# A fibrational view on computational effects

**(an overview of my PhD thesis)**

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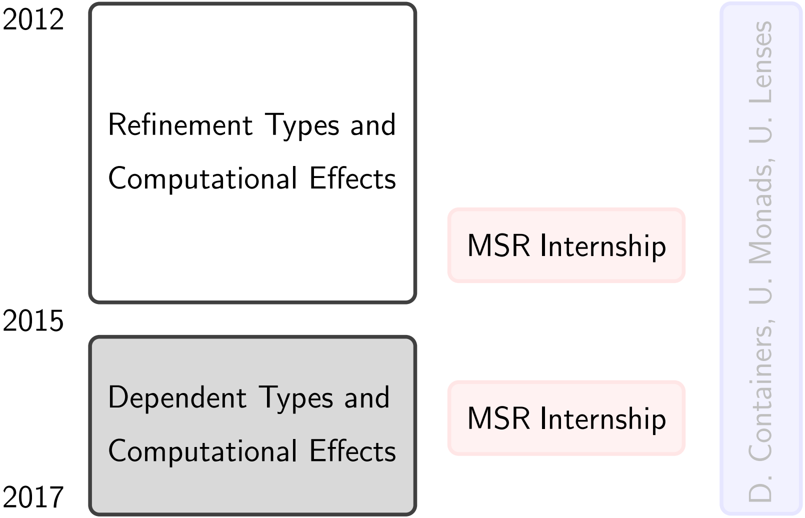
# Overview – how my PhD journey looked like



# Overview – dependent types

**The Curry-Howard correspondence:**

| | | | |
|---|---|---|---|
| Simple Types | $\sim$ | Propositional Logic | $(\text{Nat}, \text{String}, \ldots)$ |
| Dependent Types | $\sim$ | Predicate Logic | $(\Sigma, \Pi, =, \ldots)$ |

**A tiny example:** we can use dep. types to express sorted lists

$$\ell : (\text{List Nat}) \vdash \text{Sorted}(\ell) \stackrel{\text{def}}{=} \overset{\forall}{\Pi}\, i : \text{Nat}\, .\, (0 < i < \texttt{len}\ \ell) \overset{\Rightarrow}{\to} (\ell[i\text{-}1] \leq \ell[i])$$

which in turn could be used to type a sorting function

$$\texttt{sort} : \overset{\forall}{\Pi}\, \ell : (\text{List Nat})\, .\, \overset{\exists}{\Sigma}\, \ell' : (\text{List Nat})\, .\, \Big(\text{Sorted}(\ell') \overset{\wedge}{\times} \ldots\Big)$$

**Large examples:** CompCert (Coq), miTLS and HACL* (F*), ...

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# Overview – computational effects

**Examples:**

- state
- exceptions
- nondeterminism
- interactive IO
- ...

**Meta-languages and models:**

- based on monads $(T, \eta, \mu)$ (Moggi)
- based on adjunctions (Levy)

$$\begin{tikzcd} \mathcal{V} \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] & & \mathcal{C} \arrow[ll, bend left, "U"] \end{tikzcd}$$

- based on algebraic presentations (Plotkin and Power)

  get : $1 \rightharpoonup S$ put : $S \rightharpoonup 1$ + equations

# Overview – putting the two together

**We investigate the combination of**

- dependent types $(\Pi, \Sigma, V =_A W, \ldots)$
- computational effects (state, nondeterminism, IO, ...)

**Two guiding problems**

- effectful programs in types (e.g., `get` and `put` in types)
- types of effectful programs (e.g., of sequential composition)

**Our goals**

- tell a mathematically natural story
- use established math. techniques
- cover a wide range of comp. effects
- discover smth. interesting

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- cover a wide range of comp. effects (alg. effects, continuations)
- discover smth. interesting (using handlers to reason about effects)

# Effectful programs in types

**(type-dependency in the presence of effects)**

# Effectful programs in types

**Q:** Should we allow situations such as $\mathsf{Sorted}[\,\mathtt{receive}(y.\,M)/\ell\,]$?

**A1:** In this work, we say not directly

- types should only depend on static information about effects
- we allow dependency on effectful comps. via analysing thunks

**A2:** But we are also looking into the direct case

- type-dependency needs to be "homomorphic", but not only so
- intuitively, lift $\mathsf{Sorted}(\ell)$ to $\mathsf{Sorted}^{\dagger}(c)$, where $c : T(\mathsf{List}\ \mathsf{Chr})$

