LECTURE NOTES ON LINEAR PROGRAMMING

Contents

1. Affine spaces

1.1. Definitions and affine hulls.

Definition 1.1 (Affine space). *A subset E of $\mathbb{R}^n$ is an affine space if for every $x_1, x_2 \in E$ the line*

$$\{tx_1 + (1-t)x_2 : t \in \mathbb{R}\}$$

that passes through x_1 and x_2 is contained in E.

Definition 1.2. *An affine combination of vectors $x_1, \ldots, x_m \in \mathbb{R}^n$ is any point of form $\sum_{i=1}^m t_i x_i$ with $t_i \in \mathbb{R}$ and $\sum_{i=1}^m t_i = 1$.*

Remark 1.3. *An affine combination of $x_1, \ldots, x_m$ can be written as $x_1 + \sum_{i=1}^m t_i(x_i - x_1)$ where $t_i \in \mathbb{R}$.*

We obtain the following characterization of affine spaces:

Proposition 1.4. *A subset E of $\mathbb{R}^n$ is an affine space if and only if it contains all its affine combinations.*

Proof. All we have to show is that if E is an affine space then it contains all its affine combinations. We show the result by induction on the number m of points in the affine combination. If $m = 2$ the result follows by definition of an affine space. Now assume the result holds for m and take an affine combination $\sum_{i=1}^{m+1} t_i x_i$ of $m+1$ points $x_1, \ldots, x_{m+1}$ from E. Note that at least one the t_i is not equal to one. Without loss of generality assume that $t_1 \neq 1$. Note that due to the induction hypothesis, the point $x = \sum_{i=2}^{m+1} \theta_i x_i$ with $\theta_i = \frac{t_i}{1-t_1}$ belongs to E because $\sum_{i=2}^{m+1} \theta_i = 1$. Then

$$\sum_{i=1}^{m+1} t_i x_i = t_1 x_1 + (1 - t_1) x \in E$$

since $x_1, x \in E$. □

We can see an affine subspace E as a translation of a vector space (a vector space being a special case of affine space) which is uniquely defined by E:

Proposition 1.5. *Let E be a nonempty affine space in $\mathbb{R}^n$. Then*

$$V = \{x - y : x, y \in E\}$$

is a vector space and for any $x_0 \in E$ we have

$$E = x_0 + V = \{x_0 + v : v \in V\}. \tag{1.1}$$

Proof. Let us show that V is a vector space. Take $v \in V$ and $\alpha \in \mathbb{R}$. We have $v = x - y$ for some $x, y \in E$. Then

$$\alpha v = \underbrace{x}_{\in E} - \underbrace{((1-\alpha)x + \alpha y)}_{\in E} \in V.$$

Take $v_1 = x_1 - y_1, v_2 = x_2 - y_2 \in V$ for some $x_1, x_2, y_1, y_2 \in E$. Then

$$v_1 + v_2 = \underbrace{x_1}_{\in E} - \underbrace{(y_1 + y_2 - x_2)}_{\in E} \in V.$$

Therefore V is a vector space.

Take $x \in E$. Then x can be written $x = x_0 + v$ with $v = x - x_0 \in V$. Therefore $E \subset x_0 + V$. Now if $x \in x_0 + V$ then $x = x_0 + x_1 - x_2$ for some $x_1, x_2 \in E$ and by Proposition 1.4 $x \in E$, which shows (1.1). □

The proposition above offers an equivalent characterization of affine spaces.

Proposition 1.6. *An intersection of affine spaces is an affine space. More precisely, for the family of affine spaces $E_\alpha = x_\alpha + V_\alpha, \alpha \in I$ where $V_\alpha = \{x - y : x, y \in E_\alpha\}$, if the intersection $E = \bigcap_{\alpha \in I} E_\alpha$ is nonempty, then*

$$E = x_0 + \bigcap_{\alpha \in I} V_\alpha \tag{1.2}$$

where x_0 is any point in E.

Proof. Let $x_1, x_2 \in E$. Then $x_1, x_2 \in E_\alpha$ for all $\alpha \in I$ and therefore the line passing through x_1, x_2 is contained in E_α for all $\alpha \in I$ and consequently is also contained in E which shows that E is an affine space.

Now let $x \in E$. We can write $x = x_0 + x - x_0$ with $x, x_0 \in E$. Therefore $x - x_0 \in V_\alpha$ for all α, $x - x_0 \in \cap_{\alpha \in I} V_\alpha$ and $x \in x_0 + \bigcap_{\alpha \in I} V_\alpha$. On the other hand, if $x \in x_0 + \bigcap_{\alpha \in I} V_\alpha$ then $x = x_0 + x_\alpha - y_\alpha$ for $x_\alpha, y_\alpha \in E_\alpha$. Since $x_0 \in E_\alpha$ for all α, we have that $x \in E_\alpha$ for all $\alpha \in I$ and therefore $x \in E$. □

Let X be a nonempty subset of $\mathbb{R}^n$. We denote by $\mathrm{Aff}(X)$ the smallest affine space of $\mathbb{R}^n$ that contains X which means that if S is an affine space of $\mathbb{R}^n$ that contains X then *Aff*$(X) \subset S$. We have the following characterization of *Aff*(X):

Proposition 1.7. *We have* $\mathrm{Aff}(X) = \bigcap_{S \text{ affine space and } X \subset S} S.$

Proof. Let $S(X) = \bigcap_{S \text{ affine space and } X \subset S} S$. By Proposition 1.6, $S(X)$ is an affine space which containts X. Therefore $\text{Aff}(X) \subset S(X)$ by definition of $\mathit{Aff}(X)$. On the other hand, $\mathit{Aff}(X)$ is an affine space that contains X and therefore by definition of $S(X)$ we also have $S(X) \subset \mathit{Aff}(X)$. □

The set $\mathit{Aff}(X)$ is called the affine hull of X. To handle constraints of form $y \in \mathit{Aff}(X)$, we now obtain an algebraic representation of $\mathit{Aff}(X)$. Let x_0 be an arbitrary point in X. By definition of $\mathit{Aff}(X)$ this point x_0 also belongs to $\mathit{Aff}(X)$ and using Proposition 1.5, the set $\mathit{Aff}(X)$ can be written as

$$\mathit{Aff}(X) = x_0 + V_X$$

where V_X is a vector space. We now need an algebraic representation of this set V_X. The relation $X \subset \mathit{Aff}(X)$ can be re-written $X \subset x_0 + V_X$ which means that the vector space V_X must contain the set $X - x_0$. Moreover, if $V_1 \subset V_2$ then $x_0 + V_1 \subset x_0 + V_2$. Therefore, V_X is the smallest vector space which contains the set of vectors $X - x_0$ and we know that this smallest vector space is the linear span of $X - x_0$, i.e., the set of all linear combinations of vectors from $X - x_0$: $V_X = \mathit{Span}(X - x_0)$. We come to the representation

$$\mathit{Aff}(X) = x_0 + \mathit{Span}(X - x_0)$$

where x_0 is any point in X. Knowing that

$$\mathit{Span}(X - x_0) = \left\{ \sum_{i=1}^{k} \mu_i (x_i - x_0) : k \geq 1, \mu_i \in \mathbb{R}, x_i \in X \text{ for } i \geq 1 \right\},$$

we see that

$$\mathit{Aff}(X) = \left\{ x_0 + \sum_{i=1}^{k} \mu_i (x_i - x_0) : k \geq 1, \mu_i \in \mathbb{R}, x_i \in X \text{ for } i \geq 1 \right\}, \tag{1.3}$$

where x_0 is any point in X. From this characterization we obtain the following algebraic representation of $\mathit{Aff}(X)$:

Proposition 1.8. $\text{Aff}(X)$ *is the set of all affine combinations of points from X:*

$$\text{Aff}(X) = \left\{ \sum_{i=1}^{k} \lambda_i x_i : k \geq 1, x_i \in X, \lambda_i \in \mathbb{R}, i \geq 1, \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

Proof. Using (1.3), if x is in $\mathit{Aff}(X)$, we can write

$$x = (1 - \sum_{i=1}^{k} \mu_i) x_0 + \sum_{i=1}^{k} \mu_i x_i = \sum_{i=0}^{k} \lambda_i x_i$$

for point $x_i \in X, i \geq 0$, reals μ_i, $\lambda_0 = 1 - \sum_{i=1}^{k} \mu_i$, and $\lambda_i = \mu_i, i \geq 1$. Since $\sum_{i=0}^{k} \lambda_i = 1$, x is indeed an affine combination of points from X. On the other hand, if x is an affine combination of points from X, we can write $x = \sum_{i=0}^{k} \lambda_i x_i$ with $x_i \in X, i \geq 0$ and reals λ_i summing up to one. Therefore

$$x = x_0 + \sum_{i=1}^{k} \lambda_i (x_i - x_0)$$

and we know from (1.3) that such x belongs to $\mathit{Aff}(X)$. □

From the above characterization of $\mathit{Aff}(X)$, we see that the affine hull of an affine space is this affine space.

1.2. **Affinely spanning sets, affinely independent sets, affine basis and affine dimension.** Just as vector spaces have basis, we would like to define basis for affine spaces, called affine basis, such that every vector in the affine space can be uniquely written as an affine combination of vectors from this space. Recall that an affine space Y is of form

$$Y = y_0 + V$$

where y_0 is any point in Y and V is the vector space $V = \{y_2 - y_1 : y_1, y_2 \in Y\}$. Assume that the dimension of V is k. To obtain an affine basis of Y it suffices to take a basis $(v_1, \ldots, v_k)$ of V. The set of vectors $(y_0, y_0 + v_1, y_0 + v_2, \ldots, y_0 + v_k)$ is then an affine basis of Y. Indeed, denoting $y_i = y_0 + v_i, i \geq 1$, any vector $y \in Y$ can be written as $y = y_0 + v$ for some $v \in V$ and thefore $y = y_0 + \sum_{i=1}^k \alpha_i v_i$ for some reals α_i, or equivalently,

$$y = \sum_{i=0}^{k} \lambda_i y_i \text{ with } \lambda_0 = 1 - \sum_{i=1}^{k} \alpha_i, \lambda_i = \alpha_i, i \geq 1.$$

Since in the expression above $\sum_{i=0}^k \lambda_i = 1$, any point $y \in Y$ can indeed be written as an affine combination of $(y_0, y_1, \ldots, y_k)$. We can also check that the coefficients $\lambda_i, i \geq 0$, in the decomposition of $y \in Y$ as an affine combination of $(y_0, y_1, \ldots, y_k)$ is uniquely determined by y. Indeed, assume that there are two sets of coefficients $\lambda_i, i \geq 0$, and $\lambda_i', i \geq 0$ summing up to one, such that $y = \sum_{i=0}^k \lambda_i y_i = \sum_{i=0}^k \lambda_i' y_i$. Then

$$0 = (\lambda_0 - \lambda_0')y_0 + \sum_{i=1}^{k}(\lambda_i - \lambda_i')(y_0 + v_i) = \underbrace{(\sum_{i=1}^{k} \lambda_i - \sum_{i=1}^{k} \lambda_i')}_{=1-1=0} y_0 + \sum_{i=1}^{k}(\lambda_i - \lambda_i')v_i.$$

Since vectors v_i are linearly independent the above relation implies that $\lambda_i = \lambda_i' = 0, i \geq 1$. Moreover, $\lambda_0 + \sum_{i=1}^n \lambda_i = \lambda_0' + \sum_{i=1}^n \lambda_i' = 1$ which gives $\lambda_0 = \lambda_0'$. This shows that any point in Y can be uniquely written as an affine combination of $(y_0, y_1, \ldots, y_k) = (y_0, y_0 + v_1, y_0 + v_2, \ldots, y_0 + v_k)$. Therefore we have checked that for any $y_0 \in Y$ and any basis $(v_1, \ldots, v_k)$ of V the set of vectors $(y_0, y_0 + v_1, y_0 + v_2, \ldots, y_0 + v_k)$ is an affine basis of Y.

Reciprocally, consider a basis $(y_0, y_1, \ldots, y_n)$ of vectors from Y. Let us show the following property:

(P) this basis must have $n + 1 = k + 1$ vectors and that $(y_1 - y_0, \ldots, y_n - y_0)$ is a basis of V.

Before this discussion, we need to characterize vectors $(y_0, y_1, \ldots, y_n)$ such that the coefficients λ_i in the affine combination

$$y = \sum_{i=0}^{n} \lambda_i y_i \tag{1.4}$$

are uniquely determined by y. Assume that this is not the case, that is to say that we have

$$y = \sum_{i=0}^{n} \lambda_i y_i = \sum_{i=0}^{n} \lambda_i' y_i$$

for two different sets of coefficients $\lambda_i, i \geq 0$ and $\lambda_i', i \geq 0$ summing up to one. Then

$$\sum_{i=0}^{n} \theta_i y_i = 0 \text{ for } \theta_i = \lambda_i - \lambda_i', \sum_{i=0}^{n} \theta_i = \sum_{i=0}^{n} \lambda_i - \sum_{i=0}^{n} \lambda_i' = 1 - 1 = 0,$$

where coefficients θ_i are not all null. Therefore, to ensure that coefficients λ_i in the affine combination (1.4) are uniquely determined by y, we need the following:

Definition 1.9 (Affinely independent vectors)**.** *Vectors $(y_0, \ldots, y_n)$ are affinely independent if*

$$\sum_{i=0}^{n} \lambda_i y_i = 0, \sum_{i=0}^{n} \lambda_i = 0 \Rightarrow \lambda_i = 0, i = 0, \ldots, n.$$

From our discussion above, if vectors $(y_0, y_1, \ldots, y_n)$ are affinely independent then coefficients λ_i in the affine combination (1.4) are indeed uniquely determined by y. From the definition of an affine basis, vectors forming an affine basis must be affinely independent.

Let us now show (P). Let $v \in V$. The vector $y_0 + v$ belongs to Y and therefore there are $\lambda_i, i \geq 0$ summing up to one such that $y_0 + v = \sum_{i=0}^n \lambda_i y_i$ and therefore $v = \sum_{i=1}^n \lambda_i (y_i - y_0)$. It follows that the set of vectors $(v_1, \ldots, v_n)$ with $v_i = y_i - y_0$ spans V and therefore $n \geq k$. Equivalently we say that $(y_0, y_1 - y_0, \ldots, y_n - y_0)$ affinely spans Y:

Definition 1.10 (Affinely spanning set)**.** *We say that the set of vectors $(y_0, y_1, \ldots, y_n)$ affinely spans Y if* $\mathrm{Aff}(\{y_0, y_1, \ldots, y_n\}) = Y$ *or equivalently if* $\mathrm{Span}(\{y_1 - y_0, \ldots, y_n - y_0\}) = V$.

On the other hand, if reals $\lambda_i, i \geq 1$ satisfy $\sum_{i=1}^n \lambda_i (y_i - y_0) = 0$ then we have $\sum_{i=0}^n \theta_i y_i = 0$ with $\theta_i = \lambda_i, i \geq 1, \theta_0 = -\sum_{i=1}^n \lambda_i$, $\sum_{i=0}^n \theta_i = 0$, and since $(y_0, y_1, \ldots, y_n)$ are affinely independent this implies $\theta_i = \lambda_i = 0, i \geq 1$. This shows that vectors $(v_1, \ldots, v_n)$ in V with $v_i = y_i - y_0$ are linearly independant. Therefore $n \leq k$. We have shown that $n = k$ and that in fact $(v_1, \ldots, v_n)$ is a basis of V.

Summarizing our observations, we have shown that all basis of $Y = y_0 + V$ with V of dimension k are of the form $(y_0, y_1, \ldots, y_k)$ where y_0 is any point in Y and $(y_1 - y_0, \ldots, y_k - y_0)$ is a basis of V. Equivalently, an affine basis for Y is a set of affinely independent vectors in Y that affinely spans Y.

The affine dimension of the affine space $Y = y_0 + V$ is the dimension of V. The number of elements in all affine basis of Y is the maximal number of affinely independent vectors in Y or equivalently the minimal number of vectors that affinely span Y.

For any $y \in Y$ and any affine basis $(y_0, y_1, \ldots, y_k)$ of Y there exists a unique set of coefficients $\lambda_i, i \geq 0$ summing to one such that

$$y = \sum_{i=0}^k \lambda_i y_i.$$

These coefficients are called the barycentric coordinates of y in the basis.

1.3. **Outer description of affine and vector spaces.** So far we have give a direct way of constructing affine spaces. Namely given a set of vectors Y, all affine combinations of vectors from Y define an affine space, which we denoted *Aff*(Y), the smallest affine space that contains Y. Similarly, the set of all linear combinations of vectors in X is a vector space. It is possible to obtain outer representations of vector and affine spaces expressing them as solution sets to linear systems of equations. More precisely, we have the following:

Proposition 1.11. *V is a vector space if and only if V is the solution of a system of linear equations of form*

$$a_i^T x = 0, i = 1, \ldots, m. \tag{1.5}$$

Moreover, the minimal number of equations in the system above is the dimension of $V^\perp$.

Proof. Let V be a vector space and let $(a_1, \ldots, a_m)$ be a set of vectors spanning $V^\perp$: $[a_1, \ldots, a_m] = V^\perp$. Then V is the set of solutions to the system of linear equations (1.5). Indeed, if x is in V then x is orthogonal to all vectors $a_i, i = 1, \ldots, m$, and therefore x is a solution to (1.5). On the other hand, if x is a solution to (1.5) then x is orthogonal to each $a_i, i = 1, \ldots, m$ and therefore belongs to $[a_1, \ldots, a_m]^\perp = V$. Reciprocally, the solution set to (1.5) is a vector space. □

Proposition 1.12. *A set is an affine space if and only if it is the solution of a system of linear equations of form*

$$a_i^T x = b_i, i = 1, \ldots, m, \tag{1.6}$$

where $a_i^T a = b_i, i = 1, \ldots, m$. Moreover, the minimal number of equations in the system above is the dimension of $V^\perp$.

Proof. Let $Y = a + V$ be an affine space. Take $(a_1, \ldots, a_m)$ such that $[a_1, \ldots, a_m] = V^\perp$ and define b such that $a_i^T a = b_i, i = 1, \ldots, m$. Then Y is the solution set to (1.6). Indeed, if y is in Y then $y = a + v$ for some $v \in V$ with v orthogonal to each $a_i, i = 1, \ldots, m$. Therefore, $y^T a_i = a^T a_i = b_i, i = 1, \ldots, m$, and y is a solution to (1.6). On the other hand, if y is a solution to (1.6) then $y - a \in [a_1, \ldots, a_m]^\perp = V$ and $y \in Y$. Reciprocally, we know that the solution set to (1.6) is an affine space. □

Propositions 1.11 and 1.12 offer simple ways of checking if a given vector belongs or does not belong to a vector space or to an affine space: it suffices to check if this vector satisfies the corresponding linear equations given by these propositions.

1.4. **Examples of affine spaces.** Affine space of dimension 0 in $\mathbb{R}^n$ are singletons in $\mathbb{R}^n$, translations by a given vector a of the vector space $\{0\}$. They are solution to a system of n linear equations with n unknown of form (1.6) with invertible matrix $[a_1; a_2; \dots; a_n]$.

Affine spaces of dimension 1 in $\mathbb{R}^n$ are lines:

$$\{a + td : t \in \mathbb{R}\}.$$

An affine basis is given by two different points y_1, y_2 on the line. It is the set of solutions of a system of linear equations of form (1.6) with n vaiables and $n-1$ equations.

Affine spaces of dimension $n-1$ in $\mathbb{R}^n$ are called hyperplanes. They are the solution set of a single linear equation $a^T x = b$ where $a \neq 0$.

The largest possible affine space in $\mathbb{R}^n$, of dimension n, is $\mathbb{R}^n$ itself.

2. Convexity

2.1. **Basic definitions and properties.**

We start with the definition of a convex set in $\mathbb{R}^n$.

Definition 2.1. *A nonempty set $C \subset \mathbb{R}^n$ is convex if, for any $x_1, x_2 \in C$ and $\alpha \in [0,1]$, it holds $\alpha x_1 + (1-\alpha)x_2 \in C$.*

Example 2.2. *The following sets in $\mathbb{R}^n$ are convex:*

- *An affine space (the set of solutions of a linear system of equations $C := \{x \in \mathbb{R}^n \mid Ax = b\}$ where A is some $m \times n$ matrix and $b \in \mathbb{R}^m$);*
- *the solution set to an arbitrary number of linear inequalities: $a_\alpha^T x \le b_\alpha, \alpha \in I$. In particular the solution set the finite linear inequalities $Ax \le a$ for some $m \times n$ matrix A is convex and called a polyhedron;*
- *an ellipsoid:*
$$\{x : (x - x_0)^T Q (x - x_0) \le r^2\}$$
where Q is definite positive;
- *the ε-fattening X^ε of a convex set X given by*
$$X^\varepsilon = \{y : \inf_{x \in X} \|y - x\| \le \varepsilon\};$$
- *the balls $\mathbb{B}(x_0, r) = \{x : \|x - x_0\| \le r\}$;*
- *the m-dimensional simplex*

$$S = \Big\{ \sum_{i=0}^m \lambda_i x_i : \lambda_i \ge 0, i \ge 1, \sum_{i=0}^m \lambda_i = 1 \Big\} \tag{2.7}$$

for vectors $x_0, x_1, \dots, x_m$, affinely independent. This is the convex hull of points $x_1, \dots, x_m$. As a special case, the unit simplex is the n-dimensional simplex obtained taking convex hull of 0, and vectors $e_1, e_2, \dots, e_n$, of the canonical basis. It can be expressed as $\{x : x \ge 0, \sum_{i=1}^n x_i \le 1\}$. The probability simplex is the $(n-1)$-dimensional simplex given by unit vectors $e_1, \dots, e_n$. It is the set of vectors satisfying $x \ge 0$ and $\sum_{i=1}^n x_i = 1$ which correspond to the set of possible discrete probability distributions with n elements in the support.

