Solutions for Sheet 6

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Task 6.1

The time-shift operator T^k, $k \in \mathbb{Z}$ is defined as $T^k[x](n) = x^k(n) := x(n-k)$.

A system S is time invariant $\Leftrightarrow \forall k \in \mathbb{Z}, \forall x \in \ell^p(\mathbb{Z}) : S[x^k] = S[x]^k$.

(a)

The upsampling operator is defined as $(\uparrow M)[x](n) = \begin{cases} x(n/M), & \text{if } M \text{ divides } n, \\ 0, & \text{otherwise.} \end{cases}$

$$\Rightarrow (\uparrow M)[x^k](n) = \begin{cases} x^k(n/M), & \text{if } M \text{ divides } n, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} x(n/M - k), & \text{if } M \text{ divides } n, \\ 0, & \text{otherwise.} \end{cases}$$

$$\neq (\uparrow M)[x]^k(n) = (\uparrow M)[x](n-k) = \begin{cases} x((n-k)/M), & \text{if } M \text{ divides } (n-k), \\ 0, & \text{otherwise.} \end{cases}$$

$\Rightarrow$ The upsampling operator is not time invariant.

(b)

The frequency-shift operator is defined as $E_w[x](n) := e^{-2\pi iwn}x(n)$.

$\Rightarrow E_w[x^k](n) = e^{-2\pi iwn}x^k(n) = e^{-2\pi iwn}x(n-k) \neq e^{-2\pi iw(n-k)}x(n-k) = E_w[x](n-k) = E_w[x]^k(n)$

$\Rightarrow$ The frequency-shift operator is not time invariant.

6.2

(a)

$C(\mathcal{D}) = \{(1,1),(1,3),(2,4),(3,1),(3,5),(4,2),(5,3),$
$(5,5),(6,1),(6,2),(6,5),(7,4),(8,2),(9,1),(9,4)\}$
$C(\mathcal{Q}) = \{(1,3),(2,1),(2,5),(3,4),(4,2)\}$

(b)

The shifted constellation maps are given by

$$m + C(\mathcal{Q}) = \{(m+1,3),(m+2,1),(m+2,5),(m+3,4),(m+4,2)\}$$

for $m \in [-3:8]$, i.e.

$$-3 + C(\mathcal{Q}) = \{(-2,3),(-1,1),(-1,5),(0,4),(1,2)\}$$
$$-2 + C(\mathcal{Q}) = \{(-1,3),(0,1),(0,5),(1,4),(2,2)\}$$
$$\vdots$$
$$7 + C(\mathcal{Q}) = \{(8,3),(9,1),(9,5),(10,4),(11,2)\}$$
$$8 + C(\mathcal{Q}) = \{(9,3),(10,1),(10,5),(11,4),(12,2)\}$$

(c)

For $m \in \mathbb{Z} \setminus [-3:8]$, the points cannot overlap, so $\Delta_C(m) = 0$. Furthermore, we have

$$\Delta_C(-3) = \Delta_C(-2) = 0$$
$$\Delta_C(-1) = |\{(1,1),(2,4)\}| = 2$$
$$\Delta_C(0) = |\{(1,3),(4,2)\}| = 2$$
$$\Delta_C(1) = |\{(3,1),(3,5)\}| = 2$$
$$\Delta_C(2) = |\{(6,2)\}| = 1$$
$$\Delta_C(3) = |\{(5,5)\}| = 1$$

$$\Delta_C(4) = |\{(5,3),(6,1),(6,5),(7,4),(8,2)\}| = 5$$
$$\Delta_C(5) = 0$$
$$\Delta_C(6) = |\{(9,4)\}| = 1$$
$$\Delta_C(7) = |\{(9,1)\}| = 1$$
$$\Delta_C(8) = 0$$