Variational form and iterative methods

$$\rho c(T)T_t = \nabla \cdot (k(T)\nabla T) \qquad \mathbf{x} \in \Omega \tag{1}$$

With Robin boundary condition:

$$-k(T)\frac{\partial T}{\partial n} = h(T)(T - T_s(t)), \qquad \mathbf{x} \in \partial\Omega \tag{2}$$

(a) The variational form

We apply a Backward Euler discretization in time:

$$\begin{aligned}\rho c(u)^n \frac{u^n - u^{n-1}}{\Delta t} &= [\nabla \cdot (k(u)\nabla u)]^n \\ \rho c(u^n)u^n &= \rho c(u^n)u^{n-1} + \Delta t[\nabla \cdot (k(u^n)\nabla u^n)]\end{aligned} \tag{3}$$

The residual reads:

$$R = \rho c(u^n)u^n - \rho c(u^n)u^{n-1} - \Delta t[\nabla \cdot (k(u^n)\nabla u^n)] \tag{4}$$

We use the Galerkin method to derive the variational form:

$$\begin{aligned}&(R, v) = 0 \\ &(R, v) = (\rho c(u^n)u^n, v) - (\rho c(u^n)u^{n-1}, v) - \Delta t(\nabla \cdot (k(u^n)\nabla u^n), v)\end{aligned} \tag{5}$$

We reformulate the last term using integration by parts and the boundary condition

$$\begin{aligned}(\nabla \cdot (k(u^n)\nabla u^n), v) &= \int_\Omega \nabla \cdot (k(u^n)\nabla u^n) v \,\mathrm{d}x \\ &= \int_{\partial\Omega} (k(u^n)\nabla u^n) v \cdot \mathbf{n} \,\mathrm{d}x - \int_\Omega (k(u^n)\nabla u^n) \cdot \nabla v \,\mathrm{d}x \\ &= -\int_{\partial\Omega} h(u^n)(u^n - T_s(t)) v \,\mathrm{d}x - \int_\Omega (k(u^n)\nabla u^n) \cdot \nabla v \,\mathrm{d}x\end{aligned} \tag{6}$$

The variational form now looks as follows:

$$\int_\Omega \rho c(u^n)u^n v \,\mathrm{d}x = \int_\Omega \rho c(u^n)u^{n-1} v \,\mathrm{d}x - \Delta t \int_\Omega (k(u^n)\nabla u^n) \cdot \nabla v \,\mathrm{d}x - \Delta t \int_{\partial\Omega} h(u^n)(u^n - T_s(t)) v \,\mathrm{d}x \tag{7}$$

(b) Picard Iteration

We use u^- as a guess for u^n in all the nonlinear terms. To simplify the notation we use $u^n = u$ and $u^{n-1} = u_1$. The linear system we need to solve is then:

$$\int_\Omega \rho c(u^-) uv + \Delta t \int_\Omega k(u^-) \nabla u \cdot \nabla v \, \mathrm{d}x + \tag{8}$$

$$\Delta t \int_{\partial\Omega} h(u^-)(u - T_s(t)) v \, \mathrm{d}x = \int_\Omega \rho c(u^-) u_1 v \, \mathrm{d}x \tag{9}$$

u can be expressed in terms of the basisfunctions $u = \sum c_j \phi_j$:

$$\begin{aligned}
&\sum c_j \int_\Omega \rho c(u^-) \phi_j \phi_i \, \mathrm{d}x \\
&+ \Delta t \sum c_j \int_\Omega k(u^-) \nabla \phi_j \cdot \nabla \phi_i \, \mathrm{d}x \\
&+ \Delta t \sum c_j \int_{\partial\Omega} h(u^-) \phi_j \phi_i \, \mathrm{d}x \\
&= \int_\Omega \rho c(u^-) u_1 \phi_i \, \mathrm{d}x + \Delta t \int_{\partial\Omega} h(u^-) T_s(t) \phi_i \, \mathrm{d}x
\end{aligned} \tag{10}$$

(c) Newtons Method on the variational form

$$\begin{aligned}
F(u) = &\int_\Omega \rho c(u^n) u^n v \\
&- \int_\Omega \rho c(u^n) u^{n-1} v \, \mathrm{d}x \\
&+ \Delta t \int_\Omega (k(u^n) \nabla u^n) \cdot \nabla v \, \mathrm{d}x \\
&+ \Delta t \int_{\partial\Omega} h(u^n)(u^n - T_s(t)) v \, \mathrm{d}x
\end{aligned}$$