The m-dimensional simplex S given by (2.7) can be seen as a polyhedron given by the solution set to linear equalities and inequalities. Indeed, if vectors $x_0, x_1, \dots, x_m$ are affinely independent then

$$A = \begin{bmatrix} x_0 & x_1 & \cdots & x_m \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

has full column right and therefore has a left inverse B. If e is a vector in $\mathbb{R}^{m+1}$ with all components equal to one, we then have

$$\begin{aligned} y \in C \quad &\Leftrightarrow \quad \exists \theta \geq 0 : A\theta = \begin{pmatrix} y \\ 1 \end{pmatrix}, e^T\theta = 1, \\ &\Leftrightarrow \quad \exists \theta \geq 0 : \theta = B\begin{pmatrix} y \\ 1 \end{pmatrix}, e^T\theta = 1, \\ &\Leftrightarrow \quad B\begin{pmatrix} y \\ 1 \end{pmatrix} \geq 0, e^T B \begin{pmatrix} y \\ 1 \end{pmatrix} = 1. \end{aligned}$$

We have shown in the previous section that a set is affine if and only if it is an affine combination of its members. Similarly we can show that a set is convex if and only if it is a convex combination of its members, knowing that convex combination is an affine combination with nonnegative weights:

Definition 2.3. *A convex combination of vectors $x_1, \ldots, x_m$ is a vector of form*

$$x = \sum_{i=1}^{m} \lambda_i x_i$$

where

$$\lambda_i \geq 0, i \geq 1, \sum_{i=1}^{m} \lambda_i = 1.$$

We have the following analogue of Proposition 1.4 (we skip the proof which is similar to the proof of Proposition 1.4):

Proposition 2.4. *X is convex if and only if every convex combination of points from X belongs to X.*

We have seen that a nonempty intersection of affine spaces is an affine space. A nonempty intersection of convex sets is also convex:

Proposition 2.5. *A nonempty set which is the intersection of a (possibly infinite) set of convex sets is convex.*

For any nonempty convex set X, the convex hull of X is the smallest convex set which contains X. It is denoted by $\mathrm{Conv}(X)$ and due to the previous proposition it is the intersection of all convex sets that contain X. We can derive an algebraic representation of $\mathrm{Conv}(X)$:

Proposition 2.6. *Let X be a nonempty set. Then $\mathrm{Conv}(X)$ is the set of all convex combinations of points from X:*

$$\mathrm{Conv}(X) = \Big\{ \sum_{i=1}^{k} \lambda_i x_i : k \geq 1, \lambda_i \geq 0, x_i \in X, i \geq 1, \sum_{i=1}^{k} \lambda_i = 1 \Big\}.$$

Proof. Let

$$S(X) = \Big\{ \sum_{i=1}^{k} \lambda_i x_i : k \geq 1, \lambda_i \geq 0, x_i \in X, i \geq 1, \sum_{i=1}^{k} \lambda_i = 1 \Big\}.$$

It is straightforward to check that $S(X)$ is a convex set that contains X. Therefore $Conv(X) \subset S(X)$. Reciprocally, if $x \in S(X)$, then $x = \sum_{i=1}^{k} \lambda_i x_i$ for $x_i \in X$ and nonnegative coefficients λ_i summing to one. Since $x_i \in X \subset Conv(X)$ and since $Conv(X)$ is convex, x which is a convex combination of points from convex set $Conv(X)$ belongs to $Conv(X)$. Therefore $S(X) \subset Conv(X)$, which achieves the proof. $\square$

2.2. **Operations preserving convexity.** The following proposition is straightforward:

Proposition 2.7. *The following operations preserve convexity:*

- *Intersection: if $C_\alpha, \alpha \in I$ are convex sets then $\bigcap_{\alpha \in I} C_\alpha$ is convex.*
- *Image of a convex set under an affine function: if $C \subset \mathbb{R}^n$ is convex and $f : \mathbb{R}^n \to \mathbb{R}^m$ is affine then $f(C)$ is convex. Examples: if C is convex then $C + x_0$ (translation of C) and αC (scaling) are convex.*

- *Inverse image of a convex set under an affine function: if $f : \mathbb{R}^n \to \mathbb{R}^m$ is affine and C is convex then*
$$f^{-1}(C) = \Big\{x : f(x) \in C\Big\}$$
is convex.
- *Projection: the projection of a convex set onto some of its coordinates is convex: if $C \subset \mathbb{R}^{m+n}$ is convex then*
$$\Pi = \Big\{x_1 \in \mathbb{R}^m : \exists x_2 \in \mathbb{R}^n, (x_1, x_2) \in C\Big\}$$
is convex.
- *Sum of two sets: if C_1 and C_2 are convex then $C_1 + C_2 = \{c_1 + c_2 : c_1 \in C_1, c_2 \in C_2\}$ is convex.*
- *Cartesian product of two sets:if C_1 and C_2 are convex then $C_1 \times C_2$ is convex.*

Application: The ellipsoid
$$C = \Big\{x : (x - x_0)^T Q^{-1}(x - x_0) \leq 1\Big\}$$
is convex. It is the image of the unit ball $\{u : \|u\|_2 \leq 1\}$ under the affine mapping $f(u) = Q^{1/2}u + x_0$. It is also the inverse image of the unit ball under the affine mapping $f(x) = Q^{-1/2}(x - x_0)$.

We define the perspective function $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$ with domain $\mathbb{R}^n \times \{t \in \mathbb{R} : t > 0\}$ by $P(x, t) = \frac{x}{t}$.

Proposition 2.8. *If $C \subset dom(P)$ is convex then $P(C)$ is convex.*

Proof. Let $x, y \in \mathbb{R}^{n+1}$ with $x_{n+1}, y_{n+1} > 0$. Then for $0 \leq \theta \leq 1$ we have
$$P(\theta x + (1 - \theta)y) = tP(x) + (1 - t)P(y)$$
where
$$t = \frac{\theta x_{n+1}}{\theta x_{n+1} + (1 - \theta)y_{n+1}} \in [0, 1].$$
It follows that $P([x, y]) = [P(x), P(y)]$ and therefore if C is convex then $P(C)$ is convex too. □

Proposition 2.9. *If $C \subset \mathbb{R}^n$ is convex then $P^{-1}(C)$ is convex.*

Proof. Let $(x_1, t_1), (x_2, t_2) \in P^{-1}(C)$ and $0 \leq \theta \leq 1$. We want to show that
$$\frac{\theta x_1 + (1 - \theta)x_2}{\theta t_1 + (1 - \theta)t_2} \in C.$$
This follows from the fact that
$$\frac{\theta x_1 + (1 - \theta)x_2}{\theta t_1 + (1 - \theta)t_2} = \mu\frac{x_1}{t_1} + (1 - \mu)\frac{x_2}{t_2}$$
where
$$\mu = \frac{\theta t_1}{\theta t_1 + (1 - \theta)t_2}$$
and $\frac{x_1}{t_1} \in C$, $\frac{x_2}{t_2} \in C$. □

Proposition 2.10. *Let $g : \mathbb{R}^n \to \mathbb{R}^{m+1}$ given by*
$$g(x) = \begin{bmatrix} A \\ c^T \end{bmatrix} x + \begin{bmatrix} b \\ d \end{bmatrix}.$$
Let $f : \mathbb{R}^n \to \mathbb{R}^{m+1}$ be the linear fractional function given by
$$f(x) = \frac{Ax + b}{c^T x + d}$$
where $dom(f) = \{x : c^T x + d > 0\}$. From the previous propsition, we obtain that if $C \subset dom(f)$ is convex then $f(C)$ is convex and if $C \subset \mathbb{R}^m$ is convex then $f^{-1}(C)$ is convex.

We also have the following:

Proposition 2.11. *Let C be a convex set and X a random variable which belongs to C with probability one. Then $\mathbb{E}[X] \in C$.*

Definition 2.12. *The closed convex hull of a nonempty set* $A \subset \mathbb{R}^n$ *is defined as* $\overline{Conv}(A) := \bigcap_{C \in \mathcal{B}(A)} C$ *where* $\mathcal{B}(A) = \{C \,|\, C$ *is closed convex and* $A \subset C\}$.

Remark 2.13. *For future use, notice, with the notations above, that simple calculations yields* $\mathcal{B}(\{y\} + A) = \{y\} + \mathcal{B}(A)$. *Also, for any family of sets* $\mathcal{B}$, *it holds* $\bigcap_{C \in \mathcal{B}} (\{y\} + C) = \{y\} + \bigcap_{C \in \mathcal{B}} C$. *Combining the two above relations, we conclude that that*

$$\overline{Conv}\,(\{y\} + A) = \{y\} + \overline{Conv}(A).$$

Example 2.14. *Let* $X = \{x_1, \ldots, x_m\} \subset \mathbb{R}^n$. *We claim that*

$$\overline{Conv}(X) = \Big\{ \sum_{i=1}^{k} \lambda_i x_i : \lambda_i \geq 0,\ \sum_{i=1}^{k} \lambda_i = 1 \Big\}$$

which is a compact set. Notice that, in this case, the set $Conv(X)$ *is the image of the compact set* $\Delta = \Big\{ \lambda : \lambda_i \geq 0,\ \sum_{i=1}^{k} \lambda_i = 1 \Big\}$ *by the continous function* $\phi : \mathbb{R}^n \to \mathbb{R}$, $\phi(\lambda) = \sum_{i=1}^{m} \lambda_i x_i$ *and therefore it is also compact. Then, since* $Conv(X)$ *is compact, it concides with* $\overline{Conv}(X)$, *and the claim follows.*

2.3. **Projection onto a convex set.**

Proposition 2.15. *Given a nonempty closed and convex set* $C \subset \mathbb{R}^n$ *and* $\hat{x} \notin C$ *there exists a unique point* $a \in C$ *such that* $\|\hat{x} - a\| = \min\{\|\hat{x} - y\| \,|\, y \in C\}$.

Proof. Take $x_0 \in C$ and define $\delta = \|\hat{x} - x_0\|$. Since $D = B(\hat{x}, \delta) \cap C$ is a nonempty compact set and the function $f(x) = \|\hat{x} - x\|$ is continuous in $\mathbb{R}^n$, it follows that f achieves its minimum value in D, at some point $a \in D$. Then, by definition, it holds

$$\|\hat{x} - a\| = \min\{\|\hat{x} - x\| \,|\, x \in D\} \leq \delta \leq \|\hat{x} - y\| \quad \forall y \in C - D = C \cap D^c,$$

which clearly shows that $\|\hat{x} - a\| = \min\{\|\hat{x} - y\| \,|\, y \in C\}$. Let us denote $d(\hat{x}, C) = \min\{\|\hat{x} - y\| \,|\, y \in C\}$ and assume that there exists another point $\hat{a} \in C$ such that $\|\hat{x} - \hat{a}\| = d(\hat{x}, C)$. Convexity of C implies that $(a + \hat{a})/2 \in C$, which combined with the Parallelogram identity

$$\|x + y\|^2 + \|x - y\|^2 = 2\Big[\|x\|^2 + \|y\|^2\Big]$$

with

$$x = \frac{\hat{x} - a}{2}, \quad y = \frac{\hat{x} - \hat{a}}{2},$$

yields

$$d(\hat{x}, C)^2 \leq \left\| \hat{x} - \frac{(a + \hat{a})}{2} \right\|^2 = \frac{\|\hat{x} - a\|^2}{2} + \frac{\|\hat{x} - a\|^2}{2} - \frac{\|a - \hat{a}\|^2}{4} = d(\hat{x}, C)^2 - \frac{\|a - \hat{a}\|^2}{4}$$

which implies that $a = \hat{a}$ and ends the proof. □

Under the assumptions of the previous proposition, the point a will be called the projection of x onto C and it will be denoted by $P_C(x)$. Also, we will denote $d(x, C) = \|x - PC(x)\|$.

Proposition 2.16. *Consider a nonempty closed and convex set* $C \subset \mathbb{R}^n$ *and* $\hat{x} \notin C$. *Point* $x \in C$ *is the projection onto* C *of* $\hat{x}$ *if and only if*

$$\langle \hat{x} - x, y - x \rangle \leq 0 \quad \forall y \in C. \tag{2.8}$$

Proof. Let $a = P_C(\hat{x})$. By the convexity of C and the definition of the projection, it follows that, for any $y \in C$ and $t \in [0, 1]$

$$\|\hat{x} - (a + t(y - a))\|^2 \geq \|\hat{x} - a\|^2,$$

which yields

$$t\,\|y - a\|^2 \geq 2\,\langle \hat{x} - a, y - a \rangle.$$

Now, taking $t \to 0$, if follows that (2.8) holds.

Now assume now that, for some $x \in C$, it holds

$$\langle \hat{x} - x, y - x \rangle \leq 0 \quad \forall y \in C.$$

Relation

$$\|\hat{x}-y\|^2 = \|\hat{x}-x\|^2 + \|y-x\|^2 - 2\langle \hat{x}-x, y-x\rangle \geq \|\hat{x}-x\|^2 \quad \forall y \in C$$

shows that $x = P_C(\hat{x})$, which ends the proof. □

3. Cones

Definition 3.1. *A nonempty set C is conic if for every $x \in C$ and every $t \geq 0$ we have $tx \in C$.*

Definition 3.2. *A cone is a convex conic set.*

We deduce:

Proposition 3.3. *A nonempty set C is a cone if and only if it satisfies the following two properties:*

(i) *for every $x \in C$ and every $t \geq 0$ we have $tx \in C$;*
(ii) *for every $x, y \in C$ we have $x + y \in C$.*

Proof. If C satisfies (i), (ii), for every $x, y \in C$ for every $0 \leq t \leq 1$ we have $tx \in C, (1-t)y \in C$, and therefore the sum $tx + (1-t)y \in C$. Reciprocally, if C is a cone then C is conic and for every $x, y \in C$ we have

$$x + y = 2\underbrace{(\frac{1}{2}x + \frac{1}{2}y)}_{\in C \text{ by convexity}} \quad \in C.$$

□

Definition 3.4. *A conic combination of points $x_1, \ldots, x_m$ is a vector of form $\sum_{i=1}^m \lambda_i x_i$ where $\lambda_i \geq 0, i \geq 1$.*

We deduce from Proposition 3.3 that C is a cone if and only if it contains all conic combinations of points from C.

Example 3.5. *The set of solutions to the (possibly infinite) set of inequalities $a_\alpha^T x \leq 0, \alpha \in I$ is a cone. In particular, the solution set to a homogeneous finite system of m homogeneous linear inequalities $Ax \leq b$ where A is an $m \times n$ matrix is a cone, called polyhedral cone.*

Similarly to affine spaces and convex sets, a nonempty intersection of cones is a cone. For any nonempty set X we can also define the smallest cone that contains X, the conic hull of X denoted by $\mathrm{Cone}(X)$. The conic hull of X is the intersection of all cones that contain X and we obtain the following representation of the conic hull (the proof is similar to the proof of Proposition 2.6):

Proposition 3.6. *Let X be a nonempty set. Then*

$$\mathrm{Conic}(X) = \Big\{ \sum_{i=1}^k \lambda_i x_i : k \geq 1, x_i \in X, \lambda_i \geq 0, i \geq 1 \Big\}.$$

Example 3.7. *The following sets are cones:*

- *The solution set C to an homogeneous system of linear inequalities $C = \{x : a_i^T x \leq 0, i = 1, \ldots, m\}$. This cone is called polyhedral.*
- *The norm cone: $C = \{(x,t) : \|x\| \leq t\}$. For $\|\cdot\| = \|\cdot\|_2$, we obtain the second-order cone or Lorentz cone or ice-cream cone $C = \{(x,t) \in \mathbb{R}^{n+1} : \|x\|_2 \leq t\}$.*
- *The set of semidefinite positive matrices.*

Also, for any nonempty set X, the closed conical hull of of a set X is the smallest closed cone containing X. It is denoted by $\overline{Conic}(X)$.

Proposition 3.8. *Let $X = \{x_1, \ldots, x_m\}$. Then,*

$$\overline{\mathrm{Conic}}(X) = \Big\{ \sum_{i=1}^m \lambda_i x_i \;:\; \lambda_i \geq 0,\; i = 1, \ldots, m \Big\}.$$

Proof. Let $K = Conic(X) = \left\{ \sum_{i=1}^{m} \lambda_i x_i \ : \ \lambda_i \geq 0, \ i = 1, \ldots, m \right\}$. We will show that K is closed. First, notice that any linear combination of elements of X, with nonnegative linear coefficients, can be rewritten as a linear combination of all the elements of X, also with nonnegative linear coefficients. Next, take a sequence $\{x^k\} \subset K$ such that $x^k \to \hat{x}$ when $k \to +\infty$. We want to show that $\hat{x} \in K$. If $\hat{x} = 0$ then $\hat{x} \in K$. Now assume $\hat{x} \neq 0$. Without loss of generality, we can assume that $x^k \neq 0$ for all $k \geq 1$. Now, for each $k \geq 1$, we have that $x^k = \sum_{i=1}^m \lambda_i^k x_i$ where all λ_i^k are nonnegative and at least one of them is different from zero. Define $L_k = \sum_{i=1}^m \lambda_i^k$ and $\bar{x}^k = (1/L_k)\, x^k = (1/L_k) \sum_{i=1}^m \lambda_i^k x_i$ for all $k \geq 1$. Since $\{\bar{x}^k\} \subset \text{Conv}(X)$ which is compact, there exists a subsequence $\{\bar{x}^k \, : \, k \in J \subset \mathbb{N}\} \subset \{x^k\}$ converging to some $\bar{x} \in \text{Conv}(X)$. We have two cases to consider:

i) $0 \notin \text{Conv}(X)$. In this case, we have that $\bar{x} \neq 0$ and therefore $L_k = \left\|x^k\right\| / \left\|\bar{x}^k\right\| \to \bar{L} = \|\hat{x}\| / \|\bar{x}\| \in \mathbb{R}$ when $k \to +\infty, k \in J$, and it follows that $\hat{x} = \bar{L}\bar{x} \in K$.

ii) $0 \in \text{Conv}(X)$. Let us write $0 = \sum_{i=1}^m \bar{\lambda}_i x_i$ with $\bar{\lambda}_i \geq 0 \ i = 1, \ldots, m$ and $\sum_{i=1}^m \bar{\lambda}_i = 1$, and let us define $I = \left\{ i \in \{1, \ldots, m\} \, : \, \bar{\lambda}_i > 0 \right\} \neq \emptyset$. Take $x = \sum_{i=1}^m \lambda_i x_i \in K$ with $\lambda_i \geq 0 \ i = 1, \ldots, m$ and $x \neq 0$, and define $\delta = \min\left\{\lambda_i / \bar{\lambda}_i \, : \, i \in I\right\} \geq 0$ and $I_1 = \left\{ i \in I \, : \, \delta = \lambda_i / \bar{\lambda}_i \right\} \neq \emptyset$. Simple calculations show that

$$x = \sum_{i=1}^{m} (\lambda_i - \delta \bar{\lambda}_i) x_i \quad \lambda_i - \delta\bar{\lambda}_i \geq 0 \ \ i = 1, \ldots, m, \quad \lambda_\ell - \delta \bar{\lambda}_\ell = 0 \ \forall \, \ell \in I_1$$

which means that $x \in \text{Conic}(X')$ where $X' = \{x_i \, : \, i \in J = \{1 \ldots, m\} - I_1\}$. This argument shows that it holds

$$K = \bigcup_{i \in \{1, \ldots, m\}} \text{Conic}(X_i) \text{ where } X_i = X - \{x_i\}$$

Notice, by the previous argument in case i), that, if $0 \notin \text{Conv}(X_i)$ for all $i \in I$, then $\text{Conic}(X_i)$ is a closed set for all $i \in I$ and, therefore, the above realtion shows that K is closed. If $0 \in \text{Conv}(X_i)$ for some $i \in I$, then for this index i we could apply the process described above and, in this way, decompose the set $\text{Conic}(X_i)$, where $X_i = X - \{x_i\}$, as the union of conic sets of subsets of X_i, each of these subsets with one less elements than X_i. Since X is a finite set, this process, after a finite number of steps, will lead to a decomposition of K as the union of a finite family of closed sets, and the claim will follow.

□

Proposition 3.9. *If C is a closed cone and $\hat{x} \notin C$, then it holds*

$$\langle \hat{x} - P_C(\hat{x}), \hat{x} \rangle > 0 \quad \textit{and} \quad \langle \hat{x} - P_C(\hat{x}), y \rangle \leq 0 \quad \forall \, y \in C,$$

Proof. It holds

$$\begin{aligned} \langle \hat{x} - P_C(\hat{x}), \hat{x} \rangle &= \langle \hat{x} - P_C(\hat{x}), \hat{x} - P_C(\hat{x}) + P_C(\hat{x}) \rangle = \|\hat{x} - P_C(\hat{x})\|^2 + \langle \hat{x} - P_C(\hat{x}), P_C(\hat{x}) \rangle \\ &= \|\hat{x} - P_C(\hat{x})\|^2 + 2 \left\langle \hat{x} - P_C(\hat{x}), P_C(\hat{x}) - \frac{1}{2} P_C(\hat{x}) \right\rangle \geq \|\hat{x} - P_C(\hat{x})\|^2 > 0 \end{aligned}$$

where the before-last inequality follows from Proposition 2.16 and the fact that $\frac{1}{2} P_C(\hat{x}) \in C$. In addition, using again this proposition and the fact that C is a cone, we have, for any $t > 0$ and $y \in C$,

$$\langle \hat{x} - P_C(\hat{x}), y \rangle = \frac{\langle \hat{x} - P_C(\hat{x}), ty - P_C(\hat{x}) \rangle}{t} + \frac{\langle \hat{x} - P_C(\hat{x}), P_C(\hat{x}) \rangle}{t} \leq \frac{\langle \hat{x} - P_C(\hat{x}), P_C(\hat{x}) \rangle}{t}.$$

Taking $t \to +\infty$ in the relation above, yields the other part of the claim. □

4. Separation theorems

Definition 4.1. *We say that a point $x \in C$ is in the relative interior of the set $C \subset \mathbb{R}^n$ if there exists $r > 0$ such that $B(x, r) \cap \text{Aff}(C) \subset C$. We denote the set of such points by $ri(C)$ and call this set the relative interior of C.*

Definition 4.2. *An hyperplane in $\mathbb{R}^n$ is an affine space $H \subset \mathbb{R}^n$ of dimension equal to $n - 1$.*

Proposition 4.3. *A set $H \subset \mathbb{R}^n$ is an hyperplane if and only if there exists a nonzero vector $a \in \mathbb{R}^n$ and a real constant c such that $H = \{x \in \mathbb{R}^n \mid \langle x, a\rangle = c\}$.*

Proof. Any subspace V of $\mathbb{R}^n$ of dimension $n-1$ is of the form $V = \{x \in \mathbb{R}^n \mid \langle a, x\rangle = 0\}$ for some nonzero vector $a \in \mathbb{R}^n$. Hence, it follows that $H = V + \{x_0\}$ where V is a linear subspace of dimension $n-1$ and $H = \{x \in \mathbb{R}^n \mid \langle a, x\rangle = 0\} + \{x_0\}$ for some vector $a \neq 0$. Now, taking $c = \langle a, x_0\rangle$ the result follows. $\square$

Definition 4.4. *Given an hyperplane $H = \{x \in \mathbb{R}^n \mid \langle a, x\rangle = c\}$ we define the associated positive and negative closed semi-spaces $H_+ := \{x \in \mathbb{R}^n \mid \langle a, x\rangle \geq c\}$ and $H_- := \{x \in \mathbb{R}^n \mid \langle a, x\rangle \leq c\}$, respectively.*

Definition 4.5. *A convex polytope $H \subset \mathbb{R}^n$ is the set obtained by the intersection of a finite number of closed semi-spaces of $\mathbb{R}^n$, that is, $H = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{c} \langle a_1, x\rangle \leq b_1 \\ \langle a_2, x\rangle \leq b_2 \\ \vdots \\ \langle a_m, x\rangle \leq b_m \end{array} \right\}$. A polyhedral set is a nonempty bounded and convex polytope.*

Theorem 4.6. *Let $C \subset \mathbb{R}^n$ be a convex set and $y \in \overline{C}^c$. Then, there exists a vector $a \in \mathbb{R}^n$ such that $\langle a, y\rangle > \sup_{x \in C} \langle a, x\rangle$.*

Proof. In view of Proposition 2.16, we have, for any $x \in C$,

$$0 \leq \langle y - P_{\overline{C}}(y), P_{\overline{C}}(y) - x\rangle = \langle y - P_{\overline{C}}(y), P_{\overline{C}}(y) - y + y - x\rangle = \langle y - P_{\overline{C}}(y), y - x\rangle - \|y - P_{\overline{C}}(y)\|^2 .$$

Hence, defining $a = y - P_{\overline{C}}(y)$, it follows that

$$\langle a, y\rangle \geq \langle a, x\rangle + d(y, \overline{C})^2 \quad \forall x \in C,$$

which, in view of the fact that $d(y, \overline{C}) > 0$, implies the claim. $\square$

Remark 4.7. *Under the assumptions of Theorem 4.6, any hyperplane*

$$H = \{x : \langle a, x\rangle = b\}$$

with $\langle a, y\rangle > b > \sup_{x \in C} \langle a, x\rangle$ strictly separates y from C, in the sense that y and C are contained in the respective interiors of the negative and positive semi-spaces H_- and H_+. For instance, we can take $H = \left\{ x : \langle a, x\rangle = \left\langle a, \frac{y + P_{\overline{C}}(y)}{2} \right\rangle \right\}$.

