A fibrational view on computational effects

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Outline

We investigate the combination of

- dependent types ($\Pi, \Sigma, V =_A W, \ldots$)
- computational effects (state, I/O, probability, recursion, ...)

Two guiding problems

- effectful programs in types (e.g., `read` and `write` in types)
- types of effectful programs (e.g., of sequential composition)

Our goals

- tell a mathematically natural story
- use established math. techniques
- cover a wide range of comp. effects
- discover smth. interesting

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- cover a wide range of comp. effects (alg. effects, continuations)
- discover smth. interesting (using handlers to reason about effects)

Effectful programs in types

(type-dependency in the presence of effects)

Effectful programs in types

Let's assume that we have some dependent type A, e.g.:

$$\ell : (\mathsf{List\ Chr}) \vdash A(\ell) \quad \stackrel{\text{def}}{=} \quad \Sigma\, \ell' : (\mathsf{List\ Chr})\, .\, (\mathtt{length}\ \ell =_{\mathsf{Nat}} \mathtt{length}\ \ell' \times \ldots)$$

which could be used to type the dependent function

$$\mathtt{sort} : \Pi\, \ell : (\mathsf{List\ Chr})\, .\, A(\ell)$$

Q: Should we allow $A[\mathtt{receive}(y.\, M)/\ell]$?

- i.e., should we be allowed to type $\mathtt{receive}(y.\, M) : \mathsf{List\ Chr}$

A1: In this work we say no

- types should only depend on static information about effects
- we recover dependency on effectful computations via thunks

A2: We are also looking into the yes case

- type-dependency needs to be "homomorphic" (more on this later)

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Effectful programs in types

Aim: Types should only depend on static info about effects

Solution: CBPV/EEC style distinction between vals. and comps.

- value types $\Gamma \vdash A$ (MLTT + thunks + ...)
- computation types $\Gamma \vdash \underline{C}$ (dep. CBPV/EEC)
- where Γ contains **only** value variables $x_1 : A_1, \ldots, x_n : A_n$

Note: Could have also considered λ_{ML} and FGCBV

- building on CBPV/EEC gives a more general story
- especially for the treatment of sequential composition
- and also for integrating dependent- and effect-typing

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Assigning types to effectful programs

(e.g., sequential composition)

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The problem: The standard typing rule for seq. composition

$$\frac{\Gamma \vDash_{\mathsf{c}} M : F\,A \qquad \Gamma, x \!:\! A \vDash_{\mathsf{c}} N : \underline{C}}{\Gamma \vDash_{\mathsf{c}} M \ \mathtt{to}\ x \!:\! A \ \mathtt{in}\ N : \underline{C}}$$

is not correct any more because x can appear free in the type

$$\underline{C}$$

in the conclusion

Assigning types to effectful programs

Aim: To fix the typing rule of sequential composition

Option 1: We could restrict the free variables in $\underline{C}$: [Levy'04]

$$\frac{\Gamma \vDash_c M : F\,A \qquad \Gamma \vdash \underline{C} \qquad \Gamma, x\!:\!A \vDash_c N : \underline{C}}{\Gamma \vDash_c M \;\mathtt{to}\; x\!:\!A \;\mathtt{in}\; N : \underline{C}}$$

But sometimes it is useful if $\underline{C}$ can depend on x!

- if we consider

$$\mathtt{fopen}\,(\mathtt{return\;true}, \mathtt{return\;false}) \;\mathtt{to}\; x\!:\!\mathsf{Bool} \;\mathtt{in}\; N$$

- then it would be natural to let $\underline{C}$ depend on x, e.g.,

$$x\!:\!\mathsf{Bool} \vdash \underline{C}(x) \stackrel{\text{def}}{=} \mathtt{if}\; x \;\mathtt{then}\; \text{“allow } \mathtt{fread}, \mathtt{fwrite}, \text{ and } \mathtt{fclose}\text{”}$$
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Option 2: One could lift sequential composition to type level

$$\Gamma \vdash_{\mathsf{c}} M \ \mathtt{to}\ x\,{:}\,A\ \mathtt{in}\ N : M\ \mathtt{to}\ x\,{:}\,A\ \mathtt{in}\ \underline{C}$$

But then all computation types would be singleton-like!?!