We use that $u = \sum c_j \phi_j$, and $u^n = u$, $u^{n-1} = u_1$:

$$
\begin{aligned}
F_i(c) = &\sum c_j \int_\Omega \rho c(\sum c_k\phi_k)\phi_j\phi_i \,\mathrm{d}x \\
&+\Delta t \sum c_j \int_\Omega k(\sum c_k\phi_k)\nabla\phi_j \cdot \nabla\phi_i \,\mathrm{d}x \\
&+ \Delta t \sum c_j \int_{\partial\Omega} h(\sum c_k\phi_k)\phi_j\phi_i \,\mathrm{d}x \\
&- \int_\Omega \rho c(\sum c_k\phi_k)u_1\phi_i \,\mathrm{d}x \\
&-\Delta t \int_{\partial\Omega} h(\sum c_k\phi_k)T_s(t)\phi_i \,\mathrm{d}x
\end{aligned} \tag{11}
$$

We need the Jacobian $J_{i,j} = \dfrac{\partial F_i}{\partial c_j}$:

$$
\begin{aligned}
J_{i,j} = &\int_\Omega \rho c(\sum c_k\phi_k)\phi_j\phi_i \,\mathrm{d}x + \sum c_m \int_\Omega \rho c'(\sum c_k\phi_k)\phi_j\phi_m\phi_i \,\mathrm{d}x \\
&+ \Delta t \int_\Omega k(\sum c_k\phi_k)\nabla\phi_j \cdot \nabla\phi_i \,\mathrm{d}x + \Delta t \sum c_m \int_\Omega k'(\sum c_k\phi_k)\phi_j\nabla\phi_m \cdot \nabla\phi_i \,\mathrm{d}x \\
&+ \Delta t \int_{\partial\Omega} h(\sum c_k\phi_k)\phi_j\phi_i \,\mathrm{d}x + \Delta t \sum c_m \int_{\partial\Omega} h'(\sum c_k\phi_k)\phi_j\phi_m\phi_i \,\mathrm{d}x \\
&- \int_\Omega \rho c'(\sum c_k\phi_k)\phi_j u_1\phi_i \,\mathrm{d}x \\
&- \Delta t \int_{\partial\Omega} h'(\sum c_k\phi_k)\phi_j T_s(t)\phi_i \,\mathrm{d}x
\end{aligned} \tag{12}
$$

In the iteration we use $J(u^-)$

$$
\begin{aligned}
J(u^-) = &\int_\Omega \rho c(u^-)\phi_j\phi_i \,\mathrm{d}x + \int_\Omega \rho c'(u^-)u^-\phi_j\phi_i \,\mathrm{d}x \\
&+ \Delta t \int_\Omega k(u^-)\nabla\phi_j \cdot \nabla\phi_i \,\mathrm{d}x + \Delta t \int_\Omega k'(u^-)\phi_j\nabla u^- \cdot \nabla\phi_i \,\mathrm{d}x \\
&+ \Delta t \int_{\partial\Omega} h(u^-)\phi_j\phi_i \,\mathrm{d}x + \Delta t \int_{\partial\Omega} h'(u^-)u^-\phi_j\phi_i \,\mathrm{d}x \\
&- \int_\Omega \rho c'(u^-)\phi_j u_1\phi_i \,\mathrm{d}x \\
&- \Delta t \int_{\partial\Omega} h'(u^-)\phi_j T_s(t)\phi_i \,\mathrm{d}x
\end{aligned} \tag{13}
$$

One can now solve $J(u^-)\delta u = -F(u^-)$

(d) Newtons Method at PDE level

$$F(u) = \rho c(u)u - \rho c(u)u_1 - \Delta t[\nabla \cdot (k(u)\nabla u)] \tag{14}$$

We insert $u = u^- + \delta u$ in $F(u)$:

$$F(u^-+\delta u) = \rho c(u^-+\delta u)(u^-+\delta u) - \rho c(u^-+\delta u)u_1 - \Delta t[\nabla\cdot(k(u^-+\delta u)\nabla(u^-+\delta u))] \tag{15}$$

