Oxford A1 - Differential Equations

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1 Sheet 1

> 1.1 Let $[a, b]$ be a closed and bounded interval of the real line and let $\{y_n\}_{n\geq 0}$ be a sequence of real-valued functions, each of which is defined on $[a, b]$. What does it mean to say that **the sequence converges uniformly on $[a, b]$ to a limit function y?** If each y_n is continuous on $[a, b]$ show that the uniform limit y is continuous on $[a, b]$ and that, when $n \to \infty$,
>
> $$\int_a^b |y_n(x) - y(x)|dx \to 0, \quad \int_a^b y_n(x)dx \to \int_a^b y(x)dx.$$

(a) Definition of uniform convergence

The sequence of functions $\{y_n\}_{n\geq 0}$ **converges uniformly on $[a, b]$ to** y if and only if for all $\epsilon > 0$ there exists an $m \in \mathbb{N}$ such that for all $n > m$ and for all $x \in [a, b]$, $|y_n(x) - y(x)| < \epsilon$.
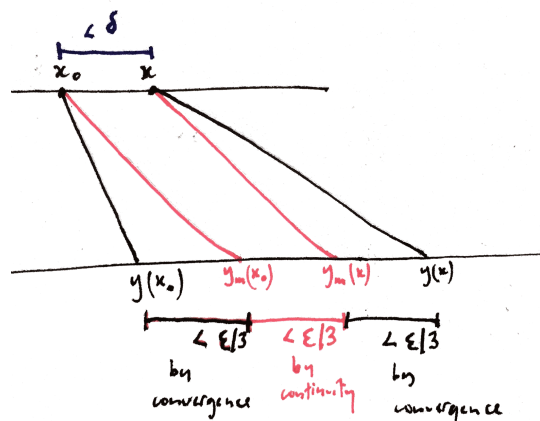
(b) Show that the limit function is continuous

The claim is that if each y_n is continuous on $[a, b]$ then y is continuous on $[a, b]$. We are told that

1. $\{y_n\}_{n\geq 0}$ converges uniformly to y, and

2. each y_n is continuous on $[a, b]$.

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Informal illustration of proof:

Fix arbitrary $\epsilon > 0$ and $x_0 \in [a, b]$.

Let $m \in \mathbb{N}$ be such that $|y_m(x_0) - y(x_0)| < \epsilon/3$. Such an m exists because the $\{y_n\}$ converge uniformly to y.

Let δ be such that $|x - x_0| < \delta \implies |y_m(x) - y_m(x_0)| < \epsilon/3$. Such a δ exists because y_m is continuous on $[a, b]$.

Fix an arbitrary x such that $|x - x_0| < \delta$.

Now we have the following:

1. $|y(x_0) - y_m(x_0)| < \epsilon/3$ by convergence of the $\{y_n\}$

2. $|y_m(x_0) - y_m(x)| < \epsilon/3$ by continuity of y_m

3. $|y_m(x) - y(x)| < \epsilon/3$ by convergence of the $\{y_n\}$

Therefore $|y(x_0) - y(x)| < \epsilon$, proving continuity of y on $[a, b]$. $\square$

(Approximate time taken for reading and producing an answer: 4hrs)

(c) Show limit of definite integral I

Let $I_n = \int_a^b |y_n(x) - y(x)|\, \mathrm{dx}$.

The claim is that $\lim_{n\to\infty} I_n = 0$.

In other words $\forall \epsilon > 0 : \exists\, m \in \mathbb{N} : \forall\, n > m : |I_n - 0| < \epsilon$.

Fix an $\epsilon > 0$.

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Since the $\{y_n\}$ converge uniformly to y, there exists an $m \in \mathbb{N}$ such that for all $n > m$ and for all $x \in [a, b]$

$$|y_n(x) - y(x)| < \epsilon/(b - a).$$

Therefore $\int_a^b |y_n(x) - y(x)| \, dx < \epsilon$ for all $n > m$, as required. $\qquad\square$

(d) Show limit of definite integral II

The claim is that $\lim_{n\to\infty} \int_a^b y_n(x) \, dx = \int_a^b y(x) \, dx$.

