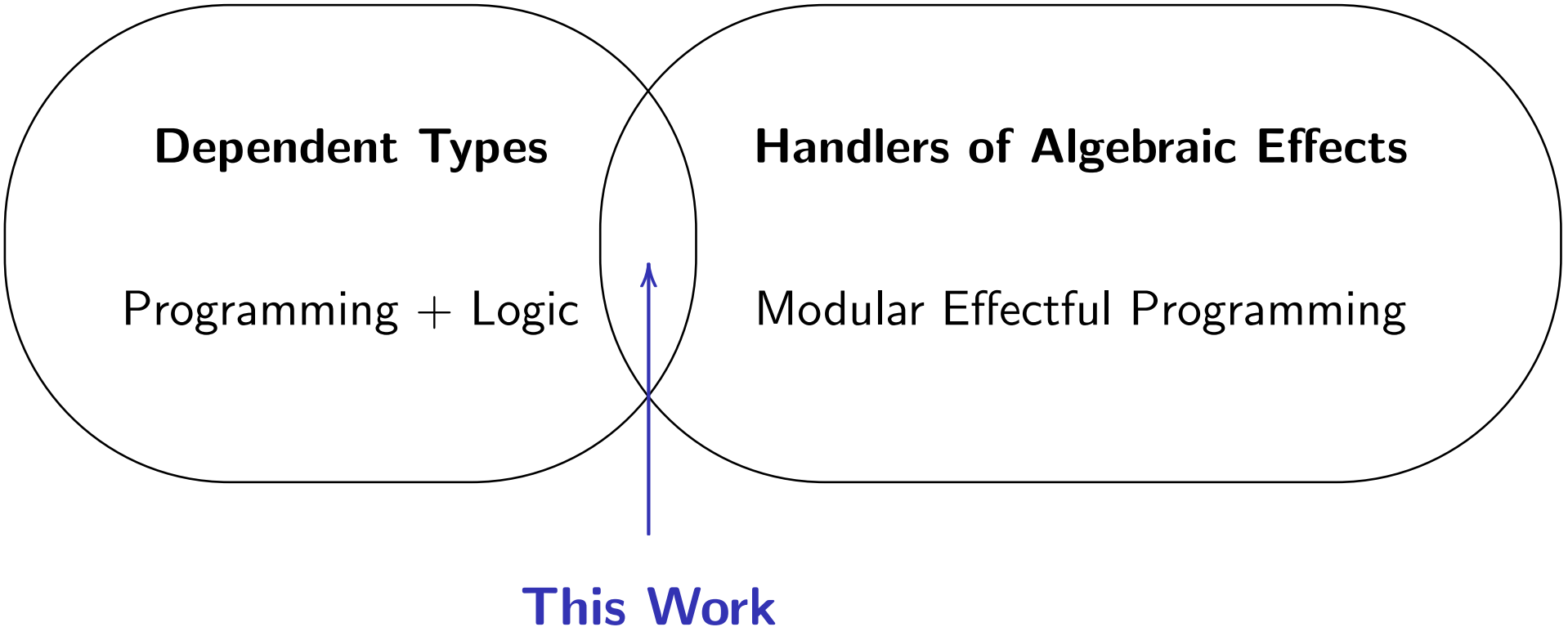
Handling Fibred Algebraic Effects

Danel Ahman

Prosecco Team, INRIA Paris

POPL 2018

January 10, 2018


Dependent Types
Programming + Logic
Handlers of Algebraic Effects
Modular Effectful Programming
This Work

Outline

- Setting the scene
  - Algebraic effects and their handlers
  - A core effectful dependently typed calculus (FoSSaCS'16)

    [A., Ghani, Plotkin'16]

- What can we gain from handlers + dependent types?
  - Programming with handlers + expressiveness of dep. types
  - Useful for defining predicates/types depending on computations

- Extending the FoSSaCS'16 calculus with alg. effects and handlers
  - Take 1: The common term-level def. of handlers (unsound)
  - Take 2: A new type-level treatment of handlers

Outline

- Setting the scene
  - Algebraic effects and their handlers
  - A core effectful dependently typed calculus (FoSSaCS'16)
    [A., Ghani, Plotkin'16]
- What can we gain from handlers + dependent types?
  - Programming with handlers + expressiveness of dep. types
  - Useful for defining predicates/types depending on computations
- Extending the FoSSaCS'16 calculus with alg. effects and handlers
  - Take 1: The common term-level def. of handlers (unsound)
  - Take 2: A new type-level treatment of handlers

Algebraic effects and their handlers

- Moggi taught us to model comp. effects using **monads** $(T, \eta, (-)^\dagger)$

$$\eta_A : A \to TA \qquad (f : A \to TB)^\dagger_{A,B} : TA \to TB$$

- Plotkin and Power showed that most of these monads arise from
  - **operation symbols** – representing the sources of effects

$$\mathsf{raise} : \mathsf{Exc} \longrightarrow 0 \qquad \mathsf{read} : \mathsf{Loc} \longrightarrow \mathsf{Val} \qquad \mathsf{write} : \mathsf{Loc} \times \mathsf{Val} \longrightarrow 1$$

  - **equations** – describing the computational behaviour

$$\ell : \mathsf{Loc} \mid w : 1 \vdash \mathtt{read}_\ell(x.\mathtt{write}_{\langle \ell, x \rangle}(w(\star))) = w(\star)$$

- The algebraic approach significantly simplifies
  - **choosing** a monad/adjunction to model a given language
  - modelling **combinations** of two or more comp. effects
  - **generic** effectful programming (via **handlers**)

Algebraic effects and their handlers

- Moggi taught us to model comp. effects using **monads** $(T, \eta, (-)^\dagger)$

$$\eta_A : A \to TA \qquad (f : A \to TB)^\dagger_{A,B} : TA \to TB$$

- Plotkin and Power showed that most of these monads arise from
  - **operation symbols** – representing the sources of effects

$$\mathsf{raise} : \mathsf{Exc} \longrightarrow 0 \qquad \mathsf{read} : \mathsf{Loc} \longrightarrow \mathsf{Val} \qquad \mathsf{write} : \mathsf{Loc} \times \mathsf{Val} \longrightarrow 1$$

  - **equations** – describing the computational behaviour

$$\ell : \mathsf{Loc} \mid w : 1 \vdash \mathtt{read}_\ell\big(x.\mathtt{write}_{\langle \ell, x\rangle}\big(w(\star)\big)\big) = w(\star)$$

- The algebraic approach significantly simplifies
  - **choosing** a monad/adjunction to model a given language
  - modelling **combinations** of two or more comp. effects
  - **generic** effectful programming (via **handlers**)

Algebraic effects and their handlers

- Moggi taught us to model comp. effects using **monads** $(T, \eta, (-)^\dagger)$

$$\eta_A : A \to TA \qquad (f : A \to TB)^\dagger_{A,B} : TA \to TB$$

- Plotkin and Power showed that most of these monads arise from
  - **operation symbols** – representing the sources of effects

$$\mathsf{raise} : \mathsf{Exc} \longrightarrow 0 \qquad \mathsf{read} : \mathsf{Loc} \longrightarrow \mathsf{Val} \qquad \mathsf{write} : \mathsf{Loc} \times \mathsf{Val} \longrightarrow 1$$

  - **equations** – describing the computational behaviour

$$\ell : \mathsf{Loc} \mid w : 1 \vdash \mathtt{read}_\ell\big(x.\mathtt{write}_{\langle \ell, x\rangle}\big(w(\star)\big)\big) = w(\star)$$

- The algebraic approach significantly simplifies
  - **choosing** a monad/adjunction to model a given language
  - modelling **combinations** of two or more comp. effects
  - **generic** effectful programming (via **handlers**)

Algebraic effects and their handlers ctd.

