Oxford M2 - Real Analysis I - Sequences and Series

1 Sheet 4

1.1

(a)

> 1. (a) Let the sequence (a_n) be defined by
> $$a_n = \left(\frac{n^2-1}{n^2+1}\right)\cos(2\pi n/3).$$
> By considering suitable subsequences prove that (a_n) diverges.

Note that:

(1) $\frac{n^2-1}{n^2+1} = \frac{1-n^{-2}}{1+n^{-2}} \to 1$

(2) $\cos(\frac{2\pi n}{3}) = 1$ for $n = 0, 3, 6, 9, \ldots$

(3) $\cos(\frac{2\pi n}{3}) = -1$ for $n = \frac{3}{2}, \frac{9}{2}, \frac{15}{2}, \ldots$

Therefore, the subsequence indicated in (2) has limit 1 whereas that in (3) has limit -1.

Therefore (a_n) diverges, since a sequence diverges if it contains two subsequences converging to different limits.

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(b)

> (b) Consider the sequence $(\cos n)$. Show that, for a suitable positive constant K, there exist subsequences (b_r) and (c_s) of $(\cos n)$ with $b_r > K$ for all r and $c_s < -K$ for all s. Deduce that $(\cos n)$ diverges.

Proof. Let $K = \sqrt{\frac{1}{2}}$, let $n \in \mathbb{N}$ and let $x \in \mathbb{R}^{>0}$.

Note that $|x - 2n\pi| < \frac{\pi}{4} \implies \cos(x) > K$. Let $f(n)$ be the smallest integer in $\left(2n\pi - \frac{\pi}{4}, 2n\pi + \frac{\pi}{4}\right)$ (the interval exceeds 1 in width therefore it contains at least one integer). Define $b_r = a_{f(r)}$. Then $b_r > K$ for all $r \in \mathbb{N}$.

Similarly, note that $|x - (2n+1)\pi| < \frac{\pi}{4} \implies \cos(x) < -K$. Let $g(n)$ be the smallest integer in $\left((2n+1)\pi - \frac{\pi}{4}, (2n+1)\pi + \frac{\pi}{4}\right)$ (the interval exceeds 1 in width therefore it contains at least one integer). Define $c_s = a_{g(s)}$. Then $c_s < -K$ for all $r \in \mathbb{N}$.

Define

$$d_n = \begin{cases} b_n & \text{n odd} \\ c_n & \text{n even.} \end{cases}$$

Then (d_n) does not converge since it is not Cauchy. But (d_n) is a subsequence of (a_n) therefore (a_n) does not converge. $\square$

1.2

> 2. (a) Let (a_n) be a sequence such that the subsequences (a_{2n}) and (a_{2n+1}) both converge to a real number L. Show that (a_n) also converges to L.

Proof. Let (a_n) be such that (a_{2n}) and (a_{2n+1}) both converge to $L \in \mathbb{R}$.

Fix $\epsilon > 0$. Let $N_1 \in \mathbb{N}$ be such that $|a_{2n} - L| < \epsilon$ for all $n \geq N_1$ and let $N_2 \in \mathbb{N}$ be such that $|a_{2n+1} - L| < \epsilon$ for all $n \geq N_2$.

Let $N = 2\max(N_1, N_2)$. Then $|a_n - L| < \epsilon$ for all $n \geq N$, therefore $a_n \to L$. □

> (b) Let (b_n) be a sequence such that each of the subsequences (b_{2n}), (b_{2n+1}), (b_{3n}) converges. Need (b_n) converge? Either provide a proof or a counterexample.

Let (b_{3n}) converge to L. Therefore (b_{6n}) and (b_{6n+3}) converge to L. Note that (b_{6n}) is a subsequence of (b_{2n}), and (b_{6n+3}) is a subsequence of (b_{2n+1}). It is given that (b_{2n}) and (b_{2n+1}) converge, therefore they converge to L also.

Therefore (b_n) converges to L by the result in part (a).

> (c) Let (c_n) be a sequence such that the subsequence (c_{kn}) converges for each $k = 2, 3, 4, \ldots$. Need (c_n) converge? Provide either a proof or a counterexample.

Proof. c_n need not converge. As a counterexample define

$$c_n = \begin{cases} 0 & n \text{ prime} \\ 1 & \text{otherwise.} \end{cases}$$

This satisfies the description given of (c_n) but is not Cauchy since the set of primes has no upper bound and the primes are interspersed with non-primes. □

1.3

3. For which of the following choices of a_n does the sequence (a_n) converge? Justify your answers, and find the value of the limit when it exists.

$$\text{(i) } \frac{n^2}{n!}; \quad \text{(ii) } \frac{2^n n^2 + 3^n}{3^n(n+1) + n^7}; \quad \text{(iii) } \frac{(n!)^2}{(2n)!}; \quad \text{(iv) } \frac{n^4 + n^3 \sin n + 1}{5n^4 - n \log n}.$$

[*You may freely make use of standard limits and inequalities, sandwiching and AOL methodology, as appropriate.*]

(i)

$$\frac{n^2}{n!} = \frac{n}{(n-1)!} = \prod_{k=1}^{n-1} \frac{1}{1 - \frac{1}{k}} \to \prod_{k=1}^{n-1} \frac{1}{1-0} = 1.$$

1.4

4. (a) Let (a_n) be a real sequence. Prove from the limit definition that $a_n \geqslant 0$ and $a_n \to L$ implies $L \geqslant 0$ and prove further that $\sqrt{a_n} \to \sqrt{L}$.

(a) Let $a_n \geq 0$ and $a_n \to L$.

