Exercises: Theorem on Alternative and cones

Exercise 1. Let K be a closed cone with nonempty interior and let $\bar{x}$ in the interior of K. Prove that x is in the interior of K if and only if there exists a positive real t such that $x - t\bar{x}$ is in K.

Exercise 2. Let

$$K_* = \{\lambda : \langle \lambda, z \rangle \leq 0 \ \forall z \in K\}.$$

Let K be a closed cone with nonempty interior. 1) Prove that if $x \neq 0$ is in K and λ is in the interior of K_* then $\langle \lambda, x \rangle > 0$.

2) Assume that $\lambda \in K_*$. Prove that λ is in the interior of K_* if and only if for every $x \neq 0$ with $x \in K$ we have $\langle \lambda, x \rangle > 0$.

Exercise 3. Derive the General Theorem on Alternative from Homogeneous Farkas Lemma.

Hint: Verify that the system

$$(\mathcal{S}) : \begin{cases} a_i^T x > b_i, \ i = 1, \ldots, m_s, \\ a_i^T x \geq b_i, \ i = m_s + 1, \ldots, m, \end{cases}$$

in variable x has no solution if and only if the homogeneous inequality

$$\varepsilon \leq 0$$

in variables x, ε, t, is a consequence of the system of homogeneous inequalities

$$(\mathcal{T}) : \begin{cases} a_i^T x - b_i t - \varepsilon & \geq & 0, \ i = 1, \ldots, m_s, \\ a_i^T x - b_i t & \geq & 0, \ i = m_s + 1, \ldots, m, \\ t & \geq & \varepsilon \end{cases}$$

in these variables.

Exercice 4. Prove the following corollaries of General Theorem on Alternative:

1) Gordan's Theorem on Alternative. One of the following systems of inequalities

$$(I) \ \ Ax < 0, \ x \in \mathbb{R}^n,$$

in variable x and

$$(II) \ \ A^T y = 0, \ y \neq 0, \ y \geq 0, y \in \mathbb{R}^m,$$

in variable y, with A an $m \times n$ matrix has a solution if and only if the other has no solution.

2) Inhomogeneous Farkas Lemma. A linear inequality $a^T x \leq p$ in variable x is a consequence of a solvable system of linear inequalities $Nx \leq q$ iff there exists $\nu \geq 0$ such that $a = N^T \nu$ and $\nu^T q \leq p$.

3) Motzkin's Theorem on Alternative. The system

$$Sx < 0, \ Nx \leq 0$$

in variables x has no solution if and only if the system

$$S^T\sigma + N^T\nu = 0,\ \sigma \geq 0,\ \nu \geq 0,\ \sigma \neq 0,$$

in variables σ, ν has a solution.