Theorem 4.8. *Let $C \subset \mathbb{R}^n$ be a convex set and y a boundary point of C, that is $y \in \overline{C} \backslash \operatorname{int}(C)$. Then, there exists a vector $a \in \mathbb{R}^n$ such that $\langle a, y\rangle \geq \sup_{x \in C} \langle a, x\rangle$.*

Proof. Take a sequence $\{y^k\} \subset \overline{C}^c$ such that $y^k \to y$ when $k \to +\infty$. From Theorem 4.6, it follows that, for each $k \in \mathbb{N}$, there exists a nonzero $a^k \in \mathbb{R}^n$ such that

$$\left\langle a^k, y^k \right\rangle > \left\langle a^k, x \right\rangle \quad \forall x \in C.$$

Let a_* be an accumulation point of the sequence $\{a^k / \left\|a^k\right\|\}$. It follows from the above relation, dividing each side of the inequality by $\left\|a^k\right\|$ and taking limits when $k \to +\infty$, that

$$\langle a_*, y\rangle \geq \langle a_*, x\rangle \quad \forall x \in C$$

which clearly implies the claim and ends the proof. $\square$

Definition 4.9. *An hyperplane H such that a set C is contained in one of its related semi-spaces and contains a point y of the boundary of C is called a supporting hyperplane of C.*

Theorem 4.10. *Let C and D be two closed convex sets such that $C \cap D = \emptyset$. Then, there exists an hyperplane that separates C and D, that is, there exists a nonzero vector $a \in \mathbb{R}^n$ such that $\inf_{c \in C} \langle a, c\rangle \geq \sup_{d \in D} \langle a, d\rangle$.*

Proof. Define set $E = D - C = \{d - c : d \in D, c \in C\}$ which, in view of Proposition 2.7, is convex. Notice that $0 \notin E$. We have two options, $0 \in \overline{E}^c$ or $0 \in \overline{E}$. In the first case by Theorem 4.6 and in the second case by Theorem 4.8 (notice that 0 is not a point of the interior of E since $0 \notin E$ hence in the second case we have $0 \in \overline{E} \backslash \mathrm{int}(E)$ and the assumptions of Theorem 4.8 apply), it follows that there exists $a \in \mathbb{R}^n$ such that

$$0 = \langle a, 0 \rangle \geq \sup_{u \in E} \langle a, u \rangle = \sup_{d \in D, c \in C} \langle a, d - c \rangle \geq \langle a, d \rangle - \langle a, c \rangle \quad \forall\, c \in C,\, d \in D,$$

which clearly implies the claim. □

Theorem 4.11. *Let C and D be two closed convex sets such that C is bounded and $C \cap D = \emptyset$. Then, there exists an hyperplane that strictly separates C and D, that is, there exists a nonzero vector $a \in \mathbb{R}^n$ such that $\inf_{c \in C} \langle a, c \rangle > \sup_{d \in D} \langle a, d \rangle$.*

Proof. Define the convex set $E = D - C$. Assume that $0 \in \overline{E}$. Then, there exist sequences $\{x^k\} \subset C$ and $\{y^k\} \subset D$ such that $u^k = x^k - y^k \to 0$ when $k \to +\infty$. Since C is compact, there exists $\hat{x}$ an accumulation point of $\{x^k\}$, which, in view of the above relation, is also an accumulation point of $\{y^k\}$. Since C and D are closed, it follows that $\hat{x} \in C \cap D$ which is a contradiction. Hence $0 \notin \overline{E}$ and from Theorem 4.6 it follows that there exist nonzero vector $a \in \mathbb{R}^n$ and a real scalar b such that,

$$0 = \langle a, 0 \rangle > b > \sup_{u \in E} \langle a, u \rangle \geq \langle a, d \rangle - \langle a, c \rangle \quad \forall\, c \in C,\, d \in D.$$

The above relation implies that

$$0 > b \geq \sup_{d \in D} \langle a, d \rangle - \inf_{c \in C} \langle a, c \rangle$$

which yields the claim. □

Theorem 4.12. *Let C and D be two convex sets such that $\mathrm{ri}(C) \cap \mathrm{ri}(D) = \emptyset$. Then, there exists an hyperplane that separates C and D, that is, there exists a nonzero vector $a \in \mathbb{R}^n$ such that $\sup_{d \in D} \langle a, d \rangle \leq \inf_{c \in C} \langle a, c \rangle$.*

Proof. Define the convex set $E = D - C$. We have two possibilities, $0 \notin \overline{E}$ or $0 \in \overline{E}$. In the second case, in view of the assumptions, we have that $0 \notin \mathrm{ri}(E) = \mathrm{ri}(D) - \mathrm{ri}(C)$ and therefore $0 \notin \mathrm{int}(E)$ (otherwise, if we had $0 \in \mathrm{int}(E)$, since $\mathrm{int}(E) \subset \mathrm{ri}(E)$, we would also have $0 \in \mathrm{int}(E)$). Therefore, 0 is a boundary point of $\overline{E}$. Thus, in the first case by Theorem 4.6 and in the second case by Theorem 4.8, it follows that there exists $a \in \mathbb{R}^n$ such that

$$0 = \langle a, 0 \rangle \geq \sup_{u \in E} \langle a, u \rangle \geq \langle a, d \rangle - \langle a, c \rangle \quad \forall\, c \in C,\, d \in D,$$

which clearly implies that $a \in \mathbb{R}^n$ satisfies the claim. □

Definition 4.13. *Given a convex set C, a point $x \in C$ is an extreme point of C if there are no $x_1, x_2 \in C$, $x_1 \neq x_2$, and $\alpha \in (0, 1)$ such that $x = \alpha x_1 + (1 - \alpha) x_2$. That is, x is not a convex combination of points in C. We will denote the set of extreme points of C by ext(C).*

Example 4.14.

i) *If $C = [a, b]$ then ext$(C) = \{a, b\}$*
ii) *If $C = (a, b)$ then ext$(C) = \emptyset$.*
iii) *If $C = [a, +\infty)$, then ext$(C) = \{a\}$.*
iv) *If $C = B(a, r) = \{y \in \mathbb{R}^n : \|y - a\|_2 \leq r\}$, then ext$(C) = \{x : \|x - a\|_2 = r\}$.*

Lemma 4.15. *Every bounded and closed convex set has at least one extreme point.*

Proof. Let C be a bounded closed convex set, $\delta = \max\left\{\|x\|^2 : x \in C\right\}$ and $\hat{x} \in$C such that $\|\hat{x}\|^2 = \delta$. We will prove that $\hat{x}$ is an extreme point of C. Assume that $\hat{x} = \alpha x_1 + (1 - \alpha) x_2$ for some $x_1, x_2 \in C$ and $\alpha \in (0, 1)$. The relation

$$\begin{aligned} \delta = \|\alpha x_1 + (1 - \alpha) x_2\|^2 &= \alpha \|x_1\|^2 + (1 - \alpha) \|x_2\|^2 - \alpha(1 - \alpha) \|x_1 - x_2\|^2 \\ &\leq \alpha \delta + (1 - \alpha)\delta - \alpha(1 - \alpha) \|x_1 - x_2\|^2 = \delta - \alpha(1 - \alpha) \|x_1 - x_2\|^2 \end{aligned}$$

implies that $x_1 = x_2$. In view of the assumptions, we conclude that $x_1 = x_2 = \hat{x}$, which yields the claim. □

Lemma 4.16. *Let C be a convex set, H a supporting hyperplane of C, and $T = C \cap H$. Then, ext$(T) \subset$ ext(C).*

Proof. Let us write $H = \{x \mid \langle a, x\rangle = c\}$ and let us assume that $C \subset H_+ = \{x \mid \langle a, x\rangle \geq c\}$. Let x_0 be an extreme point of T. Assume that $x_0 = \alpha x_1 + (1-\alpha)x_2$ for some $x_1, x_2 \in C$ and $\alpha \in (0,1)$. If we have $\max\{\langle a, x_1\rangle, \langle a, x_2\rangle\} > c$ then since $\langle a, x_2\rangle, \langle a, x_1\rangle \geq c$, we would have $\langle a, x_0\rangle > c$, which contradicts the fact that $x_0 \in H$. Therefore, it follows that $\langle a, x_1\rangle = \langle a, x_2\rangle = c$ which means that $x_1, x_2 \in T$. Recalling that x_0 is an extreme point of T, the relation $x_0 = \alpha x_1 + (1-\alpha)x_2$ for $x_1, x_2 \in T$ is only possible if $x_1 = x_2 = x_0$. Hence, we conclude that x_0 is an extreme point of C. □

Lemma 4.17. *Let $C \subset \mathbb{R}^n$ be a convex set and $y \in \mathbb{R}^n$ a given point. Then, ext$(C) + \{y\} =$ ext$(C + \{y\})$. That is, a is an extreme point of the set C if and only if $a + y$ is an extreme point of the set $\hat{C} = C + \{y\}$.*

Proof. Assume that $a + y \notin \text{ext}(C) + y$, i.e., a is not an extreme point of C. Then, there exist $x_1, x_2 \in C$ and $\alpha \in (0,1)$ such that $a = \alpha x_1 + (1-\alpha)x_2$. Hence, it follows that $a + y = \alpha(x_1 + y) + (1-\alpha)(x_2 + y)$, which shows that $a + y$ is not an extreme point of $\hat{C}$. Similarly if $a + y$ for $a \in C$ is not an extreme point of $\hat{C}$ then $a + y = \alpha(a_1 + y) + (1-\alpha)(a_2 + y)$ for $a_1 \neq a_2 \in C$ and $a = \alpha a_1 + (1-\alpha)a_2$ is not an extreme point of C and $a + y \notin \text{ext}(C) + y$. □

Theorem 4.18. *Let $C \subset \mathbb{R}^n$ be a bounded closed and convex set. Then, $C = \overline{Conv}(ext(C))$. That is, C is the closed convex hull of the set of its extreme points.*

Proof. We will prove the result using induction in n, the dimension of the space. When $n = 1$, the bounded closed and convex sets of $\mathbb{R}$ are points and closed and bounded intervals, and then the claim holds trivially. Next, assume that the claim holds for all bounded closed and convex sets of $\mathbb{R}^m$. Let C be a bounded closed and convex set of $\mathbb{R}^{m+1}$ and $K = \overline{\text{Conv}}(\text{ext}(C))$. Under the assumptions on C, it is easy to see that $K \subset C$ (every point in K is a limit of points of $\text{Conv}(\text{ext}(C))$ and therefore, by convexity, of points of C and such limit belongs to C due to the fact that C is closed). Assume that $C \neq K$ and take $y \in C \backslash K$. Since K is closed and convex, it follows that there exists an hyperplane H that strictly separates y and K, that is, there exist a nonzero vector $a \in \mathbb{R}^{m+1}$ such that

$$\langle a, y\rangle < \inf\{\langle a, k\rangle \mid k \in K\}.$$

Take $c_0 = \inf\{\langle a, x\rangle \mid x \in C\} < +\infty$, and $x_0 \in C$ such that $c_0 = \langle a, x_0\rangle$, which are well defined since C is compact. It follows that the hyperplane $H_1 = \left\{x \in \mathbb{R}^{m+1} \mid \langle a, x\rangle = c_0\right\}$ is a supporting hyperplane for C. In addition, note that for $x \in K$, we have $\langle a, x\rangle \geq \inf\{\langle a, k\rangle \mid k \in K\} > \langle a, y\rangle \geq c_0$ and therefore $x \notin H_1$, i.e., $H_1 \cap K = \emptyset$. Now define $T = C \cap H_1$ and notice that $T \neq \emptyset$ since $x_0 \in T$ and it holds trivially that T is a bounded, closed and convex set. Since $\dim(H_1) = m$, it follows by the induction assumption that T contains extreme points that, by Lemma 4.16 are extreme points of C. Thus we have found extreme points of C not in K, which is a contradiction Therefore, it holds $C = K$ which ends the proof. □

Corollary 4.19. *A polyhedron is the convex combination of its extreme points.*

5. General theorem on the alternative.

In this section, we will consider the system of inequalities

$$(\mathcal{S}:)\begin{cases} a_i^T x > b_i, \ i = 1, \dots, p, \\ a_i^T x \geq b_i, \ i = p+1, \dots, m \end{cases} \tag{5.9}$$

in variable $x \in \mathbb{R}^n$. We associate with $\mathcal{S}$ the two systems of linear equations and inequalities in variable $\lambda \in \mathbb{R}^m$:

$$\mathcal{T}_I: \begin{cases} \lambda \geq 0, \\ \sum_{i=1}^m \lambda_i a_i = 0, \\ \sum_{i=1}^p \lambda_i > 0, \\ \sum_{i=1}^m \lambda_i b_i \geq 0. \end{cases} \qquad \mathcal{T}_{II}: \begin{cases} \lambda \geq 0, \\ \sum_{i=1}^m \lambda_i a_i = 0, \\ \sum_{i=1}^m \lambda_i b_i > 0. \end{cases} \tag{5.10}$$

The next result relates the infeasibility of system $\mathcal{S}$ with the feasibility of systems $\mathcal{T}_I$ and $\mathcal{T}_{II}$.

Theorem 5.1 (General Theorem on Alternative (GTA))**.** *The system $\mathcal{S}$ is infeasible if and only if either $\mathcal{T}_I$ or $\mathcal{T}_{II}$, or both of these systems have a solution.*

To prove the GTA we will use the following results.

Lemma 5.2 (The homogeneous Farkas Lemma)**.** *The system of homogeneous linear inequalities*

$$(F): \begin{cases} a^T x & < 0, \\ a_i^T x & \geq 0, \ i = 1, \dots, m. \end{cases}$$

is infeasible if and only if there exists $\lambda \geq 0$ such that $a = \sum_{i=1}^m \lambda_i a_i$.

Proof. Let us assume that there exist $\lambda \geq 0$ such that $a = \sum_{i=1}^m \lambda_i a_i$ and let x be a solution of system (F). Then, it follows that $a^T x = \sum_{i=1}^m \lambda_i a_i^T x \geq 0$ which is a contradiction. In summary, existence of such a λ implies the infeasibility of the system (F). Next, consider the set $X = \{a_1, \dots, a_m\}$, and let $K = \mathrm{Conic}(X)$ which, by Proposition 3.8, is a closed cone. Notice that there exists $\lambda \geq 0$ satisfying $a = \sum_{i=1}^m \lambda_i a_i$ if and only if $a \in K$. Let us assume that $a \notin K$. Then, by Proposition 3.9, it follows that $x = P_K(a) - a$ satisfies

$$\langle a, x \rangle < 0 \text{ and } \langle y, x \rangle \geq 0 \quad \forall y \in K.$$

which means, taking $y = a_i$, $i = 1, \dots, m$, that system (F) is feasible and ends the proof. ☐

Proposition 5.3. *The system of linear inequalities (5.9) has no solution if and only if this is the case for the following homogeneus system*

$$(\mathcal{S}^*): \begin{cases} -s < 0, \\ t - s \geq 0, \\ a_i^T x - b_i t - s \geq 0, \ i = 1, \dots, p, \\ a_i^T x - b_i t \geq 0, \ i = p+1, \dots, m. \end{cases} \tag{5.11}$$

Proof. Let $\hat{x}$ be a solution of the system (5.9) (system $(\mathcal{S})$). Define $\hat{t} = 1$ and $\hat{s} = \min\{1, a_i^T \hat{x} - b_i \ i = \ \dots, p\}$. It follows that

$$\begin{aligned} &\hat{s} > 0, \\ &\hat{t} - \hat{s} = 1 - \hat{s} \geq 0, \\ &a_i^T \hat{x} - b_i \hat{t} - \hat{s} = a_i^T \hat{x} - b_i - \hat{s} \geq 0 \quad i = 1, \dots, p, \\ &a_i^T \hat{x} - b_i \hat{t} = a_i^T \hat{x} - b_i \geq 0 \quad i = p+1, \dots, m, \end{aligned}$$

which means that $(\hat{x}, \hat{s}, \hat{t})$ is a solution of (5.11) (system $(\mathcal{S}^*)$). Now, let $(\hat{x}, \hat{s}, \hat{t})$ be a solution of system $(\mathcal{S}^*)$. Since $\hat{t} \geq \hat{s} > 0$ it follows that we can define $\bar{x} = \hat{x}/\hat{t}$ and we get

$$a_i^T \bar{x} - b_i = \frac{1}{\hat{t}} \left(a_i^T \hat{x} - \hat{t} b_i \right) \geq \frac{\hat{s}}{\hat{t}} > 0 \quad i = 1, \dots, p,$$

and

$$a_i^T \bar{x} - b_i = \frac{1}{\hat{t}} \left(a_i^T \hat{x} - \hat{t} b_i \right) \geq 0 \quad i = p+1, \dots, m,$$

which means that $\bar{x}$ is a solution of (5.9) (system $(\mathcal{S})$). We have just proved that one system is feasible if and only if the other system is also feasible, which is equivalent to the claim. ☐

Proof of the GTA

Proof. Rewitting system (S^*) as follows

$$\begin{cases} (0, -1, 0)^T (x, s, t) < 0, \\ (0, -1, 1)^T (x, s, t) \geq 0, \\ (a_i, -1, -b_i)^T (x, s, t) \geq 0, \ i = 1, \dots, p, \\ (a_i, 0, -b_i)^T (x, s, t) \geq 0, \ i = p+1, \dots, m. \end{cases}$$

and using the Farkas Lemma, it follows that (S^*) is infeasible if, and only if, there exists $\bar{\lambda} = (\lambda_0, \lambda_1, \dots, \lambda_m) \geq 0$ such that

$$(0, -1, 0) = \lambda_0 (0, -1, 1) + \sum_{i=1}^{p} \lambda_i (a_i, -1, -b_i) + \sum_{i=p+1}^{m} \lambda_i (a_i, 0, -b_i).$$

This means that $\bar{\lambda}$ satisfies

$$\bar{\lambda} \geq 0, \quad 0 = \sum_{i=1}^{m} \lambda_i a_i, \quad \sum_{i=1}^{p} \lambda_i = 1 - \lambda_0, \quad \sum_{i=1}^{m} \lambda_i b_i = \lambda_0 \tag{5.12}$$

Since $\sum_{i=1}^{p} \lambda_i \geq 0$ it follows that $\lambda_0 \in [0, 1]$. Therefore, $\lambda = (\lambda_1, \dots, \lambda_m)$ satisfies

$$\lambda \geq 0, \quad 0 = \sum_{i=1}^{m} \lambda_i a_i, \quad \sum_{i=1}^{p} \lambda_i = 1, \quad \sum_{i=1}^{m} \lambda_i b_i = 0 \text{ if } \lambda_0 = 0$$

or

$$\lambda \geq 0, \quad 0 = \sum_{i=1}^{m} \lambda_i a_i, \quad \sum_{i=1}^{p} \lambda_i \in [0, 1), \quad \sum_{i=1}^{m} \lambda_i b_i \in (0, 1] \text{ if } \lambda_0 \in (0, 1].$$

which means that in the first case above λ satisfies system $\mathcal{T}_I$ and in the second case it satisfies system $\mathcal{T}_{II}$. Now, let x be a solution to system (S) (5.9) and let $\lambda = (\lambda_1, \dots, \lambda_m) \geq 0$ be a solution to either system $\mathcal{T}_I$ or system $\mathcal{T}_{II}$. If λ solves system $\mathcal{T}_I$, it follows that

$$\sum_{i=1}^{p} \lambda_i (a_i^T x) > \sum_{i=1}^{p} \lambda_i b_i \text{ and } \sum_{i=p+1}^{m} \lambda_i (a_i^T x) \geq \sum_{i=p+1}^{m} \lambda_i b_i.$$

Threfore, we have that

$$0 = \left(\sum_{i=1}^{m} \lambda_i a_i \right)^T x = \sum_{i=1}^{m} \lambda_i \left(a_i^T x \right) > \sum_{i=1}^{m} \lambda_i b_i^T \geq 0$$

and we obtain a contradiction. If λ solves system $\mathcal{T}_{II}$, then we have

$$0 = \left(\sum_{i=1}^{m} \lambda_i a_i \right)^T x = \sum_{i=1}^{m} \lambda_i (a_i^T x) \geq \sum_{i=1}^{m} \lambda_i b_i > 0$$

and we obtain a contradiction. Hence in both cases we obtain a contradiction, which shows that if either $\mathcal{T}_I$ or system $\mathcal{T}_{II}$ is solvable then system (S) is infeasible, and ends the proof. □

6. Linear programming problems

6.1. **Basic definitions and examples.**

The linear programming problem (LPP) consists in the minimization or maximization of a linear objective function subjected to a finite set of constraints defined by linear equalities and inequalities. The LPP, in its standard form, is formulated as follows:

$$\begin{aligned} &\text{minimize } c^T x \\ &\text{subject to } Ax = b \text{ and } x \geq 0 \end{aligned} \tag{6.13}$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and A is an $m \times n$ matrix.

Remark 6.1. *Notice that other formulations of the LPP can be cast in the form of* (6.13)*. For example,*

i) *inequality constraints can be recast as equality constraints introducing slack or surplus nonnegative variables, i.e.* $a^T x \leq d$ *can be reformulated as* $a^T x + s = b$, $s \geq 0$, *and* $a^T x \geq d$ *can be reformulated as* $a^T x - s = b$, $s \geq 0$, *respectively;*

ii) *free variables, which means variables that are not required to be nonnegative, can be rewritten as the difference of two nonnegative variables.*

Assumption 6.2. *The* $m \times n$ *matrix* A *satisfies the full rank assumption if* $m < n$ *and the* m *rows of* A *are linearly independent.*

6.2. **Basic solutions and extreme points.**

Consider the system of linear equations

$$Ax = b \tag{6.14}$$

where $b \in \mathbb{R}^m$, and A is an $m \times n$ matrix that satisfies Assumption 6.2.

Definition 6.3. *Consider an ordered set of m indexes $J \subset \{1, 2, \ldots, n\}$ such that the matrix B, formed by the columns of A indexed by J, is invertible. The basic solution of the system (6.14) defined by B is the vector x with $x_J = B^{-1}b$ and $x_I = 0$ where $I = \{1, 2, \ldots, n\} \backslash J$. The components x_j, $j \in J$ are called basic variables of the basic solution. If one or more of the basic variables is equal to zero, we call the basic solution a degenerate basic solution.*

Remark 6.4. *To simplify the exposition of the results that will follow in these notes, usually we will denote a basic solution, related to a matrix B as in the definition above, by $x = (x_B, 0) = (B^{-1}b, 0)$. Formally, this corresponds to the matrix B being equal to the submatrix formed by the first m columns of the matrix A. By a simple reordering of the variables, we can reduce any case of basic solution to this case. Also, we will denote by B the set of indexes which index the vectors of the matrix B.*

Now consider the system of constraints

$$Ax = b, \quad x \geq 0 \tag{6.15}$$

where $b \in \mathbb{R}^m$, and $A \in M(m, n)$ satisfies Assumption 6.2.

Definition 6.5. *A vector satisfying (6.14) is said to be feasible for these constraints. A feasible solution to the system (6.15) that is also basic for the related linear system (6.14) is said to be a basic feasible solution; if this solution is also a degenerate basic solution, it is called a degenerate basic feasible solution.*

Definition 6.6. *An optimal solution to LPP (6.13) that is also basic for the related linear system (6.14) is said to be an optimal basic solution; if this solution is also a degenerate basic solution, it is called a degenerate optimal basic solution.*

Theorem 6.7. *Consider the LPP (6.13) where A satisfies the Assumption 6.2. Then,*

i) *if there is a feasible solution to the system (6.15), there is a basic feasible solution;*
ii) *if there is an optimal feasible solution to the LPP (6.13), there is an optimal basic feasible solution.*

Proof. First, we prove i). Notice that if $x = 0$ is a feasible solution to (6.15) then it is a degenerate basic feasible solution and the claim is proved. Next, consider $x = (x_1, x_2, \ldots, x_n) \neq 0$, a feasible solution for (6.15), and let $J = \{i \in \{2, \ldots, n\} \,|\, x_i > 0\}$. Let $B = \{a_j \,|\, j \in J\}$ denote the set of column vectors of A indexed by J. If B is a linear independent set, then we have a basic feasible solution and the claim is proved. Otherwise, consider the vector $y = (y_1, y_2, \ldots, y_n)$ with $\{y_j \,|\, j \in J\}$ satisfying

$$\sum_{j \in J} y_j a_j = 0$$

and $y_j = 0$ for all $j \notin J$. Clearly, we can assume that at least one component of y is positive. Notice that the vector $x_\varepsilon = x + \varepsilon y$ is a solution of the linear system (6.14) for any $\varepsilon \in \mathbb{R}$. In particular, considering

$$\varepsilon = \min \left\{ -\frac{x_i}{y_i} \,|\, y_i < 0, \right\} > 0$$

we obtain a new feasible solution to (6.15), since clearly $x_\varepsilon \geq 0$, and in addition $(x_\varepsilon)_j = 0$ for all $j \notin J$ and, at least for some $j \in J$, $(x_\varepsilon)_j$ became 0. This process shows how to reduce to zero at least one of the positive components of any nonzero and nonbasic feasible solution to (6.15). Clearly, after a finite number of steps of this procedure, we will obtain a feasible solution to (6.15), whose nonzero variables are associated to linear independent column vectors of the matrix A, in summary, a basic feasible solution. Then, the claim follows.

