Solutions of Linear Equation

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BSCS3

Contents

A linear system is a set of n equations that can solve n number of unknowns. This can be written as:

$A_{11}X_1 + A_{12}X_2 + ... + A_{1n}X_n = B_1$

$A_{21}X_1 + A_{22}X_2 + ... + A_{2n}X_n = B_2$

$A_{n1}X_1 + A_{n2}X_2 + ... + A_{nn}X_n = B_n$

The set $x = X_1 X_2, ..., X_n$ is considered the solution of the linear system if they are **correct in all equations.**

Methods of Solution

Direct Method

Gauss - Elimination

- Consists of 2 phases:
  1. **Elimination**: Series or row operations are performed in the coefficient matrix until it becomes an **upper triangular matrix**.

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{bmatrix}$$

  2. **Back-substitution**: Solving each element of the solution will start at the last row and move up to the first row.

Problem #1 Determine the solution of the linear system shown below using Gauss Elimination.

$2X_1 - 4X_2 + X_3 = -11$

$X_1 + 3X_2 + 2X_3 - 4$

$3X_1 + 5X_2 + 2X_3 = -2$

Elimination Phase

$$\left|\begin{array}{ccc} 2 & -4 & 1 \\ 1 & 3 & 2 \\ 3 & 5 & 2 \end{array}\right| \left|\begin{array}{c} -11 \\ -4 \\ -2 \end{array}\right| \qquad \begin{array}{c} R_2 \to R_2 - \left(\frac{1}{2}\right) R_1 \\ R_3 \to R_3 - \left(\frac{3}{2}\right) R_1 \end{array} \qquad \left|\begin{array}{ccc} 2 & -4 & 1 \\ 0 & 5 & 1.5 \\ 0 & 11 & 0.5 \end{array}\right| \left|\begin{array}{c} -11 \\ 1.5 \\ 14.5 \end{array}\right|$$

*The goal here is to create an **upper triangular matrix**, where all elements below the main diagonal are zeros. To achieve this, we need to perform row operations on R_2 and R_3 in order to eliminate the first element of both rows (below the diagonal). The number 2 is derived from the 1st element of R_1.*

$$\left|\begin{array}{ccc} 2 & -4 & 1 \\ 0 & 5 & 1.5 \\ 0 & 11 & 0.5 \end{array}\right| \left|\begin{array}{c} -11 \\ 1.5 \\ 14.5 \end{array}\right| \qquad R_3 \to R_3 - \left(\frac{11}{5}\right) R_2 \qquad \left|\begin{array}{ccc} 2 & -4 & 1 \\ 0 & 5 & 1.5 \\ 0 & 0 & -2.8 \end{array}\right| \left|\begin{array}{c} -11 \\ 1.5 \\ 11.2 \end{array}\right|$$

Same process as above, except we only needed to change the one in R_3 now.

Back-substitution Phase

$$\left|\begin{array}{ccc} 2 & -4 & 1 \\ 0 & 5 & 1.5 \\ 0 & 0 & -2.8 \end{array}\right| \left|\begin{array}{c} -11 \\ 1.5 \\ 11.2 \end{array}\right|$$

Back substitution means solving from the last row towards the first row

R_3

Divide both sides by -2.8 to isolate X_3:

$$-2.8X_3 = 11.2 \quad \Rightarrow \quad X_3 = -4$$

R_2

Substitute X_3, transpose the result, and divide both sides by 5 to isolate X_2:

$$5X_2{+}1.5(-4) = 1.5 \quad \Rightarrow \quad 5X_2{-}6 = 1.5 \quad \Rightarrow \quad 5X_2 = 7.5$$
$$X_2 = 1.5$$

R_1

Substitute X_2 and X_3 and solve for X_1:

$$2X_1 - 4(1.5) + (-4) = -11 \quad \Rightarrow \quad 2X_1 - 10 = -11$$
$$2X_1 = -1 \quad \Rightarrow \quad X_1 = -0.5$$

Final Values:

$$X_1 = -0.5, \quad X_2 = 1.5, \quad X_3 = -4$$

Gauss Jordan

- Similar to *Elimination*, except we need to make an **identity matrix**
- No back substitution required.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 1 Determine the solution of the linear system shown below using Gauss Jordan

$2X_1 - 4X_2 + X_3 = -11$

$X_1 + 3X_2 + 2X_3 - 4$

$3X_1 + 5X_2 + 2X_3 = -2$

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & x_{32} & x_{33} \end{bmatrix}$$

$$\left|\begin{array}{ccc} 2 & -4 & 1 \\ 1 & 3 & 2 \\ 3 & 5 & 2 \end{array}\right| \left|\begin{array}{c} -11 \\ -4 \\ -2 \end{array}\right| \qquad \begin{array}{c} R_2 \to R_2 - \left(\frac{1}{2}\right) R_1 \\ R_3 \to R_3 - \left(\frac{3}{2}\right) R_1 \end{array} \qquad \left|\begin{array}{ccc} 2 & -4 & 1 \\ 0 & 5 & 1.5 \\ 0 & 11 & 0.5 \end{array}\right| \left|\begin{array}{c} -11 \\ 1.5 \\ 14.5 \end{array}\right|$$

*The goal here is to create an **identity matrix**, where the main diagonal consists of 1s, while everything else is 0.*

$$\begin{bmatrix} x_{11} & 0 & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{bmatrix}$$

$$\left|\begin{array}{ccc} 2 & -4 & 1 \\ 0 & 5 & 1.5 \\ 0 & 11 & 0.5 \end{array}\right| \left|\begin{array}{c} -11 \\ 1.5 \\ 14.5 \end{array}\right| \qquad \begin{array}{c} R_1 \to R_1 - \left(\frac{-4}{5}\right) R_2 \\ R_3 \to R_3 - \left(\frac{11}{5}\right) R_2 \end{array} \qquad \left|\begin{array}{ccc} 2 & 0 & 2.2 \\ 0 & 5 & 1.5 \\ 0 & 0 & -2.8 \end{array}\right| \left|\begin{array}{c} -9.8 \\ 1.5 \\ 11.2 \end{array}\right|$$

$$\begin{bmatrix} x_{11} & 0 & 0 \\ 0 & x_{22} & 0 \\ 0 & 0 & x_{33} \end{bmatrix}$$

$$\left|\begin{array}{ccc} 2 & 0 & 2.2 \\ 0 & 5 & 1.5 \\ 0 & 0 & -2.8 \end{array}\right| \left|\begin{array}{c} -9.8 \\ 1.5 \\ 11.2 \end{array}\right| \qquad \begin{array}{c} R_1 \to R_1 - \left(\frac{-11}{14}\right) R_3 \\ R_2 \to R_2 - \left(\frac{-15}{28}\right) R_3 \end{array} \qquad \left|\begin{array}{ccc} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2.8 \end{array}\right| \left|\begin{array}{c} -1 \\ 7.5 \\ 11.2 \end{array}\right|$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left|\begin{array}{ccc} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2.8 \end{array}\right| \left|\begin{array}{c} -1 \\ 7.5 \\ 11.2 \end{array}\right| \qquad \begin{array}{c} R_1 \to R_1 \div 2 \\ R_2 \to R_2 \div 5 \\ R_3 \to R_3 \div -2.8 \end{array} \qquad \left|\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right| \left|\begin{array}{c} -0.5 \\ 1.5 \\ -4 \end{array}\right|$$

Final Values:

$$X_1 = -0.5, \quad X_2 = 1.5, \quad X_3 = -4$$