Solutions of Linear Equation

Dan del Prado

BSCS3

Contents

A linear system is a set of n equations that can solve n number of unknowns. This can be written as:

$A_{11}X_1 + A_{12}X_2 + ... + A_{1n}X_n = B_1$

$A_{21}X_1 + A_{22}X_2 + ... + A_{2n}X_n = B_2$

$A_{n1}X_1 + A_{n2}X_2 + ... + A_{nn}X_n = B_n$

The set $x = X_1X_2, ..., X_n$ is considered the solution of the linear system if they are **correct in all equations.**

Methods of Solution

Direct Method

Gauss - Elimination

- Consists of 2 phases:
  1. **Elimination**: Series or row operations are performed in the coefficient matrix until it becomes an **upper triangular matrix**.

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ 0 & x_5 & x_6 \\ 0 & 0 & x_9 \end{bmatrix}$$

  2. **Back-substitution**: Solving each element of the solution will start at the last row and move up to the first row.

Problem #1 Determine the solution of the linear system shown below using Gauss Elimination.

$2X_1 - 4X_2 + X_3 = -11$

$X_1 + 3X_2 + 2X_3 - 4$

$3X_1 + 5X_2 + 2X_3 = -2$

Elimination Phase

$$\left|\begin{matrix} 2 & -4 & 1 \\ 1 & 3 & 2 \\ 3 & 5 & 2 \end{matrix}\right|\left|\begin{matrix} -11 \\ -4 \\ -2 \end{matrix}\right| \qquad \begin{matrix} R_2 \to R_2 - \left(\frac{1}{2}\right) R_1 \\ R_3 \to R_3 - \left(\frac{3}{2}\right) R_1 \end{matrix} \qquad \left|\begin{matrix} 2 & -4 & 1 \\ 0 & 5 & 1.5 \\ 0 & 11 & 0.5 \end{matrix}\right|\left|\begin{matrix} -11 \\ 1.5 \\ 14.5 \end{matrix}\right|$$

*The goal here is to create an **upper triangular matrix**, where all elements below the main diagonal are zeros. To achieve this, we need to perform row operations on R_2 and R_3 in order to eliminate the first element of both rows (below the diagonal). The number 2 is derived from the 1st element of R_1.*

$$\left|\begin{array}{ccc} 2 & -4 & 1 \\ 0 & 5 & 1.5 \\ 0 & 11 & 0.5 \end{array}\right|\left|\begin{array}{c} -11 \\ 1.5 \\ 14.5 \end{array}\right| \qquad R_3 \to R_3 - \left(\tfrac{11}{5}\right) R_2 \qquad \left|\begin{array}{ccc} 2 & -4 & 1 \\ 0 & 5 & 1.5 \\ 0 & 0 & -2.8 \end{array}\right|\left|\begin{array}{c} -11 \\ 1.5 \\ 11.2 \end{array}\right|$$

Same process as above, except we only needed to change the one in R_3 now.

Back-subsitution Phase

$$\left|\begin{array}{ccc} 2 & -4 & 1 \\ 0 & 5 & 1.5 \\ 0 & 0 & -2.8 \end{array}\right|\left|\begin{array}{c} -11 \\ 1.5 \\ 11.2 \end{array}\right|$$

The labels in the matrix should be as follows:

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 \end{bmatrix}$$

Back substitution means solving from the last row towards the first row

R_3

Divide both sides by -2.8 to isolate X_3:

$$-2.8X_3 = 11.2 \quad \Rightarrow \quad X_3 = -4$$

R_2

Substitute X_3, transpose the result, and divide both sides by 5 to isolate X_2:

$$5X_2 + 1.5(-4) = 1.5 \quad \Rightarrow \quad 5X_2 - 6 = 1.5 \quad \Rightarrow \quad 5X_2 = 7.5$$

$$X_2 = 1.5$$

R_1

Substitute X_2 and X_3 into R_1 and solve for X_1:

$$2X_1 - 4(1.5) + (-4) = -11 \quad \Rightarrow \quad 2X_1 - 10 = -11$$

$$2X_1 = -1 \quad \Rightarrow \quad X_1 = -0.5$$

Final Values:

$$X_1 = -0.5, \quad X_2 = 1.5, \quad X_3 = -4$$