Oxford A1 - Differential Equations

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1 Sheet 1

> 1.1 Let $[a, b]$ be a closed and bounded interval of the real line and let $\{y_n\}_{n\geq 0}$ be a sequence of real-valued functions, each of which is defined on $[a, b]$. What does it mean to say that **the sequence converges uniformly on** $[a, b]$ **to a limit function** y**?** If each y_n is continuous on $[a, b]$ show that the uniform limit y is continuous on $[a, b]$ and that, when $n \to \infty$,
>
> $$\int_a^b |y_n(x) - y(x)|dx \to 0, \quad \int_a^b y_n(x)dx \to \int_a^b y(x)dx.$$

(a) Definition of uniform convergence

The sequence of functions $\{y_n\}_{n\geq 0}$ **converges uniformly on** $[a, b]$ **to** y if and only if for all $\epsilon > 0$ there exists an $m \in \mathbb{N}$ such that for all $n > m$ and for all $x \in [a, b]$, $|y_n(x) - y(x)| < \epsilon$.

(b) Show that the limit function is continuous

The claim is that if each y_n is continuous on $[a, b]$ then y is continuous on $[a, b]$. We are told that

1. $\{y_n\}_{n\geq 0}$ converges uniformly to y, and
2. each y_n is continuous on $[a, b]$.

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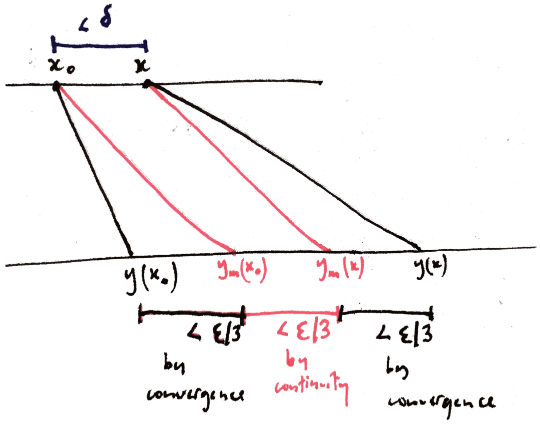
https://courses.maths.ox.ac.uk/node/5372

Informal illustration of proof:



Fix arbitrary $\epsilon > 0$ and $x_0 \in [a, b]$.

Let $m \in \mathbb{N}$ be such that $|y_m(x_0) - y(x_0)| < \epsilon/3$. Such an m exists because the $\{y_n\}$ converge uniformly to y.

Let δ be such that $|x - x_0| < \delta \implies |y_m(x) - y_m(x_0)| < \epsilon/3$. Such a δ exists because y_m is continuous on $[a, b]$.

Fix an arbitrary x such that $|x - x_0| < \delta$.

Now we have the following:

1. $|y(x_0) - y_m(x_0)| < \epsilon/3$ by convergence of the $\{y_n\}$
2. $|y_m(x_0) - y_m(x)| < \epsilon/3$ by continuity of y_m
3. $|y_m(x) - y(x)| < \epsilon/3$ by convergence of the $\{y_n\}$

Therefore $|y(x_0) - y(x)| < \epsilon$, proving continuity of y on $[a, b]$. $\square$

(Approximate time taken for reading and producing an answer: 4hrs)

(c) Show limit of definite integral I

Let $I_n = \int_a^b |y_n(x) - y(x)|\,\mathrm{d}x$.

The claim is that $\lim_{n\to\infty} I_n = 0$.

In other words $\forall \epsilon > 0 : \exists\, m \in \mathbb{N} : \forall\, n > m : |I_n - 0| < \epsilon$.

Fix an $\epsilon > 0$.

Since the $\{y_n\}$ converge uniformly to y, there exists an $m \in \mathbb{N}$ such that for all $n > m$ and for all $x \in [a, b]$

$$|y_n(x) - y(x)| < \epsilon/(b - a).$$

Therefore $\int_a^b |y_n(x) - y(x)| \, \mathrm{dx} < \epsilon$ for all $n > m$, as required. $\square$

(d) Show limit of definite integral II

The claim is that $\lim_{n\to\infty} \int_a^b y_n(x) \, \mathrm{dx} = \int_a^b y(x) \, \mathrm{dx}$.

