Practice 4 Non smooth optimization and application to image processing

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Abstract

1 Introduction

In this TP, we learn the knowledge of convex optimization. The definitions of Proximal operators. Studying the Primal algorithms(forward-backward splitting and projected gradient) and A primal-dual algorithm(Chambolle Pock Algorithm) and apply them to image processing.

2 Applications

2.1 Computation of some proximal operators

The proximity operator is defined by:

$$y = (I + h\partial F)^{-1}(x) = prox_h^F(x) = \arg\min_u \left\{ \frac{\|u - x\|_2}{2h} + F(u) \right\} \tag{1}$$

Question 1

$$prox_h^F(x) = \arg\min_u \left\{ \frac{\|u - x\|_2}{2h} + \frac{1}{2} \|u\|_2^2 \right\}. \tag{2}$$

when u is a minimizer. Then

$$\text{Let } F(u) = \frac{\|u - x\|_2}{2h} + \frac{1}{2} \|u\|_2^2, \text{ then } \nabla F = \frac{u - x}{h} + u = 0, u = \frac{x}{1 + h}. \Rightarrow prox_h^F(x) = \frac{x}{1 + h}. \tag{3}$$

Question 2

The same ideal as before, let

$$F(u) = \frac{\|u - x\|_2}{2h} + \frac{1}{2} \|u - f\|_2^2, then \nabla F = \frac{u - x}{h} + u - f = 0 \tag{4}$$

when u is a minimizer.Then

$$prox_h^F(x) = u = \frac{x+hf}{1+h} \tag{5}$$

where $h > 0$ and $h \leq \frac{1}{L}$.

Question 3

The same ideal as before,let

$$F(u) = \frac{\|u-x\|_2}{2h} + \frac{1}{2}\|Ku-f\|_2^2, \tag{6}$$

Where K is a continuous linear operator,then

$$\nabla F = \frac{u-x}{h} + K^*(Ku-f) = 0 \Rightarrow (I_d + hK^*K)u = x + K^*f \tag{7}$$

Since K is a continuous linear operator, then $I_d + hK^*K$ inverse, we have

$$prox_h^F(x) = u = (I_d + hK{*}K)^{-1}(x + hK^*f) \tag{8}$$

Question 4

$$prox_h^F(x) = \arg\min_u \left\{ \frac{\|u-x\|_2^2}{2h} + \|u\|_1 \right\} = \arg\min_u \sum_{i=1}^{n} \left(|u_i| + \frac{1}{2h}(u_i - x_i)^2 \right) \tag{9}$$

Since $|x|$ is non-smooth function,then let

$$g(x) = |u_i| + (u_i - x_i)^2 \tag{10}$$

is also a non-smooth function. If u_i is a minimizer of $g(x)$, then $0 \in \partial f(u_i) = \partial |u_i| + \frac{1}{h}(u_i - x_i)$. Since

$$\partial |u_i| = \begin{cases} 1 & \text{if } u_i > 0 \\ -1 & \text{if } u_i < 0 \\ [-1,1] & \text{if } u_i = 0 \end{cases} \tag{11}$$

then, we have

$$u_i = \begin{cases} x_i - h & \text{if } x_i > h \\ x_i + h & \text{if } x_i < -h \\ 0 & \text{if } |x_i| \leq h \end{cases} \tag{12}$$

Then, we have

$$prox_h^F(x) = ST(u,h). \tag{13}$$

Question 5

Since $\|u\| = TV(u) = max_{\|w\|_\infty \leq 1} < w, \nabla u >$, then

$$min_{u\in R^n} \|\nabla u\| + \frac{1}{2h}\|u-x\|_2^2 = min_{u\in R^n} max_{\|w\|_\infty \leq 1} < w, \nabla u > + \tag{14}$$

$$\frac{1}{2h}\left\|u-x\right\|_2^2 = max_{\|w\|_\infty\leq 1} min_{u\in R^n} < w, \nabla u > + \frac{1}{2h}\left\|u-x\right\|_2^2 \tag{15}$$

Consider: $min_{u\in R^n} < w, \nabla u > +\frac{1}{2h}\left\|u-x\right\|_2^2 = min_{u\in R^n} - < u, div(w) > +\frac{1}{2h}\left\|u-x\right\|_2^2$
Let

$$g(u) = - < u, div(w) > + \frac{1}{2h}\left\|u-x\right\|_2^2 \tag{16}$$

let $\nabla g(u) = 0$,

$$\nabla g(u) = -div(w) + \frac{1}{h}(x-u) = 0 \Rightarrow u = x - hdiv(w) \tag{17}$$