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# Effectful programs in types

**Aim:** Types should only depend on static info about effects

**Solution:** CBPV/EEC style distinction between vals. and comps.

- value types $\Gamma \vdash A$ (MLTT + thunks + ...)
- computation types $\Gamma \vdash \underline{C}$ (dep. typed CBPV/EEC)
- where $\Gamma$ contains only value variables $x_1 : A_1, \ldots, x_n : A_n$

Could have also considered Moggi's $\lambda_{\mathrm{ML}}$ and Levy's FGCBV

- building on CBPV/EEC gives a more general story
- especially for the treatment of sequential composition
- and also for integrating dependent- and effect-typing (ongoing)

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# Assigning types to effectful programs

**(e.g., sequential composition)**

# Assigning types to effectful programs

**The problem:** The standard typing rule for seq. composition

$$\frac{\Gamma \vDash_{c} M : F\,A \qquad \Gamma, x\!:\!A \vDash_{c} N : \underline{C}}{\Gamma \vDash_{c} M \ \texttt{to}\ x\!:\!A\ \texttt{in}\ N : \underline{C}}$$

is not correct any more because $x$ can appear free in the type

$$\underline{C}$$

in the conclusion

# Assigning types to effectful programs

**Aim:** To fix the typing rule of sequential composition

**Option 1:** We could restrict the free variables in $\underline{C}$: [Levy'04]

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**But:** Sometimes it is useful if $\underline{C}$ can depend on $x$!

- say we consider

$$\texttt{fopen}\,(\texttt{return true}, \texttt{return false}) \texttt{ to } x\!:\!\mathsf{Bool} \texttt{ in } N$$

- then it would be natural to let $\underline{C}$ depend on $x$, e.g.,

$$x\!:\!\mathsf{Bool} \vdash \underline{C}(x) \overset{\text{def}}{=} \begin{array}{l}\texttt{if } x \texttt{ then } \text{“allow } \texttt{fread}, \texttt{fwrite}, \text{ and } \texttt{fclose}\text{”} \\ \qquad\qquad\quad \texttt{else } \text{“allow } \texttt{fopen}\text{”}\end{array}$$

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**But:** Then comp. types would be singleton-like!?!

However, smth. like this is probably needed for the direct case.

**Option 3:** In the monadic metalanguage $\lambda_{\mathsf{ML}}$, one could try

$$\frac{\Gamma \vdash M : T\, A \qquad \Gamma, x{:}A \vdash N : T\, B(x)}{\Gamma \vdash M \ \mathtt{to}\ x{:}A \ \mathtt{in}\ N : T\, (\Sigma x : A . B)}$$

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- and combine it with restricting $\underline{C}$ in seq. comp. (**Option 1**)

E.g., consider the non-deterministic prog. $\Big(\text{for } x\,{:}\,\mathsf{Nat} \vDash_{\mathsf{c}} N : \underline{C}(x)\Big)$

$$M \quad \stackrel{\text{def}}{=} \quad \mathtt{choose}\,\big(\mathtt{return}\,4, \mathtt{return}\,2\big)\;\mathtt{to}\;x\,{:}\,\mathsf{Nat}\;\mathtt{in}\;N$$

After making the non-det. choice, this program evaluates as either

$$N[4/x] : \underline{C}[4/x] \qquad \text{or} \qquad N[2/x] : \underline{C}[2/x]$$

**Idea:** $M$ denotes an element of the coproduct of algebras

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# Putting these ideas together

**(eMLTT: a core dep.-typed language with comp. effects)**

# eMLTT – value and comp. types

**Value types:** MLTT + thunks + ...

$$A, B ::= \mathsf{Nat} \mid 1 \mid 0 \mid \Pi\, x{:}A\,.\,B \mid \Sigma\, x{:}A\,.\,B \mid V =_A W \mid U\underline{C} \mid \ldots$$

- $U\underline{C}$ is the type of thunked (i.e., suspended) computations

**Computation types:** dep.-typed version of EEC's comp. types

$$\underline{C}, \underline{D} ::= F A \mid \Pi\, x{:}A\,.\,\underline{C} \mid \Sigma\, x{:}A\,.\,\underline{C}$$

