Calculus 2

First order linear differential equation

(see sec 16.2 textbook)

Motivational example: population growth model

$P = P(t)$: population of a certain city

year	Population(thousand)
2000	100
2010	150
2020	250
…	…

$\frac{dP}{dt}$ is proportional to $P(t)$ => DE $\frac{dP}{dt} = kP$

$P(t) = Ce^{kt}$ satisfies the equation.

$P(0) = C$ (initial population)

To determine the relative growth rate, we demand

$$P(10) = 150{,}000 = 100{,}000e^{10k}$$

=> Solve for k

Compare P(20) (=predicted value) with real data. Is our model good?

1^{st} order differential equation : $F(t, y, y') = 0$

Case F is linear in y and y' => $F(t, y, y') = a(t) + b(t)y + c(t)y' = 0$

(Standard form) $\frac{dy}{dt} + P(t)y = Q(t)$ (y' has coefficient 1)

Question) How can we find solutions of DE?

Example) Getting idea from observation

$$\frac{dy}{dt} + \frac{1}{t}y = t, \quad (t > 0)$$

Multiply t both sides => $t\frac{dy}{dt} + y = t^2$

LHS => $\frac{d}{dt}(ty) = (t)'y + t(y)' = y + t\frac{dy}{dt}$

Original DE => $\frac{d}{dt}(ty) = t^2$

We can integrate to get y :

$$\int \frac{d}{dt}(ty)dt = \int t^2 dt$$

$$ty = \frac{1}{3}t^3 + C$$

$$y = \frac{1}{3}t^2 + \frac{C}{t}, \text{ (t >0)} \qquad \text{(general solution)}$$

*Generalize our observation

Multiply a proper function $\mu(t)$ to the both sides of $\frac{dy}{dt} + P(t)y = Q(t)$ such that LHS = derivative of single function :

$$\mu(t)(\frac{dy}{dt} + P(t)y) = \mu(t)Q(t)$$

Question) What is a condition that the multiplier $\mu(t)$ satisfy ?

$$\mu(t)(\frac{dy}{dt} + P(t)y) = \frac{d}{dt}(\mu(t)y)$$

=>

$$\mu y' + \mu' y = \mu y' + \mu P(t)y$$

$$\mu' y = \mu P(t)y$$

$$\mu' = P(t)\mu$$

=> multiplier is a solution of new DE

(Separable DE - first order)

$$\frac{d\mu}{dt} = P(t)\mu$$

$$\int \frac{1}{\mu} d\mu = \int P(t)dt$$

$$\ln|\mu| = \int P(t)dt$$

$$\mu = \pm e^{\int P(t)dt}$$

This is called **integrating factor of DE.**

For example

$$\frac{dy}{dt} = 3y$$

$$\frac{1}{y}dy = 3dt \quad \Rightarrow \quad \int \frac{1}{y}dy = \int 3dt$$

$$\ln|y| = 3t + C$$

How can we justify ?

Let $y = \phi(t)$ be a solution for DE. Then

$$\frac{d\phi}{dt} = \phi'(t) = f(t)g(\phi(t))$$

$$\int \frac{1}{g(y)} dy = \int \frac{1}{g(\phi(t))} \phi'(t)dt = \int \frac{1}{g(\phi(t)} f(t)g(\phi(t))dt = \int f(t)dt$$

Example) Find general solutions of $t\frac{dy}{dt} = t^2 + 3y, \; (t > 0)$

STEP 1) standard form

$$\frac{dy}{dt} - \frac{3}{t}y = t$$

STEP 2) Integrating factor

$$e^{-\int \frac{3}{t}dt} = e^{-3\ln t} = e^{\ln t^{-3}} = t^{-3} \quad (t > 0)$$

STEP 3) Multiply IF to the equation

$$t^{-3}(y' - \frac{3}{t}y) = t^{-2}$$

$$\frac{d}{dt}(t^{-3}y) = t^{-2}$$

$$t^{-3}y = \int t^{-2}dt = -\frac{1}{t} + C$$

$$y = -t^2 + Ct^3, \ (t > 0)$$

Example. Solve the following Initial value problem

$t\frac{dy}{dt} = t^2 + 3y, \quad y(1) = 1$

Solution) y(1)=1 => can assume that t >0

General solution is $y = -t^2 + Ct^3$.

