Oxford A1 - Differential Equations

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November 5, 2017

1 Sheet 1

1.1 Let $[a, b]$ be a closed and bounded interval of the real line and let $\{y_n\}_{n\geq 0}$ be a sequence of real-valued functions, each of which is defined on $[a, b]$. What does it mean to say that **the sequence converges uniformly on $[a, b]$ to a limit function y?** If each y_n is continuous on $[a, b]$ show that the uniform limit y is continuous on $[a, b]$ and that, when $n \to \infty$,

$$\int_a^b |y_n(x) - y(x)| dx \to 0, \quad \int_a^b y_n(x) dx \to \int_a^b y(x) dx.$$

(a) Definition of uniform convergence

The sequence of functions $\{y_n\}_{n\geq 0}$ **converges uniformly on $[a, b]$ to y** if and only if for every $\epsilon > 0$ there exists an $m \in \mathbb{N}$ such that for every $n > m$, y_n differs from y by no more than ϵ at every point in $[a, b]$.

(b) Show that the limit function is continuous

The claim is that if each y_n is continuous on $[a, b]$ then y is continuous on $[a, b]$. We are told that

1. $\{y_n\}_{n\geq 0}$ converges uniformly to y, and

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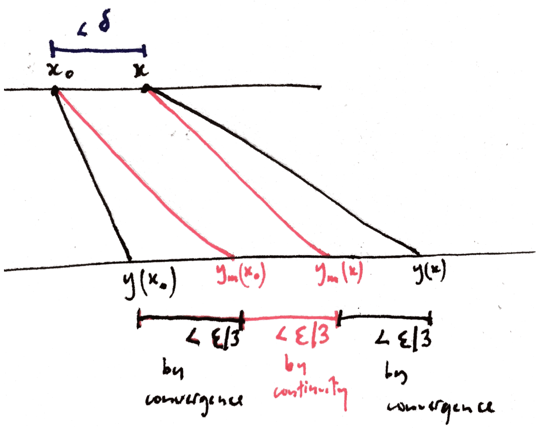
2. each y_n is continuous on $[a, b]$.

Informal illustration of proof:



Fix arbitrary $\epsilon > 0$ and $x_0 \in [a, b]$.

Let $m \in \mathbb{N}$ be such that $|y_m(x_0) - y(x_0)| < \epsilon/3$. Such an m exists because the $\{y_n\}$ converge uniformly to y.

Let δ be such that $|x - x_0| < \delta \implies |y_m(x) - y_m(x_0)| < \epsilon/3$. Such a δ exists because y_m is continuous on $[a, b]$.

Fix an arbitrary x such that $|x - x_0| < \delta$.

Now we have the following:

1. $|y(x_0) - y_m(x_0)| < \epsilon/3$ by convergence of the $\{y_n\}$
2. $|y_m(x_0) - y_m(x)| < \epsilon/3$ by continuity of y_m
3. $|y_m(x) - y(x)| < \epsilon/3$ by convergence of the $\{y_n\}$

Therefore $|y(x_0) - y(x)| < \epsilon$, proving continuity of y on $[a, b]$. □

(Approximate time taken for reading and producing an answer: 4hrs)

(c) Show limit of definite integral I

Let $I_n = \int_a^b |y_n(x) - y(x)|\,\mathrm{dx}$.

The claim is that $\lim_{n\to\infty} I_n = 0$.

In other words $\forall \epsilon > 0 : \exists\, m \in \mathbb{N} : \forall\, n > m : |I_n - 0| < \epsilon$.

Fix an $\epsilon > 0$.

Since the $\{y_n\}$ converge uniformly to y, there exists an $m \in \mathbb{N}$ such that for all $n > m$ and for all $x \in [a, b]$

$$|y_n(x) - y(x)| < \epsilon/(b-a).$$

Therefore $\int_a^b |y_n(x) - y(x)| \, \mathrm{dx} < \epsilon$ for all $n > m$, as required. □

(d) Show limit of definite integral II

The claim is that $\lim_{n\to\infty} \int_a^b y_n(x) \, \mathrm{dx} = \int_a^b y(x) \, \mathrm{dx}$.

