Cable problem with 2 P1 elements

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0.1 Setting up the problem

Consider the problem

$$u'' = 1 \tag{1}$$

with Dirichlet boundary condition

$$u(0) = u(1) = 0 \tag{2}$$

let V be given by

$$V = span\{\varphi_0 \cdots \varphi_N\}$$

where φ_i are the Lagrange elements of order 1.

0.2 Variational formulation

The variational formulation of (1, 2) is: Find $u \in V$ such that :

$$-\int_0^1 u'v'\mathrm{d}x = \int_0^1 v\mathrm{d}x \quad \forall v \in V. \tag{3}$$

with

$$a(u, v) = -\int_0^1 u'v'\mathrm{d}x$$

$$L(v) = \int_0^1 v\mathrm{d}x$$

0.3 Finite element solution

$u, v \in V$ mean that u and v can be written as the linear combination of φ_i:

$$u = \sum_{i=0}^{N} c_i\varphi_i \tag{4}$$

$$v = \varphi_j \tag{5}$$

Inserting (4, 5) in 3 gives:

$$-\int_0^1 \varphi_i'\varphi_i'\mathrm{d}x \sum_{i=0}^{N} c_i = \int_0^1 \varphi_j\mathrm{d}x \tag{6}$$

where the stiffness matrix K and the load matrix b are respectively given by:

$$K_{ij} = -\int_0^1 \varphi_i'\varphi_i'\mathrm{d}x$$

$$b_j = \int_0^1 \varphi_j\mathrm{d}x$$

The basis function ϕ_i and their derivatives are given by:

$$\varphi_i(x) = \begin{cases} 0 & \text{if } x < x_{i-1} \\ \frac{x-x_{i-1}}{h} & \text{if } x_{i-1} \leq x < x_i \\ 1 - \frac{x-x_i}{h} & \text{if } x_i \leq x < x_{i+1} \\ 0 & \text{if } x \geq x_{i+1} \end{cases}$$

$$\varphi_i'(x) = \begin{cases} 0 & \text{if } x < x_{i-1} \\ \frac{1}{h} & \text{if } x_{i-1} \leq x < x_i \\ -\frac{1}{h} & \text{if } x_i \leq x < x_{i+1} \\ 0 & \text{if } x \geq x_{i+1} \end{cases}$$

Ad we have

$$K_{ii} = \frac{2}{h}$$

$$K_{i,i-1} = A_{i,i+1} = -\frac{1}{h}$$

$$b_j = h$$

with N = 1 we have

$$K = \frac{1}{h}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \tag{7}$$

$$K^{-1} = \frac{h}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{8}$$

$$b = h\begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{9}$$

And solving the system

$$Kc = b \to c = K^{-1}b$$

$$c_0 = c_1 = \frac{h^2}{3}$$

$$u(x) = \frac{h^2}{3}\varphi_0 + \frac{h^2}{3}\varphi_1$$

0.4 Solution with fenics for N = 60

```
from dolfin import *

#define mesh
mesh = UnitIntervalMesh(60)
V = FunctionSpace(mesh,'Lagrange',1)

# set Dirichlet boundary condition
u0 = Expression('0.0')
def u0_boundary(x,on_boundary):
    return on_boundary

bc = DirichletBC(V,u0,u0_boundary)

# varional formulation
u = TrialFunction(V)
v = TestFunction(V)
f = Constant(1)
a = -inner(grad(u),grad(v))*dx
L = f*v*dx

# solution of the variatiobal problem
u = Function(V)
solve(a==L,u,bc)

#plot solution
plot(u)

# view solution
interactive()
```

Figure 1: Deflection of the cable with finite element for N = 60