Open Methods in Finding the Root of a Non-linear System

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BSCS3

Contents

Newton Raphson Method

1. Determine $f'(x_n)$ - 1st derivative
2. Solve $f(x_n)$ and $f'(x_n)$
3. Solve for x_{n+1} using $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Problem 1 - Newton Raphson Method

Determine the root of $3x^4 + 7x^3 - 15x^2 + 5x = 17$ between [0,2]. Use Newton Raphson method and perform 7 iterations. Use $x_0 = 0$.

$f(x_n) = 3x^4 + 7x^3 - 15x^2 + 5x - 17$

$f'(x_n) = 12x^3 + 21x^2 - 30x + 5$

#	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
0.	0	-17	5	3.4
1.	-3.4	502.6288	617.408	2.58590
2.	2.58590	150.81056	275.34665	2.03819
3.	2.03819	41.91987	132.69793	1.72229
4.	1.72229	9.27530	76.92883	1.60172
5.	1.60172	1.03604	60.13473	1.58449
6.	1.58449	0.01903	57.92449	1.58416
7.	1.58416	-0.00008	57.88261	1.58416

$f(x_n) = 3(0)^4 + 7(0)^3 - 15(0)^2 + 5(0) - 17 = -17$

$f'(x_n) = 12(0)^3 + 21(0)^2 - 30(0) + 5 = 5$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = 0 - \frac{-17}{5} = 3.4$

x_{n+1} becomes the new x_n in the next iteration.

First Iteration

$f(x_n) = 3(3.4)^4 + 7(3.4)^3 - 15(3.4)^2 + 5(3.4) - 17 = 502.6288$

$f'(x_n) = 12(3.4)^3 + 21(3.4)^2 - 30(3.4) + 5 = 617.408$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = 3.4 - \frac{502.6288}{617.408} = 2.58590$

Second Iteration

$f(x_n) = 3(2.58590)^4 + 7(2.58590)^3 - 15(2.58590)^2 + 5(2.58590) - 17 = 150.81056$

$f'(x_n) = 12(2.58590)^3 + 21(2.58590)^2 - 30(2.58590) + 5 = 275.34665$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = 2.58590 - \frac{150.81056}{275.34665} = 2.03819$

Third Iteration

$f(x_n) = 3(2.03819)^4 + 7(2.03819)^3 - 15(2.03819)^2 + 5(2.03819) - 17 = 41.91987$

$f'(x_n) = 12(2.03819)^3 + 21(2.03819)^2 - 30(2.03819) + 5 = 132.69793$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = 2.03819 - \frac{41.91987}{132.69793} = 1.72229$

Fourth Iteration

$f(x_n) = 3(1.72229)^4 + 7(1.72229)^3 - 15(1.72229)^2 + 5(1.72229) - 17 = 9.27530$

$f'(x_n) = 12(1.72229)^3 + 21(1.72229)^2 - 30(1.72229) + 5 = 76.92883$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = 1.72229 - \frac{9.27530}{76.92883} = 1.60172$

Fifth Iteration

$f(x_n) = 3(1.60172)^4 + 7(1.60172)^3 - 15(1.60172)^2 + 5(1.60172) - 17 = 1.03604$

$f'(x_n) = 12(1.60172)^3 + 21(1.60172)^2 - 30(1.60172) + 5 = 60.13473$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = 1.60172 - \frac{1.03604}{60.13473} = 1.58449$

Sixth Iteration

$f(x_n) = 3(1.58449)^4 + 7(1.58449)^3 - 15(1.58449)^2 + 5(1.58449) - 17 = 0.01903$

$f'(x_n) = 12(1.58449)^3 + 21(1.58449)^2 - 30(1.58449) + 5 = 57.92449$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = 1.58449 - \frac{0.01903}{57.92449} = 1.58416$

Seventh Iteration

$f(x_n) = 3(1.58416)^4 + 7(1.58416)^3 - 15(1.58416)^2 + 5(1.58416) - 17 = -0.00008$

$f'(x_n) = 12(1.58416)^3 + 21(1.58416)^2 - 30(1.58416) + 5 = 57.88261$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = 1.58416 - \frac{-0.00008}{57.88261} = 1.58416$

Problem 2 - Newton Raphson Method

Determine the root of $f_x = x^3 - 2x - 5$ between [2,3]. Newton Raphson Method and perorm 5 iterations. Use $x_0 = 0$

$f(x_n) = x^3 - 2x - 5$

$f'(x_n) = 3x^2 - 2$

#	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
0.	0	-5	-2	-2.5
1.	-2.5	15.626	16.75	-1.56716
2.	1.56716	-5.71461	5.36797	-0.50258
3.	-0.50258	-4.12179	1.24224	2.81545
4.	2.81545	11.68649	21.78028	2.27889
5.	2.27889	2.27727	13.58002	2.11120

$f(x_n) = 0^3 - 2(0) - 5 = -5$

$f'(x_n) = 3(0)^2 - 2 = -2$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = 0 - \frac{-5}{-2} = -2.5$

x_{n+1} becomes the new x_n in the next iteration.

