Spline Interpolation

- there is a separate polynomial function for every two consecutive given points.
- depends on the degree of polynomial connecting each two consecutive points, but the most common are linear (1^{st} degree), quadratic (2^{nd} degree), and cubic (3^{rd} degree) splines.

Alternative Method in Alternative Cubic Spline Interpolation

$$
\begin{aligned}
f_i = {} & \frac{f''(x_{i-1})}{6(x_i - x_{i-1})}(x_i - x)^3 + \frac{f''(x_i)}{6(x_i - x_{i-1})}(x - x_{i-1})^3 \\
& + \left[\frac{f(x_{i-1})}{x_i - x_{i-1}} - \frac{f''(x_{i-1})(x_i - x_{i-1})}{6}\right](x_i - x) \\
& + \left[\frac{f(x_i)}{x_i - x_{i-1}} - \frac{f''(x_i)(x_i - x_{i-1})}{6}\right](x - x_{i-1})
\end{aligned}
$$

Since the second derivative is unknown, we can use the following equation to find it:

$$
\begin{aligned}
& (x_i - x_{i-1})f''(x_{i-1}) + 2(x_{i+1} - x_{i-1})f''(x_i) + (x_{i+1} - x_i)f''(x_{i+1}) \\
& \quad = \frac{6}{x_{i+1} - x_i}[f(x_{i+1}) - f(x_i)] + \frac{6}{x_i - x_{i-1}}[f(x_{i-1}) - f(x_i)]
\end{aligned}
$$

Problem # 1 Determine the value of $f(7)$ for a given set of data below using the alternative cubic spline interpolation method. x should be in increasing number.

x	$f(x)$
1	12
5	-26
8	-14
10	37

The second derivative at the end points of the curve is equal to 0, therefore:

$f''(x_0) = 0$

$f''(x_1) = ?$

$f''(x_2) = ?$

$f''(x_3) = 0$

Using the equation above, we can find the value of $f''(x_1)$ and $f''(x_2)$:

Solving for $f''(x_1)$:

$$(x_i - x_{i-1})f''(x_{i-1}) + 2(x_{i+1} - x_{i-1})f''(x_i) + (x_{i+1} - x_i)f''(x_{i+1})$$
$$= \frac{6}{x_{i+1} - x_i}[f(x_{i+1}) - f(x_i)] + \frac{6}{x_i - x_{i-1}}[f(x_{i-1}) - f(x_i)]$$

$$(x_1 - x_{1-1})f''(x_{1-1}) + 2(x_{1+1} - x_{1-1})f''(x_1) + (x_{1+1} - x_1)f''(x_{1+1})$$
$$= \frac{6}{x_{1+1} - x_1}[f(x_{1+1}) - f(x_1)] + \frac{6}{x_1 - x_{1-1}}[f(x_{1-1}) - f(x_1)]$$

$$(x_1 - x_0)f''(x_0) + 2(x_2 - x_0)f''(x_1) + (x_2 - x_1)f''(x_2)$$
$$= \frac{6}{x_2 - x_1}[f(x_2) - f(x_1)] + \frac{6}{x_1 - x_0}[f(x_0) - f(x_1)]$$

Substituting the given values

$$(5-1)(0) + 2(8-1)f''(x_1) + (8-5)f''(x_2) = \frac{6}{8-5}[-14 - (-26)] + \frac{6}{5-1}[12 - (-26)]]$$

