Solutions for Sheet 1

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Pattern Matching and Machine Learning for Audio Signal Processing

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Task 1.1

(a) $z = (\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})^3 = (\frac{\sqrt{2}}{2})^3 + 3(\frac{\sqrt{2}}{2})^2(i\frac{\sqrt{2}}{2}) + 3(\frac{\sqrt{2}}{2})(i\frac{\sqrt{2}}{2})^2 + (i\frac{\sqrt{2}}{2})^3 = \frac{1}{2}(-1+i)\sqrt{2} = \frac{1}{\sqrt{2}}(-1+i) = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

$|z| = \sqrt{(-\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2} = 1$

$cos\varphi = \frac{Re[z]}{|z|} = \frac{-\frac{1}{\sqrt{2}}}{1} = -\frac{1}{\sqrt{2}}$

$\varphi = cos^{-1}(-\frac{1}{\sqrt{2}}) = \frac{3\pi}{4}$

$\Rightarrow z = |z| \cdot e^{i\varphi} = e^{\frac{3\pi i}{4}}$

(b) Write down the following complex number in Cartesian coordinates:

$$z = \frac{e^{\frac{5\pi}{3}i}}{e^{\frac{4}{3}\pi i} \cdot 2e^{\frac{11}{6}\pi i}}.$$

$$\begin{aligned} z &= \frac{e^{\frac{5\pi}{3}i}}{e^{\frac{4}{3}\pi i} \cdot 2e^{\frac{11}{6}\pi i}} = \frac{1}{2}e^{\left(\frac{5}{3}-\frac{4}{3}-\frac{11}{6}\right)\pi i} \\ &= \frac{1}{2}e^{\frac{3}{2}\pi i} \end{aligned}$$

With Euler's Formula $e^{iz} = cos(z) + i \cdot sin(z)$ and $cos\left(\frac{3\pi}{2}\right) = 0$ and $sin\left(\frac{3\pi}{2}\right) = -1$:

$$z = \frac{1}{2}e^{\frac{3}{2}\pi i} = \frac{1}{2} \cdot (-i) = 0 - \frac{1}{2}i$$

So the complex number in Cartesian coordinates is: $z = 0 - \frac{1}{2}i$

(c) Calculate explicitly: $Re[5e^{\frac{\pi}{4}i} + \sqrt{2}e^{\pi i}]$.

$$\begin{aligned} Re\left[5e^{\frac{\pi}{4}i} + \sqrt{2}e^{\pi i}\right] &= Re\left[5\left(\frac{\sqrt{2}}{2} + i \cdot \frac{\sqrt{2}}{2}\right) + \sqrt{2} \cdot (-1)\right] \\ &= Re\left[5 \cdot \frac{\sqrt{2}}{2} - \sqrt{2} + 5 \cdot \frac{\sqrt{2}}{2} i\right] = Re\left[\frac{3}{2}\sqrt{2} + \frac{5\sqrt{2}}{2} i\right] \\ &= \frac{3}{2}\sqrt{2} \end{aligned}$$

The real part of the complex number $5e^{\frac{\pi}{4}i} + \sqrt{2}e^{\pi i}$ is $\frac{3}{2}\sqrt{2}$.

(d) Using Euler's Formula, prove the following statement:

$sin(z)^2 + cos(z)^2 = 1, z \in \mathbb{C}$
Proof:
Euler's formula is as follows: $e^{iz} = cos(z) + i \cdot sin(z), z \in \mathbb{C}$
For the opposite angle $-z$, we have $e^{-iz} = cos(z) - i \cdot sin(z)$
With these two, we can derive the following for $i \cdot sin(z), cos(z)$:
$i \cdot sin(z) = e^{iz} - cos(z) = cos(z) - e^{-iz}$
$cos(z) = e^{iz} - i \cdot sin(z) = e^{-iz} + i \cdot sin(z)$
And by plugging in, we can write $sin(z), cos(z)$ only in terms of e^{iz}, e^{-iz}:
$cos(z) = e^{iz} - i \cdot sin(z) = e^{iz} - cos(z) + e^{-iz} = \frac{e^{iz}+e^{-iz}}{2}$
$sin(z) = \frac{e^{iz}-cos(z)}{i} = \frac{e^{iz}-e^{-iz}-i \cdot sin(z)}{i} = \frac{e^{iz}-e^{-iz}}{2i}$
All that's left is to square and plug in once more:
$sin(z)^2 = \frac{1}{-4}((e^{iz})^2 - 2e^{iz}e^{-iz} + (e^{-iz})^2)$
$cos(z)^2 = \frac{1}{4}((e^{iz})^2 + 2e^{iz}e^{-iz} + (e^{-iz})^2)$
$sin(z)^2 + cos(z)^2 = \frac{1}{4}4e^{iz}e^{-iz} = 1$

Task 1.2

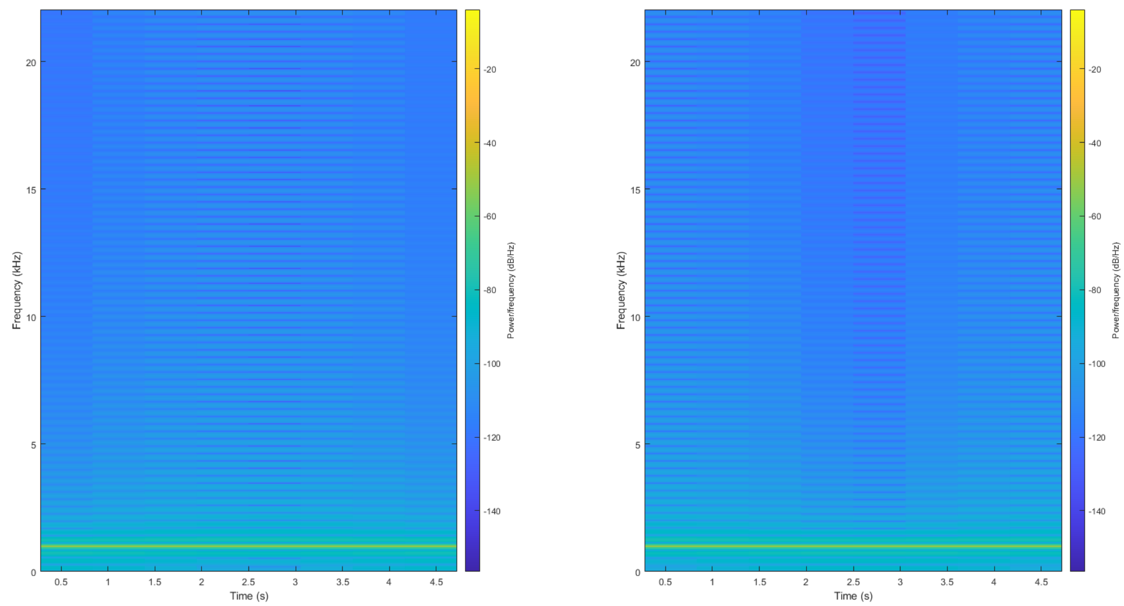

Figure 1: time-frequency representations for the sine wave (left) and cosine wave (right)

If we look at both spectrograms in Figure 1, we see that there are no differences between the two waves. This is due to the fact that a cosine wave is only a phase-shifted sine wave. Therefore, both time-frequency representations are the same.