Applied Statistical Analysis I

Bivariate regression

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Today's Agenda

(1) Lecture recap & exam review

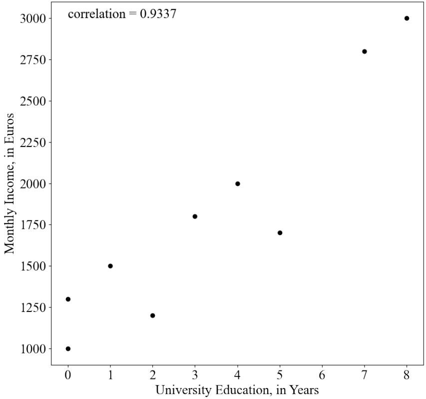
(2) Git pull

(3) Tutorial exercises

Linear regression model

What is a linear regression model? What interpretations can we make?

So far, correlation analysis



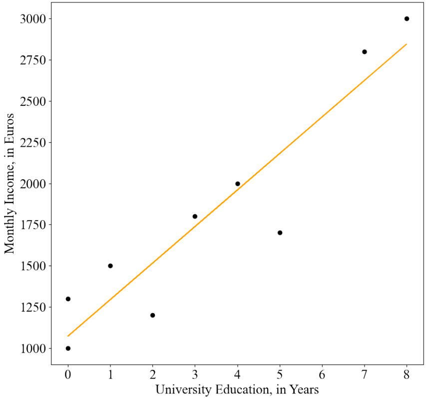
Just by looking at the plot, can you identify the straight line which best describes the joint variation between X and Y?

*This is fictional data.

Regression analysis

What is a linear regression model?

- Find linear line of best fit, $Y_i = \alpha + \beta X_i + \epsilon_i$



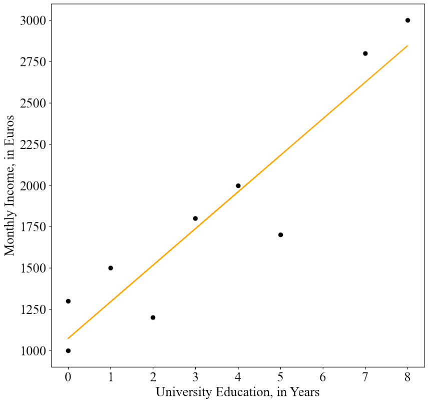
Regression analysis

What is a linear regression model?

- Find linear line of best fit, $Y_i = \alpha + \beta X_i + \epsilon_i$
- α (intercept): expected value of Y when $X = 0$
- β (slope): expected change in Y when X increases by one unit
- $\hat{Y}$ (expected value): predicted outcome based on the regression model, $\hat{Y}_i = \hat{\alpha} + \hat{\beta} X_i$
- ϵ (error/residual): difference between actual and predicted outcome, $\epsilon_i = Y_i - \hat{Y}_i$

Regression analysis

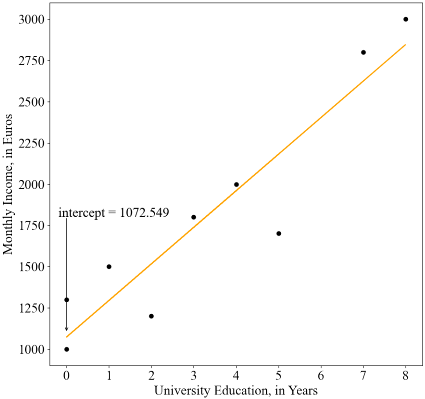
What interpretations can we make?



$$income = \alpha + \beta * education$$
$$income = 1072.5490 + 221.5686 * education$$

Regression analysis

What interpretations can we make? (intercept)

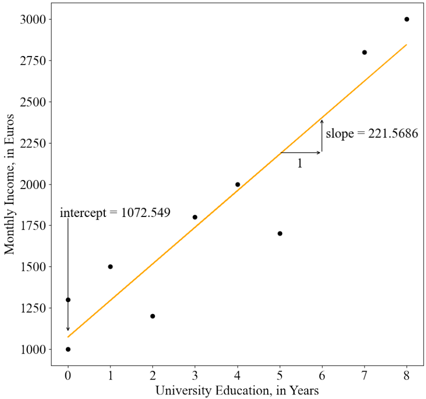


If an individual has a university education of 0 years, what income would we expect for that person?