Now, we prove ii). Observe that we can proceed as in the proof of item i) of the theorem, but considering in addition that x is optimal for the LPP (6.13). In this case, let $\varepsilon > 0$ such that $x_1 = x_\varepsilon = x + \varepsilon y \geq 0$ is feasible (taking $\varepsilon \leq \min\left\{-\frac{x_i}{y_i} \,|\, y_i < 0,\right\} > 0$)) and $x_2 = x - \varepsilon y \geq 0$ is feasible (taking $\varepsilon \leq \min\left\{\frac{x_i}{y_i} \,|\, y_i > 0,\right\} > 0$) and x_1, x_2 are feasible solutions to (6.15). Next, if $c^T y > 0$ we have

$$c^T x_2 = c^T x - \varepsilon c^T y < c^T x$$

which is not possible while if $c^T y < 0$ we have

$$c^T x_1 = c^T x + \varepsilon c^T y < c^T x$$

which is also a contradiction. Therefore, $c^T y = 0$. Hence, $c^T x = c^T x_1 = c^T x_2$. Using the same procedure as in item i), we can obtain a basic feasible solution for system (6.15) with the value of the objective function equal to $c^t x$, that is, a basic optimal solution to (6.15), and the claim follows. □

Theorem 6.8 (Equivalence of Extreme Points and Basic Solutions)**.** *Consider the system* (6.15) *where A satisfies the Assumption 6.2. Let K be the convex polytope consisting of all feasible solutions to this system. The, a vector x is an extreme point of K if, and only if, x is a basic feasible solution to* (6.15)*.*

Proof. Let x be a basic solution to (6.15) associated to a nonsingular submatrix B of A. Following the notations of Definition 6.3, consider the set of indexes J and I associated to B. We have that $x = (x_B, 0)$ and $x_I = 0$. Assume that $x = \alpha\hat{x} + (1-\alpha)\bar{x}$ for some $\alpha \in (0,1)$ and $\hat{x}, \bar{x} \in K$. Since $\hat{x}, \bar{x} \geq 0$, it follows that $\hat{x}_I = \bar{x}_I = 0$. These relations and the feasibility of $\hat{x}$ and $\bar{x}$ imply that $\hat{x}_J = \bar{x}_J = B^{-1}b$. Hence, we obtain that $x = \hat{x} = \bar{x}$ and we conclude that x is an extreme point of K. Next, let x be an extreme point of K. Define $J = \{i \in \{1, 2, \ldots, n\} \,|\, x_i > 0\}$ and $I = \{1, 2, \ldots, n\} \backslash J$, and let B be the submatrix of vectors of A indexed by J. If the column vectors of B are linearly independent, then x is a basic solution to (6.15) (a degenerated one if $|J| < m$). On the other hand, if these vectors are linearly dependent, then there exists a vector $u \in \mathbb{R}^{|J|}$ such that $Bu = 0$. Taking $y \in \mathbb{R}^n$ with $y_J = u$ and $y_I = 0$, and $\varepsilon > 0$ small enough, it is easy to see that the vectors $x_1 = x + \varepsilon y$ and $x_2 = x - \varepsilon y$ are feasible to (6.15), which, combined with the relation $x = (1/2)x_1 + (1/2)x_2$, shows that x is not an extreme point of K, yielding a contradiction and ending the proof. □

6.3. **Duality.**

In this subsection, we will consider the Linear Programming Problems (6.13)

Remark 6.9. *Other formulations of the LPP can be recast as instances of* (6.13)*. Notice that $\{x \,|\, Ax = b\} = \{x \,|\, Ax \geq b,\ -Ax \geq -b\}$.*

Definition 6.10. *The dual problem of the LPP* (6.13) *is the LPP*

$$\begin{aligned} &\textit{maximize } b^T y \\ &\textit{subject to } y^T A \leq c^T \end{aligned} \tag{6.16}$$

Remark 6.11. *Note that we can reformulate the above problem as an equivalent minimization LPP:*

$$\begin{aligned} &\textit{minimize } -b^T y \\ &\textit{subject to } y^T A \leq c^T \end{aligned} \tag{6.17}$$

Theorem 6.12 (Weak duality)**.** *If x and y are feasible for* (6.13) *and* (6.16)*, respectively, then $c^T x \geq y^T b$.*

Proof. Simple calculations show that

$$y^T b = y^T(Ax) = (y^T A)x \leq c^T x$$

since $y^T A \leq c^T$ and $x \geq 0$. □

The following result follows immediately from the previous theorem.

Corollary 6.13. *If x and y are feasible for* (6.13) *and* (6.16)*, respectively, and $c^T x = y^T b$, then x and y are optimal for* (6.13) *and* (6.16)*, respectively,*

Theorem 6.14 (Strong duality)**.** *If either of the problems* (6.13) *or* (6.16) *has a finite optimal solution, so does the other, and the corresponding values of the objective functions are equal. If either problem has an unbounded objective, the other problem has no feasible solution.*

Proof. The second statement of the theorem follows trivially from Theorem 6.12 since any feasible solution of either of the problems provides a bound for the functional value of the other problem. To prove the first statement, let x_0 be an optimal feasible solution for the LPP (6.13), set $z_0 = c^T x_0$ and define the set $C = \left\{ t(z_0 - c^T x, b - Ax) \,|\, t \geq 0,\ x \geq 0 \right\}$. Observe that C is a closed convex cone. Also, it holds $(1, 0) \notin C$, since this would imply that, for some $\hat{x} \geq 0$ and $\hat{t} > 0$, it holds $b - A\hat{x} = 0$, but in this case we must have $\hat{t}(z_0 - c^T \hat{x}) \leq 0$ since $\hat{x}$ would be feasible for the LPP problem (6.13). Now, by Theorem 4.6, it follows that there exists $a = (s, y)$ and a real scalar δ such that

$$s = (s, y)^T (1, 0) < \delta < (s, y)^T \left(t(z_0 - cx, b - Ax) \right) = t \left(s z_0 - s c^T x + y^T b - y^T A x \right) \ \forall t \geq 0,\ x \geq 0.$$

The above relation implies that $\delta \leq 0$ and, therefore, that $s < 0$. Notice that through a simple rescaling we can assume that $s = -1$. Dividing by t and making $t \to +\infty$, it follows that

$$0 \geq (z_0 - y^T b) + (-c + A^T y)^T x \quad \forall x \geq 0,$$

which implies that

$$0 \geq -c + A^T y \text{ and } 0 \geq z_0 - y^T b.$$

The first relation shows that y is feasible for the dual problem (6.16). The second relation, combined with this fact and Theorem 6.12, implies that $z_0 = y^T b$, whence it follows that y is an optimal solution for this problem and that both problems, (6.13) and (6.16), have the same optimal value. Next, notice that if the LPP (6.16) has a solution then its equivalent reformulation (6.11) has the same solution, and, in view of the first part of the proof, the the dual of this later problem also has an optimal solution with the same optimal value. It is easy to see the the dual of the LPP (6.11) is an equivalent reformulation of the LPP (6.13), which ends the proof. $\square$

We now present another proof of the duality theorem for LPs.

Theorem 6.15 (Duality Theorem in Linear Programming)**.** *Consider linear program (LP) along with its dual (D) given by:*

$$(LP) \quad c_* = \left\{ \begin{array}{l} \min c^T x \\ a_i^T x \geq b_i,\ i = 1, \ldots, m. \end{array} \right. \qquad (D) \quad \left\{ \begin{array}{l} \max\ \lambda^T b \\ \lambda \geq 0,\ A^T \lambda = c. \end{array} \right. \tag{6.18}$$

Then:

1) Duality is symmetric: the dual to the dual is the dual.

2) The value of the dual objective at every dual feasible solution is $\leq$ the value of the primal objective at every primal feasible solution.

3) The following 5 properties are equivalent:

(i) The primal is feasible and bounded below.

(ii) The dual is feasible and bounded above.

(iii) The primal is solvable.

(iv) The dual is solvable.

(v) Both the primal and the dual are feasible.

Whenever (i)-(v) hold both the primal and the dual are solvable and have the same optimal value.

Proof. 1) Writing the dual as

$$\begin{array}{l} -\min\ -b^T y \\ \left[\begin{array}{c} A^T \\ -A^T \\ I_m \end{array} \right] y \geq \left[\begin{array}{c} c \\ -c \\ 0 \end{array} \right] \end{array}$$

the dual of the dual is the linear program

$$\begin{array}{l} -\max\ (\lambda_1 - \lambda_2)^T c \\ A(\lambda_1 - \lambda_2) + \lambda_3 = -b, \\ \lambda_3 \geq 0,\ \lambda_1, \lambda_2 \geq 0 \end{array}$$

which is equivalent to (LP) setting $x = \lambda_2 - \lambda_1$.

2) is weak duality.

3) (i) $\Rightarrow$ (iv). Taking $a = c_*$ we get that there exists a dual feasible solution λ satisfying $\lambda^T b \geq c_*$. Together with 2) this implies $c_* = \lambda^T b$ and therefore $d_* = c_*$.

(iv) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (iii) comes from the primal-dual symmetry and (i) $\Rightarrow$ (iv).

(iii) $\Rightarrow$ (i) is clear.

Clearly (i)-(iv) imply (v). If (v) holds then the primal is feasible and bounded below by the value of the dual objective at any feasible dual point (see 2)). $\square$

We deduce the following necessary and sufficient optimality conditions for linear programs:

Theorem 6.16. *Consider primal problem (LP) and its dual (D) given by (6.18). Then a pair (x, λ) of primal-dual feasible solutions is optimal if and only if*

$$\lambda_i[Ax - b]_i = 0, \ i = 1, \dots, m.$$

The above condition can also be written $c^T x = \lambda^T b$ (no duality gap).

Proof. If (x, λ) is a primal-dual optimal solution then optimal values of the primal and the dual are equal, i.e., $c^T x = \lambda^T b$. On the other hand, if $c^T x = \lambda^T b$, then due to 2) in Theorem 6.15, x and λ are optimal primal and dual solutions. Finally, there is no duality gap iff $\lambda_i[Ax - b]_i = 0, \ i = 1, \dots, m$, because if (x, λ) is a primal-dual feasible solution then

$$\lambda^T(Ax - b) = x^T A^T \lambda - b^T \lambda = x^T c - b^T \lambda.$$

$\square$

Remark 6.17. *Notice that, starting with the primal-dual pair of LPPs given in (6.18) and taking into account Remark 6.9, we can obtain the following primal-dual pair of LPPs*

Primal problem (P)	$\max\ c^\top x$ $Ax = b$ $x \geq 0$	$\max\ c^\top x$ $Ax = b$ $Cx \leq d$ $x \geq 0$	$\min\ c^\top x$ $Ax = b$ $x \geq 0$	$\min\ c^\top x$ $Ax = b$ $Cx \leq d$ $x \geq 0$
Dual problem (D)	$\min\ \lambda^\top b$ $A^\top \lambda \geq c$	$\min\ \lambda^\top b + \mu^\top d$ $A^\top \lambda + C^\top \mu \geq c$ $\mu \geq 0$	$\max\ \lambda^\top b$ $A^\top \lambda \leq c$	$\max\ \lambda^\top b + \mu^\top d$ $A^\top \lambda + C^\top \mu \leq c$ $\mu \leq 0$

For each one of these primal-dual pairs of LPPs a similar result to Theorem 6.15 also holds.

6.4. **The SIMPLEX algorithm.**

6.4.1. **Simplex algorithm: compact version, examples**. The Simplex is an algorithm for solving linear programming problems. We sill consider the linear programming problem

$$\left\{ \begin{array}{l} \max\ c^\top x = \sum_{i=1}^n c_i x_i \\ \ \sum_{j=1}^n A_{ij} x_j = b_i \quad\ \ i = 1, \dots, m_1, \\ \ \sum_{j=1}^n A_{ij} x_j \geq b_i, \quad i = m_1 + 1, \dots, m_1 + m_2, \\ \ \sum_{j=1}^n A_{ij} x_j \leq b_i, \quad i = m_1 + m_2 + 1, \dots, m, \\ \ x \geq 0 \end{array} \right. \tag{6.19}$$

where $m = m_1 + m_2 + m_3$. In (6.19), we can assume that $b_i \geq 0$ for all $i = 1, \dots, m$ (if for some i it holds $b_i < 0$, then we can re-write the corresponding equality or inequality by multiplying both left and right hand sides by -1).

We introduce (non-negative) slack variables e_i in order to re-write (6.19) as follows

$$\left\{ \begin{array}{ll} \max\ c^\top x = \sum_{i=1}^n c_i x_i & \\ \ \sum_{j=1}^n A_{ij} x_j = b_i, & i = 1, \dots, m_1, \\ \ \sum_{j=1}^n A_{ij} x_j - e_i = b_i, & i = m_1 + 1, \dots, m_1 + m_2, \\ \ \sum_{j=1}^n A_{ij} x_j + e_i = b_i, & i = m_1 + m_2 + 1, \dots, m, \\ \ x \geq 0, \quad e \geq 0 & \end{array} \right. \tag{6.20}$$

where $b \geq 0$. The Simplex algorithm has two phases summarized below.

Phase I of the Simplex The objective of the first phase of the simplex is to find out whether problem (6.20) has a solution. Furthermore, if (6.20) has a solution, Phase I provides a base matrix of constraints (set of independent columns of A), used to start phase II.

Phase II of the Simplex For a problem of the form (6.20) that has a solution, phase II finds an optimal solution based on the matrix of constraints provided by phase I. Phase II then allows a problem to be solved linear, given a basis of the constraints matrix.

Details of Phase I. Phase I consists of solving the problem

$$\left\{\begin{array}{ll} \min \sum_{i=1}^{m_1+m_2} v_i & \\ \sum_{j=1}^{n} A_{ij}x_j + v_i = b_i, & i = 1, \ldots, m_1, \\ \sum_{j=1}^{n} A_{ij}x_j - e_i + v_i = b_i, & i = m_1+1, \ldots, m_1+m_2, \\ \sum_{j=1}^{n} A_{ij}x_j + e_i = b_i, & i = m_1+m_2+1, \ldots, m, \\ x \geq 0, \ \ e \geq 0, \ v \geq 0, & \end{array}\right. \tag{6.21}$$

which minimizes the sum of constraint violations. The problem above is always viable and has an optimal solution. As we have the evident base $B = (v, e_{m_1+m_2+1}, \ldots, e_m))$ for this problem, we can apply Phase II of the Simplex to solve (6.21) from this base.

Proposition 6.18. *Problem (6.20) has a solution if and only if the optimal value of problem (6.21) is 0.*

If the optimal value of problem (6.21) is 0 and if there are artificial variablesv_i in the basic solution, we can replace each of these variables with variables from the initial problem x_i or e_i.

Details of Phase II. Here, without loss of generality, we will consider the LP problem (6.13). Starting with that base B determined in Phase I, we can reformulate problem (6.21) as follows

$$\begin{array}{l} \min \ z = z_0 + \bar{c}_D^\top x_D \\ x_B + N x_D = \bar{b} \\ x_B \geq 0, \ x_D \geq 0. \end{array} \tag{6.22}$$

where, denoting $A = (B|D)$, we have $z_0 = c_B^T B^{-1} b$, $N = B^{-1}D$, $\bar{b} = B^{-1}b$ and $\bar{c}_D = c_D - N^T c_B$

To simplify the presentation we will assume that $\{1, 2, \ldots, m\}$ is the set of indexes of the columns of A which correspond to the vectors in B.

Definition 6.19. *Let $x = (x_B, x_N) = (x_B, 0)$ the basic solution associated to B. For $j \notin B$, the jth basic direction at x is the vector d defined by letting $d_j = 1$, $d_i = 0$ for $i \notin B$ and $i \neq j$, and $d_B = -B^{-1}A_j$.*

Definition 6.20. *Let x be feasible for the linear system in (6.13). A vector $d \in \mathbb{R}^n$ is said to be a feasible direction at x, if there exists a positive scalar θ_0 for which $x + \theta d$ is feasible for the linear system in (6.13) for all $\theta \in [0, \theta_0]$.*

Lemma 6.21. *Let $x = (x_B, x_N) = (x_B, 0)$ be the basic solution associated to B and d be the j-th basic direction at x. If x is nondegenerated, then d is a feasible direction at x.*

Proof. Simple calculations show that $Ad = Bd + Nd = B(-B^{-1}A_j) + A_j = 0$. Hence $A(x + \theta d) = Ax = b$ for all $\theta \in \mathbb{R}$. In addition, note that, for all $\theta > 0$,

$$(x + \theta d)_j = x_j + \theta d_j = \theta > 0 \ \text{ and } \ (x + \theta d)_i = x_i = 0 \ \forall\, i \notin B \text{ and } i \neq j.$$

Also, since $x_B > 0$, it holds

$$(x + \theta d)_B \geq 0 \text{ if } d_b \geq 0,$$

and

$$(x + \theta d)_B = x_B + \theta d_B \geq 0 \quad \forall\, \theta \in [0, \theta_0], \quad \text{otherwise}$$

where

$$\theta_0 = \min\left\{-\frac{x_i}{d_i} \ : \ d_i < 0, \, i \in B\right\} > 0. \tag{6.23}$$

In summary, we have proved that $x + \theta d$ is feasible for the linear system in (6.13) for $\theta \in [0, \theta_0]$, which proves the claim. $\square$

Lemma 6.22. *Let x be the basic solution associated to B and d be the jth basic direction at x. Assume that x is non degegerated and that $d_i < 0$ for some $i \in B$. Let $\bar{x} = x + \theta_0 d$ where $\theta_0 = \min\{-x_i/d_i \,|\, d_i < 0,\, i \in B\}$. Then, $\bar{x}$ is a feasible basic solution to the system in* (6.13)

Proof. From the proof of Lemma 6.21, it follows that $\bar{x}$ is feasible for the system in (6.13). In addition, taking $i \in B$ such that $\theta_0 = -x_i/d_i$, it follows that $\hat{x}_i = 0$. Consider now the sets of indexes $\bar{B} = B \cup \{j\} - \{i\}$. We claim that the associated matrix $\bar{B} = \left[A_{\bar{B}(1)} \dots A_{\bar{B}(m)}\right]$ is nonsingular. Notice that $d = B^{-1}A_j$ is the vector of coordinates of A_j in the basis B. We have that $A_j = \sum_{l \in B} d_l A_l$ where A_k denotes the k-th column vector of A for all $k \in \{1, 2, \dots, m\}$; and, since $d_i \neq 0$, it follows that

$$A_i = \sum_{l \in B,\, l \neq i} \frac{-d_l}{d_i} A_l + \frac{1}{d_i} A_j$$

Now, take any $w \in \mathbb{R}^n$ and let $w = \sum_{l \in B} \alpha_l A_l$. From the above relation it follows that

$$A_i = \sum_{l \in B,\, l \neq i} \frac{-d_l}{d_i} A_l + \frac{1}{d_i} A_j$$

which implies that

$$w = \sum_{l \in B,\, l \neq i} \alpha_l A_l + \alpha_i \left(\sum_{l \in B\, l \neq i} \frac{-d_l}{d_i} A_l + \frac{-1}{d_i} A_j \right) = \left(\frac{-\alpha_i}{d_i} \right) A_j + \sum_{l \in B\, l \neq i} \left(\alpha_l - \frac{d_l}{d_i} \right) A_l$$

The above relation shows how to write any vector $w \in \mathbb{R}^n$ as a linear combination of the column vectors of $\bar{B}$, which, in particular, implies the claim and ends the proof of the lemma. □

Remark 6.23. *The process described in the lemma above will be called variable x_j entering the basis.*

Definition 6.24. *Let x be a basic solution associated to a basis matrix B and let c_B be the vector of costs of the basic variables. We define the vector of reduced costs $\bar{c}$ associated to x according to the formula*

$$\bar{c}^T = c^T - c_B^T B^{-1} A. \tag{6.24}$$

Remark 6.25. *Notice that $\bar{c}_B = c_B - \left(c_B^T B^{-1} B\right)^T = 0$, that is, the reduced costs corresponding to basic variables are equal to zero.*

Remark 6.26. *Let d be the jth basic direction at x, θ_0 defined as in* (6.23)*. For $\theta \in (0, \theta_0]$, let $x^\theta = x + \theta d = (x_B + \theta d_B, \theta d_D)$, which is feasible for the system in* (6.13)*. Following the notations of* (6.22)*, we have that*

$$c^T x^\theta = z_0 + \hat{c}_D^\top x_D^\theta = z_0 + \left(c_D - \left(B^{-1}D\right)^T c_B\right)^T (\theta d_D) = z_0 + \theta\left(c - \left(\left(B^{-1}A\right)^T c_B\right)\right)_j = z_0 + \theta \bar{c}_j$$

Hence, if the reduced cost corresponding to the non basic variable x_j is negative, we can decrease the objective function by moving along the jth basic direction at x. In particular, in this case, the basic feasible solution defined in Lemma 6.22, obtained from the basic solution x, strictly decreases the value of the objective function.

