On the Phase Change in the reflected wave of a dielectric mirror (Checking Marty Fejer's math)

Edgard Bonilla (Drawing heavy inspiration from Marty Fejer)

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1 Introduction

The objective of this writeup is to check the calculations for the derivative of the phase of the reflected electromagnetic wave coming out of a quarter-wave stack of dielectric materials.

We assume that there is a scalar quantity γ that has an effect on the properties of the dielectric materials in question. We want to calculate the derivative:

$$\frac{d\phi}{d\gamma}|_{\gamma=\gamma_0} \tag{1}$$

In our case is $\frac{d\phi}{dE}$

where ϕ is the phase of the reflected wave relative to the incident one. γ_0 is the nominal value of the parameter γ, under which the system should look like a quarter-wave stack.

We also assume for most of the derivation that the electric field of the electromagnetic wave is parallel to the surface of the material. However, since we are interested in the normal incidence case this will have no effect on the results.

1.1 Notation

- The term 'nominal values' is intended to represent the ideal conditions for the dielectric material stack. There, they look like a quarter wave stack and $\gamma = \gamma_0$.
- x' represents the derivative of x with respect to γ.
- $\hat{x}$ represents the quantity x evaluated at the nominal value γ_0. This superscript will be ommited in quantities that are defined only at this value of γ
- $\hat{x'}$ represents the derivative of x with respect to γ, evaluated at the nominal values for the stack.

1.2 Definitions

We will start by assuming a result for the reflectivity of a multilayer dielectric (this is discussed, in Born and Wolf section 1.6, in a very detailed way).

The reflectivity for N alternating pairs of layers of thickness h_2 and h_3 and refractive indices n_2 and n_3, lying between a semi infinite input medium of index n_1 and an output medium with index n_f for a harmonic wave of vacuum wavelength λ_0 is given by an expression of the form:

$$\mathbf{r} = \frac{A + iB}{C + iD} \tag{2}$$

Where the quantities A, B, C, D are real and related to the transfer matrix of the stratified medium M by:

$$A = p_1 M_{11} - p_f M_{22} \tag{3}$$
$$B = p_1 p_f M_{12} - M_{21} \tag{4}$$
$$C = p_1 M_{11} + p_f M_{22} \tag{5}$$
$$D = p_1 p_f M_{12} + M_{21} \tag{6}$$
$$\tag{7}$$

The matrix elements here are defined so that all of them are real. They are given by:

$$M_{11} = \left[\cos\beta_2 \cos\beta_3 - \frac{p_3}{p_2} \sin\beta_2 \sin\beta_3\right] U_{N-1}(a) - U_{N-2}(a) \tag{8}$$
$$M_{12} = -\left[\frac{1}{p_3} \cos\beta_2 \sin\beta_3 + \frac{1}{p_2} \sin\beta_2 \cos\beta_3\right] U_{N-1}(a) \tag{9}$$
$$M_{21} = -\left[p_2 \sin\beta_2 \cos\beta_3 + p_3 \cos\beta_2 \sin\beta_3\right] U_{N-1}(a) \tag{10}$$
$$M_{22} = \left[\cos\beta_2 \cos\beta_3 - \frac{p_2}{p_3} \sin\beta_2 \sin\beta_3\right] U_{N-1}(a) - U_{N-2}(a) \tag{11}$$
$$\tag{12}$$

where

$$a = \cos\beta_2 \cos\beta_3 - \frac{1}{2}\left(\frac{p_2}{p_3} + \frac{p_3}{p_2}\right) \sin\beta_2 \sin\beta_3 \tag{13}$$

and the $U_N(a)$ are the Type II Chebyshev polynomials. Defined in the complex plane by:

$$U_N(a) = \frac{\sin\left((N+1)\arccos(a)\right)}{\sin(\arccos(a))} \tag{14}$$

finally, for a non magnetic medium, $p_j = n_j \cos\theta_j$ and $\beta_j = \frac{2\pi}{\lambda_0} n_j h_j \cos\theta_j$. Where θ_j is the incidence angle for the wave, measured from the normal to the surface and n_j is the refractive index of the material.

1.3 Values for a quarter wave stack and normal incidence:

With the condition of normal incidence $\hat{p}_j = n_j$. Adding the condition for a quarter wave stack: $\hat{\beta}_j = \frac{\pi}{2}$

1.3.1 Chebyshev polynomials:

In the nominal conditions for the stack, it is possible to give a very simple closed form expression for the Chebyshev polynomials involved in the calculation.