In other words: $\forall \epsilon > 0 : \exists\, m \in \mathbb{N} : \forall\, n > m :$

$$\left| \left(\int_a^b y_n(x) \, \mathrm{dx} \right) - \left(\int_a^b y(x) \, \mathrm{dx} \right) \right| < \epsilon.$$

This is equivalent to: $\forall \epsilon > 0 : \exists\, m \in \mathbb{N} : \forall\, n > m :$

$$A_1 := \left| \int_a^b (y_n(x) - y(x)) \, \mathrm{dx} \right| < \epsilon.$$

From part (c) above, we know that: $\forall \epsilon > 0 : \exists\, m \in \mathbb{N} : \forall\, n > m :$

$$A_2 := \int_a^b |y_n(x) - y(x)| \, \mathrm{dx} < \epsilon.$$

Now[1] if the sign of $y_n(x) - y(x)$ is constant for all $x \in [a, b]$ (i.e. the graphs do not cross over), then $A_1 = A_2 < \epsilon$. Otherwise, there is some cancellation in the integral A_1 and $0 \le A_1 < A_2 < \epsilon$. So the same choice of m as was used in part (c) works here, since for that value of m, we have $A_1 < \epsilon$ as required. $\square$

(Approximate time taken for (c) and (d): 2hrs)

[1] This is related to the triangle inequality. I should prove it properly.

If $[a,b]=[0,1]$ and $y_n(x)=nxe^{-nx^2}$ show that, for each $x\in[0,1], y_n(x)\to 0$ but $\displaystyle\int_0^1 y_n(x)dx\to\frac{1}{2}$. Thus the convergence must be non-uniform. Show that

$$\max_{0\le x\le 1} y_n(x)=\sqrt{\frac{n}{2e}}$$

and sketch the graph of $y_n(x)$ versus x.

To show that $y_n(x):=\frac{nx}{e^{nx^2}}\to 0$ for all $x\in[0,1]$, first note that it is true for $x=0$ since $y_n(0)=0$ for all $n\in\mathbb{N}$. So we have to show it is true for $x\in(0,1]$.

Fix $x\in(0,1]$ and define $f(\alpha)=\frac{\alpha x}{e^{\alpha x}}$ for $\alpha\in\mathbb{R}$. $\lim_{\alpha\to\infty}f(\alpha)$ is an indeterminate form $\frac{\infty}{\infty}$ and we can use l'Hôpital's rule, differentiating with respect to α:

$$\lim_{\alpha\to\infty}\frac{\alpha x}{e^{\alpha x^2}}=\lim_{\alpha\to\infty}\frac{x}{x^2e^{\alpha x^2}}=0.$$

Since $f(\alpha)=y_n$ at integer values of α it follows that $\lim_{n\to\infty}y_n(x)=0$ for all $x\in(0,1]$. $\square$

For the limit of the definite integral we have

$$\int_0^1 nxe^{-nx^2}\,\mathrm{dx}=\Big[-\frac{1}{2}e^{-nx^2}\Big]_0^1=\frac{1}{2}(1-e^{-n}),$$

and so $\lim_{n\to\infty}\int_0^1 y_n(x)\,\mathrm{dx}=\frac{1}{2}$. $\square$

To find the maximum value attained by $y_n(x)$ for $x\in[0,1]$, note that the derivative is

$$\frac{\mathrm{d}\,y_n(x)}{\mathrm{dx}}=nx(-2nx)e^{-nx^2}+ne^{-nx^2}=ne^{-nx^2}(1-2nx^2),$$

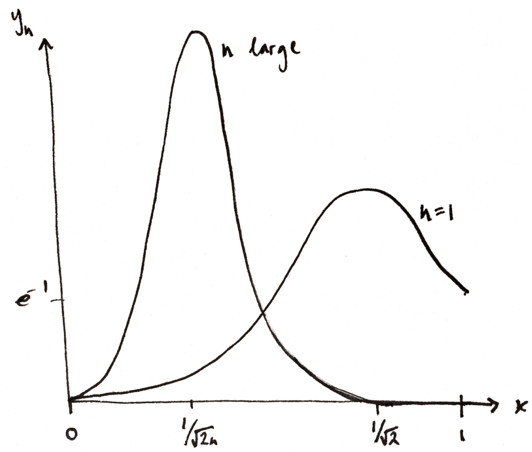
and therefore that the only solution to $\frac{\mathrm{d}\,y_n(x)}{\mathrm{dx}}=0$ for $x\in[0,1]$ is $x=\frac{1}{\sqrt{2n}}$.

The second derivative is

$$ne^{-nx^2}(-4nx)-2n^2xe^{-nx^2}(1-2nx^2)=2n^2xe^{-nx^2}\left(2nx^2-3\right).$$

This is negative at the critical point $x=\frac{1}{\sqrt{2n}}$ showing that it is a maximum. Therefore

$$\max_{x\in[0,1]}y_n(x)=n\frac{1}{\sqrt{2n}}e^{-n(\frac{1}{\sqrt{2n}})^2}=\sqrt{\frac{n}{2e}}.\quad\square$$



(Approximate time for reading and producing answer: 3 hrs)

1.2 Let $\sum_{n=0}^{\infty} u_n$ be a series of real-valued functions defined on $[a, b]$. State the **Weierstrass M-test** for the uniform convergence of the series.

Show that the series $\sum_{n=0}^{\infty} (-1)^n \frac{\cos\, nx}{1+n^2}$ converges uniformly on $[-\pi, \pi]$.

