A fibrational view on computational effects

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Background – dependent types

The Curry-Howard correspondence:

Simple Types	$\sim$	Propositional Logic	$(\text{Nat}, \text{String}, \dots)$
Dependent Types	$\sim$	Predicate Logic	$(\Sigma, \Pi, =, \dots)$

A tiny example: we can use dep. types to express sorted lists

$$\ell : (\text{List Nat}) \vdash \text{Sorted}(\ell) \stackrel{\text{def}}{=} \overset{\forall}{\Pi}\, i : \text{Nat} \,.\, (0 < i < \texttt{len}\ \ell) \overset{\Rightarrow}{\to} (\ell[i\text{-}1] \leq \ell[i])$$

which in turn could be used for typing sorting functions

$$\texttt{sort} : \overset{\forall}{\Pi}\, \ell : (\text{List Nat}) \,.\, \overset{\exists}{\Sigma}\, \ell' : (\text{List Nat}) \,.\, \Big(\text{Sorted}(\ell') \overset{\wedge}{\times} \dots\Big)$$

Large examples: CompCert (Coq), miTLS and HACL* (F*), ...

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Background – computational effects

Examples:

- state, exceptions, divergence, IO, nondeterminism, probability, ...

Meta-languages and models for comp. effects: based on

- monads (λ_c, λ_{ML}, FGCBV) (Moggi, Levy)

$$[\![\Gamma \vdash M : A]\!]_{\lambda_c} : [\![\Gamma]\!] \longrightarrow T[\![A]\!]$$

- adjunctions (CBPV, EEC) (Levy, Egger et al.)

$$[\![\Gamma \vdash_v V : A]\!]_{CBPV} : [\![\Gamma]\!] \longrightarrow [\![A]\!] \qquad [\![\Gamma \vdash_c M : \underline{C}]\!]_{CBPV} : [\![\Gamma]\!] \longrightarrow U([\![\underline{C}]\!])$$

- algebraic presentations (Plotkin and Power)

$$\mathsf{get} : 1 \rightharpoonup S \qquad \mathsf{put} : S \rightharpoonup 1 \qquad (+\ \text{equations})$$

Outline – putting the two together

We investigate the combination of

- dependent types $(\Pi, \Sigma, V =_A W, \ldots)$
- computational effects (state, nondeterminism, IO, ...)

Goals

- tell a mathematically natural story
- use established math. techniques
- cover a wide range of comp. effects
- discover smth. interesting

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Two guiding problems

- effectful programs in types (e.g., `get` and `put` in types)
- typing of effectful programs (e.g., sequential composition)

Effectful programs in types

(type-dependency in the presence of effects)

Effectful programs in types

Q: Should we allow situations such as Sorted[$\texttt{receive}(y.\,M)/\ell$]?

A1: In this work, we say not directly

- types should only depend on static information about effects
- allow dependency on effectful comps. via analysing thunks

A2: Various people are also looking at the direct case

- type-dependency needs to be "homomorphic"
- intuitively,
  - need to lift Sorted(ℓ) to Sorted†(c), where $c : T(\text{List Chr})$

$$\text{Sorted}^\dagger(\texttt{receive}(y.\,\texttt{return}\,y)) = \langle\texttt{receive}\rangle(y.\,\text{Sorted}(y))$$

  - for this Sorted needs to be a T-algebra
- (cf. recent papers by Pédrot and Tabareau; and Bowman et al.)

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Effectful programs in types

Aim: Types should only depend on static info about effects

Solution: CBPV/EEC style distinction between vals. and comps.

- value types $\Gamma \vdash A$ (MLTT + thunks + ...)
- computation types $\Gamma \vdash \underline{C}$ (dep. typed CBPV/EEC)
- where Γ contains only value variables $x_1 : A_1, \ldots, x_n : A_n$

Could have also considered Moggi's λ_{ML} or Levy's FGCBV

- building on CBPV/EEC gives a more general story
- especially for the treatment of sequential composition
- and also for (Idris-style parameterised) dependent effect-typing

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Typing of effectful programs

(e.g., sequential composition)

Assigning types to effectful programs

The problem: The standard typing rule for seq. composition

$$\frac{\Gamma \vDash_{\mathsf{c}} M : F\,A \qquad \Gamma, x\!:\!A \vDash_{\mathsf{c}} N : \underline{C}(x)}{\Gamma \vDash_{\mathsf{c}} M \;\mathtt{to}\; x\!:\!A \;\mathtt{in}\; N : \underline{C}(x)}$$

is not correct any more because it potentially allows

$$x \in FV(\underline{C})$$

in the conclusion

Assigning types to effectful programs

Aim: To fix the typing rule of sequential composition

Option 1: We could restrict the free variables in $\underline{C}$: [Levy'04]

$$\frac{\Gamma \models_c M : F\,A \qquad \Gamma \vdash \underline{C} \qquad \Gamma, x{:}A \models_c N : \underline{C}}{\Gamma \models_c M \;\mathtt{to}\; x{:}A \;\mathtt{in}\; N : \underline{C}}$$

But: Sometimes it is useful if $\underline{C}$ can depend on x!