- $F A$ is the type of computations returning values of type $A$
- $\Pi\, x{:}A\,.\,\underline{C}$ is the type of dependent effectful functions
  - generalises CBPV/EEC's comp. types $A \to \underline{C}$ and $\underline{C} \times \underline{D}$
- $\Sigma\, x{:}A\,.\,\underline{C}$ is the type of dep. pairs of values and effectful comps.
  - captures the intuition about seq. comp. and coprods. of algebras
  - generalises EEC's comp. types $!A \otimes \underline{C}$ and $\underline{C} \oplus \underline{D}$

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# eMLTT – value and comp. terms

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$$V, W \ ::= \ x \ | \ \texttt{zero} \ | \ \texttt{succ}\, V \ | \ \ldots \ | \ \texttt{thunk}\ M \ | \ \ldots$$

- equational theory based on intensional MLTT

**Comp. terms:** dep.-typed version of CBPV/EEC's comp. terms

$$\begin{array}{rcll} M, N & ::= & \texttt{force}\ V & \\ & | & \texttt{return}\ V & \\ & | & M \ \texttt{to}\ x\!:\!A\ \texttt{in}\ N & \\ & | & \lambda\, x\!:\!A\,.\,M & \\ & | & M V & \\ & | & \langle V, M \rangle & \text{(comp. } \Sigma \text{ intro.)} \\ & | & M \ \texttt{to}\ \langle x\!:\!A, z\!:\!\underline{C} \rangle\ \texttt{in}\ K & \text{(comp. } \Sigma \text{ elim.)} \end{array}$$

**But:** Value and comp. terms alone do not suffice, as in EEC!

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& \mid & \lambda\, x \!:\! A \,.\, M & \\
& \mid & M V & \\
& \mid & \langle V, M \rangle & \text{(comp. } \Sigma \text{ intro.)} \\
& \mid & M\ \texttt{to}\ \langle x \!:\! A, z \!:\! \underline{C} \rangle\ \texttt{in}\ K & \text{(comp. } \Sigma \text{ elim.)}
\end{array}$$

**But:** Value and comp. terms alone do not suffice, as in EEC!

# eMLTT – homomorphism terms

**Note:** We need to define $K$ in such a way that the intended left-to-right evaluation order is preserved, e.g., consider

$$\Gamma \vDash_{\mathsf{c}} \langle V, M \rangle \; \mathtt{to} \; \langle x : A, z : \underline{C} \rangle \; \mathtt{in} \; K = K[V/x, M/z] : \underline{D}$$

**Homomorphism terms:** dep.-typed version of EEC's linear terms

$$\begin{array}{rcll} K, L & ::= & z & \text{(linear comp. vars.)} \\ & \mid & K \; \mathtt{to} \; x : A \; \mathtt{in} \; M & \\ & \mid & \lambda \, x : A \, . \, K & \\ & \mid & K \, V & \\ & \mid & \langle V, K \rangle & \text{(comp. } \Sigma \text{ intro.)} \\ & \mid & K \; \mathtt{to} \; \langle x : A, z : \underline{C} \rangle \; \mathtt{in} \; L & \text{(comp. } \Sigma \text{ elim.)} \end{array}$$

**Typing judgments:**

- $\Gamma \vDash_{\mathsf{v}} V : A$
- $\Gamma \vDash_{\mathsf{c}} M : \underline{C}$
- $\Gamma \mid z : \underline{C} \vDash_{\mathsf{h}} K : \underline{D}$ (linear in $z$; comp. bound to $z$ happens first)

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# eMLTT – typing sequential composition

We can then account for type-dependency in seq. comp. as

$$\frac{\Gamma \vDash_{\mathsf{c}} M : F\,A \qquad \Gamma \vdash \Sigma\, x\!:\!A\,.\,\underline{C}(x) \qquad \dfrac{\Gamma, x\!:\!A \vDash_{\mathsf{c}} N : \underline{C}(x)}{\Gamma, x\!:\!A \vDash_{\mathsf{c}} \langle x, N\rangle : \Sigma\, x\!:\!A\,.\,\underline{C}(x)}}{\Gamma \vDash_{\mathsf{c}} M \;\mathtt{to}\; x\!:\!A \;\mathtt{in}\; \langle x, N\rangle : \Sigma\, x\!:\!A\,.\,\underline{C}(x)}$$

The seq. comp. rule for $\lambda_{\mathsf{ML}}$ is justified by the type isomorphism

$$\frac{\Gamma \vdash A \qquad \Gamma, x\!:\!A \vdash B(x)}{\Gamma \vdash U(\Sigma\, x\!:\!A\,.\,F\,B) \cong UF(\Sigma\, x\!:\!A\,.\,B) = T(\Sigma\, x\!:\!A\,.\,B)}$$

