1A

> **1A Vectors and Matrices**
> Consider $z \in \mathbb{C}$ with $|z| = 1$ and $\arg z = \theta$, where $\theta \in [0, \pi)$.
>
> (a) Prove algebraically that the modulus of $1+z$ is $2\cos\frac{1}{2}\theta$ and that the argument is $\frac{1}{2}\theta$. Obtain these results geometrically using the Argand diagram.
>
> (b) Obtain corresponding results algebraically and geometrically for $1 - z$.

Proof. (Algebraic)
Note that $\cos\theta = \cos^2\frac{1}{2}\theta - \sin^2\frac{1}{2}\theta = 2\cos^2\frac{1}{2}\theta - 1$.

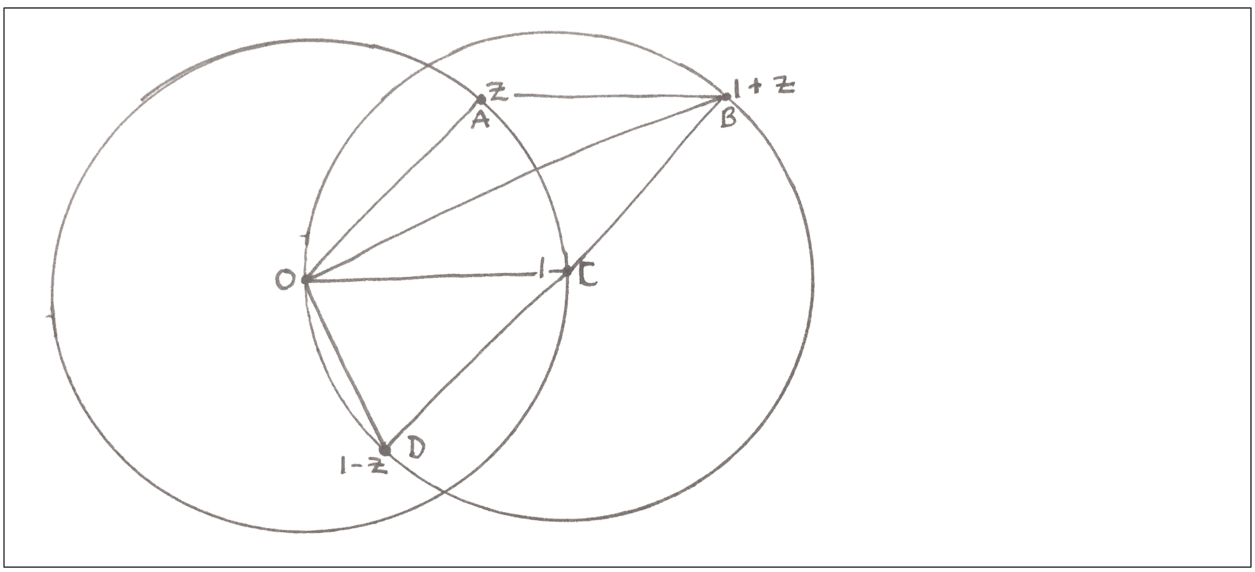
We have $z = \cos\theta + i\sin\theta$ and therefore

$$|1+z| = \sqrt{(1+\cos\theta)^2 + \sin^2\theta} = \sqrt{2(1+\cos\theta)} = \sqrt{2\left(1 + 2\cos^2\frac{1}{2}\theta - 1\right)} = 2\cos\frac{1}{2}\theta \quad (1)$$

$$|1-z| = \sqrt{(1-\cos\theta)^2 + \sin^2\theta} = \sqrt{2(1-\cos\theta)} = \sqrt{2(1 - (2\cos^2\frac{1}{2}\theta - 1))} = 2\sin\frac{1}{2}\theta. \quad (2)$$

□

Proof. (Geometric)



$OABC$ is a rhombus, with sides of length 1 and $\angle AOC = \theta$. The diagonal OB bisects $\angle AOC$, therefore $\angle OBC = \theta/2$. $\angle BOD$ is a right angle since it is formed from a triangle inscribed in a circle. Hypotenuse BD has length 2, since C is the centre of a second unit circle with radii CB and CD. Therefore the length of OB is $2\cos\frac{1}{2}\theta$ and the length of OD is $2\sin\frac{1}{2}\theta$. □

2C

2C Vectors and Matrices

Let A and B be real $n \times n$ matrices.