We Taylorexpand the functions around u^-

$$\begin{aligned} c(u^- + \delta u) &= c(u^-) + c'(u^-)\delta u + O(\delta u^2) \\ k(u^- + \delta u) &= k(u^-) + k'(u^-)\delta u + O(\delta u^2) \end{aligned} \tag{16}$$

We keep terms up to order δu:

$$\begin{aligned} F(u^- + \delta u) =& \rho c(u^-)(u^- + \delta u) + \rho c'(u^-)\delta u u^- - \rho(c(u^-) + c'(u^-)\delta u)u_1 \\ &- \Delta t[\nabla \cdot (k(u^-)\nabla(u^- + \delta u) + k'(u^-)\delta u \nabla u^-)] \end{aligned} \tag{17}$$

This gives a linear system for δu

$$\begin{aligned} &\rho[c(u^-) + c'(u^-)u^- - c'(u^-)u_1]\delta u \\ &- \Delta t[\nabla \cdot (k(u^-)\nabla \delta u + k'(u^-)\nabla u^- \delta u)] \\ &= -\rho c(u^-)u^- + \rho c(u^-)u_1 + \Delta t[\nabla \cdot (k(u^-)\nabla u^-)] \end{aligned} \tag{18}$$

Linearizing the boundary term:

$$\begin{aligned} -k(u^- + \delta u)\frac{\partial(u^- + \delta u)}{\partial n} &= h(u^- + \delta u)(u^- + \delta u - T_s(t)) \\ (-k(u^-) - k'(u^-))\delta u(\frac{\partial u^-}{\partial n} + \frac{\partial \delta u}{\partial n}) &= (h(u^-) + h'(u^-)\delta u)(u^- + \delta u - T_s(t)) \end{aligned} \tag{19}$$

We only keep the linear terms:

$$\begin{aligned} -k(u^-)(\frac{\partial u^-}{\partial n} + \frac{\partial \delta u}{\partial n}) - k'(u^-)\delta u(\frac{\partial u^-}{\partial n}) = \\ h(u^-)(u^- + \delta u - T_s(t)) + h'(u^-)\delta u(u^- - T_s(t)) \end{aligned} \tag{20}$$

Variational form:

$$\begin{aligned} &\rho([c(u^-) + c'(u^-)u^- - c'(u^-)u_1]\delta u, v) \\ &- \Delta t([\nabla \cdot (k(u^-)\nabla \delta u + k'(u^-)\nabla u^- \delta u)], v) \\ &= -\rho(c(u^-)(u^- + u_1), v) + \Delta t(\nabla \cdot (k(u^-)\nabla u^-), v) \end{aligned} \tag{21}$$

Integration by parts:

$$
\begin{aligned}
&(\nabla \cdot (k(u^-)\nabla(u^- + \delta u) + k'(u^-)\delta u \nabla u^-), v) \\
&= -((k(u^-)\nabla(u^- + \delta u) + k'(u^-)\delta u \nabla u^-), \nabla v) \\
&+ \int_{\partial\Omega} (k(u^-)\nabla(u^- + \delta u) + k'(u^-)\delta u \nabla u^-) v \cdot \mathbf{n} dx \\
&= -((k(u^-)\nabla(u^- + \delta u) + k'(u^-)\delta u \nabla u^-), \nabla v) \\
&- \int_{\partial\Omega} (h(u^-)(u^- + \delta u - T_s(t)) + h'(u^-)\delta u(u^- - T_s(t)))v dx
\end{aligned}
$$

We use this in the variational form:

$$
\begin{aligned}
&\rho([c(u^-) + c'(u^-)u^- - c'(u^-)u_1]\delta u, v) \\
&+ \Delta t([(k(u^-)\nabla \delta u + k'(u^-)\nabla u^- \delta u)], \nabla v) \\
&- dt \int_{\partial\Omega} (h(u^- + h'(u^-)(u^- - T_s(t))))\delta u v dx \\
&= -\rho(c(u^-)(u^- + u_1), v) - \Delta t(k(u^-)\nabla u^-, \nabla v) \\
&+ dt \int_{\partial\Omega} (u^- - T_s(t)) v dx
\end{aligned} \tag{22}
$$