Math 185 - Homework 6

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6. Complex Integration

VI.7.2 Derive the formula

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^{2n} t \; dt = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$$

by integrating the function $\frac{1}{z}\left(z+\frac{1}{z}\right)^{2n}$ around the unit circle, parameterized by the curve $\gamma(t) = e^{it}(0 \le t \le 2\pi)$.

Here are two slightly different attempts:

We can write the integral as

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \cos^{2n} t \; dt &= \frac{1}{2^{2n+1}\pi}\int_0^{2\pi} (e^{it}+e^{-it})^{2n} \; dt \\ &= \frac{1}{2^{2n+1}\pi}\int_\gamma (z+z^{-1})^{2n} \; dz\end{aligned}$$

Now consider the related integral

$$\begin{aligned}\int_\gamma z^{-1}\left(z+z^{-1}\right)^{2n} \; dz &= \int_\gamma z^{-1}\sum_{k=0}^{2n}\binom{2n}{k} z^{2n-k}z^{-k} \; dz \\ &= \sum_{k=0}^{2n}\binom{2n}{k}\int_\gamma z^{2(n-k)-1} \; dz.\end{aligned}$$

If $k \ne n$, then $z^{2(n-k)-1}$ is the derivative of $\frac{z^{2(n-k)}}{2(n-k)}$, in which case $\int_\gamma z^{2(n-k)-1} \; dz = 0$ since γ is a closed curve. Therefore the only terms remaining in the summation are those for which $k = n$:

$$\begin{aligned}\int_\gamma z^{-1}\left(z+z^{-1}\right)^{2n} \; dz &= \binom{2n}{n}\int_\gamma z^{-1} \; dz \\ &= \binom{2n}{n}\int_0^{2\pi} e^{-it} i e^{it} \; dz \\ &= \binom{2n}{n} 2\pi i.\end{aligned}$$

Returning to the original problem, we now know the value of a similar integral:

$$\begin{aligned}\frac{1}{2^{2n+1}\pi}\int_\gamma z^{-1}(z+z^{-1})^{2n}\ dz &= \frac{1}{2^{2n+1}\pi}\binom{2n}{n}2\pi i\\ &= \frac{1}{2^{2n+1}\pi}\ \frac{(2n)!}{2(n!)}\ 2\pi i\\ &= \frac{(n+1)\cdot(n+2)\cdots 2n}{2^{2n+1}}i\end{aligned}$$

Alternatively we can write the integral as

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\cos^{2n} t\ dt &= \frac{1}{2^{2n+1}\pi}\int_0^{2\pi}(e^{it}+e^{-it})^{2n}\ dt\\ &= \frac{1}{2^{2n+1}\pi}\int_\gamma (z+z^{-1})^{2n}\ dz\\ &= \frac{1}{2^{2n+1}\pi}\int_\gamma \sum_{k=0}^{2n}\binom{2n}{k}z^{2n-k}z^{-k}\ dz\\ &= \frac{1}{2^{2n+1}\pi}\sum_{k=0}^{2n}\binom{2n}{k}\int_\gamma z^{2(n-k)}\ dz.\end{aligned}$$

Now if $2n\neq k$, then $z^{2(n-k)}$ is the derivative of $\frac{z^{2(n-k)+1}}{2(n-k)+1}$, in which case $\int_\gamma z^{2(n-k)}\ dz = 0$.

We can view the integral on the right side as integrating the function $(z+z^{-1})^{2n}$ around the unit circle:

VI.8.1 Let z_1 and z_2 be distinct points of $\mathbb{C}$. Evaluate $\int_{[z1,z2]} z^n dz$ and $\int_{[z1,z2]} \overline{z}^n dz$ for $n = 0, 1, 2, \ldots$.

Let $\gamma(t) = z_1 + t(z_2 - z_1)$ for $t\in[0,1]$ represent the curve $[z_1, z_2]$. We have

$$\begin{aligned}\int_{[z1,z2]} z^n dz &= \int_0^1 \gamma(t)^n\gamma'(t)dt\\ &= (z_2-z_1)\int_0^1 (z_1+t(z_2-z_1))^n\ dt.\\ &= (z_2-z_1)\sum_{k=0}^{n}\binom{n}{k}z_1^{n-k}(z_2-z_1)^k\int_0^1 t^k dt\\ &= \sum_{k=0}^{n}\frac{\binom{n}{k}}{k+1}z_1^{n-k}(z_2-z_1)^{k+1}.\end{aligned}$$

And for $\overline{z}$ we have

$$\begin{aligned}\int_{[z1,z2]} \overline{z}^n dz &= \int_0^1 \left(\overline{\gamma(t)}\right)^n \gamma'(t)dt \\ &= (z_2 - z_1) \int_0^1 \left(\overline{z_1} + t(\overline{z_2} - \overline{z_1})\right)^n dt \\ &= (z_2 - z_1) \sum_{k=0}^{n} \binom{n}{k} \overline{z_1}^{n-k} (\overline{z_2} - \overline{z_1})^k \int_0^1 t^k dt \\ &= \sum_{k=0}^{n} \frac{\binom{n}{k}}{k+1} \overline{z_1}^{n-k} (z_2 - z_1)^{k+1}\end{aligned}$$

VI.8.3 Let the complex-valued function f be defined and continuous in the disc $|z - z_0| < R$. For $0 < r < R$ let C_r denote the circle $|z - z_0| = r$, with counterclockwise orientation.

VI.8.4 Assume that f is of class C^1. Prove that

$$\lim_{r\to 0} \frac{1}{r^2} \int_{C_r} f(z)dz = 2\pi i \frac{\partial f}{\partial \overline{z}}(z_0).$$

Let $\gamma(\theta) = z_0 + re^{i\theta}$ for $\theta \in [0, 2\pi]$ represent the curve C_r. Then

$$\begin{aligned}\lim_{r\to 0} \frac{1}{r^2} \int_{C_r} f(z)dz &= \lim_{r\to 0} \frac{1}{r^2} \int_0^{2\pi} f(\gamma(\theta))\gamma'(\theta)d\theta \\ &= \lim_{r\to 0} \frac{i}{r} \int_0^{2\pi} f(z_0 + re^{i\theta})e^{i\theta}d\theta\end{aligned}$$

(Not sure where to go from here.)

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

VI.12.2 Evaluate the integrals $\int_0^\infty \cos t^2 d\theta$ and$\int_0^\infty \sin t^2 d\theta$ (the Fresnel intervals) by integrating e^{-z^2} in the counterclockwise direction around the boundary of the region $\{z : |z| < R, 0 \leq \textbf{Arg}\ z \leq \frac{\pi}{4}\}$ and letting $R \to \infty$.

We represent the specified curve as $\gamma(\theta) = Re^{i\theta}$ for $\theta \in [0, \frac{\pi}{4}]$, in which case the specified integral is

$$\int_0^{\pi/4} e^{-e^{2i\theta}} Rie^{i\theta} d\theta = Ri \int_0^{\pi/4} e^{i\theta - e^{2i\theta}} d\theta.$$

Give an example of two convergent series whose product diverges.