For the sake of brevity, let $z = (n_2/n_3)$. We can then write $\hat{a}$ as:

$$\hat{a} = -\frac{1}{2}\left(\frac{n_2}{n_3} + \frac{n_3}{n_2}\right) = -\frac{1}{2}\left(z + \frac{1}{z}\right) \tag{15}$$

In order to evaluate equation 14, we can formally compute $\arccos(\hat{a})$ in the complex numbers as:

$$\arccos(\hat{a}) = -i\log\left(\hat{a} + \sqrt{\hat{a}^2 - 1}\right) \tag{16}$$

From the definition of $\hat{a}$, it can be calculated that $\hat{a}^2 - 1 = \left(\frac{1}{2}\left(z - \frac{1}{z}\right)\right)^2$.

We need to choose a branch for the square root in equation 16. It does not matter which one we choose, since U_N is going to be real valued. We select:

$$\sqrt{\hat{a}^2 - 1} = -\frac{1}{2}\left(z - \frac{1}{z}\right) \tag{17}$$

And so:

$$\arccos(\hat{a}) = -i\log\left(-\frac{1}{2}\left(z + \frac{1}{z}\right) - \frac{1}{2}\left(z - \frac{1}{z}\right)\right) = -i\log\left(-z\right) \tag{18}$$

We are finally ready to evaluate the Chebyshev polynomials. Using eqns 14 and 18:

$$U_N(\hat{a}) = \frac{\sin\left(-(N+1)i\log\left(-z\right)\right)}{\sin(-i\log\left(-z\right))} = \frac{e^{(N+1)\log(-z)} - e^{-(N+1)\log(-z)}}{e^{\log(-z)} - e^{-\log(-z)}}$$

$$\boxed{U_N(\hat{a}) = \frac{(-z)^{N+1} - \left(-\frac{1}{z}\right)^{N+1}}{-z + \frac{1}{z}} = (-1)^N\left(\frac{1}{z^{N+1}} - z^{N+1}\right)\frac{z}{1 - z^2}} \tag{19}$$

1.3.2 Matrix elements

We proceed to evaluate the values for the matrix elements for normal incidence in a quarter wave stack.

The off-diagonal elements are zero because $\hat{\beta}_2 = \hat{\beta}_3 = \frac{\pi}{2}$. $\hat{M}_{12} = \hat{M}_{21} = 0$.

The diagonal elements satisfy:

$$\hat{M}_{11} = -\frac{1}{z}U_{N-1}(\hat{a}) - U_{N-2}(\hat{a}) = (-1)^N\frac{z}{1 - z^2}\left(\frac{1}{z^{N+1}} - z^{N-1} - \frac{1}{z^{N-1}} + z^{N-1}\right)$$

$$\hat{M}_{11} = (-1)^N\frac{z}{1 - z^2}\frac{1}{z^{N+1}}(1 - z^2) = (-1)^N\frac{1}{z^N} \tag{20}$$

$$\hat{M}_{22} = -zU_{N-1}(\hat{a}) - U_{N-2}(\hat{a}) = (-1)^N\frac{z}{1 - z^2}\left(\frac{1}{z^{N-1}} - z^{N+1} - \frac{1}{z^{N-1}} + z^{N-1}\right)$$

$$\hat{M}_{22} = (-1)^N\frac{z}{1 - z^2}z^{N-1}(1 - z^2) = (-1)^N z^N \tag{21}$$

1.3.3 The reflectivity coefficients A, B, C, D

Using the previous result for the off diagonal elements of $\hat{M}$, we determine that $\hat{B} = \hat{D} = 0$ for a quarter wave stack and normal incidence. While $\hat{A}$ and $\hat{D}$ satisfy:

$$\hat{A} = (-1)^N \left(n_1 \frac{1}{z^N} - n_f z^N \right) \tag{22}$$

$$\hat{C} = (-1)^N \left(n_1 \frac{1}{z^N} + n_f z^N \right) \tag{23}$$

2 Derivation

To derive the quantities of interest, we first find the phase of $\mathbf{r}$ from equation 2. Noting that the quantities A, B, C, D in the equation are real, it follows from the properties of complex numbers that the phase ϕ is just the difference of the phases of the numerator and denominator:

$$\phi = \arctan\left(\frac{B}{A}\right) - \arctan\left(\frac{D}{C}\right) \tag{24}$$

From which it follows that:

$$\frac{d\phi}{d\gamma} = \frac{\frac{dB}{d\gamma}A - \frac{dA}{d\gamma}B}{A^2 + B^2} - \frac{\frac{dD}{d\gamma}C - \frac{dC}{d\gamma}D}{C^2 + D^2} \tag{25}$$

Eevaluating this expression at the nominal values (section 1.3.3) yields:

$$\phi' = \frac{\hat{B}'}{\hat{A}} - \frac{\hat{D}'}{\hat{C}} = \frac{\hat{B}'\hat{C} - \hat{D}'\hat{A}}{\hat{A}\hat{C}} \tag{26}$$

2.1 The derivatives B', D'

We compute B', D' should be analogous to it.