First Iteration

$f(x_n) = -2.5^3 - 2(-2.5) - 5 = 15.625$

$f'(x_n) = 3(-2.5)^2 - 2 = 16.75$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = -2.5\frac{-15.625}{16.75} = -1.56716$

Second Iteration

$f(x_n) = -1.56716^3 - 2(-1.56716) - 5 = -5.71461$

$f'(x_n) = 3(-1.56716)^2 - 2 = 5.36797$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = -1.56716\frac{-5.71461}{5.36797} = -0.50258$

Third Iteration

$f(x_n) = -0.50258^3 - 2(-0.50258) - 5 = -4.12179$

$f'(x_n) = 3(-0.50258)^2 - 2 = 1.24224$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = -0.50258\frac{-4.12179}{1.24224} = 2.81545$

Fourth Iteration

$f(x_n) = 2.81545^3 - 2(2.81545) - 5 = 11.68649$

$f'(x_n) = 3(2.81545)^2 - 2 = 21.78028$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = 2.81545\frac{11.68649}{21.78028} = 2.27889$

Fifth Iteration

$f(x_n) = 2.27889^3 - 2(2.27889) - 5 = 2.27727$

$f'(x_n) = 3(2.27889)^2 - 2 = 13.58002$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = 2.27889\frac{2.27727}{13.58002} = 2.11120$

Secant Method

1. Solve $f(x_{n-1})$ and $f(x_n)$
2. Solve for new x_{n+1} using $x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1}}{f(x_n) - f(x_{n-1}}$

Problem 1 - Secant Method

Determine the root of $3x^4 + 7x^3 - 15x^2 + 5x = 17$ between [1,2]. Use Secant Method and perform 7 iterations.

#	x_{n-1}	x_n	$f(x_{n-1})$	$f(x_n)$	x_{n+1}
0.	1	2	-17	37	1.31481
1.	2	1.31481	37	-11.48074	1.47707
2.	1.31481	1.47707	-11.48074	-5.50274	1.62643
3.	1.47707	1.62643	-5.50274	2.56194	1.57898
4.	1.62643	1.57898	2.56194	-0.29821	1.58393
5.	1.57898	1.58393	-0.29821	-0.01339	1.58416
6.	1.58393	1.58416	-0.01339	-0.00008	1.58416

x_n is the new x_{n-1} in the next iteration

x_{n+1} is the new x_n in the next iteration

$f(x_{n-1}) = 3(1)^4 + 7(1)^3 - 15(1)^2 + 5(1) - 17 = -17$

$f(x_n) = 3(2)^4 + 7(2)^3 - 15(2)^2 + 5(2) - 17 = 37$

$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1}}{f(x_n) - f(x_{n-1}} = 2 - \frac{(37)(2-1)}{37-(-17)} = 1.31481$

First Iteration

$f(x_{n-1}) = 3(2)^4 + 7(2)^3 - 15(2)^2 + 5(2) - 17 = 37$

$f(x_n) = 3(1.31481)^4 + 7(1.31481)^3 - 15(1.31481)^2 + 5(1.31481) - 17 = -11.48074$

$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1}}{f(x_n) - f(x_{n-1}} = 1.31481 - \frac{(-11.48074)(1.31481-2)}{-11.48074-37} = 1.47707$

Second Iteration

$f(x_{n-1}) = 3(1.31481)^4 + 7(1.31481)^3 - 15(1.31481)^2 + 5(1.31481) - 17 = -11.48074$

$f(x_n) = 3(1.47707)^4 + 7(1.47707)^3 - 15(1.47707)^2 + 5(1.47707) - 17 = -5.50274$

$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1}}{f(x_n) - f(x_{n-1}} = 1.47707 - \frac{(-5.50274)(1.47707-1.31481)}{-5.50274-(-11.48074)} = 1.62643$

Third Iteration

$f(x_{n-1}) = 3(1.47707)^4 + 7(1.47707)^3 - 15(1.47707)^2 + 5(1.47707) - 17 = -5.50274$

$f(x_n) = 3(1.62643)^4 + 7(1.62643)^3 - 15(1.62643)^2 + 5(1.62643) - 17 = 2.56194$

$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1}}{f(x_n) - f(x_{n-1}} = 1.62643 - \frac{(2.56194)(1.62643-1.47707)}{2.56194-(-5.50274)} = 1.57898$

Fourth Iteration

$f(x_{n-1}) = 3(1.62643)^4 + 7(1.62643)^3 - 15(1.62643)^2 + 5(1.62643) - 17 = 2.56194$