Weierstrass M-test

Suppose

1. there exists a sequence $(M_n)_{n\geq 0}$ such that $|u_n(x)| \leq M_n$ for all $n \geq 0$ and for all $x \in [a, b]$, and
2. the series $\sum_{n=0}^{\infty} M_n$ converges.

Then the series of functions $\sum_{n=0}^{\infty} u_n$ converges uniformly on $[a, b]$.

Define $u_n(x) = (-1)^n \frac{\cos nx}{1+n^2}$.

Let $M_n = \frac{1}{1+n^2}$ and note that $|u_n| \leq M_n$ for all $x \in [-\pi, \pi]$.

Note that the integral $\int_1^{\infty} \frac{1}{x^2}\,\mathrm{dx} = [-\frac{1}{x}]_1^{\infty} = 1$ converges, therefore the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the integral test for convergent series.

Now $M_n < \frac{1}{n^2}$ for $n > 0$, so the series $\sum_{n=1}^{\infty} M_n$ converges. Therefore the series $\sum_{n=0}^{\infty} M_n$ also converges, since its tail converges.

Therefore the series $\sum_{n=0}^{\infty} u_n$ converges uniformly on $[-\pi, \pi]$.

1.3 Consider the initial-value problems

$$\begin{aligned} y' &= x^2+y^2, \quad y(0)=0, \qquad (1) \\ y' &= (1-2x)y, \quad y(0)=1. \qquad (2) \end{aligned}$$

In each case find y_0, y_1, y_2, y_3, where $\{y_n\}_{n\geq 0}$ is the sequence of Picard approximations. By considering the behaviour of x^2+y^2 on the square $\{(x,y) : |x| \leq \frac{1}{\sqrt{2}},\ |y| \leq \frac{1}{\sqrt{2}}\}$ and appealing to Picard's theorem show that in case (1) the sequence converges uniformly for $|x| \leq \frac{1}{\sqrt{2}}$.

In case (2), use Picard's theorem to show that the problem has a unique solution for all x. Now find the solution explicitly and, by expanding as a series, show that the sequence $\{y_n\}_{n\geq 0}$ converges to the solution.

Consider an ODE $y' = f\Big(x, y(x)\Big)$ with initial condition $y(a) = b$.

The sequence of Picard approximations are defined by

$$\begin{aligned} y_0(x) &= b \\ y_{n+1}(x) &= b + \int_a^x f\Big(t, y_n(t)\Big)\,\mathrm{dt}\,. \end{aligned}$$

(1)

$$\begin{aligned} y_0(x) &= 0 \\ y_1(x) &= 0 + \int_0^x t^2 + 0^2\,\mathrm{dt} \\ &= \frac{x^3}{3} \\ y_2(x) &= 0 + \int_0^x t^2 + \left(\frac{t^3}{3}\right)^2 \mathrm{dt} = 0 + \int_0^x t^2 + \frac{t^6}{9} \\ &= \frac{x^3}{3} + \frac{x^7}{63} \\ y_3(x) &= 0 + \int_0^x t^2 + \left(\frac{t^3}{3} + \frac{t^7}{63}\right)^2 \mathrm{dt} = 0 + \int_0^x t^2 + \frac{t^6}{9} + \frac{2t^{10}}{189} + \frac{t^{14}}{3969}\,\mathrm{dt} \\ &= \frac{x^3}{3} + \frac{x^7}{63} + \frac{2x^{11}}{2079} + \frac{x^{15}}{59535} \end{aligned}$$

We need to show that this situation satisfies the requirements of Picard's theorem.

Define $v(x, y) = x^2 + y^2$ and let $h = \frac{1}{\sqrt{2}}$ be half the width of the square, which is centered at $(0, 0)$.

1. $|v|$ **must be bounded by some** $M > 0$ **in the rectangle, with** $Mh \leq h$
   True. The maximum value attained by $|v|$ in the rectangle is $M = 1$.

2. v **must be Lipschitz continuous in** y
   True. The maximum value of $|\frac{\partial v}{\partial y}|$ on the rectangle is $2 \cdot \frac{1}{\sqrt{2}} = \sqrt{2}$. Let (x, y_0) and (x, y_1) be two points lying on a line parallel to the y axis in the rectangle. By the Mean Value Theorem, the partial derivative at some point in this line is equal to the slope of the line joining $v(x, y_0)$ and $v(x, y_1)$. Therefore the slope of this line cannot exceed $\sqrt{2}$. I.e. $|v(x, y_1) - v(x, y_0)| \leq \sqrt{2}|y_1 - y_0|$; v is Lipschitz continuous in the y direction within the rectangle.

Therefore the sequence of functions given by the Picard iterates $y_0, y_1, \ldots$ converge uniformly to a solution of the ODE on $|x| \leq \frac{1}{\sqrt{2}}$.

