Oxford M2 - Real Analysis I - Sequences and Series

Contents

https://courses.maths.ox.ac.uk/node/37482

1 Sheet 6

1.1 (COMPLETISH)

1. For the following choices of a_k, use the indicated test to establish whether or not $\sum a_k$ converges.

   (a) $\dfrac{(2k+1)(3k-1)}{(k+1)(k+2)^2}$ (Comparison Test, limit form);

   (b) $\dfrac{1}{k^{1/k}k}$ (Comparison Test, limit form);

   (c) $\dfrac{k!}{k^k}$ (Ratio Test);

   (d) $\binom{2k}{k}^{-1}(4-10^{-23})^k$ (Ratio Test);

   (e) $2^{-k}k$ if k is of the form 2^m and 0 otherwise (Ratio Test).

   Write brief comments on possible alternative approaches, if any, you might have tried on these examples.

(a)

Claim. *Let* $a_k = \frac{(2k+1)(3k-1)}{(k+1)(k+2)^2}$. *Then* $\sum a_k$ *diverges.*

Proof. Let $b_k = \frac{1}{k+1}$. Then

$$\begin{aligned} \frac{a_k}{b_k} &= \frac{(2k+1)(3k-1)}{(k+2)^2} = \frac{6k^2+k-1}{k^2+4k+4} = \frac{6+k^{-1}-k^{-2}}{1+4k^{-1}+4k^{-2}} \\ &\to 6. \end{aligned}$$

We have $b_k > 0$ and $a_k > 0$ for $k > 0$. By the Comparison Test (limit form) $\sum a_k$ diverges, since $\sum b_k$ diverges.

To show that $\sum b_k$ diverges, define $c_k = \frac{1}{k}$ and note that $\frac{b_k}{c_k} = \frac{k}{k+1} \to 1$. We know that the harmonic series $\sum c_k$ diverges, so $\sum b_k$ diverges by the Comparison Test (limit form).

```
#+begin_src mathematica
SumConvergence[(2 k + 1) (3 k - 1)/((k + 1) (k + 2)^2), k]
#+end_src

#+RESULTS:
: False
```

$\square$

(b)

Claim. *Let* $a_k = \frac{1}{k^{1/k}k}$. *Then* $\sum a_k$ *diverges.*

Remark. We know that $\sum \frac{1}{k \log k}$ diverges. Also $k^{1/k} < \log k$ for large k. Therefore $\frac{1}{k \log k} < \frac{1}{k^{1/k}k}$ for large k. Therefore we expect $\sum \frac{1}{k^{1/k}k}$ to diverge also. So we expect to be able to exhibit a divergent series $\sum b_k$ such that $\frac{a_k}{b_k} \to L$.

Proof. Let $a_k = \frac{1}{k^{1/k}k}$ and let $b_k = \frac{1}{k}$. Then

$$\frac{a_k}{b_k} = \frac{1}{k^{1/k}} = \left(\frac{1}{k}\right)^{1/k}.$$

Note that $0 < \left(\frac{1}{k}\right)^{1/k} < 1$ for $k > 1$.

Note that $\log \frac{a_k}{b_k} = -\frac{\log k}{k} \to 0$ as $k \to \infty$. TODO: proof (l'Hopital's rule?)

Want to show that $\left(\frac{1}{k}\right)^{1/k} \to L$ where $0 \leq L \leq 1$.

```
#+begin_src mathematica
SumConvergence[1/(k(k^(1/k))), k]
#+end_src

#+RESULTS:
: False
```

$\square$

Theorem. *Let f be continuous.* (x_n) *converges iff* $(f(x_n))$ *converges.*

(c)

Claim. *Let* $a_k = \frac{k!}{k^k}$*. Then* $\sum a_k$ *converges.*

Proof. Let $a_k = \frac{k!}{k^k}$. We have

$$\begin{aligned}
\lim_{k\to\infty} \frac{a_{k+1}}{a_k} &= \lim_{k\to\infty} \frac{(k+1)!}{(k+1)^{k+1}} \frac{k^k}{k!} \\
&= \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^k \\
&= \lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^{-k} \\
&= e^{-1} < 1,
\end{aligned}$$

therefore $\sum a_k$ converges by the Ratio Test.

```
#+begin_src mathematica
SumConvergence[(k!)/(k^k), k]
#+end_src

#+RESULTS:
: True
```

□

(d)

