Math 185 - Complex Analysis
Homework 5

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1 Power Series

V.6.2 Prove that the sequence $(g_n)_{n=0}^{\infty}$ converges locally uniformly in the open set G if and only if it converges uniformly on each compact subset of G.

First, terminology: Sarason states that $(g_n)_{n=0}^{\infty}$ converges locally uniformly in G if each point of G has a neighborhood in which the sequence converges uniformly. I'm going to take that to mean "if and only if".

For the forward direction, we need to show that if

(A): each point of G has a neighborhood in which (g_n) converges uniformly

then

(B): (g_n) converges uniformly on each compact subset of G.

I don't have a proof, but a suggested approach for how to prove this is by contradiction:

1. Suppose (A) is true but that (B) is not, so that there exists some compact subset S of G on which (g_n) does not converge uniformly.
2. Show that there exists a point of S which lacks any neighborhood within which convergence is uniform. $\square$

For the reverse direction, we need to show that if

(B): (g_n) converges uniformly on each compact subset of G.

then

(A): each point of G has a neighborhood in which (g_n) converges uniformly

Again I don't have a proof, but a suggested approach for how to prove this is:

1. Consider a point z of G.
2. Show that there is a compact subset S of G that contains z.
3. Show that z has a neighborhood which is a subset of S. $\square$

V.7.2 Prove that the series $\sum_{n=0}^{\infty}\left(\frac{z-1}{z+1}\right)^n$ converges locally uniformly in the half-plane $\mathrm{Re}\, z > 0$, and find the sum.
(No attempt)

V.14.1(b) Find the radius of convergence of the following series:

$$\sum_{n=0}^{\infty} \frac{(n!)^3}{(3n)!} z^{3n}$$

We use the ratio test:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| &= \lim_{n\to\infty} \left| \frac{(n!)^3 z^{3n}}{(3n)!} \frac{(3n+3)!}{((n+1)!)^3 z^{3n+3}} \right| \\
&= |z^{-3}| \lim_{n\to\infty} \left| \frac{(3n+3)(3n+2)(3n+1)}{(n+1)^3} \right| \\
&= |z^{-3}| \lim_{n\to\infty} \left| \frac{27 + o(n^{-1})}{1 + o(n^{-1})} \right| \\
&= 27|z^{-3}|
\end{aligned}$$

So the series converges when $|z^3| > 27$, i.e. outside a disc of radius 3 centered at the origin. The radius of convergence is infinite.

V.16.2 What function is represented by the power series $\sum_{n=1}^{\infty} n^2 z^n$?
(No attempt)

V.18.1 Use the scheme above to determine the power series with center 0 representing the function $f(z) = \frac{1}{1+z+z^2}$ near 0. What is the radius of convergence of this series?

Assume $f(z)$ can be represented as a power series $\sum_{n=0}^{\infty} a_n z^n$. We can write f as the ratio

$$f(z) = \frac{1}{1+z+z^2} = \frac{\sum_{n=0}^{\infty} b_n z^n}{\sum_{n=0}^{\infty} c_n z^n} =: \frac{g(z)}{h(z)},$$

where

$$b_n = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$c_n = \begin{cases} 1, & 0 \le n \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned}
g(z) = \sum_{n=0}^{\infty} b_n z^n &= f(z)h(z) \\
&= \left(\sum_{n=0}^{\infty} a_n z^n \right) \left(\sum_{n=0}^{\infty} c_n z^n \right) \\
&= \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} a_k c_{n-k}
\end{aligned}$$