Solutions for Sheet 3

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Pattern Matching and Machine Learning
for Audio Signal Processing

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Task 3.1

(a) We have the following formulas for $cos(z)$ and u_k:

$$cos(z) = \frac{1}{2}e^{iz} + e^{-iz}$$
$$u_k = e^{\frac{2\pi ikn}{N}}$$

So for f, we get:

$$\begin{aligned} f(t) &= cos(4\pi t) + 4cos(20\pi t) + 8cos(2\pi 20t) = cos(2\pi 2t) + 4cos(2\pi 10t) + 8cos(2\pi 20t) \\ &= \frac{1}{2}\left(e^{2\pi 2ti} + e^{-2\pi 2ti}\right) + \frac{4}{2}\left(e^{2\pi 10ti} + e^{-2\pi 10ti}\right) + \frac{8}{2}\left(e^{2\pi 20ti} + e^{-2\pi 20ti}\right) \\ &= \frac{1}{2}\left(e^{2\pi 2ti} + e^{-2\pi 2ti}\right) + 2\left(e^{2\pi 10ti} + e^{-2\pi 10ti}\right) + 4\left(e^{2\pi 20ti} + e^{-2\pi 20ti}\right) \end{aligned}$$

With $t = \frac{k}{N}$ we get:

$$f = \frac{1}{2}\left(u_2 + u_{N-2}\right) + 2\left(u_{10} + u_{N-10}\right) + 4\left(u_{20} + u_{N-20}\right)$$

(b) We obtain from f:

$$|\hat{f}(k)| = w_k \cdot \frac{N}{2}$$

with $w_2 = w_{N-2} = 1$, $w_{10} = w_{N-10} = 2$, $w_{20} = w_{N-20} = 4$ and otherwise $w_k = 0$.

c)