- Plotkin and Pretnar's **handlers** of algebraic effects
  - generalisation of exception handlers
  - given by redefining the given ops. (handlers denote **algebras**)
  - many uses – rollbacks, stream redirection, concurrency, ...
- Usually included in languages using the **handling** construct

$$M \texttt{ handled with } \{\mathrm{op}_{x_v}(x_k) \mapsto N_{\mathrm{op}}\}_{\mathrm{op} \in \mathcal{S}_{\mathrm{eff}}} \texttt{ to } y : A \texttt{ in}_{\underline{C}} N_{\mathrm{ret}}$$

interpreted using the **homomorphism** $FA \longrightarrow \langle U\underline{C}, \overrightarrow{N_{\mathrm{op}}} \rangle$

$$(\mathrm{op}_V(y.M)) \texttt{ handled with } \{\ldots\}_{\mathrm{op} \in \mathcal{S}_{\mathrm{eff}}} \texttt{ to } y : A \texttt{ in}_{\underline{C}} N_{\mathrm{ret}}$$

$$=$$

$$N_{\mathrm{op}}[V/x_v][\lambda y : O \,.\, \texttt{thunk}(M \texttt{ handled with } \ldots)/x_k]$$

and

$$(\texttt{return } V) \texttt{ handled with } \{\ldots\}_{\mathrm{op} \in \mathcal{S}_{\mathrm{eff}}} \texttt{ to } y : A \texttt{ in}_{\underline{C}} N_{\mathrm{ret}} \;=\; N_{\mathrm{ret}}[V/y]$$

Algebraic effects and their handlers ctd.

- Plotkin and Pretnar's **handlers** of algebraic effects
  - generalisation of exception handlers
  - given by redefining the given ops. (handlers denote **algebras**)
  - many uses – rollbacks, stream redirection, concurrency, ...
- Usually included in languages using the **handling** construct

$$M \texttt{ handled with } \{\mathrm{op}_{x_v}(x_k) \mapsto N_{\mathrm{op}}\}_{\mathrm{op} \in \mathcal{S}_{\mathrm{eff}}} \texttt{ to } y : A \texttt{ in}_{\underline{C}} \; N_{\mathrm{ret}}$$

interpreted using the **homomorphism** $FA \longrightarrow \langle U\underline{C}, \overrightarrow{N_{\mathrm{op}}} \rangle$

$$(\mathrm{op}_V(y.M)) \texttt{ handled with } \{\ldots\}_{\mathrm{op} \in \mathcal{S}_{\mathrm{eff}}} \texttt{ to } y : A \texttt{ in}_{\underline{C}} \; N_{\mathrm{ret}}$$

$$=$$

$$N_{\mathrm{op}}[V/x_v][\lambda y : O \,.\, \texttt{thunk}(M \texttt{ handled with } \ldots)/x_k]$$

and

$$(\texttt{return } V) \texttt{ handled with } \{\ldots\}_{\mathrm{op} \in \mathcal{S}_{\mathrm{eff}}} \texttt{ to } y : A \texttt{ in}_{\underline{C}} \; N_{\mathrm{ret}} \;=\; N_{\mathrm{ret}}[V/y]$$

Algebraic effects and their handlers ctd.

- Plotkin and Pretnar's **handlers** of algebraic effects
  - generalisation of exception handlers
  - given by redefining the given ops. (handlers denote **algebras**)
  - many uses – rollbacks, stream redirection, concurrency, ...
- Usually included in languages using the **handling** construct

$$M \texttt{ handled with } \{\mathrm{op}_{x_v}(x_k) \mapsto N_{\mathrm{op}}\}_{\mathrm{op} \in \mathcal{S}_{\mathrm{eff}}} \texttt{ to } y : A \texttt{ in}_{\underline{C}} \; N_{\mathrm{ret}}$$

interpreted using the **homomorphism** $FA \longrightarrow \langle U\underline{C}, \overrightarrow{N_{\mathrm{op}}} \rangle$

$$(\mathrm{op}_V(y.M)) \texttt{ handled with } \{\ldots\}_{\mathrm{op} \in \mathcal{S}_{\mathrm{eff}}} \texttt{ to } y : A \texttt{ in}_{\underline{C}} \; N_{\mathrm{ret}}$$

$$=$$

$$N_{\mathrm{op}}[V/x_v][\lambda y : O \,.\, \texttt{thunk}(M \texttt{ handled with } \ldots)/x_k]$$

and

$$(\texttt{return } V) \texttt{ handled with } \{\ldots\}_{\mathrm{op} \in \mathcal{S}_{\mathrm{eff}}} \texttt{ to } y : A \texttt{ in}_{\underline{C}} \; N_{\mathrm{ret}} \;=\; N_{\mathrm{ret}}[V/y]$$

Algebraic effects and their handlers ctd.

- Plotkin and Pretnar's **handlers** of algebraic effects
  - generalisation of exception handlers
  - given by redefining the given ops. (handlers denote **algebras**)
  - many uses – rollbacks, stream redirection, concurrency, ...
- Usually included in languages using the **handling** construct

$$M \texttt{ handled with } \{\mathrm{op}_{x_v}(x_k) \mapsto N_{\mathrm{op}}\}_{\mathrm{op} \in \mathcal{S}_{\mathrm{eff}}} \texttt{ to } y : A \texttt{ in}_{\underline{C}} \; N_{\mathrm{ret}}$$

interpreted using the **homomorphism** $FA \longrightarrow \langle U\underline{C}, \overrightarrow{N_{\mathrm{op}}} \rangle$

$$(\mathrm{op}_V(y.M)) \texttt{ handled with } \{\ldots\}_{\mathrm{op} \in \mathcal{S}_{\mathrm{eff}}} \texttt{ to } y : A \texttt{ in}_{\underline{C}} \; N_{\mathrm{ret}}$$
$$=$$
$$N_{\mathrm{op}}[V/x_v][\lambda\, y : O \,.\, \texttt{thunk}\,(M \texttt{ handled with } \ldots)/x_k]$$

and

$$(\texttt{return } V) \texttt{ handled with } \{\ldots\}_{\mathrm{op} \in \mathcal{S}_{\mathrm{eff}}} \texttt{ to } y : A \texttt{ in}_{\underline{C}} \; N_{\mathrm{ret}} \;=\; N_{\mathrm{ret}}[V/y]$$

Outline

- Setting the scene
  - Algebraic effects and their handlers
  - A core effectful dependently typed calculus (FoSSaCS'16)

    [A., Ghani, Plotkin'16]

- What can we gain from handlers + dependent types?
  - Programming with handlers + expressiveness of dep. types
  - Useful for defining predicates/types depending on computations

- Extending the FoSSaCS'16 calculus with alg. effects and handlers
  - Take 1: The common term-level def. of handlers (unsound)
  - Take 2: A new type-level treatment of handlers