Claim. *Let* $a_k = \binom{2k}{k}^{-1}(4-10^{-23})^k$. *Then* $\sum a_k$ *converges.*

Proof. Let $c = 4 - 10^{-23} > 1$ so that

$$a_k = \binom{2k}{k}^{-1} c^k = \frac{c^k (k!)^2}{(2k)!}.$$

Then

$$\begin{aligned}
\frac{a_{k+1}}{a_k} &= \frac{c^{k+1}\left((k+1)!\right)^2}{\left(2(k+1)\right)!}\frac{(2k)!}{c^k(k!)^2} \\
&= c\frac{(k+1)^2}{(2k+2)(2k+1)} \\
&= c\frac{1+2k^{-1}+k^{-2}}{4+5k^{-1}+2k^{-2}}.
\end{aligned}$$

Therefore $\frac{a_{k+1}}{a_k} \to \frac{c}{4} < 1$, therefore $\sum a_k$ converges by the Ratio Test.

```
#+begin_src mathematica
SumConvergence[Binomial[2k, k]^(-1) * (4 - 10^(-23))^k, k]
#+end_src

#+RESULTS:
: True


#+begin_src mathematica
SumConvergence[Binomial[2k, k]^(-1) * (4 + 10^(-23))^k, k]
#+end_src

#+RESULTS:
: False
```

□

(e)

Claim. *Let* $a_k = \begin{cases} 2^{-k}k & \textit{if } k \textit{ is of the form } 2^m \textit{ for integer } m \\ 0 & \textit{otherwise} \end{cases}.$

Proof. Define $b_m = a_{2^m}$ for $m = 1, 2, \ldots$. We have

$$\begin{aligned}
\frac{b_{m+1}}{b_m} &= \frac{2^{-2^{m+1}}2^{m+1}}{2^{-2^m}2^m} \\
&= 2^{2^m - 2^{m+1} + 1} \\
&= 2^{2^m(1-2)+1} \\
&= 2^{1-2^m} \\
&= \frac{2}{2^m} \to 0.
\end{aligned}$$

Therefore $\sum b_m$ converges. But $\sum_{k=1}^{\infty} a_k = \sum_{m=1}^{\infty} \sum_{k=2^{m-1}}^{2^m-1} a_k = \sum_{m=1}^{\infty} b_m$. Therefore $\sum a_m$ converges. $\square$

1.2 (COMPLETISH)

2. (a) Use the Integral Test to prove that $\sum_{k\geqslant 3} \frac{1}{k(\log k)^p}$ converges if $p > 1$ and diverges if $0 < p \leqslant 1$ ($p \in \mathbb{R}$).

Claim. $\sum_{k\geq 3} \frac{1}{k(\log k)^p}$ *converges if* $p > 1$ *and diverges otherwise, for* $p \in \mathbb{R}$.

Proof. Note that $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1-p}(\log x)^{1-p} = \frac{1}{x(\log x)^p}$. Therefore

$$\int_3^\infty \frac{1}{x(\log x)^p}\,\mathrm{d}x = \left[\frac{1}{1-p}(\log x)^{1-p}\right]_3^\infty.$$

For $p > 1$ this integral converges to $\frac{\log 3}{p-1}$ and for $p \leq 1$ it diverges. ☐

```
#+begin_src mathematica
SumConvergence[1/((k+2) Log[k+2]^p),k]
#+end_src

#+RESULTS:
: p > 1
```

(b) For which positive values of α (α not assumed to be a natural number) does $\sum \frac{k^{-\alpha}}{1+\alpha^{-k}}$ converge?

INCOMPLETE

Proof. Let $a_k = \frac{k^{-\alpha}}{1+\alpha^{-k}}$.

Note that $a_k < k^{-\alpha}$ and that $\sum k^{-\alpha}$ converges for $\alpha > 1$ (Proof:).

Therefore $\sum a_k$ converges for $\alpha > 1$, by the Comparison Test (Simple Form).

For $\alpha = 1$ we have $a_k = \frac{1}{2k}$, which diverges [proof].

For $\alpha < 1$ we know that $\frac{1}{k^\alpha}$ diverges, but $\frac{\alpha^k}{1+\alpha^k} \to 0$, so it's possible that the product converges for some values of $\alpha < 1$. ☐

Apparently Mathematica can't solve this:

```
#+begin_src mathematica
SumConvergence[k^(-a)/(1 + a^(-k)), k]
#+end_src
```

```
#+RESULTS:
: SumConvergence[1/((1 + a^(-k))*k^a), k]
```

1.3

3. [A miscellany of examples. Standard convergence tests may be assumed where required] For which of the following choices of a_k is $\sum a_k$ convergent? Justify your answers briefly.