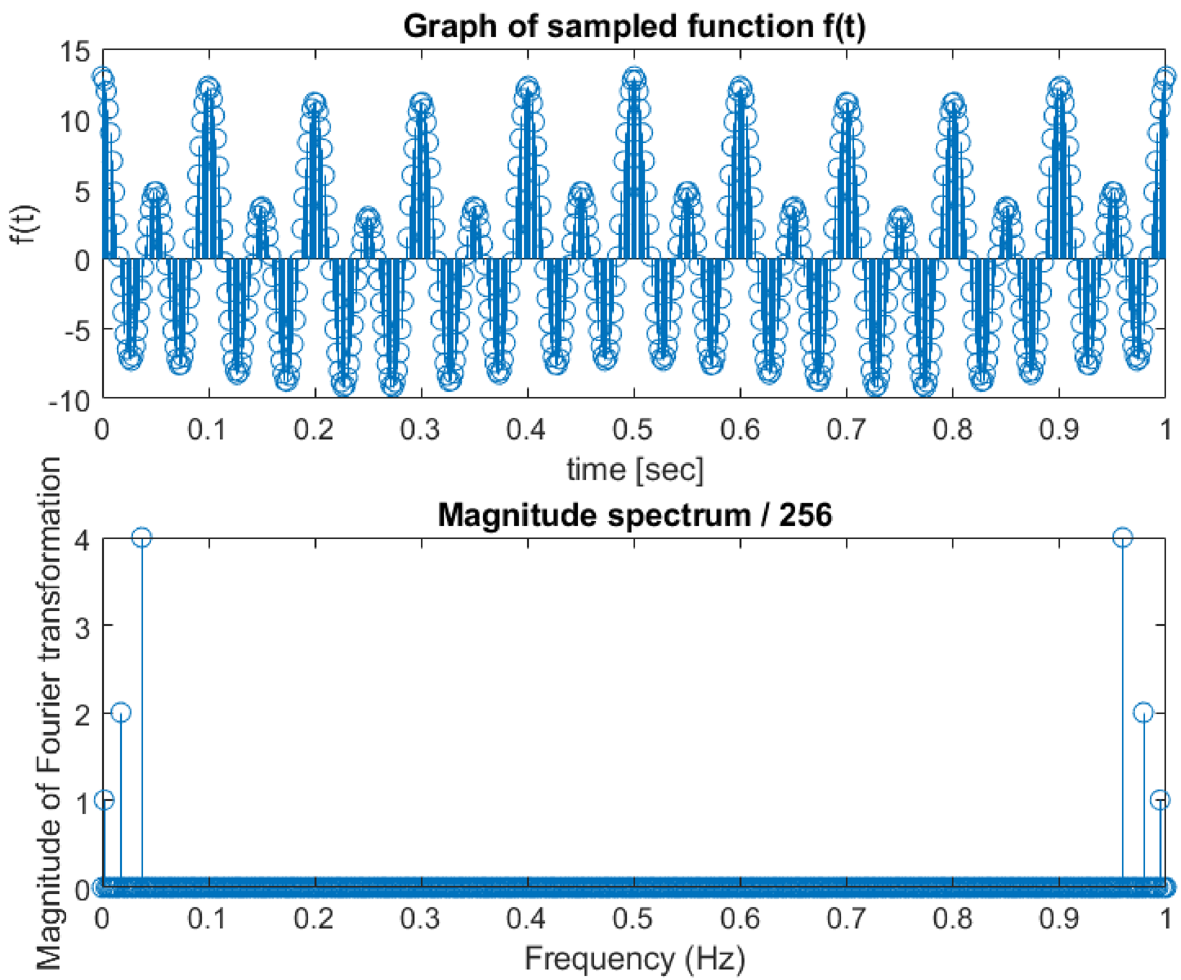

Abbildung 1: $f(t)$(top) and$|\hat{f}(k)|$(bottom)

Task 3.2

The spectrum X is defined as $X := (X(0), X(1), ..., X(N-1))^\top \in \mathbb{C}^N$.

$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-2\pi ikn/N}$

The vector $x \in \mathbb{C}^4$ is given as $x := [-2, 1, -1, 3]^\top$

Therefore $N = 4$ and
$x(0) = -2$
$x(1) = 1$
$x(2) = -1$
$x(3) = 3$
Now we can calculate the spectrum X as follows:

$X(0) = \sum_{n=0}^{3} x(n) \cdot e^{-2\pi i \cdot 0 \cdot n/4} = \sum_{n=0}^{3} x(n) = -2 + 1 - 1 + 3 = 1$

$X(1) = \sum_{n=0}^{3} x(n) \cdot e^{-2\pi i \cdot 1 \cdot n/4} = x(0) \cdot e^{-2\pi i \cdot 0/4} + x(1) \cdot e^{-2\pi i \cdot 1/4} + x(2) \cdot e^{-2\pi i \cdot 2/4} + x(3) \cdot e^{-2\pi i \cdot 3/4}$

$= 1 \cdot (-2) - i \cdot 1 + 1 + i \cdot 3 = -1 + 2i$

$X(2) = \sum_{n=0}^{3} x(n) \cdot e^{-2\pi i \cdot 2 \cdot n/4}$

$= x(0) \cdot e^{-2\pi i \cdot 2 \cdot 0/4} + x(1) \cdot e^{-2\pi i \cdot 2 \cdot 1/4} + x(2) \cdot e^{-2\pi i \cdot 2 \cdot 2/4} + x(3) \cdot e^{-2\pi i \cdot 2 \cdot 3/4}$

$= -2 \cdot e^{-2\pi i \cdot 0/4} + 1 \cdot e^{-2\pi i \cdot 2/4} - 1 \cdot e^{-2\pi i \cdot 4/4} + 3 \cdot e^{-2\pi i \cdot 6/4}$

$= -2 + e^{-2\pi i \cdot 2/4} - e^{-2\pi i \cdot 4/4} + 3 \cdot e^{-2\pi i \cdot 6/4}$

$= -2 - 1 - 1 + 3 \cdot (-1)$

$= -7$

$X(3) = \sum_{n=0}^{3} x(n) \cdot e^{-2\pi i \cdot 3 \cdot n/4}$

$= x(0) \cdot e^{-2\pi i \cdot 3 \cdot 0/4} + x(1) \cdot e^{-2\pi i \cdot 3 \cdot 1/4} + x(2) \cdot e^{-2\pi i \cdot 3 \cdot 2/4} + x(3) \cdot e^{-2\pi i \cdot 3 \cdot 3/4}$

$= -2 \cdot e^{-2\pi i \cdot 0/4} + 1 \cdot e^{-2\pi i \cdot 3/4} - 1 \cdot e^{-2\pi i \cdot 6/4} + 3 \cdot e^{-2\pi i \cdot 9/4}$

$= -2 + e^{-2\pi i \cdot 3/4} - e^{-2\pi i \cdot 6/4} + 3 \cdot e^{-2\pi i \cdot 9/4}$

$= -2 + i + 1 + 3 \cdot (-i)$

$= -1 - 2i$

The spectrum is $X = (1, -1 + 2i, -7, -1 - 2i)^\top$.