Theorem 6.27. *Let x be a basic solution associated to a basis matrix B and $\bar{c}$ the corresponding vector of reduced costs. Then*

(a) *if $\bar{c} \geq 0$, then x is optimal;*
(b) *if x is optimal and non degenerate, then $\bar{c} \geq 0$.*

Proof. First we prove item (a). Note that $\lambda = \left(B^{-1}\right)^T c_B$ satisfies

$$\bar{c} = c - A^T \lambda.$$

Hence, if $\bar{c} \geq 0$, then λ is a feasible for the dual LPP (6.16). Additionally, we have that

$$\lambda^T b = b c_B^T B^{-1} b = c_b^T x_B = c^T x$$

which means, in view of Theorem 6.14, that λ and x are optimal solutions for problems (6.13) and (6.16), respectively. Item (b) follows straightforwardly from Remark 6.26. □

An iteration of the simplex method:

(1) We start with a basis consisting of the basic columns $A_{B(1)}, \ldots.., A_{B(m)}$, and an associated basic feasible solution x, with value of the objective function equal to $z_0 = c_B B^{-1} b$.
(2) Compute the reduced costs $\bar{c}_l = c_l - c_B^T B^{-1} A_l$, for al nonbasic indices l. If they are all nonnegative, the current basic feasible solution is optimal, and the algorithm terminates; else, choose some j for which $\bar{c}_j < 0$.
(3) Compute $u = B^{-1} A_j$, If no component of u is positive, the problem is unbounded and the algorithm terminates, i.e., the optimal cost is $-\infty$.
(4) If some component of u is positive, let
$$\theta_0 = \min \left\{ \frac{x_{B(i)}}{u_i} \ : \ u_i > 0 \right\}$$
(5) Let l be such that $\bar{\theta} = x_{B(i)}/u_l$. Form a new basis by replacing $A_{B(l)}$ with A_j. If y is the new basic feasible solution, the values of the new basic variables are $y_j = \theta_0$ and $y_{B(i)} = x_{B(i)} - \theta_0 u_i$ $i \neq l$, and the value fo the objective function at y is equal to $z_0 + (x_{B(i)}/u_l)\bar{c}_j$

Remark 6.28. *Note that we need to calculate $B^{-1}A_j$ where x_j is a nonbasic variable. If we consider all the nonbasic variables, we would need to perform $n - m$ of such calculations. In general, we can calculate $d = B^{-1}u$ in two different ways: 1) solving the linear system $Bd = u$ or 2) calculating the inverse B^{-1} and performing the multiplcation*

Naive implementation: In steps 2) and 3) of the implementation above: first, we calculate the so called *vector of multipliers* $p = c_B^T B^{-1}$ by solving the linear system $pB = c_B$, which requires $\mathcal{O}(m^3)$ arithmetic operations; then we calculate the reduced $n - m$ costs $\bar{c}_l = c_l - c_B^T B^{-1} A_l = c_l - p^T A_l$ for $j \notin B$, which requires $\mathcal{O}(mn)$ arithmetic operations, due to the calculation of $n = m$ inner products of vectors in $\mathbb{R}^n$; next, once we have determined the non basic variable x_j that will enter the base, we calculate vector $u = B^{-1} A_j$ by solving the linear system $Bu = A_j$, adding another $\mathcal{O}\left(m^3\right)$ arithmetic operations. Thus, in total, this naive implementation requires $\mathcal{O}\left(m^3 + mn\right)$ arithmetic operations

Revised method: We can improve the complexity estimation, in terms of number of arithmetic operations, presented in the analysis of the naive implementation above, by making available at the beginning of the iteration the matrix B^{-1} and then devising an efficient way of calculating $\bar{B}^{-1}$ where $\bar{B}$ denotes the new base. Let
$$B = \left[A_{B(1)} \ldots A_{B(m)}\right]$$
be the basis at the beginning of the iteration and let
$$\bar{B} = \left[A_{B(1)} \ldots A_{B(l-1)} A_{B(j)} A_{B(l+1)} \ldots A_{B(m)}\right]$$
the basis that results from x_j replacing x_l in the basis, at the end of the iteration. Notice that this implies that for $u = B^{-1}A_j$ it holds $u_l > 0$. Now, assuming that we have B^{-1}, we will present an efficient way of calculating $\bar{B}^{-1}$.

Definition 6.29. *Given a matrix, not necessarily square, the operation of adding a constant multiple of one row to the same row or to another row is called an elementary row operation.*

Lemma 6.30. *The elementary row operation of multiplying the ith row of a matrix A by a constant β and adding the result to the j-th row of A can be obtained by multiplying the matrix A by the nonsingulat matrix $Q_{ij} = I + D_{ij}$ where D_{ij} is the matrix with all entries equal to zero except the (i, j)-th entry which is equal to β.*

Proof. First, note that the determinant of Q_{ij} is equal to one, which proves that it is a nonsingular matrix. The proof of the second part of the claim follows straightforwardly from direct calculations, which ends the proof of the lemma. □

Remark 6.31. *It follows trivially that any sequence of elementary operations applied to a matrix can be obtained by multiplying this matrix by the nonsingular matrix equal to the product of all matrices representing each elementary operation, in the same order that these operations are applied. Also, notice that performing one elementary operation on a $k \times p$ matrix requires, at most, $2p$ arithmetic operations.*

Under the initial assumptions, simple calculations show that

$$B^{-1}\bar{B} = [e_1|\dots|e_{l-1}|\,u\,|e_{l+1}|\dots|e_m]$$

where $\{e_1, e_2, \dots, e_m\}$ denotes the canonical base of $\mathbb{R}^m$ and $u = B^{-1}A_j$ with $u_l > 0$. Now let Q be the nonsingular matrix corresponding to the sequence of elementary operations which transforms the vector u in the vector e_l. Note that that we can obtain this transformation performing m elementary operations on u as follows: first we multiply the l-th row of u by $1/u_l$ and then we add the l-th row of u multiplied by $-u_j$ to the j-th row of u, for all $j \neq i$. Now it follows

$$\begin{aligned} QB^{-1}\bar{B} &= Q\left([e_1|\dots|e_{l-1}|\,u\,|e_{l+1}|\dots|e_m]\right) = \left([Qe_1|\dots|Qe_{l-1}|\,Qu\,|Qe_{l+1}|\dots|Qe_m]\right) \\ &= \left([e_1|\dots|e_{l-1}|\,e_l\,|e_{l+1}|\dots|e_m]\right), \end{aligned}$$

since, the definition of Q and simple calculations show that $Qe_k = e_k$ for all $k \neq l$. The above relation implies that $QB^{-1} = \bar{B}^{-1}$ which means that applying the sequence of m elementary operations that defined Q to the matrix B^{-1} allows to compute $\bar{B}^{-1}$. Hence, given B^{-1}, we need $\mathcal{O}(m(2m)) = \mathcal{O}\left(m^2\right)$ to compute $\bar{B}^{-1}$.

An iteration of the Revised Simplex method:

(1) We start with a basis consisting of the basic columns $A_{B(1)}, \dots, A_{B(m)}$, the associated basic feasible solution x, and the inverse B^{-1} of the basis matrix.
(2) Compute the reduced costs $\bar{c}_l = c_l - c_B^T B^{-1} A_l$, for a nonbasic indices l. If they are all nonnegative, the current basic feasible solution is optimal, and the algorithm terminates; else, choose some j for which $\bar{c}_j < 0$.
(3) Compute $u = B^{-1}A_j$, If no component of u is positive, the problem is unbounded and the algorithm terminates, i.e., the optimal cost is $-\infty$.
(4) If some component of u is positive, let

$$\theta_0 = \min\left\{\frac{x_{B(i)}}{u_i} \,:\, u_i > 0\right\}$$

(5) Let l be such that $\bar{\theta} = x_{B(l)}/u_l$. Form a new basis $\bar{B}$ by replacing $A_{B(l)}$ with A_j. If y is the new basic feasible solution, the values of the new basic variables are $y_j = \theta_0$ and $y_{B(i)} = x_{B(i)} - \theta_0 u_i$ $i \neq l$.
(6) Form the $m \times (m+1)$ matrix $[B|u]$. Apply the elementary operations previously describe to transform the vector u in the vector e_l. The first m columns of the result is the matrix $\bar{B}^{-1}$

Notice that, in steps 2) and 3) of the implementation of the revised method, presented above: direct calculation of $p = c_B^T B^{-1}$ and $u = B^{-1}A_j$ requires $\mathcal{O}\left(m^2\right)$ arithmetic operations; again, the calculation of the reduced costs for the $(n-m)$ non basic variables requires $\mathcal{O}\left(mn\right)$ arithmetic operations; and, finally, updating the inverse of the matrix of the basis requires $\mathcal{O}\left(m^2\right)$ arithmetic operations. Hence, the cost of the implementation of revised variant of the Simplex requires $\mathcal{O}\left(m^2 + mn\right)$ arithmetic operations, in contrast with the estimation for the naive implementation, showed above.

Remark 6.32. *Another option to improve the complexity estimations of the naive implementation of the Simplex method is to calculate the expressions of the form $B^{-1}d$ by solving the related system $Bu = d$ in a more efficient way. Notice that, given LU decomposition of the matrix B, solving the linear system above would require only $\mathcal{O}(m^2)$ arithmetic operations. A more efficient procedure would be the updating of the LU decomposition of the basis matrix to obtain a similar decomposition for the new basis matrix.*

Full tableau implementation of the Simplex method:
Given the basic basis $B = \left\{A_{B(1)}, \dots, A_{B(m)}\right\}$ (assuming to easy the presentation that $B = \{1, 2, \dots, m\}$) and the related reduced costs vector $\bar{c}$, we can rewrite problem (6.22) as follows

$$\begin{aligned} &\min\ z = c_B^T B^{-1} b + \bar{c}^T x = z_0 + (0, \bar{c}_D)^\top (x_B, x_D) \\ &B^{-1}Ax = \left[I \,|\, B^{-1}D\right] x = B^{-1}b \\ &x = (x_B, x_D) \geq 0. \end{aligned} \tag{6.25}$$

where $z_0 = c_B^T B^{-1} b$ and $\bar{c}_D^T = c_D^T - c_B^T B^{-1} D$. We can represent (6.25) using the tableau below

(6.26)

Basic variables	z	x_B	x_D	
x_B	0	I	$B^{-1}D$	$\bar{b}$
	1	0	$-\bar{c}_D$	z_0

with $x_B = \{x_1, \ldots, x_m\}$, $x_D = \{x_{m+1}, \ldots, x_n\}$, $\bar{c}_D = \{\bar{c}_{m+1}, \ldots, \bar{c}_n\}$ and $\bar{b} = B^{-1}b$. Notice that considering, as above, the nonsingular matrix Q, associated to a certain sequence of elementary operations on the matrix B^{-1}, that satisfies $QB^{-1} = \bar{B}^{-1}$, where $\bar{B}$ is the new basis, it follows that

$$QB^{-1}\left[A \,|\, b\right] = \bar{B}^{-1}\left[A \,|\, b\right].$$

This implies that applying these elementary operations to the second row of the tableau above, (starting with the second column) we obtain the second row of the updated tableau that corresponds to the new basis $\bar{B}$. Notice that we would need some additional permutations of the columns, in the new tableau, for the identity matrix to appear. In practice, this is not necessary, since this is equivalent to do a reordering of the variables. To easy the notations, will assume that the variable x_s enters the basis and that the variable x_r exists the basis. Hence, the elementary operations will use the pivot A_{rs} to make $B^{-1}A_s = e_r$. More specifically, first we will multiply the r-th row by $1/A_{rs}$ and then, for $i = i, 2, \ldots, m$ with $i \neq r$, we will multiply it by $-A_{is}$ and add it to the i-th row. In addition, we will prove that the applying a similar elementary operation to the last row of the tableau allows us to obtain the updated vector of reduced costs, that is, the vectors of reduced costs of the new basic solution associated to the basis $\bar{B}$, and also the updated value of the objective function at the new basic solution. For the transformation of the last row, we will multiply the r-th row for $\bar{c}_s/A_{sr}$ and then we will add it to the last row. The purpose here is to set the reduced cost of x_s equal to zero, which is the value of the reduced costs of the basic variables. Notice that the last row of the tableau has the form

$$L_{m+1} = \left(-\bar{c}^T, z_0\right) = \left(-c^T + c_B^T B^{-1} A, z_0\right)$$

where $z_0 = c_B^T B^{-1} b$, while the r-th row of the matrix $B^{-1}A$ has the expression

$$L_r = \left(e_r^T B^{-1} A, e_r^T B^{-1} b\right)$$

Hence, after applying the elementary operations we obtain the following new expression for the last row

$$\begin{aligned}\left(\frac{\bar{c}_s}{N_{rs}}\right) L_r + L_{m+1} &= \left(-c^T + \left(\left(\frac{\bar{c}_s}{N_{rs}}\right) e_r^T + c_B^T\right) B^{-1} A, \left(\frac{\bar{c}_s}{N_{rs}}\right) \left(B^{-1}b\right)_r + z_0\right) \\ &= \left(-c^T + p^T \bar{B}^{-1} A, \left(\frac{\bar{c}_s}{N_{rs}}\right) \left(B^{-1}b\right)_r + z_0\right)\end{aligned}$$

where $p^T = \left(\left(\bar{c}_s/N_{rs}\right) e_r + c_B\right)^T B^{-1}\bar{B}$. First, notice that Remark 6.26 with $\theta = 1$ shows that, after the Simplex iteration, the value of the objective function changes from $z_0 = c_B^T B^{-1} b$ to $z_0 + (x_r/A_{sr})c_s = z_0 + \left((B^{-1}b)_r/A_{sr}\right) c_s$, which coincides with the value of the last component of the vector above. Next, notice that relations $B^{-1}\bar{B}_i = e_i$ for all $i \in B$ with $i \neq s$ and $B^{-1}\bar{B}_s = N_s$ imply that

$$p_i = \left(\left(\frac{\bar{c}_s}{N_{rs}}\right) e_r + c_B\right)_i = \left(c_B\right)_i = c_i \quad \forall\, i \in B \text{ with } i \neq r$$

and

$$p_s = \left(\left(\frac{\bar{c}_s}{N_{rs}}\right) e_r^T + c_B^T\right) N_s = \left(\frac{\bar{c}_s}{N_{rs}}\right) N_{rs} + c_B^T N_s = \bar{c}_s + c_B^T N_s = \bar{c}_s + c_B^T B^{-1} A_s = c_s.$$

Therefore, $p = c_{\bar{B}}$ and it holds $-c^T + p^T \bar{B}^{-1} A = -c + c_{\bar{B}}^T \bar{B}^{-1} A$, which is the vector of reduced costs corresponding to the new basis $\bar{B}$. Hence, we have that

$$\left(\frac{\bar{c}_s}{N_{rs}}\right) L_r + L_{m+1} = \left(-c + c_{\bar{B}} \bar{B}^{-1} A, \bar{B}^{-1} b\right),$$

and the claim follows.

A more detailed version of the tableau and of the procedure follows:

$$
(6.27)\qquad
\begin{array}{lcc||c|ccccc|ccccc|c}
 & & \text{Basic variables} & z & x_1 & \dots & x_r & \dots & x_m & x_{m+1} & \dots & x_s & \dots & x_n & \\
\hline
L_1 & & x_1 & 0 & & & & & & & & & & & \bar{b}_1 \\
 & & \vdots & & & & & & & & & & & & \vdots \\
L_r & \leftarrow & x_r & 0 & & & I & & & & & N_{rs} & & & \bar{b}_r \\
 & & \vdots & & & & & & & & & & & & \vdots \\
L_m & & x_m & 0 & & & & & & & & & & & \bar{b}_m \\
\hline
L_{m+1} & & & 1 & 0 & \dots & 0 & \dots & 0 & -\bar{c}_{m+1} & \dots & -\bar{c}_s & \dots & -\bar{c}_n & z_0 \\
 & & & & & & & & & & & \uparrow & & &
\end{array}
$$

If $\bar{c}_N \leq 0$, i.e., is $-\bar{c}_i \geq 0, i = m+1, \dots, m$, in the table (6.26), then we find the optimal solution $x_B = \bar{b}$ and $x_N = 0$, being z_0 the optimal value. Otherwise, we introduce the index s satisfying

$$-\bar{c}_s = \min\{-\bar{c}_i,\ i = m+1, \dots, n\}.$$

If $N_{is} \leq 0$ for $i = 1, \dots, m$, then the optimal value of (6.20) is $+\infty$. Otherwise, we introduce the index r defined by

$$\frac{b_r}{N_{rs}} = \min\left\{\frac{b_i}{N_{is}}\ :\ i = 1, \dots, m,\ \text{e}\ N_{is} > 0\right\}.$$

The new basis $\bar{B}$ is $\bar{B} = B \cup \{s\} - \{r\}$ and we update the table (6.27) (i.e. we re-write s (6.20) using the base $\bar{B}$) with the following sequence of operations:

$$
\begin{array}{lcl}
L_r & \leftarrow & \dfrac{L_r}{N_{rs}} \\
L_i & \leftarrow & L_i - N_{is}L_r, \quad i = 1, \dots, m,\ i \neq r, \\
L_{m+1} & \leftarrow & L_{m+1} + c_s L_r.
\end{array}
$$

The element N_{rs} is called pivot.

6.5. **Duality and sensitivity analysis.** Recall the primal-dual pairs of LPPs (see Remark 6.17) and the duality Theorem 6.14.

Interpretation of the dual variables. We consider the primal problem of profit maximization

$$\max\ c^\top x, \quad Cx \leq d,\ x \geq 0 \tag{6.28}$$

where d_i represents quantity of resource$i$ available. The dual problem is written

$$\min\ \mu^\top d, \quad C^\top \mu \geq c,\ \mu \geq 0. \tag{6.29}$$

We consider a perturbation of the amount of available resource in (6.28):

$$\max\ c^\top x, \quad Cx \leq d + \varepsilon,\ x \geq 0. \tag{6.30}$$

and the corresponding dual problem

$$\min\ \mu^\top (d + \varepsilon), \quad C^\top \mu \geq c,\ \mu \geq 0. \tag{6.31}$$

We assume that the solutions to the initial dual problem (6.29) are also solutions to the perturbed dual problem (6.31). In this case, denoting by μ^* a solution of the dual problem (6.29), the optimal value of the perturbed problem (6.30) is the optimal value of the initial problem plus $d^\top \mu^*$. In other words, μ_i^* represents the increase in profit when the quantity of resource i increases by one unit. This interpretation is valid for perturbations given by d_i. A solution of the dual problem (graphically or using simplex) allows us to determine for which values of the perturbations the set of solutions of the dual problem does not change.

Example 6.33. *We want to produce strawberry and rose jelly. The amount of sugar available is 8 tons and the capital is 40,000 reais. Strawberries are bought for 2 reais per kg and roses for 15 reais per kg. Strawberry jam (resp. rose) is sold for 4.50 reais per kg (resp. 12.60 reais per kg). Strawberry jam (resp. rose) is obtained by mixing 50% strawberries (resp. 40% roses) and 50% (resp. 60%) sugar. Model the problem and write the dual problem. Provide an interpretation of the dual solution and study the variation* *in profit when the amount of available sugar changes.*

Model the problem and write the dual problem. Provide an interpretation of the dual solution and study the variation in profit when the amount of available sugar changes.

Let x_1 and x_2 be the quantities of sugar destined respectively for the production of strawberry and rose jelly. The problem of maximizing the profit is modeled by the LPP

$$\begin{cases} \max\ 7x_1 + 11x_2 \\ \begin{array}{ccccc} x_1 & + & x_2 & \leq & 8000 \\ 2x_1 & + & 10x_2 & \leq & 40000 \end{array} \\ x_1 \geq 0,\ \ x_2 \geq 0. \end{cases}$$

If the amount of sugar were $8000+\varepsilon$ the problem to be solved would be

$$\begin{cases} \max\ 7x_1 + 11x_2 \\ \begin{array}{ccccc} x_1 & + & x_2 & \leq & 8000+\varepsilon \\ 2x_1 & + & 10x_2 & \leq & 40000 \end{array} \\ x_1 \geq 0,\ \ x_2 \geq 0. \end{cases} \tag{6.32}$$

Let's find a solution to the perturbed problem (6.32) graphically and using the Simplex algorithm. We write the (6.32) problem in standard form:

$$\begin{cases} \max\ 7x_1 + 11x_2 \\ \begin{array}{ccccccc} x_1 & + & x_2 & + & e_1 & = & 8000+\varepsilon \\ 2x_1 & + & 10x_2 & + & e_2 & = & 40000 \end{array} \\ x_1 \geq 0,\ x_2 \geq 0,\ e_1 \geq 0,\ e_2 \geq 0. \end{cases}$$

We do not seed Phase I. We initiate Phase II with $x_B = \{e_1, e_2\}$:

		z	x_1	x_2	e_1	e_2	
	e_1	0	1	1	1	0	$8000+\varepsilon$
$\leftarrow$	e_2	0	2	$\underline{10}$	0	1	40000
	z	1	-7	-11	0	0	0
				$\uparrow$			

Variable x_2 enter the basis and e_2 leaves the basis if $4000 \leq 8000+\varepsilon$, i.e., if $\varepsilon \geq -4000$. In this case, in the first iteration we have

		z	x_1	x_2	e_1	e_2	
$\leftarrow$	e_1	0	$\underline{0.8}$	0	1	-0.1	$4000+\varepsilon$
	x_2	0	0.2	1	0	0.1	4000
	z	1	-4.8	0	0	1.1	44000
			$\uparrow$				

In the secon iteration, x_1 enter the basis and e_1 leaves the basis if $1.25(4000+\varepsilon) \leq 5\times 4000$, i.e., if $\varepsilon \leq 12000$:

	z	x_1	x_2	e_1	e_2	
x_1	0	1	0	1.25	-0.125	$5000+1.25\varepsilon$
x_2	0	0	1	-0.25	0.125	$3000-0.25\varepsilon$
z	1	0	0	6	0.5	$68000+6\varepsilon$

We have found an optimal solution : $x_1 = 5000 + 1.25\varepsilon$ kg of sugar for the strawberry jelly, $x_2 = 3000 - 0.25\varepsilon$ kg of sugar for the rose jelly, and $x_3 = x_4 = 0$. The optimal value is $68000 + 6\varepsilon$. We can ferivy graphically this result

The dual problem of the perturbed problem (6.32) is

$$\begin{cases} \min\ (8000+\varepsilon)\mu_1 + 40000\mu_2 \\ \mu_1 + 2\mu_2 \geq 7 \\ \mu_1 + 10\mu_2 \geq 11 \\ \mu_1 \geq 0,\ \ \mu_2 \geq 0. \end{cases}$$

We can check graphically or using the simplex algorithm that the optimal value of this perturbed dual problem is

- $\mu_1^* = 6$ *e* $\mu_2^* = 0.5$ *para* $-4000 \leq \varepsilon \leq 12000$*;*
- $\mu_1^* = 0$ *e* $\mu_2^* = 3.5$ *para* $\varepsilon > 12000$*;*
- $\mu_1^* = 11$ *e* $\mu_2^* = 0$ *para* $\varepsilon < -4000$.

The dual variable μ_1^* *represents the increase in the profit when the quantity of sugar increases by 1kg. Consequently:*

- *If the initial amount of sugar is between* $8,000 - 4,000 = 4,000$ *and* $8,000 + 12,000 = 20,000$*,* $\mu_1^* = 6$*, i.e., the increase in profit for each kg of sugar added is 6 reais. Then it is interesting to buy sugar until the amount of 20,000kg of sugar is reached if the purchase price of the sugar is less than 6 reais per kg. Any ammount of s ugar added above 20,000 kg does not produce any additional profit, since in this case* $\mu_1^* = 0$ *(with the available capital we cannot purchase more than 20,000 kg of strawberries).*
- *If the initial amount of sugar is less than* $8,000 - 4,000 = 4,000$*, each kg of sugar added until the total ammount reaches* 4000 *kg can be used to make rose jelly and produces a profit increase of 11 reais. Ánd sõinteresting buy sugar if the price of sugar is less than 11 reais per kg.*

7. Column generation for LP

Consider the linear program (P) and its dual (D) given by

$$(P)\left\{\begin{array}{l}\max c^T x\\ Ax=b, x\geq 0\end{array}\right. \quad (D)\left\{\begin{array}{l}\min \pi^T b\\ A^T\pi \geq c.\end{array}\right.$$

Assume that the primal problem is bounded from below and feasible. Then the optimal values of (P) and (D) are equal. Let f_* be this optimal value.

The column generation algorithm works as follows. At iteration $k \geq 1$, we select a set S_k of columns of A and denote by x_k the vector whose variables have indexes in $\mathcal{S}_k$. For $k=1$, we can take a known feasible solution of $Ax=b, x\geq 0$ (for instance given by a heuristic applied to (P) in several applications of column generation) and take the set S_1 to be the set of columns of A corresponding to positive entries of this solution. We denote by A_k (resp. c_k) the submatrix of A (resp. subvector of c) whose columns (resp. entries) are indexed by S_k. We solve the following relaxed problem P_k along with its dual D_k given by

$$(P_k)\left\{\begin{array}{l}\max c_k^T x_k\\ A_k x_k=b, x_k\geq 0\end{array}\right. \quad (D_k)\left\{\begin{array}{l}\min \pi^T b\\ A_k^T\pi \geq c_k.\end{array}\right.$$

The common (by the duality theorem for bounded feasible LPs) optimal value f_k of (P_k) and (D_k) satisfies $f_k \leq f_*$ (the feasible set of P_k is contained in the feasible set of (P)). Therefore, if π_k is an optimal solution of (D_k) and x_k is an optimal solution of (P_k) then if π_k is feasible for (D), i.e., if $A^T\pi \geq c$ then we both have $\pi_k^T b \geq f_*$ (by feasibility of π_k) and $\pi_k^T b = f_k \leq f_*$ which shows that $\pi_k^T b = f_k = f_*$ and therefore π_k is optimal for (D) and x obtained adding null components to x_k is an optimal solution to (P) since x is feasible and $c^T x = c_k^T x_k = f_k = f_*$.