(a) $\dfrac{k+2^k}{2^k k}$; (b) $\dfrac{1}{k}\sin\dfrac{1}{k}$; (c) $\dfrac{\sinh k}{2^k}$;

(d) $(-1)^{k-1}\dfrac{\log k}{\sqrt{k}}$; (e) $\begin{cases} 1/k^2 & \text{if } k \text{ is odd,} \\ -(\log k)/k^2 & \text{if } k \text{ is even;} \end{cases}$ (f) $\left(1-\dfrac{1}{k}\right)^{k^2}$.

(a)

Claim. *Let* $a_k = \frac{k+2^k}{2^k k}$. *Then* $\sum a_k$ *diverges.*

Proof. Note that

(a) $a_k = \frac{1}{2^k} + \frac{1}{k} > \frac{1}{k}$, and

(b) $\sum \frac{1}{k}$ diverges.

Therefore $\sum a_k$ is divergent by the Comparison Test (simple form).

```
#+begin_src mathematica
SumConvergence[(k + 2^k)/(k * 2^k), k]
#+end_src

#+RESULTS:
: False
```

✓ □

(b)

Claim. *Let* $a_k = \frac{1}{k} \sin \frac{1}{k}$. *Then* $\sum a_k$ *converges.*

Proof. Note that $\sin x < x$ for $x > 0$. Therefore $a_k < \frac{1}{k^2}$. Therefore $\sum a_k$ converges by the Comparison Test (simple form).

```
#+begin_src mathematica
SumConvergence[Sin[1/k]/k, k]
#+end_src

#+RESULTS:
: SumConvergence[Sin[k^(-1)]/k, k]
```

Why doesn't Mathematica solve this?

□

(c) TODO: don't know anything about sinh or cosh.

INCOMPLETE

Claim. *Let* $a_k = \frac{\sinh k}{2^k}$*. Then* $\sum a_k$ *converges/diverges?*

Proof. By definition, $\sinh k = \sum_{n=0}^{\infty} \frac{k^{2n+1}}{(2n+1)!}$. Therefore

$$\frac{a_{k+1}}{a_k} = \frac{2^k}{2^{k+1}} \frac{\sum_{n=0}^{\infty} \frac{(k+1)^{2n+1}}{(2n+1)!}}{\sum_{n=0}^{\infty} \frac{k^{2n+1}}{(2n+1)!}}.$$

```
#+begin_src mathematica
SumConvergence[Sinh[k]/2^k, k]
#+end_src

#+RESULTS:
: False
```

□

(d)

Claim. *Let* $a_k = (-1)^{k-1}\frac{\log k}{\sqrt{k}}$. *Then* $\sum a_k$ *converges.*

Proof. INCOMPLETE For the AST we need to show that $\frac{\log k}{\sqrt{k}} \to 0$ monotonically.

Note that

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\log x}{\sqrt{x}} = \frac{1}{x}\cdot x^{-1/2} + (\log x)(-1/2)x^{-3/2} = \frac{2-\log x}{2x^{3/2}}.$$

Therefore $\frac{\mathrm{d}}{\mathrm{d}x}\frac{\log x}{\sqrt{x}} < 0$ for $x > e^2 = 7.38906....$

Therefore a tail of $\frac{\log k}{\sqrt{k}}$ is monotonic decreasing.

(Also that $2 < \log 9 < 3$, therefore $\frac{\log 9}{\sqrt{9}} < 1$. Therefore $0 < \frac{\log k}{\sqrt{k}} < 1$ for $k \in \{9, 10, 11, \ldots\}$.)

But does $\frac{\log k}{\sqrt{k}} \to 0$?