A core dependently typed effectful calculus

- (Model-theoretically) natural extension of type theory
  - clear distinction between **values** and **computations** (CBPV, EEC)
- **Value types** $(\Gamma \vdash A)$ and **computation types** $(\Gamma \vdash \underline{C})$

$$A, B ::= \ldots \mid U\underline{C} \qquad \underline{C}, \underline{D} ::= FA \mid \Pi x{:}A.\,\underline{C} \mid \boxed{\Sigma x{:}A.\,\underline{C}}$$

- **Value terms** $(\Gamma \vdash V : A)$

$$V, W ::= \ldots \mid \mathtt{thunk}\, M$$

- **Computation terms** $(\Gamma \vdash M : \underline{C})$

$$M, N ::= \mathtt{return}\, V \mid M \,\mathtt{to}\, x{:}A \,\mathtt{in}_{\underline{C}}\, N \mid \lambda x{:}A.\, M \mid M\, V$$
$$\mid \langle V, M\rangle \mid \boxed{M \,\mathtt{to}\, (x{:}A, z{:}\underline{C}) \,\mathtt{in}_{\underline{D}}\, K} \mid \mathtt{force}_{\underline{C}}\, V$$

- **Homomorphism terms** $(\Gamma \mid z : \underline{C} \vdash K : \underline{D})$

$$K, L ::= z \mid K \,\mathtt{to}\, x{:}A \,\mathtt{in}_{\underline{C}}\, M \mid \ldots \qquad \text{(stack terms, eval. ctxs.)}$$

A core dependently typed effectful calculus

- (Model-theoretically) natural extension of type theory
  - clear distinction between **values** and **computations** (CBPV, EEC)
- **Value types** $(\Gamma \vdash A)$ and **computation types** $(\Gamma \vdash \underline{C})$

$$A, B \;::=\; \ldots \mid U\underline{C} \qquad \underline{C}, \underline{D} \;::=\; FA \mid \Pi x{:}A\,.\,\underline{C} \mid \boxed{\Sigma x{:}A\,.\,\underline{C}}$$

- **Value terms** $(\Gamma \vdash V : A)$

$$V, W \;::=\; \ldots \mid \mathtt{thunk}\, M$$

- **Computation terms** $(\Gamma \vdash M : \underline{C})$

$$M, N \;::=\; \mathtt{return}\, V \mid M \;\mathtt{to}\; x{:}A \;\mathtt{in}_{\underline{C}}\; N \mid \lambda x{:}A\,.\,M \mid M\,V$$
$$\mid \langle V, M\rangle \mid \boxed{M \;\mathtt{to}\; (x{:}A, z{:}\underline{C})\;\mathtt{in}_{\underline{D}}\; K} \mid \mathtt{force}_{\underline{C}}\, V$$

- **Homomorphism terms** $(\Gamma \mid z{:}\underline{C} \vdash K : \underline{D})$

$$K, L \;::=\; z \mid K \;\mathtt{to}\; x{:}A \;\mathtt{in}_{\underline{C}}\; M \mid \ldots \qquad \text{(stack terms, eval. ctxs.)}$$

A core dependently typed effectful calculus

- (Model-theoretically) natural extension of type theory
  - clear distinction between **values** and **computations** (CBPV, EEC)
- **Value types** $(\Gamma \vdash A)$ and **computation types** $(\Gamma \vdash \underline{C})$

$$A, B \ ::= \ \ldots \ \mid \ U\underline{C} \qquad \underline{C}, \underline{D} \ ::= \ FA \mid \Pi x{:}A\,.\,\underline{C} \mid \boxed{\Sigma x{:}A\,.\,\underline{C}}$$

- **Value terms** $(\Gamma \vdash V : A)$

$$V, W \ ::= \ \ldots \ \mid \ \mathtt{thunk}\ M$$

- **Computation terms** $(\Gamma \vdash M : \underline{C})$

$$M, N \ ::= \ \mathtt{return}\ V \ \mid \ M \ \mathtt{to}\ x{:}A\ \mathtt{in}_{\underline{C}}\ N \ \mid \ \lambda x{:}A\,.\,M \ \mid \ M\,V$$
$$\mid \ \langle V, M\rangle \ \mid \ \boxed{M \ \mathtt{to}\ (x{:}A, z{:}\underline{C})\ \mathtt{in}_{\underline{D}}\ K} \ \mid \ \mathtt{force}_{\underline{C}}\ V$$

- **Homomorphism terms** $(\Gamma \mid z{:}\underline{C} \vdash K : \underline{D})$

$$K, L \ ::= \ z \ \mid \ K \ \mathtt{to}\ x{:}A\ \mathtt{in}_{\underline{C}}\ M \ \mid \ \ldots \qquad \text{(stack terms, eval. ctxs.)}$$

A core dependently typed effectful calculus

- (Model-theoretically) natural extension of type theory
  - clear distinction between **values** and **computations** (CBPV, EEC)
- **Value types** $(\Gamma \vdash A)$ and **computation types** $(\Gamma \vdash \underline{C})$

$$A, B \;::=\; \ldots \mid U\underline{C} \qquad \underline{C}, \underline{D} \;::=\; FA \mid \Pi\, x{:}A\,.\,\underline{C} \mid \boxed{\Sigma\, x{:}A\,.\,\underline{C}}$$

- **Value terms** $(\Gamma \vdash V : A)$

$$V, W \;::=\; \ldots \mid \mathtt{thunk}\ M$$

- **Computation terms** $(\Gamma \vdash M : \underline{C})$

$$M, N \;::=\; \mathtt{return}\ V \mid M\ \mathtt{to}\ x{:}A\ \mathtt{in}_{\underline{C}}\ N \mid \lambda\, x{:}A\,.\,M \mid M\,V$$
$$\mid \langle V, M\rangle \mid \boxed{M\ \mathtt{to}\ (x{:}A, z{:}\underline{C})\ \mathtt{in}_{\underline{D}}\ K} \mid \mathtt{force}_{\underline{C}}\ V$$

- **Homomorphism terms** $(\Gamma \mid z{:}\underline{C} \vdash K : \underline{D})$

$$K, L \;::=\; z \mid K\ \mathtt{to}\ x{:}A\ \mathtt{in}_{\underline{C}}\ M \mid \ldots \qquad \text{(stack terms, eval. ctxs.)}$$

A core dependently typed effectful calculus

- (Model-theoretically) natural extension of type theory
  - clear distinction between **values** and **computations** (CBPV, EEC)
- **Value types** $(\Gamma \vdash A)$ and **computation types** $(\Gamma \vdash \underline{C})$

$$A, B \;::=\; \ldots \mid U\underline{C} \qquad \underline{C}, \underline{D} \;::=\; FA \mid \Pi\, x{:}A\,.\,\underline{C} \mid \boxed{\Sigma\, x{:}A\,.\,\underline{C}}$$

- **Value terms** $(\Gamma \vdash V : A)$

$$V, W \;::=\; \ldots \mid \mathtt{thunk}\ M$$

- **Computation terms** $(\Gamma \vdash M : \underline{C})$

$$M, N \;::=\; \mathtt{return}\ V \mid M\ \mathtt{to}\ x{:}A\ \mathtt{in}_{\underline{C}}\ N \mid \lambda\, x{:}A\,.\,M \mid M\ V$$
$$\mid \langle V, M\rangle \mid \boxed{M\ \mathtt{to}\ (x{:}A, z{:}\underline{C})\ \mathtt{in}_{\underline{D}}\ K} \mid \mathtt{force}_{\underline{C}}\ V$$

- **Homomorphism terms** $(\Gamma \mid z{:}\underline{C} \vdash K : \underline{D})$

$$K, L \;::=\; z \mid K\ \mathtt{to}\ x{:}A\ \mathtt{in}_{\underline{C}}\ M \mid \ldots \qquad \text{(stack terms, eval. ctxs.)}$$

Outline

- Setting the scene
  - Algebraic effects and their handlers
  - A core effectful dependently typed calculus (FoSSaCS'16)
    [A., Ghani, Plotkin'16]

- What can we gain from handlers + dependent types?
  - Programming with handlers + expressiveness of dep. types
  - Useful for defining predicates/types depending on computations

- Extending the FoSSaCS'16 calculus with alg. effects and handlers
  - Take 1: The common term-level def. of handlers (unsound)
  - Take 2: A new type-level treatment of handlers

Defining predicates on effectful comps.

