Solutions for Sheet 4

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Pattern Matching and Machine Learning for Audio Signal Processing

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Task 4.1

(a)

$X_0 = (\omega_N^{0\cdot 0}, ..., \omega_N^{0\cdot(N-1)}) \cdot x$

$= \sum_{k=0}^{N-1} \omega_N^0 \cdot x_k$

$= \sum_{k=0}^{N-1} x_k \quad \Rightarrow X_0 \in \mathbb{R}$

$X_M = (\omega_N^{0\cdot M}, ..., \omega_N^{(N-1)\cdot M}) \cdot x$

$= \sum_{k=0}^{N-1} \omega_N^{kM} \cdot x_k$

$= \sum_{k=0}^{N-1} exp(\frac{-2\pi i M}{N})^k \cdot x_k$

$= \sum_{k=0}^{N-1} exp(\frac{-2\pi i M k}{2M}) \cdot x_k$

$= \sum_{k=0}^{N-1} exp(-\pi i k) \cdot x_k$

$= \sum_{k=0}^{N-1} (-1)^k \cdot x_k \quad \Rightarrow X_M \in \mathbb{R}$

$X_{N-k} = \sum_{l=0}^{N-1} \omega_N^{l(N-k)} \cdot x_l$

$= \sum_{l=0}^{N-1} exp(\frac{-2\pi i l(N-k)}{N}) \cdot x_l$

$= \sum_{l=0}^{N-1} exp(\frac{-2\pi i l N}{N} - (\frac{-2\pi i l k}{N})) \cdot x_l$

$= \sum_{l=0}^{N-1} exp(\frac{2\pi ilk}{N} - 2\pi ilN) \cdot x_l$

$= \sum_{l=0}^{N-1} exp(\frac{2\pi ilk}{N}) \cdot exp(-2\pi ilN) \cdot x_l$

$= \sum_{l=0}^{N-1} exp(\frac{2\pi ilk}{N}) \cdot x_l$

$= \overline{\sum_{l=0}^{N-1} exp(\frac{-2\pi ilk}{N}) \cdot x_l}$

$= \overline{\sum_{l=0}^{N-1} \omega_N^{lk} \cdot x_l}$

$= \overline{X_k}$

(b)

Task 4.2

(a)

For A we got:

$$A = \mathrm{DFT}_N \cdot a = \mathrm{DFT}_N \cdot \left(sin\left(\frac{2\pi 4k}{N}\right)\right)$$

And with Eulers Formula:

$$a = sin\left(\frac{2\pi 4k}{N}\right) = \frac{1}{2i} \cdot \left(e^{\frac{2\pi 4ki}{N}} - e^{\frac{-2\pi 4ki}{N}}\right)$$
$$= \frac{1}{2i} \cdot (u_4 - u_{N-4})$$

With $\langle u_k | u_l \rangle = N \cdot \delta_{k,l}$ we get:

$$|a(k)| = \frac{N}{2} \cdot w_k$$

with $w_4 = w_{N-4} = 1$ and else $w_k = 0$.
That means, for $A = \mathrm{DFT}_N \cdot a$ the only nonzero entries of the matrix A are at $k = 4$ and $k = N - 4 = 508$. For this entries we get:

$$A(k) = \mathrm{DFT}_N(k) \cdot \frac{N}{2}$$

(b)

We know, that $|A(k)| > 2\theta$ for all nonzero $A(k)$ and $R(k) \leq \theta$. So $|A + R| > \theta$ for all $k \in \{0, ..., N-1\}$. So for B_θ we get:

$$B_\theta(k) = \begin{cases} A(k) + R(k), \mathrm{for} k = 4 \wedge k = 508 \\ 0, \mathrm{else} \end{cases}$$

