Oxford A1 - Differential Equations

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1 Sheet 1

> 1.1 Let $[a, b]$ be a closed and bounded interval of the real line and let $\{y_n\}_{n\geq 0}$ be a sequence of real-valued functions, each of which is defined on $[a, b]$. What does it mean to say that **the sequence converges uniformly on** $[a, b]$ **to a limit function** y**?** If each y_n is continuous on $[a, b]$ show that the uniform limit y is continuous on $[a, b]$ and that, when $n \to \infty$,
>
> $$\int_a^b |y_n(x) - y(x)|dx \to 0, \quad \int_a^b y_n(x)dx \to \int_a^b y(x)dx.$$

(a) Definition of uniform convergence

The sequence of functions $\{y_n\}_{n\geq 0}$ **converges uniformly on** $[a, b]$ **to** y if and only if for all $\epsilon > 0$ there exists an $m \in \mathbb{N}$ such that for all $n > m$ and for all $x \in [a, b]$, $|y_n(x) - y(x)| < \epsilon$.

(b) Show that the limit function is continuous

The claim is that if each y_n is continuous on $[a, b]$ then y is continuous on $[a, b]$. We are told that

1. $\{y_n\}_{n\geq 0}$ converges uniformly to y, and
2. each y_n is continuous on $[a, b]$.

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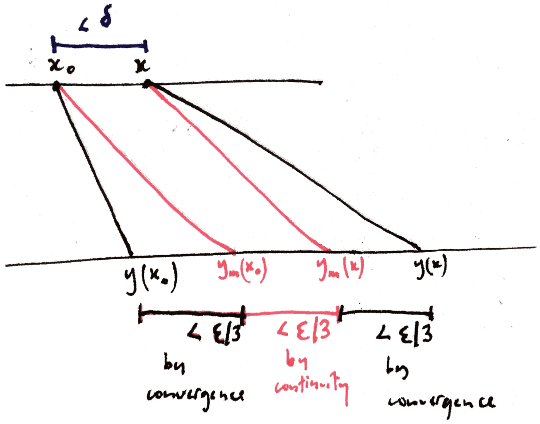
https://courses.maths.ox.ac.uk/node/5372

Informal illustration of proof:



Fix arbitrary $\epsilon > 0$ and $x_0 \in [a, b]$.

Let $m \in \mathbb{N}$ be such that $|y_m(x_0) - y(x_0)| < \epsilon/3$. Such an m exists because the $\{y_n\}$ converge uniformly to y.

Let δ be such that $|x - x_0| < \delta \implies |y_m(x) - y_m(x_0)| < \epsilon/3$. Such a δ exists because y_m is continuous on $[a, b]$.

Fix an arbitrary x such that $|x - x_0| < \delta$.

Now we have the following:

1. $|y(x_0) - y_m(x_0)| < \epsilon/3$ by convergence of the $\{y_n\}$
2. $|y_m(x_0) - y_m(x)| < \epsilon/3$ by continuity of y_m
3. $|y_m(x) - y(x)| < \epsilon/3$ by convergence of the $\{y_n\}$

Therefore $|y(x_0) - y(x)| < \epsilon$, proving continuity of y on $[a, b]$. $\square$

(Approximate time taken for reading and producing an answer: 4hrs)

(c) Show limit of definite integral I

Let $I_n = \int_a^b |y_n(x) - y(x)| \, \mathrm{dx}$.

The claim is that $\lim_{n\to\infty} I_n = 0$.

In other words $\forall \epsilon > 0 : \exists\, m \in \mathbb{N} : \forall\, n > m : |I_n - 0| < \epsilon$.

Fix an $\epsilon > 0$.

Since the $\{y_n\}$ converge uniformly to y, there exists an $m \in \mathbb{N}$ such that for all $n > m$ and for all $x \in [a, b]$

$$|y_n(x) - y(x)| < \epsilon/(b - a).$$

Therefore $\int_a^b |y_n(x) - y(x)|\,\mathrm{dx} < \epsilon$ for all $n > m$, as required. $\square$

(d) Show limit of definite integral II

The claim is that $\lim_{n\to\infty} \int_a^b y_n(x)\,\mathrm{dx} = \int_a^b y(x)\,\mathrm{dx}$.