1.3 Consider the initial-value problems

$$
\begin{aligned}
y' &= x^2 + y^2, \quad & y(0) = 0, \qquad & (1)\\
y' &= (1-2x)y, \quad & y(0) = 1. \qquad & (2)
\end{aligned}
$$

In each case find y_0, y_1, y_2, y_3, where $\{y_n\}_{n\geq 0}$ is the sequence of Picard approximations. By considering the behaviour of x^2+y^2 on the square $\{(x,y) : |x| \leq \frac{1}{\sqrt{2}},\ |y| \leq \frac{1}{\sqrt{2}}\}$ and appealing to Picard's theorem show that in case (1) the sequence converges uniformly for $|x| \leq \frac{1}{\sqrt{2}}$.

In case (2), use Picard's theorem to show that the problem has a unique solution for all x. Now find the solution explicitly and, by expanding as a series, show that the sequence $\{y_n\}_{n\geq 0}$ converges to the solution.

(2)

Show that a unique solution exists for all x

The ODE is

$$y'(x) = (1-2x)y.$$

Let $v(x,y) = (1-2x)y$ and define an arbitrary rectangle $\{(x,y)\ :\ |x| \leq h, |y| \leq k\}$.

1. **$|v|$ must be bounded by some $M > 0$ in the rectangle, with $Mh \leq k$**
   False. The maximum value attained by $|v|$ in the rectangle is $M = (1+2h)k$. Therefore $Mh = h(1+2h)k > k$. But this contradicts the question.

2. **v must be Lipschitz continuous in y**
   The maximum value of $|\frac{\partial v}{\partial y}|$ on the rectangle is $1+2h$, so v is Lipschitz continuous in the y direction.

Find the solution via Picard's theorem

Picard iterates are

$$
\begin{aligned}
y_0(x) &= 1 \\
y_1(x) &= 1 + \int_0^x (1-2t)\cdot 1 \,\mathrm{dt} \\
&= 1 + [t - t^2]_0^x \\
&= 1 + x - x^2 \\
y_2(x) &= 1 + \int_0^x (1-2t)(1+t-t^2)\,\mathrm{dt} \\
&= 1 + \int_0^x 1 + t - t^2 - 2t - 2t^2 + 2t^3 \,\mathrm{dt} \\
&= 1 + \int_0^x 1 - t - 3t^2 + 2t^3 \,\mathrm{dt} \\
&= 1 + x - \frac{1}{2}x^2 - x^3 + 8x^4 \\
y_3(x) &= 1 + \int_0^x (1-2t)(1 + t - \frac{1}{2}t^2 - t^3 + \frac{1}{2}t^4) \\
&= 1 + \int_0^x 1 + t - \frac{1}{2}t^2 - t^3 + \frac{1}{2}t^4 - 2t - 2t^2 + t^3 + 2t^4 - t^5 \\
&= 1 + \int_0^x 1 - t - \frac{5}{2}t^2 + \frac{5}{2}t^4 - t^5 \\
&= 1 + x - \frac{1}{2}x^2 - \frac{5}{6}x^3 + \frac{1}{2}x^5 - \frac{1}{6}x^6
\end{aligned}
$$

Check solution with sympy

```python
from sympy import latex, integrate, symbols
t, y, tau = symbols('t y tau')

def picard(f, y_prev, a, b):
    return b + integrate(f.subs([(t, tau), (y, y_prev)]), (tau, a, t))

a, b = 0, 1

y = b
for i in [1, 2, 3]:
    f = (1 - 2*t) * y
    y_next = picard(f, y, a, b)
    print(latex(y_next))
    y = y_next
```

$$-t^2 + t + 1$$
$$\frac{t^4}{2} - t^3 - \frac{t^2}{2} + t + 1$$
$$-\frac{t^6}{6} + \frac{t^5}{2} - \frac{5t^3}{6} - \frac{t^2}{2} + t + 1$$

Find the solution explicitly

Find general solution using separation of variables:

$$\begin{aligned} \frac{\mathrm{dy}}{\mathrm{dx}} &= (1-2x)y \\ \frac{1}{y}\frac{\mathrm{dy}}{\mathrm{dx}} &= (1-2x) \\ \int \frac{1}{y}\frac{\mathrm{dy}}{\mathrm{dx}}\,\mathrm{dx} &= \int (1-2x)y\,\mathrm{dx} \end{aligned}$$