- say we consider

$$\mathtt{fopen}\,(\mathtt{return\ true}, \mathtt{return\ false}) \;\mathtt{to}\; x{:}\mathsf{Bool} \;\mathtt{in}\; N$$

- then it would be natural to let $\underline{C}$ depend on x, e.g.,

$$x{:}\mathsf{Bool} \vdash \underline{C}(x) \;\stackrel{\text{def}}{=}\; \mathtt{if}\; x \;\mathtt{then}\; \text{“allow } \mathtt{fread}, \mathtt{fwrite}, \text{ and } \mathtt{fclose}\text{”} \;\mathtt{else}\; \text{“allow } \mathtt{fopen}\text{”}$$

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Option 2: One could lift sequential composition to type level

$$\Gamma \vdash_c M \ \mathtt{to}\ x{:}A\ \mathtt{in}\ N : M\ \mathtt{to}\ x{:}A\ \mathtt{in}\ \underline{C}$$

But: Then comp. types would be singleton-like!?!

Option 3: In the monadic metalanguage λ_{ML}, one could try

$$\frac{\Gamma \vdash M : T\,A \qquad \Gamma, x{:}A \vdash N : T\,B(x)}{\Gamma \vdash M\ \mathtt{to}\ x{:}A\ \mathtt{in}\ N : T\,(\Sigma x : A.B)}$$

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Our solution: We draw inspiration from algebraic effects

- and combine this with restricting $\underline{C}$ in seq. comp. (**Option 1**)

E.g., consider the non-deterministic prog. $\left(\text{for } x\!:\!\mathsf{Nat} \vDash_c N : \underline{C}(x)\right)$

$$M \overset{\text{def}}{=} \texttt{choose}\,(\texttt{return}\;4, \texttt{return}\;2)\;\texttt{to}\;x\!:\!\mathsf{Nat}\;\texttt{in}\;N$$

After making the non-det. choice, this program evaluates as either

$$N[4/x] : \underline{C}[4/x] \qquad \text{or} \qquad N[2/x] : \underline{C}[2/x]$$

Idea: M denotes an element of the coproduct of algebras

$$\underline{C}[4/x] + \underline{C}[2/x] \overset{\text{def}}{=} F\left(U\left(\underline{C}[4/x]\right) + U\left(\underline{C}[2/x]\right)\right)_{/\equiv}$$

which we generalise to A-indexed coproducts, i.e., a comp. Σ-type

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Putting these ideas together

(eMLTT: a core dep.-typed calculus with comp. effects)

eMLTT – value and comp. types

Value types: MLTT + thunks + ...

$$A, B ::= \mathsf{Nat} \mid 1 \mid 0 \mid \Pi\, x : A \,.\, B \mid \Sigma\, x : A \,.\, B \mid V =_A W \mid U\underline{C} \mid \ldots$$

- $U\underline{C}$ is the type of thunked (i.e., suspended) computations

Computation types: dep.-typed version of EEC's comp. types

$$\underline{C}, \underline{D} ::= F A \mid \Pi\, x : A \,.\, \underline{C} \mid \Sigma\, x : A \,.\, \underline{C}$$

- $F A$ is the type of computations returning values of type A
- $\Pi\, x : A \,.\, \underline{C}$ is the type of dependent effectful functions
  - generalises CBPV/EEC's comp. types $A \to \underline{C}$ and $\underline{C} \times \underline{D}$
- $\Sigma\, x : A \,.\, \underline{C}$ is the type of dep. pairs of values and effectful comps.
  - captures the intuition about seq. comp. and coprods. of algebras
  - generalises EEC's comp. types $!A \otimes \underline{C}$ and $\underline{C} \oplus \underline{D}$

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$$V, W ::= x \mid \mathtt{zero} \mid \mathtt{succ}\, V \mid \ldots \mid \mathtt{thunk}\ M \mid \ldots$$