Option 3: In the monadic metalanguage λ_{ML}, one could also try

$$\frac{\Gamma \vdash M : T\,A \qquad \Gamma, x\,{:}\,A \vdash N : T\,B(x)}{\Gamma \vdash M\ \mathtt{to}\ x\,{:}\,A\ \mathtt{in}\ N : T\,(\Sigma x : A.B)}$$

But what makes this a principled solution? Why is it correct?

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Option 4: We draw inspiration from algebraic effects

- and combine it with restricting $\underline{C}$ in seq. comp. (**Option 1**)

E.g., consider the non-det. program $\left(\text{for } x : \mathsf{Nat} \vDash_{c} N : \underline{C}(x)\right)$

$$M \quad \stackrel{\text{def}}{=} \quad \mathtt{choose}\,(\mathtt{return}\,4, \mathtt{return}\,2)\ \mathtt{to}\ x : \mathsf{Nat}\ \mathtt{in}\ N$$

After tossing the coin, this program evaluates as either

$$N[4/x] : \underline{C}[4/x] \qquad \text{or} \qquad N[2/x] : \underline{C}[2/x]$$

Idea: M denotes an element of the coproduct of algebras

$$\underline{C}[4/x] + \underline{C}[2/x] \quad \text{“}\stackrel{\text{def}}{=}\text{”} \quad F\left(U\left(\underline{C}[4/x]\right) + U\left(\underline{C}[2/x]\right)\right)_{/\equiv}$$

and thus we would like to type M at the type $\Sigma x : \mathsf{Nat}.\underline{C}$

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Putting these ideas together

(eMLTT: a core dep.-typed language with comp. effects)

eMLTT – types

Value types: MLTT + thunks + ...

$$A, B \ ::= \ \mathsf{Nat} \mid 1 \mid 0 \mid \Pi x \colon A.B \mid \Sigma x \colon A.B \mid V =_A W \mid U\,\underline{C} \mid \ \ldots$$

- $U\,\underline{C}$ is the type of thunked (i.e., suspended) computations

Computation types: dep.-typed version of EEC's comp. types

$$\underline{C}, \underline{D} \ ::= \ F\,A \mid \Pi x \colon A.\underline{C} \mid \Sigma x \colon A.\underline{C}$$

- $F\,A$ is the type of computations returning values of type A
- $\Pi x \colon A.\underline{C}$ is the type of dependent effectful functions
  - generalises CBPV/EEC's comp. types $A \to \underline{C}$ and $\underline{C} \times \underline{D}$
- $\Sigma x \colon A.\underline{C}$ is the type of dep. pairs of values and effectful comps.
  - captures the intuition about seq. comp. and coprods. of algebras
  - generalises EEC's comp. types $!A \otimes \underline{C}$ and $\underline{C} \oplus \underline{D}$

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Value terms: MLTT + thunks + ...

$$V, W ::= x \mid \texttt{zero} \mid \texttt{succ}\, V \mid \ldots \mid \texttt{thunk}\ M \mid \ldots$$

- equational theory based on intensional MLTT

Comp. terms: dep.-typed version of CBPV/EEC's comp. terms

$$\begin{array}{rcll} M, N & ::= & \texttt{force}\ V & \\ & \mid & \texttt{return}\, V & \\ & \mid & M \texttt{ to } x : A \texttt{ in } N & \\ & \mid & \lambda x : A . M & \\ & \mid & M V & \\ & \mid & \langle V, M \rangle & \text{(comp. } \Sigma \text{ intro.)} \\ & \mid & M \texttt{ to } \langle x : A, z : \underline{C} \rangle \texttt{ in } K & \text{(comp. } \Sigma \text{ elim.)} \end{array}$$

But: Value and comp. terms alone do not suffice, as in EEC!

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Note: We need to define K in such a way that the intended left-to-right evaluation order is preserved, e.g., consider

$$\Gamma \vdash_{\mathsf{c}} \langle V, M \rangle \; \mathtt{to} \; \langle x : A, z : \underline{C} \rangle \; \mathtt{in} \; K = K[V/x, M/z] : \underline{D}$$

Homomorphism terms: dep.-typed version of EEC's linear terms

$$\begin{array}{rcll} K, L & ::= & z & \text{(linear comp. vars.)} \\ & | & K \; \mathtt{to} \; x : A \; \mathtt{in} \; M & \\ & | & \lambda x : A . K & \\ & | & K \, V & \\ & | & \langle V, K \rangle & \text{(comp. } \Sigma \text{ intro.)} \\ & | & K \; \mathtt{to} \; \langle x : A, z : \underline{C} \rangle \; \mathtt{in} \; L & \text{(comp. } \Sigma \text{ elim.)} \end{array}$$

