DANDONG TU

a) $X = \begin{pmatrix} \underset{\sim}{x}_1^T \\ \underset{\sim}{x}_2^T \\ \vdots \\ \underset{\sim}{x}_n^T \end{pmatrix}$ $\quad X\hat{\beta} = \begin{pmatrix} \underset{\sim}{x}_1^T\hat{\beta} \\ \vdots \\ \underset{\sim}{x}_n^T\hat{\beta} \end{pmatrix}$ $\quad Y - X\hat{\beta} = \begin{pmatrix} y_1 - \underset{\sim}{x}_1^T\hat{\beta} \\ \vdots \\ y_n - \underset{\sim}{x}_n^T\hat{\beta} \end{pmatrix}$ $\quad (Y - X\hat{\beta})^T =$
$(y_1 - \underset{\sim}{x}_1^T\hat{\beta},\ y_2 - x_2^T\hat{\beta} \cdots y_n - x_n^T\hat{\beta})$

Therefore, $(Y - X\hat{\beta})^T$ times $(Y - X\hat{\beta}) = (y_1 - \underset{\sim}{x}_1^T\hat{\beta} \cdots y_n\ x_n^T\hat{\beta}) \cdot \begin{pmatrix} y_1 - x_1^T\hat{\beta} \\ y_2 - \underset{\sim}{x}_2^T\hat{\beta} \\ \vdots \\ y_n - \underset{\sim}{x}_n^T\hat{\beta} \end{pmatrix}$

So that, the result for $(Y - X\hat{\beta})^T (Y - X\hat{\beta})$ is following:

$$\sum_{i=1}^{n} (y_i - x_i^T\hat{\beta})^2$$

Finally, we have $(Y - X\hat{\beta})^T(Y - X\hat{\beta}) = \sum_{i=1}^{n} (y_i - x_i^T\hat{\beta})^2$

b) $H^T = [X(X^TX)^{-1}X^T]^T = X[(X^TX)^{-1}]^T X^T = X[(X^TX)^T]^{-1}X^T = X(X^TX)^{-1}X^T = H$
$HH = [X(X^TX)^{-1}X^T][X(X^TX)^{-1}X^T]$
$= X(X^TX)^{-1}(X^TX)(X^TX)^{-1}X^T$
$= X(X^TX)^{-1}X^T$
$= H$
So that, H is symmetric and idempotent

c) $(I - H)^T = I^T - H^T = I - H$
$(I - H)(I - H) = I - 2H + HH = I - H$
so that, $I - H$ is symmetric and idempotent

d) $HX = X(X^TX)^{-1}X^TX = X(X^TX)^{-1}(X^TX) = X$ So that, $HX = X$

e) $(I-H)(Y-X\hat{\beta}) = Y - X\hat{\beta} - HY + HX\hat{\beta}$

$= Y - HY - X\hat{\beta} + X\hat{\beta}$

$= (I-H)Y$

f) $(Y-X\hat{\beta})^T(I-H)(Y-X\hat{\beta}) = (I-H)(Y-X\hat{\beta})^T(I-H)(Y-X\hat{\beta})$

$= (I-H)^T(Y-X\hat{\beta})^T(I-H)(I-H)(Y-X\hat{\beta})$

$= Y^T(I-H)Y$

(using part c and e)

g) $X\hat{\beta} = HY$

$$RSS(\hat{\beta}) = \sum_{i=1}^{n}(y_i - X_i^T\hat{\beta})^2$$

$= (Y-X\hat{\beta})^T(Y-X\hat{\beta})$

$= (Y-HY)^T(Y-HY)$

$= Y^T(I-H)^T(I-H)Y$

$= Y^T(I-H)Y$