# Categorical semantics of eMLTT

## (fibrations + adjunctions)

# Fibred adjunction models – value part

Given by a split closed comprehension category $p$, as in

$$
\begin{tikzcd}
\mathcal{V} \arrow[d, bend right=60, "p"'] \arrow[d, bend left=60, "\{-\}"] \\
\mathcal{B} \arrow[u, "1"' near start, "\dashv"' near end, "\dashv" near end]
\end{tikzcd}
$$

allowing us to define a partial interpretation fun. $[\![-]\!]$, that maps:

- a context $\Gamma$ to and object $[\![\Gamma]\!]$ in $\mathcal{B}$, with
  - $[\![\diamond]\!] \stackrel{\text{def}}{=} 1$
  - $[\![\Gamma, x\!:\!A]\!] \stackrel{\text{def}}{=} \{[\![\Gamma; A]\!]\}$ (if $x \notin \mathit{Vars}(\Gamma)$ and $[\![\Gamma; A]\!]$ is defined)
- a context $\Gamma$ and a value type $A$ to an object $[\![\Gamma; A]\!]$ in $\mathcal{V}_{[\![\Gamma]\!]}$
- a context $\Gamma$ and a value term $V$ to $[\![\Gamma; V]\!] : 1_{[\![\Gamma]\!]} \longrightarrow A$ in $\mathcal{V}_{[\![\Gamma]\!]}$

# Fibred adjunction models – value part

Given by a split closed comprehension category $p$, as in

$$\begin{tikzcd}
\mathcal{V} \arrow[d, bend right=50, "p"'] \arrow[d, bend left=50, "\{-\}"] \arrow[d, phantom, shift right=4, "\dashv"] \arrow[d, phantom, shift left=4, "\dashv"] \\
\mathcal{B} \arrow[u, "1"']
\end{tikzcd}$$

such that

- $p$ has split fibred strong colimits of shape $\mathbf{0}$ and $\mathbf{2}$ [Jacobs'99]
  - (in thesis, also Jacobs-style axiomatisation for arbitrary shapes)
- $p$ has weak split fibred strong natural numbers
  - (axiomatisation is given in the style of fibrational induction)
- $p$ has split intensional propositional equality
  - (currently very synthetic ax., would like a weak form of adjoints)

# Fibred adjunction models – effects part

Given by a split fibration $q$ and a split fib. adjunction $F \dashv U$, as in

$$\begin{tikzcd}
\mathcal{V} \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] \arrow[d, bend right=60, "p"'] \arrow[d, bend left=60, "\{-\}"] & & \mathcal{C} \arrow[ll, bend left, "U"] \arrow[dll, bend left, "q"] \\
\mathcal{B} \arrow[u, "1"'] & &
\end{tikzcd}$$

we extend the partial interpretation fun. $[\![-]\!]$ so that it maps:

- a ctx. $\Gamma$ and a comp. type $\underline{C}$ to an object $[\![\Gamma; \underline{C}]\!]$ in $\mathcal{C}_{[\![\Gamma]\!]}$
- a ctx. $\Gamma$ and a comp. term $M$ to $[\![\Gamma; M]\!] : 1_{[\![\Gamma]\!]} \longrightarrow U(\underline{C})$ in $\mathcal{V}_{[\![\Gamma]\!]}$
- a ctx. $\Gamma$, a comp. var. $z$, a comp. type $\underline{C}$, and a hom. term $K$ to $$[\![\Gamma; z : \underline{C}; K]\!] : [\![\Gamma; \underline{C}]\!] \longrightarrow \underline{D} \text{ in } \mathcal{C}_{[\![\Gamma]\!]}$$

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$$

such that

- $q$ has split dependent $p$-products (comp. $\Pi$-type; r. adj. to wk.)
- $q$ has split dependent $p$-coproducts (comp. $\Sigma$-type; l. adj. to wk.)

and to account for the full calculus presented in the thesis,

- $q$ admits split fibred pre-enrichment in $p$ (hom. function type $\multimap$)