Typing judgments:

- $\Gamma \vdash_{\mathsf{v}} V : A$
- $\Gamma \vdash_{\mathsf{c}} M : \underline{C}$
- $\Gamma \,|\, z : \underline{C} \vdash_{\mathsf{h}} K : \underline{D}$ (linear in z; comp. bound to z happens first)

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eMLTT – typing sequential composition

We can then account for type-dependency in seq. comp. as

$$\frac{\Gamma \vDash_{\mathsf{c}} M : F\,A \qquad \Gamma \vdash \Sigma y\!:\!A.\underline{C}(y) \qquad \dfrac{\Gamma, x\!:\!A \vDash_{\mathsf{c}} N : \underline{C}(x)}{\Gamma, x\!:\!A \vDash_{\mathsf{c}} \langle x, N\rangle : \Sigma y\!:\!A.\underline{C}(y)}}{\Gamma \vDash_{\mathsf{c}} M \;\texttt{to}\; x\!:\!A \;\texttt{in}\; \langle x, N\rangle : \Sigma y\!:\!A.\underline{C}(y)}$$

The seq. comp. rule for λ_{ML} is justified by the type isomorphism

$$\frac{\Gamma \vdash A \qquad \Gamma, x\!:\!A \vdash B(x)}{\Gamma \vdash U(\Sigma x\!:\!A.F(B)) \cong UF(\Sigma x\!:\!A.B) = T(\Sigma x\!:\!A.B)}$$

Categorical semantics of eMLTT

(fibrations + adjunctions)

Categorical semantics – MLTT part

We define fibred adjunction models $(\mathcal{B}, \mathcal{V}, p, q, F \dashv U)$

- **Theorem:** a sound and complete class of models for eMLTT

given by: i) a split closed comprehension cat. p (with s. fib. 0, ...)

$$\begin{tikzcd}
\mathcal{V} \arrow[d, bend right=60, "p"'] \arrow[d, bend left=60, "\{-\}"] \\
\mathcal{B} \arrow[u, "1" description]
\end{tikzcd}$$

$\dashv$ $\dashv$

- we define a partial interpretation fun. $[\![-]\!]$, that (if defined) maps:
  - a context Γ to and object $[\![\Gamma]\!]$ in $\mathcal{B}$, with $[\![\Gamma, x : A]\!] \stackrel{\text{def}}{=} \{[\![\Gamma; A]\!]\}$
  - a context Γ and a value type A to an object $[\![\Gamma; A]\!]$ in $\mathcal{V}_{[\![\Gamma]\!]}$
  - a context Γ and a value term V to $[\![\Gamma; V]\!] : 1_{[\![\Gamma]\!]} \longrightarrow A$ in $\mathcal{V}_{[\![\Gamma]\!]}$

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- the display maps $\pi_{[\![\Gamma;A]\!]} : [\![\Gamma, x : A]\!] \longrightarrow [\![\Gamma]\!]$ in $\mathcal{B}$

  induce the weakening functors $\pi^*_{[\![\Gamma;A]\!]} : \mathcal{V}_{[\![\Gamma]\!]} \longrightarrow \mathcal{V}_{[\![\Gamma,x:A]\!]}$, and
- the value Σ- and Π-types are interpreted as adjoints

  $\Sigma_{[\![\Gamma;A]\!]} \dashv \pi^*_{[\![\Gamma;A]\!]} : \mathcal{V}_{[\![\Gamma]\!]} \longrightarrow \mathcal{V}_{[\![\Gamma,x:A]\!]}$ (such that Σ is strong)

  $\pi^*_{[\![\Gamma;A]\!]} \dashv \Pi_{[\![\Gamma;A]\!]} : \mathcal{V}_{[\![\Gamma,x:A]\!]} \longrightarrow \mathcal{V}_{[\![\Gamma]\!]}$

Categorical semantics – effects part

We define fibred adjunction models $(\mathcal{B}, \mathcal{V}, p, q, F \dashv U)$

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$$
\begin{tikzcd}
\mathcal{V} \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] \arrow[d, bend right=60, "p"'] \arrow[d, bend left=60, "\{-\}"] & & \mathcal{C} \arrow[ll, bend left, "U"] \arrow[dll, bend left, "q"] \\
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$$