- In our extension of the FoSSaCS'16 calculus, we have
  - a Tarski-style **value universe** $\mathcal{U}$ (with codes $\widehat{\Pi}, \widehat{\Sigma}, \widehat{0}, \widehat{1}, \ldots$)
  - fibred **algebraic effects** $\mathsf{op} : (x_v : I) \longrightarrow O(x_v)$
  - a derivable "into-values" variant of **handlers** and **handling**

$$M \texttt{ handled with } \{\mathsf{op}_{x_v}(x_k) \mapsto V_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathsf{eff}}} \texttt{ to } y : A \texttt{ in}_B \; V_{\mathsf{ret}}$$

- Computations of interest to us have type $M : FA$
- Using this, we can define **predicates** $P : UFA \to \mathcal{U}$ by
  1) equipping $\mathcal{U}$ (or a corresponding type) with an **algebra** structure
  2) **handling** the given computation using that algebra
- Intuitively, $P\,(\texttt{thunk}\, M)$ computes a **proof obligation** for M

Defining predicates on effectful comps.

- In our extension of the FoSSaCS'16 calculus, we have
  - a Tarski-style **value universe** $\mathcal{U}$ (with codes $\widehat{\Pi}, \widehat{\Sigma}, \widehat{0}, \widehat{1}, \ldots$)
  - fibred **algebraic effects** $\mathsf{op} : (x_v : I) \longrightarrow O(x_v)$
  - a derivable "into-values" variant of **handlers** and **handling**

$$M \texttt{ handled with } \{\mathsf{op}_{x_v}(x_k) \mapsto V_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathsf{eff}}} \texttt{ to } y : A \texttt{ in}_B \; V_{\mathsf{ret}}$$

- Computations of interest to us have type $M : FA$
- Using this, we can define **predicates** $P : UFA \to \mathcal{U}$ by
  1) equipping $\mathcal{U}$ (or a corresponding type) with an **algebra** structure
  2) **handling** the given computation using that algebra
- Intuitively, $P\ (\texttt{thunk}\, M)$ computes a **proof obligation** for M

Three examples of such predicates

- **Ex1:** Lifting predicates from return values to computations (I/O)
- **Ex2:** Dijkstra's weakest precondition semantics (state)
- **Ex3:** Specifying allowed patterns of effects (I/O)

Ex1: Lifting predicates to effectful comps.

- Given a predicate $P : A \to \mathcal{U}$ on **return values**,

  we define a predicate $\Box P : UFA \to \mathcal{U}$ on **(I/O)-comps.** as

$$\Box P \stackrel{\text{def}}{=} \lambda\, y : UFA\,.\,(\texttt{force}\, y)\ \texttt{handled with}\ \{\ldots\}_{\texttt{op} \in \mathcal{S}_{\text{I/O}}}\ \texttt{to}\ y' : A\ \texttt{in}_{\mathcal{U}}\ P\, y'$$

  using the **handler** given by

$$\begin{array}{lcll} \text{read}(x_k) & \mapsto & \widehat{\Pi}\, y : \text{El}(\widehat{\text{Chr}})\,.\, x_k\ y & (\text{where } x_k : \text{Chr} \to \mathcal{U}) \\ \text{write}_{x_v}(x_k) & \mapsto & x_k\ \star & (\text{where } x_v : \text{Chr},\ x_k : 1 \to \mathcal{U}) \end{array}$$

- $\Box P$ is similar to the **necessity modality** from Evaluation Logic

$$\Gamma \vdash \Box P\,(\texttt{thunk}\,(\texttt{read}(x\,.\,\texttt{write}_{e'}(\texttt{return}\, V)))) = \widehat{\Pi}\, x : \text{El}(\widehat{\text{Chr}})\,.\, P\ V$$

- To get $\Diamond P$, we only have to replace $\widehat{\Pi}$ with $\widehat{\Sigma}$ in the handler

Ex1: Lifting predicates to effectful comps.

- Given a predicate $P : A \to \mathcal{U}$ on **return values**,

  we define a predicate $\Box P : \mathit{UFA} \to \mathcal{U}$ on **(I/O)-comps.** as

$$\Box P \stackrel{\text{def}}{=} \lambda\, y : \mathit{UFA}\,.\,(\mathtt{force}\, y)\ \mathtt{handled\ with}\ \{\ldots\}_{\mathsf{op} \in \mathcal{S}_{\mathrm{I/O}}}\ \mathtt{to}\ y' : A\ \mathtt{in}_{\mathcal{U}}\ P\, y'$$

  using the **handler** given by

$$\begin{array}{lcll} \mathsf{read}(x_k) & \mapsto & \widehat{\Pi}\, y : \mathsf{El}(\widehat{\mathsf{Chr}})\,.\, x_k\ y & (\text{where } x_k : \mathsf{Chr} \to \mathcal{U}) \\ \mathsf{write}_{x_v}(x_k) & \mapsto & x_k\ \star & (\text{where } x_v : \mathsf{Chr},\ x_k : 1 \to \mathcal{U}) \end{array}$$

- $\Box P$ is similar to the **necessity modality** from Evaluation Logic

$$\Gamma \vdash \Box P\,(\mathtt{thunk}\,(\mathtt{read}(x\,.\,\mathtt{write}_{e'}(\mathtt{return}\, V)))) = \widehat{\Pi}\, x : \mathsf{El}(\widehat{\mathsf{Chr}})\,.\, P\ V$$

- To get $\Diamond P$, we only have to replace $\widehat{\Pi}$ with $\widehat{\Sigma}$ in the handler

Ex1: Lifting predicates to effectful comps.

- Given a predicate $P : A \to \mathcal{U}$ on **return values**,

  we define a predicate $\Box P : \mathit{UFA} \to \mathcal{U}$ on **(I/O)-comps.** as

$$\Box P \stackrel{\text{def}}{=} \lambda\, y : \mathit{UFA}\,.\,(\texttt{force}\; y)\;\texttt{handled with}\;\{\ldots\}_{\mathsf{op} \in \mathcal{S}_{\mathsf{I/O}}}\;\texttt{to}\; y' : A\;\texttt{in}_{\mathcal{U}}\; P\, y'$$

  using the **handler** given by

$$\begin{array}{lcll} \mathsf{read}(x_k) & \mapsto & \widehat{\Pi}\, y : \mathsf{El}(\widehat{\mathsf{Chr}})\,.\, x_k\; y & (\text{where } x_k : \mathsf{Chr} \to \mathcal{U}) \\ \mathsf{write}_{x_v}(x_k) & \mapsto & x_k\; \star & (\text{where } x_v : \mathsf{Chr},\; x_k : 1 \to \mathcal{U}) \end{array}$$

- $\Box P$ is similar to the **necessity modality** from Evaluation Logic

$$\Gamma \vdash \Box P\,(\texttt{thunk}\,(\texttt{read}(x\,.\,\texttt{write}_{e'}(\texttt{return}\, V)))) = \widehat{\Pi}\, x : \mathsf{El}(\widehat{\mathsf{Chr}})\,.\, P\; V$$

- To get $\Diamond P$, we only have to replace $\widehat{\Pi}$ with $\widehat{\Sigma}$ in the handler

Ex1: Lifting predicates to effectful comps.