- equational theory based on intensional MLTT

Comp. terms: dep.-typed version of CBPV/EEC's comp. terms

$$\begin{array}{rcll} M, N & ::= & \mathtt{force}\ V & \\ & \mid & \mathtt{return}\, V & \\ & \mid & M \ \mathtt{to}\ x : A\ \mathtt{in}\ N & \\ & \mid & \lambda\, x : A\, .\, M & \\ & \mid & M V & \\ & \mid & \langle V, M \rangle & \text{(comp. } \Sigma \text{ intro.)} \\ & \mid & M \ \mathtt{to}\ \langle x : A, z : \underline{C} \rangle\ \mathtt{in}\ K & \text{(comp. } \Sigma \text{ elim.)} \end{array}$$

But: Value and comp. terms alone do not suffice, as in EEC!

eMLTT – value and comp. terms

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eMLTT – homomorphism terms

Note: We need to define K in such a way that the intended left-to-right evaluation order is preserved, e.g., consider

$$\Gamma \models_{\mathsf{c}} \langle V, M \rangle \ \texttt{to} \ \langle x \!:\! A, z \!:\! \underline{C} \rangle \ \texttt{in} \ K = K[V/x, M/z] : \underline{D}$$

Homomorphism terms: dep.-typed version of EEC's linear terms

$$
\begin{array}{rcll}
K, L & ::= & z & \text{(linear comp. vars.)} \\
& \mid & K \ \texttt{to} \ x \!:\! A \ \texttt{in} \ M & \\
& \mid & \lambda \, x \!:\! A \, . \, K & \\
& \mid & K \, V & \\
& \mid & \langle V, K \rangle & \text{(comp. } \Sigma \text{ intro.)} \\
& \mid & K \ \texttt{to} \ \langle x \!:\! A, z \!:\! \underline{C} \rangle \ \texttt{in} \ L & \text{(comp. } \Sigma \text{ elim.)}
\end{array}
$$

Typing judgments:

- $\Gamma \models_{\mathsf{v}} V : A$
- $\Gamma \models_{\mathsf{c}} M : \underline{C}$
- $\Gamma \mid z \!:\! \underline{C} \models_{\mathsf{h}} K : \underline{D}$ (linear in z; comp. bound to z happens first)

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$$\Gamma \vDash_{\mathsf{c}} \langle V, M \rangle \texttt{ to } \langle x : A, z : \underline{C} \rangle \texttt{ in } K = K[V/x, M/z] : \underline{D}$$

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& | & K\, V & \\
& | & \langle V, K \rangle & \text{(comp. } \Sigma \text{ intro.)} \\
& | & K \texttt{ to } \langle x : A, z : \underline{C} \rangle \texttt{ in } L & \text{(comp. } \Sigma \text{ elim.)}
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eMLTT – typing sequential composition

- We can then account for type-dependency in seq. comp. as

$$\dfrac{\Gamma \vDash_{c} M : F\,A \qquad \Gamma \vdash \Sigma\, x\!:\!A\,.\,\underline{C}(x) \qquad \dfrac{\Gamma, x\!:\!A \vDash_{c} N : \underline{C}(x)}{\Gamma, x\!:\!A \vDash_{c} \langle x, N\rangle : \Sigma\, x\!:\!A\,.\,\underline{C}(x)}}{\Gamma \vDash_{c} M \;\mathtt{to}\; x\!:\!A \;\mathtt{in}\; \langle x, N\rangle : \Sigma\, x\!:\!A\,.\,\underline{C}(x)}$$

- As a bonus, the comp. Σ-type can also be used to explain Idris's

$$\dfrac{\Gamma \vDash_{v} \varepsilon_1 : \mathsf{Effect} \qquad \Gamma \vdash A \qquad \Gamma \vDash_{v} \varepsilon_2 : A \to \mathsf{Effect}}{\Gamma \vdash T\,\varepsilon_1\,A\,\varepsilon_2}$$

in terms of standard parameterised effect-typing as

$$T\,\varepsilon_1\,A\,\varepsilon_2 \stackrel{\text{def}}{=} U_{\varepsilon_1}(\Sigma\, x\!:\!A\,.\,F_{\varepsilon_2\,x}\,1)$$

and thus naturally accommodate examples like

$$\mathtt{fopen}\,(\mathtt{return\ true}, \mathtt{return\ false})\;\mathtt{to}\; x\!:\!\mathsf{Bool}\;\mathtt{in}\; N$$

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Fibred adjunction models

(categorical semantics of eMLTT)

Fibred adjunction models – value part

Given by a split closed comprehension category p, as in

$$\begin{tikzcd}
\mathcal{V} \arrow[d, bend right=60, "p"'] \arrow[d, bend left=60, "\{-\}"] \\
\mathcal{B} \arrow[u, "\dashv"', "1 \;\dashv"]
\end{tikzcd}$$

allowing us to define a partial interpretation fun. $[\![-]\!]$, that maps:

- a context Γ to and object $[\![\Gamma]\!]$ in $\mathcal{B}$, with
  - $[\![\diamond]\!] \overset{\text{def}}{=} 1$
  - $[\![\Gamma, x : A]\!] \overset{\text{def}}{=} \{[\![\Gamma; A]\!]\}$ (if $x \notin \mathit{Vars}(\Gamma)$ and $[\![\Gamma; A]\!]$ is defined)
- a context Γ and a value type A to an object $[\![\Gamma; A]\!]$ in $\mathcal{V}_{[\![\Gamma]\!]}$
- a context Γ and a value term V to $[\![\Gamma; V]\!] : 1_{[\![\Gamma]\!]} \longrightarrow A$ in $\mathcal{V}_{[\![\Gamma]\!]}$

Fibred adjunction models – value part

Given by a split closed comprehension category p, as in

$$\begin{tikzcd}
\mathcal{V} \arrow[d, bend right=60, "p"'] \arrow[d, bend left=60, "\{-\}"] \arrow[d, phantom, shift right=4ex, "\dashv"] \arrow[d, phantom, shift left=4ex, "\dashv"] \\
\mathcal{B} \arrow[u, "1"']
\end{tikzcd}$$

such that

- p has split fibred strong colimits of shape $\mathbf{0}$ and $\mathbf{2}$ [Jacobs'99]
  - (in thesis, also Jacobs-style characterisation for arbitrary shapes)
- p has weak split fibred strong natural numbers
  - (axiomatisation is given in the style of fibrational induction)
- p has split intensional propositional equality
  - (currently very synthetic ax., would like a weak form of adjoints)

Fibred adjunction models – effects part

Given by a split fibration q and a split fib. adjunction $F \dashv U$, as in

$$\begin{tikzcd}
\mathcal{V} \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] \arrow[d, bend right=60, "p"'] \arrow[d, bend left=60, "\{-\}"] & & \mathcal{C} \arrow[ll, bend left, "U"] \arrow[dll, bend left, "q"] \\
\mathcal{B} \arrow[u, "\dashv\ 1\ \dashv" description]
\end{tikzcd}$$

we extend the partial interpretation fun. $[\![-]\!]$ so that it maps:

- a ctx. Γ and a comp. type $\underline{C}$ to an object $[\![\Gamma; \underline{C}]\!]$ in $\mathcal{C}_{[\![\Gamma]\!]}$
- a ctx. Γ and a comp. term M to $[\![\Gamma; M]\!] : 1_{[\![\Gamma]\!]} \longrightarrow U(\underline{C})$ in $\mathcal{V}_{[\![\Gamma]\!]}$
- a ctx. Γ, a comp. var. z, a comp. type $\underline{C}$, and a hom. term K to
$$[\![\Gamma; z : \underline{C}; K]\!] : [\![\Gamma; \underline{C}]\!] \longrightarrow \underline{D} \text{ in } \mathcal{C}_{[\![\Gamma]\!]}$$

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$$\begin{tikzcd}
\mathcal{V} \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] \arrow[d, bend right=60, "p"'] \arrow[d, bend right=30, phantom, "\dashv"] \arrow[d, bend left=30, phantom, "\dashv"] \arrow[d, bend left=60, "\{-\}"] & & \mathcal{C} \arrow[ll, bend left, "U"] \arrow[dll, bend left, "q"] \\
\mathcal{B} \arrow[u, "1"] & &
\end{tikzcd}$$

such that

- q has split dependent p-products (comp. Π-type; r. adj. to wk.)
- q has split dependent p-coproducts (comp. Σ-type; l. adj. to wk.)

and to account for the full calculus presented in the thesis,

- q admits split fibred pre-enrichment in p (hom. function type $\multimap$)

Fibred adjunction models – correctness

Theorem (Soundness):

- If $\Gamma \vdash \underline{C}$, then $[\![\Gamma; \underline{C}]\!] \in \mathcal{C}_{[\![\Gamma]\!]}$
- If $\Gamma \vdash_{\mathsf{c}} M : \underline{C}$, then $[\![\Gamma; M]\!] : 1_{[\![\Gamma]\!]} \longrightarrow U([\![\Gamma; \underline{C}]\!])$
- If $\Gamma \,|\, z : \underline{C} \vdash_{\mathsf{h}} K : \underline{D}$, then $[\![\Gamma; z : \underline{C}; K]\!] : [\![\Gamma; \underline{C}]\!] \longrightarrow [\![\Gamma; \underline{D}]\!]$
- If $\Gamma \vdash \underline{C} = \underline{D}$, then $[\![\Gamma; \underline{C}]\!] = [\![\Gamma; \underline{D}]\!] \in \mathcal{C}_{[\![\Gamma]\!]}$
- ...

