Math 185 - Complex Analysis - Homework 4

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IV.5.2 Describe the curves $|f| =$ constant and $\arg f =$ constant for the function

$$f(z) = \exp(z^2).$$

Let $z = x + iy$, so

$$\begin{aligned} f(z) &= \exp((x^2 - y^2) + 2ixy) \\ &= e^{x^2-y^2}(\cos 2xy + i \sin 2xy) \end{aligned}$$

Therefore $|f| = k$ for some constant $k \in \mathbb{R}$ implies that $e^{x^2-y^2} = k > 0$, i.e. $y = \pm\sqrt{x^2 - \log k}$. In other words, the preimage of a circle of radius k centered on the origin is the union of the two curves $y = \pm\sqrt{x^2 - \log k}$.

$\arg f = \theta$ for some constant $0 \le \theta < 2\pi$ implies that $2xy = \theta$, i.e. $y = \frac{\theta}{2x}$. In other words, the preimage of a ray at angle θ is the graph of $y = \frac{\theta}{2x}$.

IV.9.2 Find all values of $\log(\log i)$

$\log i$ is the following set of image points lying on the imaginary axis:

$$\log i = \left\{ i\left(\frac{\pi}{2} + 2\pi k\right) : k \in \mathbb{Z} \right\}.$$

Fix a particular k. The log of the corresponding image point is the following set of secondary image points, lying on the vertical line through $\frac{\pi}{2} + 2\pi k$:

$$\log\left(i\left(\frac{\pi}{2} + 2\pi k\right)\right) = \left\{ \log\left(\frac{\pi}{2} + 2\pi k\right) + i\left(\frac{\pi}{2} + 2\pi l\right) : l \in \mathbb{Z} \right\},$$

Therefore the set of all values of $\log(\log i)$ is the following rectangular grid of points

$$\log(\log i) = \left\{ \log\left(\frac{\pi}{2} + 2\pi k\right) + i\left(\frac{\pi}{2} + 2\pi l\right) : k \in \mathbb{Z}, l \in \mathbb{Z} \right\}.$$

IV.13.3 [Not in homework.] Let G be the open set one obtains by removing from $\mathbb{C}$ the interval $[-1, 1]$ on the real axis. Prove that there is a branch of the function $\sqrt{\frac{z+1}{z-1}}$ in G. (Suggestion: What is the image of G under the map $z \mapsto \frac{z+1}{z-1}$?)

IV.13.4 Let G be as in Exercise IV.13.3. Prove that there is a branch of the function $\sqrt{z^2 - 1}$ in G.

Let $f(z) = \sqrt{z^2 - 1}$, using the principal square root function defined by

$$\sqrt{w} = \sqrt{|w|}\left(\cos\frac{\operatorname{Arg} w}{2} + i \sin\frac{\operatorname{Arg} w}{2}\right).$$

The principal square root function is discontinuous at points w in $(-\infty, 0]$. Therefore f will be continuous for all $z \in \mathbb{C}$ except where $z^2 - 1 \in (-\infty, 0]$, i.e. $-1 \leq z \leq 1$. Therefore f will be continuous in $G = \mathbb{C} \backslash [-1, 1]$.

IV.16.1 Find all the values of $(1+i)^i$.

$(1+i)^i$ is the set of values

$$\begin{aligned}\exp\left(i \log(1+i)\right) &= \exp\left(i\left(\log\sqrt{2} + i\left(\frac{\pi}{4} + 2\pi k\right)\right)\right) \\ &= \exp\left(-\pi\left(2k + \frac{1}{4}\right) + i\frac{\log 2}{2}\right) \\ &= e^{-\pi\left(2k+\frac{1}{4}\right)}\left(\cos\frac{\log 2}{2} + i\sin\frac{\log 2}{2}\right)\end{aligned}$$

for $k \in \mathbb{Z}$.

IV.16.3 Prove that if f is a branch of z^c in an open set not containing 0, then f is holomorphic and f' is a branch of cz^{c-1}.

(No solution)