$f(x_n) = 3(1.57898)^4 + 7(1.57898)^3 - 15(1.57898)^2 + 5(1.57898) - 17 = -0.29821$

$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1}}{f(x_n) - f(x_{n-1}} = 1.57898 - \frac{(-0.29821)(1.57898-1.62643)}{-0.29821-2.56194} = 1.58393$

Fifth Iteration

$f(x_{n-1}) = 3(1.57898)^4 + 7(1.57898)^3 - 15(1.57898)^2 + 5(1.57898) - 17 = -0.29821$

$f(x_n) = 3(1.58393)^4 + 7(1.58393)^3 - 15(1.58393)^2 + 5(1.58393) - 17 = -0.01339$

$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1}}{f(x_n) - f(x_{n-1}} = 1.58393 - \frac{(-0.01339)(1.58393-1.57898)}{-0.01339-(-0.29821)} = 1.58416$

Sixth Iteration

$f(x_{n-1}) = 3(1.58393)^4 + 7(1.58393)^3 - 15(1.58393)^2 + 5(1.58393) - 17 = -0.01339$

$f(x_n) = 3(1.58416)^4 + 7(1.58416)^3 - 15(1.58416)^2 + 5(1.58416) - 17 = -0.00008$

$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1}}{f(x_n) - f(x_{n-1}} = 1.58416 - \frac{(-0.00008)(1.58416-1.58393)}{-0.00008-(-0.01339)} = 1.58416$

Problem 2 - Secant Method

Determine the root of $f(x) = x^3 - 2x - 5$ between [2,3]. Use the Secant Method and perform 5 iterations.

#	x_{n-1}	x_n	$f(x_{n-1})$	$f(x_n)$	x_{n+1}
0.	2	3	-1	16	2.05882
1.	3	2.05882	16	-0.39084	2.08126
2.	2.05882	2.08126	-0.39084	-0.14724	2.09482
3.	2.08126	2.09482	-0.14724	-0.00300	2.09510
4.	2.09482	2.09510	-0.0300	-0.00612	2.09455
5.	2.09510	2.09455	-0.00612	-0.00002	2.09455

x_n is the new x_{n-1} in the next iteration

x_{n+1} is the new x_n in the next iteration

$f(x_{n-1}) = 2^3 - 2(2) - 5 = -1$

$f(x_n) = 3^3 - 2(3) - 5 = 16$

$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1}}{f(x_n) - f(x_{n-1}} = 3 - \frac{(16)(3-2)}{16-(-1)} = 2.05882$

First Iteration

$f(x_{n-1}) = 3^3 - 2(3) - 5 = 16$

$f(x_n) = 2.05882^3 - 2(2.05882) - 5 = -0.39084$

$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1}}{f(x_n) - f(x_{n-1}} = 2.05882 - \frac{(-0.39084)(2.05882-3)}{-0.39084-16} = 2.08126$

Second Iteration $f(x_{n-1}) = 2.05882^3 - 2(2.05882) - 5 = -0.39084$

$f(x_n) = 2.08126^3 - 2(2.08126) - 5 = -0.14724$

$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1}}{f(x_n) - f(x_{n-1}} = 2.08126 - \frac{(-0.14724)(2.08126-2.05882)}{-0.14724-(-0.39084)} = 2.09482$

Third Iteration $f(x_{n-1}) = 2.08126^3 - 2(2.08126) - 5 = -0.14724$

$f(x_n) = 2.09482^3 - 2(2.09482) - 5 = -0.00300$

$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1}}{f(x_n) - f(x_{n-1}} = 2.09482 - \frac{(-0.00300)(2.09482-2.08126)}{-0.00300-(-0.14724)} = 2.09510$

Fourth Iteration

$f(x_{n-1}) = 2.09482^3 - 2(2.09482) - 5 = -0.00300$

$f(x_n) = 2.09510^3 - 2(2.09510) - 5 = -0.00612$

$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1}}{f(x_n) - f(x_{n-1}} = 2.09510 - \frac{(-0.00612)(2.09510-2.09482)}{-0.00612-(-0.00300)} = 2.09455$

Fifth Iteration

$f(x_{n-1}) = 2.09510^3 - 2(2.09510) - 5 = -0.00612$

$f(x_n) = 2.09455^3 - 2(2.09455) - 5 = -0.00002$

$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1}}{f(x_n) - f(x_{n-1}} = 2.09455 - \frac{(-0.00002)(2.09455-2.09510)}{-0.00002-(-0.00612)} = 2.09455$

Comparison of Root Values Between the 2 Methods:

1. Problem 1

- Newton Raphson Method: 1.58416
- Secant Method: 1.58416

2. Problem 2

- Newton Raphson Method: 2.11120
- Secant Method: 2.09455