-free if every non-isolated vertex of G has some color appearing exactly once on its open neighborhood.



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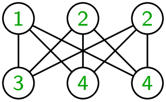

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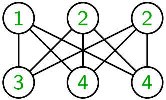

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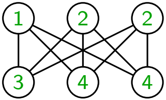

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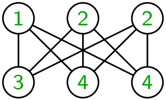

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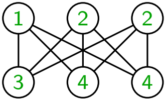

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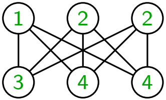

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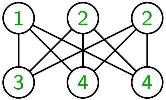

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Rem: Liu and Reed showed that $\chi_{pcf}(G) \leq \Delta(1 + o(1))$. This bound is stronger than ours, but much less general.

Key Rosenfeld Counting Lemma

Defn: For an integer t, a graph G, and a hypergraph $\mathcal{H}$ with $V(\mathcal{H}) = V(G)$, a coloring φ is a proper t-conflict-free coloring of $(G, \mathcal{H})$ if φ is a proper coloring of G such that for every $f \in E(\mathcal{H})$, some color is used k times by φ on f for some $k \in \{1, \ldots, t\}$.

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Defn: Fix $t, i, d \in \mathbb{Z}^+$. The t-associated Stirling number of second kind, $S_t(d, i)$, is the number of partitions of the set $\{1, \ldots, d\}$ into i parts, each of size at least t.

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Defn: For an integer t, a graph G, and a hypergraph $\mathcal{H}$ with $V(\mathcal{H}) = V(G)$, a coloring φ is a proper t-conflict-free coloring of $(G, \mathcal{H})$ if φ is a proper coloring of G such that for every $f \in E(\mathcal{H})$, some color is used k times by φ on f for some $k \in \{1, \ldots, t\}$.

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Key Lem: Fix G, $\mathcal{H}$, t as above. Let β be a real number. If a is a real number such that

$$a \geq \Delta(G) + \beta + \sum_{f \in E(\mathcal{H}), f \ni v} \sum_{i=1}^{\lfloor |f|/(t+1) \rfloor} S_{t+1}(|f|, i) \cdot \beta^{i-|f|+1}$$

for every $v \in V(G)$, then for every a-assignment L of G, there are at least $\beta^{|V(G)|}$ proper t-conflict-free L-colorings of $(G, \mathcal{H})$.

Bounding $S_2(d, i)$

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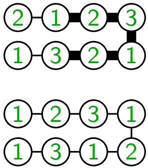

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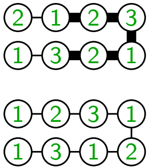

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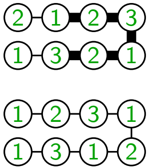

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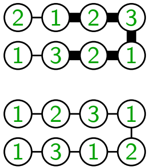

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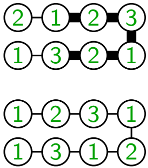

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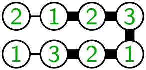

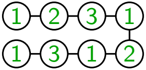

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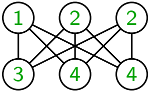

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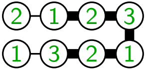

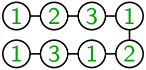

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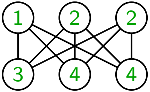

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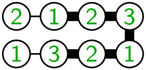

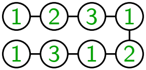

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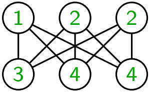

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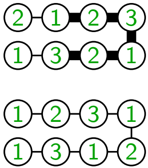

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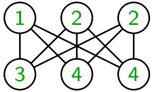

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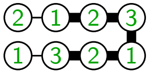