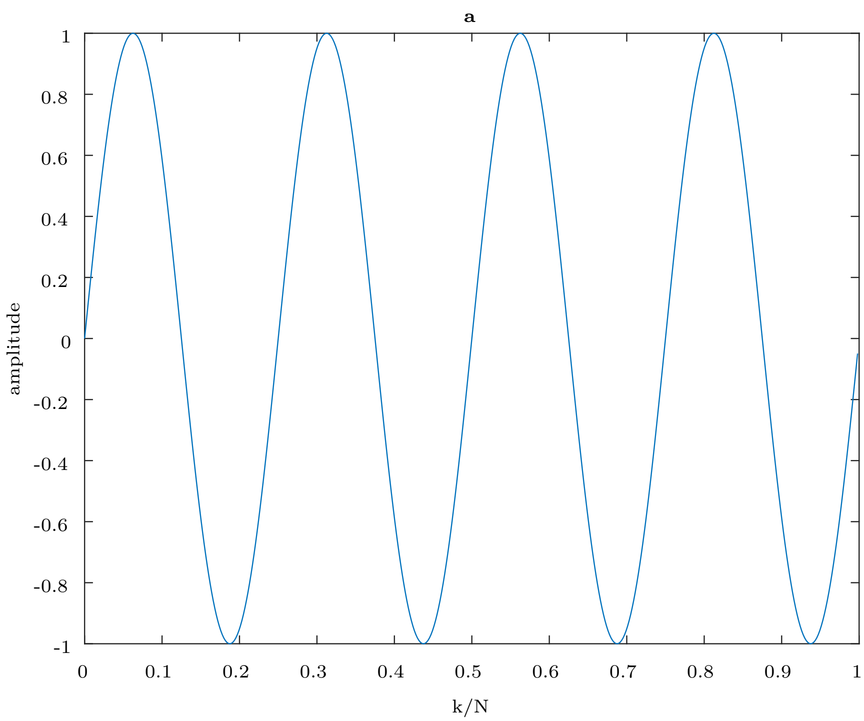
If we now define us a new R_θ with:

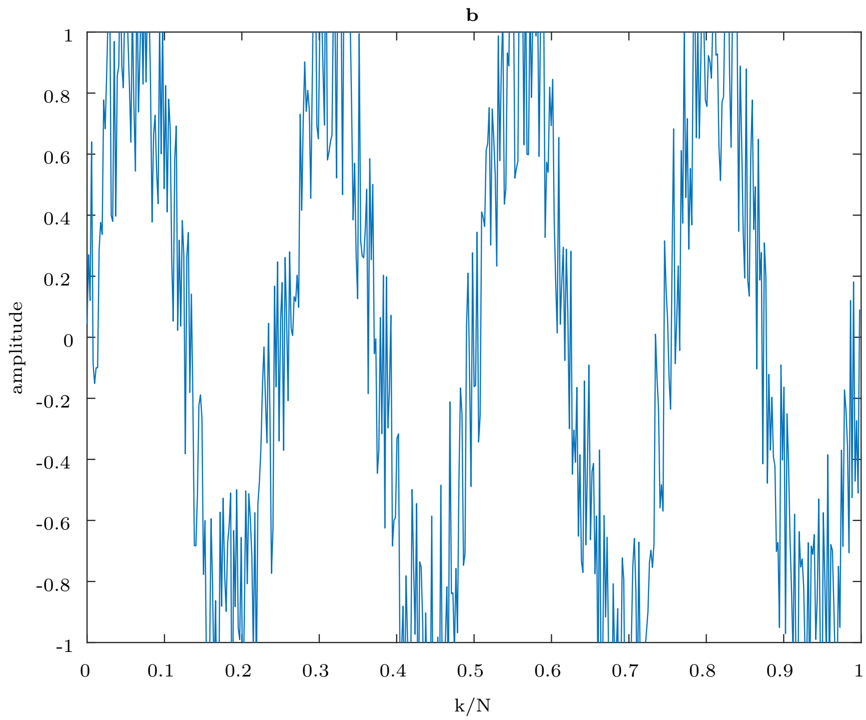
$$R_\theta(k) = \begin{cases} R(k), \text{for} k = 4 \wedge k = 508 \\ 0, \text{else} \end{cases}$$