In other words: $\forall \epsilon > 0 : \exists\, m \in \mathbb{N} : \forall\, n > m :$

$$\Big| \left(\int_a^b y_n(x)\,\mathrm{dx} \right) - \left(\int_a^b y(x)\,\mathrm{dx} \right) \Big| < \epsilon.$$

This is equivalent to: $\forall \epsilon > 0 : \exists\, m \in \mathbb{N} : \forall\, n > m :$

$$A_1 := \Big| \int_a^b (y_n(x) - y(x))\,\mathrm{dx} \Big| < \epsilon.$$

From part (c) above, we know that: $\forall \epsilon > 0 : \exists\, m \in \mathbb{N} : \forall\, n > m :$

$$A_2 := \int_a^b |y_n(x) - y(x)|\,\mathrm{dx} < \epsilon.$$

Now[1] if the sign of $y_n(x) - y(x)$ is constant for all $x \in [a, b]$ (i.e. the graphs do not cross over), then $A_1 = A_2 < \epsilon$. Otherwise, there is some cancellation in the integral A_1 and $0 \le A_1 < A_2 < \epsilon$. So the same choice of m as was used in part (c) works here, since for that value of m, we have $A_1 < \epsilon$ as required. $\square$

(Approximate time taken for (c) and (d): 2hrs)

[1] This is related to the triangle inequality. I should prove it properly.

If $[a,b]=[0,1]$ and $y_n(x)=nxe^{-nx^2}$ show that, for each $x\in[0,1], y_n(x)\to 0$ but $\int_0^1 y_n(x)dx \to \frac{1}{2}$. Thus the convergence must be non-uniform. Show that

$$\max_{0\le x\le 1} y_n(x)=\sqrt{\frac{n}{2e}}$$

and sketch the graph of $y_n(x)$ versus x.

To show that $y_n(x) := \frac{nx}{e^{nx^2}} \to 0$ for all $x\in[0,1]$, first note that it is true for $x=0$ since $y_n(0)=0$ for all $n\in\mathbb{N}$. So we have to show it is true for $x\in(0,1]$.

Fix $x\in(0,1]$ and define $f(\alpha)=\frac{\alpha x}{e^{\alpha x}}$ for $\alpha\in\mathbb{R}$. $\lim_{\alpha\to\infty} f(\alpha)$ is an indeterminate form $\frac{\infty}{\infty}$ and we can use l'Hôpital's rule, differentiating with respect to α:

$$\lim_{\alpha\to\infty}\frac{\alpha x}{e^{\alpha x^2}}=\lim_{\alpha\to\infty}\frac{x}{x^2e^{\alpha x^2}}=0.$$

Since $f(\alpha)=y_n$ at integer values of α it follows that $\lim_{n\to\infty} y_n(x)=0$ for all $x\in(0,1]$. $\square$

For the limit of the definite integral we have

$$\int_0^1 nxe^{-nx^2}\,\mathrm{dx}=\Big[-\frac{1}{2}e^{-nx^2}\Big]_0^1=\frac{1}{2}(1-e^{-n}),$$

and so $\lim_{n\to\infty}\int_0^1 y_n(x)\,\mathrm{dx}=\frac{1}{2}$. $\square$

To find the maximum value attained by $y_n(x)$ for $x\in[0,1]$, note that the derivative is

$$\frac{\mathrm{d}\,y_n(x)}{\mathrm{dx}}=nx(-2nx)e^{-nx^2}+ne^{-nx^2}=ne^{-nx^2}(1-2nx^2),$$