Theorem (Classifying model):

- The well-formed syntax of eMLTT forms a fib. adjunction model.

Theorem (Completeness):

- If two types or terms are equal in all fibred adjunction models, then they are also equal in the equational theory of eMLTT.

Examples of fibred adjunction models

$$\begin{tikzcd}
\mathcal{V} \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] \arrow[d, bend right=60, "p"'] \arrow[d, bend left=60, "\{-\}"] \arrow[d, phantom, shift right=3ex, "\dashv"] \arrow[d, phantom, shift left=3ex, "\dashv"] & & \mathcal{C} \arrow[ll, bend left, "U"] \arrow[dll, bend left, "q"] \\
\mathcal{B} \arrow[u, "1"'] & &
\end{tikzcd}$$

Examples of fibred adjunction models

Example 1 (identity adjunctions):

- sound as long as no actual comp. effects in the calculus

Example 2 (simple fibrations from enriched adj. models of EEC):

- given an adj. model of EEC $F \dashv U : \mathcal{C} \longrightarrow \mathcal{V}$ ($\mathcal{V}$ a CCC, ...),

  we can lift it to simple fibrations $\widehat{F} \dashv \widehat{U} : \mathsf{s}(\mathcal{V},\mathcal{C}) \longrightarrow \mathsf{s}(\mathcal{V})$

  where

  $$\mathsf{s}_{\mathcal{V},\mathcal{C}} : \mathsf{s}(\mathcal{V},\mathcal{C}) \longrightarrow \mathcal{V}$$

  is defined as

  $$\mathsf{s}_{\mathcal{V},\mathcal{C}}\Big(X \in \mathcal{V}\,,\,\underline{C} \in \mathcal{C}\Big) \overset{\text{def}}{=} X$$

  $$\mathsf{s}_{\mathcal{V},\mathcal{C}}\Big(f : X \longrightarrow Y\,;\,h : X \otimes \underline{C} \longrightarrow \underline{D}\Big) \overset{\text{def}}{=} f \qquad : \mathsf{s}_{\mathcal{V},\mathcal{C}}(X,\underline{C}) \longrightarrow \mathsf{s}_{\mathcal{V},\mathcal{C}}(Y,\underline{D})$$

- doesn't support any real type dependency (constant families)

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- doesn't support any real type dependency (constant families)

Examples of fibred adjunction models

Example 3 (families fibrations and lifting of adjunctions):

- given a suitable adjunction $F_{\mathcal{D}} \dashv U_{\mathcal{D}} : \mathcal{D} \longrightarrow \mathsf{Set}$,

  we can lift it to $\widehat{F_{\mathcal{D}}} \dashv \widehat{U_{\mathcal{D}}} : \mathsf{Fam}(\mathcal{D}) \longrightarrow \mathsf{Fam}(\mathsf{Set})$

  between

$$\mathsf{fam}_{\mathsf{Set}} : \mathsf{Fam}(\mathsf{Set}) \longrightarrow \mathsf{Set}$$

$$\mathsf{fam}_{\mathcal{D}} : \mathsf{Fam}(\mathcal{D}) \longrightarrow \mathsf{Set}$$

- resulting in
  - $[\![\Gamma; A]\!] = ([\![\Gamma]\!], [\![A]\!]) \in \mathsf{Fam}(\mathsf{Set})$ $\quad ([\![\Gamma]\!] \in \mathsf{Set}, [\![A]\!] \in [\![\Gamma]\!] \longrightarrow \mathsf{Set})$
  - $[\![\Gamma; \underline{C}]\!] = ([\![\Gamma]\!], [\![\underline{C}]\!]) \in \mathsf{Fam}(\mathcal{D})$ $\quad ([\![\underline{C}]\!] \in [\![\Gamma]\!] \longrightarrow \mathcal{D})$

- examples
  - $F^{\mathbf{T}} \dashv U^{\mathbf{T}} : \mathsf{Set}^{\mathbf{T}} \longrightarrow \mathsf{Set}$
  - $(-) \times S \;\dashv\; (-)^S : \mathsf{Set} \longrightarrow \mathsf{Set}$
  - $R^{(-)} \;\dashv\; R^{(-)} : \mathsf{Set}^{op} \longrightarrow \mathsf{Set}$