In other words: $\forall \epsilon > 0 : \exists\, m \in \mathbb{N} : \forall\, n > m :$

$$\left| \left(\int_a^b y_n(x) \, dx \right) - \left(\int_a^b y(x) \, dx \right) \right| < \epsilon.$$

This is equivalent to: $\forall \epsilon > 0 : \exists\, m \in \mathbb{N} : \forall\, n > m :$

$$A_1 := \left| \int_a^b (y_n(x) - y(x)) \, dx \right| < \epsilon.$$

From part (c) above, we know that: $\forall \epsilon > 0 : \exists\, m \in \mathbb{N} : \forall\, n > m :$

$$A_2 := \int_a^b |y_n(x) - y(x)| \, dx < \epsilon.$$

Now[1] if the sign of $y_n(x) - y(x)$ is constant for all $x \in [a, b]$ (i.e. the graphs do not cross over), then $A_1 = A_2 < \epsilon$. Otherwise, there is some cancellation in the integral A_1 and $0 \le A_1 < A_2 < \epsilon$. So the same choice of m as was used in part (c) works here, since for that value of m, we have $A_1 < \epsilon$ as required. $\qquad\square$

(Approximate time taken for (c) and (d): 2hrs)

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[1] How do I prove this section properly?

If $[a, b] = [0, 1]$ and $y_n(x) = nxe^{-nx^2}$ show that, for each $x \in [0, 1]$, $y_n(x) \to 0$ but $\int_0^1 y_n(x)dx \to \frac{1}{2}$. Thus the convergence must be non-uniform. Show that

$$\max_{0 \le x \le 1} y_n(x) = \sqrt{\frac{n}{2e}}$$

and sketch the graph of $y_n(x)$ versus x.

To show that $y_n(x) := \frac{nx}{e^{nx^2}} \to 0$ for all $x \in [0, 1]$, first note that it is true for $x = 0$ since $y_n(0) = 0$ for all $n \in \mathbb{N}$. So we have to show it is true for $x \in (0, 1]$.

Fix $x \in (0, 1]$ and define $f(\alpha) = \frac{\alpha x}{e^{\alpha x}}$ for $\alpha \in \mathbb{R}$. $\lim_{\alpha \to \infty} f(\alpha)$ is an indeterminate form $\frac{\infty}{\infty}$ and we can use l'Hôpital's rule, differentiating with respect to α:

$$\lim_{\alpha \to \infty} \frac{\alpha x}{e^{\alpha x^2}} = \lim_{\alpha \to \infty} \frac{x}{x^2 e^{\alpha x^2}} = 0.$$

Since $f(\alpha) = y_n$ at integer values of α it follows that $\lim_{n \to \infty} y_n(x) = 0$ for all $x \in (0, 1]$. $\square$

For the limit of the definite integral we have

$$\int_0^1 nxe^{-nx^2} \, \mathrm{dx} = \left[-\frac{1}{2}e^{-nx^2} \right]_0^1 = \frac{1}{2}(1 - e^{-n}),$$

and so $\lim_{n \to \infty} \int_0^1 y_n(x) \, \mathrm{dx} = \frac{1}{2}$. $\square$

To find the maximum value attained by $y_n(x)$ for $x \in [0, 1]$, note that the derivative is

$$\frac{\mathrm{d}\, y_n(x)}{\mathrm{dx}} = nx(-2nx)e^{-nx^2} + ne^{-nx^2} = ne^{-nx^2}(1 - 2nx^2),$$

and therefore that the only solution to $\frac{\mathrm{d}\, y_n(x)}{\mathrm{dx}} = 0$ for $x \in [0, 1]$ is $x = \frac{1}{\sqrt{2n}}$.

The second derivative is

$$ne^{-nx^2}(-4nx) - 2n^2 xe^{-nx^2}(1 - 2nx^2) = 2n^2 xe^{-nx^2}\left(2nx^2 - 3\right).$$

This is negative at the critical point $x = \frac{1}{\sqrt{2n}}$ showing that it is a maximum. Therefore

$$\max_{x \in [0,1]} y_n(x) = n\frac{1}{\sqrt{2n}}e^{-n(\frac{1}{\sqrt{2n}})^2} = \sqrt{\frac{n}{2e}}. \quad \square$$

(Approximate time for reading and producing answer: 3 hrs)

1.2 Let $\displaystyle\sum_{n=0}^{\infty} u_n$ be a series of real-valued functions defined on $[a, b]$. State the **Weierstrass M-test** for the uniform convergence of the series.

Show that the series $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\cos nx}{1+n^2}$ converges uniformly on $[-\pi, \pi]$.

Weierstrass M-test

The series of functions $(u_n)_{n \geq 0}$ converges uniformly on $[a, b]$ if

1. there exists a sequence $(M_n)_{n \geq 0}$ such that $|u_n(x)| \leq M_n$ for all $n \geq 0$ and for all $x \in [a, b]$, and

2. the series $\sum_{n=0}^{\infty} M_n$ converges.

Define $u_n(x) = (-1)^n \frac{\cos nx}{1+n^2}$.