- Given a predicate $P : A \to \mathcal{U}$ on **return values**,

  we define a predicate $\Box P : \mathit{UFA} \to \mathcal{U}$ on **(I/O)-comps.** as

$$\Box P \stackrel{\text{def}}{=} \lambda\, y : \mathit{UFA}\,.\,(\texttt{force}\; y)\;\texttt{handled with}\;\{\ldots\}_{\mathsf{op} \in \mathcal{S}_{\mathsf{I/O}}}\;\texttt{to}\; y' : A\;\texttt{in}_{\mathcal{U}}\; P\, y'$$

  using the **handler** given by

$$\begin{array}{lcll} \mathsf{read}(x_k) & \mapsto & \widehat{\Pi}\, y : \mathsf{El}(\widehat{\mathsf{Chr}})\,.\, x_k\; y & (\text{where } x_k : \mathsf{Chr} \to \mathcal{U}) \\ \mathsf{write}_{x_v}(x_k) & \mapsto & x_k\; \star & (\text{where } x_v : \mathsf{Chr},\; x_k : 1 \to \mathcal{U}) \end{array}$$

- $\Box P$ is similar to the **necessity modality** from Evaluation Logic

$$\Gamma \vdash \Box P\,(\texttt{thunk}\,(\texttt{read}(x\,.\,\texttt{write}_{e'}(\texttt{return}\; V)))) = \widehat{\Pi}\, x : \mathsf{El}(\widehat{\mathsf{Chr}})\,.\, P\; V$$

- To get $\Diamond P$, we only have to replace $\widehat{\Pi}$ with $\widehat{\Sigma}$ in the handler

Ex2: Dijkstra's weakest precondition sem.

- Given a postcondition on **return values** and **final states**

$$Q : A \to S \to \mathcal{U} \qquad (S \stackrel{\text{def}}{=} \Pi\, \ell : \mathrm{Loc}.\mathrm{Val}(\ell))$$

we define a precondition for **stateful comps.** on **initial states**

$$\mathrm{wp}_Q : UFA \to S \to \mathcal{U}$$

by

1) handling the given comp. into a **state-passing function** using

$$V_{\mathsf{get}}\,, V_{\mathsf{put}} \quad \text{on} \quad S \to (\mathcal{U} \times S) \qquad \text{and} \qquad V_{\mathsf{ret}} \;\text{“}=\text{”}\; Q$$

2) feeding in the **initial state**; and **3)** projecting out the **value of** $\mathcal{U}$

- Then, wp_Q satisfies the **expected properties**, such as

$$\Gamma \vdash \mathrm{wp}_Q\,(\mathrm{thunk}\,(\mathtt{return}\,V)) \;=\; \lambda x_S : S\,.\,Q\;V\;x_S$$

$$\Gamma \vdash \mathrm{wp}_Q\,(\mathrm{thunk}\,(\mathtt{put}_{\langle \ell, V\rangle}(M))) \;=\; \lambda x_S : S\,.\,\mathrm{wp}_Q\,(\mathrm{thunk}\,M)\;x_S[\ell \mapsto V]$$

Ex2: Dijkstra's weakest precondition sem.

- Given a postcondition on **return values** and **final states**

$$Q : A \to S \to \mathcal{U} \qquad (S \stackrel{\text{def}}{=} \Pi\, \ell : \mathsf{Loc}\,.\,\mathsf{Val}(\ell))$$

  we define a precondition for **stateful comps.** on **initial states**

$$\mathsf{wp}_Q : \mathit{UFA} \to S \to \mathcal{U}$$

  by

  1) handling the given comp. into a **state-passing function** using

$$V_{\mathsf{get}}\,,\, V_{\mathsf{put}} \quad \text{on} \quad S \to (\mathcal{U} \times S) \qquad \text{and} \qquad V_{\mathsf{ret}} \;\text{“}=\text{”}\; Q$$

  2) feeding in the **initial state**; and 3) projecting out the **value of** $\mathcal{U}$

- Then, wp_Q satisfies the **expected properties**, such as

$$\Gamma \vdash \mathsf{wp}_Q\,(\mathsf{thunk}\,(\mathtt{return}\,V)) \;=\; \lambda x_S : S\,.\,Q\;V\;x_S$$

$$\Gamma \vdash \mathsf{wp}_Q\,(\mathsf{thunk}\,(\mathtt{put}_{\langle \ell, V \rangle}(M))) \;=\; \lambda x_S : S\,.\,\mathsf{wp}_Q\,(\mathsf{thunk}\,M)\;x_S[\ell \mapsto V]$$

Ex2: Dijkstra's weakest precondition sem.

- Given a postcondition on **return values** and **final states**

$$Q : A \to S \to \mathcal{U} \qquad (S \stackrel{\text{def}}{=} \Pi\, \ell : \mathsf{Loc}\,.\,\mathsf{Val}(\ell))$$

  we define a precondition for **stateful comps.** on **initial states**

$$\mathsf{wp}_Q : \mathit{UFA} \to S \to \mathcal{U}$$

  by

  1) handling the given comp. into a **state-passing function** using

$$V_{\mathsf{get}}\,, V_{\mathsf{put}} \quad \text{on} \quad S \to (\mathcal{U} \times S) \qquad \text{and} \qquad V_{\mathsf{ret}} \;\text{“}=\text{”}\; Q$$

  2) feeding in the **initial state**; and **3)** projecting out the **value of** $\mathcal{U}$

- Then, wp_Q satisfies the **expected properties**, such as

$$\Gamma \vdash \mathsf{wp}_Q\,(\mathsf{thunk}\,(\mathtt{return}\,V)) \;=\; \lambda x_S : S\,.\,Q\;V\;x_S$$

$$\Gamma \vdash \mathsf{wp}_Q\,(\mathsf{thunk}\,(\mathtt{put}_{\langle \ell, V\rangle}(M))) \;=\; \lambda x_S : S\,.\,\mathsf{wp}_Q\,(\mathsf{thunk}\,M)\;x_S[\ell \mapsto V]$$

Ex2: Dijkstra's weakest precondition sem.

