Reconfiguration of Colorings and List Colorings: Proofs and Conjectures

Daniel W. Cranston
dcransto@gmail.com

CanaDAM
21 May 2025

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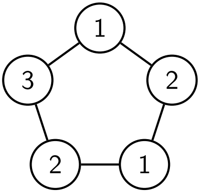

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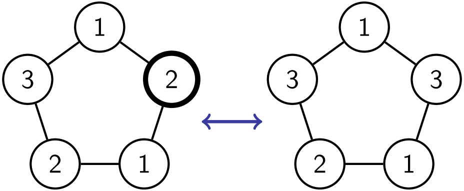

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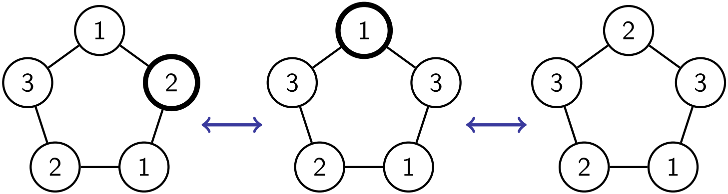

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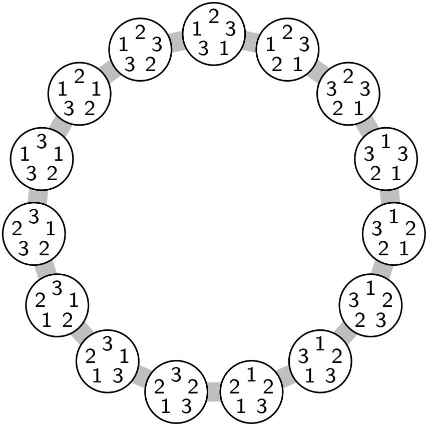

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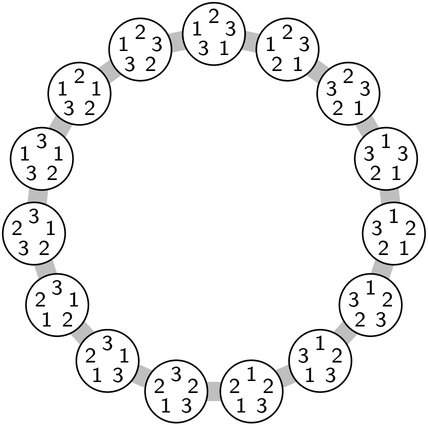

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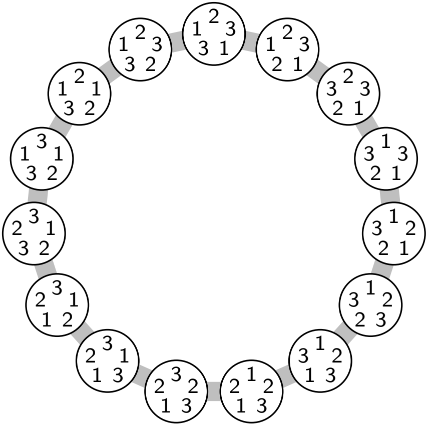

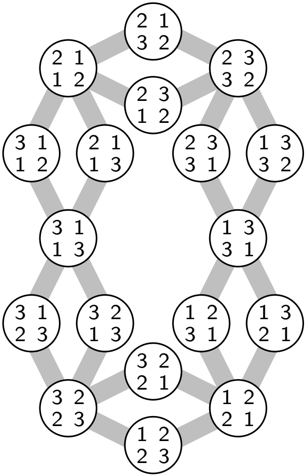

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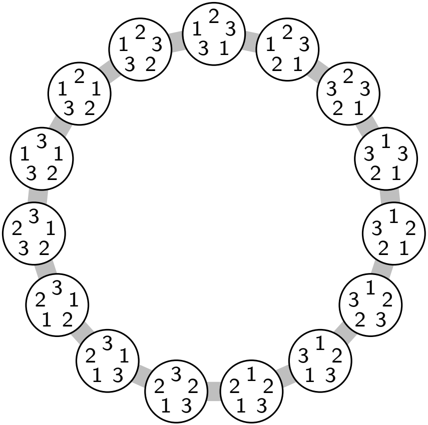

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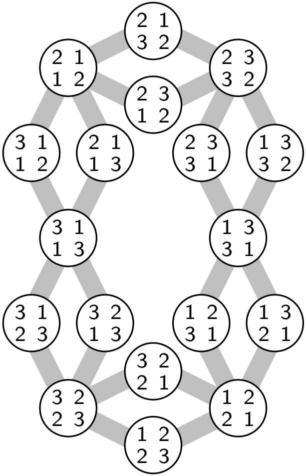