$$B' = (p_1 p_f)' M_{12} + (p_1 p_f) M'_{12} - M'_{21}$$

since at the nominal conditions $M_{12} = M_{21} = 0$, and we assume $(p_1 \hat{p}_f)'$ is finite, then we only need the derivatives for the off-diagonal matrix elements. We evaluate B' at the nominal values to obtain:

$$\hat{B}' = (n_1 n_f) \hat{M}'_{12} - \hat{M}'_{21} \tag{27}$$

Similarly, for D' we would have:

$$\hat{D}' = (n_1 n_f) \hat{M}'_{12} + \hat{M}'_{21} \tag{28}$$

Once more we compute $\hat{M}'_{12}$ explicitly and $\hat{M}'_{21}$ is analogous.

By inspecting equation 9 we observe that the only surviving terms after taking a derivative and evaluating at the nominal conditions are the ones where we remove the simple zero of the cosines at $\beta_2 = \beta_3 = \frac{\pi}{2}$. In consequence:

$$\hat{M}'_{12} = -\left[\frac{1}{\hat{p}_3} (\cos\hat{\beta}_2)' \sin\hat{\beta}_3 + \frac{1}{\hat{p}_2} \sin\hat{\beta}_2 (\cos\hat{\beta}_3)' \right] U_{N-1}(\hat{a})$$

Using the chain rule and evaluating at the corresponding conditions:

$$\hat{M}'_{12} = \left[\frac{1}{n_3}\hat{\beta}'_2 + \frac{1}{n_2}\hat{\beta}'_3\right] U_{N-1}(\hat{a}) \tag{29}$$

similarly:

$$\hat{M}'_{21} = \left[n_2\hat{\beta}'_3 + n_3\hat{\beta}'_2\right] U_{N-1}(\hat{a}) \tag{30}$$

2.2 Putting it all together:

We will put together equation 26 by parts:

First, the denominator. Using the values for A and C from equations 1.3.3 and 23:

$$\hat{A}\hat{C} = \left((n_1)^2 \frac{1}{z^{2N}} - (n_f)^2 z^{2N}\right) \tag{31}$$

Second, the numerator. Using equations 2.1 and 2.1 together with the equations for A and C.

$$\hat{B}'\hat{C} = (-1)^N \left[\left((n_1 n_f)\hat{M}'_{12} - \hat{M}'_{21}\right)\left(n_1 \frac{1}{z^N} + n_f z^N\right)\right]$$
$$\hat{D}'\hat{A} = (-1)^N \left[\left((n_1 n_f)\hat{M}'_{12} + \hat{M}'_{21}\right)\left(n_1 \frac{1}{z^N} - n_f z^N\right)\right]$$

Identifying similar terms with different signs in the previous two equations we arrive at:

$$\hat{B}'\hat{C} - \hat{D}'\hat{A} = 2(-1)^N \left[n_1 (n_f)^2 z^N \hat{M}'_{12} - n_1 \frac{1}{z^N}\hat{M}'_{21}\right] \tag{32}$$

Using the identity for the Chebyshev polynomials (19), we parse the z dependent terms of the equation to expand the solution:

$$z^N M'_{12} = \left[\frac{1}{n_3}\hat{\beta}'_2 + \frac{1}{n_2}\hat{\beta}'_3\right](-1)^{N-1}(1 - z^{2N})\frac{z}{1-z^2}$$
$$\frac{1}{z^N} M'_{21} = \left[n_2\hat{\beta}'_3 + n_3\hat{\beta}'_2\right](-1)^{N-1}\left(\frac{1}{z^{2N}} - 1\right)\frac{z}{1-z^2}$$

Plugging these into equation 32, we find:

$$\hat{B}'\hat{C} - \hat{D}'\hat{A} = \frac{-2n_1 z}{1-z^2}(1 - z^{2N})\left[\frac{(n_f)^2}{n_2 n_3}(n_2\hat{\beta}'_2 + n_3\hat{\beta}'_3) - \frac{1}{z^{2N}}(n_2\hat{\beta}'_3 + n_3\hat{\beta}'_2)\right] \tag{33}$$

Combining equations 31 and 33, multiplying by z^{2N} to simplify the fractions:

$$\boxed{\hat{\phi}' = \frac{2n_1 z}{1-z^2}(z^{2N} - 1)\frac{z^{2N}\frac{(n_f)^2}{n_2 n_3}(n_2\hat{\beta}'_2 + n_3\hat{\beta}'_3) - (n_2\hat{\beta}'_3 + n_3\hat{\beta}'_2)}{(n_1)^2 - (n_f)^2 z^{4N}}} \tag{34}$$

with $\boxed{z = \frac{n_2}{n_3}}$ and $\boxed{\beta_j = \frac{2\pi}{\lambda_0} n_j h_j \cos\theta_j}$