- we extend $[\![-]\!]$ so that (if defined) it maps:
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  - a ctx. Γ and a comp. term M to $[\![\Gamma; M]\!] : 1_{[\![\Gamma]\!]} \longrightarrow U(\underline{C})$ in $\mathcal{V}_{[\![\Gamma]\!]}$
  - a ctx. Γ, a c. var. z, a c. type $\underline{C}$, and a hom. term K to
    $[\![\Gamma; z : \underline{C}; K]\!] : [\![\Gamma; \underline{C}]\!] \longrightarrow \underline{D}$ in $\mathcal{C}_{[\![\Gamma]\!]}$

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- we again have weakening functors $\pi^*_{[\![\Gamma;A]\!]} : \mathcal{C}_{[\![\Gamma]\!]} \longrightarrow \mathcal{C}_{[\![\Gamma,x:A]\!]}$, and
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Digression: dep. elimination of 0 and $+$

The coproduct type $A + B$: [Jacobs'99]

- require $p : \mathcal{V} \longrightarrow \mathcal{B}$ to have split fibred coproducts $A +_X B$, and
- $\langle \{\mathsf{inl}_A\}^*, \{\mathsf{inr}_B\}^* \rangle : \mathcal{V}_{\{A +_X B\}} \longrightarrow \mathcal{V}_{\{A\}} \times \mathcal{V}_{\{B\}}$ to be fully-faith.

- allows one to interpret dependent case analysis, i.e.,

$$\mathcal{V}_{\{A\}}\Big(1_{\{A\}}, \{\mathsf{inl}_A\}^*(C)\Big) \times \mathcal{V}_{\{B\}}\Big(1_{\{B\}}, \{\mathsf{inr}_B\}^*(C)\Big)$$
$$\cong$$
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provides semantics for

$$\frac{\Gamma, y_1 \mathbin{:} A \vDash W_1 : C[\mathtt{inl}_A\ y_1/x] \quad \Gamma, y_2 \mathbin{:} B \vDash W_2 : C[\mathtt{inr}_B\ y_2/x]}{\Gamma, x \mathbin{:} A + B \vDash \mathtt{case}\ x\ \mathtt{of}\ \big(\mathtt{inl}(y_1) \mapsto W_1, \mathtt{inr}(y_2) \mapsto W_2\big) : C}$$

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Digression: dep. elimination of colimits

A generalisation: [Ahman'17]

- **Idea:** fully-faith. for cocones $A \longrightarrow A \circledast_X B \longleftarrow B$ is enough, and we can generalise this to all split fibred colimits

- **Theorem:**
  - if for every object $X \in \mathcal{B}$ and diagram $J : \mathcal{D} \longrightarrow \mathcal{V}_X$ there exists a cocone $\underline{\mathsf{in}}^J : J \longrightarrow \Delta(\underline{\mathsf{colim}}(J))$ in $\mathcal{V}_X$,
  - such that $f^*(\underline{\mathsf{in}}^J_D) = \underline{\mathsf{in}}^{f^* \circ J}_D$, for any $f : X \longrightarrow Y$, and such that the unique mediating functor

    $$\langle \{\underline{\mathsf{in}}^J_D\}^*_{D \in \mathcal{D}} \rangle : \mathcal{V}_{\{\underline{\mathsf{colim}}(J)\}} \longrightarrow \lim(\widehat{J})$$

    is fully-faithful (for $\widehat{J} : \mathcal{D}^{op} \longrightarrow \mathsf{Cat}$, where $\widehat{J}(D) = \mathcal{V}_{\{J(D)\}}$),
  - then p has split fibred colimits of shape $\mathcal{D}$, and p supports dependent elimination for them (analogously to $+_X$)

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Examples of fibred adjunction models

$$\begin{tikzcd}
\mathcal{V} \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] \arrow[d, bend right, "p"'] \arrow[d, bend left, "\{-\}"] & & \mathcal{C} \arrow[ll, bend left, "U"] \arrow[dll, bend left, "q"] \\
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Examples of fibred adjunction models

Example 1 (identity adjunctions): $\mathsf{Id}_{\mathcal{V}} \dashv \mathsf{Id}_{\mathcal{V}} : \mathcal{V} \longrightarrow \mathcal{V}$

