

Solutions for Sheet 4

Raphael Wude, Martin Brückmann, Claude Jordan, Daniel Degenstein

PATTERN MATCHING AND MACHINE LEARNING FOR AUDIO SIGNAL PROCESSING

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Task 4.1

(a)

$$X_0 = (\omega_N^{0 \cdot 0}, \dots, \omega_N^{0 \cdot (N-1)}) \cdot x$$

$$= \sum_{k=0}^{N-1} \omega_N^0 \cdot x_k$$

$$= \sum_{k=0}^{N-1} x_k \quad \Rightarrow X_0 \in \mathbb{R}$$

$$X_M = (\omega_N^{0 \cdot M}, \dots, \omega_N^{(N-1) \cdot M}) \cdot x$$

$$= \sum_{k=0}^{N-1} \omega_N^{kM} \cdot x_k$$

$$= \sum_{k=0}^{N-1} \exp\left(\frac{-2\pi i M}{2M}\right)^k \cdot x_k$$

$$= \sum_{k=0}^{N-1} \exp(-\pi i k) \cdot x_k$$

$$= \sum_{k=0}^{N-1} (-1)^k \cdot x_k \quad \Rightarrow X_M \in \mathbb{R}$$

$$X_{N-k} = \sum_{l=0}^{N-1} \omega_N^{l(N-k)} \cdot x_l$$

$$= \sum_{l=0}^{N-1} \exp\left(\frac{-2\pi i l(N-k)}{N}\right) \cdot x_l$$

$$= \sum_{l=0}^{N-1} \exp\left(\frac{2\pi i l k}{N} - 2\pi i l N\right) \cdot x_l$$

$$= \sum_{l=0}^{N-1} \exp\left(\frac{2\pi i l k}{N}\right) \cdot \exp(2\pi i l N) \cdot x_l$$

$$\begin{aligned}
&= \sum_{l=0}^{N-1} \overline{\exp\left(\frac{-2\pi i l k}{N}\right) \cdot x_l} \\
&= \sum_{l=0}^{N-1} \overline{\omega_N^{lk} \cdot x_l} \\
&= \overline{X_k}
\end{aligned}$$

(b)

Task 4.2

(a)

For A we got:

$$A = \text{DFT}_N \cdot a = \text{DFT}_N \cdot \left(\sin\left(\frac{2\pi 4k}{N}\right) \right)$$

And with Eulers Formula:

$$\begin{aligned}
a &= \sin\left(\frac{2\pi 4k}{N}\right) = \frac{1}{2i} \cdot \left(e^{\frac{2\pi 4ki}{N}} - e^{\frac{-2\pi 4ki}{N}} \right) \\
&= \frac{1}{2i} \cdot (u_4 - u_{N-4})
\end{aligned}$$

With $\langle u_k | u_l \rangle = N \cdot \delta_{k,l}$ we get:

$$|a(k)| = \frac{N}{2} \cdot w_k$$

with $w_4 = w_{N-4} = 1$ and else $w_k = 0$.

That means, for $A = \text{DFT}_N \cdot a$ the only nonzero entries of the matrix A are at $k = 4$ and $k = N - 4 = 508$. For this entries we get:

$$A(k) = \text{DFT}_N(k) \cdot \frac{N}{2}$$

(b)

We know, that $|A(k)| > 2\theta$ for all nonzero $A(k)$ and $R(k) \leq \theta$. So $|A + R| > \theta$ for all $k \in \{0, \dots, N-1\}$. So for B_θ we get:

$$B_\theta(k) = \begin{cases} A(k) + R(k), & \text{for } k = 4 \wedge k = 508 \\ 0, & \text{else} \end{cases}$$

If we now define us a new R_θ with:

$$R_\theta(k) = \begin{cases} R(k), & \text{for } k = 4 \wedge k = 508 \\ 0, & \text{else} \end{cases}$$

we can write $B_\theta = A + R_\theta$

So for \tilde{a} and $\tilde{a} - a$ we get:

$$\begin{aligned}\tilde{a} &= \text{DFT}_n^{-1} B_\theta = \text{DFT}_n^{-1} (A + R_\theta) = \text{DFT}_n^{-1} A + \text{DFT}_n^{-1} R_\theta = a + \text{DFT}_n^{-1} R_\theta \\ \tilde{a} - a &= a + \text{DFT}_n^{-1} R_\theta - a = \text{DFT}_n^{-1} R_\theta\end{aligned}$$