Simplifying

$$14f''(x_1) + 3f''(x_2) = 81(equation1)$$

Solving for $f''(x_2)$:

$$(x_i - x_{i-1})f''(x_{i-1}) + 2(x_{i+1} - x_{i-1})f''(x_i) + (x_{i+1} - x_i)f''(x_{i+1})$$
$$= \frac{6}{x_{i+1} - x_i}[f(x_{i+1}) - f(x_i)] + \frac{6}{x_i - x_{i-1}}[f(x_{i-1}) - f(x_i)]$$

$$(x_2 - x_{2-1})f''(x_{2-1}) + 2(x_{2+1} - x_{2-1})f''(x_2) + (x_{2+1} - x_2)f''(x_{2+1})$$
$$= \frac{6}{x_{2+1} - x_2}[f(x_{2+1}) - f(x_2)] + \frac{6}{x_2 - x_{2-1}}[f(x_{2-1}) - f(x_2)]$$

$$(x_2 - x_1)f''(x_1) + 2(x_3 - x_1)f''(x_2) + (x_3 - x_2)f''(x_3)$$
$$= \frac{6}{x_3 - x_2}[f(x_3) - f(x_2)] + \frac{6}{x_2 - x_1}[f(x_1) - f(x_2)]$$

Substituting the given values

$$(8-5)f''(x_1) + 2(10-5)f''(x_2) + (10-8)f''(x_3) = \frac{6}{10-8}[37 - (-14)] + \frac{6}{8-5}[-26 - (-14)]$$

Simplifying $\quad$ *Automatically,* $f''(x_3) = 0$

$$3f''(x_1) + 10f''(x_2) = 3[51] + 2[-12]$$

$$3f''(x_1) + 10f''(x_2) = 129(equation2)$$

Finding the value of the second derivatives $f''(x_1)$ and $f''(x_2)$

$$14f''(x_1) + 3f''(x_2) = 81$$
$$= \frac{14f''x_1}{14} = \frac{81 - 3f''(x_2)}{14}$$
$$= f''(x_1) = \frac{81 - 3f''(x_2)}{14}$$

$$3f''(x_1) + 10f''(x_2) = 129$$
$$= 3(\frac{81 - 3f''(x_2)}{14}) + 10f''(x_2) = 129$$

Multiply the second term by 14 to be able to combine the two terms

$$\frac{243 - 9f''x_2}{14} + \frac{140f''x_2}{14} = 129$$

$$\frac{243 - 131f''x_2}{14} = 129$$

Multiply both sides by 14 again to cancel the denominator because $\frac{14}{14}$ = 1

$$243 - 131f''x_2 = 1806$$

Transpose and get rid of the denominator

$$f''x_2 = 11.93130$$

$$f''x_1 = \frac{81 - 3f''(x_2)}{14}$$
$$= \frac{81 - 3(11.93130)}{14}$$
$$= \frac{81 - 35.7939}{14}$$
$$= \frac{45.2061}{14}$$
$$f''x_1 = 3.22901$$

Finding the value of $f(7)$

$$\begin{aligned} f_7 = {} & \frac{f''(x_{i-1})}{6(x_i - x_{i-1})}(x_i - x)^3 + \frac{f''(x_i)}{6(x_i - x_{i-1})}(x - x_{i-1})^3 \\ & + \left[\frac{f(x_{i-1})}{x_i - x_{i-1}} - \frac{f''(x_{i-1})(x_i - x_{i-1})}{6}\right](x_i - x) \\ & + \left[\frac{f(x_i)}{x_i - x_{i-1}} - \frac{f''(x_i)(x_i - x_{i-1})}{6}\right](x - x_{i-1}) \end{aligned}$$

$$\begin{aligned} f_7 = {} & \frac{3.22901}{18}(1)^3 + \frac{11.93130}{18}(2)^3 \\ & + \left[\frac{-26}{3} - \frac{3.22901(3)}{6}\right](8 - 7) \\ & + \left[\frac{-14}{3} - \frac{11.93130(3)}{6}\right](7 - 5) \end{aligned}$$

$$\begin{aligned} f_7 = 0.17939 + 5.3028 + (-8.66667 - 1.61450) + (-4.66667 - 5.96565)(2) \\ = 0.17939 + 5.3028 - 10.28117 - 21.26464 \\ f_7 = -26.06362 \end{aligned}$$