# Fibred adjunction models – correctness

**Theorem** (Soundness):

- If $\Gamma \vdash \underline{C}$, then $[\![\Gamma; \underline{C}]\!] \in \mathcal{C}_{[\![\Gamma]\!]}$
- If $\Gamma \vDash_{\mathsf{c}} M : \underline{C}$, then $[\![\Gamma; M]\!] : 1_{[\![\Gamma]\!]} \longrightarrow U([\![\Gamma; \underline{C}]\!])$
- If $\Gamma \,|\, z : \underline{C} \vDash_{\mathsf{h}} K : \underline{D}$, then $[\![\Gamma; z : \underline{C}; K]\!] : [\![\Gamma; \underline{C}]\!] \longrightarrow [\![\Gamma; \underline{D}]\!]$
- If $\Gamma \vdash \underline{C} = \underline{D}$, then $[\![\Gamma; \underline{C}]\!] = [\![\Gamma; \underline{D}]\!] \in \mathcal{C}_{[\![\Gamma]\!]}$
- ...

**Theorem** (Classifying model):

- The well-formed syntax of eMLTT forms a fib. adjunction model.

**Theorem** (Completeness):

- If two types or terms are equal in all fibred adjunction models, then they are also equal in the equational theory of eMLTT.

# Examples of fibred adjunction models

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# Examples of fibred adjunction models

**Example 1** (identity adjunctions):

- sound as long as we haven't included any actual comp. effects

**Example 2** (simple fibrations from enriched adj. models of EEC):

- doesn't support any real type dependency (constant families)

**Example 3** (families fibrations and lifting of adjunctions):

- $(\llbracket \Gamma \rrbracket, \llbracket A \rrbracket) \in \mathrm{Fam}(\mathrm{Set})$ (where $\llbracket A \rrbracket \in \llbracket \Gamma \rrbracket \longrightarrow \mathrm{Set}$)
- $(\llbracket \Gamma \rrbracket, \llbracket \underline{C} \rrbracket) \in \mathrm{Fam}(\mathcal{D})$ (where $\llbracket \underline{C} \rrbracket \in \llbracket \Gamma \rrbracket \longrightarrow \mathcal{D}$)

**Example 4** (continuous families and CPO-enriched monads):

- $(\llbracket \Gamma \rrbracket, \llbracket A \rrbracket) \in \mathrm{CFam}(\mathrm{CPO})$ (where $\llbracket A \rrbracket \in \llbracket \Gamma \rrbracket \longrightarrow \mathrm{CPO}^{EP}$)
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# Examples of fibred adjunction models

**Example 5** (EM-resolutions of split fibred monads):

- given a split fibred monad $\mathbf{T} = (T, \eta, \mu)$ on $p$, i.e.,

$$\begin{tikzcd} \mathcal{V} \arrow[rr, "T"] \arrow[dr, "p"'] & & \mathcal{V} \arrow[dl, "p"] \\ & \mathcal{B} & \end{tikzcd}$$

and $\quad p(\eta_A) = \mathrm{id}_{p(A)} \qquad p(\mu_A) = \mathrm{id}_{p(A)}$

- we consider models based on the EM-resolution of $\mathbf{T}$

$$\begin{tikzcd} \mathcal{V} \arrow[rr, bend left, "F^{\mathbf{T}}"] \arrow[rr, phantom, "\perp"] \arrow[dr, "p"'] & & \mathcal{V}^{\mathbf{T}} \arrow[ll, bend left, "U^{\mathbf{T}}"] \arrow[dl, "p^{\mathbf{T}}"] \\ & \mathcal{B} & \end{tikzcd}$$

where $\quad \Big(A \in \mathcal{V}\ ,\ \alpha : T(A) \longrightarrow A\Big) \in \mathcal{V}^{\mathbf{T}}$

- and show that three familiar results hold for this situation

# Examples of fibred adjunction models

**Example 5** (EM-resolutions of split fibred monads):

- **Theorem 1:** If $p$ supports $\Pi$-types, then $p^{\mathbf{T}}$ also supports $\Pi$-types

$$\Pi^{\mathbf{T}}_A(B,\beta) \stackrel{\text{def}}{=} \left(\Pi_A(B), \beta_{\Pi^{\mathbf{T}}_A}\right)$$

- **Prop.:** If $p$ supports $\Sigma$-types, then $\mathbf{T}$ has a dependent strength

$$\sigma_A : \Sigma_A \circ T \longrightarrow T \circ \Sigma_A \qquad\qquad (A \in \mathcal{V})$$

- **Theorem 2:** If $\sigma_A$ are natural isos., then $p^{\mathbf{T}}$ supports $\Sigma$-types

$$\Sigma^{\mathbf{T}}_A(B,\beta) \stackrel{\text{def}}{=} \left(\Sigma_A(B), \beta_{\Sigma^{\mathbf{T}}_A}\right)$$