$$income = 1072.5490 + 221.5686 * 0 = 1072.5490$$

Regression analysis

What interpretations can we make? (slope)

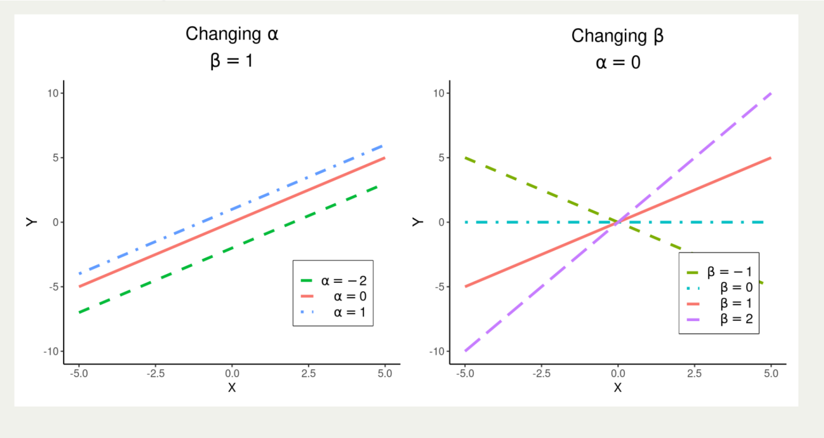


If the university education increases by one year, how much more Euros would we expect an individual to earn? $income = 1072.5490 + 221.5686 * 1 = 1294.1176$

$\rightarrow$ With every additional year of university education, the expected income increases by 221.5686 Euros.

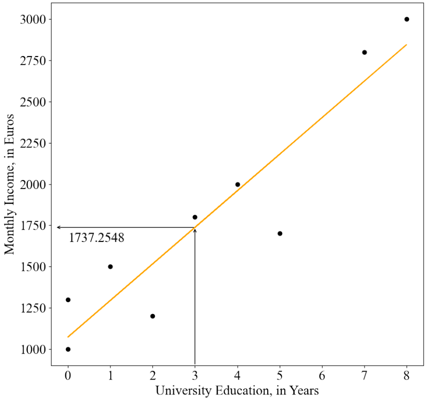
Regression analysis

Varieties of linear relationships



Regression analysis

What interpretations can we make? (expected value)

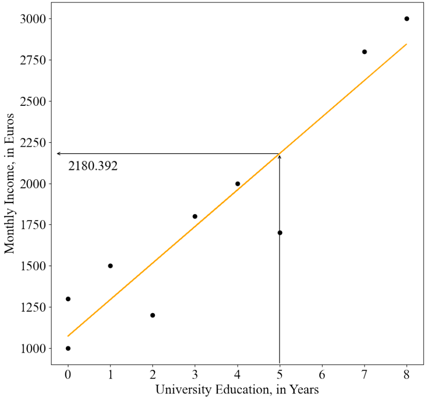


If an individual has 3 university education years, what income would we expect for that person?

$$income = 1072.5490 + 221.5686 * 3 = 1737.2548$$

Regression analysis

What interpretations can we make? (residual)



$income = 1072.5490 + 221.5686 * 5 = 2180.392$

Residual $=$ Actual $-$ Predicted

Residual $= 1700 - 2180.392 = -480.392$

Binary independent variables

How to include binary independent variables in simple linear regression?

Binary independent variables

How to include binary independent variables in simple linear regression?

$$Y_i = \alpha + \beta X_i + \epsilon_i \text{ } (X_i\text{=binary/dummy variable})$$

- α (intercept): expected value of Y when $X = 0$
- β (slope): expected change in Y for $X = 1$, in comparison to $X = 0$ (the reference category).