(Note[2])

$$\begin{aligned} \log y &= x - x^2 + C \\ y &= Ae^{x(1-x)} \end{aligned}$$

Use initial values to find particular solution:

$$\begin{aligned} 1 &= Ae^0 = A \\ y(x) &= e^{x-x^2} \end{aligned}$$

The first few derivatives, evaluated at $x = 0$, are

$$\begin{aligned} y^{(1)}(0) &= (1-2x)e^{x-x^2} \\ &= 1 \\ y^{(2)}(0) &= (1-2x)^2 e^{x-x^2} + (-2)e^{x-x^2} \\ &= (4x^2 - 4x - 1)e^{x-x^2} \\ &= -1 \\ y^{(3)}(0) &= (4x^2 - 4x - 1)(1-2x)e^{x-x^2} + (8x-4)e^{x-x^2} \\ &= (4x^2 - 4x - 1 - 8x^3 + 8x^2 + 2x + 8x - 4)e^{x-x^2} \\ &= (-8x^3 + 12x^2 + 6x - 5)e^{x-x^2} \\ &= -5 \end{aligned}$$

[2] We don't use the Leibnitz notation to perform "cancellations". This is asking for the antiderivative, with respect to x, of $\frac{1}{y(x)}y'(x)$, the answer to which is $\log(y(x)) + C$.

The Taylor series expansion of the solution around $x = 0$ is

$$\begin{aligned} y(x) &= e^{x-x^2} \\ &= \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n \\ &= 1 + x - \frac{x^2}{2} - \frac{5}{6}x^3 + \dots. \end{aligned}$$

So the first few terms appear to match the Picard iterates.

TODO: prove that the Picard iterates converge to the Taylor series.

> 1.4 Consider the initial-value problem
>
> $$y' = xy^{1/3}, \quad y(0) = b,$$
>
> a) (i) Does the function $F(x, y) = xy^{1/3}$ satisfy a Lipschitz condition on the rectangle $\{(x, y) : |x| \le h,\ |y| \le k\}$, where $h > 0$ and $k > 0$?

The partial derivative with respect to y is $\frac{\partial F}{\partial y} = \frac{x}{3} y^{-2/3}$.

Note that the rectangle necessarily includes the origin. But $\frac{\partial F}{\partial y} \to \infty$ as $y \to 0$. So F does not satisfy a Lipschitz condition in the y direction.

(ii) If $b > 0$ use Picard's theorem to show that there is a unique solution on an interval $[-h, h]$, for a suitable $h > 0$ which you should specify (you must check carefully that the assumptions of Picard's theorem are satisfied).

First note that we can solve this by separation-of-variables:

$$\begin{aligned}
y' &= xy^{1/3} \\
\int y^{-1/3}\,\mathrm{dy} &= \int x\,\mathrm{dx} \\
\frac{3}{2}y^{2/3} &= \frac{1}{2}x^2 + C \\
y &= \left(\frac{1}{3}x^2 + C\right)^{3/2} \\
y(0) &= C^{3/2} = b \\
y &= \left(\frac{1}{3}x^2 + b^{2/3}\right)^{3/2}
\end{aligned}$$

iii) If $b = 0$, show that for any $c > 0$ there is a solution y which is identically zero on $[-c, c]$ and positive when $|x| > c$.

b) [Optional] Now return to the case $b > 0$. Consider the set $R = \{(x, y) : y \geq b, |x| \leq h\}$. By working in this R, and adapting the proof of Picard's theorem, prove that in fact there is a unique solution of the problem on $|x| \leq h$ for any h and hence that there is global existence of solutions.

1.5 Suppose that $f : [a, b] \to \mathbb{R}$ and $K : [a, b] \times [a, b] \to \mathbb{R}$ are continuous. Consider the integral equation for $y(x)$

$$y(x) = f(x) + \int_a^x K(x, t)y(t)dt, \quad x \in [a, b].$$

For $x \in [a, b]$ define

$$\begin{aligned} y_0(x) &= f(x) \\ y_{n+1}(x) &= f(x) + \int_a^x K(x, t)y_n(t)dt. \end{aligned}$$

Adapt the proof of Picard's theorem to show that y_n converges uniformly to a solution of the integral equation for all $x \in [a, b]$. [You may assume that if $y : [a, b] \to \mathbb{R}$ is continuous then so too is $f(x) + \int_a^x K(x, t)y(t)dt$ for $x \in [a, b]$.]

Now show that the solution is unique.

Prove also that the solution depends continuously on f. [You will need to decide what this means.]

We have to show the following:

1. that the sequence of functions converges uniformly to a limiting function,
2. that the limiting function is a solution, and
3. that the solution is unique, and
4. that the solution "depends continuously on f".

1. Proof that the sequence converges uniformly to a limiting function

Proof. Restrict attention to a rectangle with width $2h$ and height $2k$, centered on $(a, f(a))$.

Note that, since f and K are continuous, there exist bounds $B, C \in \mathbb{R}$ such that $\left|f(x)\right| < B$ and $\left|K(x, x')\right| \leq C$, for $x, x' \in [a, b]$.

Define $y_\infty = \lim\limits_{n\to\infty} y_n$.

Define $e_n(x) = y_{n+1}(x) - y_n(x)$.