- Given a postcondition on **return values** and **final states**

$$Q : A \to S \to \mathcal{U} \qquad (S \stackrel{\text{def}}{=} \Pi\, \ell : \mathsf{Loc}\,.\,\mathsf{Val}(\ell))$$

we define a precondition for **stateful comps.** on **initial states**

$$\mathsf{wp}_Q : \mathit{UFA} \to S \to \mathcal{U}$$

by

1) handling the given comp. into a **state-passing function** using

$$V_{\mathsf{get}}\,,\, V_{\mathsf{put}} \quad \text{on} \quad S \to (\mathcal{U} \times S) \qquad \text{and} \qquad V_{\mathsf{ret}} \;\text{“}=\text{”}\; Q$$

2) feeding in the **initial state**; and **3)** projecting out the **value of** $\mathcal{U}$

- Then, wp_Q satisfies the **expected properties**, such as

$$\Gamma \vdash \mathsf{wp}_Q\ (\mathtt{thunk}\,(\mathtt{return}\ V)) \quad = \quad \lambda\, x_S : S\,.\, Q\ V\ x_S$$

$$\Gamma \vdash \mathsf{wp}_Q\ (\mathtt{thunk}\,(\mathtt{put}_{\langle \ell, V \rangle}(M))) \quad = \quad \lambda\, x_S : S\,.\, \mathsf{wp}_Q\ (\mathtt{thunk}\ M)\ x_S[\ell \mapsto V]$$

Ex3: Allowed patterns of (I/O)-effects

- Assuming an inductive type of **I/O-protocols**, given by

$$\mathtt{e} : \mathsf{Protocol} \qquad \mathtt{r} : (\mathsf{Chr} \to \mathsf{Protocol}) \to \mathsf{Protocol}$$

$$\mathtt{w} : (\mathsf{Chr} \to \mathcal{U}) \times \mathsf{Protocol} \to \mathsf{Protocol}$$

- We can define a **relation** between **comps.** and **protocols**

$$\mathsf{Allowed} : \mathit{UFA} \to \mathsf{Protocol} \to \mathcal{U}$$

by handling the given computation using a **handler** on

$$\mathsf{Protocol} \to \mathcal{U}$$

given by (using pattern-matching lambda notation)

$$\begin{aligned}
\mathsf{read}(x_k) \quad &\mapsto \quad \lambda\{(\mathtt{r}\ x_{pr}) \quad \to \widehat{\Pi}\, y : \mathsf{El}(\widehat{\mathsf{Chr}})\,.\, x_k\ y\ (x_{pr}\ y)\ ; \\
&\qquad\quad\ _ \qquad\quad \to \widehat{0}\,\} \\
\mathsf{write}_{x_v}(x_k) \quad &\mapsto \quad \lambda\{(\mathtt{w}\ P\ x_{pr}) \to \widehat{\Sigma}\, y : \mathsf{El}(P\, x_v)\,.\, x_k\ \star\ x_{pr}\ ; \\
&\qquad\quad\ _ \qquad\quad \to \widehat{0}\,\}
\end{aligned}$$

Ex3: Allowed patterns of (I/O)-effects

- Assuming an inductive type of **I/O-protocols**, given by

$$\texttt{e} : \text{Protocol} \qquad \texttt{r} : (\text{Chr} \to \text{Protocol}) \to \text{Protocol}$$

$$\texttt{w} : (\text{Chr} \to \mathcal{U}) \times \text{Protocol} \to \text{Protocol}$$

- We can define a **relation** between **comps.** and **protocols**

$$\text{Allowed} : \mathit{UFA} \to \text{Protocol} \to \mathcal{U}$$

by handling the given computation using a **handler** on

$$\text{Protocol} \to \mathcal{U}$$

given by (using pattern-matching lambda notation)

$$\begin{aligned}
\text{read}(x_k) \quad &\mapsto \quad \lambda\{(\texttt{r}\ x_{pr}) \quad \to \widehat{\Pi}\, y : \text{El}(\widehat{\text{Chr}})\,.\, x_k\ y\ (x_{pr}\ y)\ ; \\
&\qquad\quad _ \qquad\quad \to \widehat{0}\,\} \\
\\
\text{write}_{x_v}(x_k) \quad &\mapsto \quad \lambda\{(\texttt{w}\ P\ x_{pr}) \to \widehat{\Sigma}\, y : \text{El}(P\, x_v)\,.\, x_k\ \star\ x_{pr}\ ; \\
&\qquad\quad _ \qquad\quad \to \widehat{0}\,\}
\end{aligned}$$

Ex3: Allowed patterns of (I/O)-effects

- Assuming an inductive type of **I/O-protocols**, given by

$$\mathtt{e} : \mathsf{Protocol} \qquad \mathtt{r} : (\mathsf{Chr} \to \mathsf{Protocol}) \to \mathsf{Protocol}$$

$$\mathtt{w} : (\mathsf{Chr} \to \mathcal{U}) \times \mathsf{Protocol} \to \mathsf{Protocol}$$

- We can define a **relation** between **comps.** and **protocols**

$$\mathsf{Allowed} : \mathit{UFA} \to \mathsf{Protocol} \to \mathcal{U}$$

by handling the given computation using a **handler** on

$$\mathsf{Protocol} \to \mathcal{U}$$

given by (using pattern-matching lambda notation)

$$\begin{aligned}
\mathsf{read}(x_k) \quad &\mapsto \quad \lambda\{(\mathtt{r}\ x_{pr}) \quad \to \widehat{\Pi}\, y : \mathsf{El}(\widehat{\mathsf{Chr}})\,.\, x_k\ y\ (x_{pr}\ y)\ ; \\
&\qquad\qquad\ \ _ \qquad\quad \to \widehat{0}\,\} \\
\mathsf{write}_{x_v}(x_k) \quad &\mapsto \quad \lambda\{(\mathtt{w}\ P\ x_{pr}) \to \widehat{\Sigma}\, y : \mathsf{El}(P\, x_v)\,.\, x_k\ \star\ x_{pr}\ ; \\
&\qquad\qquad\ \ _ \qquad\quad \to \widehat{0}\,\}
\end{aligned}$$

Outline

- Setting the scene
  - Algebraic effects and their handlers
  - A core effectful dependently typed calculus (FoSSaCS'16)
    [A., Ghani, Plotkin'16]
- What can we gain from handlers + dependent types?
  - Programming with handlers + expressiveness of dep. types
  - Useful for defining predicates/types depending on computations
- Extending the FoSSaCS'16 calculus with alg. effects and handlers
  - Take 1: The common term-level def. of handlers (unsound)
  - Take 2: A new type-level treatment of handlers

Extending the FoSSaCS'16 calculus

- We assume given a **fibred effect theory** $\mathcal{T} = (\mathcal{S}, \mathcal{E})$
- First, we extend the calculus with **algebraic effects** as follows:
  - we extend the **computation terms** with

$$M, N ::= \ldots \mid \mathrm{op}_V^C(y : O[V/x_v] \,.\, M) \qquad (\mathrm{op} : (x_v : I) \longrightarrow O \in \mathcal{S})$$

  - we extend the **equational theory** with equations given in $\mathcal{E}$
  - we capture the **interaction** of comp. terms and ops. with the eq.

$$\frac{\Gamma \vdash V : I \quad \Gamma, x : O[V/x_v] \vdash M : \underline{C} \quad \Gamma \mid z : \underline{C} \vdash K : \underline{D}}{\Gamma \vdash K[\mathrm{op}_V^C(x.M)/z] = \mathrm{op}_V^D(x.K[M/z]) : \underline{D}} \; (\mathrm{op} : (x_v : I) \longrightarrow O \in \mathcal{S})$$

- Second, we extend the calculus with a support for **handlers** ...