Let $\epsilon > 0$. We require $K \in \mathbb{N}$ such that $\frac{\log k}{\sqrt{k}} < \epsilon$ whenever $k \geq K$.

$$\begin{aligned}\frac{\log k}{\sqrt{k}} &< \epsilon \\ \log k &< \epsilon k^{1/2} \\ k &< e^{\epsilon k^{1/2}}\end{aligned}$$

$$\int_1^n \frac{\log x}{\sqrt{x}}\,\mathrm{d}x = 2\sqrt{x}(\log x - 2)\Big|_1^n = 2\sqrt{n}(\log n - 2) + 4$$

□

```
#+begin_src mathematica
SumConvergence[(-1)^(k-1)(Log[k])/(Sqrt[k]), k]
#+end_src

#+RESULTS:
: True
```

1.4

4. For each of the following power series $\sum c_k x^k$ establish which of the following is true: (i) $\sum |c_k x^k|$ converges for all $x \in \mathbb{R}$; (ii) $\sum |c_k x^k|$ converges only for $x = 0$; (iii) $\sum |c_k x^k|$ converges for $|x| < R$ and diverges for $|x| > R$ for some $R \in \mathbb{R}^{>0}$, which you should determine.

$$\text{(a)} \quad \sum k^{2015} x^k; \qquad \text{(b)} \quad \sum \frac{x^k}{2^k k^4}; \qquad \text{(c)} \quad \sum 2^k x^{k!};$$
$$\text{(d)} \quad \sum k^k x^k; \qquad \text{(e)} \quad \sum \frac{1}{(4k)!} x^{2k}; \qquad \text{(f)} \quad \sum \sin(k) x^k.$$

What can you say in each case about the values of x for which $\sum c_k x^k$ converges?

[*This is a familiarisation exercise on the notion of the radius of convergence of a power series. The intention is that you should make use of your knowledge of convergence tests for series to discover how the given power series behave, not that you should attempt to call on general results from Section 13 of the webnotes or from elsewhere.*]

(a)

Problem. *For what values of x does $\sum k^{2015} x^k$ converge?*

Proof.
Let $f_k(x) = k^{2015} x^k$. Note that

$$\begin{aligned} \lim_{k\to\infty} \left|\frac{f_{k+1}(x)}{f_k(x)}\right| &= \lim_{k\to\infty} |x| \frac{(k+1)^{2015}}{k^{2015}} \\ &= \lim_{k\to\infty} |x|(1 + k^{-1} + \ldots + k^{-2015}) \\ &= |x|. \end{aligned}$$

Note also that if $|x| = 1$ then $f_k(x) = k^{2015} \not\to 0$.

Therefore $\sum_k f_k(x)$ converges iff $|x| < 1$.

```
#+begin_src mathematica
SumConvergence[k^{2015}x^k, k]
#+end_src

#+RESULTS:
Abs (x) < 1
```

✓

□

(b)

Problem. *For what values of x does $\sum \frac{x^k}{2^k k^4}$ converge?*

Proof. Let $f_k(x) = \frac{x^k}{2^k k^4}$. Note that

$$\begin{aligned}
\lim_{k\to\infty} \left|\frac{f_{k+1}(x)}{f_k(x)}\right| &= \lim_{k\to\infty} \left|\frac{x^{k+1}}{x^k}\right| \frac{2^k k^4}{2^{k+1}(k+1)^4} \\
&= \lim_{k\to\infty} \frac{|x|}{2} \frac{k^4}{(k+1)^4} \\
&= \frac{|x|}{2}.
\end{aligned}$$

Further, if $|x| = 2$ then $f_k(x) = \pm\frac{1}{k^4}$.

Therefore $\sum f_k(x)$ converges for $|x| \leq 2$.

```
#+begin_src mathematica
SumConvergence[x^k / (2^k k^4), k]
#+end_src

#+RESULTS:
: Abs[x] < 2
```

TODO: Mathematica wrong?? Surely this converges for $|x| = 2$?

□

(c)

Problem. *For what values of x does $\sum 2^k x^{k!}$ converge?*

Proof. Let $f_k(x) = 2^k x^{k!}$. Note that

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{f_{k+1}(x)}{f_k(x)}\right| &= \lim_{k\to\infty}\frac{2^{k+1}}{2^k}\left|\frac{x^{(k+1)!}}{x^{k!}}\right| \\ &= 2\lim_{k\to\infty}|x^{kk!}|.\end{aligned}$$

Therefore $\sum f_k(x)$ converges iff $|x| < 1$.

```
#+begin_src mathematica
SumConvergence[2^k x^(k!), k]
#+end_src

#+RESULTS:
: SumConvergence[2^k*x^k!, k]
```

Why can't Mathematica solve this? □

(d)

Problem. *For what values of x does $\sum k^k x^k$ converge?*

Proof.

$$\lim_{k\to\infty} \frac{(k+1)^{k+1}}{k^k}\left|\frac{x^{k+1}}{x^k}\right| = \lim_{k\to\infty} |x|(k^1 + k^0 + k^{-1} + \ldots + k^{-k}),$$

so this converges for $x = 0$ only.

```
#+begin_src mathematica
SumConvergence[k^k x^k, k]
#+end_src

#+RESULTS:
: False
```

Why does Mathematica say this doesn't converge for $x = 0$?