- **Theorem 3:** If $p$ supports $\Sigma$-types and $p^{\mathbf{T}}$ has split fibred reflexive coequalizers, then $p^{\mathbf{T}}$ also supports $\Sigma$-types

$$\Sigma^{\mathbf{T}}_A(B,\beta) \stackrel{\text{def}}{=} F^{\mathbf{T}}(\Sigma_A(B))_{/\equiv}$$

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# Examples of fibred adjunction models

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# Algebraic effects
## (operations and equations)

# Algebraic effects – ops. and eqs.

**Fibred effect theories $\mathcal{T}_{\mathsf{eff}}$:**

- signatures of dependently typed operation symbols

$$\frac{\cdot \vdash I \qquad x_i : I \vdash O \qquad I \text{ and } O \text{ are pure value types}}{\mathsf{op} : (x_i : I) \rightharpoonup O}$$

- equipped with equations on derivable effect terms

**In eMLTT:**

$$M \;::=\; \ldots \;\mid\; \mathsf{op}^{\underline{C}}_{V}(x.M)$$

**General algebraicity equations** (in addition to eff. th. eqs.):

$$\frac{\Gamma \vDash_{\mathsf{v}} V : I \quad \Gamma, x : O[V/x_i] \vDash_{\mathsf{c}} M : \underline{C} \quad \Gamma \mid z : \underline{C} \vDash_{\mathsf{h}} K : \underline{D}}{\Gamma \vDash_{\mathsf{c}} K[\mathsf{op}^{\underline{C}}_{V}(x.M)/z] = \mathsf{op}^{\underline{D}}_{V}(x.K[M/z]) : \underline{D}} \; (\mathsf{op} : (x_i : I) \rightharpoonup O)$$

**Sound semantics:** Based on families fibrations and Law. theories

- $p : \mathrm{Fam}(\mathrm{Set}) \longrightarrow \mathrm{Set}$ and $q : \mathrm{Fam}(\mathrm{Mod}(\mathcal{L}_{\mathcal{T}_{\mathsf{eff}}}, \mathrm{Set})) \longrightarrow \mathrm{Set}$

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# Algebraic effects – examples

**Example 1** (interactive IO):

- $\mathsf{read} : 1 \rightharpoonup \mathsf{Chr}$ $\qquad (\mathsf{Chr} \stackrel{\text{def}}{=} 1 + \ldots + 1)$

  $\mathsf{write} : \mathsf{Chr} \rightharpoonup 1$

- no equations

**Example 2** (global state with location-dependent store type):

- $\diamond \vdash \mathsf{Loc}$

  $\ell : \mathsf{Loc} \vdash \mathsf{Val}$

  $\diamond \vdash_{\mathsf{V}} \mathsf{isDec}_{\mathsf{Loc}} : \Pi\, \ell : \mathsf{Loc} \,.\, \Pi\, \ell' : \mathsf{Loc} \,.\, (\ell =_{\mathsf{Loc}} \ell') + (\ell =_{\mathsf{Loc}} \ell' \to 0)$

- $\mathsf{get} : (\ell : \mathsf{Loc}) \rightharpoonup \mathsf{Val}$

  $\mathsf{put} : (\Sigma\, \ell : \mathsf{Loc} \,.\, \mathsf{Val}) \rightharpoonup 1$

- five equations (two of them branching on $\mathsf{isDec}_{\mathsf{Loc}}$)

**Example 3** (dep. typed update monads $T\, X \stackrel{\text{def}}{=} \Pi_{s:S} \,.\, P\, s \times X$)

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- get : $(\ell \,{:}\, \mathsf{Loc}) \rightharpoonup \mathsf{Val}$

  put : $(\Sigma\, \ell \,{:}\, \mathsf{Loc} \,.\, \mathsf{Val}) \rightharpoonup 1$

- five equations (two of them branching on $\mathsf{isDec}_{\mathsf{Loc}}$)

**Example 3** (dep. typed update monads $T\,X \stackrel{\text{def}}{=} \Pi_{s:S} \,.\, P\,s \times X$)

# Algebraic effects – examples

**Example 1** (interactive IO):

- $\mathsf{read} : 1 \rightharpoonup \mathsf{Chr}$ $\qquad (\mathsf{Chr} \stackrel{\text{def}}{=} 1 + \ldots + 1)$

