Exercise 7
Discretize a 1D problem with a nonlinear coefficient

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We consider the problem

$$((1+u^2)u')' = 1, \qquad x \in (0,1), \qquad u(0) = u(1) = 0.$$

and will analyse different ways of discretizating it in a function space V with basis functions $\{\psi_i\}$.

a) In the first part, we are going to apply a centered finite difference method on a uniform mesh with points $x_i = i\Delta x$, $i = 0, \ldots, N_n$. The problem is equivalent to

$$\frac{\delta}{\delta x}(1+u^2)\,\frac{\delta}{\delta x}u = 1$$

and in operator notation expressed as:

$$[D_x(1+u^2)D_x u = 1]_i$$

Applying the D_x operator step by step on the mesh, we get:

$$[\,\frac{\big((1+u^2_{i+1/2})D_x u_{i+1/2}\big) - \big((1+u^2_{i-1/2})D_x u_{i-1/2}\big)}{\Delta x} = 1]_i$$

$$[\,\frac{(1+u^2_{i+1/2})\,\frac{u_{i+1}-u_i}{\Delta x} - (1+u^2_{i-1/2})\,\frac{u_i-u_{i-1}}{\Delta x}}{\Delta x} = 1]_i$$

$$[\,\frac{1}{\Delta x^2}\big((1+u^2_{i+1/2})(u_{i+1}-u_i) - (1+u^2_{i-1/2})(u_i-u_{i-1})\big) = 1]_i$$

$\alpha(u) = (1+u^2)$ is a function of u and therefore only known at the mesh points. We need to express $\alpha_{i+1/2}$ and $\alpha_{i-1/2}$ in terms of u_i and u_{i+1}: we can apply the geometric mean so that, in general,

$$\alpha_{i+1/2} \approx \alpha(\,\frac{1}{2}(u_i + u_{i+1})) := [\alpha(\overline{u}^x)]^{i+1/2}$$

and in our case,

$$1 + u^2_{i+1/2} \approx 1 + (\,\frac{1}{2}(u_i+u_{i+1}))^2 = 1 + \,\frac{(u_i+u_{i+1})^2}{4} := [1 + (\overline{u}^x)^2]^{i+1/2}$$

This gives the difference approximation

$$\big[\frac{1}{\Delta x^2}[\big(1+\frac{(u_i+u_{i+1})^2}{4}\big)(u_{i+1}-u_i)-\big(1+\frac{(u_{i-1}+u_i)^2}{4}\big)(u_i-u_{i-1})]=1\big]_i$$

corresponding to, in operator notation:

$$[D_x(1+(\overline{u}^x)^2)D_x u=1]_i$$

b) Next we will discretize the problem by a finite element method with P1 elements, and choosing the Trapezoidal rule to compute the integrals, we will discover an interesting relationship with part a.

The variational formulation. This is found applying Galierkin's method: let $v\in \text{Span}\{\psi_0,\dots,\psi_N\}$ for some basis function ψ_i, or more simply let v be equal to one arbitrary ψ_i. Then multiply both parts of our equation by v and integrate:

$$\int_0^1((1+u^2)u')'v\,\mathrm{d}x=\int_0^1 1v\,\mathrm{d}x$$

integration by parts gives

$$[(1+u^2)u'v]_0^1-\int_0^1(1+u^2)u'v'\,\mathrm{d}x=\int_0^1 v\,\mathrm{d}x$$

We must require that all basis functions agree with the Dirichlet boundary conditions (since these are "unnatural" boundary conditions), and hence also v: $v(0)=v(1)=0$. Hence the first term disappears and we get

$$\underbrace{-\int_0^1(1+u^2)u'v'\,\mathrm{d}x}_{a(u,v)}=\underbrace{\int_0^1 v\,\mathrm{d}x}_{L(v)}$$

as variational formulation.

Derivation of the system. Next we derive the algebraic equations arising from this formulation: set $v=\psi_i$ and write the approximate solution as $u=B+\sum_k c_k\psi_k$, where $B(x)$ can be viewed as a boundary function implementing the Dirichlet boundary conditions $u(0)=u(1)=0$:

$$-\int_0^1(1+(B+\sum_k c_k\psi_k)^2)(B+\sum_k c_k\psi_k)'\psi_i'\,\mathrm{d}x=\int_0^1\psi_i\,\mathrm{d}x$$
$$-\sum_k\big(\int_0^1(1+(B+\sum_k c_k\psi_k)^2)\psi_k'\psi_i'\,\mathrm{d}x\big)c_k=\int_0^1\psi_i\,\mathrm{d}x$$

We now have a system of equations nonlinear in c, and this gives many difficulties in the integration. Hence, in order to derive the structure of the nonlinear algebraic equations, we must apply either numerical integration, or the group finite element method. The first one is used in this exercise.

The basis functions. First we introduce finite element basis functions: let $\psi_i = \phi_{\nu(i)}$, and expand u as

$$\sum_{j \in I_s} c_j \phi_{\nu(i)} + \sum_{j \in I_b} b_j \phi_j$$

where I_s is the index set of interior points and I_b is the index set of boundary points where Dirichlet conditions are applied (in this 1D case only $i = 0$ and $i = N_x$).

