## Exercises: convexity

**Exercice 1.** Let $A$ be an $m\times n$ matrix, let $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. Solve the optimization problem: minimize $c^T x$ subject to the constraints $Ax = b$.

**Exercice 2.** Show that the following sets are convex:

a) $\{\sum_{i=1}^n \lambda_i x_i : \lambda \geq 0, \sum_{i=1}^n \lambda_i = 1\}$ where $x_1, \ldots, x_n$ are given in $\mathbb{R}^n$.
b) $\{x \in \mathbb{R}^n : \|x - x_0\| \leq r\}$ where $r > 0$, $x_0 \in \mathbb{R}^n$, and $\|\cdot\|$ is a norm on $\mathbb{R}^n$.
c) $\{x \in \mathbb{R}^n : a^T x = b\}$ and $\{x \in \mathbb{R}^n : a^T x \leq b\}$ where $a \in \mathbb{R}^n, b \in \mathbb{R}$.
d) $\{x \in \mathbb{R}^n : Ax = b, Cx \leq d\}$.

**Exercice 3.** Show that if a nonempty set is an intersection of convex sets then it is convex. Deduce that the set of semidefinite positive matrices is convex.

**Exercice 4.** Show that the set of optimal solutions of a convex optimization problem is convex.

**Exercice 5.** a) Show that if $X$ is convex its interior is convex too.
b) We define the $\varepsilon$-enlargement of set $X$ by

$$X^\varepsilon = \{y : \mathrm{dist}(y, X) \leq \varepsilon\}$$

for $\varepsilon > 0$ where dist is the distance function:

$$\mathrm{dist}(y, X) = \left\{ \begin{array}{l} \inf \ \|y - x\| \\ x \in X. \end{array} \right.$$

Show that if $X$ is convex then the set $X^\varepsilon$ is convex too.
c) Let $B_F(0, \varepsilon) = \{x : \|x\| \leq \varepsilon\}$. Show that if $X$ is convex and closed then $X^\varepsilon = X + \mathbb{B}_F(0, \varepsilon)$. If $X$ is closed and convex, deduce another proof of the convexity of $X^\varepsilon$.

**Exercice 6.** Let $C$ be a convex set and let $\alpha_1, \alpha_2 > 0$. Show that

$$(\alpha_1 + \alpha_2)C = \alpha_1 C + \alpha_2 C.$$

Find a set $C$ for which the above relation does not hold.

**Exercice 7.** Let $S, T$ be two nonempty convex sets in $\mathbb{R}^n$. Show that $a$ separates $S, T$ if and only if

$$\sup_{x \in S} a^T x \leq \inf_{y \in T} a^T y$$

and

$$\inf_{x \in S} a^T x < \sup_{y \in T} a^T y.$$

Show that the separation is strong if and only if

$$\sup_{x \in S} a^T x < \inf_{y \in T} a^T y.$$