Closed Methods in Finding the Root of a Non-linear System

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BSCS3

Contents

Bisection Method

1. Determine $x_m = \frac{x_l + x_u}{2}$
2. Find $f(x_l), f(x_m), f(x_u)$
3. Choose new interval (*find the interval with 2 different signs (+,-)*)

Problem 1 - Bisection Method

Determine the root of $3x^4 + 7x^3 - 15x^2 + 5x = 17$ between $[0, 2]$. Use Bisection Method for 7 iterations. The equation becomes $f(x) = 3x^4 + 7x^3 - 15x^2 + 5x - 17$. *17 is transposed so it became -17*

Count	x_l	x_m	x_u	$f(x_l)$	$f(x_m)$	$f(x_u)$	*New*
1.	0	1	2	-17	-17	37	[1,2]
2.	1	1.5	2	-17	-4.43750	37	[1.5,2]
3.	1.5	1.75	2	-4.43750	11.46484	37	[1.5,1.75]
4.	1.5	1.625	1.75	-4.43750	2.47144	11.46484	[1.5,1.625]
5.	1.5	1.5625	1.625	-4.43750	-1.22432	2.47144	[1.5625,1.625]
6.	1.5625	1.59375	1.625	-1.22432	0.56087	2.47144	[1.5625,1.59375]
7.	1.5625	1.57813	1.59375	-1.22432	-0.34681	0.56087	[1.57813,1.59375]

To get the root after the 7th iteration, use the formula of getting x_m. So, $x_m = \frac{1.57813+1.59375}{2} = 1.58594$

First Iteration

$x_m = \frac{x_l + x_u}{2} = \frac{0+2}{2} = 1$

$f(x_l) = 3(0)^4 + 7(0)^3 - 15(0)^2 + 5(0) - 17 = -17$

$f(x_m) = 3(1)^4 + 7(1)^3 - 15(1)^2 + 5(1) - 17 = -17$

$f(x_u) = 3(2)^4 + 7(2)^3 - 15(2)^2 + 5(2) - 17 = 37$

Second Iteration

$x_m = \frac{x_l + x_u}{2} = \frac{1+2}{2} = 1.5$

$f(x_l) = 3(1)^4 + 7(1)^3 - 15(1)^2 + 5(1) - 17 = -17$

$f(x_m) = 3(1.5)^4 + 7(1.5)^3 - 15(1.5)^2 + 5(1.5) - 17 = -4.43750$

$f(x_u) = 3(2)^4 + 7(2)^3 - 15(2)^2 + 5(2) - 17 = 37$

Third Iteration

$x_m = \frac{x_l + x_u}{2} = \frac{1.5+2}{2} = 1.75$

$f(x_l) = 3(1.5)^4 + 7(1.5)^3 - 15(1.5)^2 + 5(1.5) - 17 = -4.43750$

$f(x_m) = 3(1.75)^4 + 7(1.75)^3 - 15(1.75)^2 + 5(1.75) - 17 = 11.46484$

$f(x_u) = 3(2)^4 + 7(2)^3 - 15(2)^2 + 5(2) - 17 = 37$

Fourth Iteration

$x_m = \frac{x_l + x_u}{2} = \frac{1.5+1.75}{2} = 1.625$

$f(x_l) = 3(1.5)^4 + 7(1.5)^3 - 15(1.5)^2 + 5(1.5) - 17 = -4.43750$

$f(x_m) = 3(1.625)^4 + 7(1.625)^3 - 15(1.625)^2 + 5(1.625) - 17 = 2.47144$

$f(x_u) = 3(1.75)^4 + 7(1.75)^3 - 15(1.75)^2 + 5(1.75) - 17 = 11.46484$

Fifth Iteration

$x_m = \frac{x_l + x_u}{2} = \frac{1.5+1.625}{2} = 1.5625$

$f(x_l) = 3(1.5)^4 + 7(1.5)^3 - 15(1.5)^2 + 5(1.5) - 17 = -4.43750$

$f(x_m) = 3(1.5625)^4 + 7(1.5625)^3 - 15(1.5625)^2 + 5(1.5625) - 17 = -1.22432$

$f(x_u) = 3(1.625)^4 + 7(1.625)^3 - 15(1.625)^2 + 5(1.625) - 17 = 2.47144$

Sixth Iteration

$x_m = \frac{x_l + x_u}{2} = \frac{1.5625+1.625}{2} = 1.59375$

$f(x_l) = 3(1.5625)^4 + 7(1.5625)^3 - 15(1.5625)^2 + 5(1.5625) - 17 = -1.22432$

$f(x_m) = 3(1.59375)^4 + 7(1.59375)^3 - 15(1.59375)^2 + 5(1.59375) - 17 = 0.56087$