On the contrary, if there is i such that the reduced cost $c(i) - a_i^T \pi_k$ is > 0 where a_i is ith column of A then we add one such column a_i to S_k (for instance the column providing the largest reduced cost) and continue the process, solving the relaxed problems (P_{k+1}) and (D_{k+1}). If A has a finite number of columns, this process stops in a finite number of iterations finding an optimal solution to (P). In practice, column generation is useful for problems with a large number of variables and such that a large number of variables is null at an optimum. Observe that since (P_1) is feasible and the feasible set of (P_k) contains the feasible set of (P_1) all problems (P_k) are bounded and feasible.

8. Modeling

Example 8.1. *The problem of localization of fire stations.*

- *Objective: determine in which city to build fire stations in order to minimize the number of stations, knowing that every city must be able to be reached from at least one fire point in a maximum of 15 min.*

Example with 5 cities and the times between each city below:

	C_1	C_2	C_3	C_4	C_5
C_1	0	10	25	15	30
C_2	10	0	20	50	10
C_3	25	20	0	30	10
C_4	15	50	30	0	70
C_5	30	10	10	70	0

- *x_i takes the value 1 if a station is built in city i and 0 otherwise.*
- *We can reach city 1 in a maximum of 15 minutes from 1, 2 or 4. To reach city 1 in a maximum of 15min, we then have to build a station in 1, 2 ou 4, i.e.,*

$$x_1 + x_2 + x_4 \geq 1.$$

Reasoning in the same way for the other cities, we obtain the PLI model:

$$\begin{cases} \min \sum_{i=1}^{5} x_i \\ x_1 + x_2 + x_4 \geq 1, \ x_1 + x_2 + x_5 \geq 1 \\ x_3 + x_5 \geq 1, \ x_1 + x_4 \geq 1 \\ x_2 + x_3 + x_5 \geq 1 \\ x_1, \, x_2, \, x_3, \, x_4, \, x_5 \in \{0, 1\}. \end{cases}$$

Example 8.2. *Problem of locating plants and managing transport between plants and customers:*

- *n candidate cities.*
- *Fix cust of placing a plant in city i: f_i.*
- *Daily production capacity of a plant located in city i: a_i.*
- *m customers: customer j with daily demand d_j.*
- *Transport cost between city i and customer j: c_{ij}.*
- *Maximum daily capacity that can be transported between i and j: K_{ij}.*

Decision variables:

- *$y_i = 1$ if a plant is built in city i.*
- *x_{ij}: quantity transported between city i and customer j.*

The plant location model is given by:

$$\begin{cases} \max \sum_{i=1}^{n} f_i y_i + \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij} \\ 0 \leq x_{ij} \leq y_i \, K_{ij} \\ \sum_{i \,|\, (i,j) \in E} x_{ij} \geq d_j, \ \forall \, j = 1, \ldots, m, \\ \sum_{j \,|\, (i,j) \in E} x_{ij} \leq a_i, \ \forall \, i = 1, \ldots, n, \\ y_i \in \{0, 1\}, \ \forall \, i = 1, \ldots, n. \end{cases}$$

Example 8.3. *Investment problem.*

- *We have 4 projects. We want to decide which projects to invest in based on the following data*

Projeto	*Custo*	*VPLN*
P_1	5000	8000
P_2	7000	11000
P_3	4000	6000
P_4	3000	4000

and knowing that we have a capital of 14,000 to invest today.

Model for choosing projects: using the variables x_i, with value equal to 1 if we invest in project i and 0 otherwise, the model is written

$$\begin{cases} \max \ 8x_1 + 11x_2 + 6x_3 + 4x_4 \\ 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14 \\ x_i \in \{0, 1\}, \ i = 1, \ldots, 4. \end{cases}$$

We consider an integer linear programming (ILP) problem:

$$\begin{cases} \max \ c^\top x \\ Ax = b \\ x \geq 0 \\ x \in \mathbb{Z}^n. \end{cases} \quad ou \quad \begin{cases} \min \ c^\top x \\ Ax = b \\ x \geq 0 \\ x \in \mathbb{Z}^n. \end{cases}$$

In the particular case of binary variables: $x_i \in \{0, 1\}$:

- *We can assume $c \geq 0$. If $c_i < 0$, we make the change of variables $y_i = 1 - x_i \in \{0, 1\}$.*
- *Observe that $x_i^2 = x_i$.*
- *If the objective function is not linear but has terms of the type $x_1 x_2$, we can transform the problem into a PLI exchanging $x_1 x_2$ by y with the restrictions*

$$\begin{array}{l} x_2 + x_3 - y \leq 1 \\ -x_2 - x_3 + 2y \leq 0 \\ x_2, \, x_3, \ y \in \{0, 1\}. \end{array}$$

Example 8.4. *A Warehousing Problem. Consider the problem of operating a warehouse, by buying and selling a certain commodity which can be stored in the warehouse, in order to maximize profit over a certain time window. The warehouse has a fixed capacity C and there is a cost r per unit for holding stock for one period. The price, p_i of the commodity is known to fluctuate over a number of time periods—say months, indexed by i. In any period the same price holds for both purchase or sale. The warehouse is originally empty and is required to be empty at the end of the last period. To formulate this problem, variables are introduced for each time period. In particular, let x_i denote the level of stock in the warehouse at the beginning of period i. Let u_i denote the amount bought during period i, and let s_i denote the amount sold during period i. If there are n periods, the problem is*

$$\begin{array}{lc} \textit{maximize} & \sum_i^n (p_i(s_i - u_i) - rx_i) \\ \textit{subject to} & x_{i+1} = x_i + u_i - s_i \quad i = 1, 2, \ldots, n-1 \\ & 0 = x_n + u_n - s_n \\ & x_i + z_i = C \quad i = 2, \ldots, n \\ & x_1 = 0,\ x_i \geq 0,\ u_i \geq 0, \quad s_i \geq 0,\ z_i \geq 0, \end{array}$$

where z_i is a slack variable. If the constraints are written out explicitly for the case $n = 3$, they take the form

$-x_1 - u_1 + s_1$	$+\ x_2$		$= 0$
	$-x_2 - u_2 + s_2$	$+x_3$	$= 0$
	$x_2 + z_2$		$= C$
		$x_3 + u_3 - s_3$	$= 0$
		$x_3 + z_3$	$= C$

Note that the coefficient matrix can be partitioned into blocks corresponding to the variables of the different time periods. The only blocks that have nonzero entries are the diagonal ones and the ones immediately above the diagonal. This structure is common for problems involving time.

Example 8.5. Manufacturing Problem. *Suppose we own a facility that is capable of manufacturing n different products, each of which may require various amounts of m different resources. Each product j can be produced at any level $x_j \geq 0$, $j = 1, 2, \ldots, n$, and each unit of the jth product can sell for p_j dollars and needs a_{ij} units of the i-th resource, $i = 1, 2, \ldots, m$. Assuming linearity of the production costs, if we are given a set of m numbers $b_1, b_2, \ldots, b_m$ describing the available quantities of the m resources, and we wish to manufacture products to maximize the revenue, we need to maximize*

$$p_1x_1 + p_2x_2 + \ldots + p_nx_n$$

subject to the resource constraints

$$a_{i1}x_1 + a_{12}x_2 + \ldots + a_{in}x_n \leq b_i, \quad i = 1, \ldots, m$$

and the nonnegativity constraints on all production variables.

Example 8.6. Linear Classifier and Support Vector Machine. *Suppose several d-dimensional data points are classified into two distinct classes. For example, two-dimensional data points may be grade averages in science and humanities for different students. We also know the academic major of each student, as being in science or humanities, which serves as the classification. In general, we have vectors $a_i \in \mathbb{R}^d$ for $i = 1, 2, \ldots, n_1$ and vectors $b_j \in \mathbb{R}^d$ for $j = 1, 2, \ldots, n_2$. We wish to find a hyperplane that separates the a_i's from the b_j's. Mathematically, we wish to find $y \in \mathbb{R}^d$ and a number β such that*

$$a_i^T y + \beta \geq 1 \quad \textit{for all } i,$$
$$b_j^T y + \beta \leq -1 \quad \textit{for all } j,$$

where $\{x \ : \ x^T y + \beta = 0\}$ is the desired hyperplane, and the separation is defined by the values $+1$ and -1. This is a linear program.

- **Network flow problems:**

Example 8.7. The general network flow problem: *Network-flow problems deal with the distribution of a single homogeneous product from plants (origins) to consumer markets (destinations). The total number of units produced at each plant and the total number of units required at each market are assumed to be known. The product need not be sent directly from source to destination, but may be routed through intermediary points reflecting warehouses or distribution centers. Further, there may be capacity restrictions that limit some of the shipping links. The objective is to minimize the variable cost of producing and shipping the products to meet the consumer demand. The sources, destinations, and intermediate points are collectively called nodes of the network, and the transportation links connecting nodes are termed arcs.*

	Urban transportation	*Communication systems*	*Water resources*
Product	*Buses, autos, etc.*	*Messages*	*Water*
Nodes	*Bus stops, street intersections*	*Communication centers, relay station*	*Lakes, reservoirs, pumping stations*
Arcs	*Street (lanes)*	*Communication channels*	*Pipelines, canals, rivers.*

TABLE 1. Examples of Network Flow Problems

Example 8.8. A special network flow problem: The Transportation Problem. *Quantities* $a_1, a_2, \ldots, a_m$*, respectively, of a certain product are to be shipped from each of* m *locations and received in amounts* $b_1, b_2, \ldots, b_n$ *, respectively, at each of* n *destinations. Associated with the shipping of a unit of product from origin* i *to destination* j *is a shipping cost* c_{ij}*. It is desired to determine the amounts* x_{ij} *to be shipped between each origin–destination pair* $i = 1, 2, \ldots, m$*;* $j = 1, 2, \ldots, n$*; so as to satisfy the shipping requirements and minimize the total cost of transportation. To formulate this problem as a linear programming problem, we set up the array shown below:*

$$
\begin{array}{|cccc|c}
x_{11} & x_{12} & \ldots & x_{1n} & a_1 \\
x_{21} & x_{22} & \ldots & x_{2n} & a_2 \\
\vdots & & & \vdots & \vdots \\
x_{m1} & x_{m2} & \ldots & x_{mn} & a_m \\
\hline
b_1 & b_2 & \ldots & b_n &
\end{array}
$$

The i*th row in this array defines the variables associated with the* i*th origin, while the* j*th column in this array defines the variables associated with the* j*th destination. The problem is to place nonnegative variables* x_{ij} *in this array so that the sum across the* i*th row is* a_i*, the sum down the* j*th column is* b_j*, and the weighted sum* $\sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij} x_{ij}$*, representing the transportation cost, is minimized. Thus, we have the linear programming problem:*

$$
\begin{aligned}
&\textit{minimize} && \textstyle\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} && \\
&\textit{subject to} && \textstyle\sum_{j=1}^{n} x_{ij} = a_i && \textit{for } i = 1, 2, \ldots, m \qquad (8.33)\\
& && \textstyle\sum_{i=1}^{m} x_{ij} = b_j && \textit{for } j = 1, 2, \ldots, n \qquad (8.34)\\
& && x_{ij} \geq 0 && \textit{for } i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n.
\end{aligned}
$$

For constraints (8.33) and (8.34) to be consistent, we must, of course, assume that $\sum_{i=1}^{m} a_i = \sum_{j=1}^{m} b_j$ *which corresponds to assuming that the total amount shipped is equal to the total amount received.*

Let us consider a simple example. A compressor company has plants in three locations: Cleveland, Chicago, and Boston. During the past week the total production of a special compressor unit out of each plant has been 35, 50, and 40 units respectively. The company wants to ship 45 units to a distribution center in Dallas, 20 to Atlanta, 30 to San Francisco, and 30 to Philadelphia. The unit production and distribution costs from each plant to each distribution center are given in the is presented in Table 2. What is the best shipping strategy to follow?

Plants	*Distribution centers*				*Availability (units)*
	Dallas	*Atlanta*	*San Francisco*	*Philadelfia*	
Cleveland	*8*	*6*	*10*	*9*	*35*
Chicago	*9*	*12*	*13*	*7*	*50*
Boston	*14*	*9*	*16*	*5*	*40*
Requirements (units)	*45*	*20*	*30*	*30*	*125*

TABLE 2. Unit Production and Shipping Costs

The linear-programming formulation of the corresponding transportation problem is:

$$\text{Minimize } z = 8x_{11} + 6x_{12} + 10x_{13} + 9x_{14} + 9x_{21} + 12x_{22} + 13x_{23} + 7x_{24} + 14x_{31} + 9x_{32} + 16x_{33} + 5x_{34},$$

subject to

$$\begin{array}{cccccccccccccc}
x_{11} & +x_{12} & +x_{13} & +x_{14} & & & & & & & & & = & 35, \\
 & & & & x_{21} & +x_{22} & +x_{23} & +x_{24} & & & & & = & 50, \\
 & & & & & & & & x_{31} & +x_{32} & +x_{33} & +x_{34} & = & 40, \\
-x_{11} & & & & -x_{21} & & & & -x_{31} & & & & = & -45, \\
 & -x_{12} & & & & -x_{22} & & & & -x_{32} & & & = & -20, \\
 & & -x_{13} & & & & -x_{23} & & & & -x_{33} & & = & -30, \\
 & & & -x_{14} & & & & -x_{24} & & & & -x_{34} & = & -30,
\end{array}$$

$$x_{ij} \geq 0 \quad (i = 1, 2, 3;\ j = 1, 2, 3, 4).$$

Example 8.9. A special network flow: The Assignment Problem. *n people are to be assigned to n jobs and that c_{ij} measures the performance of person i in job j. If we let*

$$x_{ij} = \begin{cases} 1 & \text{if person } i \text{ is assigned to job } j, \\ 0 & \text{otherwise,} \end{cases}$$

we can find the optimal assignment by solving the optimization problem:

$$\text{Maximize } z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij},$$

subject to:

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad (i = 1, \dots, n)$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad (j = 1, \dots, n)$$

$$x_{ij} = 0 \quad \text{or} \quad 1 \quad (i = 1, \dots, n;\ j = 1, \dots, n)$$

The first set of constraints shows that each person is to be assigned to exactly one job and the second set of constraints indicates that each job is to be performed by one person. If the second set of constraints were multiplied by minus one, the equations of the model would have the usual network interpretation. As stated, this assignment problem is formally an integer program, since the decision variables x_{ij} are restricted to be zero or one. However, if these constraints are replaced by $x_{ij} \geq 0$, the model becomes a special case of the transportation problem, with one unit available at each source (person) and one unit required by each destination (job). Network-flow problems have integer solutions, and therefore formal specification of integrality constraints is unnecessary. Consequently, application of the simplex method, or most network-flow algorithms, will solve such integer problems directly.

Example 8.10. A special network flow: The Shortest-Path Problem. *Given a network with distance c_{ij} (or travel time, or cost, etc.) associated with each arc, find a path through the network from a particular origin (source) to a particular destination (sink) that has the shortest total distance. A number of important applications can be formulated as shortest- (or longest-) path problems where this formulation is not obvious at the outset. Further, the shortest-path problem often occurs as a subproblem in more complex situations, such as the subproblems in applying decomposition to traffic-assignment problems or the group-theory problems that arise in integer programming. In general, the formulation of the shortest-path problem is as follows:*

$$\textit{Minimize } z = \sum_i \sum_j c_{ij} x_{ij}$$

subject to:

$$\sum_j x_{ij} - \sum_k x_{ki} = \begin{cases} 1 & \textit{if } i = s \ (\textit{source}), \\ -1 & \textit{if } i = t \ (\textit{sink}) \\ 0 & \textit{otherwise}, \end{cases} \quad .$$

$$x_{ij} \geq 0 \textit{ for all arcs } i - j \textit{ in the network}$$

We can interpret the shortest-path problem as a network-flow problem very easily. We simply want to send one unit of flow from the source to the sink at minimum cost. At the source, there is a net supply of one unit; at the sink, there is a net demand of one unit; and at all other nodes there is no net inflow or outflow.

- **Linear integer programming problems:** Consider the problem

$$\begin{array}{ll} & \text{maximize } \sum_{j=1}^n c_j x_j \\ \text{subject to} & \\ & \sum_{j=1}^n a_{ij} x_j = b_i \quad (i = 1, 2, \ldots, m), \\ & x \geq 0 \quad (j = 1, 2, \ldots, n) \\ & x_j \text{ integer} \qquad (\text{for some or all } j) \end{array} \tag{8.35}$$

This problem is called the (linear) integer-programming problem. It is said to be a mixed integer program when some, but not all, variables are restricted to be integer, and is called a pure integer program when all decision variables must be integers.

Example 8.11. Capital Budgeting. *In a typical capital-budgeting problem, decisions involve the selection of a number of potential investments. The investment decisions might be to choose among possible plant locations, to select a configuration of capital equipment, or to settle upon a set of research-and-development projects. Often it makes no sense to consider partial investments in these activities, and so the problem becomes a go–no-go integer program, where the decision variables are taken to be $x_j = 0$ or 1, indicating that the j-th investment is rejected or accepted. Assuming that c_j is the contribution resulting from the j-th investment and that a_{ij} is the amount of resource i, such as cash or manpower, used on the j-th investment, we can state the problem formally as:*

$$\begin{array}{ll} & \textit{maximize } \sum_{j=1}^n c_j x_j \\ \textit{subject to} & \\ & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad (i = 1, 2, \ldots, m), \\ & x_j = 0 \ \textit{ or } \ 1 \quad (j = 1, 2, \ldots, n) \end{array}$$

The objective is to maximize total contribution from all investments without exceeding the limited availability of any resource. One important special scenario for the capital-budgeting problem involves cash-flow constraints. In this case, the constraints

$$\sum_{j=1}^n a_{ij} x_j \leq b_i$$

reflect incremental cash balance in each period. The coefficients a_{ij} represent the net cash flow from investment j in period i. If the investment requires additional cash in period i, then $a_{ij} > 0$, while if the investment generates cash in period i , then $a_{ij} < 0$. The righthand-side coefficients $b_i$represent the incremental exogenous cash flows. If additional funds are made available in period i, then $b_i > 0$, while if funds are withdrawn in period i, then $b_i < 0$. These constraints state that the funds required for investment must be less than or equal to the funds generated from prior investments plus exogenous funds made available (or minus exogenous funds withdrawn).

The capital-budgeting model can be made much richer by including logical considerations. Suppose, for example, that investment in a new product line is contingent upon previous investment in a new plant. This contingency is modeled simply by the constraint

$$x_j \geq x_i,$$

which states that if $x_i = 1$ and project i (new product development) is accepted, then necessarily $x_j = 1$ and project j (construction of a new plant) must be accepted. Another example of this nature concerns conflicting projects. The constraint

$$x_1 + x_2 + x_3 + x_4 \leq 1,$$

for example, states that only one of the first four investments can be accepted. Constraints like this commonly are called multiple-choice constraints. By combining these logical constraints, the model can incorporate many complex interactions between projects, in addition to issues of resource allocation.

A particular case: the 0–1 knapsack problem. It is stated as

$$\begin{aligned} &\textit{maximize } \textstyle\sum_{j=1}^{n} c_j x_j \\ &\textit{subject to} \\ &\textstyle\sum_{j=1}^{n} a_j x_j \leq b, \\ &x_j = 0 \ \textit{ or } \ 1 \quad (j = 1, 2, \ldots, n) \end{aligned}$$

This problem models the situation when a hiker must decide which goods to include on his trip. Here c_j is the "value" or utility of including good j, which weighs $a_j > 0$ pounds; the objective is to maximize the "pleasure of the trip," subject to the weight limitation that the hiker can carry no more than b pounds. The model can be altered by allowing more than one unit of any good to be taken, by writing $x_j \geq 0$ and x_j integer in place of the 0–1 restrictions on the variables.

Example 8.12. Scheduling problems: *Consider the scheduling of airline flight personnel. The airline has a number of routing "legs" to be flown, such as 10 A.M. New York to Chicago, or 6 P.M. Chicago to Los Angeles. The airline must schedule its personnel crews on routes to cover these flights. One crew, for example, might be scheduled to fly a route containing the two legs just mentioned. The decision variables, then, specify the scheduling of the crews to routes:*

$$x_j = \begin{cases} 1 & \textit{if a crew is assgined to route } j, \\ 0 & \textit{otherwise.} \end{cases}$$

Let

$$a_{ij} = \begin{cases} 1 & \textit{if leg } i \textit{ is included on route } j, \\ 0 & \textit{otherwise,} \end{cases}$$

and

$$c_j = \textit{Cost for assigning a crew to route } j$$

The coefficients a_{ij} define the acceptable combinations of legs and routes, taking into account such characteristics as sequencing of legs for making connections between flights and for including in the routes ground time for maintenance. The model becomes:

$$\begin{aligned} &\textit{maximize } \textstyle\sum_{j=1}^{n} c_j x_j \\ &\textit{subject to} \\ &\textstyle\sum_{j=1}^{n} a_{ij} x_j = 1 \quad (i = 1, 2, \ldots, m), \\ &x_j = 0 \ \textit{ or } \ 1 \quad (j = 1, 2, \ldots, n) \end{aligned} \tag{8.36}$$

The ith constraint requires that one crew must be assigned on a route to fly leg i. An alternative formulation permits a crew to ride as passengers on a leg. Then the constraints (8.36) become:

$$\sum_{j=1}^{n} a_{ij}x_j \geq 1 \quad (i = 1, 2, \ldots, m)$$

If, for example,

$$\sum_{j=1}^{n} a_{1j}x_j = 3$$

then two crews fly as passengers on leg 1, possibly to make connections to other legs to which they have been assigned for duty.

Example 8.13. The traveling salesman problem. *Starting from his home, a salesman wishes to visit each of $(n-1)$ other cities and return home at minimal cost. He must visit each city exactly once and it costs c_{ij} to travel from city i to city j. What route should he select? If we let*

$$x_{ij} = \begin{cases} 1 & \textit{if he goes from city } i \textit{ to city } j, \\ 0 & \textit{otherwise.} \end{cases}$$

We may formulate this problem as the assignment problem:

$$\textit{Minimize } z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij},$$

subject to:

$$\begin{aligned} \sum_{i=1}^{n} x_{ij} &= 1 \qquad (j = 1, \ldots, n) \\ \sum_{j=1}^{n} x_{ij} &= 1 \qquad (i = 1, \ldots, n) \\ x_{ij} &\geq 0 \quad (i = 1, \ldots, n;\ j = 1, \ldots, n) \end{aligned}$$

The constraints require that the salesman must enter and leave each city exactly once. Unfortunately, the assignment model can lead to infeasible solutions (see Figure 1). It is possible in a six-city problem, for example, for the assignment solution to route the salesman through two disjoint subtours of the cities instead of on a single trip or tour (see Figure 1). In this particular example, we can avoid the subtour solution of Figure 1 by including the constraint:

$$x_{14} + x_{15} + x_{16} + x_{24} + x_{25} + x_{26} + x_{34} + x_{35} + x_{36} \geq 1.$$

This inequality ensures that at least one leg of the tour connects cities 1, 2, and 3 with cities 4, 5, and 6. In general, if a constraint of this form is included for each way in which the cities can be divided into two groups, then subtours will be eliminated. The problem with this and related approaches is that, with n cities, $(2^n - 1)$ constraints of this nature must be added, so that the formulation becomes a very large integer-programming problem. For this reason the traveling salesman problem generally is regarded as difficult when there are many cities. The traveling salesman model is used as a central component of many vehicular routing and scheduling models. It also arises in production scheduling. For example, suppose that we wish to sequence $(n-1)$ jobs on a single machine, and that c_{ij} is the cost for setting up the machine for job j, given that job i has just been completed. What scheduling sequence for the jobs gives the lowest total setup costs? The problem can be interpreted as a traveling salesman problem, in which the "salesman" corresponds to the machine which must "visit" or perform each of the jobs. "Home" is the initial setup of the machine, and, in some applications, the machine will have to be returned to this initial setup after completing all of the jobs. That is, the "salesman" must return to "home" after visiting the "cities."