Examples of fibred adjunction models

Example 4 (continuous families and CPO-enriched monads):

- given the EM-adjunction $F^{\mathbf{T}} \dashv U^{\mathbf{T}} : \mathrm{CPO}^{\mathbf{T}} \longrightarrow \mathrm{CPO}$,

  we can lift it to $\widehat{F_{\mathcal{D}}} \dashv \widehat{U_{\mathcal{D}}} : \mathrm{CFam}(\mathrm{CPO}^{\mathbf{T}}) \longrightarrow \mathrm{CFam}(\mathrm{CPO})$

  between

$$\mathrm{cfam}_{\mathrm{CPO}} : \mathrm{CFam}(\mathrm{CPO}) \longrightarrow \mathrm{CPO}$$

$$\mathrm{cfam}_{\mathrm{CPO}^{\mathbf{T}}} : \mathrm{CFam}(\mathrm{CPO}^{\mathbf{T}}) \longrightarrow \mathrm{CPO}$$

- resulting in
  - $(\llbracket \Gamma \rrbracket, \llbracket A \rrbracket) \in \mathrm{CFam}(\mathrm{CPO})$ $\quad (\llbracket \Gamma \rrbracket \in \mathrm{CPO}, \llbracket A \rrbracket \in \llbracket \Gamma \rrbracket \longrightarrow \mathrm{CPO}^{EP})$
  - $(\llbracket \Gamma \rrbracket, \llbracket \underline{C} \rrbracket) \in \mathrm{CFam}(\mathrm{CPO}^{\mathbf{T}})$ $\quad (\llbracket \underline{C} \rrbracket \in \llbracket \Gamma \rrbracket \longrightarrow (\mathrm{CPO}^{\mathbf{T}})^{EP})$

- if $\mathbf{T}$ supports a least zero-ary op., then it also models recursion

$$M \ ::= \ \ldots \ \mid \ \mu\, x : U\underline{C} \,.\, M$$

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- resulting in
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  - $(\llbracket \Gamma \rrbracket, \llbracket \underline{C} \rrbracket) \in \mathrm{CFam}(\mathrm{CPO}^{\mathbf{T}})$ $\quad(\llbracket \underline{C} \rrbracket \in \llbracket \Gamma \rrbracket \longrightarrow (\mathrm{CPO}^{\mathbf{T}})^{EP})$

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Examples of fibred adjunction models

Example 5 (EM-resolutions of split fibred monads):

- given a split fibred monad $\mathbf{T} = (T, \eta, \mu)$ on p, i.e.,

$$\begin{tikzcd} \mathcal{V} \arrow[rr, "T"] \arrow[dr, "p"'] & & \mathcal{V} \arrow[dl, "p"] \\ & \mathcal{B} & \end{tikzcd}$$

and $\quad p(\eta_A) = \mathrm{id}_{p(A)} \qquad p(\mu_A) = \mathrm{id}_{p(A)}$

- we consider models based on the EM-resolution of $\mathbf{T}$

$$\begin{tikzcd} \mathcal{V} \arrow[rr, bend left, "F^{\mathbf{T}}"] \arrow[rr, phantom, "\perp"] \arrow[dr, "p"'] & & \mathcal{V}^{\mathbf{T}} \arrow[ll, bend left, "U^{\mathbf{T}}"] \arrow[dl, "p^{\mathbf{T}}"] \\ & \mathcal{B} & \end{tikzcd}$$

where $\quad \Big(A \in \mathcal{V}\ ,\ \alpha : T(A) \longrightarrow A\Big) \in \mathcal{V}^{\mathbf{T}}$

- and show that three familiar results hold for this situation

Examples of fibred adjunction models

Example 5 (EM-resolutions of split fibred monads):

- **Theorem 1:** If p supports Π-types, then $p^{\mathbf{T}}$ also supports Π-types

$$\Pi^{\mathbf{T}}_A(B, \beta) \stackrel{\text{def}}{=} \left(\Pi_A(B), \beta_{\Pi^{\mathbf{T}}_A}\right)$$

- **Prop.:** If p supports Σ-types, then $\mathbf{T}$ has a dependent strength

$$\sigma_A : \Sigma_A \circ T \longrightarrow T \circ \Sigma_A \qquad (A \in \mathcal{V})$$

- **Theorem 2:** If σ_A are natural isos., then $p^{\mathbf{T}}$ supports Σ-types

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- **Theorem 3:** If p supports Σ-types and $p^{\mathbf{T}}$ has split fibred reflexive coequalizers, then $p^{\mathbf{T}}$ also supports Σ-types

$$\Sigma^{\mathbf{T}}_A(B, \beta) \stackrel{\text{def}}{=} F^{\mathbf{T}}\left(\Sigma_A(B)\right)_{/\equiv}$$

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Algebraic effects

(operations and equations)

Algebraic effects – ops. and eqs.