$f(x_u) = 3(1.625)^4 + 7(1.625)^3 - 15(1.625)^2 + 5(1.625) - 17 = 2.47144$

Seventh Iteration

$x_m = \frac{x_l + x_u}{2} = \frac{1.5625+1.59375}{2} = 1.57813$

$f(x_l) = 3(1.5625)^4 + 7(1.5625)^3 - 15(1.5625)^2 + 5(1.5625) - 17 = -1.22432$

$f(x_m) = 3(1.57813)^4 + 7(1.57813)^3 - 15(1.57813)^2 + 5(1.57813) - 17 = -0.34681$

$f(x_u) = 3(1.59375)^4 + 7(1.59375)^3 - 15(1.59375)^2 + 5(1.59375) - 17 = 0.56087$

Problem 2 - Bisection Method

Determine the root of $f(x) = x^3 - 2x - 5$ between [2,3]. Use Bisection Method for 5 iterations.

$Count$	x_l	x_m	x_u	$f(x_l)$	$f(x_m)$	$f(x_u)$	New
1.	2	2.5	3	-1	5.625	16	[2,2.5]
2.	2	2.25	2.5	-1	6.89063	5.625	[2,2.25]
3.	2	2.125	2.25	-1	0.34570	6.89063	[2,2.125]
4.	2	2.0625	2.125	-1	8.96289	0.34570	[2,2.0625]
5.	2	2.03125	2.0625	-1	-0.68161	8.96289	[2.03125,2.0625]

To get the root after the 5th iteration, use the formula of getting x_m. So, $x_m = \frac{2.03125+2.0625}{2} = 2.04688$

First Iteration

$x_m = \frac{x_l + x_u}{2} = \frac{2+3}{2} = 1$

$f(x_l) = 2^3 - 2(2) - 5 = -1$

$f(x_m) = 2.5^3 - 2(2.5) - 5 = 5.625$

$f(x_u) = 3^3 - 2(3) - 5 = 16$

Second Iteration

$x_m = \frac{x_l + x_u}{2} = \frac{2+2.5}{2} = 2.25$

$f(x_l) = 2^3 - 2(2) - 5 = -1$

$f(x_m) = 2.25^3 - 2(2.25) - 5 = 6.89063$

$f(x_u) = 2.5^3 - 2(2.5) - 5 = 5.625$

Third Iteration

$x_m = \frac{x_l + x_u}{2} = \frac{2+2.25}{2} = 2.125$

$f(x_l) = 2^3 - 2(2) - 5 = -1$

$f(x_m) = 2.125^3 - 2(2.125) - 5 = 0.34570$

$f(x_u) = 2.25^3 - 2(2.25) - 5 = 6.89063$

Fourth Iteration

$x_m = \frac{x_l + x_u}{2} = \frac{2+2.125}{2} = 2.0625$

$f(x_l) = 2^3 - 2(2) - 5 = -1$

$f(x_m) = 2.0625^3 - 2(2.0625) - 5 = 8.96289$

$f(x_u) = 2.125^3 - 2(2.125) - 5 = 0.34570$

Fifth Iteration

$x_m = \frac{x_l + x_u}{2} = \frac{2+2.0625}{2} = 2.03125$

$f(x_l) = 2^3 - 2(2) - 5 = -1$

$f(x_m) = 2.03125^3 - 2(2.03125) - 5 = -0.68161$

$f(x_u) = 2.0625^3 - 2(2.0625) - 5 = 8.96289$

False Position Method

1. Determine $f(x_u)$ and $f(x_l)$
2. Determine x_m
3. Determine $f(x_m)$
4. Choose new interval

The formula for finding x_m is: $x_m = \frac{x_l(f(x_u)) - x_u(f(x_l))}{f(x_u) - f(x_l)}$

Problem 1 - False Position Method

Determine the root of $3x^4 + 7x^3 - 15x^2 + 5x = 17$ between $[0, 2]$. Use False Position Method for 7 iterations. The equation becomes $f(x) = 3x^4 + 7x^3 - 15x^2 + 5x - 17$. *17 is transposed so it became -17*

Count	x_l	x_m	x_u	$f(x_l)$	$f(x_m)$	$f(x_u)$	*New*
1.	0	0.62963	2	-17	-17.57963	37	[0.62963,2]
2.	0.62963	1.07101	2	-17.57963	-16.30402	37	[1.07101,2]
3.	1.07101	1.35516	2	-16.30402	-10.23247	37	[1.35516,2]
4.	1.35516	1.49486	2	-10.23247	-4.68143	37	[1.49486,2]
5.	1.49486	1.55159	2	-4.68143	-1.81892	37	[1.55159,2]
6.	1.55159	1.57260	2	-1.81892	-0.66077	37	[1.57260,2]
7.	1.57260	1.58010	2	-0.66077	-0.23404	37	[1.58010,2]

To get the root, use the formula for x_m.