- $\mathcal{C} \stackrel{\text{def}}{=} \mathcal{V}$
- $q \stackrel{\text{def}}{=} p$
- $F \stackrel{\text{def}}{=} \mathsf{Id}_{\mathcal{V}}$
- $U \stackrel{\text{def}}{=} \mathsf{Id}_{\mathcal{V}}$

$$
\begin{tikzcd}
\mathcal{V} \arrow[rr, bend left=20, "\mathsf{Id}_{\mathcal{V}}"] \arrow[rr, phantom, "\perp"] \arrow[d, bend right=60, "p"'] \arrow[d, bend left=60, "\{-\}"] & & \mathcal{V} \arrow[ll, bend left=20, "\mathsf{Id}_{\mathcal{V}}"] \arrow[dll, bend left=20, "p"] \\
\mathcal{B} \arrow[u, "1\ \dashv"'] \arrow[u, phantom, shift left=4ex, "\dashv"] & &
\end{tikzcd}
$$

- **Note:** sound as long as we haven't included any effects

Examples of fibred adjunction models

Example 2 (models of Egger et al.'s EEC):

- given an adjunction $F_{\mathrm{EEC}} \dashv U_{\mathrm{EEC}} : \mathcal{E} \longrightarrow \mathcal{D}$, such that
  - $\mathcal{D}$ is a CCC (with 0, . . .), and
  - $F_{\mathrm{EEC}} \dashv U_{\mathrm{EEC}}$ and $\mathcal{E}$ are $\mathcal{D}$-enriched, and
  - $\mathcal{E}$ has all $\mathcal{D}$-tensors ($A \otimes \underline{C}$) and $\mathcal{D}$-cotensors ($A \Rightarrow \underline{C}$)
- we use the simple fibration $p : \mathrm{s}(\mathcal{D}) \longrightarrow \mathcal{D}$, where

$$p(X, A) \overset{\mathrm{def}}{=} X \qquad p(f, g) \overset{\mathrm{def}}{=} f \qquad \text{where} \quad f : X \longrightarrow Y \quad g : X \times A \longrightarrow B$$

- then, we define the simpl. $\mathcal{D}$-enrich. fib. $q : \mathrm{s}(\mathcal{D}, \mathcal{E}) \longrightarrow \mathcal{D}$, where

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- finally, we define $F \dashv U$ as the lifting of $F_{\mathrm{EEC}} \dashv U_{\mathrm{EEC}}$

$$F(X, A) \overset{\mathrm{def}}{=} (X, F_{\mathrm{EEC}}(A)) \qquad U(X, \underline{C}) \overset{\mathrm{def}}{=} (X, U_{\mathrm{EEC}}(\underline{C}))$$

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Examples of fibred adjunction models

Example 3 (families fibrations):

- given an adjunction $F_{\mathcal{D}} \dashv U_{\mathcal{D}} : \mathcal{D} \longrightarrow \mathsf{Set}$, such that
  - $\mathcal{D}$ has set-indexed products and coproducts

- we use the families of sets fibration $p : \mathsf{Fam}(\mathsf{Set}) \longrightarrow \mathsf{Set}$

$$p(X, A) \overset{\mathrm{def}}{=} X \qquad p(f, \{g_x\}_{x \in X}) \overset{\mathrm{def}}{=} f$$

  where

$$X, Y \in \mathsf{Set} \qquad A : X \longrightarrow \mathsf{Set} \qquad B : Y \longrightarrow \mathsf{Set}$$

$$f : X \longrightarrow Y \qquad g_x : A(x) \longrightarrow (B \circ f)(x)$$

- analogously, we use the $\mathcal{D}$-valued fam. fib. $q : \mathsf{Fam}(\mathcal{D}) \longrightarrow \mathsf{Set}$

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- **Ex.:** EM-cats. ($\mathcal{D} \stackrel{\text{def}}{=} \mathsf{Set}^{\mathbf{T}}$) and Lawvere ths. ($\mathcal{D} \stackrel{\text{def}}{=} \mathsf{Mod}(\mathcal{L}, \mathsf{Set})$)

Examples of fibred adjunction models

Example 3 (families fibrations):

- given an adjunction $F_{\mathcal{D}} \dashv U_{\mathcal{D}} : \mathcal{D} \longrightarrow \mathrm{Set}$, such that
  - $\mathcal{D}$ has set-indexed products and coproducts