Fibred effect theories $\mathcal{T}_{\mathsf{eff}}$:

- signatures of dependently typed operation symbols

$$\frac{\cdot \vdash I \qquad x_i : I \vdash O \qquad I \text{ and } O \text{ are pure value types}}{\mathsf{op} : (x_i : I) \rightharpoonup O}$$

- equipped with equations on derivable effect terms

In eMLTT:

$$M \;::=\; \ldots \;\mid\; \mathsf{op}^{\underline{C}}_{V}(x.M)$$

General algebraicity equations (in addition to eff. th. eqs.):

$$\frac{\Gamma \vdash_{\mathsf{v}} V : I \qquad \Gamma, x : O[V/x_i] \vdash_{\mathsf{c}} M : \underline{C} \qquad \Gamma \,|\, z : \underline{C} \vdash_{\mathsf{h}} K : \underline{D}}{\Gamma \vdash_{\mathsf{c}} K[\mathsf{op}^{\underline{C}}_{V}(x.M)/z] = \mathsf{op}^{\underline{D}}_{V}(x.K[M/z]) : \underline{D}} \; (\mathsf{op} : (x_i : I) \rightharpoonup O)$$

Sound semantics: Based on families fibrations and Law. theories

- $p : \mathsf{Fam}(\mathsf{Set}) \longrightarrow \mathsf{Set}$ and $q : \mathsf{Fam}(\mathsf{Mod}(\mathcal{L}_{\mathcal{T}_{\mathsf{eff}}}, \mathsf{Set})) \longrightarrow \mathsf{Set}$

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Fibred effect theories $\mathcal{T}_{\text{eff}}$:

- signatures of dependently typed operation symbols

$$\frac{\cdot \vdash I \qquad x_i : I \vdash O \qquad I \text{ and } O \text{ are pure value types}}{\mathsf{op} : (x_i : I) \rightharpoonup O}$$

- equipped with equations on derivable effect terms

In eMLTT:

$$M \ ::= \ \ldots \ \mid \ \mathsf{op}^{\underline{C}}_{V}(x.M)$$

General algebraicity equations (in addition to eff. th. eqs.):

$$\frac{\Gamma \vDash_{\mathsf{v}} V : I \quad \Gamma, x : O[V/x_i] \vDash_{\mathsf{c}} M : \underline{C} \quad \Gamma \,|\, z : \underline{C} \vDash_{\mathsf{h}} K : \underline{D}}{\Gamma \vDash_{\mathsf{c}} K[\mathsf{op}^{\underline{C}}_{V}(x.M)/z] = \mathsf{op}^{\underline{D}}_{V}(x.K[M/z]) : \underline{D}} \ (\mathsf{op} : (x_i : I) \rightharpoonup O)$$

Sound semantics: Based on families fibrations and Law. theories

- $p : \mathrm{Fam}(\mathrm{Set}) \longrightarrow \mathrm{Set}$ and $q : \mathrm{Fam}(\mathrm{Mod}(\mathcal{L}_{\mathcal{T}_{\text{eff}}}, \mathrm{Set})) \longrightarrow \mathrm{Set}$

Algebraic effects – examples

Example 1 (interactive IO):

- read : $1 \rightharpoonup \mathsf{Chr}$ $\qquad\qquad$ $(\mathsf{Chr} \stackrel{\mathrm{def}}{=} 1 + \ldots + 1)$

  write : $\mathsf{Chr} \rightharpoonup 1$

- no equations

Example 2 (global state with location-dependent store type):

- $\diamond \vdash \mathsf{Loc}$

  $\ell : \mathsf{Loc} \vdash \mathsf{Val}$

  $\diamond \vdash_{\mathrm{v}} \mathsf{isDec}_{\mathsf{Loc}} : \Pi\, \ell : \mathsf{Loc} \,.\, \Pi\, \ell' : \mathsf{Loc} \,.\, (\ell =_{\mathsf{Loc}} \ell') + (\ell =_{\mathsf{Loc}} \ell' \to 0)$

- get : $(\ell : \mathsf{Loc}) \rightharpoonup \mathsf{Val}$

  put : $(\Sigma\, \ell : \mathsf{Loc} \,.\, \mathsf{Val}) \rightharpoonup 1$

- five equations (two of them branching on $\mathsf{isDec}_{\mathsf{Loc}}$)

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Handlers of algebraic effects