Proof. Suppose for a contradiction that $L < 0$. Put $\epsilon = -\frac{L}{2} > 0$. Then there exists $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq N$. Let $n \geq N$. Then $a_n < L + \epsilon = L - \frac{L}{2} = \frac{L}{2} < 0$. But $a_n \geq 0$, a contradiction. Therefore $L \geq 0$. $\square$

(b) Let (a_n), (b_n) and (c_n) be sequences of real numbers converging to L_1, L_2, L_3, respectively. Let $d_n = \max\{a_n, b_n, c_n\}$. Assuming any standard AOL results that you require, prove that $d_n \to \max\{L_1, L_2, L_3\}$.

Proof. Let (a_n), (b_n), (c_n) be real sequences with $(a_n) \to L_1$, $(b_n) \to L_2$, $(c_n) \to L_3$, and let $d_n = \max\ \{a_n, b_n, c_n\}$.

First suppose $L_1 = L_2 = L_3 = L$. Fix $\epsilon > 0$. Put $N = \max\{N_1, N_2, N_3\}$ where N_1, N_2, N_3 "work" for (a_n), (b_n), (c_n) respectively, for this choice of ϵ. Then $|d_n - L| < \epsilon$ for $n \geq N$, so $d_n \to L = \max\{L_1, L_2, L_3\}$.

Finally, WLOG, suppose $L_1 > L_2 \geq L_3$. Let $\epsilon = L_1 - L_2$. Put $N = \max\{N_1, N_2, N_3\}$ where N_1, N_2, N_3 "work" for $(a_n), (b_n), (c_n)$ respectively, for this choice of ϵ. Then $a_n > b_n$ and $a_n > c_n$ for $n \geq N$. Therefore $d_n = a_n$ for $n \geq N$. Therefore $d_n \to L_1 = \max\{L_1, L_2, L_3\}$.

Right? But why does the question suggest AOL is needed? $\square$

1.5

> 5. Let $r > 0$. Let $a_n = r^n/n!$.
>
> (a) By considering a_{n+1}/a_n show that the tail $(a_n)_{n\geqslant N}$ is monotonic decreasing if N is sufficiently large. [*You should specify a suitable value of* N.]

Proof. Let $r > 0$ and let $a_n = \frac{r^n}{n!}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{r^{n+1}}{(n+1)!}\frac{n!}{r^n} = \frac{r}{n+1}.$$

Therefore $(a_n)_{n\geq N}$ is monotonic decreasing for $N = \lceil r \rceil$. □

> (b) Show that (a_n) converges to a limit L and find the value of L.

Proof. Note that $a_n > 0$. Since (a_n) is bounded below by zero and the tail $(a_n)_{n\geq N}$ is monotonic decreasing for $N = \lceil r \rceil$, we have that a_n converges to a limit $L > 0$ by the Monotonic Sequence Theorem. □

Claim. $a_n \to 0$.

Proof. If $r \leq 1$ then $a_n \leq \frac{1}{n!} \to 0$.

So let $r > 1$. Note that for $n > r$

$$a_n = \prod_{k=1}^{n} \frac{r}{k} = \left(\prod_{k=1}^{\lceil r \rceil - 1} \frac{r}{k}\right)\left(\prod_{k=\lceil r \rceil}^{n} \frac{r}{k}\right).$$

The first factor is a product of terms each of which is greater than 1, and we have

$$\left(\prod_{k=1}^{\lceil r \rceil - 1} \frac{r}{k}\right) < r^{\lceil r \rceil - 1}.$$

The second factor is a product of terms each of which is not greater than 1, and we have

$$\left(\prod_{k=\lceil r \rceil}^{n} \frac{r}{k}\right) < \left(\frac{r}{\lceil r \rceil}\right)^{n - \lceil r \rceil + 1}.$$

What about if $r = \lceil r \rceil$?

Therefore

$$a_n < r^{\lceil r \rceil - 1}\left(\frac{r}{\lceil r \rceil}\right)^{n - \lceil r \rceil + 1} = \left(\frac{r}{\lceil r \rceil}\right)^{n} \lceil r \rceil^{\lceil r \rceil - 1}$$

Scratch work:

$$\begin{aligned}\left(\frac{r}{\lceil r\rceil}\right)^n \lceil r\rceil^{\lceil r\rceil-1} &< \epsilon\\ \left(\frac{r}{\lceil r\rceil}\right)^n &< \epsilon\lceil r\rceil^{1-\lceil r\rceil}\\ n\log\frac{r}{\lceil r\rceil} &< \log\epsilon + (1-\lceil r\rceil)\lceil r\rceil\\ n &> \frac{\log\epsilon + (1-\lceil r\rceil)\lceil r\rceil}{\log r - \log\lceil r\rceil}.\end{aligned}$$

Fix $\epsilon > 0$. Let $N = \left\lceil \frac{\log\epsilon+(1-\lceil r\rceil)\log\lceil r\rceil}{\log r-\log\lceil r\rceil}\right\rceil$. Then $a_n < \epsilon$ for $n \geq N$. Therefore $a_n \to 0$. $\square$

1.6

6. The real sequence (a_n) is defined by

$$a_1 = c, \qquad (\alpha + \beta)a_{n+1} = a_n^2 + \alpha\beta,$$

where $0 < \alpha < \beta$ and $c > \alpha$.

(a) Prove that if (a_n) converges to a limit L then necessarily $L = \alpha$ or $L = \beta$.

(b) Prove that $a_{n+1} - \gamma$ and $a_n - \gamma$ have the same sign, where γ denotes either α or β.

(c) Prove that, if $c < \beta$ then (a_n) converges monotonically to α. Discuss the limiting behaviour of (a_n) when $c \geqslant \beta$.

(d) Prove that, if $\alpha < c < \beta$,

$$|a_n - \alpha| \leqslant \left(\frac{\alpha + c}{\alpha + \beta}\right)^{(n-1)} (c - \alpha).$$