Note that $y_\infty(x) = \sum_{i=0}^\infty e_n(x) + y_0(x) = \sum_{i=0}^\infty e_n(x) + f(x)$.

Therefore, to show that $(y_n)_{n\geq 0}$ converges uniformly, it suffices to show that $\sum_{i=0}^\infty e_n(x)$ converges uniformly.

We will use the Weierstrass M-test for this. Therefore, for each n we need to find a constant bound W_n such that $|y_n(x)| \le W_n$ for all $x \in [a, b]$, and we need to show that the sequence W_n converges.

Note that for $n \ge 1$

$$\begin{aligned}
|e_n(x)| &= \left| \int_a^x K(x,t)\Big(y_n(x) - y_{n-1}(x)\Big)\,\mathrm{dt} \right| \\
&\le \left| \int_a^x \Big|K(x,t)\Big|\Big|y_n(x) - y_{n-1}(x)\Big|\,\mathrm{dt} \right| \\
&\le \left| \int_a^x \Big|K(x,t)\Big|\Big|e_{n-1}(t)\Big|\,\mathrm{dt} \right|.
\end{aligned}$$

The first two terms are

$$\begin{aligned}
|e_0(x)| &= y_1(x) - f(x) \\
&= \int_a^x K(x,t)f(t)\,\mathrm{dt} \\
&\le BC|x-a| \\
|e_1(x)| &\le \left| \int_a^x \Big|K(x,t)\Big|\Big|e_0(t)\Big|\,\mathrm{dt} \right| \\
&\le BC\left| \int_a^x \Big|K(x,t)\Big|\Big|t-a\Big|\,\mathrm{dt} \right| \\
&\le BC^2\frac{|x-a|^2}{2}.
\end{aligned}$$

Let $W_n = BC^{n+1}\frac{h^{n+1}}{(n+1)!}$.

It seems that $|e_n(x)| \le W_n$ for all $n \ge 0$. To prove this, note that it is true for $n = 0$. For induction, suppose that it is true for n. Then

$$\begin{aligned}
|e_{n+1}(x)| &\le \left| \int_a^x \Big|K(x,t)\Big|\Big|e_n(t)\Big|\,\mathrm{dt} \right| \\
&\le \left| \int_a^x \Big|K(x,t)\Big|BC^{n+1}\frac{(t-a)^{n+1}}{(n+1)!}\,\mathrm{dt} \right| \\
&\le BC^{n+2}\frac{(x-a)^{n+2}}{(n+2)!} \\
&\le BC^{n+2}\frac{h^{n+2}}{(n+2)!} \\
&= W_{n+1},
\end{aligned}$$

as required. Therefore $|e_n| \leq W_n$ holds for all $n \geq 0$. The Ratio Test shows that the sequence $(W_n)_{n\geq 0}$ converges to zero:

$$\lim_{n\to\infty} \frac{W_{n+1}}{W_n} = \lim_{n\to\infty} \frac{BC^{n+2}\frac{h^{n+2}}{(n+2)!}}{BC^{n+1}\frac{h^{n+1}}{(n+1)!}} = \lim_{n\to\infty} \frac{Ch}{n+2} = 0.$$

Therefore the series $\sum_{i=0}^{\infty} e_n(x)$ converges uniformly by the Weierstrass M-test, and therefore $(y_n)_{n\geq 0}$ converges uniformly to a limiting function, which we will denote as y_∞. □

2. Proof that the limiting function is a solution

Proof. To prove that y_∞ is a solution we need to show that

$$y_\infty(x) = \lim_{n\to\infty} y_n(x) = f(x) + \int_a^x K(x,t)y_\infty(t)\,\mathrm{dt} \qquad \text{and} \qquad y_\infty(a) = f(x).$$

The second requirement, $y_\infty(a) = f(x)$, is clearly true.

The definition of y_n is

$$y_n(x) = f(x) + \int_a^x K(x,t)y_{n-1}(t)\,\mathrm{dt}\,.$$

If it were valid to take the limit inside the integral then we would have

$$y_\infty(x) = \lim_{n\to\infty} y_n(x) = f(x) + \int_a^x K(x,t)y_\infty(t)\,\mathrm{dt}$$

as required. To justify taking the limit inside the integral it's sufficient to prove that $K(x,t)y_n(t)$ converges uniformly to $K(x,t)y_\infty(t)$. But this simply requires that $y_n(t)$ converges uniformly to $y_\infty(t)$, which has been proved already. □

3. Proof that the solution is unique

Proof. Suppose that Y is a solution and define $e_n(x) = Y(x) - y_n(x)$. We will show that $\lim_{n\to\infty} |e_n(x)| = 0$.

Recall that we are working in a rectangle with width $2h$ and height $2k$, centered on $(a, f(a))$.