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Example 4 (continuous families of cpos for $\mu\, x\!:\!U\underline{C}.M$):

- given a CPO-enriched monad $\mathbf{T}$ on CPO, such that
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  - consider the epimorphism $e \stackrel{\text{def}}{=} n \mapsto n : \mathbb{N}_{=} \longrightarrow \mathbb{N}_{\omega}$ in CPO, and
  - assume given a non-empty cpo X, and
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  - then we have the following situation

$$\begin{tikzcd}
(\emptyset, =) \arrow[r, "f_\omega^*(e)"] \arrow[dr, bend right] & (\{\langle x, \omega\rangle \mid x \in |X|\}, \leq) \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] \arrow[d] & Y \arrow[dl, bend left, "h"] \\
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Example 5 (EM-resolutions of split fibred monads):

- given a split fibred monad $\mathbf{T} = (T, \eta, \mu)$ on p, i.e.,

$$\begin{tikzcd} \mathcal{V} \arrow[rr, "T"] \arrow[dr, "p"'] & & \mathcal{V} \arrow[dl, "p"] \\ & \mathcal{B} & \end{tikzcd} \qquad \text{and} \qquad p(\eta_A) = \mathrm{id}_{p(A)} \qquad p(\mu_A) = \mathrm{id}_{p(A)}$$

- we consider models based on the EM-resolution of $\mathbf{T}$

$$\begin{tikzcd} \mathcal{V} \arrow[rr, bend left, "F^{\mathbf{T}}"] \arrow[rr, phantom, "\perp"] \arrow[dr, "p"'] & & \mathcal{V}^{\mathbf{T}} \arrow[ll, bend left, "U^{\mathbf{T}}"] \arrow[dl, "p^{\mathbf{T}}"] \\ & \mathcal{B} & \end{tikzcd}$$

- and show that three familiar results hold for this situation

Examples of fibred adjunction models

Example 5 (EM-resolutions of split fibred monads):

- **Theorem 1:** if p supports Π-types, then $p^{\mathbf{T}}$ also supports Π-types

- **Theorem 2:** if p supports Σ-types and the dependent strength

$$\sigma_A : \Sigma_A \circ T \longrightarrow T \circ \Sigma_A$$

  is a natural isomorphism, then $p^{\mathbf{T}}$ also supports Σ-types

- **Theorem 3:** if p supports Σ-types and $p^{\mathbf{T}}$ has split fibred reflexive coequalizers, then $p^{\mathbf{T}}$ also supports Σ-types

(for corresponding simply typed results, see [Borceux'94] and [Linton'69])

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Algebraic effects

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Fibred effect theories $\mathcal{T}_{\mathsf{eff}}$:

- signatures of dep. typed operation symbols

$$\frac{\cdot \vdash I \qquad x_i : I \vdash O \qquad I \text{ and } O \text{ are pure value types}}{\mathsf{op} : (x_i : I) \longrightarrow O}$$

- equipped with equations on derivable effect terms

In eMLTT:

$$M ::= \ldots \mid \mathsf{op}^{\underline{C}}_{V}(x.M)$$

General algebraicity equations (in addition to eff. th. eqs.):

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- $\mathsf{read} : 1 \longrightarrow \mathsf{Chr}$ $\qquad (\mathsf{Chr} \stackrel{\text{def}}{=} 1 + \ldots + 1)$

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- no equations

Example 2 (global state with location-dependent store type):

- $\diamond \vdash \mathsf{Loc}$

  $x : \mathsf{Loc} \vdash \mathsf{Val}$

  $\diamond \vdash_{\mathsf{v}} \mathsf{isDec}_{\mathsf{Loc}} : \Pi x : \mathsf{Loc} \,.\, \Pi y : \mathsf{Loc} \,.\, (x =_{\mathsf{Loc}} y) + (x =_{\mathsf{Loc}} y \to 0)$

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- five equations (two of them branching on $\mathsf{isDec}_{\mathsf{Loc}}$)

Example 3 (dep. typed update monads $T\,X \stackrel{\text{def}}{=} \Pi_{s:S} \,.\, P\,s \times X$)

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- five equations (two of them branching on $\mathsf{isDec}_{\mathsf{Loc}}$)

Example 3 (dep. typed update monads $T\,X \stackrel{\text{def}}{=} \Pi_{s:S} \,.\, P\,s \times X$)