Binary independent variables

How to include binary independent variables in simple linear regression?

$$\textit{Regime Longevity}_i = \alpha + \beta_1 * \textit{Regime Type}_i$$

```
Call:
lm(formula = democracy_duration ~ democracy, data = democracy_gdp_2020)

Residuals:
    Min      1Q  Median      3Q     Max
-52.610 -24.610 -10.051   7.949 175.949

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   54.610      4.975  10.976   <2e-16 ***
democracy     -9.560      6.396  -1.495    0.137
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 43.66 on 193 degrees of freedom
Multiple R-squared:  0.01144,	Adjusted R-squared:  0.00632
F-statistic: 2.234 on 1 and 193 DF,  p-value: 0.1366
```

In comparison to autocracies (= reference category), democracies last 9.56 years fewer.

Binary independent variables

How to include binary independent variables in simple linear regression?

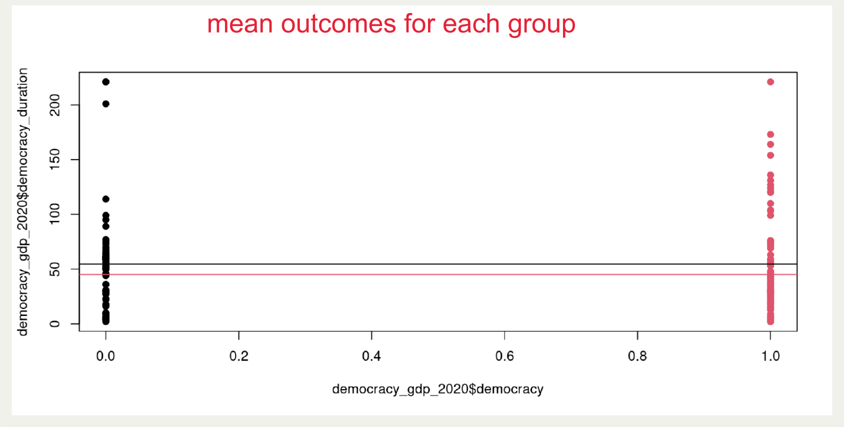
$\hat{Y}_i = \alpha + \beta_1 * Regime\ Type_i$

Model for Autocracies:
$\hat{Y}_i = 54.610 + (-9.560 * Regime\ Type_i)$
$\hat{Y}_i = 54.610 + (-9.560 * 0)$
$\hat{Y}_i = 54.610$

Model for Democracies:
$\hat{Y}_i = 54.610 + (-9.560 * Regime\ Type_i)$
$\hat{Y}_i = 54.610 + (-9.560 * 1)$
$\hat{Y}_i = 45.05$

Binary independent variables

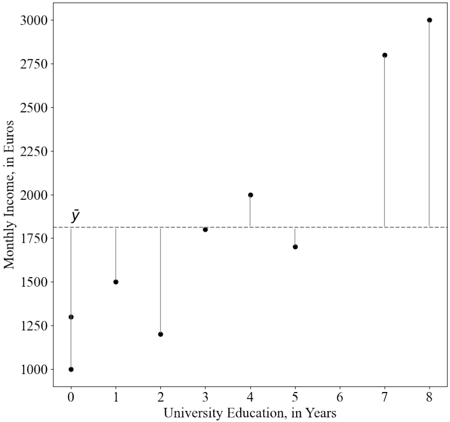
How to include binary independent variables in simple linear regression?



Partitioning the error

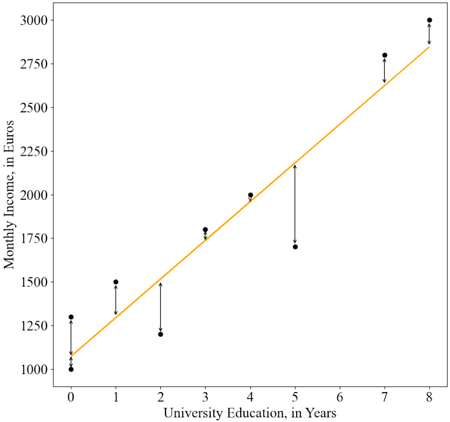
How can be partition the error?