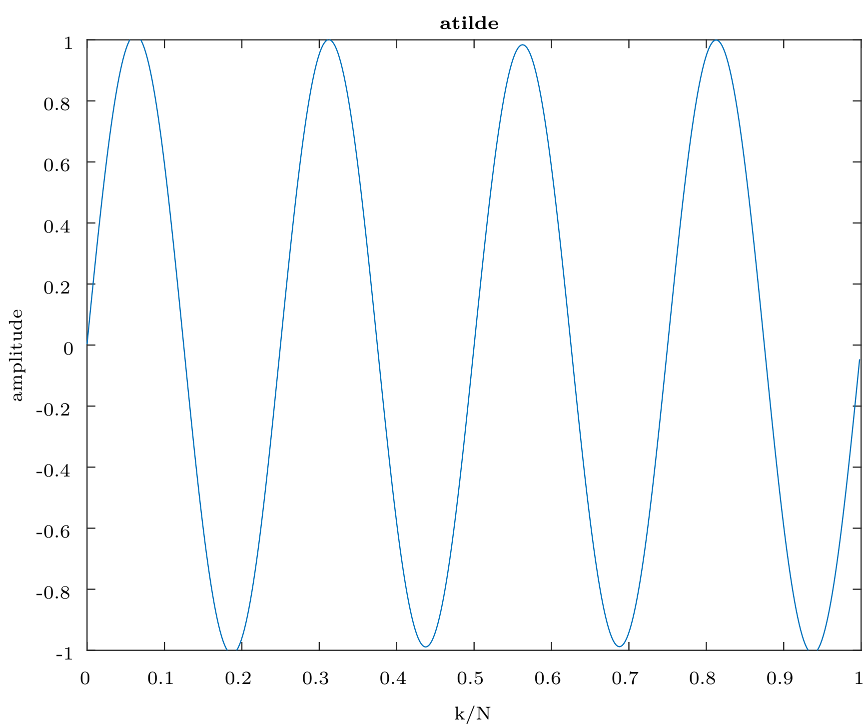
we can write $B_\theta = A + R_\theta$
So for $\tilde{a}$ and $\tilde{a} - a$ we get:

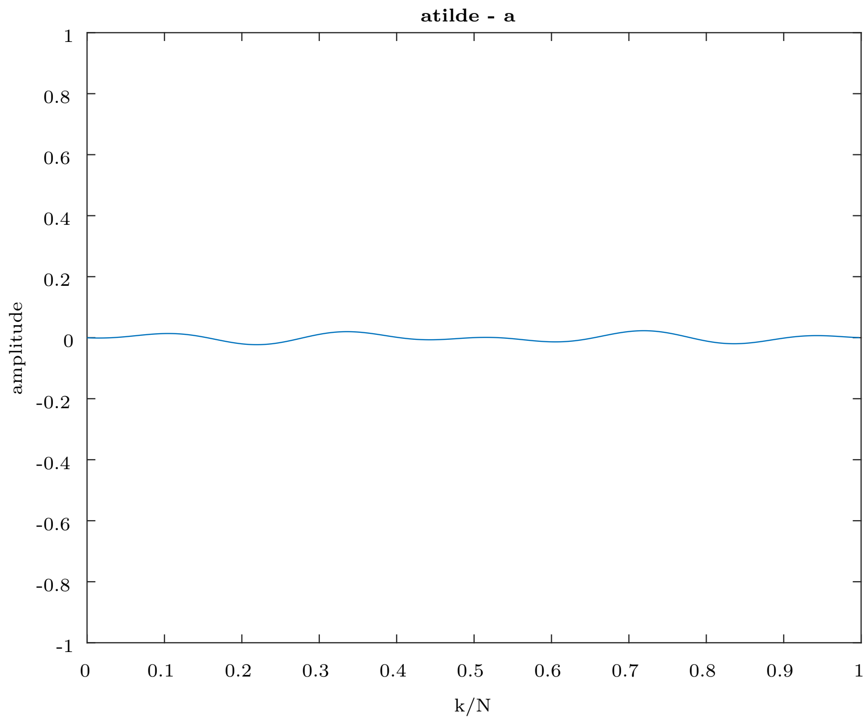
$$\tilde{a} = \mathrm{DFT}_n^{-1} B_\theta = \mathrm{DFT}_n^{-1}(A + R_\theta) = \mathrm{DFT}_n^{-1} A + \mathrm{DFT}_n^{-1} R_\theta = a + \mathrm{DFT}_n^{-1} R_\theta$$
$$\tilde{a} - a = a + \mathrm{DFT}_n^{-1} R_\theta - a = \mathrm{DFT}_n^{-1} R_\theta$$

Task 4.3




b
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
amplitude
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
k/N


atilde
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
amplitude
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
k/N


atilde - a
amplitude
k/N
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1