$f(x_l) = 3(1.58010)^4 + 7(1.58010)^3 - 15(1.58010)^2 + 5(1.58010) - 17 = -0.23404$

$f(x_u) = 3(2)^4 + 7(2)^3 - 15(2)^2 + 5(2) - 17 = 37$

$x_m = \frac{x_l(f(x_u)) - x_u(f(x_l))}{f(x_u) - f(x_l)} = \frac{1.58010(37) - 2(-0.23404)}{37 - (-0.23404)} = 1.58274$

The root is **1.58274**.

First Iteration

$f(x_l) = 3(0)^4 + 7(0)^3 - 15(0)^2 + 5(0) - 17 = -17$

$f(x_u) = 3(2)^4 + 7(2)^3 - 15(2)^2 + 5(2) - 17 = 37$

$x_m = \frac{x_l(f(x_u)) - x_u(f(x_l))}{f(x_u) - f(x_l)} = \frac{0(37) - 2(-17)}{37 - (-17)} = 0.62963$

$f(x_m) = 3(0.62963)^4 + 7(0.62963)^3 - 15(0.62963)^2 + 5(0.62963) - 17 = -17.57963$

Second Iteration

$f(x_l) = 3(0.62963)^4 + 7(0.62963)^3 - 15(0.62963)^2 + 5(0.62963) - 17 = -17.57963$

$f(x_u) = 3(2)^4 + 7(2)^3 - 15(2)^2 + 5(2) - 17 = 37$

$x_m = \frac{x_l(f(x_u)) - x_u(f(x_l))}{f(x_u) - f(x_l)} = \frac{0.62963(37) - 2(-17.57963)}{37 - (-17.57963)} = 1.07101$

$f(x_m) = 3(1.07101)^4 + 7(1.07101)^3 - 15(1.07101)^2 + 5(1.07101) - 17 = -16.30402$

Third Iteration

$f(x_l) = 3(1.07101)^4 + 7(1.07101)^3 - 15(1.07101)^2 + 5(1.07101) - 17 = -16.30402$

$f(x_u) = 3(2)^4 + 7(2)^3 - 15(2)^2 + 5(2) - 17 = 37$

$x_m = \frac{x_l(f(x_u)) - x_u(f(x_l))}{f(x_u) - f(x_l)} = \frac{1.07101(37) - 2(-16.30402)}{37 - (-16.30402)} = 1.35516$

$f(x_m) = 3(1.35516)^4 + 7(1.35516)^3 - 15(1.35516)^2 + 5(1.35516) - 17 = -10.23247$

Fourth Iteration

$f(x_l) = 3(1.35516)^4 + 7(1.35516)^3 - 15(1.35516)^2 + 5(1.35516) - 17 = -10.23247$

$f(x_u) = 3(2)^4 + 7(2)^3 - 15(2)^2 + 5(2) - 17 = 37$

$x_m = \frac{x_l(f(x_u)) - x_u(f(x_l))}{f(x_u) - f(x_l)} = \frac{1.35516(37) - 2(-10.23247)}{37 - (-10.23247)} = 1.49486$

$f(x_m) = 3(1.49486)^4 + 7(1.49486)^3 - 15(1.49486)^2 + 5(1.49486) - 17 = -4.68143$

Fifth Iteration

$f(x_l) = 3(1.49486)^4 + 7(1.49486)^3 - 15(1.49486)^2 + 5(1.49486) - 17 = -4.68143$

$f(x_u) = 3(2)^4 + 7(2)^3 - 15(2)^2 + 5(2) - 17 = 37$

$x_m = \frac{x_l(f(x_u)) - x_u(f(x_l))}{f(x_u) - f(x_l)} = \frac{1.49486(37) - 2(-4.68143)}{37 - (-4.68143)} = 1.55159$

$f(x_m) = 3(1.55159)^4 + 7(1.55159)^3 - 15(1.55159)^2 + 5(1.55159) - 17 = -1.81892$

Sixth Iteration

$f(x_l) = 3(1.55159)^4 + 7(1.55159)^3 - 15(1.55159)^2 + 5(1.55159) - 17 = -1.81892$

$f(x_u) = 3(2)^4 + 7(2)^3 - 15(2)^2 + 5(2) - 17 = 37$

$x_m = \frac{x_l(f(x_u)) - x_u(f(x_l))}{f(x_u) - f(x_l)} = \frac{1.55159(37) - 2(-1.81892)}{37 - (-1.81892)} = 1.57260$

