

# Oxford A1 - Differential Equations

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## 1 Sheet 1

1.1 Let  $[a, b]$  be a closed and bounded interval of the real line and let  $\{y_n\}_{n \geq 0}$  be a sequence of real-valued functions, each of which is defined on  $[a, b]$ . What does it mean to say that **the sequence converges uniformly on  $[a, b]$  to a limit function  $y$** ? If each  $y_n$  is continuous on  $[a, b]$  show that the uniform limit  $y$  is continuous on  $[a, b]$  and that, when  $n \rightarrow \infty$ ,

$$\int_a^b |y_n(x) - y(x)| dx \rightarrow 0, \quad \int_a^b y_n(x) dx \rightarrow \int_a^b y(x) dx.$$

### (a) Definition of uniform convergence

The sequence of functions  $\{y_n\}_{n \geq 0}$  **converges uniformly on  $[a, b]$  to  $y$**  if and only if for all  $\epsilon > 0$  there exists an  $m \in \mathbb{N}$  such that for all  $n > m$  and for all  $x \in [a, b]$ ,  $|y_n(x) - y(x)| < \epsilon$ .

### (b) Show that the limit function is continuous

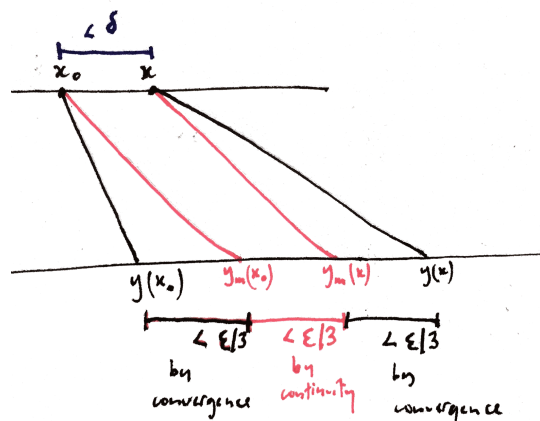
The claim is that if each  $y_n$  is continuous on  $[a, b]$  then  $y$  is continuous on  $[a, b]$ . We are told that

1.  $\{y_n\}_{n \geq 0}$  converges uniformly to  $y$ , and
2. each  $y_n$  is continuous on  $[a, b]$ .

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<https://courses.maths.ox.ac.uk/node/5372>

Informal illustration of proof:



Fix arbitrary  $\epsilon > 0$  and  $x_0 \in [a, b]$ .

Let  $m \in \mathbb{N}$  be such that  $|y_m(x_0) - y(x_0)| < \epsilon/3$ . Such an  $m$  exists because the  $\{y_n\}$  converge uniformly to  $y$ .

Let  $\delta$  be such that  $|x - x_0| < \delta \implies |y_m(x) - y_m(x_0)| < \epsilon/3$ . Such a  $\delta$  exists because  $y_m$  is continuous on  $[a, b]$ .

Fix an arbitrary  $x$  such that  $|x - x_0| < \delta$ .

Now we have the following:

1.  $|y(x_0) - y_m(x_0)| < \epsilon/3$  by convergence of the  $\{y_n\}$
2.  $|y_m(x_0) - y_m(x)| < \epsilon/3$  by continuity of  $y_m$
3.  $|y_m(x) - y(x)| < \epsilon/3$  by convergence of the  $\{y_n\}$

Therefore  $|y(x_0) - y(x)| < \epsilon$ , proving continuity of  $y$  on  $[a, b]$ . □

(Approximate time taken for reading and producing an answer: 4hrs)

**(c) Show limit of definite integral I**

Let  $I_n = \int_a^b |y_n(x) - y(x)| dx$ .

The claim is that  $\lim_{n \rightarrow \infty} I_n = 0$ .

In other words  $\forall \epsilon > 0 : \exists m \in \mathbb{N} : \forall n > m : |I_n - 0| < \epsilon$ .

Fix an  $\epsilon > 0$ .

Since the  $\{y_n\}$  converge uniformly to  $y$ , there exists an  $m \in \mathbb{N}$  such that for all  $n > m$  and for all  $x \in [a, b]$

$$|y_n(x) - y(x)| < \epsilon/(b - a).$$

Therefore  $\int_a^b |y_n(x) - y(x)| \, dx < \epsilon$  for all  $n > m$ , as required.  $\square$

#### (d) Show limit of definite integral II

The claim is that  $\lim_{n \rightarrow \infty} \int_a^b y_n(x) \, dx = \int_a^b y(x) \, dx$ .

In other words:  $\forall \epsilon > 0 : \exists m \in \mathbb{N} : \forall n > m :$

$$\left| \left( \int_a^b y_n(x) \, dx \right) - \left( \int_a^b y(x) \, dx \right) \right| < \epsilon.$$

This is equivalent to:  $\forall \epsilon > 0 : \exists m \in \mathbb{N} : \forall n > m :$

$$A_1 := \left| \int_a^b (y_n(x) - y(x)) \, dx \right| < \epsilon.$$

From part (c) above, we know that:  $\forall \epsilon > 0 : \exists m \in \mathbb{N} : \forall n > m :$

$$A_2 := \int_a^b |y_n(x) - y(x)| \, dx < \epsilon.$$

Now<sup>1</sup> if the sign of  $y_n(x) - y(x)$  is constant for all  $x \in [a, b]$  (i.e. the graphs do not cross over), then  $A_1 = A_2 < \epsilon$ . Otherwise, there is some cancellation in the integral  $A_1$  and  $0 \leq A_1 < A_2 < \epsilon$ . So the same choice of  $m$  as was used in part (c) works here, since for that value of  $m$ , we have  $A_1 < \epsilon$  as required.  $\square$

(Approximate time taken for (c) and (d): 2hrs)

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<sup>1</sup>How do I prove this section properly?