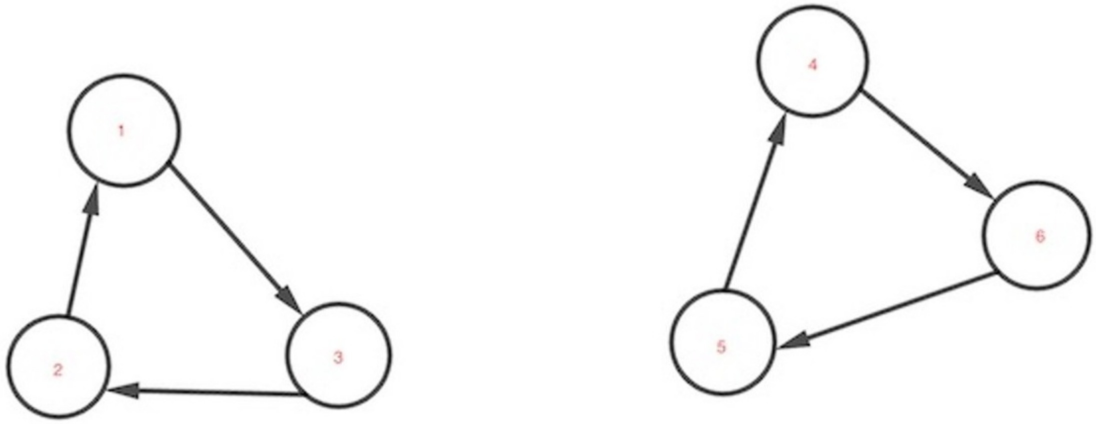


FIGURE 1. Disjoint subtours.

9. ALGORITHM *Branch and bound*

9.1. **Description.** For the optimization problem

$$(P)\begin{cases} \min\ f(x) \\ x \in X, \end{cases}$$

a relaxation is a problem of the form

$$(RP)\begin{cases} \min\ f(x) \\ x \in Y, \end{cases}$$

where $X \subset Y$.

- If (RP) is infeasible then (P) is infeasible.
- The optimal value of (RP) is smaller or equal to the optimal value of (P).

We will assume that the coefficients c are integers. We can easily adapt the algorithm if this is not the case.

Relaxations are used when the initial problem is complicated and allow for lower quotas (for minimization) over or optimal value solving a simpler problem..

Relaxations we will use for PLI:

(A) $x_i \in \mathbb{Z}$ relaxed in $x_i \in \mathbb{R}$ (suppressed restriction);
(B) $x_i \in \mathbb{Z}_+$ relaxed in $x_i \geq 0$;
(C) $x_i \in \{0, 1\}$ relaxed in $0 \leq x_i \leq 1$.

Using relaxations (A), (B), and (C) for PLI, we end up with a linear programming problem.

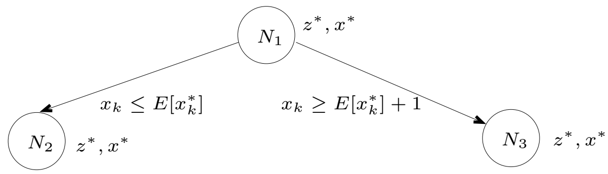
- The objective of Branch and Bound is to replace the solution of the initial problem with a sequence of simpler problems, which in the case of PLI will be linear problems corresponding to relaxations of the initial problem.
- After solving each relaxation, a lower bound L (in the case of a maximization problem) or a upper bound U (in the case of a minimization problem) on the optimal value of the problem is updated.

In our case, we solve the following relaxations:

PLI	$\begin{cases} \max\ c^\top x \\ Ax = b \\ x \geq 0 \\ x \in \mathbb{Z}^n. \end{cases}$	$\begin{cases} \min\ c^\top x \\ Ax = b \\ x \geq 0 \\ x \in \mathbb{Z}^n. \end{cases}$
Relaxação RPLI	$\begin{cases} \max\ c^\top x \\ Ax = b \\ x \geq 0. \end{cases}$	$\begin{cases} \min\ c^\top x \\ Ax = b \\ x \geq 0. \end{cases}$

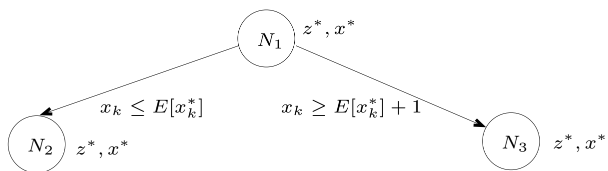
- If the RPLI solution is integer, we find an optimal solution for the PLI.
- If RPLI is infeasible, then PLI is infeasible.

In the case where x^*, the solution of the RPLI, is not integer, we choose the variable x_k with the largest fractional value and solve two new PLI relaxations: in the first one one $x_k \in \mathbb{Z}$ is replaced by $x_k \leq E[x_k^*]$ and in the second one $x_k \geq E[x_k^*] + 1$. For a minimization problem:



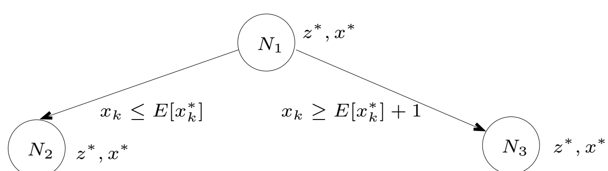
Em N_1, resolvemos $\begin{cases} \max\ c^\top x \\ Ax = b,\ x \geq 0. \end{cases}$

In the case where x^*, the solution of the RPLI, is not integer, we choose the variable x_k with the largest fractional value and solve two new PLI relaxations: in the first one one $x_k \in \mathbb{Z}$ is replaced by $x_k \leq E[x_k^*]$ and in the second one $x_k \geq E[x_k^*] + 1$. For a minimization problem:



Em N_2, resolvemos $\begin{cases} \max\ c^\top x \\ Ax = b,\ x \geq 0 \\ x_k \leq E[x_k^*]. \end{cases}$

In the case where x^*, the solution of the RPLI, is not integer, we choose the variable x_k with the largest fractional value and solve two new PLI relaxations: in the first one one $x_k \in \mathbb{Z}$ is replaced by $x_k \leq E[x_k^*]$ and in the second one $x_k \geq E[x_k^*] + 1$. For a minimization problem:



Em N_3, resolvemos $\begin{cases} \max\ c^\top x \\ Ax = b,\ x \geq 0 \\ x_k \geq E[x_k^*] + 1. \end{cases}$

For each of the 2 new leaves of the tree:

- For a maximization problemção, if the solutionção x^* in the sheet is integer and if $c^\top x^* > L$ then $L \leftarrow c^\top x^*$.
- For a minimization problemção, if the solutionção x^*in the sheet is integer and if $c^\top x^* < U$ then $U \leftarrow c^\top x^*$.

For a problem of minimizationção (resp. maximizeção) U (resp. L) ánd the smallest (resp. largest) value of the objective function found so far.

For each of the 2 new leaves of the tree:

- If the problem on the sheet is infeasible, then the sheet ís said to be inactive (the problems for all children will be infeasible).
- If the solutionção x^* is entire, then the sheet ís said to be inactive: for a problem of minimizationção (resp. minimizationção) all children would have greater óptimal values (resp. lower).

Procedure of updating of the active leaves: for each one of the leaves of the treee still active (including the two last ones):

- For a problem of maximization, if the solution x^* satisfies $E[c^\top x^*] \leq L$ then the leaf becomes inactive.

- For a problem of minimization, if the solution x^* satisfies $E[c^\top x^*] \geq U$ then the leaf becomes inactive.

The leaves that are not inactive are called active.

Branch and Bound algorithm for PLI: summarized version As long as an active sheet exists:

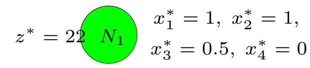
- choose an active leaf with the largest value of z^*;
- choose a variable for the "branching" with largest fractionary value;
- solve the two corresponding relaxed linear problems;
- call the process of updating the active leaves.

9.2. **Example I: binary variables.**

$L = -\infty.$ $\qquad z^* = 22$ N_1 $\quad x_1^* = 1,\ x_2^* = 1,$ $x_3^* = 0.5,\ x_4^* = 0$

$$\text{Em } N_1, \text{ resolvemos} \quad \begin{cases} \max\ 8x_1 + 11x_2 + 6x_3 + 4x_4 \\ 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14 \\ 0 \leq x_i \leq 1,\ i = 1, \ldots, 4. \end{cases}$$

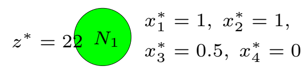
$L = -\infty.$



Em N_1, resolvemos $\begin{cases} \max\ 8x_1 + 11x_2 + 6x_3 + 4x_4 \\ 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14 \\ 0 \leq x_i \leq 1,\ i = 1, \ldots, 4. \end{cases}$

A solução não é viável para PLI e $E[z^*] > L$: N_1 é ativo.

$L = -\infty.$



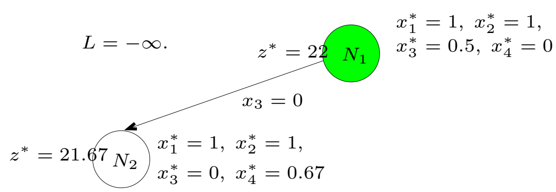
Escolhemos uma variável com maior valor fracionário

para o "branching": x_3

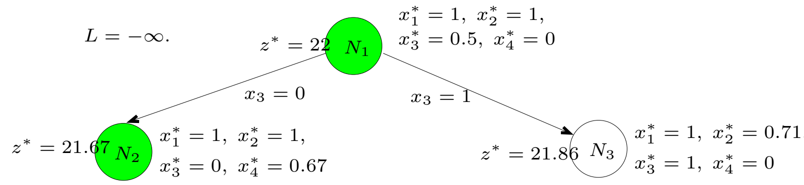
Em N_1, resolvemos $\begin{cases} \max\ 8x_1 + 11x_2 + 6x_3 + 4x_4 \\ 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14 \\ 0 \leq x_i \leq 1,\ i = 1, \ldots, 4. \end{cases}$

A solução não é viável para PLI e $E[z^*] > L$: N_1 é ativo.

$L = -\infty.$

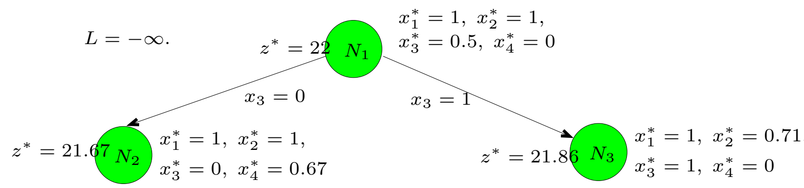


Em N_2, resolvemos $\begin{cases} \max\ 8x_1 + 11x_2 + 6x_3 + 4x_4 \\ 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14 \\ 0 \leq x_i \leq 1,\ i \neq 3 \\ x_3 = 0. \end{cases}$



$$\text{Em } N_3\text{, resolvemos} \begin{cases} \max\ 8x_1 + 11x_2 + 6x_3 + 4x_4 \\ 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14 \\ 0 \leq x_i \leq 1,\ i \neq 3 \\ x_3 = 1. \end{cases}$$

Folha N_2 : a solução não é viável para PLI e $E[z^*] > L$: N_2 é ativo.

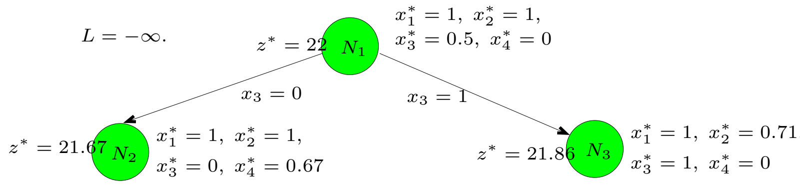


$$\text{Em } N_3\text{, resolvemos} \begin{cases} \max\ 8x_1 + 11x_2 + 6x_3 + 4x_4 \\ 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14 \\ 0 \leq x_i \leq 1,\ i \neq 3 \\ x_3 = 1. \end{cases}$$

Folha N_3 : a solução não é viável para PLI e $E[z^*] > L$: N_3 é ativo.

9.3. **Example II.** We will consider the minimization problem

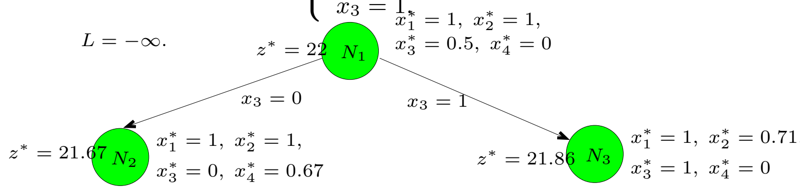
$$\begin{cases} \min\ -2x_1 - 3x_2 \\ \frac{2}{9}x_1 + \frac{1}{4}x_2 \leq 1 \\ \frac{1}{7}x_1 + \frac{1}{3}x_2 \leq 1 \\ x_1,\ x_2 \in \mathbb{Z}_+. \end{cases}$$



Escolhemos uma folha ativa com

maior valor de z^*: N_3

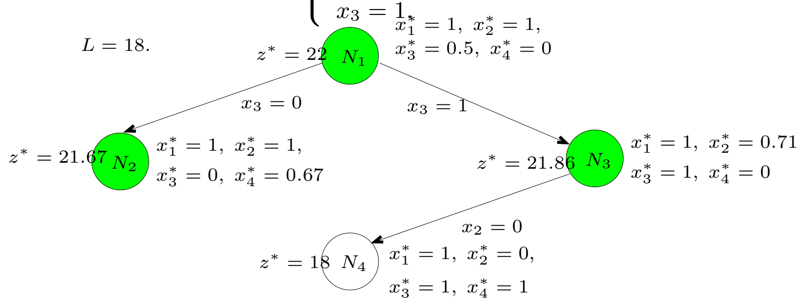
$$\text{Em } N_3\text{, resolvemos} \begin{cases} \max\ 8x_1 + 11x_2 + 6x_3 + 4x_4 \\ 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14 \\ 0 \leq x_i \leq 1,\ i \neq 3 \\ x_3 = 1. \end{cases}$$



Escolhemos uma variável com maior valor fracionário

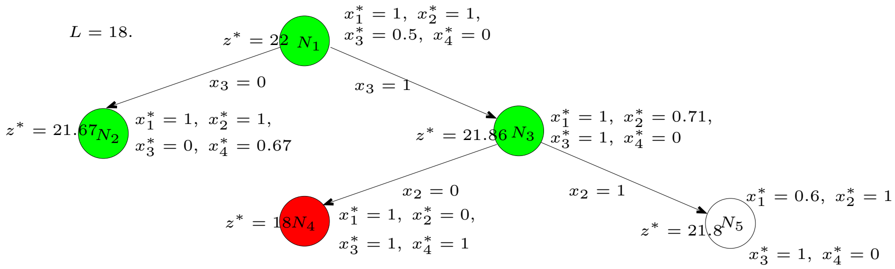
para o "branching": x_2

$$\text{Em } N_3\text{, resolvemos} \begin{cases} \max\ 8x_1 + 11x_2 + 6x_3 + 4x_4 \\ 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14 \\ 0 \leq x_i \leq 1,\ i \neq 3 \\ x_3 = 1. \end{cases}$$



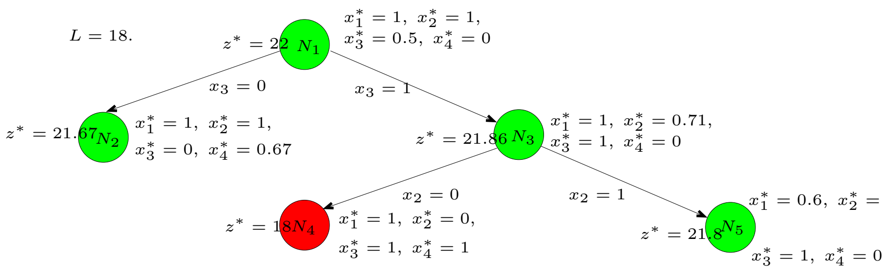
Como $z^* = 18 > L$, $L \leftarrow z^*$.

$$\text{Em } N_4\text{, resolvemos} \begin{cases} \max\ 8x_1 + 11x_2 + 6x_3 + 4x_4 \\ 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14 \\ 0 \leq x_1 \leq 1,\ 1 \leq x_4 \leq 1 \\ x_3 = 1,\ x_2 = 0. \end{cases}$$



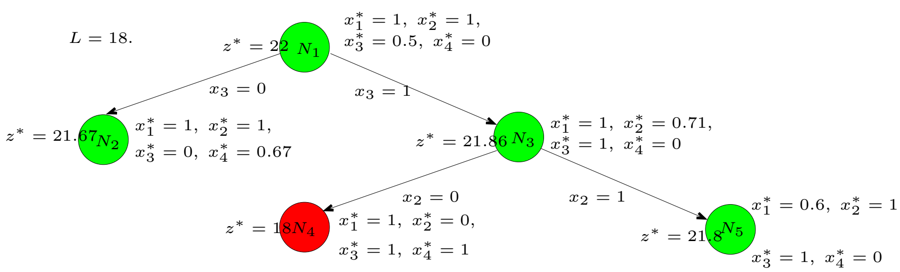
$$\text{Em } N_5\text{, resolvemos}\begin{cases} \max\ 8x_1 + 11x_2 + 6x_3 + 4x_4 \\ 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14 \\ 0 \leq x_1 \leq 1,\ 1 \leq x_4 \leq 1 \\ x_3 = 1,\ x_2 = 1. \end{cases}$$

Folha N_4: a solução é viável para PLI: N_4 não é ativo.



$$\text{Em } N_5\text{, resolvemos}\begin{cases} \max\ 8x_1 + 11x_2 + 6x_3 + 4x_4 \\ 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14 \\ 0 \leq x_1 \leq 1,\ 1 \leq x_4 \leq 1 \\ x_3 = 1,\ x_2 = 1. \end{cases}$$

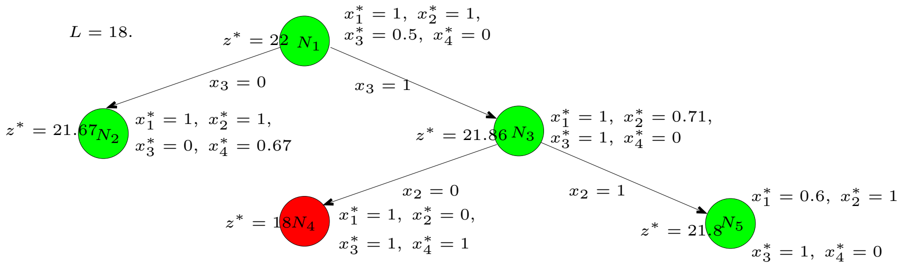
Folha N_5: a solução não 'e viável para PLI e $L < E[z^*]$: N_5 é ativo.



Escolhemos uma folha ativa com maior valor de z^*: N_5

$$\text{Em } N_5\text{, resolvemos}\begin{cases} \max\ 8x_1 + 11x_2 + 6x_3 + 4x_4 \\ 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14 \\ 0 \leq x_1 \leq 1,\ 1 \leq x_4 \leq 1 \\ x_3 = 1,\ x_2 = 1. \end{cases}$$

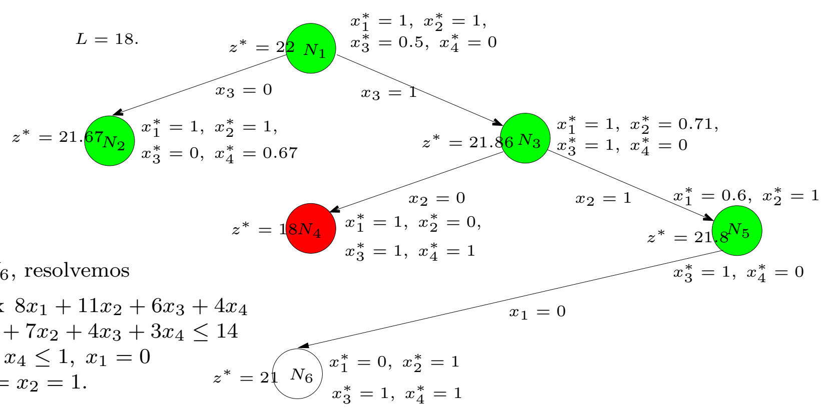
Folha N_2: a solução não 'e viável para PLI e $L < E[z^*]$: N_2 é ativo.



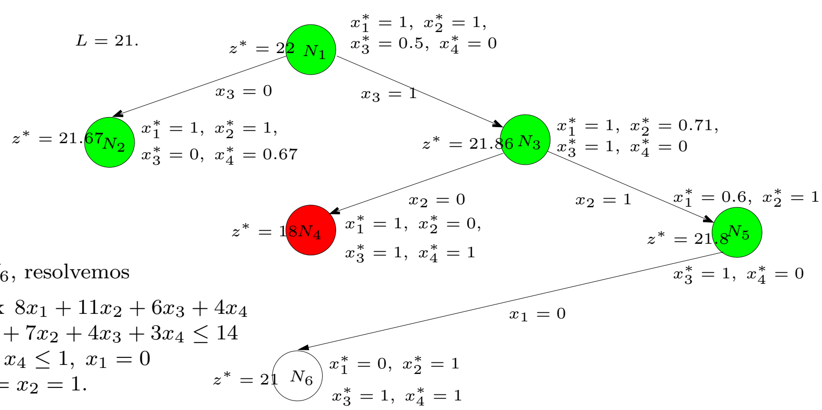
Escolhemos uma variável com maior valor fracionario: x_1

Em N_5, resolvemos
$$\begin{cases} \max\ 8x_1 + 11x_2 + 6x_3 + 4x_4 \\ 5x_1 + 7x_2 + 4x_3 + 3x_4 \le 14 \\ 0 \le x_1 \le 1,\ 1 \le x_4 \le 1 \\ x_3 = 1,\ x_2 = 1. \end{cases}$$

Folha N_2: a solução não 'e viável para PLI e $L < E[z^*]$: N_2 é ativo.

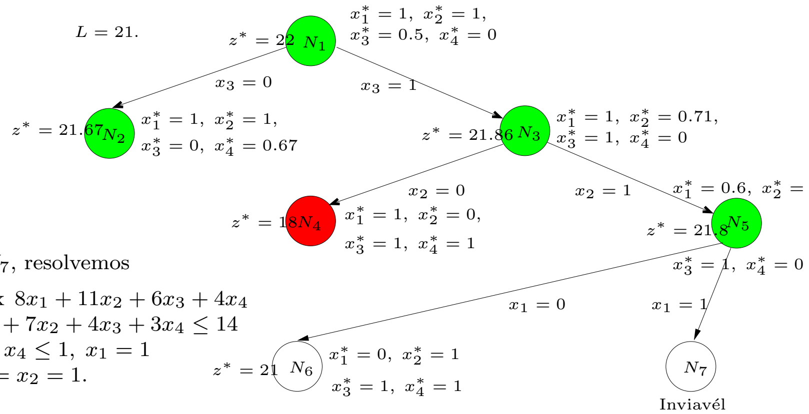


Em N_6, resolvemos
$$\begin{cases} \max\ 8x_1 + 11x_2 + 6x_3 + 4x_4 \\ 5x_1 + 7x_2 + 4x_3 + 3x_4 \le 14 \\ 0 \le x_4 \le 1,\ x_1 = 0 \\ x_3 = x_2 = 1. \end{cases}$$



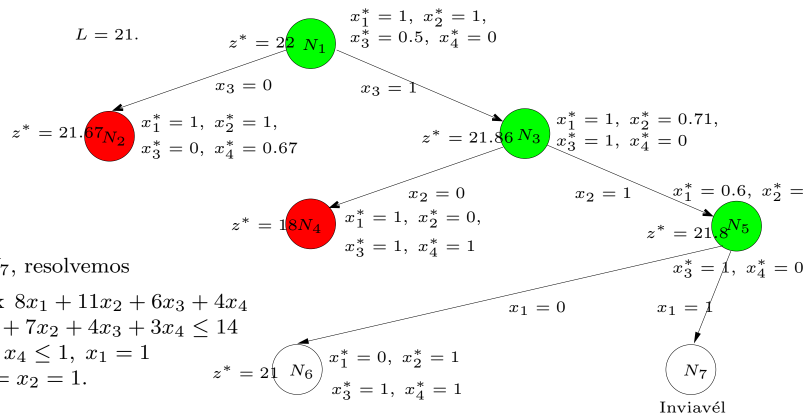
Em N_6, resolvemos
$$\begin{cases} \max\ 8x_1 + 11x_2 + 6x_3 + 4x_4 \\ 5x_1 + 7x_2 + 4x_3 + 3x_4 \le 14 \\ 0 \le x_4 \le 1,\ x_1 = 0 \\ x_3 = x_2 = 1. \end{cases}$$

A solução é viavél e $z^* = 21 > L$: $L \leftarrow z^*$.