Show that $(AB)^T = B^T A^T$.

For any square matrix, the *matrix exponential* is defined by the series

$$e^A = I + \sum_{k=1}^{\infty} \frac{A^k}{k!}.$$

Show that $(e^A)^T = e^{A^T}$. [You are not required to consider issues of convergence.]

Calculate, in terms of A and A^T, the matrices Q_0, Q_1 and Q_2 in the series for the matrix product

$$e^{tA}\, e^{tA^T} = \sum_{k=0}^{\infty} Q_k t^k, \quad \text{where } t \in \mathbb{R}.$$

Hence obtain a relation between A and A^T which necessarily holds if e^{tA} is an orthogonal matrix.

Claim. $(AB)^T = B^T A^T$

Proof. Let $i, j \in \{1, \ldots, n\}$. Then

$$\left((AB)^T\right)_{ij} = (AB)_{ji} = \sum_{l=1}^{n} A_{jl} B_{li} = \sum_{l=1}^{n} (A^T)_{lj} (B^T)_{il} = (B^T A^T)_{ij} \tag{3}$$

□

Lemma. $(A^k)^T = (A^T)^k$.

Proof. Note that $(A^m)^T (A^T)^n = (AA^{m-1})^T (A^T)^n = (A^{m-1})^T (A^T)^{n+1}$. By iterating this formal manipulation k times, we have $(A^k)^T = (A^k)^T (A^T)^0 = (A^0)^T (A^T)^k = (A^T)^k$. □

Claim. $(e^A)^T = e^{(A^T)}$

Proof. Note that $(e^A)_{ij} := \sum_{k=0}^{\infty} \frac{(A^k)_{ij}}{k!}$, where $A^0 := I$ and $0! := 1$. Therefore

$$\left((e^A)^T\right)_{ij} = \sum_{k=0}^{\infty} \frac{((A^k)^T)_{ij}}{k!} = \sum_{k=0}^{\infty} \frac{((A^T)^k)_{ij}}{k!} = \left(e^{(A^T)}\right)_{ij}. \tag{4}$$

□

Problem. *For $t \in \mathbb{R}$ we define matrices Q_k such that $e^{tA}e^{tA^T} = \sum_{k=0}^{\infty} Q_k t^k$. Calculate Q_0, Q_1, Q_2.*

Solution.
We switch notation, so that Q_k becomes $Q^{(k)}$. We have

$$\begin{aligned}
\left(e^{tA}e^{tA^T}\right)_{ij} &= \left(\sum_{k=0}^{\infty} \frac{((tA)^k)_{ij}}{k!}\right)\left(\sum_{k=0}^{\infty} \frac{((tA^T)^k)_{ij}}{k!}\right) &(5)\\
&= \left(\sum_{k=0}^{\infty} \frac{t^k}{k!}(A^k)_{ij}\right)\left(\sum_{k=0}^{\infty} \frac{t^k}{k!}(A^k)_{ji}\right) &(6)\\
&= \delta_{ij}\delta_{ji}t^0 + (A_{ij}\delta_{ji} + \delta_{ij}A_{ji})t + \left(A_{ij}A_{ji} + \frac{1}{2}\delta_{ij}A_{ji}^2 + \frac{1}{2}A_{ij}^2\delta_{ji}\right)t^2 + \dots &(7)\\
&= Q_{ij}^{(0)}t^0 + Q_{ij}^{(1)}t + Q_{ij}^{(2)}t^2 + \dots. &(8)
\end{aligned}$$

Therefore

$$\begin{aligned}
Q_0 &= I &(9)\\
Q_1 &= 2\,\mathrm{diag}(A) &(10)\\
Q_2 &= A \times A^T + \mathrm{diag}(A^2) \qquad \text{(where } \times \text{ is elementwise).} &(11)
\end{aligned}$$

If e^{tA} is orthogonal, then $e^{tA}e^{tA^T} = e^{tA}(e^{tA})^{-1} = I = \sum_{k=0}^{\infty} Q_k t^k$. □

3F

> **3F Analysis I**
> Given an increasing sequence of non-negative real numbers $(a_n)_{n=1}^{\infty}$, let
>
> $$s_n = \frac{1}{n}\sum_{k=1}^{n} a_k.$$
>
> Prove that if $s_n \to x$ as $n \to \infty$ for some $x \in \mathbb{R}$ then also $a_n \to x$ as $n \to \infty$.