Numerical integration, right-hand side. When we apply a numerical integration method that samples the integrand at the node points only, as the Trapezoidal rule does, we get very simplified expressions because the sums will collapse:

$$\phi_j(x_i) = \delta_{ij} = 0 \text{ if } j \neq i$$

then the right-hand side $\langle f, v \rangle$ simplifies to

$$\int_0^1 1\psi_i \,\mathrm{d}x \approx h \sum_{l=0}^{N_x} 1 \underbrace{\phi_i(x_l)}_{\delta_{il}} - h/21 \underbrace{\phi_i(0)}_{=0} - h/21 \underbrace{\phi_i(1)}_{=0} = h$$

Numerical integration, left-hand side. while for the left-hand side $\langle (1 + u^2)u', v' \rangle$, we must consider another way because we are working with discontinuous derivatives.
It is then convenient to map the element domain $[x_i, x_{i+1}]$ to a standardized reference domain $[-1, 1]$. Let $X \in [-1, 1]$ be the coordinate in the reference element. Integrating on the reference element is just a matter of changing the integration variable from x to X, and introduce a stretch factor dx/dX (which is equal to $h/2$ in 1D).
The basis functions are simply the Lagrange polynomials defined through the local nodes in the reference element. For P1 elements we have that

$$\begin{aligned} &\phi_0(X) = 1/2(1 - X) \qquad \phi_1(X) = 1/2(1 + X) \\ &\phi_0'(X) = -1/2 \qquad \phi' = 1/2 \end{aligned}$$

The integral in the variational form corresponding to $((1 + u^2)u')'$ is

$$\int_0^1 (1 + u^2)u'v' \,\mathrm{d}x = \int_0^1 (1 + (B + \sum_k c_k \psi_k)^2)\psi_i' \psi_j' \,\mathrm{d}x$$

Introduce the mapping functions $i = q(e, r)$, $j = q(e, s)$, $k = q(e, g)$ from global elements i, j, k to local elements r, s, g. Since we use P1 elements, $r, s, g = 0, 1$.

Then at the cell level this will correspond to

$$\int_{-1}^{1}(1+(B+\sum_{g}\tilde{c}_g\tilde{\psi}_g)^2)\tilde{\psi}'_r\tilde{\psi}'_s\frac{h}{2}\,\mathrm{d}X$$
$$=\int_{-1}^{1}(1+(B+\sum_{0}^{1}\tilde{c}_g\tilde{\psi}_g)^2)\frac{2}{h}\frac{\mathrm{d}\tilde{\psi}_r}{\mathrm{d}X}\frac{2}{h}\frac{\tilde{\psi}_s}{\mathrm{d}X}\frac{h}{2}\,\mathrm{d}X$$
$$=\frac{1}{2h}(-1)^r(-1)^s\int_{-1}^{1}(1+(B+\sum_{0}^{1}\tilde{c}_g\tilde{\psi}_g(X))^2)\,\mathrm{d}X$$
$$\approx\frac{1}{2h}(-1)^r(-1)^s(1+(B+\sum_{0}^{1}\tilde{c}_g\psi(\tilde{-1})_g)^2)+(1+(B+\sum_{0}^{1}\tilde{c}_g\psi(\tilde{1})_g)^2)$$
$$=\frac{1}{2h}(-1)^r(-1)^s((1+(\tilde{u_0})^2)+(1+(\tilde{u_1})^2))$$
$$=\frac{1}{2}((1+(\tilde{u_0})^2)+(1+(\tilde{u_1})^2))\frac{1}{h}\left[\begin{smallmatrix}1 & -1\\ -1 & 1\end{smallmatrix}\right]$$

The global matrix entry will become

$$\Big(-\frac{(1+u_{i-1}^2)+(1+u_i^2)}{2h}\ ,\ -\frac{(1+u_{i-1}^2)+2(1+u_i^2)+(1+u_{i+1}^2)}{2h}\ ,$$
$$-\frac{(1+u_{i+1}^2)+(1+u_i^2)}{2h}\Big)$$

Multiplying by the vector of unknowns u_i we get

$$\frac{1}{h}\Big(\frac{1}{2}((1+u_{i+1}^2)+(1+u_i^2))(u_{i+1}-u_i)-\frac{1}{2}((1+u_{i-1}^2)+(1+u_i^2))(u_i-u_{i-1})\Big)$$
$$\Updownarrow$$
$$[hD_x(1+(\overline{u}^x)^2)D_xu]_i$$

Hence the Trapezoidal rule, applied with P1 elements to the nonlinear Laplace term, corresponds to a standard finite discretization with the use of arithmetic mean, as we did in a).

Conclusion. In general, P1 finite elements result in difference equations where the nonlinear coefficient $((1+u^2)u')'$ becomes $h[D_x(1+(\overline{u}^x)^2)D_xu]_i$ if the group finite element method or Trapezoidal integration is applied, and the right-hand side $f(u)=1$ becomes $hf(u_i)=h$ with Trapezoidal integration or group finite element method (since the expression $h[f(u)-\frac{h}{6}D_xD_xf(u)]_i$ boils down to $h[f(u)]_i$ when f is given by a constant).
All together, the finite element discretization will give

$$[hD_x(1+(\overline{u}^x)^2)D_xu=h]_i$$

or

$$[D_x(1+(\overline{u}^x)^2)D_xu=1]_i$$

which is exactly the same expression as in a): this happends exactly when the right-hand side f is constant.