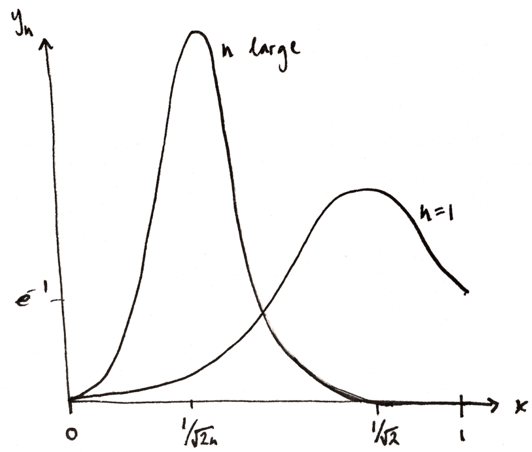
and therefore that the only solution to $\frac{\mathrm{d}\,y_n(x)}{\mathrm{dx}}=0$ for $x\in[0,1]$ is $x=\frac{1}{\sqrt{2n}}$.

The second derivative is

$$ne^{-nx^2}(-4nx)-2n^2xe^{-nx^2}(1-2nx^2)=2n^2xe^{-nx^2}\left(2nx^2-3\right).$$

This is negative at the critical point $x=\frac{1}{\sqrt{2n}}$ showing that it is a maximum. Therefore

$$\max_{x\in[0,1]} y_n(x)=n\frac{1}{\sqrt{2n}}e^{-n(\frac{1}{\sqrt{2n}})^2}=\sqrt{\frac{n}{2e}}.\quad\square$$



(Approximate time for reading and producing answer: 3 hrs)

1.2 Let $\sum_{n=0}^{\infty} u_n$ be a series of real-valued functions defined on $[a, b]$. State the **Weierstrass M-test** for the uniform convergence of the series.

Show that the series $\sum_{n=0}^{\infty} (-1)^n \frac{\cos nx}{1+n^2}$ converges uniformly on $[-\pi, \pi]$.

Weierstrass M-test

The series of functions $(u_n)_{n\geq 0}$ converges uniformly on $[a, b]$ if

1. there exists a sequence $(M_n)_{n\geq 0}$ such that $|u_n(x)| \leq M_n$ for all $n \geq 0$ and for all $x \in [a, b]$, and
2. the series $\sum_{n=0}^{\infty} M_n$ converges.

Define $u_n(x) = (-1)^n \frac{\cos nx}{1+n^2}$.

Let $M_n = \frac{1}{1+n^2}$ and note that $|u_n| \leq M_n$ for all $x \in [-\pi, \pi]$.

Note that the integral $\int_1^{\infty} \frac{1}{x^2}\,\mathrm{dx} = [-\frac{1}{x}]_1^{\infty} = 1$ converges, therefore the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the integral test for convergent series.

Now $M_n < \frac{1}{n^2}$ for $n > 0$, so the series $\sum_{n=1}^{\infty} M_n$ converges. Therefore the series $\sum_{n=0}^{\infty} M_n$ also converges, since its tail converges.

Therefore the series $\sum_{n=0}^{\infty} u_n$ converges uniformly on $[-\pi, \pi]$.

1.3 Consider the initial-value problems

$$\begin{aligned} y' &= x^2 + y^2, \quad y(0) = 0, \qquad (1) \\ y' &= (1-2x)y, \quad y(0) = 1. \qquad (2) \end{aligned}$$

In each case find y_0, y_1, y_2, y_3, where $\{y_n\}_{n\geq 0}$ is the sequence of Picard approximations. By considering the behaviour of x^2+y^2 on the square $\{(x,y) : |x| \leq \frac{1}{\sqrt{2}},\ |y| \leq \frac{1}{\sqrt{2}}\}$ and appealing to Picard's theorem show that in case (1) the sequence converges uniformly for $|x| \leq \frac{1}{\sqrt{2}}$.

In case (2), use Picard's theorem to show that the problem has a unique solution for all x. Now find the solution explicitly and, by expanding as a series, show that the sequence $\{y_n\}_{n\geq 0}$ converges to the solution.

