Oxford A1 - Differential Equations

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1 Sheet 1

https://courses.maths.ox.ac.uk/node/5372

1.1 Let $[a, b]$ be a closed and bounded interval of the real line and let $\{y_n\}_{n \geq 0}$ be a sequence of real-valued functions, each of which is defined on $[a, b]$. What does it mean to say that **the sequence converges uniformly on** $[a, b]$ **to a limit function** y**?** If each y_n is continuous on $[a, b]$ show that the uniform limit y is continuous on $[a, b]$ and that, when $n \to \infty$,

$$\int_a^b |y_n(x) - y(x)| dx \to 0, \quad \int_a^b y_n(x) dx \to \int_a^b y(x) dx.$$

(a) Definition of uniform convergence

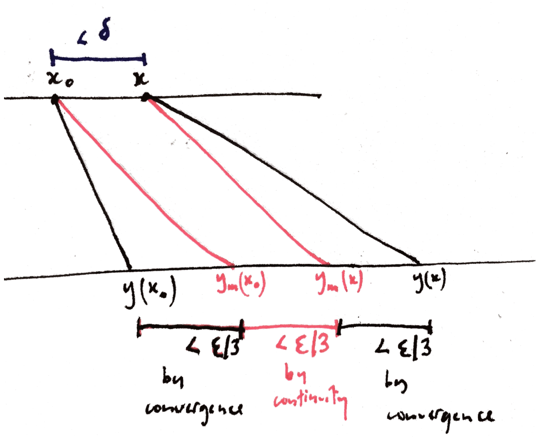
The sequence of functions $\{y_n\}_{n \geq 0}$ **converges uniformly on** $[a, b]$ **to** y if and only if for every $\epsilon > 0$ there exists an $m \in \mathbb{N}$ such that for every $n > m$, y_n differs from y by no more than ϵ at every point in $[a, b]$.

(b) Show that the limit function is continuous

The claim is that if each y_n is continuous on $[a, b]$ then y is continuous on $[a, b]$. We are told that

1. $\{y_n\}_{n \geq 0}$ converges uniformly to y, and
2. each y_n is continuous on $[a, b]$.

Diagram of proof:



Fix arbitrary $\epsilon > 0$ and $x_0 \in [a, b]$.

Let $m \in \mathbb{N}$ be such that $|y_m(x_0) - y(x_0)| < \epsilon/3$. Such an m exists because the $\{y_n\}$ converge uniformly to y.

Let δ be such that $|x - x_0| < \delta \implies |y_m(x) - y_m(x_0)| < \epsilon/3$. Such a δ exists because y_m is continuous on $[a, b]$.

Fix an arbitrary x such that $|x - x_0| < \delta$.

Now we have the following:

1. $|y(x_0) - y_m(x_0)| < \epsilon/3$ by convergence of the $\{y_n\}$
2. $|y_m(x_0) - y_m(x)| < \epsilon/3$ by continuity of y_m
3. $|y_m(x) - y(x)| < \epsilon/3$ by convergence of the $\{y_n\}$

Therefore $|y(x_0) - y(x)| < \epsilon$, proving continuity of y on $[a, b]$. □

(Approximate time taken for reading and producing an answer: 4hrs)

(c) Show limit of definite integral I

Let $I_n = \int_a^b |y_n(x) - y(x)| \, \mathrm{dx}$.

The claim is that $\lim_{n\to\infty} I_n = 0$.

In other words $\forall \epsilon > 0 : \exists\, m \in \mathbb{N} : \forall\, n > m : |I_n - 0| < \epsilon$.

Fix an $\epsilon > 0$.

Since the $\{y_n\}$ converge uniformly to y, there exists an $m \in \mathbb{N}$ such that for all $n > m$ and for all $x \in [a, b]$

$$|y_n(x) - y(x)| < \epsilon/(b - a).$$

Therefore $\int_a^b |y_n(x) - y(x)| \, \mathrm{dx} < \epsilon$ for all $n > m$, as required. □

(d) Show limit of definite integral II

The claim is that $\lim_{n\to\infty} \int_a^b y_n(x)\,\mathrm{dx} = \int_a^b y(x)\,\mathrm{dx}$.

In other words: $\forall \epsilon > 0 : \exists\ m \in \mathbb{N} : \forall\ n > m :$

$$\Big| \left(\int_a^b y_n(x)\,\mathrm{dx} \right) - \left(\int_a^b y(x)\,\mathrm{dx} \right) \Big| < \epsilon.$$

This is equivalent to: $\forall \epsilon > 0 : \exists\ m \in \mathbb{N} : \forall\ n > m :$

$$A_1 := \Big| \int_a^b (y_n(x) - y(x))\,\mathrm{dx} \Big| < \epsilon.$$

From part (c) above, we know that: $\forall \epsilon > 0 : \exists\ m \in \mathbb{N} : \forall\ n > m :$

$$A_2 := \int_a^b |y_n(x) - y(x)|\,\mathrm{dx} < \epsilon.$$

Now[a] if the sign of $y_n(x) - y(x)$ is constant for all $x \in [a, b]$ (i.e. the graphs do not cross over), then $A_1 = A_2 < \epsilon$. Otherwise, there is some cancellation in the integral A_1 and $0 \leq A_1 < A_2 < \epsilon$. So the same choice of m as was used in part (c) works here, since for that value of m, we have $A_1 < \epsilon$ as required.

(Approximate time taken for (c) and (d): 2hrs)

[a]How do I prove this section properly?

If $[a,b] = [0,1]$ and $y_n(x) = nxe^{-nx^2}$ show that, for each $x \in [0,1], y_n(x) \to 0$ but $\displaystyle\int_0^1 y_n(x)dx \to \frac{1}{2}$. Thus the convergence must be non-uniform. Show that

$$\max_{0 \leq x \leq 1} y_n(x) = \sqrt{\frac{n}{2e}}$$

and sketch the graph of $y_n(x)$ versus x.