Then,
$max_{\|w\|_\infty\leq 1} - < u, div(w) > +\frac{1}{2h}\left\|u-x\right\|_2^2 = max_{\|w\|_\infty\leq 1} - < x+hdiv(w), div(w) > +\frac{h}{2}\left\|hdiv(w)\right\|_2^2$
$= max_{\|w\|_\infty\leq 1} - < x, div(w) > -\frac{h}{2}\left\|div(w)\right\|_2^2 = max_{\|w\|_\infty\leq 1} - \frac{h}{2}\left\|div(w)+\frac{x}{h}\right\|_2^2 - \frac{1}{h^2} < x, x >$
$\Leftrightarrow max_{\|w\|_\infty\leq 1} - \frac{h}{2}\left\|div(w)+\frac{x}{h}\right\|_2^2$
$\Leftrightarrow min_{\|w\|_\infty\leq 1} \frac{h}{2}\left\|div(w)+\frac{x}{h}\right\|_2^2$.
So, we have

$$y = prox_h^F(x) \Leftrightarrow y = x - hdiv(z) \text{ with z is the solution of } min_{\|z\|_\infty\leq 1}\left\|div(z)+\frac{x}{h}\right\|_2^2 \tag{18}$$

2.2 Image restoration (gaussian noise)

In this part, we use the model:

$$\inf_u \lambda\left\|f-u\right\|_2^2 + \left\|\nabla u\right\|_1 \tag{19}$$

By using the Forward-Backward algorithm, I tried with two different noise and have the following results.

Symbol description

In my model, u is the unknown real image, x is the minimization of the likelihood of the image u with respect to the data noisy f.f is the noise image.K is the number of iteration in Forward-Backward algorithm,k is the iteration of the gradient descent algorithm which solving the value z in $prox_F 5.m$ file.λ is the parameter in this model, h is the parameter in the Forward-Backward algorithm which is the same as the h in $prox_F 5.m$ file, τ is the parameter in the gradient descent algorithm, which satify the condition that $\tau \leq \frac{2}{L}$ and Here I tried $\tau = 0.1$.Reference to the value in TP 2

Initialization

For the initialization, $X^0 = f$, by setting $\lambda = 0.00001 h = 150; K = 200; k = 30; \tau = 0.1$, by adding gaussian noise=10, we have the following results for FB and FISTA:

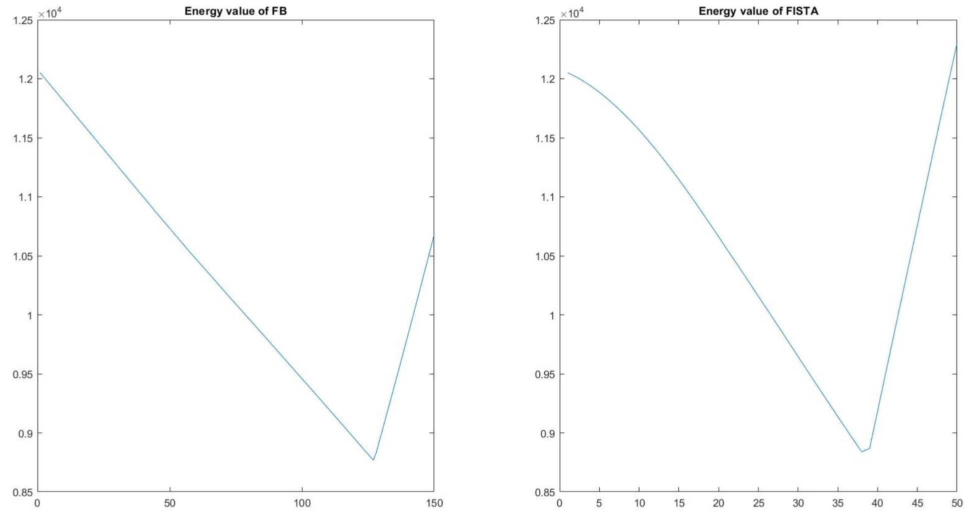

Figure 1: $EnergyJ$

As we can see here, the optimize $K = 127, k = 37$
And the crosspending image is

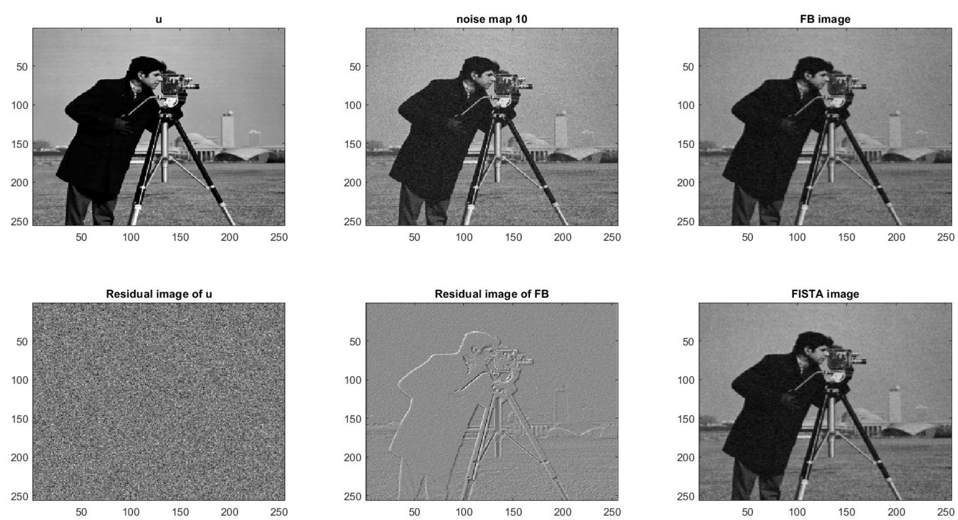

Figure 2: $solution$

Here the residual image is just to compare the resduial between f and u and x and u.
Results with the noise equals to 30