Total sum of squares (SS_T)



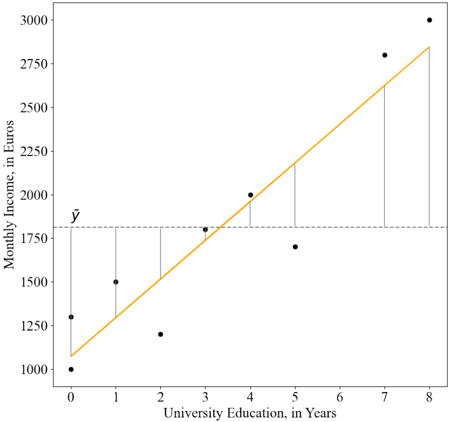
SS_T = Sum of squared differences between observed values of Y and the mean, $SS_T = \sum_{i=n}^{n}(y_i - \bar{y})^2$

Residual sum of squares (SS_R)



SS_R = Sum of squared differences between observed values of Y and the regression line, $SS_R = \sum_{i=n}^{n}(y_i - \hat{y}_i)^2$

Model sum of squares (SS_M)



SS_M = Sum of squared differences between the regression line and the mean of Y, $SS_M = \sum_{i=n}^{n} (\hat{y}_i - \bar{y})^2$

$\rightarrow$ Improvement if regression model is used rather than the mean.

Regression analysis

What interpretations can we make? (model performance)

- R^2: the proportion of variation of Y explained by X. Varies between 0 and 1. If X explains all the variation in Y, then $R^2 = 1$.

$$SS_T = SS_M + SS_R \ \underline{\text{and}}\ SS_M = SS_T - SS_R$$

$$R^2 = 1 - \frac{SS_R}{SS_T} =$$

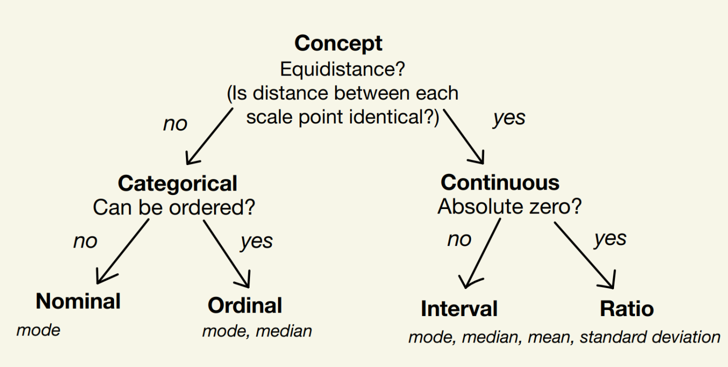
$$1 - \frac{\text{Variation not explained by model}}{\text{Total variation in y}} =$$

$$\frac{SS_M}{SS_T} = \frac{\text{Variation explained by model}}{\text{Total variation in y}}$$

Week 1–Introduction & stats review

Measurement Scales

How can we measure concepts? And why does it matter?



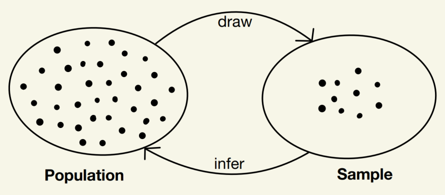
(Kellstedt and Whitten 2018, Chap. 5)

Discrete: finite set of possible values.
Continuous: infinite set of possible values.

Population, sample, parameter, statistic

What is the relationship between population and sample?

- Population: "the total set of subjects of interest in a study" (Agresti and Finlay 2009, 5).
- Parameter: "numerical summary of the population" (Agresti and Finlay 2009, 5).
- Sample: "the subset of the population on which the study collects data" (Agresti and Finlay 2009, 5).
- Statistic: "a numerical summary of the sample data" (Agresti and Finlay 2009, 5).
- Observation: single subject/unit, one row in dataset



Measures of central tendency and variability (dispersion)

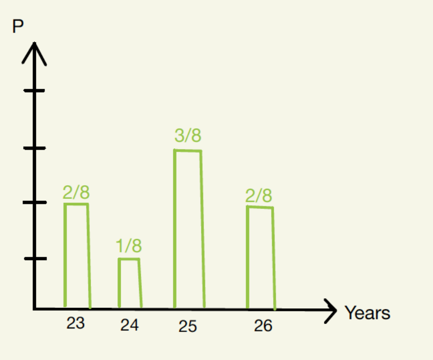
How can we describe variables?

