Oxford M2 - Real Analysis I - Sequences and Series

1 Sheet 1

https://courses.maths.ox.ac.uk/node/37482

1.1

1. Prove, *from the given Axioms for the Real Numbers*, that, for real numbers a, b, c, d:

   (a) $a(bc) = c(ba)$;

   (b) $-(a+b) = (-a) + (-b)$;

   (c) if $ab = ac$ and $a \neq 0$, then $b = c$;

   (d) if $a < b$ and $c < d$ then $a + c < b + d$;

   (e) if $a \leqslant b$ and $c \leqslant d$ then $a + c = b + d$ only if $a = b$ and $c = d$.

   [*You should write out fully detailed answers, justifying each line of your argument by appeal to one, and only one, of the axioms.*

(a)

Theorem. $a(bc) = c(ba)$

Proof.

$$\begin{aligned} a(bc) &= a(cb) && \text{(M1 commutativity of multiplication)} \\ &= (ac)b && \text{(M2 associativity of multiplication)} \\ &= (ca)b && \text{(M1 commutativity of multiplication)} \\ &= c(ab) && \text{(M2 associativity of multiplication)} \\ &= c(ba) && \text{(M1 commutativity of multiplication)} \end{aligned}$$

□

(b)

Theorem. $-(a+b) = (-a) + (-b)$

Proof.

$$
\begin{aligned}
(a+b)+(-(a+b)) &= 0 && \text{definition of negative}\\
\Big((-a)+(-b)\Big)+\Big((a+b)+(-(a+b))\Big) &= (-a)+(-b) && \text{add to both sides what axiom is this?}\\
\Big((-b)+(-a)\Big)+\Big((a+b)+(-(a+b))\Big) &= (-a)+(-b) && \text{S1 commutativity of sum}\\
\Big(((-b)+(-a))+(a+b)\Big)+(-(a+b)) &= (-a)+(-b) && \text{S2 associativity of sum}\\
\Big((-b)+((-a)+a)+b)\Big)+(-(a+b)) &= (-a)+(-b) && \text{S2 associativity of sum}\\
\Big((-b)+(0+b)\Big)+(-(a+b)) &= (-a)+(-b) && \text{definition of negative}\\
\Big((-b)+b\Big)+(-(a+b)) &= (-a)+(-b) && \text{definition of 0}\\
0+(-(a+b)) &= (-a)+(-b) && \text{definition of negative}\\
-(a+b) &= (-a)+(-b) && \text{definition of 0}
\end{aligned}
$$

$\square$

1.2

2. Prove the following assertions, for real numbers a, b, c:

(a) if $a < b$, then $ac > bc$ if and only if $c < 0$;

(b) $a^2 + b^2 = 0$ if and only if $a = b = 0$;

(c) $a^3 < b^3$ if and only if $a < b$.

[*Less detailed answers are required than in Q. 1, but you should justify each step using axioms or results which have already been proved from the axioms.*]

(a)

Theorem. *If $a < b$ then $ac > bc$ iff $c < 0$.*

Intuition. Multiplication by a scalar flips orientation iff the scalar is negative.

Proof.
TODO Prove carefully being explicit about which axioms are used.
Let $\mathbb{P}$ be the strictly positive reals. We have $a < b$, i.e. $b - a \in \mathbb{P}$.
$\Longrightarrow$:
We have $ac > bc$ i.e. $ac - bc \in \mathbb{P}$.

Therefore

$$\begin{aligned}
(a-b)c &\in \mathbb{P} \\
(b-a)(-c) &\in \mathbb{P} \\
\frac{1}{b-a}(b-a)(-c) &\in \mathbb{P} \\
(-c) &\in \mathbb{P}
\end{aligned}$$

$\Longleftarrow$:
We have $c < 0$, i.e. $-c \in \mathbb{P}$. Therefore $(-c)(b-a) \in \mathbb{P}$ (by **P2**). Therefore $ac - bc \in \mathbb{P}$, i.e. $ac > bc$. $\square$

1.3

3. (a) Prove that $(a^m)^{-1} = (a^{-1})^m$, for all $a \in \mathbb{R} \setminus \{0\}$ $(m = 1, 2, \ldots)$.