If  $[a, b] = [0, 1]$  and  $y_n(x) = nxe^{-nx^2}$  show that, for each  $x \in [0, 1]$ ,  $y_n(x) \rightarrow 0$  but  $\int_0^1 y_n(x) dx \rightarrow \frac{1}{2}$ . Thus the convergence must be non-uniform. Show that

$$\max_{0 \leq x \leq 1} y_n(x) = \sqrt{\frac{n}{2e}}$$

and sketch the graph of  $y_n(x)$  versus  $x$ .

To show that  $y_n(x) := \frac{nx}{e^{nx^2}} \rightarrow 0$  for all  $x \in [0, 1]$ , first note that it is true for  $x = 0$  since  $y_n(0) = 0$  for all  $n \in \mathbb{N}$ . So we have to show it is true for  $x \in (0, 1]$ .

Fix  $x \in (0, 1]$  and define  $f(\alpha) = \frac{\alpha x}{e^{\alpha x^2}}$  for  $\alpha \in \mathbb{R}$ .  $\lim_{\alpha \rightarrow \infty} f(\alpha)$  is an indeterminate form  $\frac{\infty}{\infty}$  and we can use l'Hôpital's rule, differentiating with respect to  $\alpha$ :

$$\lim_{\alpha \rightarrow \infty} \frac{\alpha x}{e^{\alpha x^2}} = \lim_{\alpha \rightarrow \infty} \frac{x}{x^2 e^{\alpha x^2}} = 0.$$

Since  $f(\alpha) = y_n$  at integer values of  $\alpha$  it follows that  $\lim_{n \rightarrow \infty} y_n(x) = 0$  for all  $x \in (0, 1]$ .  $\square$

For the limit of the definite integral we have

$$\int_0^1 nxe^{-nx^2} dx = \left[ -\frac{1}{2}e^{-nx^2} \right]_0^1 = \frac{1}{2}(1 - e^{-n}),$$

and so  $\lim_{n \rightarrow \infty} \int_0^1 y_n(x) dx = \frac{1}{2}$ .  $\square$

To find the maximum value attained by  $y_n(x)$  for  $x \in [0, 1]$ , note that the derivative is

$$\frac{dy_n(x)}{dx} = nx(-2nx)e^{-nx^2} + ne^{-nx^2} = ne^{-nx^2}(1 - 2nx^2),$$

and therefore that the only solution to  $\frac{dy_n(x)}{dx} = 0$  for  $x \in [0, 1]$  is  $x = \frac{1}{\sqrt{2n}}$ .

The second derivative is

$$ne^{-nx^2}(-4nx) - 2n^2xe^{-nx^2}(1 - 2nx^2) = 2n^2xe^{-nx^2}(2nx^2 - 3).$$

This is negative at the critical point  $x = \frac{1}{\sqrt{2n}}$  showing that it is a maximum. Therefore

$$\max_{x \in [0, 1]} y_n(x) = n \frac{1}{\sqrt{2n}} e^{-n(\frac{1}{\sqrt{2n}})^2} = \sqrt{\frac{n}{2e}}. \quad \square$$

(Approximate time for reading and producing answer: 3 hrs)

1.2 Let  $\sum_{n=0}^{\infty} u_n$  be a series of real-valued functions defined on  $[a, b]$ . State the **Weierstrass M-test** for the uniform convergence of the series.

Show that the series  $\sum_{n=0}^{\infty} (-1)^n \frac{\cos nx}{1+n^2}$  converges uniformly on  $[-\pi, \pi]$ .

### Weierstrass M-test

The series of functions  $(u_n)_{n \geq 0}$  converges uniformly on  $[a, b]$  if

1. there exists a sequence  $(M_n)_{n \geq 0}$  such that  $|u_n(x)| \leq M_n$  for all  $n \geq 0$  and for all  $x \in [a, b]$ , and
2. the series  $\sum_{n=0}^{\infty} M_n$  converges.

Define  $u_n(x) = (-1)^n \frac{\cos nx}{1+n^2}$ .

Let  $M_n = \frac{1}{1+n^2}$  and note that  $|u_n| \leq M_n$  for all  $x \in [-\pi, \pi]$ .

Note that the integral  $\int_1^{\infty} \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_1^{\infty} = 1$  converges, therefore the series  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges by the integral test for convergent series.

Now  $M_n < \frac{1}{n^2}$  for  $n > 0$ , so the series  $\sum_{n=1}^{\infty} M_n$  converges. Therefore the series  $\sum_{n=0}^{\infty} M_n$  also converges, since its tail converges.

Therefore the series  $\sum_{n=0}^{\infty} u_n$  converges uniformly on  $[-\pi, \pi]$ .