  $\mathsf{write} : \mathsf{Chr} \rightharpoonup 1$

- no equations

**Example 2** (global state with location-dependent store type):

- $\diamond \vdash \mathsf{Loc}$

  $\ell : \mathsf{Loc} \vdash \mathsf{Val}$

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# Handlers of algebraic effects

**(for programming and extrinsic reasoning)**

# Handlers of alg. effects – for programming

**Idea:** Generalisation of exception handlers [Plotkin,Pretnar'09]

Handler $\sim$ Algebra and Handling $\sim$ Homomorphism

**Usual term-level presentation:**

$$\Gamma \vdash_{\mathsf{c}} M \;\texttt{handled with}\; \{\mathsf{op}_{x_v}(x_k) \mapsto N_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{T}_{\mathsf{eff}}} \;\texttt{to}\; y\!:\!A \;\texttt{in}_{\underline{C}}\; N_{\mathsf{ret}} : \underline{C}$$

satisfying

$$(\texttt{return}\; V)\;\texttt{handled with}\; \{\ldots\}_{\mathsf{op} \in \mathcal{T}_{\mathsf{eff}}} \;\texttt{to}\; y\!:\!A \;\texttt{in}\; N_{\mathsf{ret}} = N_{\mathsf{ret}}[V/x]$$

$$(\mathsf{op}^{\underline{C}}_{V}(x.M))\;\texttt{handled with}\; \{\ldots\}_{\mathsf{op} \in \mathcal{T}_{\mathsf{eff}}} \;\texttt{to}\; y\!:\!A \;\texttt{in}\; N_{\mathsf{ret}} = N_{\mathsf{op}}[V/x_v][\ldots/x_k]$$

**Typical use case for programming:**

- write your programs using alg. ops. (e.g., get and put)
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**Idea:** Using a derived handle-into-values handling construct

$$M \texttt{ handled with } \{\mathtt{op}_{x_v}(x_k) \mapsto V_{\mathtt{op}}\}_{\mathtt{op} \in \mathcal{T}_{\mathsf{eff}}} \texttt{ to } y\,{:}\,A \texttt{ in}_{\underline{B}}\ V_{\mathsf{ret}}$$

we can define natural predicates (essentially, dependent types)

$$\Gamma \vDash_{\mathsf{v}} P : UFA \to \mathcal{U}$$

by

- equipping a universe $\mathcal{U}$ with an algebra for $\mathcal{T}_{\mathsf{eff}}$, and
- using the above handle-into-values construct to define $P$

**Note 1:** $P\,(\texttt{thunk}\,M)$ computes a proof obligation for $M$

**Note 2:** Formally, this is done in an extension of eMLTT with

- a universe $\mathcal{U}$ closed under Nat, 1, 0, $+$, $\Sigma$, and $\Pi$
- a type-based treatment of handlers $\quad \underline{C} \ ::= \ \ldots \mid \langle A; \overrightarrow{V_{\mathsf{op}}}; \overrightarrow{W_{\mathsf{eq}}} \rangle$
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# Handlers of alg. effects – for reasoning

**Example 1** (Evaluation Logic style modalities):

- Given a predicate $P : A \to \mathcal{U}$ on return values,

  we define a predicate $\Diamond P : UFA \to \mathcal{U}$ on IO-computations as

$$\Diamond P \stackrel{\text{def}}{=} \lambda\, x \,{:}\, UFA \,.\, (\texttt{force}\, x)\ \texttt{handled with}\ \{\ldots\}_{\mathsf{op} \in \mathcal{T}_{\mathsf{IO}}}\ \texttt{to}\ y \,{:}\, A\ \texttt{in}_{\mathcal{U}}\ P\ y$$

  using the handler given by

$$V_{\text{read}} \stackrel{\text{def}}{=} \lambda\, x \,{:}\, (\Sigma\, x_v \,{:}\, 1 \,.\, \mathsf{Chr} \to \mathcal{U}) \,.\, \widehat{\Sigma}\, y \,{:}\, \mathsf{El}(\widehat{\mathsf{Chr}}) \,.\, (\texttt{snd}\ x)\ y$$

$$V_{\text{write}} \stackrel{\text{def}}{=} \lambda\, x \,{:}\, (\Sigma\, x_v \,{:}\, \mathsf{Chr} \,.\, 1 \to \mathcal{U}) \,.\, (\texttt{snd}\ x)\ \star$$