Handlers of algebraic effects
(for programming and extrinsic reasoning)

Handlers of alg. effects (for programming)

Idea: Generalisation of exception handlers [Plotkin,Pretnar'09]

Handler = Algebra and Handling = Homomorphism

Usual term-level presentation:

$$\frac{\Gamma \models_{\mathsf{c}} M : FA \qquad \Gamma, x_v : I, x_k : O[x_v/x_i] \to U\underline{C} \models_{\mathsf{c}} N_{\mathsf{op}} : \underline{C} \qquad \Gamma, y : A \models_{\mathsf{c}} N_{\mathsf{ret}} : \underline{C}}{\Gamma \models_{\mathsf{c}} M \ \mathtt{handled\ with}\ \{\mathsf{op}_{x_v}(x_k) \mapsto N_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{T}_{\mathsf{eff}}}\ \mathtt{to}\ y : A\ \mathtt{in}_{\underline{C}}\ N_{\mathsf{ret}} : \underline{C}}$$

satisfying

$$(\mathtt{return}\ V)\ \mathtt{handled\ with}\ \{\ldots\}_{\mathsf{op} \in \mathcal{T}_{\mathsf{eff}}}\ \mathtt{to}\ y : A\ \mathtt{in}\ N_{\mathsf{ret}} = N_{\mathsf{ret}}[V/x]$$

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Typical use case for programming:

- write your programs using alg. ops. (e.g., `get` and `put`)
- use handlers to provide fit-for-purpose impl. (e.g., $S \to X \times S$)

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Handlers of alg. effects (for reasoning)

Idea: Assuming we were able to handle into values

$$M \; \texttt{handled with} \; \{\mathtt{op}_{x_v}(x_k) \mapsto V_{\mathtt{op}}\}_{\mathtt{op} \in \mathcal{T}_{\mathsf{eff}}} \; \texttt{to} \; y : A \; \texttt{in}_{B} \; V_{\mathsf{ret}}$$

we can define predicates (essentially, dependent types)

$$\Gamma \vDash_{\mathsf{v}} P : UFA \to \mathcal{U}$$

by

- equipping a universe $\mathcal{U}$ with an algebra for $\mathcal{T}_{\mathsf{eff}}$, and
- using the above handle-into-values construct to define P

Note 1: $P\,(\texttt{thunk}\,M)$ computes a proof obligation for M

Note 2: Formally, we work in an extension of eMLTT with

- a universe $\mathcal{U}$ closed under $\mathsf{Nat}, 1, 0, +, \Sigma, \Pi$
- a type-based treatment of handlers $\quad \underline{C} ::= \ldots \mid \langle A; \overrightarrow{V_{\mathsf{op}}}; \overrightarrow{W_{\mathsf{eq}}} \rangle$
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Handlers of alg. effects (for reasoning)

Example 1 (Evaluation Logic style modalities):

- Given a predicate $P : A \to \mathcal{U}$ on return values,

  we define a predicate $\Diamond P : UFA \to \mathcal{U}$ on I/O-computations as

$$\Diamond P \stackrel{\text{def}}{=} \lambda\, x \!:\! UFA \,.\, (\mathtt{force}\, x)\ \mathtt{handled\ with}\ \{\ldots\}_{\mathsf{op} \in \mathcal{T}_{\mathsf{IO}}}\ \mathtt{to}\ y \!:\! A\ \mathtt{in}_{\mathcal{U}}\ P\, y$$

  using the handler given by

$$V_{\mathsf{read}} \stackrel{\text{def}}{=} \lambda\, x \!:\! (\Sigma\, x_v \!:\! 1 \,.\, \mathsf{Chr} \to \mathcal{U}) \,.\, \widehat{\Sigma}\, y \!:\! \mathsf{El}(\widehat{\mathsf{Chr}}) \,.\, (\mathtt{snd}\ x)\ y$$

$$V_{\mathsf{write}} \stackrel{\text{def}}{=} \lambda\, x \!:\! (\Sigma\, x_v \!:\! \mathsf{Chr} \,.\, 1 \to \mathcal{U}) \,.\, (\mathtt{snd}\ x)\ \star$$

- $\Diamond P$ is the possibility modality

$$\Diamond P\ (\mathtt{thunk}\,(\mathtt{read}(x \,.\, \mathtt{write}_{\text{'}e\text{'}}(\mathtt{return}\, V)))) = \widehat{\Sigma}\, x \!:\! \mathsf{El}(\widehat{\mathsf{Chr}}) \,.\, P\ V$$