Extending the FoSSaCS'16 calculus

- We assume given a **fibred effect theory** $\mathcal{T} = (\mathcal{S}, \mathcal{E})$
- First, we extend the calculus with **algebraic effects** as follows:
  - we extend the **computation terms** with

$$M, N \;::=\; \ldots \;\mid\; \mathrm{op}_V^C(y : O[V/x_v] \,.\, M) \qquad (\mathrm{op} : (x_v : I) \longrightarrow O \in \mathcal{S})$$

  - we extend the **equational theory** with equations given in $\mathcal{E}$
  - we capture the **interaction** of comp. terms and ops. with the eq.

$$\frac{\Gamma \vdash V : I \quad \Gamma, x : O[V/x_v] \vdash M : \underline{C} \quad \Gamma \mid z : \underline{C} \vdash K : \underline{D}}{\Gamma \vdash K[\mathrm{op}_V^C(x.M)/z] = \mathrm{op}_V^D(x.K[M/z]) : \underline{D}} \; (\mathrm{op} : (x_v : I) \longrightarrow O \in \mathcal{S})$$

- Second, we extend the calculus with a support for **handlers** . . .

Extending the FoSSaCS'16 calculus

- We assume given a **fibred effect theory** $\mathcal{T} = (\mathcal{S}, \mathcal{E})$
- First, we extend the calculus with **algebraic effects** as follows:
  - we extend the **computation terms** with

$$M, N \;::=\; \ldots \;\mid\; \mathrm{op}_V^C(y : O[V/x_v] \,.\, M) \qquad (\mathrm{op} : (x_v : I) \longrightarrow O \in \mathcal{S})$$

  - we extend the **equational theory** with equations given in $\mathcal{E}$
  - we capture the **interaction** of comp. terms and ops. with the eq.

$$\frac{\Gamma \vdash V : I \quad \Gamma, x : O[V/x_v] \vdash M : \underline{C} \quad \Gamma \mid z : \underline{C} \vdash K : \underline{D}}{\Gamma \vdash K[\mathrm{op}_V^C(x.M)/z] = \mathrm{op}_V^D(x.K[M/z]) : \underline{D}} \; (\mathrm{op} : (x_v : I) \longrightarrow O \in \mathcal{S})$$

- Second, we extend the calculus with a support for **handlers** ...

Take 1: Term-level definition of handlers

- Begin by extending the FoSSaCS'16 **computation terms** with

$$M, N \;::=\; \ldots \;\mid\; M \;\texttt{handled with}\; \{\mathtt{op}_{x_v}(x_k) \mapsto N_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathsf{eff}}} \;\texttt{to}\; y : A \;\texttt{in}_{\underline{C}}\; N_{\mathsf{ret}}$$

- But as handling denotes a **homomorphism**, then perhaps also

$$K, L \;::=\; \ldots \;\mid\; K \;\texttt{handled with}\; \{\mathtt{op}_{x_v}(x_k) \mapsto N_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathsf{eff}}} \;\texttt{to}\; y : A \;\texttt{in}_{\underline{C}}\; N_{\mathsf{ret}}$$

- However, this leads to an **inconsistent** system, e.g.,

$$\Gamma \vdash \mathtt{write}_{\mathtt{a}}(\mathtt{return}\star) = \mathtt{write}_{\mathtt{z}}(\mathtt{return}\star) : F1$$

- At a very high-level, the problem is (see the paper for details)
  - interaction between Ks and ops. is governed by comp. types
  - but the type of `handled with` does not mention the handler

Take 1: Term-level definition of handlers

- Begin by extending the FoSSaCS'16 **computation terms** with

$$M, N \;::=\; \ldots \;\mid\; M \;\texttt{handled with}\; \{\mathrm{op}_{x_v}(x_k) \mapsto N_{\mathrm{op}}\}_{\mathrm{op} \in \mathcal{S}_{\mathrm{eff}}} \;\texttt{to}\; y : A \;\texttt{in}_{\underline{C}}\; N_{\mathrm{ret}}$$

- But as handling denotes a **homomorphism**, then perhaps also

$$K, L \;::=\; \ldots \;\mid\; K \;\texttt{handled with}\; \{\mathrm{op}_{x_v}(x_k) \mapsto N_{\mathrm{op}}\}_{\mathrm{op} \in \mathcal{S}_{\mathrm{eff}}} \;\texttt{to}\; y : A \;\texttt{in}_{\underline{C}}\; N_{\mathrm{ret}}$$

- However, this leads to an **inconsistent** system, e.g.,

$$\Gamma \vdash \mathtt{write}_{\mathtt{a}}(\mathtt{return}\star) = \mathtt{write}_{\mathtt{z}}(\mathtt{return}\star) : F1$$

- At a very high-level, the problem is (see the paper for details)
  - interaction between Ks and ops. is governed by comp. types
  - but the type of `handled with` does not mention the handler

Take 1: Term-level definition of handlers

- Begin by extending the FoSSaCS'16 **computation terms** with

$$M, N \;::=\; \ldots \;\mid\; M \;\texttt{handled with}\; \{\mathrm{op}_{x_v}(x_k) \mapsto N_{\mathrm{op}}\}_{\mathrm{op} \in \mathcal{S}_{\mathrm{eff}}} \;\texttt{to}\; y : A \;\texttt{in}_{\underline{C}}\; N_{\mathrm{ret}}$$

- But as handling denotes a **homomorphism**, then perhaps also

$$K, L \;::=\; \ldots \;\mid\; K \;\texttt{handled with}\; \{\mathrm{op}_{x_v}(x_k) \mapsto N_{\mathrm{op}}\}_{\mathrm{op} \in \mathcal{S}_{\mathrm{eff}}} \;\texttt{to}\; y : A \;\texttt{in}_{\underline{C}}\; N_{\mathrm{ret}}$$

- However, this leads to an **inconsistent** system, e.g.,

$$\Gamma \vdash \texttt{write}_{\mathbf{a}}(\texttt{return}\star) = \texttt{write}_{\mathbf{z}}(\texttt{return}\star) : F1$$

- At a very high-level, the problem is (see the paper for details)
  - interaction between Ks and ops. is governed by comp. types
  - but the type of `handled with` does not mention the handler

Take 1: Term-level definition of handlers

- Begin by extending the FoSSaCS'16 **computation terms** with

$$M, N \;::=\; \ldots \;\mid\; M \;\texttt{handled with}\; \{\mathrm{op}_{x_v}(x_k) \mapsto N_{\mathrm{op}}\}_{\mathrm{op} \in \mathcal{S}_{\mathrm{eff}}} \;\texttt{to}\; y : A \;\texttt{in}_{\underline{C}}\; N_{\mathrm{ret}}$$

- But as handling denotes a **homomorphism**, then perhaps also

$$K, L \;::=\; \ldots \;\mid\; K \;\texttt{handled with}\; \{\mathrm{op}_{x_v}(x_k) \mapsto N_{\mathrm{op}}\}_{\mathrm{op} \in \mathcal{S}_{\mathrm{eff}}} \;\texttt{to}\; y : A \;\texttt{in}_{\underline{C}}\; N_{\mathrm{ret}}$$

- However, this leads to an **inconsistent** system, e.g.,

$$\Gamma \vdash \texttt{write}_{\mathbf{a}}(\texttt{return}\,\star) = \texttt{write}_{\mathbf{z}}(\texttt{return}\,\star) : F1$$

- At a very high-level, the problem is (see the paper for details)
  - interaction between Ks and ops. is governed by comp. types
  - but the type of `handled with` does not mention the handler

How to proceed?