Consider an ODE $y' = f\Big(x, y(x)\Big)$ with initial condition $y(a) = b$.

The sequence of Picard approximations are given by

$$\begin{aligned} y_0(x) &= b \\ y_{n+1}(x) &= b + \int_0^x f\Big(t, y_n(t)\Big)\,\mathrm{dt}\,. \end{aligned}$$

(1)

$$\begin{aligned} y_0(x) &= 0 \\ y_1(x) &= 0 + \int_0^x t^2 + 0^2\,\mathrm{dt} \\ &= \frac{x^3}{3} \\ y_2(x) &= 0 + \int_0^x t^2 + \left(\frac{t^3}{3}\right)^2 \mathrm{dt} = 0 + \int_0^x t^2 + \frac{t^6}{9} \\ &= \frac{x^3}{3} + \frac{x^7}{63} \\ y_3(x) &= 0 + \int_0^x t^2 + \left(\frac{t^3}{3} + \frac{t^7}{63}\right)^2 \mathrm{dt} = 0 + \int_0^x t^2 + \frac{t^6}{9} + \frac{2t^{10}}{189} + \frac{t^{14}}{3969}\,\mathrm{dt} \\ &= \frac{x^3}{3} + \frac{x^7}{63} + \frac{2x^{11}}{2079} + \frac{x^{15}}{59535} \end{aligned}$$

We need to show that this situation satisfies the requirements of Picard's theorem.

Are the y_n contained within the square?
We need to show that $\frac{-1}{\sqrt{2}} \le y_n(x) \le \frac{1}{\sqrt{2}}$ for all $n > 0$ and for all $x \in [\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$.

It is true for $y_0(x) = 0$.

Suppose it is true for y_n. For induction we require that

$$y_{n+1}(x) = \frac{x^3}{3} + \int_0^x y_n(t)^2 \, \mathrm{d}t$$

is bounded by $\frac{-1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$.

A lower bound for $y_{n+1}(x)$ is given by taking $x = \frac{-1}{\sqrt{2}}$ and $y_n(x) = 0$ (i.e. a constant function). In this case $y_{n+1}(x) = \left(\frac{-1}{\sqrt{2}}\right)^3 > \frac{-1}{\sqrt{2}}$.

An upper bound for $y_{n+1}(x)$ is given by taking $x = \frac{1}{\sqrt{2}}$ and $y_n(x) = \frac{1}{\sqrt{2}}$ (i.e. a constant function). In this case

$$y_{n+1}(x) = \left(\frac{1}{\sqrt{2}}\right)^3 + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{1}{2^{3/2}} + \frac{1}{2^{3/2}} = \frac{1}{\sqrt{2}}.$$

Therefore by induction, the y_n are contained within the vertical bounds of the square.

Does $f(x, y) = x^2 + y^2$ satisfy a Lipschitz condition?

Let $S = \{(x, y) : |x| \le \frac{1}{\sqrt{2}}, |y| \le \frac{1}{\sqrt{2}}\}$ denote the square.

A Lipschitz condition requires that $\exists L$ such that $|f(x, u) - f(x, v)| \le L|u - v|$ for all $(x, u) \in S$, $(x, v) \in S$.

Note that f is differentiable on S and that the image of $f_y = x^2 + 2y$ on S is $\left[\frac{1}{2} - \sqrt{2}, \frac{1}{2} + \sqrt{2}\right]$.

Therefore by the Mean Value Theorem, for all $(x, u) \in S$, $(x, v) \in S$, there exists $w \in [u, v]$ such that

$$f(x, v) - f(x, u) = f_y'(x, w) \cdot (v - u),$$

and therefore

$$|f(x, v) - f(x, u)| \le \left(\frac{1}{2} + \sqrt{2}\right)|v - u|.$$

So f satisfies a Lipschitz condition.

Therefore the sequence of functions given by the Picard iterates $y_0, y_1, \ldots$ converge uniformly to a solution of the ODE on $|x| \le \frac{1}{\sqrt{2}}$.