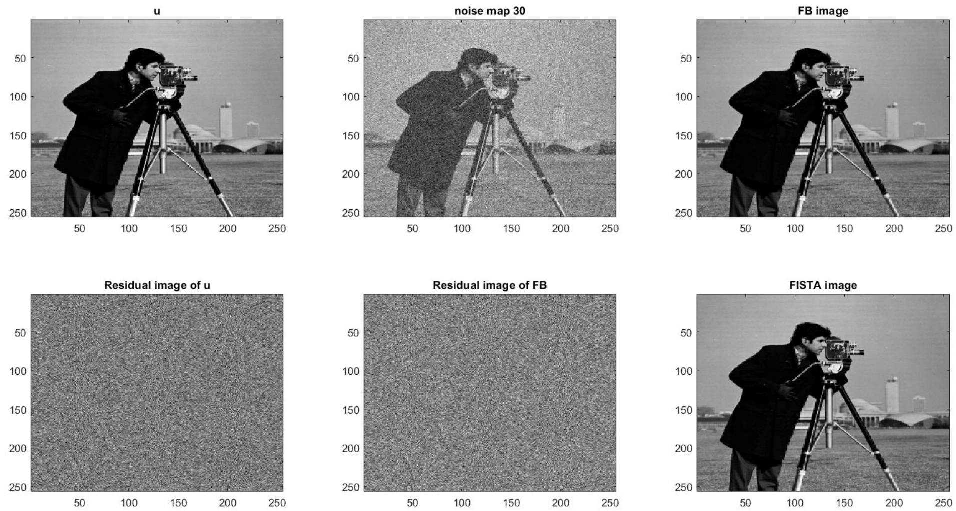

Figure 3: $solution with noise = 30$

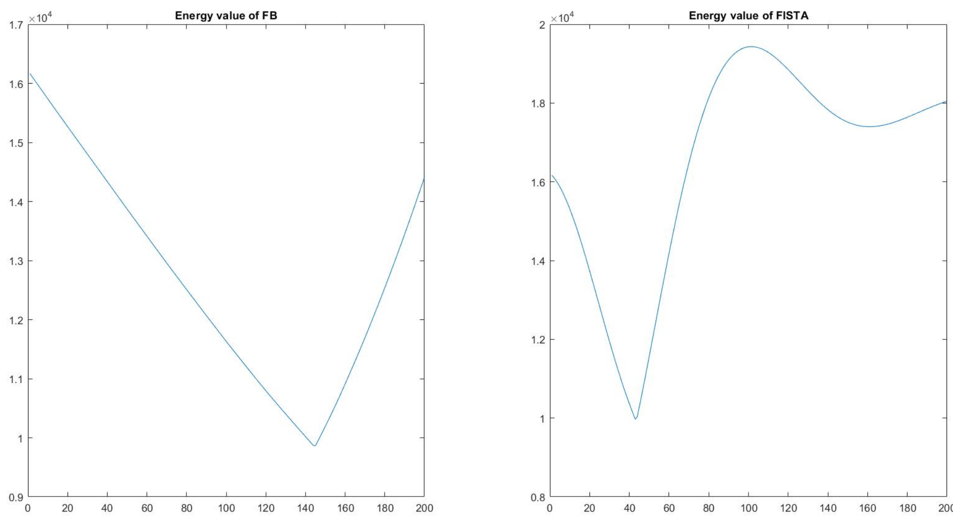

Figure 4: $Energy J with noise = 30$

Here, we can see that the optical K is the same as before when the other paramater are the same, and for FISTA, if $h = 15$, the value of optical K is the same as FB in this case.
Remark The energy here we can see is not stable with K, It has only one minimum value and only one optical K. It make sense, Because of the strict convexity of the objective function.

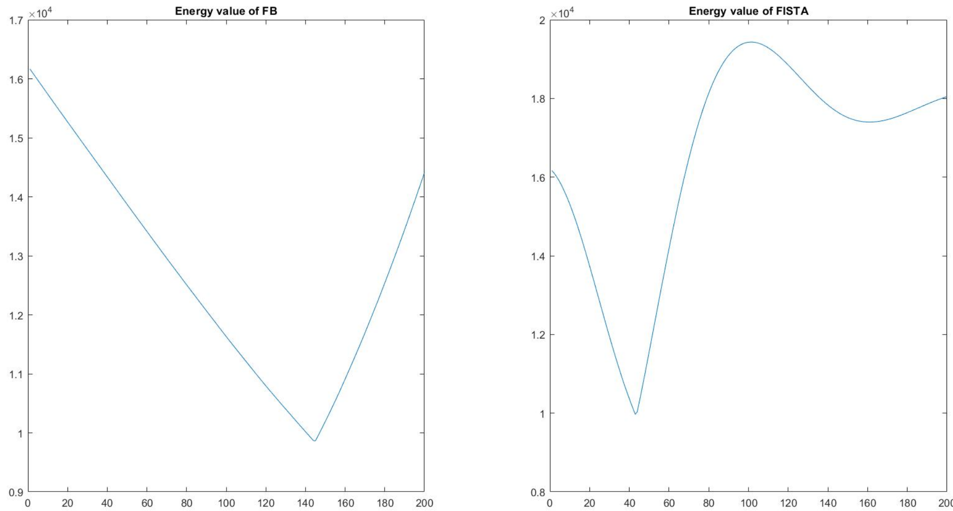

Figure 5: $EnergyJwithnoise = 30$

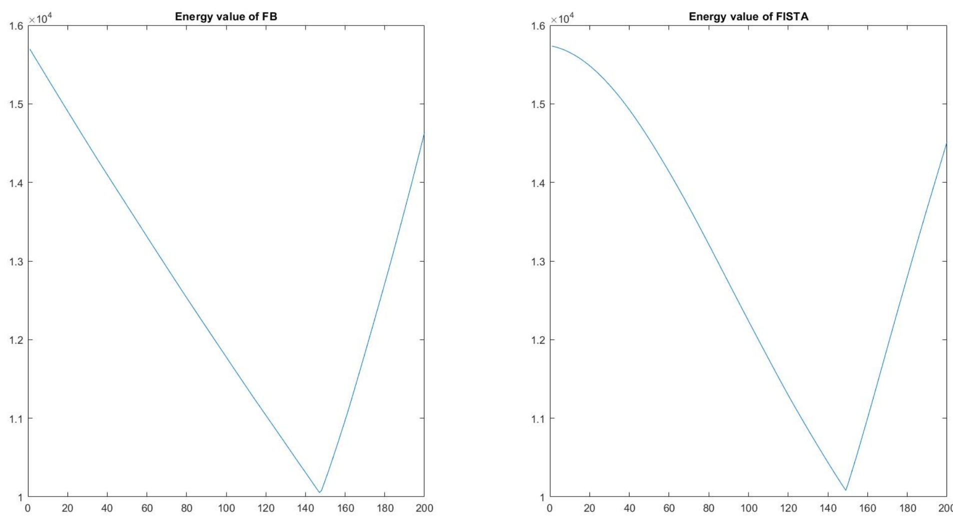

Figure 6: $EnergyJwithnoise = 30$

2.3 Image restoration (salt and peper noise)

In this part, we use the model:

$$\inf_u \lambda \|f-u\|_1^1 + \|\nabla u\|_1 \tag{20}$$

By using the primal dual algorithm by Chambolle-Pock, to get the solution of this model. According to the a primal-dual algorithm in TP 4 , the Legendre Fenchel transform of F, and $\|\nabla u\|_1 = TV(u) = \max_{x \in X1 \times 2X} < \nabla u, x >_X -I_c(x)$, we have $G(u) = \lambda \|f-u\|_1^1$ and $F^*(x) = I_c(x)$. where$I_c(x)$ is indictor funvtion. By using a projection algorithm, we have $prox_h^{F*}(x) = \frac{x}{\max(1,|x|)}$, Then we can write the a primal-dual algorithm like what it was in TP4. Notes that $\tau\sigma.8\lambda^2 < 1$

Results

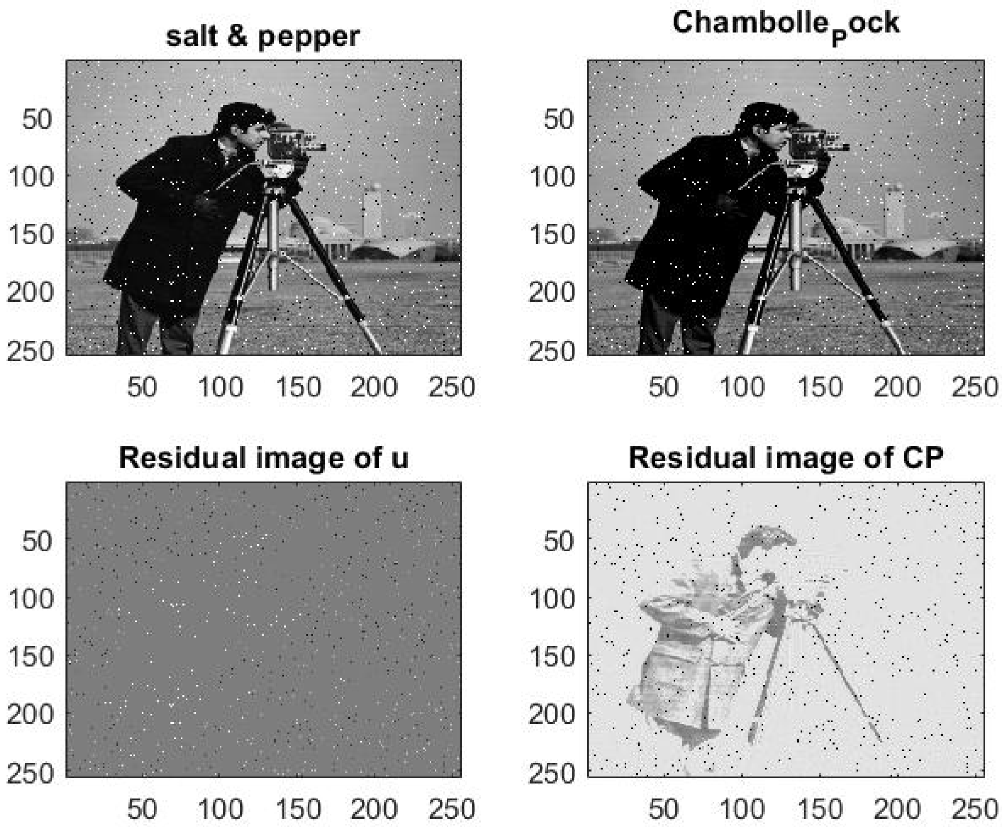

Figure 7: salt and peper noise

Problems

Here the results is false, which is not what we want, It's strange that the original image change but the noise didn't change. but I didn't find out the error.

