Solutions for Sheet 2

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Task 2.1

Theorem 1. *Let $1 \neq q \in \mathbb{C}$, then for all positive integers n:* $\sum_{n=0}^{N-1} q^n = \frac{1-q^N}{1-q}$

Proof. Base Case ($N = 1$)

$$\sum_{n=0}^{1-1} q^n = \sum_{n=0}^{0} q^n = q^0 = 1 = \frac{1-q}{1-q} = \frac{1-q^1}{1-q}$$

Induction Hypothesis: Assume that for a given N the following statement holds:

$$\sum_{n=0}^{N-1} q^n = \frac{1-q^N}{1-q}$$

Inductive Step ($N \longrightarrow N+1$)

$$\begin{aligned}
\sum_{n=0}^{N+1-1} q^n = \sum_{n=0}^{N} q^n &= \sum_{n=0}^{N-1} q^n + q^N \\
&\overset{\text{IH}}{=} \frac{1-q^N}{1-q} + q^N \\
&= \frac{1-q^N}{1-q} + \frac{q^N \cdot (1-q)}{1-q} \\
&= \frac{1-q^N}{1-q} + \frac{q^N - q^{N+1}}{1-q} \\
&= \frac{1-q^N + q^N - q^{N+1}}{1-q} \\
&= \frac{1-q^{N+1}}{1-q}
\end{aligned}$$

□

Task 2.3

Theorem 2. *Let $B = b_0, ..., b_{N-1}$ denote an ON-basis of $\mathbb{C}^N$. Every $x \in \mathbb{C}^N$ has the following Fourier representation with respect to this ON-basis:*

$$x = \sum_{k=0}^{N-1} \langle x|b_k \rangle b_k$$

.

Proof. Since B is a basis of $\mathbb{C}^N$, every vector $x \in \mathbb{C}^N$ can be written as follows:

$$\begin{aligned} x &= x_0 \cdot b_0 + x_1 \cdot b_1 + ... + x_{N-1} \cdot b_{N-1} \\ &= \sum_{k=0}^{N-1} x_k \cdot b_k \end{aligned}$$

additionally for x$_i$, *because* $\langle b_i|b_i \rangle = 1$ for normalized vectors and $\langle b_j|b_i \rangle = 0$ for $i, j \in \{0, ..., N-1\} \wedge i \neq j$, stands:

$$\begin{aligned} x_i &= \langle b_i|b_i \rangle x_i = \langle b_i|b_i \rangle x_i + \sum_{k=0 \wedge k \neq i}^{N-1} \langle b_i|b_k \rangle x_k = \sum_{k=0}^{N-1} x_k \cdot \langle b_i|b_k \rangle \\ &= \sum_{k=0}^{N-1} \langle b_i|x_k \cdot b_k \rangle = \left\langle b_i \middle| \sum_{k=0}^{N-1} x_k \cdot b_k \right\rangle \end{aligned}$$

And with the way to write a vector $x \in \mathbb{C}^N$ we get:

$$x_i = \langle b_i|x \rangle$$

So all in all, we get for every vector $x \in \mathbb{C}^N$ the following representation:

$$x = \sum_{k=0}^{N-1} x_k \cdot b_k = \sum_{k=0}^{N-1} \langle x|b_k \rangle b_k$$

□