- Mean: $\bar{y}$ = Sum of all values divided by the number of observations, $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$
- Variance: $s^2(y)$ = Sum of squared deviations divided by number of observations (deviation is the difference between observed value and the mean, $y_i - \bar{y}$), $s^2 = \frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{n-1}$
- Standard Deviation: Return original units by taking square root, $s = \sqrt{\frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{n-1}}$

Distributions and probability distributions

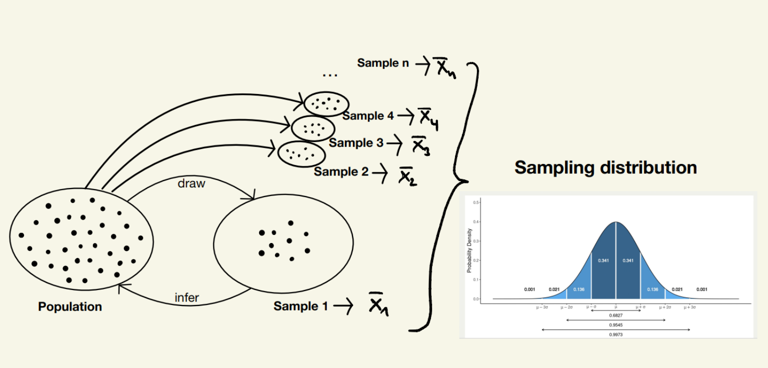
What is a probability distribution?

- Probability distribution "lists the possible outcomes and their probabilities" (Agresti and Finlay 2009, 75).



Sampling distribution

theoretically...



Sampling distribution

What is a sampling distribution?

- Sampling distribution "A sampling distribution of a statistic is the probability distribution that specifies probabilities for the possible values the statistic can take" (Agresti and Finlay 2009, 87).
- In other words, a probability distribution for a statistic rather than values of observations $\rightarrow$ What is the probability of $\bar{Y} = 0.5$, rather than what is the probability of $Y = 3$?

Sampling distribution

Why is this important?

- The corresponding probability theory "helps us predict how close a statistic falls to the parameter it estimates" (Agresti and Finlay 2009, 87). $\rightarrow$ how close is $\bar{y}$ to μ?
- Usually only one sample/one estimate $\rightarrow$ Point estimate: "is a single number that is the best guess for the parameter value" (Agresti and Finlay 2009, 107).

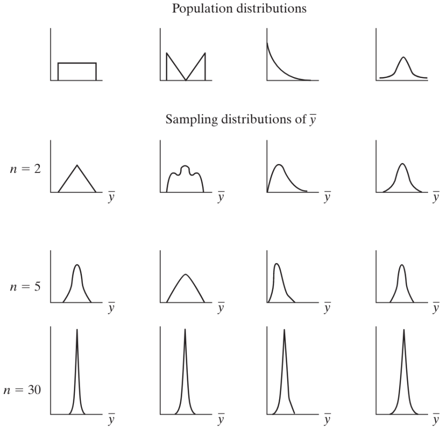
The sampling distribution of the mean, $\bar{y}$

- "If we repeatedly took samples, then in the long run, the mean of the sample means would equal the population mean μ" (Agresti and Finlay 2009, 90). $\rightarrow$ mean of the sampling distribution of $\bar{y}$ equals the population mean, hence, $\mu = \bar{y}$
- "The standard error describes how much $\bar{y}$ varies from sample to sample" (Agresti and Finlay 2009, 90). $\rightarrow$ standard error is estimated based on standard deviation, hence, $\sigma_{\bar{y}} = \frac{\sigma}{\sqrt{n}}$
- *Why does this work?*

Central Limit Theorem

What is the Central Limit Theorem?