- To get the necessity modality, use $\widehat{\Pi}\, x \!:\! \mathsf{El}(\widehat{\mathsf{Chr}})$ in V_{read}

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Handlers of alg. effects (for reasoning)

Example 2 (Dijkstra's weakest precondition semantics):

- Given a postcondition on return values and final states

$$Q : A \to S \to \mathcal{U} \qquad\qquad (S \stackrel{\text{def}}{=} \Pi x : \mathrm{Loc} \,.\, \mathrm{Val})$$

  we define a precondition for stateful comps. on initial states

$$\mathrm{wp}_Q : UFA \to S \to \mathcal{U}$$

  by

  i) handling the given comp. into a state-passing function using

$$V_{\mathrm{get}}, V_{\mathrm{put}} \quad \text{on} \quad S \to (\mathcal{U} \times S) \qquad \text{and} \qquad V_{\mathrm{ret}} \ \text{“} = \text{”} \ Q$$

  ii) feeding in the initial state; and *iii*) projecting out $\mathcal{U}$

- **Theorem:** wp_Q satisfies expected properties of WPs, e.g.,

$$\mathrm{wp}_Q\, (\mathtt{thunk}\, (\mathtt{return}\, V)) \quad = \quad \lambda\, x_S : S \,.\, Q\ V\ x_S$$

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Example 3 (Patterns of allowed effects):

- Assuming an inductive type Protocol, given by

$$\mathtt{e} : \mathsf{Protocol} \qquad \mathtt{r} : (\mathsf{Chr} \to \mathsf{Protocol}) \to \mathsf{Protocol}$$

$$\mathtt{w} : (\mathsf{Chr} \to \mathcal{U}) \to \mathsf{Protocol} \to \mathsf{Protocol}$$

  and potentially also by $\wedge$, $\vee$, ...

- Then, given a protocol $\mathsf{Pr} : \mathsf{Protocol}$, we define

$$\underline{\mathsf{Pr}} : \mathit{UFA} \to \mathcal{U}$$

  by handling the given comp. using

$$V_{\mathsf{read}} \, , V_{\mathsf{write}} \qquad \text{on} \qquad \mathsf{Protocol} \to \mathcal{U}$$

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Conclusion

In work we told a mathematically natural story of combining

- dependent types and computational effects

In particular, we saw

- a clean core language of dependent types and comp. effects
- a natural category-theoretic semantics
- alg. effects and handlers, in particular, for reasoning using
  - Evaluation Logic style modalities
  - Dijkstra's weakest precondition semantics
  - patterns of allowed (I/O)-effects

Future work involves

- type-dependency on computations
- local effects
- more expressive computation types

Thank you!

D. Ahman.
Fibred Computational Effects. (PhD Thesis, 2017)

D. Ahman, N. Ghani, G. Plotkin.
Dependent Types and Fibred Computational Effects. (FoSSaCS'16)

D. Ahman.
Handling Fibred Computational Effects. (POPL'18)

Future work (type-dependency on comps.)

- How to accommodate $\underline{D}(\mathtt{read}(x.M))$
- That is, how to avoid restricting the typing of seq. comp.?
- $M \ \mathtt{to}\ x\!:\!A \ \mathtt{in}\ N : \underline{C}[\mathtt{thunk}\, M/y]$ (where $y\!:\!UFA$) [Vákár'17]
- $\alpha : \widehat{T}(\mathcal{U}_{\mathsf{comp}}) \longrightarrow \mathcal{U}_{\mathsf{comp}}$ [Pédrot, Tabareau'17]
- for eMLTT, one possible way forward
  - **i)** build on Vákár's proposal
  - **ii)** but force type-dep. to be homomorphic
    - $\underline{D}[\mathtt{thunk}\,(M \ \mathtt{to}\ x\!:\!A \ \mathtt{in}\ N)/y] = M \ \mathtt{to}\ x\!:\!A \ \mathtt{in}\ \underline{D}[\mathtt{thunk}\, N/y]$
    - $\underline{D}[M \ \mathtt{to}\ x\!:\!A \ \mathtt{in}\ N/z] = M \ \mathtt{to}\ x\!:\!A \ \mathtt{in}\ \underline{D}[N/z]$