- $\Diamond P$ corresponds to Evaluation Logic's possibility modality

$$\Diamond P\,(\texttt{thunk}\,(\texttt{read}(x \,.\, \texttt{write}_{e'}(\texttt{return}\, V)))) = \widehat{\Sigma}\, x \,{:}\, \mathsf{El}(\widehat{\mathsf{Chr}}) \,.\, P\ V$$

- To get the necessity modality $\Box P$, we use $\widehat{\Pi}\, x \,{:}\, \mathsf{El}(\widehat{\mathsf{Chr}})$ in $V_{\text{read}}$

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**Example 2** (Dijkstra's weakest precondition semantics for state):

- Given a postcondition on return values and final states

  $$Q : A \to S \to \mathcal{U} \qquad\qquad (S \stackrel{\text{def}}{=} \Pi\, \ell : \mathsf{Loc}\,.\, \mathsf{Val})$$

  we define a precondition for stateful comps. on initial states

  $$\mathsf{wp}_Q : UFA \to S \to \mathcal{U}$$

  by

  1) handling the given comp. into a state-passing function using

  $$V_{\text{get}}, V_{\text{put}} \quad \text{on} \quad S \to (\mathcal{U} \times S) \qquad \text{and} \qquad V_{\text{ret}} \text{ ``$=$'' } Q$$

  2) feeding in the initial state; and 3) projecting out $\mathcal{U}$

- **Theorem:** $\mathsf{wp}_Q$ satisfies expected properties of WPs, e.g.,

  $$\mathsf{wp}_Q\,(\mathtt{thunk}\,(\mathtt{return}\,V)) \quad = \quad \lambda\, x_S : S\,.\, Q\; V\; x_S$$

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**Example 3** (Patterns of allowed (IO-)effects):

- Assuming an inductive type of IO-protocols, given by

$$\mathtt{e} : \mathsf{Protocol} \qquad\qquad \mathtt{r} : (\mathsf{Chr} \to \mathsf{Protocol}) \to \mathsf{Protocol}$$

$$\mathtt{w} : (\mathsf{Chr} \to \mathcal{U}) \to \mathsf{Protocol} \to \mathsf{Protocol}$$

  and potentially also by $\wedge$, $\vee$, ...

- Then, we define the predicate (rel. between comps. and protocols)

$$\mathsf{Allowed} : UFA \to \mathsf{Protocol} \to \mathcal{U}$$

  by handling the given computation using

$$V_{\mathsf{read}}\,,\, V_{\mathsf{write}} \qquad \text{on} \qquad \mathsf{Protocol} \to \mathcal{U}$$

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$$\begin{array}{lcl} V_{\mathsf{read}}\ \langle \_\ , V_{\mathsf{rk}} \rangle\ (\mathtt{r}\ \mathsf{Pr}') & \stackrel{\mathrm{def}}{=} & \widehat{\Pi}\, x \,{:}\, \mathsf{El}(\widehat{\mathsf{Chr}})\,.\,(V_{\mathsf{rk}}\ x)\ (\mathsf{Pr}'\ x) \\ V_{\mathsf{write}}\ \langle V\ , V_{\mathsf{wk}} \rangle\ (\mathtt{w}\ P\ \mathsf{Pr}') & \stackrel{\mathrm{def}}{=} & \widehat{\Sigma}\, x \,{:}\, \mathsf{El}(P\ V)\,.\,V_{\mathsf{wk}}\ \star\ \mathsf{Pr}' \\ \_ & \stackrel{\mathrm{def}}{=} & \widehat{0} \end{array}$$

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# Conclusion

At a high-level, the presented work was about combining

dependent types and computational effects

In particular, you saw

- a clean core language of dependent types and comp. effects
- a natural category-theoretic semantics
- alg. effects and handlers, in particular, for reasoning using
  - Evaluation Logic style modalities
  - Dijkstra's weakest precondition semantics for state
  - patterns of allowed (IO-)effects

Ongoing and future work:

- uniform account of the various handler-defined predicates
- more expressive comp. types (par. adjunctions, Dijkstra monads)
- type-dependency on computations (e.g., in seq. composition)

# Thank you!

D. Ahman.
**Fibred Computational Effects.** (PhD Thesis, 2017)

D. Ahman, N. Ghani, G. Plotkin.
**Dependent Types and Fibred Computational Effects.** (FoSSaCS'16)

D. Ahman.
**Handling Fibred Computational Effects.** (POPL'18)