2.4 Image deconvolution(FISTA)(gaussian noise))

$$\inf_u \lambda \|f - Ku\|_2^2 + \|\nabla u\|_1 \tag{21}$$

Here I set $lambda = 0,00001, k = 30$, for FB of deconvolution, I set $K = 180, h = 150$ and the other one are $K = 160, h = 15$. Here are the results:

Gaussion noise at level 10

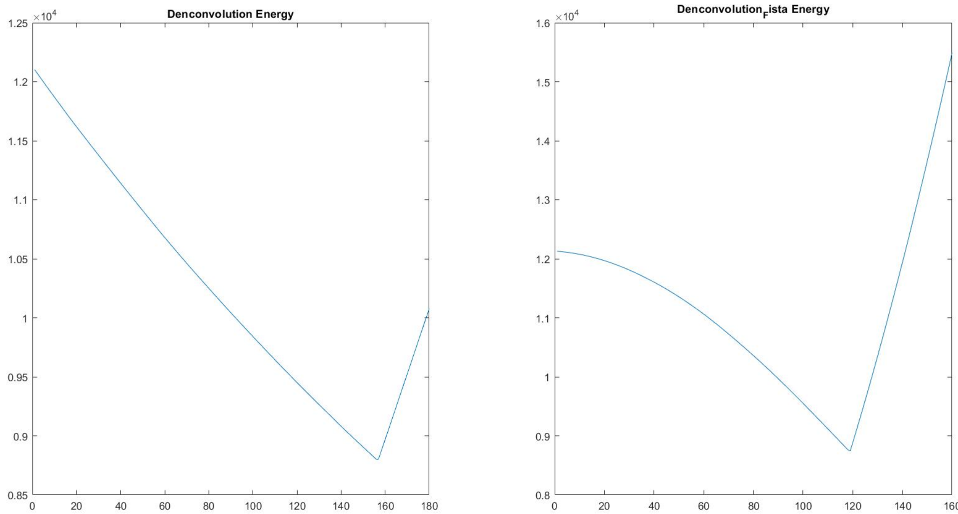

Figure 8: Energy

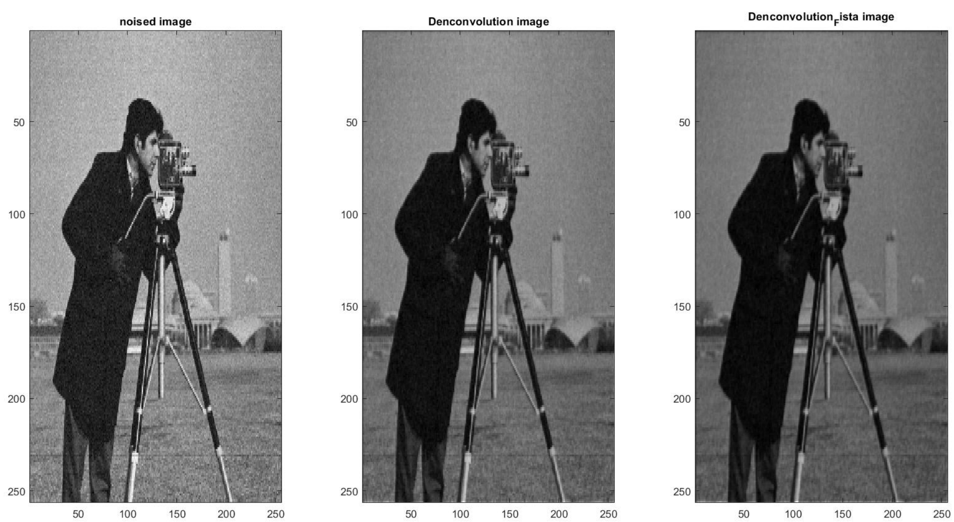

Figure 9: solution

Gaussion noise at level 30 Here, I set $\lambda = 0.001$, the other paramaters are the same as before