   (b) Prove that $a^{k+1} = a^k a$ for $a \neq 0$ and $k = -1, -2, -3, \ldots$.

   (c) Derive the law of indices: $a^m a^n = a^{m+n}$ for $a \neq 0$ and $m, n \in \mathbb{Z}$.

1.4

4. [*In this question you may use familiar results about arithmetic and order in the real numbers. You are not expected to justify each line of your argument by citing axioms or properties derived from the axioms.*] For $n = 1, 2, 3, \ldots$, let

$$a_n := \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad b_n := \left(1 + \frac{1}{n}\right)^{n+1}.$$

   (a) Show that the inequality $a_n \leqslant a_{n+1}$ can be rearranged as

$$\left(\frac{n\,(n+2)}{(n+1)^2}\right)^{n+1} \geqslant \frac{n}{n+1}.$$

   By applying Bernoulli's inequality to the left-hand side, verify this inequality.

   (b) Show that $b_{n+1} \leqslant b_n$ for all n.

   (c) Note that $a_n \leqslant a_{n+1} \leqslant b_{n+1} \leqslant b_n$ for all n. Deduce that $a_n < 3$ for all n.

   [*The significance of the result in this exercise will be revealed later.*]

1.5

5. Define $\max(a, b)$, the *maximum* of two real numbers a and b, saying which axiom(s) show that your specification is well-defined. Using your definition prove that $\max(a, b) = \frac{1}{2}(a+b)+\frac{1}{2}|a-b|$, and write down an analogous formula for $\min(a, b)$.

Definition. *Let $a, b \in \mathbb{R}$ with $a \neq b$.*

$$\max(a, b) = \begin{cases} b & \text{if } a < b \\ a & \text{if } b < a. \end{cases}$$

This is well-defined (a valid function) by trichotomy of order properties.

Theorem.

$$\max(a, b) = \frac{1}{2}(a + b) + \frac{1}{2}|a - b|.$$

2 Sheet 2

2.1

1. For each of the following conditions, find and sketch the set of all pairs $(x, y) \in \mathbb{R}^2$ satisfying the condition:

   (a) $|x| + |y| = 1$; (b) $|x - y| + |x + y| = 1$; (c) $\max(|x|, |y|) = 1$.

2.2

2. A student claims erroneously that

$$\frac{1}{|a+b|} \leqslant \frac{1}{|a|+|b|} \quad \text{if } a+b \neq 0.$$

Find an example to show that this is false in general. How might the student have come to make the error? Find an argument based on the Reverse Triangle Law which gives a valid upper bound for $1/|a+b|$ in terms of $|a|$ and $|b|$.

2.3 (COMPLETE)

3. Prove that $\{ 2^n \mid n \in \mathbb{N} \}$ is not bounded above. [*You may not make any use of logarithms.*]

Proof. Let $S = \{2^n \mid n \in \mathbb{N}\}$ and suppose an upper bound for S exists.

Then $\sup S$ exists by completeness of the reals.

By the Approximation Property there exists $2^k \in S$ such that

$$\sup S - \frac{1}{2} < 2^k \leq \sup S.$$

Note that $k + 1 \in \mathbb{N}$ therefore $2^{k+1} = 2^k + 2^k \in S$. Therefore

$$\sup S - \frac{1}{2} + 2^k < 2^{k+1} \leq \sup S$$
$$\sup S \leq \sup S + \frac{1}{2} - 2^k < \sup S,$$

a contradiction. Therefore no upper bound for S exists. ☐

2.4 (COMPLETE)

4. For each of the following sets, decide which of the following exist: (i) supremum; (ii) infimum; (iii) maximum; (iv) minimum.

   (a) $\mathbb{Q} \cap [0, \sqrt{2}]$;

   (b) $\{ (-1)^n + 1/n \mid n = 1, 2, \dots \}$;

   (c) $\{ 3^n \mid n \in \mathbb{Z} \}$;

   (d) $\bigcup_{n=1}^{\infty} \left[\frac{1}{2n}, \frac{1}{2n-1}\right]$.

   In each case, either write down the value (without proof) or indicate why it is not defined.

