Oxford M2 - Real Analysis II - Continuity and Differentiability

1 Sheet 2

> 1. (a) Let $f\colon (0, \infty) \to \mathbb{R}$ be defined by
>
> $$f(x) = \begin{cases} 1/x & \text{if } x \text{ is rational,} \\ x & \text{if } x \text{ is irrational.} \end{cases}$$
>
> Determine at which points f is continuous.

Theorem. *f is continuous at 1 and -1 only.*

Proof. Let $q \in \mathbb{Q} \cap (0, \infty)$.

Note that there exists a sequence (q_n) such that $q_n \in \mathbb{Q}$, $q_n \neq q$ and $\lim_{n\to\infty} q_n = q$.

From Theorem 1.13 (Algebra of Limits) we have $\lim_{x\to q} x^{-1} = q^{-1}$.

Therefore from Theorem 1.10 we have

Also, there exists a sequence (p_n) such that $p_n \in \mathbb{R} \setminus \mathbb{Q}$, and $\lim_{n\to\infty} p_n = q$.

□

Definition (continuity)**.** *f is continuous at a if $\lim_{x\to a} f(x) = f(a)$.*

Definition (limit)**.** *$\lim_{x\to a} f(x) = f(a)$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that $0 < |x-a| < \delta \implies |f(x) - f(a)| < \epsilon$.*

Theorem. *f is continuous nowhere. I.e. for all $a \in (0, \infty)$ there exists $\epsilon > 0$ such that for all $\delta > 0$ there exists $x \in (0, \infty)$ such that $|x - a| < \delta$ and yet $|f(x) - f(a)| \geq \epsilon$.*

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Intuition. However small we make δ, an interval of radius δ centered at a rational point will contain irrational points, and vice versa.

Proof.
Let $x \in (0, \infty)$.

Let $q \in \mathbb{Q} \cap (0, \infty)$. Then there exists a sequence:

Fix $\delta > 0$. Note that there are both rational x and irrational x satisfying $0 < |x - q| < \delta$. Therefore the maximum value attained by $|f(x) - f(q)|$ is $||$ $\square$

Let $p \in (0, \infty) \setminus \mathbb{Q}$ be an

Let $p, q \in (0, \infty)$ with $p \notin \mathbb{Q}$ and $q \in \mathbb{Q}$.