Seminar class 3

Exercise 1 (exercise 6.7 in the book)

We have shown that for a finite hypothesis class $\mathcal{H}$, $\text{VCdim}(\mathcal{H}) \le \lfloor \log(|\mathcal{H}|) \rfloor$. However, this is just an upper bound. The VC-dimension of a class can be much lower than that:

1. Find an example of a class $\mathcal{H}$ of functions over the real interval $\mathcal{X} = [0,1]$ such that $\mathcal{H}$ is infinite while $\text{VCdim}(\mathcal{H}) = 1$.
2. Give an example of a finite hypothesis class $\mathcal{H}$ over the domain $\mathcal{X} = [0,1]$, where $\text{VCdim}(\mathcal{H}) = \lfloor \log_2(|\mathcal{H}|) \rfloor$.

Exercise 2 (exercise 6.5 in the book)

VC-dimension of axis aligned rectangles in $\mathbb{R}^d$: Let $\mathcal{H}^d_{\text{rec}}$ be the class of axis aligned rectangles in $\mathbb{R}^d$. We have already seen that $\text{VCdim}(\mathcal{H}^2_{\text{rec}}) = 4$. Prove that in general, $\text{VCdim}(\mathcal{H}^d_{\text{rec}}) = 2d$.

Exercise 3 (exercise 6.6 in the book)

VC-dimension of Boolean conjunctions: Let $\mathcal{H}^d_{con}$ be the class of Boolean conjunctions over the variables $x_1, \ldots, x_d$ ($d \ge 2$). We already know that this class is finite and thus (agnostic) PAC learnable. In this question we calculate $\text{VCdim}(\mathcal{H}^d_{con})$.

1. Show that $|\mathcal{H}^d_{con}| \le 3^d + 1$.
2. Conclude that $\text{VCdim}(\mathcal{H}) \le d \log 3$.
3. Show that $\mathcal{H}^d_{con}$ shatters the set of unit vectors $\{\mathbf{e}_i : i \le d\}$.
4. (**) Show that $\text{VCdim}(\mathcal{H}^d_{con}) \le d$.
   Hint: Assume by contradiction that there exists a set $C = \{c_1, \ldots, c_{d+1}\}$ that is shattered by $\mathcal{H}^d_{con}$. Let $h_1, \ldots, h_{d+1}$ be hypotheses in $\mathcal{H}^d_{con}$ that satisfy
   $$\forall i, j \in [d+1],\ h_i(c_j) = \begin{cases} 0 & i = j \\ 1 & \text{otherwise} \end{cases}$$
   For each $i \in [d+1]$, h_i (or more accurately, the conjunction that corresponds to h_i) contains some literal ℓ_i which is false on c_i and true on c_j for each $j \ne i$. Use the Pigeonhole principle to show that there must be a pair $i < j \le d+1$ such that ℓ_i and ℓ_j use the same x_k and use that fact to derive a contradiction to the requirements from the conjunctions h_i, h_j.
5. Consider the class $\mathcal{H}^d_{mcon}$ of monotone Boolean conjunctions over $\{0,1\}^d$. Monotonicity here means that the conjunctions do not contain negations. As in $\mathcal{H}^d_{con}$, the empty conjunction is interpreted as the all-positive hypothesis. We augment $\mathcal{H}^d_{mcon}$ with the all-negative hypothesis h^-. Show that $\text{VCdim}(\mathcal{H}^d_{mcon}) = d$.

Exercise 4 (exercise 6.9 in the book)

Let $\mathcal{H}$ be the class of signed intervals, that is,
$\mathcal{H} = \{h_{a,b,s} : a \le b, s \in \{-1,1\}\}$ where

$$h_{a,b,s}(x) = \begin{cases} s & \text{if } x \in [a,b] \\ -s & \text{if } x \notin [a,b] \end{cases}$$

Calculate $\text{VCdim}(\mathcal{H})$.

Exercise 5 (exercise 3.13 in the book "Foundations of Machine Learning", 2nd edition)

3.13 **VC-dimension of union of k intervals.** What is the VC-dimension of subsets of the real line formed by the union of k intervals?

Exercise 6 (exercise 3.15 in the book "Foundations of Machine Learning", 2nd edition)

VC-dimension of subsets. What is the VC-dimension of the set of subsets I_α of the real line parameterized by a single parameter α: $I_\alpha = [\alpha, \alpha+1] \cup [\alpha+2, +\infty)$?