□

(e)

Problem. *For what values of x does $\sum \frac{1}{(4k)!} x^{2k}$ converge?*

Proof. Let $f_k(x) = \frac{1}{(4k)!} x^{2k}$. Note that

$$\begin{aligned}
\lim_{k\to\infty} \left| \frac{f_{k+1}(x)}{f_k(x)} \right| &= \lim_{k\to\infty} \left| \frac{x^{2(k+1)}}{x^{2k}} \right| \frac{(4k)!}{(4(k+1))!} \\
&= \lim_{k\to\infty} x^2 \frac{1}{(4k+4)(4k+3)(4k+2)} \\
&= 0
\end{aligned}$$

Therefore the series $\sum_k f_k(x)$ converges for all $x \in \mathbb{R}$.

```
#+begin_src mathematica
SumConvergence[x^(2k) / (4k)!, k]
#+end_src

#+RESULTS:
: True
```

✓

□

(f)

Problem. *For what values of x does $\sum \sin(k)x^k$ converge?*

Proof. Let $f_k(x) = \sin(k)x^k$. Note that for $|x| \geq 1$ we have $f_k(x) \not\to 0$ therefore $\sum f_k(x)$ does not converge.

For $|x| < 1$ we have $|f_k(x)| \leq |x|^k$ which converges by the Comparison Test since $\sum |x|^k$ is a Geometric Series with ratio less than unity.

```
#+begin_src mathematica
SumConvergence[Sin[k] x^k, k]
#+end_src

#+RESULTS:
: Abs[x] < 1
```

✓ □

1.5

5. Give either a justification or a counterexample for each of the following statements about real series.

   (a) If $ka_k \to 0$ as $k \to \infty$ then $\sum a_k$ converges.

   (b) If $\lim\limits_{k\to\infty} \dfrac{a_{k+1}}{a_k}$ exists and equals L, where $L > 1$, then $\sum a_k$ diverges.

   (c) If $\sum a_k$ converges and $a_k/b_k \to 1$ then $\sum b_k$ converges.

(a)

Claim. *If $\lim\limits_{k\to\infty} ka_k = 0$ then $\sum a_k$ converges.*

Proof. This is False.

Let $a_k = \frac{1}{k \log k}$. Then $ka_k = \frac{1}{\log k} \to 0$ yet $\sum a_k$ diverges. □

(b)

Claim. *If $\lim\limits_{k\to\infty} \frac{a_{k+1}}{a_k}$ exists and equals $L > 1$ then $\sum a_k$ diverges.*

Proof. This is true by the Ratio Test, since if $\lim\limits_{k\to\infty} \frac{a_{k+1}}{a_k} = L > 1$ then $\lim\limits_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = L$. □

(c)

Claim. *If $\sum a_k$ converges and $a_k/b_k \to 1$ then $\sum b_k$ converges.*

Proof. $\sum b_k = \sum \frac{b_k}{a_k} a_k$. For large k we have $\frac{1}{2} < \frac{a_k}{b_k} < 2$. Therefore $\sum b_k$ converges iff □

1.6

6. [Optional, or for use for consolidation later)]

   (a) Prove that the series $\sum_{k \geqslant 2} (\log k)^{-k}$ and $\sum_{k \geqslant 2} (\log k)^{-\log k}$ converge.

   (b) Prove that, for any constant $\alpha > 0$, the series $\sum_{k \geqslant 2} (\log k)^{-\alpha}$ diverges.

1.7

A. A student is doing a homework exercise which asks for a statement of the simple Comparison Test and writes

$$0 \leqslant \sum a_k \leqslant \sum b_k \implies \sum a_k \text{ converges if } \sum b_k \text{ converges.}$$

Is this a valid formulation of the test and if not, why not?

1.8

B. Why is it not admissible to establish the convergence or divergence of a geometric series by using the Ratio Test?

1.9

C. Consider a real series $\sum(-1)^{k-1}u_k$ with $u_k \geqslant 0$. The Alternating Series Test imposes two further conditions on (u_k) to guarantee that the series will converge. Exhibit two divergent alternating series, one of which fails the first of these conditions and satisfies the second and another which satisfies the first condition and fails to satisfy the second. Check how the standard AST proof would fail on your examples.