Let $M_n = \frac{1}{1+n^2}$ and note that $|u_n| \leq M_n$ for all $x \in [-\pi, \pi]$.

Note that the integral $\int_1^{\infty} \frac{1}{x^2}\, dx = [-\frac{1}{x}]_1^{\infty} = 1$ converges, therefore the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the integral test for convergent series.

Now $M_n < \frac{1}{n^2}$ for $n > 0$, so the series $\sum_{n=1}^{\infty} M_n$ converges. Therefore the series $\sum_{n=0}^{\infty} M_n$ also converges, since its tail converges.

Therefore the series $\sum_{n=0}^{\infty} u_n$ converges uniformly on $[-\pi, \pi]$.

1.3 Consider the initial-value problems

$$y' = x^2 + y^2, \quad y(0) = 0, \qquad (1)$$
$$y' = (1 - 2x)y, \quad y(0) = 1. \qquad (2)$$

In each case find y_0, y_1, y_2, y_3, where $\{y_n\}_{n\geq 0}$ is the sequence of Picard approximations.

By considering the behaviour of $x^2 + y^2$ on the square $\{(x, y) : |x| \leq \frac{1}{\sqrt{2}}, |y| \leq \frac{1}{\sqrt{2}}\}$ and appealing to Picard's theorem show that in case (1) the sequence converges uniformly for $|x| \leq \frac{1}{\sqrt{2}}$.

In case (2), use Picard's theorem to show that the problem has a unique solution for all x. Now find the solution explicitly and, by expanding as a series, show that the sequence $\{y_n\}_{n\geq 0}$ converges to the solution.

Consider an ODE $y' = f\left(x, y(x)\right)$ with initial condition $y(a) = b$.

The sequence of Picard approximations are given by

$$y_0(x) = b$$

$$y_{n+1}(x) = b + \int_0^x f\left(t, y_n(t)\right) dt.$$

(1)

$$y_0(x) = 0$$

$$y_1(x) = 0 + \int_0^x t^2 + 0^2 \, dt$$

$$= \frac{x^3}{3}$$

$$y_2(x) = 0 + \int_0^x t^2 + \left(\frac{t^3}{3}\right)^2 dt = 0 + \int_0^x t^2 + \frac{t^6}{9}$$

$$= \frac{x^3}{3} + \frac{x^7}{63}$$

$$y_3(x) = 0 + \int_0^x t^2 + \left(\frac{t^3}{3} + \frac{t^7}{63}\right)^2 dt = 0 + \int_0^x t^2 + \frac{t^6}{9} + \frac{2t^{10}}{189} + \frac{t^{14}}{3969} \, dt$$

$$= \frac{x^3}{3} + \frac{x^7}{63} + \frac{2x^{11}}{2079} + \frac{x^{15}}{59535}$$

We need to show that this situation satisfies the requirements of Picard's theorem.

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Are the y_n contained within the square?
TODO: It looks plausible.

Does $f(x, y) = x^2 + y^2$ satisfy a Lipschitz condition?

Let $S = \{(x, y) : |x| \leq \frac{1}{\sqrt{2}}, |y| \leq \frac{1}{\sqrt{2}}\}$ denote the square.

A Lipschitz condition requires that $\exists L$ such that $|f(x, u) - f(x, v)| \leq L|u - v|$ for all $(x, u) \in S$, $(x, v) \in S$.

Note that f is differentiable on S and that $f_y = x^2 + 2y \in \left[\frac{1}{2} - \sqrt{2}, \frac{1}{2} + \sqrt{2}\right]$.

Therefore by the Mean Value Theorem, for all $(x, u) \in S$, $(x, v) \in S$, there exists $w \in [u, v]$ such that

$$f(x, v) - f(x, u) = f'_y(x, w) \cdot (v - u),$$

and therefore

$$|f(x, v) - f(x, u)| \leq \left(\frac{1}{2} + \sqrt{2}\right)|v - u|.$$

So f satisfies a Lipschitz condition.

Questions

1. I'm struggling to make sense of Picard criterion $\mathbf{P(i)}$ on page 10 of the lecture notes[2]

> *has a solution in the rectangle $R := \{(x, y) : |x - a| \leq h, |y - b| \leq k\}$ provided:*
>
> $\mathbf{P(i)}$: *(a) f is continuous in R, with bound M (so $|f(x, y)| \leq M$) and (b) $Mh \leq k$.*
> $\mathbf{P(ii)}$: *f satisfies a Lipschitz condition in R.*
>
> *Furthermore, this solution is unique.*

   M is a distance in the output space, but h and k are distances in the input rectangle, so why are they being combined and compared arithmetically?

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[2] https://courses.maths.ox.ac.uk/node/download_material/5375