1. $S = \mathbb{Q} \cap [0, \sqrt{2}]$

$$\begin{aligned} \sup S &= \sqrt{2} \\ \max S &= \sqrt{2} \\ \min S &= 0 \\ \inf S &= 0 \end{aligned}$$

2. $S = \{(-1)^n + 1/n \mid n = 1, 2, \dots\}$

$$\begin{aligned} \sup S &= 3/2 \\ \max S &= 3/2 \\ \min S &\text{ does not exist since } \inf S \notin S \\ \inf S &= -1 \end{aligned}$$

3. $S = \{3^n \mid n \in \mathbb{Z}\}$

$$\begin{aligned} \sup S &\text{ does not exist, } S \text{ has no upper bound} \\ \max S &\text{ does not exist, } S \text{ has no upper bound} \\ \min S &\text{ does not exist, } S \text{ has no lower bound} \\ \inf S &= 0 \end{aligned}$$

4. $S = \cup_{n=1}^{\infty}[\frac{1}{2n}, \frac{1}{2n-1}]$

$$\begin{aligned} \sup S &= 1 \\ \max S &= 1 \\ \min S &\text{ does not exist, } S \text{ has no lower bound} \\ \inf S &= 0 \end{aligned}$$

2.5 (COMPLETE)

5. Let S, T be non-empty subsets of $\mathbb{R}$.

   (a) Assume that S and T are bounded above. Prove that $S \cup T$ is bounded above and that $\sup(S \cup T) = \max(\sup S, \sup T)$.

Proof. Let $S, T \subset \mathbb{R}$ with $S, T \neq \varnothing$. Assume that S and T are bounded above. Then $\sup S$ and $\sup T$ exist. Let $b = \max(\sup S, \sup T)$.

Then $b \geq s$ for all $s \in S$ and $b \geq t$ for all $t \in T$. Therefore b is an upper bound for $S \cup T$.

Suppose there exists $a < b$ such that a is an upper bound of $S \cup T$. Then either $a < \sup S$ or $a < \sup T$. Without loss of generality, suppose $a < \sup S$. Then a is not an upper bound of S. Therefore there exists $s \in S$ such that $s > a$. But $s \in S \cup T$, therefore a is not an upper bound of $S \cup T$, a contradiction. $\square$

(b) Assume that S and T are bounded below. Prove that the set

$$S + T := \{\, s + t \mid s \in S, t \in T \,\}$$

is bounded below and that $\inf(S + T) = \inf S + \inf T$.

Proof. Let $S, T \subset \mathbb{R}$ with $S, T \neq \varnothing$. Assume that S and T are bounded below. Then $\inf S$ and $\inf T$ exist. Define

$$S + T := \{s + t \mid s \in S, t \in T\}.$$

Let $a = \inf S + \inf T$.

We claim that a is a lower bound for $S + T$, i.e. $u \geq a$ for all $u \in S + T$.

Let $u \in S + T$. Then $u = s + t$ for some $s \in S, t \in T$. Therefore

$$u = (\inf S + v) + (\inf T + w) = a + (v + w) \geq a$$

for some $v, w \geq 0$. Therefore a is a lower bound for $S + T$ as claimed.

We further claim that $a = \inf S + T$.

Suppose for a contradiction that there exists $b > a$ such that b is a lower bound for $S + T$. Then

$$b = a + v + w = (\inf S + v) + (\inf T + w)$$

for some $v, w > 0$. Fix $0 < \delta < \min(v, w)$. By **4.9** (Approximation Property of the supremum/infimum) there exist $s \in S$ and $t \in T$ such that $\inf S \leq s < \inf S + \delta$ and $\inf T \leq t < \inf T + \delta$. But then $s + t \in S + T$ and $s + t < b$, so b is not a lower bound for $S + T$, a contradiction. Therefore $a = \inf S + T$ as claimed. $\square$

2.6

6. Prove that there exists a unique real number a such that $a^3 = 2$. [*Hint: adapt the proof of the existence of* $\sqrt{2}$ *from lectures.*]

Proof. Let $S = \{x \in \mathbb{R} \mid x^3 < 2\}$. Since S is bounded above, $\sup S$ exists. Let $a = \sup S$. By trichotomy it suffices to show that $a^3 < 2$ and $a^3 > 2$ lead to contradictions.