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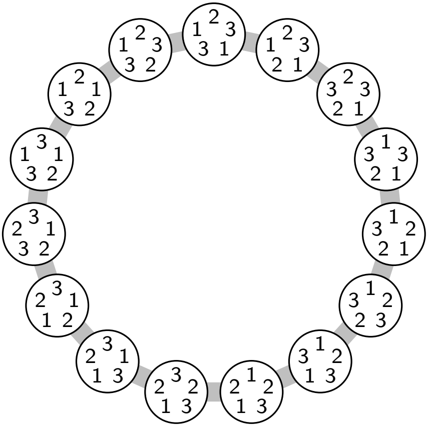

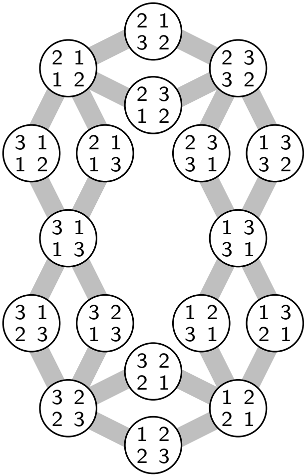

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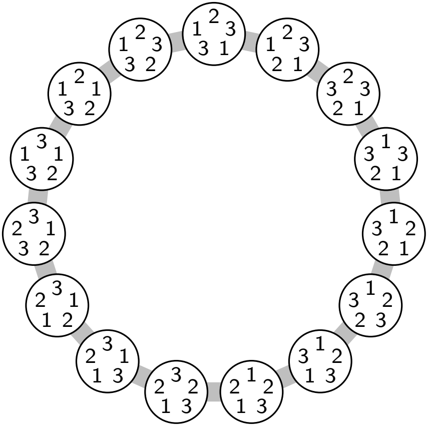

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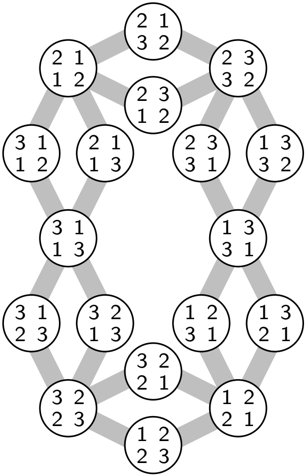

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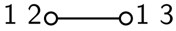

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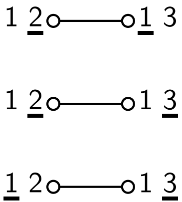

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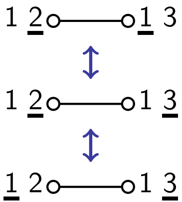

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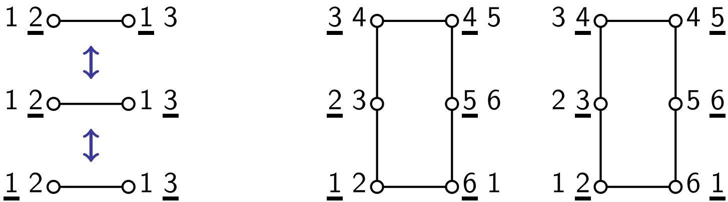

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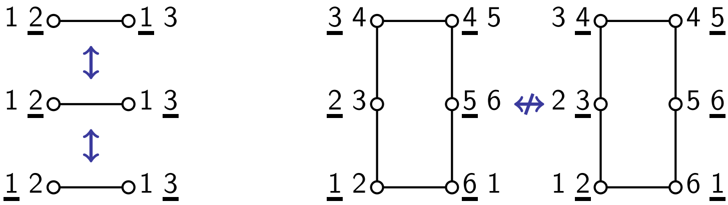

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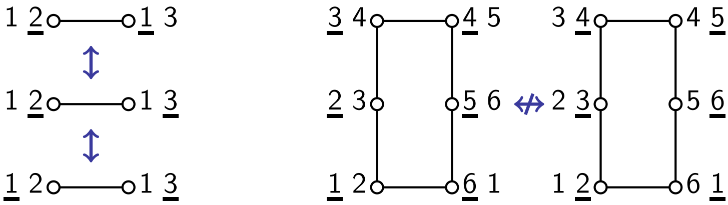

