Exercise 19
The Trapezoidal rule and P1 elements

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We consider the approximation of a general function $f(x)$ on a general domain Ω with the least squares method. We will work on the reference element $[-1, 1]$ and map the element domain $[x_i, x_{i+1}]$ to this domain, for the global variable $i = 0, 1, ..., N-1$.

Hence we define $X \in [-1, 1]$ to be the coordinate in the reference element, and we use the affine mapping from the subdomain $\Omega^{(i)}$ to be

$$x = x_m + \frac{h}{2}X$$

where $x_m = \frac{x_i + x_{i+1}}{2}$ and $h = x_{i+1} - x_i$.
The stretch factor $\delta x/\delta X$ between x and X coordinates is the determinant of the Jacobi matrix and in 1D is equal to $h/2$.

The Trapezoidal rule is used to compute the integrals. On the reference element we thus get the approximation

$$\int_{-1}^{1} g(X)dX \approx g(-1) + g(1)$$

for a generic function $g(x)$.

We will work with P1 elements, ie. Lagrange polynomial basis functions of degree 1, so the basis functions are given by $\phi_0 = \frac{1}{2}(1 - X)$ and $\phi_1 = \frac{1}{2}(1 + X)$.

We want to show that the linear system resulting from the least squares method is equal to a linear system with entries $c_i = f(x_i)$ when the Trapezoidal rule is used in the integral approximation together with P1 elements.

We start deriving the element matrix $A^{(e)}_{r,s}$ and the element vector $b^{(e)}_r$, where r and s are the local element variables and $r, s = 0, 1$ since we are working with P1 elements. Then we will assemble all local contributions and derive the global matrix $A_{i,j}$ and the global vector b_i.

In general,

$$\begin{aligned}
A_{r,s}^{(e)} &= \int_{-1}^{1} \phi_r(X)\phi_s(X)\frac{\delta x}{\delta X}\delta X \\
&= \int_{-1}^{1} \phi_r(X)\phi_s(X)\frac{h}{2}\delta X \\
&\approx \frac{h}{2}[\phi_r(-1)\phi_s(-1) + \phi_r(1)\phi_s(1)]
\end{aligned}$$

$$\begin{aligned}
b_r^{(e)} &= \int_{-1}^{1} \phi_r(X)f(x(X))\frac{\delta x}{\delta X}\delta X \\
&= \frac{h}{2}\int_{-1}^{1} \phi_r(X)f(\frac{1}{2}(x_i - x_{i+1}) + \frac{h}{2}X)\delta X \\
&\approx \frac{h}{2}[\phi_r(-1)f(\frac{1}{2}(x_i - x_{i+1}) - \frac{h}{2}) + \phi_r(1)f(\frac{1}{2}(x_i - x_{i+1}) + \frac{h}{2})]
\end{aligned}$$

Hence we get

$$\begin{aligned}
A_{0,0}^{(e)} &= \frac{h}{2}[\phi_0(-1)\phi_0(-1) + \phi_1(1)\phi_1(1)] \\
&= \frac{h}{2} \\
A_{1,0}^{(e)} &= \frac{h}{2}[\phi_0(-1)\phi_1(-1) + \phi_0(1)\phi_1(1)] \\
&= 0 = A_{0,1}^{(e)} \\
A_{1,1}^{(e)} &= \frac{h}{2}[\phi_1(-1)\phi_1(-1) + \phi_1(1)\phi_1(1)] \\
&= \frac{h}{2} \\
b_0 &= \frac{h}{2}[\phi_0(-1)f(\frac{1}{2}(x_i - x_{i+1}) - \frac{h}{2}) + \phi_0(1)f(\frac{1}{2}(x_i - x_{i+1}) + \frac{h}{2})] \\
&= \frac{h}{2}f(x_i) \\
b_1 &= \frac{h}{2}[\phi_1(-1)f(\frac{1}{2}(x_i - x_{i+1}) - \frac{h}{2}) + \phi_1(1)f(\frac{1}{2}(x_i - x_{i+1}) + \frac{h}{2})] \\
&= \frac{h}{2}f(x_{i+1})
\end{aligned}$$

hence for every element:

$$A^{(e)} = \frac{h}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$b^{(e)} = \frac{h}{2}\begin{bmatrix}f(x_i)\\ f(x_{i+1})\end{bmatrix}$$

and all together, using that
$A_{i,j} = \sum_e A_{i,j}^{(e)}$ and $b_i = \sum_e b_i^{(e)}$:

$$A = \frac{h}{2}\begin{bmatrix}\frac{h}{2} & & & & & & \\ & h & & & & & \\ & & h & & & & \\ & & \cdots & & & & \\ & & & & h & & \\ & & & & & h & \\ & & & & & & \frac{h}{2}\end{bmatrix}$$

$$b = \frac{h}{2}\begin{bmatrix}\frac{h}{2}f(x_0)\\ hf(x_1)\\ .\\ .\\ .\\ hf(x_{n-1})\\ \frac{h}{2}f(x_n)\end{bmatrix}$$

The linear system $Ac = b$ can be solved directly since A is tridiagonal:

$$c = A^{-1}b \quad = \begin{bmatrix}\frac{2}{h} & & & & & & \\ & \frac{1}{h} & & & & & \\ & & \frac{1}{h} & & & & \\ & & \cdots & & & & \\ & & & & \frac{1}{h} & & \\ & & & & & \frac{1}{h} & \\ & & & & & & \frac{2}{h}\end{bmatrix}\begin{bmatrix}\frac{h}{2}f(x_0)\\ hf(x_1)\\ .\\ .\\ .\\ hf(x_{n-1})\\ \frac{h}{2}f(x_n)\end{bmatrix}$$

so $c_i = f(x_i)$, as we wanted to show because:

$$f(x_i) = \begin{bmatrix} f(x_0) \\ f(x_1) \\ . \\ . \\ . \\ f(x_{n-1}) \\ f(x_n) \end{bmatrix} \tag{1}$$