First suppose $a^3 < 2$. We seek $h > 0$ such that $(a+h)^3 < 2$ since then $a + h \in S$, which would contradict the definition $a := \sup S$. Note that

$$\begin{aligned}
(a+h)^3 &= a^3 + 3a^2h + 3ah^2 + h^3 - 2 \\
&< a^3 + 7a^2h - 2 && \text{if } h < a \\
&< 0 && \text{if } h < \frac{2-a^3}{7a^2},
\end{aligned}$$

therefore if we take $h < \min\left(a, \frac{2-a^3}{7a^2}\right)$ then we have the desired contradiction.

Alternatively suppose that $a^3 > 2$. By the Approximation Property for all $0 < h < a$ we can find $s \in S$ such that $a - h < s$, therefore $(a-h)^3 < s^3$. We seek an h for which $(a-h)^3 > 2$ since then we would have $s^3 > 2$ which would contradict the definition of S. Note that

$$\begin{aligned}
(a-h)^3 - 2 &= a^3 - 3a^2h + 3ah^2 - h^3 - 2 \\
&> 0 && \text{if } 3ah^2 > 3a^2h + h^3.
\end{aligned}$$

So we require $3ah > 3a^2 + h^2 \iff 3a^2 - 3ah + h^2 < 0$.

Incomplete $\square$

2.7

7. (a) Prove that there are uncountably many irrational numbers.

Definition (Countable)**.** *A set S is countable if $S \preceq N$. I.e. there exists an injection $f : S \to \mathbb{N}$.*

Definition (Uncountable)**.** *A set S is uncountable if it is not countable.*

Definition (Injection)**.** *A function $A \to B$ is an injection if $f(a_1) = f(a_2) \implies a_1 = a_2$.*

Proof. Suppose for a contradiction that $\mathbb{R} \setminus \mathbb{Q}$ is countable. Then an injection $f : \mathbb{R} \setminus \mathbb{Q} \to \mathbb{N}$ exists. Since $\mathbb{Q}$ is countable, an injection $g : \mathbb{Q} \to \mathbb{N}$ exists. Consider the function $h : \mathbb{R} \to \mathbb{N}$ defined by

$$h(x) = \begin{cases} 2f(x) & x \in \mathbb{R} \setminus \mathbb{Q} \\ 2g(x) + 1 & x \in \mathbb{Q}. \end{cases}$$

We claim that $h : \mathbb{R} \to \mathbb{N}$ is an injection. Note that

(i) $h(\mathbb{R} \setminus \mathbb{Q}) \cap h(\mathbb{Q}) = \varnothing$

(ii) The $\mathbb{R} \to \mathbb{R}$ functions defined by $x \mapsto 2x$ and $x \mapsto 2x + 1$ are both injections.

(iii) The composition of two injections is an injection.

Suppose $h(x_1) = h(x_2)$. Then either $x_1, x_2 \in \mathbb{Q}$ or $x_1, x_2 \in \mathbb{R} \setminus \mathbb{Q}$ by (i). In both cases we have $x_1 = x_2$ by (ii) and (iii). Therefore $h : \mathbb{R} \to \mathbb{N}$ is an injection, hence $\mathbb{R}$ is countable.

But $\mathbb{R}$ is not countable, so we have a contradiction. Therefore no such injection f exists, i.e. $\mathbb{R} \setminus \mathbb{Q}$ is uncountable. □

Lemma. *Let a, b be real numbers with $a < b$. Let $\delta \in \mathbb{R}$ with $0 < \delta < b - a$. Then there exists $m \in \mathbb{N}$ such that $m\delta \in (a, b)$.*

Proof. Let $m \in \mathbb{N}$, $\delta \in \mathbb{R}$ with $0 < \delta < b - a$, and define $S = \{m\delta \mid m\delta < b\}$. Note that $\delta \in S$. Therefore S is non-empty, bounded above, and finite, hence $\max S$ exists.