- Possible ways to solve this unsoundness problem
  - **Option 1:** Change the FoSSaCS'16 calculus
    - change the equational theory of homomorphism terms
    - hom. terms would not denote homomorphisms any more
    - investigated for exceptions in CBPV with stacks by [Levy'06]
  - **Option 2:** Keep the FoSSaCS'16 calculus unchanged
    - extend it so that handling for comp. terms is derivable
    - while making sure that the calculus remains sound
    - **key idea:** comp. types and handlers both denote **algebras**
    - extended calculus admits a natural denotational semantics

How to proceed?

- Possible ways to solve this unsoundness problem
  - **Option 1:** Change the FoSSaCS'16 calculus
    - change the equational theory of homomorphism terms
    - hom. terms would not denote homomorphisms any more
    - investigated for exceptions in CBPV with stacks by [Levy'06]
  - **Option 2:** Keep the FoSSaCS'16 calculus unchanged
    - extend it so that handling for comp. terms is derivable
    - while making sure that the calculus remains sound
    - **key idea:** comp. types and handlers both denote **algebras**
    - extended calculus admits a natural denotational semantics

How to proceed?

- Possible ways to solve this unsoundness problem
  - **Option 1:** Change the FoSSaCS'16 calculus
    - change the equational theory of homomorphism terms
    - hom. terms would not denote homomorphisms any more
    - investigated for exceptions in CBPV with stacks by [Levy'06]
  - **Option 2:** Keep the FoSSaCS'16 calculus unchanged
    - extend it so that handling for comp. terms is derivable
    - while making sure that the calculus remains sound
    - **key idea:** comp. types and handlers both denote **algebras**
    - extended calculus admits a natural denotational semantics

Take 2: A type-level treatment of handlers

- Instead, we extend the FoSSaCS'16 **computation types** with
  - a **user-defined algebra type**

$$\underline{C}, \underline{D} \;::=\; \ldots \;\mid\; \langle A; \overrightarrow{V_{\mathsf{op}}}; \overrightarrow{W_{\mathsf{eq}}} \rangle$$

    where
    - A is the **carrier** value type
    - $\overrightarrow{V_{\mathsf{op}}}$ is a set of user-defined **operations**
    - $\overrightarrow{W_{\mathsf{eq}}}$ is a set of **witnesses** of equational proof obligations

- As a result, we can derive the **handing construct** as

$$M \;\mathtt{handled\ with}\; \{\mathsf{op}_{x_v}(x_k) \mapsto N_{\mathsf{op}}; \overrightarrow{W_{\mathsf{eq}}}\}_{\mathsf{op} \in S_{\mathsf{eff}}} \;\mathtt{to}\; y : A \;\mathtt{in}_{\underline{C}}\; N_{\mathsf{ret}}$$

$$\stackrel{\text{def}}{=}$$

$$\mathtt{force}_{\underline{C}}(\mathtt{thunk}(\underbrace{M \;\mathtt{to}\; y : A \;\mathtt{in}\; \mathtt{force}_{\langle U\underline{C}; \overrightarrow{V_{N_{\mathsf{op}}}}; \overrightarrow{W_{\mathsf{eq}}} \rangle}(\mathtt{thunk}\, N_{\mathsf{ret}})}_{\text{temporarily working at type } \langle U\underline{C}; \overrightarrow{V_{N_{\mathsf{op}}}}; \overrightarrow{W_{\mathsf{eq}}} \rangle}))$$

  and similarly for the "**into-values**" variant of it

Take 2: A type-level treatment of handlers

- Instead, we extend the FoSSaCS'16 **computation types** with
  - a **user-defined algebra type**

$$\underline{C}, \underline{D} \;::=\; \ldots \;\mid\; \langle A; \overrightarrow{V_{\mathsf{op}}}; \overrightarrow{W_{\mathsf{eq}}} \rangle$$

    where
    - A is the **carrier** value type
    - $\overrightarrow{V_{\mathsf{op}}}$ is a set of user-defined **operations**
    - $\overrightarrow{W_{\mathsf{eq}}}$ is a set of **witnesses** of equational proof obligations

- As a result, we can derive the **handing construct** as

$$M \;\texttt{handled with}\; \{\mathsf{op}_{x_v}(x_k) \mapsto N_{\mathsf{op}}; \overrightarrow{W_{\mathsf{eq}}}\}_{\mathsf{op} \in \mathcal{S}_{\mathsf{eff}}} \;\texttt{to}\; y : A \;\texttt{in}_{\underline{C}}\; N_{\mathsf{ret}}$$

$$\stackrel{\mathsf{def}}{=\!=}$$

$$\texttt{force}_{\underline{C}}(\texttt{thunk}(\underbrace{M \;\texttt{to}\; y : A \;\texttt{in}\; \texttt{force}_{\langle U\underline{C}; \overrightarrow{V_{N_{\mathsf{op}}}}; \overrightarrow{W_{\mathsf{eq}}} \rangle}(\texttt{thunk}\, N_{\mathsf{ret}})}_{\text{temporarily working at type } \langle U\underline{C}; \overrightarrow{V_{N_{\mathsf{op}}}}; \overrightarrow{W_{\mathsf{eq}}} \rangle}))$$

and similarly for the "**into-values**" variant of it

Take 2: A type-level treatment of handlers

- Instead, we extend the FoSSaCS'16 **computation types** with
  - a **user-defined algebra type**

$$\underline{C}, \underline{D} \;::=\; \ldots \;\mid\; \langle A; \overrightarrow{V_{\mathsf{op}}}; \overrightarrow{W_{\mathsf{eq}}} \rangle$$

    where
    - A is the **carrier** value type
    - $\overrightarrow{V_{\mathsf{op}}}$ is a set of user-defined **operations**
    - $\overrightarrow{W_{\mathsf{eq}}}$ is a set of **witnesses** of equational proof obligations

- As a result, we can derive the **handing construct** as

$$M \;\texttt{handled with}\; \{\mathsf{op}_{x_v}(x_k) \mapsto N_{\mathsf{op}}; \overrightarrow{W_{\mathsf{eq}}}\}_{\mathsf{op} \in \mathcal{S}_{\mathsf{eff}}} \;\texttt{to}\; y : A \;\texttt{in}_{\underline{C}}\; N_{\mathsf{ret}}$$

$$\stackrel{\mathsf{def}}{=\!=}$$

$$\texttt{force}_{\underline{C}}(\texttt{thunk}(\underbrace{M \;\texttt{to}\; y : A \;\texttt{in}\; \texttt{force}_{\langle U\underline{C}; \overrightarrow{V_{N_{\mathsf{op}}}}; \overrightarrow{W_{\mathsf{eq}}} \rangle}(\texttt{thunk}\, N_{\mathsf{ret}})}_{\text{temporarily working at type } \langle U\underline{C}; \overrightarrow{V_{N_{\mathsf{op}}}}; \overrightarrow{W_{\mathsf{eq}}} \rangle}))$$

  and similarly for the "**into-values**" variant of it

Conclusion

- In conclusion
  - handlers are natural for defining **predicates on computations**
    - lifting predicates from return values to computations
    - Dijkstra's weakest precondition semantics of state
    - specifying patterns of allowed (I/O)-effects
  - they admit a principled **type-based treatment**

- See the paper for
  - **formal details** of what I have shown you today
  - families fibrations based **denotational semantics** of the calculus
  - discussion about the calculus's inherent **extensional nature**
  - **Agda code** for the example predicates $P : \mathit{UFA} \to \mathcal{U}$

Thank you!

Questions?