We have

$$\begin{aligned}
|e_0(x)| &= \left| \int_a^x K(x,t)Y(t)\,\mathrm{dt} \right| \\
&\leq \left| \int_a^x \Big|K(x,t)\Big| \Big|Y(t)\Big|\,\mathrm{dt} \right| \\
&\leq Ck|x-a|.
\end{aligned}$$

For $n \geq 1$ we have

$$\begin{aligned}
|e_n(x)| &= \left| \int_a^x K(x,t)\Big(Y(t) - y_{n-1}(t)\Big)\,\mathrm{dt} \right| \\
&\leq \left| \int_a^x \Big|K(x,t)\Big|\Big|d_{n-1}(t)\Big|\,\mathrm{dt} \right|
\end{aligned}$$

For induction, suppose that $|e_n(x)| \leq C^{n+1}k\frac{|t-a|^{n+1}}{(n+1)!}$. This is true for $n = 0$. For $n+1$, we have

$$\begin{aligned}
|e_{n+1}(x)| &\leq \left| \int_a^x \Big|K(x,t)\Big|\Big|d_n(t)\Big|\,\mathrm{dt} \right| \\
&= \left| \int_a^x \Big|K(x,t)\Big|C^{n+1}k\frac{|t-a|^{n+1}}{(n+1)!}\,\mathrm{dt} \right| \\
&= C^{n+2}k\frac{|t-a|^{n+2}}{(n+2)!} \\
&\leq C^{n+2}k\frac{h^{n+2}}{(n+2)!},
\end{aligned}$$

which, as shown above, converges to 0 as $n \to \infty$. Therefore $\lim\limits_{n\to\infty} |e_n(x)| = 0$ and therefore if Y is a solution then $y_\infty = Y$. $\square$

3. Proof that the solution "depends continuously on f"

I've attempted to prove two similar theorems that match the description: "depends continuously on f".

Let $f, g : [a, b] \to \mathbb{R}$ and let $y_f(x)$ and $y_g(x)$ be solutions to the respective IVPs:

$$\begin{aligned} y_f(x) &= f(x) + \int_a^x K(x,t) y_f(t) \, \mathrm{dt} \\ y_g(x) &= g(x) + \int_a^x K(x,t) y_g(t) \, \mathrm{dt} \,. \end{aligned}$$

Theorem (Version 1: "pointwise" continuity)**.** *For all $x \in [a, b]$ and for all $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\Big| f(x) - g(x) \Big| < \delta \implies \Big| y_f(x) - y_g(x) \Big| < \epsilon.$$

Proof. We have

$$\begin{aligned} \Big| y_f(x) - y_g(x) \Big| &\leq \Big| f(x) - g(x) \Big| + \left| \int_a^x K(x,t) y_f(t) - y_g(t) \, \mathrm{dt} \right| \\ &\leq \Big| f(x) - g(x) \Big| + C \left| \int_a^x \Big| y_f(t) - y_g(t) \Big| \, \mathrm{dt} \right|, \end{aligned}$$

therefore by Gronwall's inequality

$$\begin{aligned} \Big| y_f(x) - y_g(x) \Big| &\leq \Big| f(x) - g(x) \Big| e^{C|x-a|} \\ &\leq \Big| f(x) - g(x) \Big| e^{Ch}. \end{aligned}$$

Therefore we can choose $\delta(\epsilon) = e^{-Ch}\epsilon$. $\square$

Let $\|g\|$ be the sup norm for a real-valued function g defined on $[a,b]$:

$$\|g\| := \sup_{x\in[a,b]} \Big|g(x)\Big|.$$

Theorem (Version 2: continuity in function space)**.** *For all* $\epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\|f-g\| < \delta \implies \|y_f - y_g\| < \epsilon.$$

Proof. We have

$$(y_f - y_g)(x) = (f-g)(x) + \int_a^x K(x,t)(y_f - y_g)(t)\,\mathrm{dt}\,.$$

Therefore

$$\begin{aligned}
\|y_f - y_g\| &= \sup_{x\in[a,b]} \left|(f-g)(x) + \int_a^x K(x,t)(y_f-y_g)(t)\,\mathrm{dt}\right| \\
&\le \sup_{x\in[a,b]} \Big|(f-g)(x)\Big| + \sup_{x\in[a,b]} \left|\int_a^x K(x,t)(y_f-y_g)(t)\,\mathrm{dt}\right| \\
&= \|f-g\| + \left\|\int_a^x K(x,t)(y_f-y_g)(t)\,\mathrm{dt}\right\|.
\end{aligned}$$

$\square$

2 Sheet 2

Systems of non-linear ODEs.

2.1 The aim of this question is to fill in the details of the proof of Theorem 1.6 in the lecture notes of Picard's theorem for a system of two first order ODEs via the CMT.