Let $M\delta = \max S$. We have $M\delta < b$. Suppose for a contradiction that $M\delta \leq a$. Recall that $\delta < b - a$. Summing these inequalities gives $(M + 1)\delta < b$. But then $M\delta \neq \max S$ since $(M + 1)\delta > M\delta$ and $(M + 1)\delta \in S$. This is a contradiction, therefore $M\delta \in (a, b)$. □

(b) Let a, b be real numbers with $a < b$. Show that there is a natural number n such that $\frac{1}{n} < b - a$. Deduce that there is a rational number in the interval (a, b).

Proof. Let $a, b \in \mathbb{R}$ with $a < b$. Then $b - a > 0$. Since $\mathbb{N}$ is not bounded above (Archimedean property) there exists $n \in \mathbb{N}$ such that $n > 1/(b - a)$, therefore $1/n < b - a$. Therefore, by the lemma with $\delta = 1/n$, there exists $m \in \mathbb{N}$ such that $m/n \in (a, b) \cap \mathbb{Q}$. $\square$

(c) Show further that between any two real numbers there is an irrational number.

Proof. Let $a, b \in \mathbb{R}$ with $a < b$. By the Archimedean property of $\mathbb{N}$, there exists $n \in \mathbb{N}$ be such that $1/n < (b - a)/\pi$, therefore $\pi/n < (b - a)$. Therefore, by the lemma with $\delta = \pi/n$, there exists $m \in \mathbb{N}$ such that $m\pi/n \in (a, b) \cap (\mathbb{R} \setminus \mathbb{Q})$. $\square$

2.8

Lemma. *The set of polynomials of degree n with integer coefficients is countable.*

Proof. Let $\{\alpha_1, \alpha_2, \ldots\}$ be the positive prime numbers and let

$$P_n := \left\{ \sum_{i=0}^{n} a_i z^i \mid a_0, \ldots, a_n \in \mathbb{Z}, a_n \neq 0 \right\}$$

be the set of polynomials with integer coefficients of degree n. Note that $|P_n| = \left| \left(\mathbb{Z} \setminus \{0\}\right) \times \mathbb{Z}^{n-1} \right|$ and that $f : \left(\mathbb{Z} \setminus \{0\}\right) \times \mathbb{Z}^{n-1} \to \mathbb{N}$ given by

$$f(a_0, a_1, \ldots, a_n) = \prod_{i=0}^{n} \alpha_i^{a_i}$$

is a bijection by uniqueness of prime factorization of the natural numbers. Therefore P_n is countable. □

> 8. [Optional] A complex number is said to be *algebraic* if it is the root of a polynomial with integer coefficients and otherwise is said to be *transcendental*.
>
>   (a) Show that there are only countably many cubic polynomials with integer coefficients.

Proof. This follows from the lemma with $n = 3$. □

Lemma (Countable union of countable sets is countable)**.**
Let $I \subseteq \mathbb{N}$ and let S_i be a countable set for all $i \in I$. Then $\bigcup_{i \in I} S_i$ is countable.

Proof. Let s_{ij} be the j-th element of S_i. The $f : \bigcup_{i \in I} S_i \to \mathbb{N}$ given by

$$f(s_{ij}) = 2^i 3^j$$

is a bijection, proving that $\bigcup_{i \in I} S_i$ is countable. But what about repeated elements? □

> (b) Show that there are only countably many complex numbers which are roots of some integer-coefficient cubic polynomial.

Lemma. *Let P_n be the set of polynomials with integer coefficients of degree n. Then the set of complex numbers that are roots of a polynomial in P_n is countable.*

Proof. Let $P_{n,k}$ be the set of polynomials of degree n that have $k \leq n$ distinct roots. Since $P_{n,k} \subseteq P_n$, we have that $P_{n,k}$ is countable. Therefore For each polynomial in $P_{n,k}$ order the k distinct roots and label them $1, \ldots, k$. Then $P_{n,k}$ is countable since... □

Proof. Note that a cubic polynomial has at most 3 distinct roots.

Let U be the set of complex numbers that are roots of some integer-coefficient cubic polynomial. For each such polynomial, assign a distinct label from $\{0, 1, 2\}$ to each of the distinct roots. Then the cardinality of U is equal(*) to the cardinality of $B := \mathbb{Z} \setminus \{0\} \times \mathbb{Z}^3 \times \{0, 1, 2\}$. The set B is countable since $f : T \to \mathbb{N}$ given by $f(a, b, c, d, e) = 2^a 3^b 5^c 7^d 11^e$ is an injection. (* How to properly deal with the fact that U is smaller than B due to some polynomials having fewer than 3 distinct roots?) □

(c) Show further that there are only countably many algebraic numbers but uncountably many transcendental numbers.