2.3 Large N limit approximations

2.3.1 Case $n_2 < n_3$:

with the overall assumption that $\frac{(n_f)^2 z^{4N}}{(n_1)^2} < 1$ we can expand the denominator with a geometric series, which gives:

$$\hat{\phi}' = \frac{2z(z^{2N}-1)}{(1-z^2)n_1}\left[z^{2N}\frac{(n_f)^2}{n_2 n_3}(n_2\hat{\beta}_2' + n_3\hat{\beta}_3') - (n_2\hat{\beta}_3' + n_3\hat{\beta}_2')\right]\sum_{j=0}^{\infty}\left(\frac{n_f}{n_1}z^{2N}\right)^{2j} \tag{35}$$

The leading order term for this expansion is:

$$\hat{\phi}_0' = \frac{2n_2 n_3}{((n_3)^2 - (n_2)^2)\,n_1}(n_2\hat{\beta}_3' + n_3\hat{\beta}_2') \tag{36}$$

With the first correction being of order z^{2N} and given by:

$$\hat{\phi}_1' = \frac{-2n_2 n_3}{((n_3)^2 - (n_2)^2)\,n_1}\left[\frac{(n_f)^2}{n_2 n_3}(n_2\hat{\beta}_2' + n_3\hat{\beta}_3') + (n_2\hat{\beta}_3' + n_3\hat{\beta}_2')\right]\left(\frac{n_2}{n_3}\right)^{2N} \tag{37}$$

2.3.2 Case $n_2 > n_3$:

TBD.

3 Application to different effects (assumes $n2 < n3$)

By setting the parameter γ to different physical properties that change uniformly on the stack, we can use equation 36 to compute the leading order term of the phase change caused by this parameter variation.

we only need to compute:

$$\frac{d\beta_j}{d\gamma}|_{\gamma=\gamma_0} = \frac{2\pi}{\lambda_0}\frac{d}{d\gamma}(n_j h_j \cos(\theta_j))|_{\gamma=\gamma_0} = \beta_j\left(\frac{\hat{n_j'}}{n_j} + \frac{\hat{h_j'}}{h_j} + \tan(\hat{\theta}_j)\hat{\theta}_j'\right) \tag{38}$$

In general, we expect the the derivative of the incidence angle θ_j' to be finite, which drops the last term ($\hat{\theta}_j = 0$). If we define:

$$\boxed{\kappa_{\gamma j} = \frac{d}{d\gamma}\log(n_j h_j)|_{\gamma=\gamma_0} = \left(\frac{\hat{n_j'}}{n_j} + \frac{\hat{h_j'}}{h_j}\right)} \tag{39}$$

Then for a quarter wave stack and normal incidence:

$$\beta_j' = \frac{\pi}{2}\kappa_{\gamma j} \tag{40}$$

$\kappa_\gamma \Delta\gamma$ is meant to be understood as the fractional change in the phase difference β as we vary the scalar γ uniformly in the dielectric medium.

With these definitions, the leading order term for ϕ' reads:

$$\boxed{\hat{\phi}'_0 = \frac{\pi}{\left(1-(n_2/n_3)^2\right)n_1}(n_3\kappa_{\gamma 2} + n_2\kappa_{\gamma 3})\left(\frac{n_2}{n_3}\right)} \tag{41}$$

The first correction being:

$$\boxed{\hat{\phi}'_1 = \frac{-\pi}{\left(1-(n_2/n_3)^2\right)n_1}\left[\frac{(n_f)^2+(n_3)^2}{n_3}\kappa_{2\gamma} + \frac{(n_f)^2+(n_2)^2}{n_2}\kappa_{3\gamma}\right]\left(\frac{n_2}{n_3}\right)^{2N+1}} \tag{42}$$

3.1 Notes on κ:

- The quantity $\frac{h'}{h}$ is the fractional change on the thickness of the dielectric layer. If γ represents temperature, this is the thermal expansion coefficient at temperature T_0.

3.2 Notes on general assumptions:

This is a short list of the assumptions that I believe this derivation hinges on:

- The material is nonmagnetic. this would change the definition of p, but probably can be relaxed.
- There is no loss of energy from the electromagnetic field to the material. This is no longer true if the conductivity of the material cannot be neglected.
- None of the quantities involved in the definition for $\mathbf{r}$ change abruptly with respect to the parameter γ
- The boundary conditions for Maxwell's equations remain the same even when we change γ (no free charges accumulate, etc.). this is to ensure that the solution for $\mathbf{r}(\gamma + \delta\gamma)$ has the same functional form.
- Similar to the point above, the quantity γ changes uniformly on the whole stack, there are no spatial gradients for it or they are negligible.
- The material properties (h_j and n_j) do not change significantly with the electromagnetic wave going inside them.