Consider the system of first order ODEs, for the functions y_1 and y_2

$$
\begin{aligned}
y_1'(x) &= f_1(x, y_1(x), y_2(x)) && (1)\\
y_2'(x) &= f_2(x, y_1(x), y_2(x)) && (2)\\
\text{with initial condition} \quad y_1(a) &= b_1, \quad y_2(a) = b_2. && (3)
\end{aligned}
$$

If we write

$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \underline{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix};$$

then we can write the problem (1), (2), (3) in vector form as

$$
\begin{aligned}
\underline{y}'(x) &= \underline{f}(x, \underline{y}(x)), && (4)\\
\underline{y}(a) &= \underline{b}, && (5)
\end{aligned}
$$

We will use the l^1 norm in $\mathbb{R}^2$, $||(y_1, y_2)||_1 = |y_1| + |y_2|$. Let $B_k(\underline{b})$ be the disc in $\mathbb{R}^2$, centre $\underline{b}$, radius k. Define the set $S = \{(x, \underline{y}) \in \mathbb{R}^3 : |x - a| \le h,\ \underline{y} \in B_k(\underline{b})\}$. We assume $\underline{f}$ is continuous on the set S, with $\sup\limits_S ||\underline{f}(x, \underline{y})||_1 \le M$, and for $x \in [a-h, a+h]$, $\underline{f}(x, \underline{y})$ is Lipschitz continuous with respect to $\underline{y}$ on S. That is, there exists L such that for $x \in [a-h, a+h]$ and $\underline{u}, \underline{v} \in B_k(\underline{b})$,

$$||\underline{f}(x, \underline{u}) - \underline{f}(x, \underline{v})||_1 \le L||\underline{u} - \underline{v}||_1. \qquad (6)$$

We will work in the space $\mathcal{C}_h = \mathcal{C}([a-h, a+h]; B_k(\underline{b}))$ of continuous functions from $[a-h, a+h]$ to the disc $B_k(\underline{b})$ in $\mathbb{R}^2$, with the sup norm defined for $\underline{y} \in C_h$ by

$$||\underline{y}||_{\sup} := \sup_{x \in [a-h, a+h]} ||\underline{y}(x)||_1.$$

We can write the initial value problem (4), (5) as an integral equation

$$\underline{y}(x) = \underline{b} + \int_a^x \underline{f}(s, \underline{y}(s))ds \qquad (7)$$

where by the integral we mean that we integrate componentwise.

Now we define

$$(T\underline{y})(x) = \underline{b} + \int_a^x \underline{f}(s, \underline{y}(s))ds$$

so we can write equation (6) as a fixed point problem in $\mathcal{C}_\eta$, for $0 < \eta \le h$.

$$\underline{y} = T\underline{y}.$$

(i) Prove that for $\underline{g} \in \mathcal{C}_h$,

$$\left\| \int_a^x \underline{g}(t) dt \right\|_1 \leq \left| \int_a^x \|\underline{g}(t)\|_1 \, dt \right| .$$

[You may assume that if $h : [a, x] \to \mathbb{R}$ is continuous then $\left| \int_a^x h(t) dt \right| \leq \left| \int_a^x |h(t)| \, dt \right|$.]

$$\begin{aligned}
\left\| \int_a^x \underline{g}(t) \, \mathrm{dt} \right\|_1 &:= \left| \int_a^x g_1(t) \, \mathrm{dt} \right| + \left| \int_a^x g_2(t) \, \mathrm{dt} \right| \\
&\leq \left| \int_a^x |g_1(t)| \, \mathrm{dt} \right| + \left| \int_a^x |g_2(t)| \, \mathrm{dt} \right| \\
&\leq \left| \int_a^x |g_1(t)| \, \mathrm{dt} + \int_a^x |g_2(t)| \, \mathrm{dt} \right| \\
&=: \left| \int_a^x \|\underline{g}(t)\|_1 \, \mathrm{dt} \right|
\end{aligned}$$

(ii) Prove that for suitable $0 < \eta \leq h$, T satisfies the conditions of the CMT so has a unique fixed point. Explain why this solution is also the unique solution of (4), (5).

We need to show that $T : \mathcal{C}_h \to \mathcal{C}_h$ and that T is a contraction.

Let $u, v \in \mathcal{C}_h$.

Claim. *$T(u) \in \mathcal{C}_h$ for all $u \in \mathcal{C}_h$.*

Claim. *There exists $0 < K < 1$ such that $\|T(y)\|_1 \leq K \|y\|_1$ for all $y \in \mathcal{C}_h$.*

Proof.

$$\begin{aligned}
\|T(\underline{y})\|_1 &= \sup_{x \in [a,b]} \left\| \int_a^x \underline{f}(s, \underline{y}(s)) \, \mathrm{ds} \right\|_1 \\
&= \left| \int_a^x f_1(s, y_1(s)) \, \mathrm{ds} \right| + \left| \int_a^x f_2(s, y_2(s)) \, \mathrm{ds} \right|
\end{aligned}$$

□