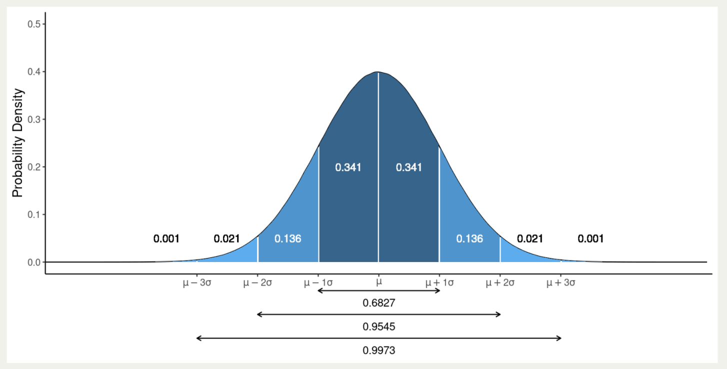
- "For random sampling with a large sample size n, the sampling distribution of the sample mean $\bar{y}$ is approximately a normal distribution" (Agresti and Finlay 2009, 93). $\rightarrow$ regardless of the population distribution



Central Limit Theorem

What is the Central Limit Theorem?

- "Knowing that the sampling distribution of $\bar{y}$ can be approximated by a normal distribution helps us to find probabilities for possible values of $\bar{y}$ (Agresti and Finlay 2009, 94). $\rightarrow$ key in inferential statistics



Confidence intervals

What are confidence intervals?

- Confidence interval: "an interval of numbers around the point estimate that we believe contains the parameter value" (Agresti and Finlay 2009, 110). $\rightarrow$ Point estimate $\pm$ Margin of error
- Confidence level: "The probability that this method produces an interval that contains the parameter" (usually 0.95, 0.99) (Agresti and Finlay 2009, 110).
- Margin of error = multiple of the standard error, $\sigma_{\bar{y}} = \frac{\sigma}{\sqrt{n}}$ (Agresti and Finlay 2009, 117).
- For example, for 95% confidence level, the margin of error is $\pm 1.96\sigma_{\bar{y}}$ (have a look at the normal distribution).

Week 2–Hypothesis testing, experiments, difference in means

Null-hypothesis significance testing

TABLE 6.1: The Five Parts of a Statistical Significance Test

1. **Assumptions**
   Type of data, randomization, population distribution, sample size condition
2. **Hypotheses**
   Null hypothesis, H_0 (parameter value for "no effect")
   Alternative hypothesis, H_a (alternative parameter values)
3. **Test statistic**
   Compares point estimate to H_0 parameter value
4. ***P*-value**
   Weight of evidence against H_0; smaller P is stronger evidence
5. **Conclusion**
   Report *P*-value
   Formal decision (optional; see Section 6.4)

(Agresti and Finlay 2009, 147)

Significance test for a mean (t-test)

TABLE 6.3: The Five Parts of Significance Tests for Population Means

1. **Assumptions**
   Quantitative variable
   Randomization
   Normal population (robust, especially for two-sided H_a, large n)
2. **Hypotheses**
   H_0: $\mu = \mu_0$
   H_a: $\mu \neq \mu_0$ (or H_a: $\mu > \mu_0$ or H_a: $\mu < \mu_0$)
3. **Test statistic**
   $t = \dfrac{\bar{y} - \mu_0}{se}$ where $se = \dfrac{s}{\sqrt{n}}$
4. ***P*-value**
   In t curve, use
   P = Two-tail probability for H_a: $\mu \neq \mu_0$
   P = Probability to right of observed t-value for H_a: $\mu > \mu_0$
   P = Probability to left of observed t-value for H_a: $\mu < \mu_0$
5. **Conclusion**
   Report P-value. Smaller P provides stronger evidence against H_0 and supporting H_a. Can reject H_0 if $P \leq \alpha$-level.

(Agresti and Finlay 2009, 155)

Significance test for a difference in means (t-test)

What is a t-test for the difference in means?

- Null and alternative hypothesis: (Step 2) The means of two groups are identical, $\bar{y}_1 = \bar{y}_2$ or $\bar{y}_1 - \bar{y}_2 = 0$ (H_0), the means of two groups are different, $\bar{y}_1 \neq \bar{y}_2$ (H_a).
- Test statistics: (Step 3) "measures the number of standard errors between the estimate and the H_0 value" (Agresti and Finlay 2009, 192).