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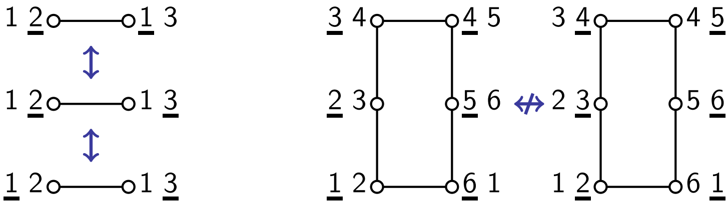

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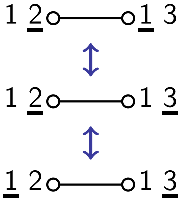

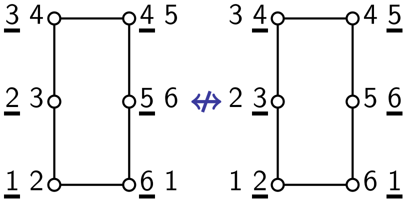

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- Given list-assignment L, can we transform every L-coloring α into every L-coloring β?
- If so, how many steps are needed in the worst case?

Lists of Size $d(v)+2$

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Prop: For every G and every f, with $f(v) \geq 2$ for all v, there is list assignment L with $|L(v)| = f(v)$ for all v and L-colorings α and β where changing α to β needs $n(G) + \mu(G)$ moves.

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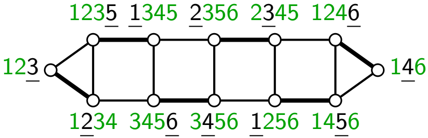

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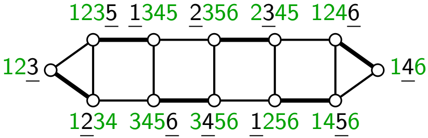

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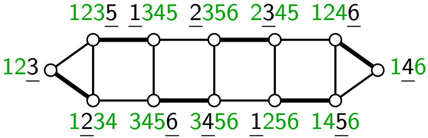

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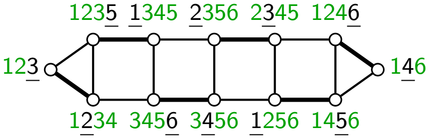

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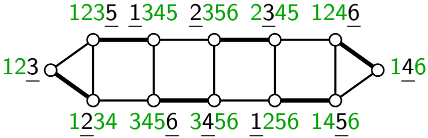

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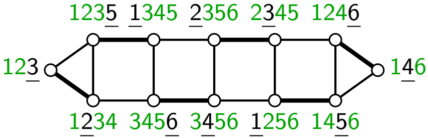

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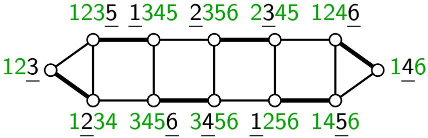

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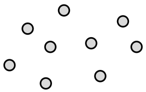

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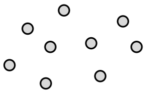

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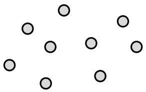

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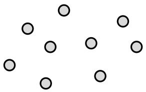

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Thm: Let a graph G be 3-connected and regular. If α and β are unfrozen $(\Delta+1)$-colorings of G, then α can reach β.

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Bonus Lem: Say G is 3-connected with $|L(v)| \geq d(v) + 1$ for all v. Let x_1, x_2 be at distance 2. If α and β are L-colorings with $\alpha(x_1) = \alpha(x_2) = \beta(x_1) = \beta(x_2)$, then α can reach β.

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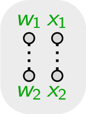
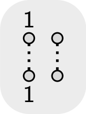
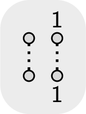




α



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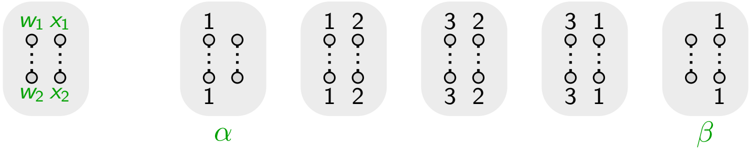

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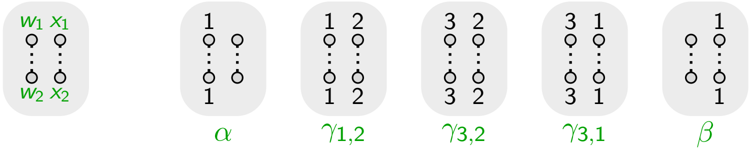