Proof. Let A_n be the set of complex numbers which are roots of a polynomial of degree n.

Claim: A_n is countable for all $n \in \mathbb{N}$, since there are countably many polynomials of degree n and each has a finite number of roots.

The set of algebraic numbers is $A := \cup_{n \in \mathbb{N}} A_n$. Let a_{ijk} be the k-th distinct root of the j-th polynomial of degree i. Then f :

(This differs because whereas previously we were restricted to cubics, now the polynomials can be of any degree $n \in \mathbb{N}$.)

The cardinality of the set of algebraic numbers is Incomplete. □

3 Sheet 3

3.1

1. For each of the following choices of a_n, and for arbitrary $\varepsilon > 0$, find N such that $|a_n| < \varepsilon$ whenever $n \geqslant N$. [*You only need to find a value of* N *that works and not necessarily the smallest such* N.]

$$\text{(i)} \quad \frac{1}{n^2+3}, \qquad \text{(ii)} \quad \frac{1}{n(n-\pi)}, \qquad \text{(iii)} \quad \frac{1}{\sqrt{5n-1}}.$$

3.2

2. Use sandwiching arguments to prove that, for each of the following choices of a_n, the sequence (a_n) converges to 0:

$$\text{(i)} \quad \frac{n+1}{n^2+n+1}, \qquad \text{(ii)} \quad 2^{-n}\cos(n^2), \qquad \text{(iii)} \quad \sin\frac{1}{n}, \qquad \text{(iv)} \quad \begin{cases} 1/2^n & \text{if } n \text{ is prime,} \\ -1/3^n & \text{otherwise.} \end{cases}$$

3.3

3. (a) Prove that $\sqrt{n+1} - \sqrt{n} \to 0$ by using the identity $a - b = (\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$ (for $a, b \in \mathbb{R}^{\geqslant 0}$).

   (b) Prove that $n^{1/n} \geqslant 1$ for $n = 1, 2, \ldots$. Let $a_n = n^{1/n} - 1$. Prove, by applying the binomial theorem to $(1 + a_n)^n$, that

$$a_n \leqslant \sqrt{\frac{2}{n-1}} \quad \text{for } n > 1.$$

   Deduce that $n^{1/n} \to 1$.

3.4

4. (a) Write down, carefully quantified, what it means for a real sequence (a_n) (i) to be convergent; (ii) to be bounded; (iii) to tend to infinity. Write down, carefully quantified, the negations of (i), (ii), (iii).

   (b) Formulate and prove an analogue of the Sandwiching Lemma applicable to real sequences which tend to infinity.

   (c) For each of the following choices of a_n decide whether or not the sequence tends to infinity:

$$\text{(i)}\quad \frac{n^2+n+1}{n+1}, \quad \text{(ii)}\quad n^2\sin n, \quad \text{(iii)}\quad \frac{n^{3/4}}{\sqrt{5n-1}}, \quad \text{(iv)}\quad \left(1+\frac{1}{n}\right)^n.$$

   Justify your answers briefly.

3.5

5. Assume that (a_n) is a sequence such that $a_n \to 0$. Let (b_n) be a bounded sequence. Prove that $a_n b_n \to 0$.

   Give an example of a single sequence (a_n) such that $a_n \to 0$ and of appropriate sequences (c_n) to demonstrate that each of the following possibilities can occur:

   (i) $a_n c_n \to 0$ and (c_n) is unbounded;
   (ii) $a_n c_n \to \infty$;
   (iii) $(a_n c_n)$ converges to a non-zero limit;
   (iv) $(a_n c_n)$ is bounded and divergent;
   (v) $a_n c_n \to -\infty$.

3.6

6. For each of the following choices of z_n decide whether or not (z_n) converges:

$$\text{(i)}\quad \left(\frac{1}{1+\mathrm{i}}\right)^n, \quad \text{(ii)}\quad \frac{(1-\mathrm{i})n}{n+\mathrm{i}}, \quad \text{(iii)}\quad (-1)^n\frac{n+\mathrm{i}}{n}.$$

   Give brief justifications for your answers.