  $t = \frac{\textit{Estimate of parameter} - \textit{Null hypothesis value of parameter}}{\textit{Standard error of estimate}}$

  $t = \frac{(\bar{y}_1 - \bar{y}_2) - 0}{se}$, H_0 assumes $\bar{y}_2 - \bar{y}_1 = 0$, $se = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$

Causal effect

What is a causal effect?

- Causal effect: "change in some feature of the world that would result from a change to some other feature of the world", $Y_{T=1,i} - Y_{T=0,i} = Y_i^1 - Y_i^0$
- Counterfactual comparison: "outcome would be different in a counterfactual world in which the action was different" $\rightarrow$ what would be the state of Y, had X not occurred?
- Fundamental problem of causal inference: "we can only observe, at most, one of the two quantities–Y_{1i} or Y_{0i}–for any individual at a particular point in time" (Bueno de Mesquita and Fowler 2021, 164). $\rightarrow$ causal effect is unobservable

(Bueno de Mesquita and Fowler 2021, 159)

Sample average treatment effect, Difference in means

- Sample average treatment effect (SATE) is unobservable due to fundamental problem of causal inference → we only observe sample difference in means
- Sample difference in means is biased estimate of the true SATE

  → **Correlation does not imply causation**
- Baseline differences: "[d]ifference in the average potential outcome between two groups (e.g., the treated and untreated groups), even when those two groups have the same treatment status" → Confounders may cause baseline differences, which may cause bias (*omitted variable bias*)

(Bueno de Mesquita and Fowler 2021, 187)

Week 3–Contingency tables, correlation & bivariate regression

Week 4–Bivariate regression, inference & prediction

Chi-square test of independence

What is the Chi-square test of independence?

- Null and alternative hypothesis: Two variables are independent, $f_o = f_e$ (H_0), two variables are dependent, $f_o \neq f_e$ (H_a).
- Test statistics: "compares the observed frequencies in the contingency table with values that satisfy the null hypothesis of independence" (Agresti and Finlay 2009, 225), $X^2 = \sum \frac{(f_o - f_e)^2}{f_e}$

Correlation

How can we measure correlation?

- Covariance: covariance is the average of the product of deviations of two quantitative variables from the mean, $cov(X,Y) = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{n}$ (only interpret sign)
- Correlation: (correlation coefficient, Pearson correlation coefficient, Pearson's r, r) standardized average of the product of deviations of two variables from the mean (=standardized covariance), $r_{xy} = \frac{covariance(XY)}{S_X S_Y}$ (interpret magnitude, range -1 and 1)

Correlation

How can we test the statistical significance of correlation?

- Null and alternative hypotheses:
  - there is no association between X and Y, $\rho_{xy} = 0$ (H_0)
  - there is an association between X and Y, $\rho_{xy} \neq 0$ (H_a)
- Test statistic: $t = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}}$ (in R)
- Test statistic: $t = \frac{r}{\sqrt{1-r^2/n-2}}$ (in Agresti and Finlay 2009)

Ordinary least squares (OLS)

How are intercept and slope estimated?

- How do we find the line which best fits the data?
- Apply the OLS (Ordinary Least Squares) method, which minimizes the sum of squared errors (SSE).
- Sum of squared errors = the sum of squared differences between actual and predicted values of Y.
- $SSE = \sum_{i=1}^{n}(\hat{\epsilon}_i)^2 = \sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2 = \sum_{i=1}^{n}(Y_i - (\hat{\alpha} - \hat{\beta}X_i))^2$
  $\rightarrow$ minimize this!

