Exercises for 20th of November

Exercise 1

1. $b^2 = 1$ *linear*
2. $a + b = 1, a - 2b = 0$ *linear*
3. $mu'' + \beta|u'|u' + cu = F(t)$ *nonlinear*
4. $u_t = \alpha u_{xx}$ *linear*
5. $u_{tt} = c^2\nabla^2 u$ *linear*
6. $u_t = \nabla \cdot (\alpha(u)\nabla u) + f(x, y)$ *nonlinear*
7. $u_t + f(u)_x = 0$ *linear/nonlinear depending on* f
8. $\mathbf{u}_t + \mathbf{u} \cdot \nabla\mathbf{u} = -\nabla p + r\nabla^2\mathbf{u}, \nabla \cdot \mathbf{u} = 0$ *nonlinear*
9. $u' = f(u, t)$ *linear/nonlinear depending on* f
10. $\nabla^2 u = \lambda e^u$ *nonlinear*

Exercise 4

We have a nonlinear vibration ploblem:

$$mu'' + bu'|u'| + s(u) = F(t) \tag{1}$$

where $m > 0$ is a constant, $b \geq 0$ is a constant, $s(u)$ a possibly nonlinear function of u, and $F(t)$ is a prescribed function. Such models arise from Newton's second law of motion in mechanical vibration problems where $s(u)$ is a spring or restoring force, mu'' is mass times acceleration, and $bu'|u'|$ models water or air drag.

a) We rewrite equation 1 as a system of two first-order ODEs. We set $v = u'$ which implies $v' = u''$, and rewrite equation 1 to get:

$$mv' + bv|v| + s(u) = F(t) \tag{2}$$

Rearranging equation 2 we get the 2x2 ODE system:

$$\begin{aligned} u' &= v, \\ v' &= \frac{F}{m} - \frac{b}{m}v|v| - \frac{s(u)}{m}, \end{aligned} \tag{3}$$

We discretize the system in time with a Crank-Nicolson scheme and get:

$$
\begin{aligned}
\frac{u^{n+1}-u^n}{\Delta t} &= v^{n+\frac{1}{2}}, \\
\frac{v^{n+1}-v^n}{\Delta t} &= \frac{F^{n+\frac{1}{2}}}{m} - \frac{b}{m}[v|v|]^{n+\frac{1}{2}} - \frac{s(u^{n+\frac{1}{2}})}{m}
\end{aligned}
\tag{4}
$$

We now have a nonlinear term $(v|v|)^{n+\frac{1}{2}}$ by using an geometric mean for $v^{n+\frac{1}{2}} \approx \sqrt{v^n v^{n+1}}$ the term becomes linearized, and we get $(v|v|)^{n+\frac{1}{2}} \approx v^{n+1}|v^n|$. Applying arithmetic mean on the other terms evaluated at $n+\frac{1}{2}$ gives the system:

$$
\begin{aligned}
\frac{u^{n+1}-u^n}{\Delta t} &= \sqrt{v^n v^{n+1}}, \\
\frac{v^{n+1}-v^n}{\Delta t} &= \frac{\frac{1}{2}(F^n + F^{n+1})}{m} - \frac{b}{m}v^{n+1}|v^n| - \frac{s(\frac{1}{2}(u^n + u^{n+1}))}{m}
\end{aligned}
\tag{5}
$$

Introduce v for v^{n+1}, $v^{(1)}$ for v^n, u for u^{n+1}, $u^{(1)}$ for u^n. After rewriting the system we get:

$$
\begin{aligned}
u &= u^{(1)} + \Delta t\sqrt{vv^{(1)}} \\
v &= \frac{v^{(1)}m + \Delta t(\frac{1}{2}(F + F^{(1)}) - s(\frac{1}{2}(u + u^{(1)})))}{m + b|v^{(1)}|\Delta t}
\end{aligned}
\tag{6}
$$

b) The only possibly nonlinear term in the system of equations (6) is in the $s(u)$ function. By assuming that we have approximations u^- for u and v^- for v we can formulate a Picard iteration to solve the system:

$$
\begin{aligned}
u &= u^{(1)} + \Delta t\sqrt{v^- v^{(1)}} \\
v &= \frac{v^{(1)}m + \Delta t(\frac{1}{2}(F + F^{(1)}) - s(\frac{1}{2}(u^- + u^{(1)})))}{m + b|v^{(1)}|\Delta t}
\end{aligned}
\tag{7}
$$

For the first step of the Picard iteration the approximations of u and v is put equal to the numerical solution of u and v at the previous timestep (e.g. $u^- = u^{(1)}$ and $v^- = v^{(1)}$). Notice also that by setting $\Delta t\sqrt{vv^{(1)}} \approx \Delta t\sqrt{v^- v^{(1)}}$ the system is decoupled.

c) To apply the Newton's method the system of equations 6 can be written as a nonlinear vector equation $F(w) = 0$, with $F = (F_u, F_v)$ and $w = (u, v)$. F_u and F_v is then given as:

$$
F_u = u - u^{(1)} - \Delta t\sqrt{vv^{(1)}}, \tag{8}
$$

$$
F_v = v - \frac{v^{(1)}m + \Delta t(\frac{1}{2}(F - F^{(1)}) - s(\frac{1}{2}(u + u^{(1)})))}{m + b|v^{(1)}|\Delta t}, \tag{9}
$$

We approximate F around a known value w^- by a linear function $\hat{F}$:

$$
\hat{F}(w) = F(w^-) + J(w^-)\cdot(w - w^-) = 0 \quad \Rightarrow \quad J(w^-)\cdot(w - w^-) = -F(w^-),
$$

where J is the Jacobian of F given as:

$$J = \begin{pmatrix} \frac{\partial F_u}{\partial u} & \frac{\partial F_u}{\partial v} \\ \frac{\partial F_v}{\partial u} & \frac{\partial F_v}{\partial u} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{\Delta t}{2\sqrt{vv^{(1)}}} \\ -\frac{s'(\frac{1}{2}(u+u^{(1)}))}{2(m+b|v^{(1)}|\Delta t)} & 1 \end{pmatrix} \tag{10}$$

The Newton system $J(w^-)\delta w = -F(w^-)$ to be solved in each iteration is then given as

$$\begin{pmatrix} 1 & -\frac{\Delta t}{2\sqrt{vv^{(1)}}} \\ -\frac{s'(\frac{1}{2}(u+u^{(1)}))}{2(m+b|v^{(1)}|\Delta t)} & 1 \end{pmatrix} \begin{pmatrix} u - u^- \\ v - v^- \end{pmatrix} = \tag{11}$$

$$\begin{pmatrix} u^- - u^{(1)} - \Delta t\sqrt{v^- v^{(1)}} \\ v^- - \frac{v^{(1)}m+\Delta t(\frac{1}{2}(F-F^{(1)})-s(\frac{1}{2}(u^-+u^{(1)})))}{m+b|v^{(1)}|\Delta t} \end{pmatrix} \tag{12}$$

For each iteration the linear system above is solved and the u^- is updated to contain the calculated u. This is done until convergence is achieved.