Problem Set 4

1.7 Group Actions

18. **Let H be a group acting on a set A. Prove that the relation $\sim$ on A defined by**

$$a \sim b \quad \text{if and only if} \quad a = hb \quad \text{for some } h \in H$$

is an equivalence relation. (For each $x \in A$ the equivalence class of x under $\sim$ is called the *orbit* of x under the action of H. The orbits under the action of H partition the set A.)

Intuition 1. Think of the elements of A as vertices of a graph. There's an edge from a_1 to a_2 if some element of h sends $a_1 \mapsto a_2$. Every multi-edge path has a corresponding single edge that goes to the same destination. To see that, imagine some path that involves multiple edges; but then there's also a single edge connecting the start and end point, given by the composition of the multiple edges (group closure means the composition is some other element in the group).

It's reflexive because every group has an identity and this

It's symmetric because h^{-1} is the same edge as h but with the opposite direction. It's transitive because as we said above, there's always a single edge that gets you to the same destination as any two-edge path.

Proof. First note that h acts on A as a permutation, and h^{-1} acts on A as the the inverse of that permutation. Also note that the identity element acts as the identity permutation because the mapping from group elements to the permutations that they act as is a homomorphism, and a homomorphism always sends identity to identity.

reflexive
$a \sim a$ since $a = 1a$.

symmetric
If $a \sim b$ then $a = hb$ for some $h \in H$. Therefore $b = h^{-1}a$ hence $b \sim a$.

transitive
If $a \sim b$ and $b \sim c$ then $a = h_1 b$ and $b = h_2 c$ for some $h_1, h_2 \in H$. Therefore $a = (h_1 h_2)c$ hence $a \sim c$. □

19. **Let H be a subgroup (cf. Exercise 26 of Section 1) of the finite group G and let H act on G (here $A = G$) by left multiplication. Let $x \in G$ and let $\mathcal{O}$ be the orbit of x under the action of H. Prove that the map**

$$H \to \mathcal{O} \quad \text{defined by} \quad h \mapsto hx$$

is a bijection (hence all orbits have cardinality $|H|$). From this and the preceding exercise deduce *Lagrange's Theorem*:

if G is a finite group and H is a subgroup of G then $|H|$ divides $|G|$.

Intuition 2. The map φ in question sends h to some element hx of the orbit of x under H. To get back, right-multiply by x^{-1}.

Proof. Let $\varphi : H \to \mathcal{O}$ be the map defined by $h \mapsto hx$.

The map $\psi : \mathcal{O} \to H$ defined by $g \mapsto gx^{-1}$ is a left inverse for φ, since $(\psi\varphi)(h) = \psi(\varphi(h)) = (hx)x^{-1} = h$, showing that $\psi\varphi = 1_H$, and also a right inverse for φ, since $(\varphi\psi)(g) = \varphi(\psi(g)) = (gx^{-1})x = g$, showing that $\varphi\psi = 1_G$.

Therefore φ is a bijection, since it has an inverse.

Therefore the orbit of x under the action of H has cardinality $|H|$ and, since x was arbitrary, this is true for the orbit of any $x \in G$ under the action of H.

Since the orbits are equivalence classes, they partition G. Therefore $|G| = k|H|$ where k is the number of distinct orbits.

This argument required no assumptions beyond the premise that H is a subgroup of a finite group G, hence Lagrange's Theorem:

if H is a subgroup of a finite group G, then H divides $|G$

. ☐

21. Show that the group of rigid motions of a cube is isomorphic to S_4. [This group acts on the set of four pairs of opposite vertices.]

In case the terminology is unfamiliar, a "rigid motion" means a rotation or a reflection; this group has the same relationship to the cube that the group D_{2n} has to a regular n-gon.

Proof. Let B be the set of 4 pairs of opposite vertices of a cube, let G be the group of rigid motions of a cube, acting on B, and let $\psi : G \to S_B$ be the permutation representation of the action of G on B.

We have $|S_B| = 4! = 24$ and also $|G| = 24$ (look at the cube face-on; after a rigid motion you are looking at one of the 6 faces and that face is in one of its 4 rotational configurations).

Furthermore, every rigid motion other than the identity acts on B as a non-identity permutation. To see this, consider the 4 vertices of the face you are initially looking at. Unless the rigid motion is the identity then all 4 of them have moved. [...needs more...]

Therefore $\ker \psi = \{1\}$, hence ψ is injective, and also an isomorphism since $|G| = |S_B|$, and since $S_B \cong S_4$ and isomorphism is a transitive relation, we have the desired result. ☐

23. Explain why the action of the group of rigid motions of a cube on the set of three pairs of opposite faces is not faithful. Find the kernel of this action.

Proof. Let B be a set of 3 pairs of opposite vertices of a cube. Then $|S_B| = 3! = 6 < 24 = |G|$, hence any map $G \to S_B$ is non-injective, hence the permutation representation of G acting on B is non-injective, which is the definition of a non-faithful action.

The kernel of this action is the set of pairs (g, b) such that $gb = b$.

union of $\{(I_G, b) \mid b \in B\}$ and $\{(r_b,)\}$ ☐

5.

(a) Prove that D_{14} can't act faithfully on any set with fewer than 7 elements.

(b) Find a faithful action of D_{12} on a set with 5 elements.

(c) Why didn't your argument in part (a) apply to the situation in part (b)?

6. We say that an action of a group g on a set A is **transitive** if, for any elements $a, b \in A$, there is some $g \in G$ such that $g.a = b$. (In other words, using the language from the first problem, an action is transitive if there is only one orbit.)

(a) Prove that, if G acts transitively on A, then all of the stabilizers are isomorphic to each other. *[See Exercise 4 on p. 44 for the definition of "stabilizer".]*

(b) Provide a counterexample to demonstrate that the transitivity hypothesis was necessary.