$f(x_m) = 3(1.57260)^4 + 7(1.57260)^3 - 15(1.57260)^2 + 5(1.57260) - 17 = -0.66077$

Seventh Iteration

$f(x_l) = 3(1.57260)^4 + 7(1.57260)^3 - 15(1.57260)^2 + 5(1.57260) - 17 = -0.66077$

$f(x_u) = 3(2)^4 + 7(2)^3 - 15(2)^2 + 5(2) - 17 = 37$

$x_m = \frac{x_l(f(x_u)) - x_u(f(x_l))}{f(x_u) - f(x_l)} = \frac{1.57260(37) - 2(-0.66077)}{37 - (-0.66077)} = 1.58010$

$f(x_m) = 3(1.58010)^4 + 7(1.58010)^3 - 15(1.58010)^2 + 5(1.58010) - 17 = -0.23404$

Problem 2 - False Position Method

Determine the root of $f(x) = x^3 - 2x - 5$ between [2,3]. Use False Position Method for 5 iterations.

$Count$	x_l	x_m	x_u	$f(x_l)$	$f(x_m)$	$f(x_u)$	New
1.	2	2.05882	3	-1	-0.39084	16	[2.05882,3]
2.	2.05882	2.08126	3	-0.39084	-0.14725	16	[2.08126,3]
3.	2.08126	2.08964	3	-0.14724	-0.05467	16	[2.08964,3]
4.	2.08964	2.09274	3	-0.05467	-0.02020	16	[2.09274,3]
5.	2.09274	2.09388	3	-0.02020	-0.00749	16	[2.09388,3]

To get the root, use the formula for x_m.

$f(x_l) = 2.09388^3 - 2(2.09388) - 5 = -0.00749

$f(x_u) = 3^3 - 2(3) - 5 = 16$

$x_m = \frac{x_l(f(x_u)) - x_u(f(x_l))}{f(x_u) - f(x_l)} = \frac{2.09388(16) - 3(-0.00749)}{16 - (-0.00749)} = 2.09430$

The root is **2.09430**.

First Iteration

$f(x_l) = 2^3 - 2(2) - 5 = -1$

$f(x_u) = 3^3 - 2(3) - 5 = 16$

$x_m = \frac{x_l(f(x_u)) - x_u(f(x_l))}{f(x_u) - f(x_l)} = \frac{2(16) - 3(-1)}{16 - (-1)} = 2.05882$

$f(x_m) = 2.05882^3 - 2(2.05882) - 5 = -0.39084$

Second Iteration

$f(x_l) = 2.05882^3 - 2(2.05882) - 5 = -0.39084$

$f(x_u) = 3^3 - 2(3) - 5 = 16$

$x_m = \frac{x_l(f(x_u)) - x_u(f(x_l))}{f(x_u) - f(x_l)} = \frac{2.05882(16) - 3(-0.39084)}{16 - (-0.39084)} = 2.08126$

$f(x_m) = 2.08126^3 - 2(2.08126) - 5 = -0.14724$

Third Iteration

$f(x_l) = 2.08126^3 - 2(2.08126) - 5 = -0.14724$

$f(x_u) = 3^3 - 2(3) - 5 = 16$

$x_m = \frac{x_l(f(x_u)) - x_u(f(x_l))}{f(x_u) - f(x_l)} = \frac{2.08126(16) - 3(-0.14724)}{16 - (-0.14724)} = 2.08964$

$f(x_m) = 2.08964^3 - 2(2.08964) - 5 = -0.05467$

Fourth Iteration

$f(x_l) = 2.08964^3 - 2(2.08964) - 5 = -0.05467$

$f(x_u) = 3^3 - 2(3) - 5 = 16$

$x_m = \frac{x_l(f(x_u)) - x_u(f(x_l))}{f(x_u) - f(x_l)} = \frac{2.08964(16) - 3(-0.05467)}{16 - (-0.05467)} = 2.09274$

$f(x_m) = 2.09274^3 - 2(2.09274) - 5 = -0.02020$

Fifth Iteration

$f(x_l) = 2.09274^3 - 2(2.09274) - 5 = -0.02020$

$f(x_u) = 3^3 - 2(3) - 5 = 16$

$x_m = \frac{x_l(f(x_u)) - x_u(f(x_l))}{f(x_u) - f(x_l)} = \frac{2.09274(16) - 3(-0.02020)}{16 - (-0.02020)} = 2.09388$

$f(x_m) = 2.09388^3 - 2(2.09388) - 5 = -0.00749$

Comparison of Root Values Between the 2 Methods:

1. Problem 1

- Bisection Method: 1.58594
- False Position Method: 1.58274

2. Problem 2

- Bisection Method: 2.04688
- False Position Method: 2.09430