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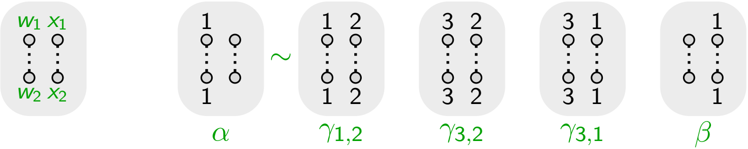

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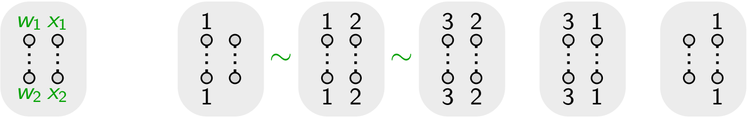

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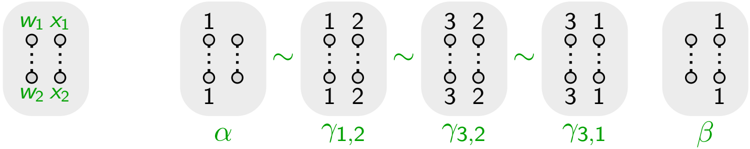

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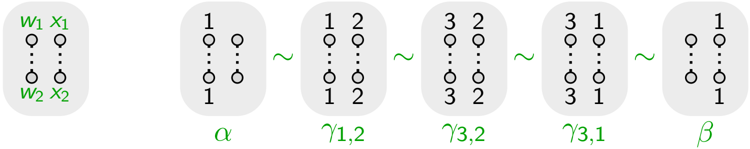

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- $\mathcal{G}$ is graphs with $\text{mad}(G) < a$, for some a; or
- $\mathcal{G}$ is planar graphs with girth at least g, for some g.

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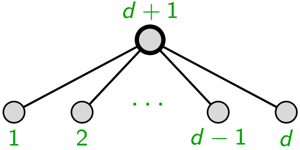

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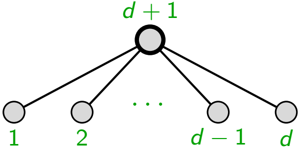

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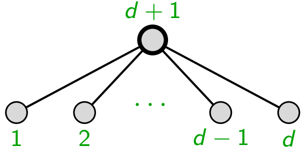

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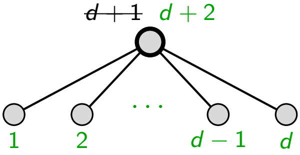

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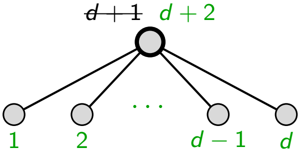

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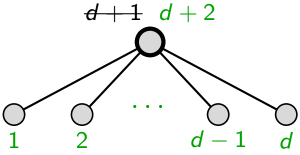

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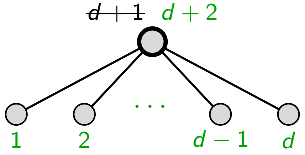

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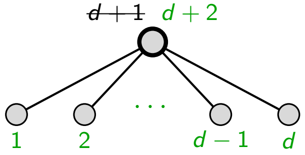

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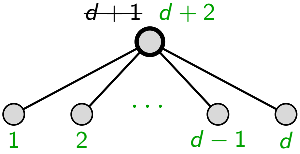

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Thm: If $|L(v)| \geq d(v) + 2$, then diam $\mathcal{C}(G, L) \leq n + 2\mu(G)$.
Thm: If $|L(v)| \geq 2d(v) + 1$, then diam $\mathcal{C}(G, L) \leq n + \mu(G)$.
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Correspondence: $\mu(G) \to \tau(G)$.

Summary

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Thm: If $|L(v)| \geq d(v) + 1$ and $\Delta \geq 3$, then $\text{diam}\, \widehat{\mathcal{C}}(G, L) \leq O(n^2)$.

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Correspondence: Analogue is false (as shown by cliques).

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Thm: If planar and $|L(v)| = 10$, then $\operatorname{diam} \mathcal{C}(G, L) = O(n)$.

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Thm: If planar and $|L(v)| = 10$, then $\operatorname{diam} \mathcal{C}(G, L) = O(n)$.
Thm: If planar, ∇-free and $|L(v)| = 7$, then $\operatorname{diam} \mathcal{C}(G, L) = O(n)$.

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Conj: If d-degen and $|L(v)| \geq d + 3$, then diam $\mathcal{C}(G, L) = O(n)$.

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Conj: If d-degen and $|L(v)| \geq d + 3$, then $\text{diam}\ \mathcal{C}(G, L) = O(n)$.
Correspondence: Analogues of theorems are true.