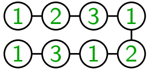

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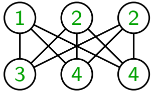
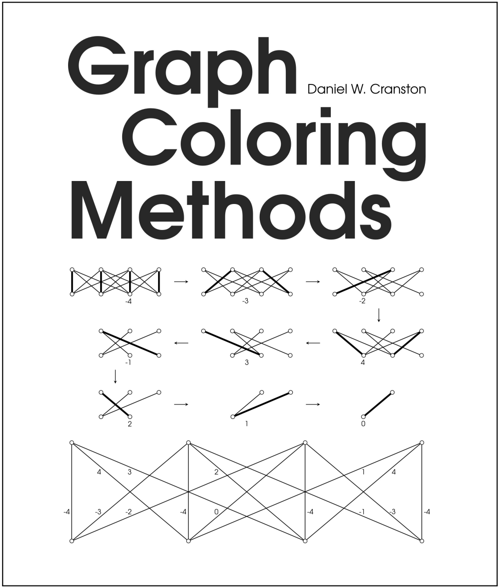


Learn More about Rosenfeld Counting



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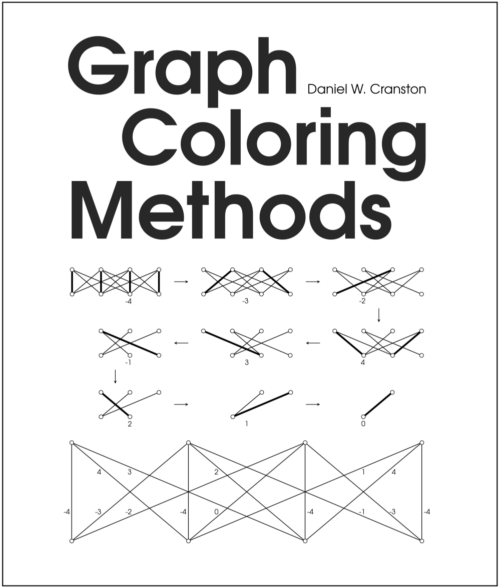

https://graphcoloringmethods.com

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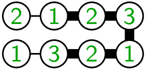

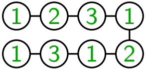

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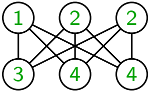

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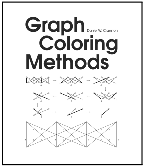

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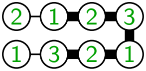

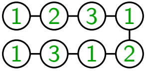

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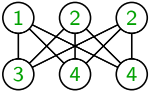

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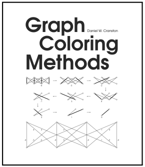

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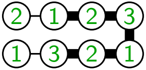

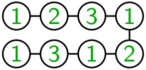

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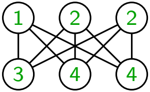

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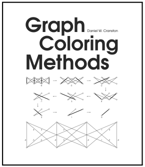

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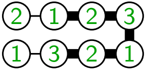

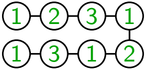

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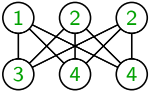

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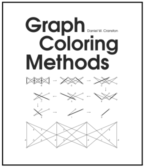

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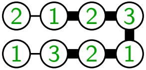
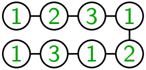
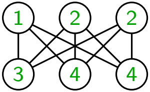
  - Color iteratively
  - $\#$ partial colorings exponential in $|$subgraph$|$
    - Bad colorings $\rightarrow$ good colorings of subgraph
    - Exponentially many good colorings remain





- Proper Conflict-free Coloring
  - CPS conjectured $\chi_{pcf}(G) \leq \Delta + 1$
  - We proved $\chi^{\ell}_{pcf}(G) \leq 1.6551\Delta(1 + o(1))$
  - Corollary of General hypergraph framework
  - Key step is bounding $S_2(d, i)$



- Graph Coloring Methods
  - Graduate Textbook (450 pages)
  - Chapter on Rosenfeld Counting
  - 11 other chapters (each on 1 method)
  - Free at `graphcoloringmethods.com`