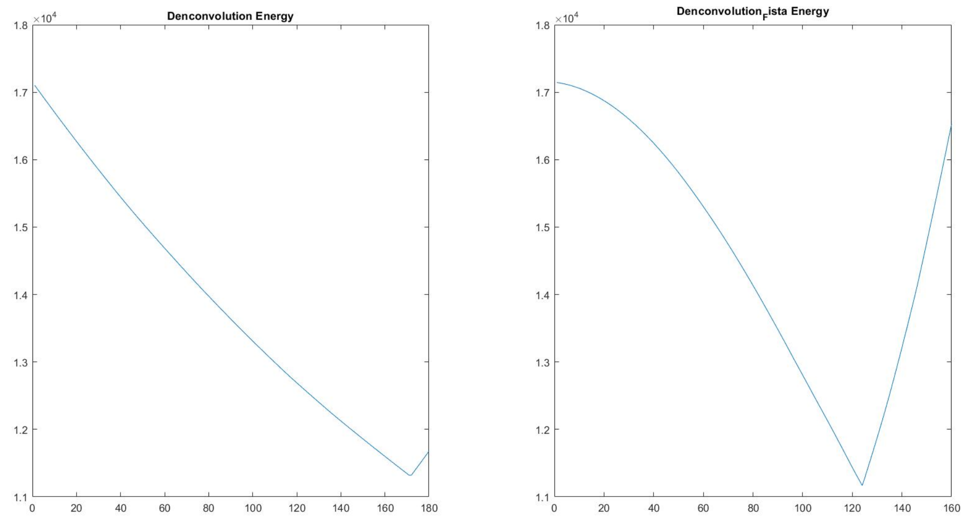

Figure 10: Energy

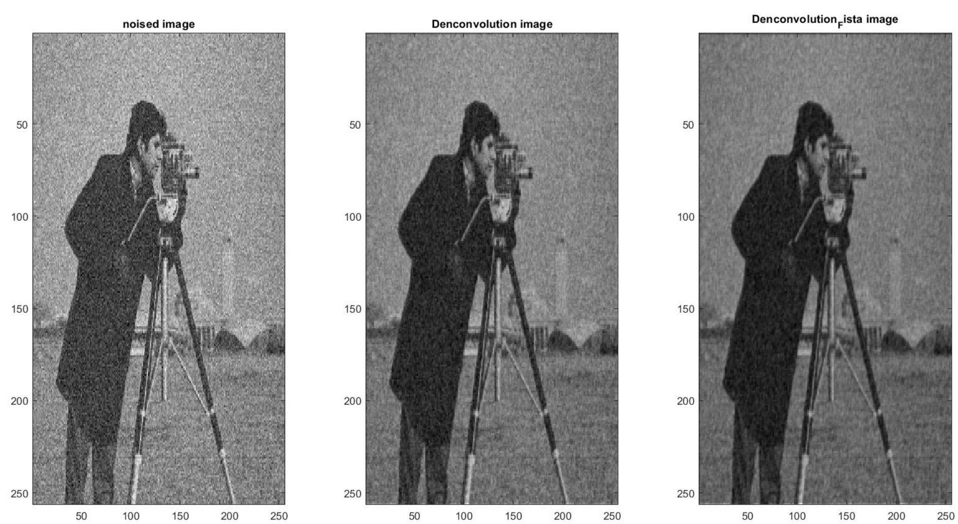

Figure 11: Solution

Comments By comparison, we can see that the more noise we added, the value of optical iteration bigger, the smaller lambda we need. Also, the higher noise is, the minimizer energy bigger.

2.5 Inpainting with FB L2

By using the $Im1.png$ and $Im1_mask.png$, $Im2.png$ and $Im2_mask.png$. The paremeters are $\lambda = 0.005, \tau = 0.01.h = 10$ Here are the results:

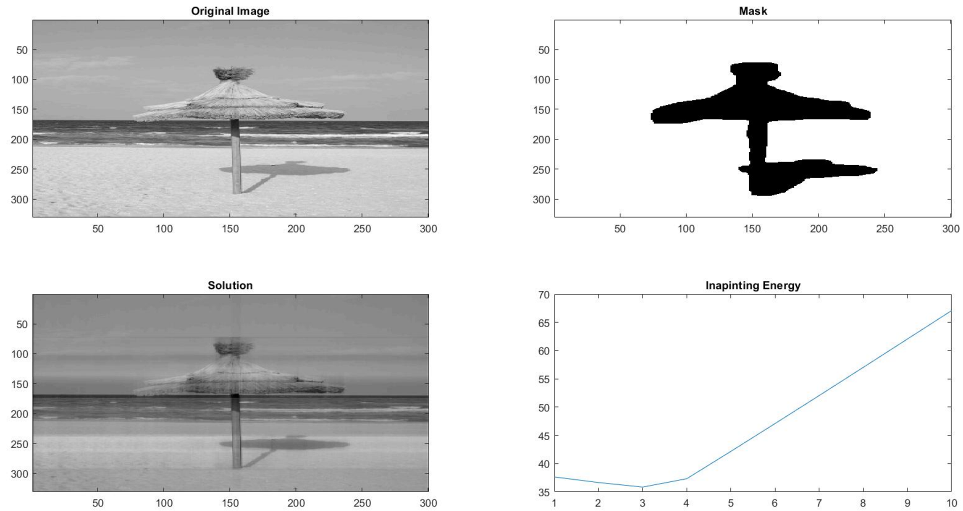

Figure 12: Im2.png

Comments

Here, when $\lambda = 0.005, \tau = 0.01.h = 10$,the optimical $K = 3$

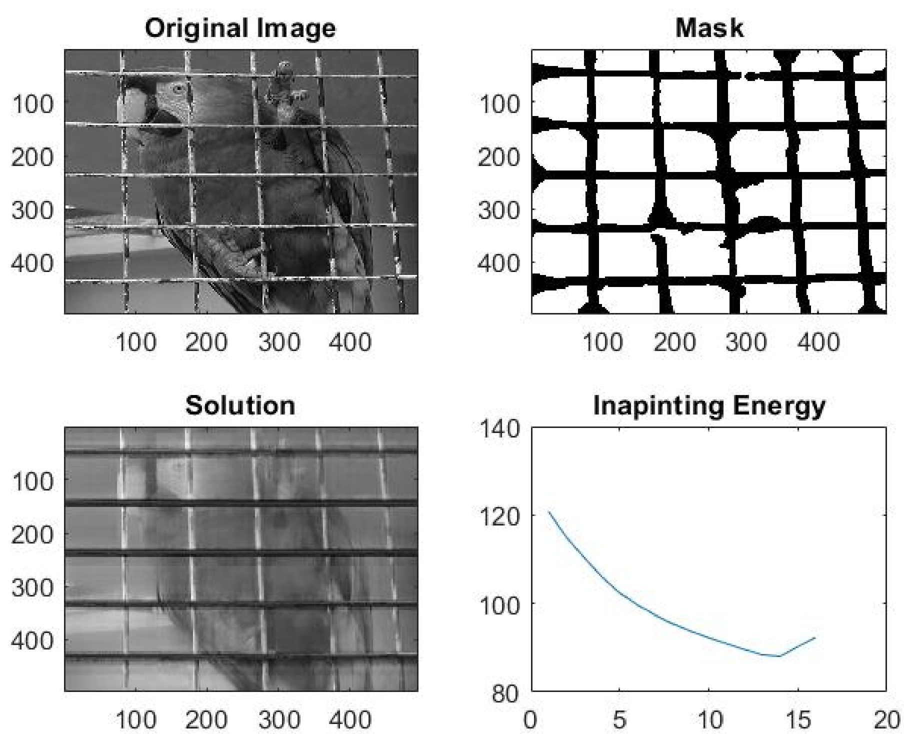

Figure 13: Im1.png

Comments

Here, when $\lambda = 0.005, \tau = 0.01.h = 10$,the optimical $K = 14$
When $K = 14$

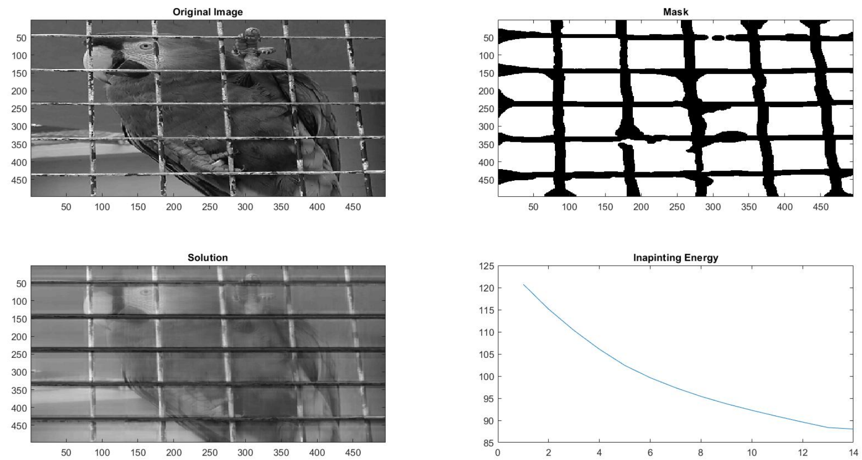

Figure 14: Im1.png

2.6 Conclusion

Trying with TV-L2 method seems not that good. We have to try TV-L1 which might be more better.

References

[1] Jean-Franois Aujol *Non smooth optimization and application to image processing.*