Assumptions of linear regression

Assumptions about the error (ϵ_i), $Y_i = \alpha + \beta X_i + \epsilon_i$

$$\epsilon_i \sim N(0, \sigma^2)$$

* ϵ_i is normally distributed $\rightarrow$ needed for inference
* $E(\epsilon_i) = 0$, no bias $\rightarrow$ violated if error is not random, but correlated with omitted variable
* ϵ_i has constant variance σ^2 (Homoscedasticity $\leftrightarrow$ Heteroscedasticity)
* No autocorrelation, "Autocorrelation occurs when the stochastic terms for any two or more cases are systematically related to each other".
* X values are measured without error

(Kellstedt and Whitten 2018, 190–194)

Assumptions of linear regression

Assumptions about the model specification, $Y_i = \alpha + \beta X_i + \epsilon_i$

* No causal variables left out and no noncausal variables included
* Parametric linearity

(Kellstedt and Whitten 2018, 190–194)

Assumptions of linear regression

Minimal mathematical requirements, $Y_i = \alpha + \beta X_i + \epsilon_i$

* X must vary
* Number of observations must be larger than the number of predictors
* In multiple regression: No perfect multicollinearity

(Kellstedt and Whitten 2018, 190–194)

Inference about the slope

What is the t-test for the slope of a regression line?

- Null and alternative hypotheses:
  - there is no association between X and Y, $\beta = 0$ (H_0)
  - there is an association between X and Y, $\beta \neq 0$ (H_a)
- Test statistic: "measures the number of standard errors between the estimate and the H_0 value" (Agresti and Finlay 2009, 192).

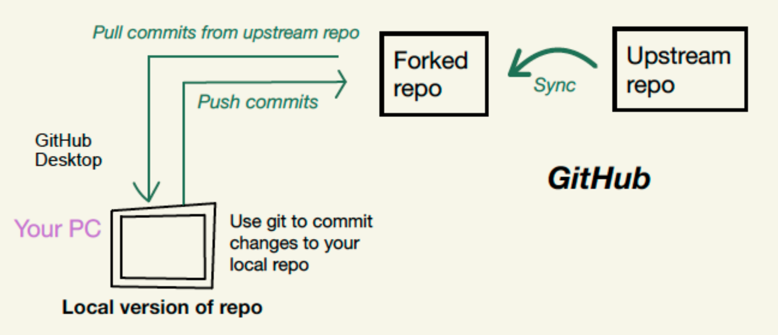
  $t = \frac{\textit{Estimate of parameter} - \textit{Null hypothesis value of parameter}}{\textit{Standard error of estimate}}$

  $t = \frac{\hat{\beta} - \beta_{H_0}}{se_{\hat{\beta}}} = \frac{\hat{\beta}}{se_{\hat{\beta}}}$, H_0 assumes $\beta = 0$

Software check

How to update your local repository? How to git pull?

Software check

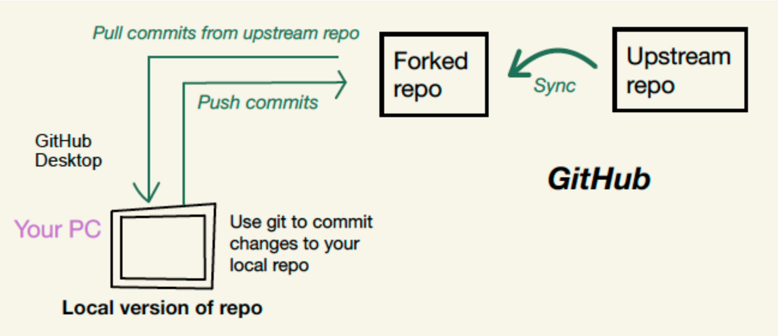


1. Synchronize fork
2. Fetch origin

Software check

How to update your repository on GitHub? How to git push?

Software check



1. Make sure your local repository is updated:
   * Synchronize fork
   * Fetch origin
2. Commit changes in file (local)
3. Push commits to fork

Software check

What to do in the event of merging conflicts?

Software check

What to do in the event of merging conflicts?

1. Find out which files are causing the conflict in GitHub Desktop
2. Decide which version to keep, and delete the other one (either in GitHub Desktop or manually)

References I

Agresti, Alan, and Barbara Finlay. 2009. *Statistical methods for the social sciences.* Essex: Pearson Prentice Hall.

Bueno de Mesquita, Ethan, and Anthony Fowler. 2021. *Thinking clearly with data: A guide to quantitative reasoning and analysis.* Princeton: Princeton University Press.

Kellstedt, Paul M., and Guy D. Whitten. 2018. *The fundamentals of political science research.* Cambridge: Cambridge University Press.