Solutions for Sheet 4

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PATTERN MATCHING AND MACHINE LEARNING
FOR AUDIO SIGNAL PROCESSING

10. Mai 2022

Task 4.1

(a)

$X_0 = (\omega_N^{0 \cdot 0}, ..., \omega_N^{0 \cdot (N-1)}) \cdot x$

$= \sum_{k=0}^{N-1} \omega_N^0 \cdot x_k$

$= \sum_{k=0}^{N-1} x_k \quad \Rightarrow X_0 \in \mathbb{R}$

$X_M = (\omega_N^{0 \cdot M}, ..., \omega_N^{(N-1) \cdot M}) \cdot x$

$= \sum_{k=0}^{N-1} \omega_N^{kM} \cdot x_k$

$= \sum_{k=0}^{N-1} exp(\frac{-2\pi i M}{N})^k \cdot x_k$

$= \sum_{k=0}^{N-1} exp(\frac{-2\pi i M k}{2M}) \cdot x_k$

$= \sum_{k=0}^{N-1} exp(-\pi i k) \cdot x_k$

$= \sum_{k=0}^{N-1} (-1)^k \cdot x_k \quad \Rightarrow X_M \in \mathbb{R}$

$X_{N-k} = \sum_{l=0}^{N-1} \omega_N^{l(N-k)} \cdot x_l$

$= \sum_{l=0}^{N-1} exp(\frac{-2\pi i l(N-k)}{N}) \cdot x_l$

$= \sum_{l=0}^{N-1} exp(\frac{-2\pi i l N}{N} - (\frac{-2\pi i l k}{N})) \cdot x_l$

$$= \sum_{l=0}^{N-1} exp(\tfrac{2\pi ilk}{N} - 2\pi ilN) \cdot x_l$$

$$= \sum_{l=0}^{N-1} exp(\tfrac{2\pi ilk}{N}) \cdot exp(-2\pi ilN) \cdot x_l$$

$$= \sum_{l=0}^{N-1} exp(\tfrac{2\pi ilk}{N}) \cdot x_l$$

$$= \overline{\sum_{l=0}^{N-1} exp(\tfrac{-2\pi ilk}{N}) \cdot x_l}$$

$$= \overline{\sum_{l=0}^{N-1} \omega_N^{lk} \cdot x_l}$$

$$= \overline{X_k}$$

(b)

Task 4.2

(a)

For A we got:

$$A = \text{DFT}_N \cdot a = \text{DFT}_N \cdot \left(sin\left(\frac{2\pi 4k}{N} \right) \right)$$

And with Eulers Formula:

$$a = sin\left(\frac{2\pi 4k}{N} \right) = \frac{1}{2i} \cdot \left(e^{\frac{2\pi 4ki}{N}} - e^{\frac{-2\pi 4ki}{N}} \right)$$

$$= \frac{1}{2i} \cdot (u_4 - u_{N-4})$$

With $\langle u_k | u_l \rangle = N \cdot \delta_{k,l}$ we get:

$$|a(k)| = \frac{N}{2} \cdot w_k$$

with $w_4 = w_{N-4} = 1$ and else $w_k = 0$.
That means, for $A = \text{DFT}_N \cdot a$ the only nonzero entries of the matrix A are at $k = 4$ and $k = N - 4 = 508$. For this entries we get:

$$A(k) = \text{DFT}_N(k) \cdot \frac{N}{2}$$

(b)

We know, that $|A(k)| > 2\theta$ for all nonzero $A(k)$ and $R(k) \le \theta$. So $|A + R| > \theta$ for all $k \in \{0, ..., N-1\}$. So for B_θ we get:

$$B_\theta(k) = \begin{cases} A(k) + R(k), \text{for} k = 4 \wedge k = 508 \\ 0, \text{else} \end{cases}$$

If we now define us a new R_θ with:

$$R_\theta(k) = \left\{ \begin{array}{l} R(k), \text{for} k = 4 \wedge k = 508 \\ 0, \text{else} \end{array} \right.$$

we can write $B_\theta = A + R_\theta$
So for $\tilde{a}$ and $\tilde{a} - a$ we get:

$$\tilde{a} = \text{DFT}_n^{-1} B_\theta = \text{DFT}_n^{-1}(A + R_\theta) = \text{DFT}_n^{-1} A + \text{DFT}_n^{-1} R_\theta = a + \text{DFT}_n^{-1} R_\theta$$
$$\tilde{a} - a = a + \text{DFT}_n^{-1} R_\theta - a = \text{DFT}_n^{-1} R_\theta$$

Task 4.3