# Exercise 8
# Linearize a 1D problem with a nonlinear coefficient

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As in exercise 7, we consider the problem

$$((1+u^2)u')' = 1, \qquad x \in (0,1), \qquad u(0) = u(1) = 0.$$

In 7 part b, when applying the finite element method, we found explicit expressions for the system of nonlinear algebraic equations, so now it is possible to define a Picard or a Newton method in order to solve these. These methods will be applied directly to the variational form in part a and b, avoiding discretization in space. In part c and d, we will work on the system arising from the space discretization.

The variational form of the problem reads: find $u \in V$ such that

$$\underbrace{-\int_0^1 (1+u^2)u'v' \,\mathrm{d}x}_{a(u,v)} = \underbrace{\int_0^1 v \,\mathrm{d}x}_{L(v)}$$

**a) Construct a Picard iteration without discretizing in space** We now don't aim at computing integrals symbolically, but at defining a generic iteration in the Picard method. The idea is to use a previously computed $u$ value in the functions creating nonlinearity- in this case, the nonlinear coefficient $(1+u^2)$. Let $u^-$ be the available approximation to $u$ from the previous iteration. The linearized variational form for Picard iteration is then

$$-\int_0^1 (1+((u^-)^2)u'v' \,\mathrm{d}x = \int_0^1 v \,\mathrm{d}x$$

**b) Apply Newton's method without discretizing in space** In order to apply Newton, we must identify the nonlinear algebraic equations $F_i$. Let the unknowns be $c_0, \ldots, c_{Nx}$, and $v = \psi_i$. Then

$$F_i = \int_0^1 [(1+u^2)u'\psi_i' - \psi_i] \,\mathrm{d}x = 0 \qquad i \in I_s$$

In order to derive the Jacobian $J_{ij} = \frac{\delta F_i}{\delta c_j}$, we can use that

$$\frac{\delta}{\delta c_j}\sum_k c_k\psi_k = \psi_j \qquad \frac{\delta}{\delta c_j}\sum_k c_k\psi_k' = \psi_j'$$

Let $u \in V$ be given by $\sum_k c_k \phi_k$.
Then

$$
\begin{aligned}
J_{ij} &= \frac{\delta}{\delta c_j} \int_0^1 [(1+u^2)u'\psi_i' - \psi_i] \,\mathrm{d}x \\
&= \int_0^1 \frac{\delta}{\delta c_j}(1+u^2)u'\psi_i' \,\mathrm{d}x \\
&= \int_0^1 [\frac{\delta(1+u^2)}{\delta u} \underbrace{\frac{\delta u}{\delta c_j}}_{\psi_i} u' + (1+u^2) \underbrace{\frac{\delta u'}{\delta c_j}}_{\psi_i'}]\psi_i' \,\mathrm{d}x \\
&= \int_0^1 [2uu'\psi_j + (1+u^2)\psi_j']\psi_i' \,\mathrm{d}x
\end{aligned}
$$

$F_i$ and $J_{ij}$ must be evaluated at a previously computed $u$ value, denoted by $u^-$.

$$
\begin{aligned}
\tilde{F}_i &= \int_0^1 [(1+(u^-)^2)(u^-)'\psi_i' - \psi_i] \,\mathrm{d}x = 0 \\
\tilde{J}_{ij} &= \int_0^1 [2(u^-)(u^-)'\psi_j + (1+(u^-)^2)\psi_j']\psi_i' \,\mathrm{d}x
\end{aligned}
$$

**Discretize by a centered finite difference scheme**

$$
[D_x(1+(\overline{u}^x)^2)D_x u = 1]_i
$$

$$
\Updownarrow
$$

$$
\frac{1}{2\Delta x^2}[((1+u_i^2)+(1+u_{i+1}^2))(u_{i+1}-u_i) - ((1+u_i^2)+(1+u_{i-1}^2))(u_i - u_{i-1})] = 1
$$

$$
\Updownarrow
$$

$$
\underbrace{\frac{((1+u_i^2)+(1+u_{i+1}^2))}{2\Delta x^2}u_{i+1}}_{A_{i,i+1}} - \underbrace{\frac{((1+u_{i+1}^2)+2(1+u_i^2)+(1+u_{i-1}^2))}{2\Delta x^2}u_i}_{A_{i,i}} +
$$

$$
\underbrace{\frac{((1+u_i^2)+(1+u_{i-1}^2))}{2\Delta x^2}u_{i-1}}_{A_{i,i}} = 1
$$

$$
i \in [1, N_x - 1]
$$

Hence we can derive a system $A(u)u = b(u)$ for the problem. The system is nonlinear and will therefore be solved by Picard or Newton's methods.

Let $F(u) = A(u)u - b(u)$, $F = (F_0, F_1, \dots, F_{Nx})$ and $u = (u_0, u_1, \dots, u_{Nx})$.

**c) Picard** Construct a Picard method for the resulting system of nonlinear algebraic equations: The approximation

$$
F(u) \approx \hat{F}(u) = A(u^- u^- - b(u^-)
$$

makes the system linear, and thus we can use standard Gaussian elimination to solve the system.

**d) Newton** Define a system of nonlinear algebraic equations, calculate the Jacobian and set up Newton's method for solving the system: The nonlinear equation nr. $i$ has the form

$$
F_i = A_{i,i-1}u_{i-1} + A_{i,i}u_i + A_{i,i+1}u_{i+1} - b_i
$$

Then

$$J_{ij} = \frac{\delta A_{i,i-1}}{\delta u_j} u_{i-1} + \frac{\delta A_{i,i}}{\delta u_j} u_i + \frac{\delta A_{i,i+1}}{\delta u_j} u_{i+1} - \frac{\delta b_i}{\delta u_j}$$

$j = i-1, i, i+1$