What is the moral of this question?

2 Sheet 7

2.1

1. (a) Prove that

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots = \frac{3}{2}\log 2.$$

(This looks like a + + - + + - pattern with odd-numbered harmonic series terms in the + positions and power-of-two, or even-numbered, harmonic terms in the - positions).

(b) [Optional, for extra practice] Calculate the value of

$$\sum_{k=1}^{\infty} \frac{1}{k(9k^2 - 1)}.$$

2.2

2. [Consolidation] Find the radius of convergence of the following real power series (assume $k \geqslant 1$ where necessary):

   (a) $\sum(-1)^k k^2 x^k$; (b) $\sum \dfrac{(2k-1)(2k-3)\cdots 3\cdot 1}{k!} x^k$; (c) $\sum (x/k)^k$; (d) $\sum k^{1/k} x^k$.

2.3

3. (a) Give examples of real power series $\sum c_k x^k$ with the specified radius of convergence R and specified behaviour at $\pm R$:

   (i) $R = 1$ and the power series converges at -1 but diverges at 1;

   (ii) $R = 1$ and the power series converges at 1 and diverges at -1;

   (iii) $R = 2$ and the power series converges at 2 and -2;

   (iv) $R = 2$ and the power series diverges at 2 and -2.

(b) Let the real power series $\sum a_k x^k$, $\sum b_k x^k$ and $\sum c_k x^k$ have radius of convergence R, S and T, respectively, where $c_k = a_k + b_k$. Obtain a lower bound for T involving R and S. Provide examples to illustrate what possibilities can arise.

2.4

4. For which real values of x does $\sum x^k$ converge? Use the Differentiation Theorem for power series to evaluate

$$\text{(i)} \quad \sum_{k=1}^{\infty} kx^k; \qquad \text{(ii)} \quad \sum_{k=1}^{\infty} k^2 x^k,$$

specifying where the formulae you obtain are valid.

2.5

5. (a) Prove that the power series

$$\sum \frac{x^k}{(2k)!} \quad \text{and} \quad \sum \frac{x^k}{(2k+1)!}$$

have infinite radius of convergence.

(b) Define

$$p(x) = \sum_{k=0}^{\infty} \frac{x^k}{(2k)!} \quad \text{and} \quad q(x) = \sum_{k=0}^{\infty} \frac{x^k}{(2k+1)!}.$$

Use the Differentiation Theorem to compute $p'(x)$ and $q'(x)$ and prove that $2p'(x) = q(x)$ and $p(x) - q(x) = 2xq'(x)$. Hence prove that, for all x,

$$(p(x))^2 = 1 + x(q(x))^2.$$

2.6

6. Find the radius of convergence of the power series defining the function J_0, where

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k}.$$

Assuming term-by-term differentiation is allowed within the interval of convergence show that $y = J_0(x)$ is a solution of the equation

$$xy'' + y' + xy = 0.$$

2.7

7. [Optional] The *Fibonacci numbers* F_n are defined by $F_0 = 0$, $F_1 = 1$ and $F_{k+2} = F_{k+1} + F_k$ for $k \geqslant 0$. Define

$$F(x) = \sum_{k=0}^{\infty} F_k x^k.$$

Determine the radius of convergence the series defining $F(x)$. By summing the identity

$$F_{k+2}x^k = F_{k+1}x^k + F_k x^k$$

from $k = 0$ to ∞, or otherwise, find $F(x)$ in closed form.

2.8

8. [Optional: addition formula for sinh] Show that each of the power series $\sum \frac{x^{2k}}{(2k)!}$ and $\sum \frac{x^{2k+1}}{(2k+1)!}$ converges for all $x \in \mathbb{R}$. Define

$$C(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \quad \text{and} \quad S(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}.$$

   (a) Assuming the Differentiation Theorem for power series, calculate the derivatives of $C(x)$ and $S(x)$.

2.9

(b) For fixed $d \in \mathbb{R}$, let

$$f_d(x) = S(d+x)C(d-x) + S(d-x)C(d+x).$$

By considering the derivative of $f_d(x)$, prove that, for all $a, b \in \mathbb{R}$,

$$S(a+b) = S(a)C(b) + S(b)C(a).$$