Em N_7, resolvemos

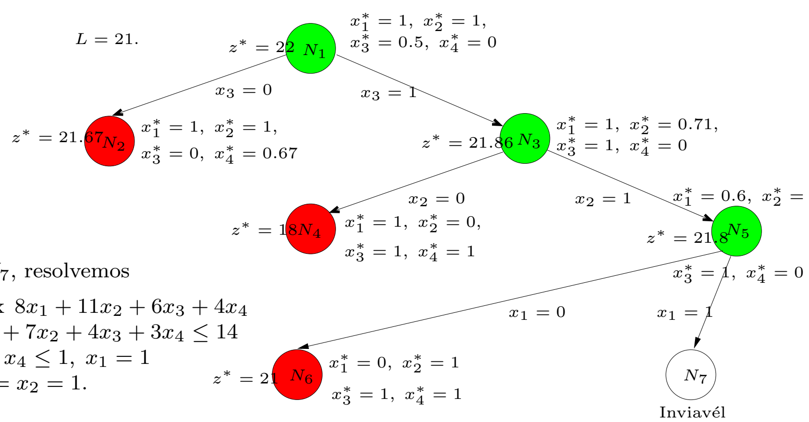
$$\begin{cases} \max\ 8x_1 + 11x_2 + 6x_3 + 4x_4 \\ 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14 \\ 0 \leq x_4 \leq 1,\ x_1 = 1 \\ x_3 = x_2 = 1. \end{cases}$$



Em N_7, resolvemos

$$\begin{cases} \max\ 8x_1 + 11x_2 + 6x_3 + 4x_4 \\ 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14 \\ 0 \leq x_4 \leq 1,\ x_1 = 1 \\ x_3 = x_2 = 1. \end{cases}$$

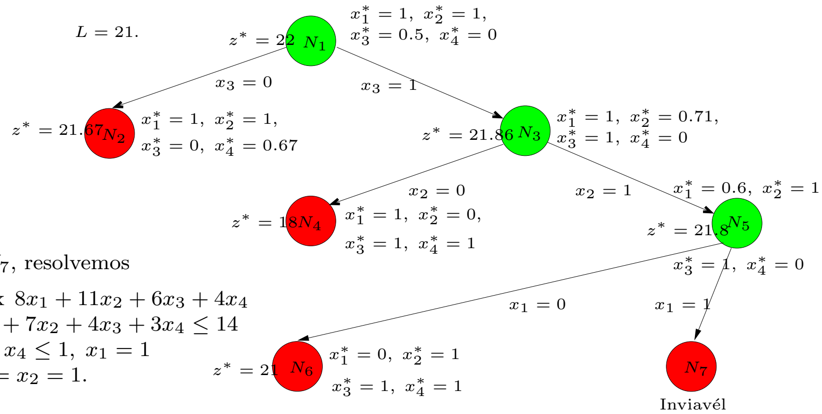
Folha N_2: $E[z^*] \leq L$. N_2 não é ativo.



Em N_7, resolvemos

$$\begin{cases} \max\ 8x_1 + 11x_2 + 6x_3 + 4x_4 \\ 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14 \\ 0 \leq x_4 \leq 1,\ x_1 = 1 \\ x_3 = x_2 = 1. \end{cases}$$

Folha N_6: solução viavél para PLI. N_6 não é ativo.

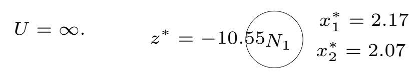


Em N_7, resolvemos

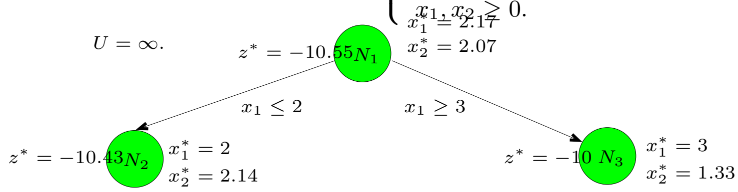
$$\begin{cases} \max\ 8x_1 + 11x_2 + 6x_3 + 4x_4 \\ 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14 \\ 0 \leq x_4 \leq 1,\ x_1 = 1 \\ x_3 = x_2 = 1. \end{cases}$$

Folha N_7: problema inviavél. N_7 não é ativo.

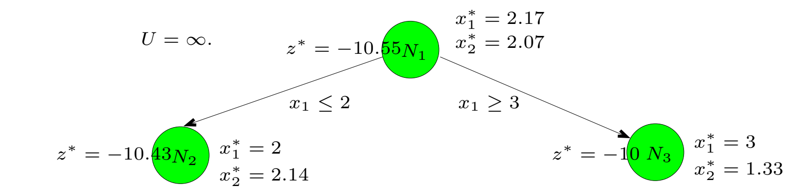
Não tem folha ativa: achamos uma solução ótima.



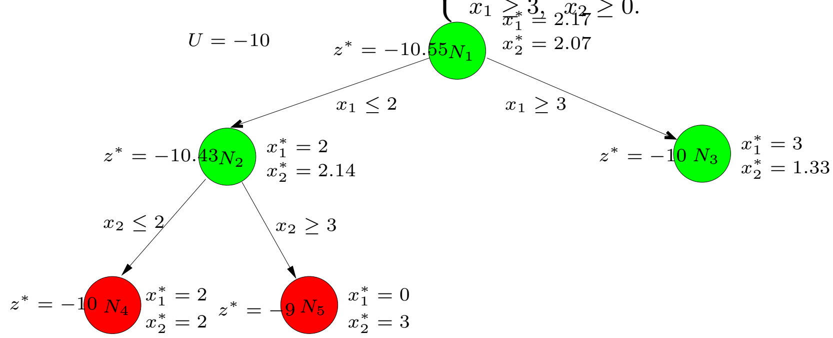
Em N_1, resolvemos $\begin{cases} \min\ -2x_1 - 3x_2 \\ \frac{2}{9}x_1 + \frac{1}{4}x_2 \leq 1 \\ \frac{1}{7}x_1 + \frac{1}{3}x_2 \leq 1 \\ x_1, x_2 \geq 0. \end{cases}$



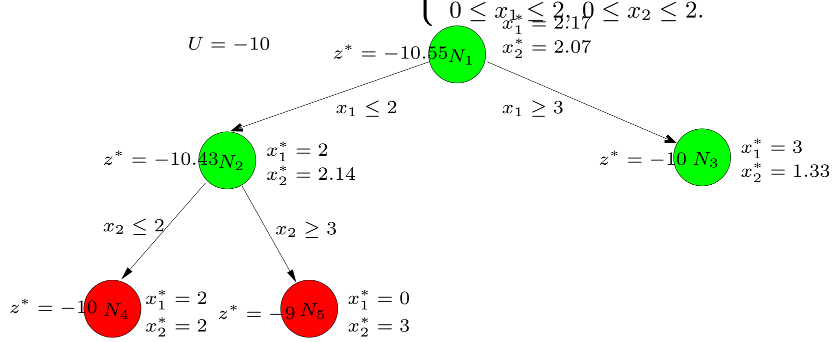
Em N_2, resolvemos $\begin{cases} \min\ -2x_1 - 3x_2 \\ \frac{2}{9}x_1 + \frac{1}{4}x_2 \leq 1 \\ \frac{1}{7}x_1 + \frac{1}{3}x_2 \leq 1 \\ 0 \leq x_1 \leq 2,\ x_2 \geq 0. \end{cases}$



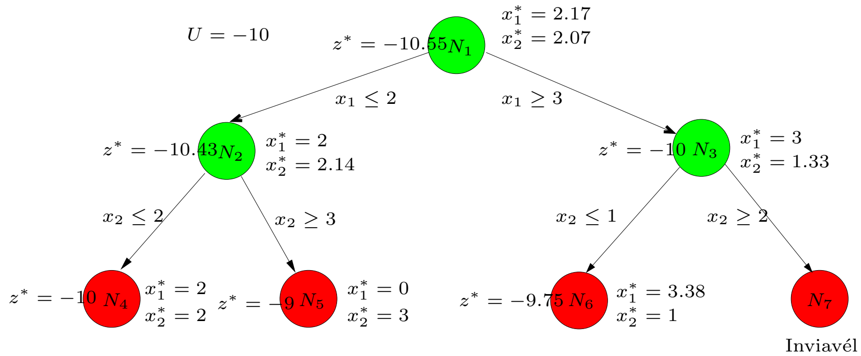
Em N_3, resolvemos $\begin{cases} \min\ -2x_1 - 3x_2 \\ \frac{2}{9}x_1 + \frac{1}{4}x_2 \leq 1 \\ \frac{1}{7}x_1 + \frac{1}{3}x_2 \leq 1 \\ x_1 \geq 3,\ \ x_2 \geq 0. \end{cases}$



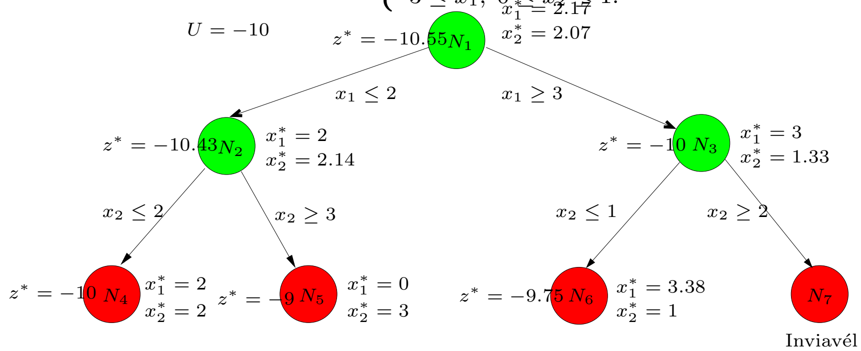
Em N_4, resolvemos $\begin{cases} \min\ -2x_1 - 3x_2 \\ \frac{2}{9}x_1 + \frac{1}{4}x_2 \leq 1 \\ \frac{1}{7}x_1 + \frac{1}{3}x_2 \leq 1 \\ 0 \leq x_1 \leq 2,\ 0 \leq x_2 \leq 2. \end{cases}$



Em N_5, resolvemos $\begin{cases} \min\ -2x_1 - 3x_2 \\ \frac{2}{9}x_1 + \frac{1}{4}x_2 \leq 1 \\ \frac{1}{7}x_1 + \frac{1}{3}x_2 \leq 1 \\ 0 \leq x_1 \leq 2,\ x_2 \geq 3. \end{cases}$



$$\text{Em } N_6 \text{, resolvemos} \quad \begin{cases} \min\ -2x_1 - 3x_2 \\ \frac{2}{9}x_1 + \frac{1}{4}x_2 \leq 1 \\ \frac{1}{7}x_1 + \frac{1}{3}x_2 \leq 1 \\ 3 \leq x_1,\ 0 \leq x_2 \leq 1. \end{cases}$$



$$\text{Em } N_7 \text{, resolvemos} \quad \begin{cases} \min\ -2x_1 - 3x_2 \\ \frac{2}{9}x_1 + \frac{1}{4}x_2 \leq 1 \\ \frac{1}{7}x_1 + \frac{1}{3}x_2 \leq 1 \\ 3 \leq x_1,\ 2 \leq x_2. \end{cases}$$

Não tem mais folhas ativas: achamos uma solução ótima
Uma solução ótima é (2;2) e o valor ótimo é -10.

Theorem 10.1. *Suppose that the polyhedron $P = \{x \in \mathbb{R}^n \,|\, a_i^T x \geq b_i,\ i = \ldots, m\}$ is nonempty. Then, the following are equivalent:*

(a) *The polyhedron P has at least one extreme point.*
(b) *The polyhedron P does not contain a line.*
(c) *There exist n vectors out of the family $a_1, \ldots, a_m$ which are linearly independent.*

Proof. (b)$\Rightarrow$(a). Let $x \in P$ and $I(x) = \{i \in \{1, \ldots, m\} \,|\, a^T x = b_i\}$. If $\{a_i,\ i \in I(x)\}$ is a l.i. set of vectors it follows that x is a basic solution and therefore is an extreme point of P. If $\{a_i,\ i \in I(x)\}$ is a l.d. set of vectors then let $n(x)$ be the cardinality of the largest (in terms of number of vectors) l.i. set of vectors that can be extracted from $\{a_i,\ i \in I(x)\}$. It follows that there exists $d \neq 0$ such that $a_i^T d = 0$ for $i = 1, \ldots, m$. Consider the point $x(t) = x + td$ with $t > 0$. Clearly it holds $a_i^T x(t) = a_i^T x = b_i$ for $i = 1, \ldots, m$ and all $t \geq 0$. Now, let us define $t_1 = \max\{t > 0 \,|\, x + td \in P\}$. Clearly, from this definition, it follows that at least one inequality will become active at $x(t_1)$. Hence, for some $j \notin I(x)$ it holds $a_j^T (x + t_1 d) < b_j$ which implies that $a_j^T d < 0$ since $a_j^T x \geq b_j$. Since $< \{a_i\ i \in I(x)\} > \subset < d >^{\perp}$ it follows that $a_j \notin < \{a_i\ i \in I(x)\} >$ which means that $n(x(t_1)) \geq n(x) + 1$. Now, repeating this process as long as required we will end up with some point in $\hat{x} \in P$ with $n(\hat{x}) = n$ which ends the proof.

(a)$\Rightarrow$(c). Let $x \in P$ be an extreme point of P and $I(x) = \{i \in \{1, \ldots, m\} \,|\, a^T x = b_i\}$. Let us assume that $\{a_i,\ i \in I(x)\}$ is an l.d. set of vectors. Then it follows that there exists $d \neq 0$ such that $a_i^T d = 0$ for $i = 1, \ldots, m$. Consider the point $x(t) = x + td$ and notice that for $\varepsilon > 0$ small enough it holds $x(\varepsilon), x(-\varepsilon) \in P$. Since $x = (1/2)(x(\varepsilon) + x(-\varepsilon))$ we obtain a contradiction. Hence it follows that $\{a_i\ i \in I(x)\}$ is a l.i. set of vectors and ends the proof.

(c)$\Rightarrow$(b). Let us assume that for some $x \in P \subset \mathbb{R}^n$ and $d \neq 0$, the line $\{x + td \,|\, t \geq 0\}$ is contained in P. It follows that $a_i^T (x + td) \geq b$ for all $t \geq 0$ and $i = 1, \ldots, m$. This implies that $a_i^T d = 0$ for $i = 1, \ldots, m$. If we have a l.i. set of n vectors $\{a + i_1, a_{i_2}, \ldots, a_{i_n}\}$ it would follow that $d = 0$ which is a contradiction. This ends the proof. $\square$

Definition 10.2. *Given a nonempty polyhedron $P = \{x \in \mathbb{R}^n \,|\, Ax \geq b\}$ and a point $y \in P$, we define the recession cone at y as the set of all directions d along which we can move indefinitely away from y, without leaving the set P. More formally, the recession cone at y is defined as the set $\{d \in A \,|\, A(y + td) \geq b,\ \forall t \geq 0\}$.*

Lemma 10.3. *Given a nonempty polyhedron $P = \{x \in \mathbb{R}^n \,|\, Ax \geq b\}$ and a point $y \in P$, it holds that the recession cone at y is a closed and convex cone.*

Proof. The convexity follows trivially from the definition and the fact that P is defined by linear inequalities. Also from the definition it follows trivially that is a cone. If $\{d^k\}$ is a sequence of directions in the recession cone converging to some $\hat{d}$, it follows that, for any $t \geq 0$,

$$b \leq A(x + td^k) \underset{k \to +\infty}{\rightarrow} A(x + t\hat{d}) \Rightarrow b \leq A(x + t\hat{d})$$

which means that $\hat{d}$ is also in the recession cone and ends the proof. $\square$

Definition 10.4.

(a) *A nonzero element x of a polyhedral cone $C \subset \mathbb{R}^n$ is called an extreme ray if there are $n-1$ linearly independent constraints that are active at x.*
(b) *An extreme ray of the recession cone associated with a nonempty polyhedron P is also called an extreme ray of P.*

Theorem 10.5. *Consider the problem of minimizing $c^T x$ over a* **pointed** *polyhedral cone $C = \{x \in \mathbb{R}^n \,|\, a_i^T x \geq 0\ i = 1, \ldots m\}$. The optimal cost is equal to $-\infty$ if and only if some extreme ray d of C satisfies $c^T d < 0$.*

Proof. Let d be an extreme ray of C satisfying $c^T d < 0$. It follows that $a + td \in C$ for all $t \geq 0$ and, since $c^T(a + td) = c^T a + tc^T d \to -\infty$ when $t \to +\infty$, we conclude that the linear optimization problem considered above is unbounded below. Now assume that optimal cost of $c^T x$ over C is equal to $-\infty$. Then it follows that for some $\hat{x} \in C$ it holds $b := c^T \hat{x} < 0$. Now consider the (non empty) polyhedral

$P = \{x \in R^n \mid a_i^T x \geq 0,\ c^T x = b\} \subset C$. Since C is pointed it follows that C and, therefore, P do not contain lines. By Theorem 10.1, it follows that P contains at least one extreme point $\bar{x}$ at which we have n active linear constraints whose associated coefficient vectors are l.i. Hence, it follows that at least $n-1$ constraints of the form $a_i^T x \geq 0$ should be active at $\bar{x}$ which means that this point is an extreme ray of C and ends the proof. $\square$

Theorem 10.6. *Consider the problem of minimizing $c^T x$ subject to $Ax \geq b$, and assume that the feasible set has at least one extreme point. The optimal cost is equal to $-\infty$ if and only if some extreme ray d of the feasible set satisfies $c^T d < 0$.*

Proof. For the proof will denote the feasible set as $S = \{x \mid Ax \geq b\}$. Let d extreme ray of satisfying $c^T d < 0$. Since $x + td \in S$ for all $t > 0$ and $c^T(x + td) = c^T x + tc^T d \to -\infty$ when $t \to +\infty$ the "if" part of the claim follows.

Next, assume that the optimal cost of the LLP mentioned above is $-\infty$. This means that the associated dual problem

$$\begin{aligned} &\max\ b^T p \\ &\quad s.t. \quad A^T p = c, \quad p \geq 0. \end{aligned}$$

is unfeasible. Therefore the related LPP

$$\begin{aligned} &\max\ 0 \\ &\quad s.t. \quad A^T p = c, \quad p \geq 0. \end{aligned}$$

is also unfeasible which implies that its dual,

$$\begin{aligned} &\min\ c^T x \\ &\quad s.t. \quad Ax \geq 0, \end{aligned}$$

is unbounded below. Since $\{x \mid Ax \geq b\}$ has at least one extreme point it follows that the rows of A spam $\mathbb{R}^n$ and therefore $\{x \mid Ax \geq 0\}$ is a pointed polyhedral cone. Now, by Theorem 10.5, it follows that there exists an extreme ray d of this set that satisfies $c^T d < 0$. Since the extreme rays of a polyhedral are the extreme rays of its recession cone the result follows. $\square$

Remark 10.7. *We will show that when the optimal cost of problem 6.18 is $-\infty$, the simplex method provides us at termination with an extreme ray. Notice that at termination we have a basis matrix B, a nonbasic variable x_j; with negative reduced cost $\hat{c}_j$ and with the j-th column of the tableau $B^{-1}A_j$ which has no positive elements. Consider the jth basic direction d ate the basic solution associated to B (see Definition 6.20). We already saw in the proof of Lemma 6.21 that d satisfies $Ad = 0$ and, in addition, by the criteria of termination in the unbounded case, we have that $d \geq 0$. In summary, the vector d belongs to the recession cone of the feasible set, that is $d \in \{w \in \mathbb{R}^n \mid Aw = 0,\ w \geq 0\}$. Since the reduced cost $\hat{c}_j$ is negative, we have that d is also a direction of cost decrease. In addition, relation $Ad = 0$ means that it satisfies m linearly independent constraints with equality, and relations $d_i = 0$ for $i \notin B$, $i \neq j$ means that d satisfies additional $n - m - 1$ l.i. linear constraints, totaling $n-1$ l.i. linear constraints. We conclude that d is an extreme ray of the feasible set.*

Theorem 10.8 (Representation theorem)**.** *Let $P = \{x \in \mathbb{R}^n \mid Ax \geq b\}$ be a nonempty polyhedron with at least one extreme point. Let $x^1, \ldots, x^k$ be the extreme points of P, and let $w^1, \ldots, w^r$ be a complete set of extreme rays of P. Let*

$$Q = \left\{ \sum_{i=1}^{k} \lambda_i x^i + \sum_{j=1}^{r} \theta_j w^j \mid \lambda_i \geq 0\ , \theta_j \geq 0,\ i = 1, \ldots, k,\ j = 1, \ldots, r,\ \sum_{i=1}^{k} \lambda_k = 1 \right\}.$$

Then, $Q = P$.

Proof. First we prove that $Q \subset P$. Take any λ_i $i = 1, \ldots, k$ and θ_j $j = 1, \ldots, r$ as in the definition of Q. By convexity it follows that $\sum_{i=1}^{k} \lambda_i x^i \in P$ and by Lemma 10.3 it follows that $\sum_{j=1}^{r} \theta_j w^j$ is a direction in the recession cone. These properties imply that $\sum_{i=1}^{k} \lambda_i x^i + \sum_{j=1}^{r} \theta_j w^j \in P$ which ends the proof of the statement. Next we prove that $P \subset Q$. Let us assume that there exists $z \in P - Q$. Note that Q is a closed and convex set since it is the sum of two closed and convex sets: the convex hull of the (finite) set of extreme points of P and the conic hull of the (finite) set of extreme rays of P. Then, by Theorem 4.10, there exists an hyperplane that strictly separate z from Q. That is, there exists $(p, q) \in \mathbb{R}^n \times \mathbb{R}$ such that $p^T z + q < 0$ and $p^T w + q \geq 0$ for all $w \in Q$. In particular, it follows that

$$p^T x^i \geq -q > p^T z \quad \text{for} \quad i = 1, \ldots, k \tag{10.37}$$

and

$$p^T(\lambda w^j) + q \geq 0 \quad \text{for} \quad j = 1, \ldots, r \quad \text{and all} \quad \lambda > 0 \quad \Rightarrow \quad p^T w^j \geq 0 \quad \text{for} \quad j = 1, \ldots, r. \tag{10.38}$$

Now consider the LPP

$$\begin{aligned} \min \ & p^T x \\ s.t. \quad & Ax \geq b \end{aligned}$$

which has at least on feasible point, z. If its optimal value is finite then there exists an extreme point x^i which is optimal, and it follows that $p^T x^i < p^T z$ which is a contradiction with (10.37). On the other hand, it its optimal value is $-\infty$ then, by Theorem 10.6, there exists an extreme ray w^j satisfying $p^T w^j < 0$ which is a contradiction with (10.38). Hence, it follows that $Q \subset P$ which ends the proof of the theorem. □

11. References

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