Task 3.3

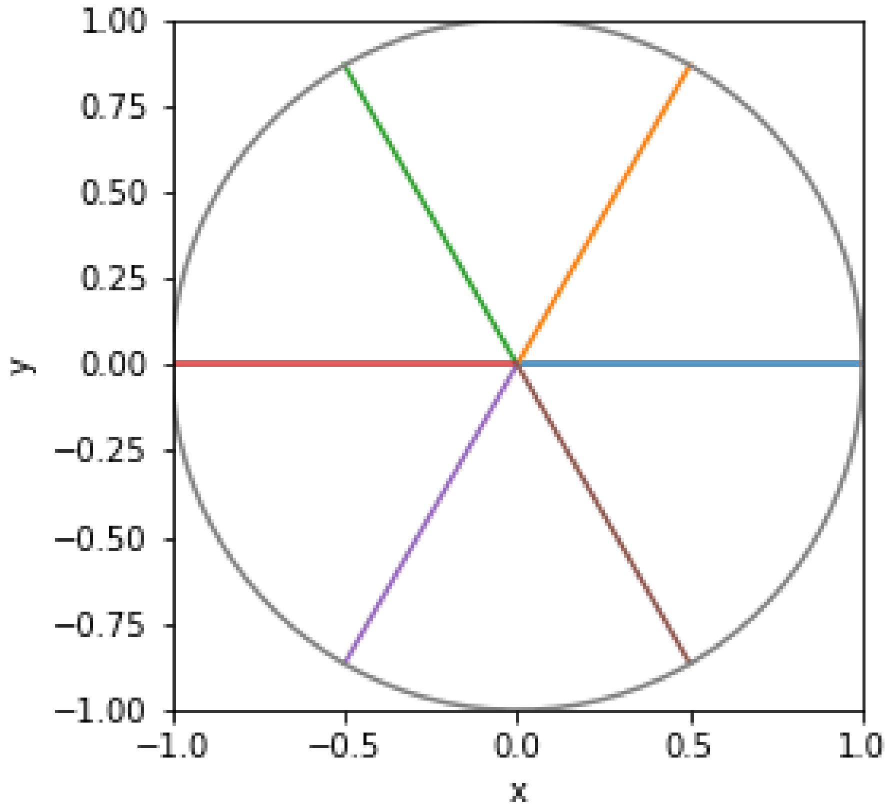

Abbildung 2: 6th root of unity

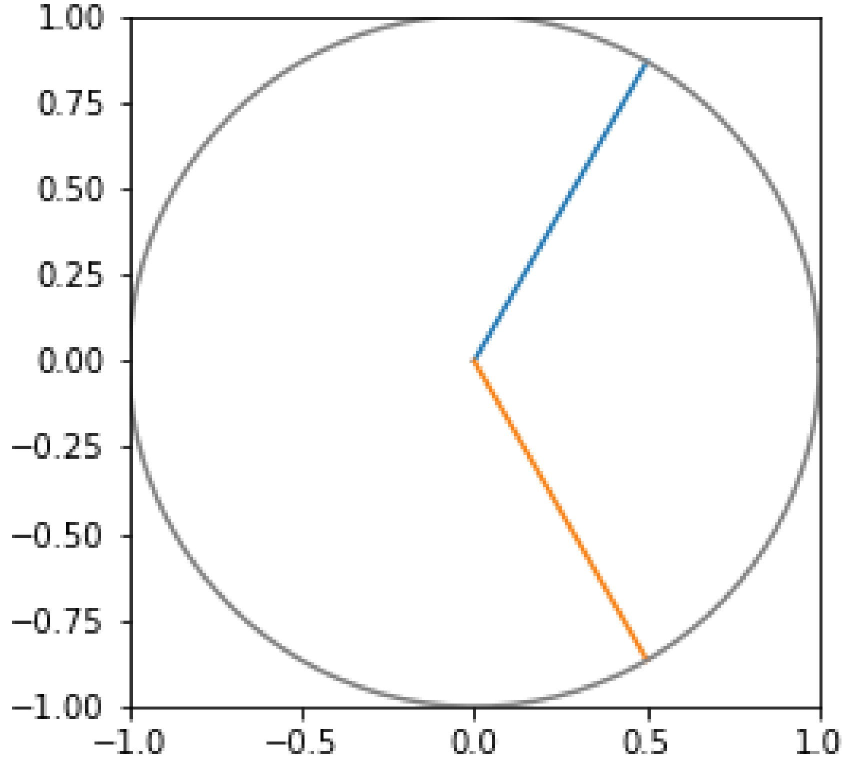

Abbildung 3: primitive 6th root of unity