In other words: $\forall \epsilon > 0 : \exists \; m \in \mathbb{N} : \forall \; n > m :$

$$\left| \left(\int_a^b y_n(x) \, \mathrm{dx} \right) - \left(\int_a^b y(x) \, \mathrm{dx} \right) \right| < \epsilon.$$

This is equivalent to: $\forall \epsilon > 0 : \exists \; m \in \mathbb{N} : \forall \; n > m :$

$$A_1 := \left| \int_a^b (y_n(x) - y(x)) \, \mathrm{dx} \right| < \epsilon.$$

From part (c) above, we know that: $\forall \epsilon > 0 : \exists \; m \in \mathbb{N} : \forall \; n > m :$

$$A_2 := \int_a^b |y_n(x) - y(x)| \, \mathrm{dx} < \epsilon.$$

Now[a] if the sign of $y_n(x) - y(x)$ is constant for all $x \in [a, b]$ (i.e. the graphs do not cross over), then $A_1 = A_2 < \epsilon$. Otherwise, there is some cancellation in the integral A_1 and $0 \leq A_1 < A_2 < \epsilon$. So the same choice of m as was used in part (c) works here, since for that value of m, we have $A_1 < \epsilon$ as required. □

(Approximate time taken for (c) and (d): 2hrs)

[a] How do I prove this section properly?

If $[a,b] = [0,1]$ and $y_n(x) = nxe^{-nx^2}$ show that, for each $x \in [0,1], y_n(x) \to 0$ but $\displaystyle\int_0^1 y_n(x)dx \to \frac{1}{2}$. Thus the convergence must be non-uniform. Show that

$$\max_{0 \leq x \leq 1} y_n(x) = \sqrt{\frac{n}{2e}}$$

and sketch the graph of $y_n(x)$ versus x.

To show that $y_n(x) := \frac{nx}{e^{nx^2}} \to 0$ for all $x \in [0,1]$, first note that it is true for $x = 0$ since $y_n(0) = 0$ for all $n \in \mathbb{N}$. So we have to show it is true for $x \in (0,1]$.

Fix $x \in (0,1]$ and define $f(\alpha) = \frac{\alpha x}{e^{\alpha x}}$ for $\alpha \in \mathbb{R}$. $\lim_{\alpha \to \infty} f(\alpha)$ is an indeterminate form $\frac{\infty}{\infty}$ and we can use l'Hôpital's rule, differentiating with respect to α:

$$\lim_{\alpha \to \infty} \frac{\alpha x}{e^{\alpha x^2}} = \lim_{\alpha \to \infty} \frac{x}{x^2 e^{\alpha x^2}} = 0.$$

Since $f(\alpha) = y_n$ at integer values of α it follows that $\lim_{n \to \infty} y_n(x) = 0$ for all $x \in (0,1]$. $\square$

For the limit of the definite integral we have

$$\int_0^1 nxe^{-nx^2}\,\mathrm{dx} = \Big[-\frac{1}{2}e^{-nx^2}\Big]_0^1 = \frac{1}{2}(1-e^{-n}),$$

and so $\lim_{n \to \infty} \int_0^1 y_n(x)\,\mathrm{dx} = \frac{1}{2}$. $\square$

To find the maximum value attained by $y_n(x)$ for $x \in [0,1]$, note that the derivative is

$$\frac{\mathrm{d}\,y_n(x)}{\mathrm{dx}} = nx(-2nx)e^{-nx^2} + ne^{-nx^2} = ne^{-nx^2}(1-2nx^2),$$

and therefore that the only solution to $\frac{\mathrm{d}\,y_n(x)}{\mathrm{dx}} = 0$ for $x \in [0,1]$ is $x = \frac{1}{\sqrt{2n}}$.

The second derivative is

$$ne^{-nx^2}(-4nx) - 2n^2xe^{-nx^2}(1-2nx^2) = 2n^2xe^{-nx^2}\left(2nx^2 - 3\right).$$

This is negative at the critical point $x = \frac{1}{\sqrt{2n}}$ showing that it is a maximum. Therefore

$$\max_{x \in [0,1]} y_n(x) = n\frac{1}{\sqrt{2n}}e^{-n(\frac{1}{\sqrt{2n}})^2} = \sqrt{\frac{n}{2e}}. \quad \square$$

(Approximate time for reading and producing answer: 3 hrs)