0.1 Exercise 14

0.1.1 Mathematical equation

The model equation is the nonlinear time dependent diffusion equation

$$\rho C(T)T_t = \nabla \cdot (k(T)\nabla T) \tag{1}$$

with Robin boundary condition

$$-k(T)\frac{\partial T}{\partial n} = h(T)(T - T_s(T)) \tag{2}$$

0.2 Varitional formulation with backward scheme

The discretisation of the time derivative with backward Euler gives

$$T_t \approx \frac{T^n - T^{n-1}}{\Delta t} \tag{3}$$

given a solution T^n at time level n (1) becomes

$$\rho C(T^n)\frac{T^n - T^{n-1}}{\Delta t} = \nabla \cdot (k(T^n)\nabla T^n) \tag{4}$$

Let $v \in V$ be a test function. The variational formulation of (1, 2) is: find $T^n \in V$ such that

$$a(u, v) = L(v)$$

with

$$a(u, v) = \int_\Omega \left(T^n v + \frac{\Delta t}{\rho C(T^n)} K(T^n)\nabla T^n \cdot \nabla v - \frac{\Delta t}{\rho C(T^n)} \frac{\partial T^n}{\partial n} v \right) \mathrm{d}x \tag{5}$$

$$L(v) = \int_\Omega T^{n-1} v \mathrm{d}x \tag{6}$$

with

$$\frac{\partial T}{\partial n} = -\frac{1}{k(T^n)}(h(T^n)(T^n - T_s(T^n))) \tag{7}$$

0.3 Picard iteration for the variational form

Given T^{n-1} at time level $n-1$ we want to find T^n

$$\int_\Omega \left(T^n v + \frac{\Delta t}{\rho C(T^n)} K(T^{n-1})\nabla T^n \cdot \nabla v - \frac{\Delta t}{\rho C(T^n)} \frac{\partial T^n}{\partial n} v \right) \mathrm{d}x = \int_\Omega T^{n-1} v \mathrm{d}x \tag{8}$$

0.4 Newton method

let

$$F(u^n; v) = \int_\Omega \left(T^n v + \frac{\Delta t}{\rho C(T^n)} K(T^n)\nabla T^n \cdot \nabla v - \frac{\Delta t}{\rho C(T^n)} \frac{\partial T^n}{\partial n} v \right) \mathrm{d}x - \int_\Omega T^{n-1} v \mathrm{d}x \tag{9}$$

be the variational form for the newton method. Newton Newton for the system $F(C_1^n, ..., C_j^n) = 0, j1, ..., N$ can be formulated as

$$\sum_{j=1}^{N} \frac{\partial}{\partial C_j} F_i(C_1^n, ..., C_N^n)\delta C_j = -F_i(C_1^n, ..., C_N^n), \quad j = 1, ..., N \tag{10}$$

$$C_j^{n+1} = C_j^n + \omega \delta C_j, \quad j = 1, ..., N \tag{11}$$

were $\omega \in [0, 1]$ is a relaxation parameter and with

$$T^n = \sum_j C_j^n \phi_j \tag{12}$$

$$T^n = \phi_j \tag{13}$$

The Jacobian matrix is then given by

$$\frac{\partial F_i}{\partial C_j} = \int_\Omega \phi_i\phi_j \mathrm{d}x + \Delta t \int_\Omega \left(k'\left(\sum_j C_j^n \phi_j\right)\phi_j \nabla\left(\sum_j C_j^n\phi_j\right)\cdot\nabla\phi_i + k'\left(\sum_j C_j^n\phi_j\right)\nabla\phi_j\cdot\nabla\phi_i\right)\mathrm{d}x + db \tag{14}$$

where db is the derivative of the boundary term:

$$db = \frac{\partial}{\partial C_j}\left(\frac{1}{k(T^n)}(h(T^n)(T^n - T_s))\right) \tag{15}$$

with again

$$T^n = \sum_j C_j^n \phi_j \tag{16}$$

0.5 Newton method at the PDE level

applying the backware Euler to the time derivative we get

$$T^{n+1} - T^n = \frac{\Delta t}{\rho C(T)} \nabla\cdot(k(T^{n+1})\nabla T^{n+1}) \tag{17}$$

Given an approximation to the solution field T^n we seek a perturbation δT such that

$$T^{n+1} = T^n + \delta T \tag{18}$$

fulfills the nonlinear equation (17). Assuming that δT is sufficiently small we can linearise the nonlinear term as

$$k(T^{n+1}) \approx k(T^n) + k(T^n)\delta T \tag{19}$$

and dropping other nonlinear terms in δ, we get

$$T^{n+1} - T^n = \frac{\Delta t}{\rho C(T^n)}\nabla\cdot(k(T^n\nabla T^n)) + \frac{\Delta t}{\rho C(T^n)}\nabla\cdot(k(T^n)\nabla\delta T) + \frac{\Delta t}{\rho C(T^n)}\nabla\cdot(k'(T^n\delta T)\nabla T^n) \tag{20}$$

collecting the terms with the unknown δ on the right-hand side we get

$$\frac{\Delta t}{\rho C(T^n)}\nabla\cdot(k(T^n)\nabla\delta T) + \frac{\Delta t}{\rho C(T^n)}\nabla\cdot(k'(T^n\delta T)\nabla T^n) = -T^{n+1} + T^n - \frac{\Delta t}{\rho C(T^n)}\nabla\cdot(k(T^n\nabla T^n)) \tag{21}$$

and the variational formulation of the resulting equation is then find $\delta T \in V$ such that

$$\frac{\Delta t}{\rho C(T^n)}\int_\Omega \cdot(k(T^n)\nabla\delta T)\cdot\nabla v \mathrm{d}x + \frac{\Delta t}{\rho C(T^n)}\int_\Omega \cdot(k'(T^n\delta T)\nabla T^n)\cdot\nabla v\mathrm{d}x = \int_\Omega(-T^{n+1}+T^n)v\mathrm{d}x - \frac{\Delta t}{\rho C(T^n)}\int_\Omega (k(T^n\nabla T^n))\cdot\nabla v\mathrm{d}x \tag{22}$$

$$a(\delta T, v) = L(v) \tag{23}$$

$$a(\delta T, v) = \frac{\Delta t}{\rho C(T^n)}\int_\Omega \cdot(k(T^n)\nabla\delta T)\cdot\nabla v\mathrm{d}x + \frac{\Delta t}{\rho C(T^n)}\int_\Omega\cdot(k'(T^n\delta T)\nabla T^n)\cdot\nabla v\mathrm{d}x \tag{24}$$

$$L(v) = \int_\Omega (-T^{n+1} + T^n)v\mathrm{d}x - \frac{\Delta t}{\rho C(T^n)}\int_\Omega (k(T^n\nabla T^n))\cdot\nabla v\mathrm{d}x \tag{25}$$