3.7

7. [Later parts may be treated as optional] Let c be a complex number. The complex numbers $z_n(c)$ are defined recursively by

$$z_1(c) = c, \qquad z_{n+1}(c) = (z_n(c))^2 + c \ \text{ for } \ n \geqslant 1.$$

The *Mandelbrot set* is defined by

$$M = \{\, c \in \mathbb{C} \mid \text{ the sequence } (z_n(c)) \text{ is bounded} \,\}.$$

(a) Show that each of $-2, -1, 0, \mathrm{i}$ lies in M but that $1 \notin M$.

(b) Show that if $c \in M$ then $\overline{c} \in M$, where $\overline{c}$ denotes the conjugate of c.

(c) Show that if $|c| \leqslant 1/4$ then $|z_n(c)| < 1/2$ for all n. (So if $|c| \leqslant 1/4$ then $c \in M$.)

(d) Show that if $|c| = 2 + \varepsilon$ where $\varepsilon > 0$, then $|z_n(c)| \geqslant 2 + a_n\varepsilon$ for $n \geqslant 1$ where $a_n = (4^n + 2)/6$. Deduce that the Mandelbrot set lies entirely within the disc $|z| \leqslant 2$.

4 Misc

(Question from Alex Coward)

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & x \in (\mathbb{R} \setminus \mathbb{Q}) \cup \{0\} \\ \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q}, \gcd(p,q) = 1, q > 0. \end{cases}$$

Where is f continuous?

Definition (Stride)**.** *Let* $S = \left\{ \frac{1}{q} \mid q \in \mathbb{N} \right\}$*. For* $x = \frac{p}{q} \in \mathbb{Q}$*, where* $\gcd(p,q) = 1$*,* $q > 0$*, define the stride function* $s : \mathbb{Q} \to S$ *by* $x \mapsto \frac{1}{q}$*. We say that the stride of* x *is* $s(x)$*.*

Claim. f *is continuous at* 0*, i.e.* $\lim_{x \to 0} f(x) = 0$*.*

Proof. Fix $\epsilon > 0$ and take $\delta = \epsilon$. We claim that $0 < |x| < \delta \implies |f(x)| < \epsilon$.

At irrational values $0 < |x| < \delta$, we have $|f(x)| = 0 < \epsilon$.

At rational values $0 < |x| < \delta$, we have $x = \frac{p}{q} \in \mathbb{Q}, \gcd(p,q) = 1$ for some $p \in \mathbb{Z}, q \in \mathbb{N}$ with $p, q \neq 0$. Therefore

$$|f(x)| = \left| \frac{1}{q} \right| \leq \left| \frac{p}{q} \right| = |x| < \delta = \epsilon.$$

$\square$

Claim. f *is continuous on* $\mathbb{R} \setminus \mathbb{Q}$*.*

Proof. Let $a \in \mathbb{R} \setminus \mathbb{Q}$ and fix $0 < \epsilon < 1$. Note that $f(a) = 0$. We now construct an interval centered at a that excludes rational numbers with stride greater than or equal to ϵ.

Formally, we claim that there exists $\delta > 0$ such that $0 < |x - a| < \delta \implies |f(x)| < \epsilon$.

Define $T = \{\frac{1}{n} \mid n \in \mathbb{N}, \frac{1}{n} > \epsilon\}$. Note that T is non-empty since $\epsilon < 1$ therefore $1 \in T$. Since T is non-empty, finite, and bounded below, $\min T$ exists. Let $\frac{1}{m} = \min T$.

Define $R \subset \mathbb{R}$ by $R = \{x \in \mathbb{Q} \mid x > a, s(x) \geq \epsilon\}$. Note that $\frac{1}{m} \in R$ therefore R is non-empty. Also R is bounded below, therefore $\inf R$ exists. We claim that $\inf R \in R$. By the Approximation theorem, for all $\gamma > 0$ there exists $r \in R$ such that $\inf R \leq r < \inf R + \gamma$. Suppose

$\square$