IC $y(1) = 1$ => $C = 2$

solution for IVP => $y = 2t^3 - t^2$

Example

Find the general solution of $\frac{dy}{dt} + \frac{1}{2}y = e^{t/3}$

SOL) integrating factor

=> $\exp(\int \frac{1}{2}dt) = \exp(\frac{1}{2}t) = e^{t/2}$

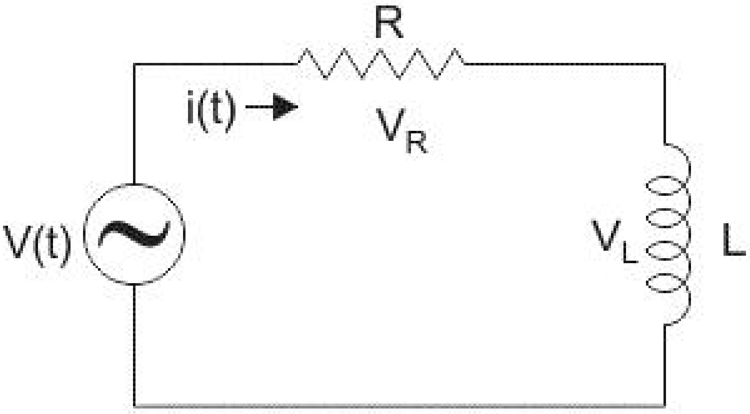
$$e^{t/2}(y' + 1/2y) = e^{t/2 + t/3}$$

$$\frac{d}{dt}(e^{t/2}y) = e^{5/6t}$$

$$e^{t/2}y = \frac{6}{5}e^{\frac{5}{6}t} + C$$

$$y = \frac{6}{5}e^{t/3} + Ce^{-t/2}$$

Application (see also section 16.2 textbook)

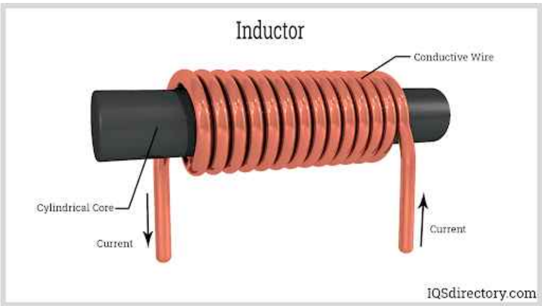


RL-circuit

Question : How does the current change after the switch is turned on?

R= resistance (for instance, a light bulb)

L= inductance (How densely coil is winded ?)



V=V(t) voltage of electric power (for instance battery)

I=I(t) = electric current

Kirchhoff's law

Total voltage drop through the electric devices along the circuit equals to voltage of electric source.

V=V_R + V_L

V_R =? and V_L=?

Ohm's law => V_R $=RI$

Voltage drop at the coil is V_L= $L\frac{dI}{dt}$이다.

$$L\frac{dI}{dt}+RI=V$$ (1st order linear DE)

Solve DE

standard form $\frac{dI}{dt}+\frac{R}{L}I=\frac{V}{L}$

Integrating factor $\mu=e^{\int\frac{R}{L}dt}=e^{\frac{R}{L}t}$

Multiply to the EQ

$$e^{\frac{R}{L}t}\left(\frac{dI}{dt}+\frac{R}{L}I\right)=\frac{1}{L}e^{\frac{R}{L}t}V$$

$$\frac{d}{dt}\left(e^{\frac{R}{L}t}I\right)=\frac{1}{L}e^{\frac{R}{L}t}V$$

Integrate both sides with respect to t variable

$$e^{\frac{R}{L}t}I=\frac{1}{L}\int e^{\frac{R}{L}t}V\,dt=\frac{V}{L}\frac{L}{R}e^{\frac{R}{L}t}+C$$

General solution is $I(t)=\frac{V}{R}+Ce^{-\frac{R}{L}t}$

IC $I(0)=I_0$ => determine C : $I_0=V/R+C$

=>Solution for IVP

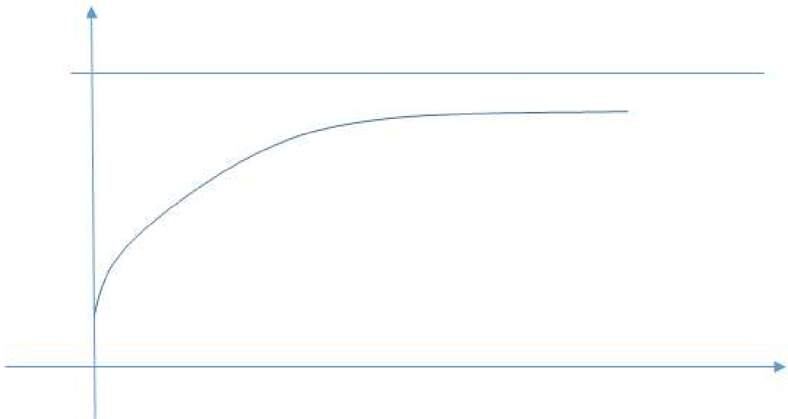
$$I(t)=\frac{V}{R}+\left(I_0-\frac{V}{R}\right)e^{-\frac{R}{L}t}$$

Q: Behavior of current depending on the initial current

(1) $I_0 < \frac{V}{R}$

$I(t) = \frac{V}{R} + (I_0 - \frac{V}{R})e^{-\frac{R}{L}t}$ is increasing function

- In the graph below the horizontal axis is time and vertical axis is current
- V/R is the horizontal asymptote as t -> ∞



(2) $I_0 > \frac{V}{R}$

$I(t) = \frac{V}{R} + (I_0 - \frac{V}{R})e^{-\frac{R}{L}t}$ is decreasing

- I(t) -> V/R as t -> ∞

-Current is rapidly decreasing once the power is turn on.