Proof. Assume that $s_n \to x$ as $n \to \infty$. We will show that $s_n \le a_n \le x$, for all n, therefore $a_n \to x$ as $n \to \infty$, as required.

First note that $s_{n+1} = \frac{n}{n+1}s_n + \frac{1}{n+1}a_{n+1}$.

It's intuitively obvious that $s_n \le a_n$ for all n. To prove this, note that it's true for $n = 1$; assume for induction that it's true for $n = k$. Then we have $s_{k+1} = \frac{k}{k+1}s_k + \frac{1}{k+1}a_{k+1} \le \frac{k}{k+1}a_k + \frac{1}{k+1}a_{k+1} \le a_{k+1}$, as required.

Finally we show that $a_n \le x$ for all n. Seeking a contradiction, assume that there exists M such that $a_M > x$. Let $\epsilon = a_M - x > 0$ (see diagram below).

Define $\Delta_n := s_{n+1} - s_n = \frac{n}{n+1}s_n + \frac{1}{n+1}a_{n+1} - s_n = \frac{1}{n+1}(a_{n+1} - s_n) > \frac{1}{n+1}(x + \epsilon - s_n)$.

Now, we seek $N \ge M$ such that $x - s_N < \Delta_N$, since then we will have $s_{N+1} > x$, a contradiction. Supposing that such an N exists and solving for it, we have

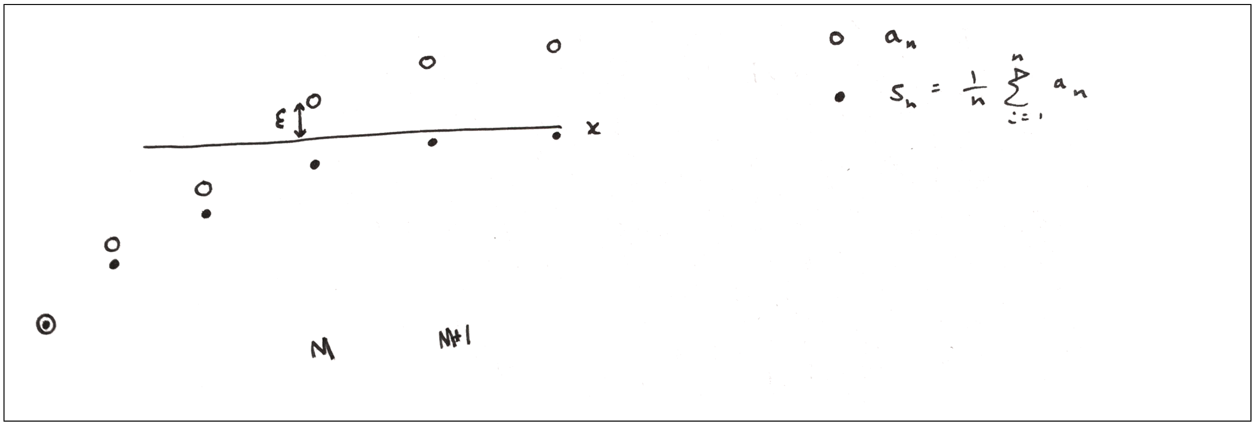
$$\begin{aligned} x - s_N &< \frac{1}{N+1}(x + \epsilon - s_N) && (12)\\ s_N\frac{N}{N+1} &> x\frac{N}{N+1} - \frac{\epsilon}{N+1} && (13)\\ s_N &> x - \epsilon/N. && (14) \end{aligned}$$

Hm, but we require an expression for s_N that does not depend on N! Let's try ϵ/M.

Let N be such that $x - s_N < \epsilon/M$. Then

$$\begin{aligned} \Delta_N &> \frac{1}{N+1}(\epsilon + \epsilon/M) && (15)\\ &= \epsilon/M - \frac{N}{N+1}\epsilon/M + \frac{1}{N+1}\epsilon && (16)\\ &= \epsilon/M - \epsilon\left(\frac{N}{M(N+1)} - \frac{1}{N+1}\right), && (17) \end{aligned}$$

which fails to prove the desired $\Delta_N > \epsilon/M$.



ε
x
M
$N+1$
a_n
$S_n = \frac{1}{n} \sum_{i=1}^{n} a_n$

□