(for programming and extrinsic reasoning)

Handlers of alg. effects – for programming

Idea: Generalisation of exception handlers [Plotkin,Pretnar'09]

Handler $\sim$ Algebra and Handling $\sim$ Homomorphism

Usual term-level presentation:

$$\Gamma \vdash_{\mathsf{c}} M \ \mathtt{handled\ with}\ \{\mathtt{op}_{x_v}(x_k) \mapsto N_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{T}_{\mathsf{eff}}}\ \mathtt{to}\ y : A\ \mathtt{in}_{\underline{C}}\ N_{\mathsf{ret}} : \underline{C}$$

satisfying

$$(\mathtt{return}\ V)\ \mathtt{handled\ with}\ \{\ldots\}_{\mathsf{op} \in \mathcal{T}_{\mathsf{eff}}}\ \mathtt{to}\ y : A\ \mathtt{in}\ N_{\mathsf{ret}} = N_{\mathsf{ret}}[V/x]$$

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Example use case for programming:

- write your programs using alg. ops. (e.g., `get` and `put`)
- use handlers to provide fit-for-purpose impl. (e.g., $S \to X \times S$)

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Handlers of alg. effects – for reasoning

Idea: Using a derived handle-into-values handling construct

$$M \;\texttt{handled with}\; \{\texttt{op}_{x_v}(x_k) \mapsto V_{\texttt{op}}\}_{\texttt{op} \in \mathcal{T}_{\text{eff}}} \;\texttt{to}\; y\!:\!A \;\texttt{in}_{\underline{B}}\; V_{\text{ret}}$$

we can define natural predicates (essentially, dependent types)

$$\Gamma \vDash_{\mathsf{v}} P : UFA \to \mathcal{U}$$

by

- equipping a universe $\mathcal{U}$ with an algebra for $\mathcal{T}_{\text{eff}}$, and
- using the above handle-into-values construct to define P

Note 1: $P\,(\texttt{thunk}\,M)$ computes a proof obligation for M

Note 2: Formally, this is done in an extension of eMLTT with

- a universe $\mathcal{U}$ closed under Nat, 1, 0, $+$, Σ, and Π
- a type-based treatment of handlers $\quad \underline{C} \;::=\; \ldots \mid \langle A; \overrightarrow{V_{\text{op}}}; \overrightarrow{W_{\text{eq}}} \rangle$
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Handlers of alg. effects – for reasoning

Example 1 (Evaluation Logic style modalities):

- Given a predicate $P : A \to \mathcal{U}$ on return values, we define a predicate $\Diamond P : UFA \to \mathcal{U}$ on IO-computations as

$$\Diamond P \stackrel{\text{def}}{=} \lambda x{:}UFA\,.\,(\mathtt{force}\,x)\ \mathtt{handled\ with}\ \{\ldots\}_{\mathsf{op}\in\mathcal{T}_{\mathsf{IO}}}\ \mathtt{to}\ y{:}A\ \mathtt{in}_{\mathcal{U}}\ P\,y$$

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$$V_{\mathsf{read}} \stackrel{\text{def}}{=} \lambda x{:}(\Sigma\, x_v{:}1\,.\,\mathsf{Chr} \to \mathcal{U})\,.\,\widehat{\Sigma}\, y{:}\mathsf{El}(\widehat{\mathsf{Chr}})\,.\,(\mathtt{snd}\ x)\ y$$

$$V_{\mathsf{write}} \stackrel{\text{def}}{=} \lambda x{:}(\Sigma\, x_v{:}\mathsf{Chr}\,.\,1 \to \mathcal{U})\,.\,(\mathtt{snd}\ x)\ \star$$

- $\Diamond P$ corresponds to Evaluation Logic's possibility modality

$$\Diamond P\,(\mathtt{thunk}\,(\mathtt{read}(x\,.\,\mathtt{write}_{e'}(\mathtt{return}\,V)))) = \widehat{\Sigma}\, x{:}\mathsf{El}(\widehat{\mathsf{Chr}})\,.\,P\ V$$

- To get the necessity modality $\Box P$, just use $\widehat{\Pi}\, x{:}\mathsf{El}(\widehat{\mathsf{Chr}})$ in V_{read}

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Handlers of alg. effects – for reasoning

Example 2 (Dijkstra's weakest precondition semantics for state):

- Given a postcondition on return values and final states

  $$Q : A \to S \to \mathcal{U} \qquad\qquad (S \stackrel{\text{def}}{=} \Pi\, \ell : \mathrm{Loc} \,.\, \mathrm{Val})$$

  we define a precondition for stateful comps. on initial states

  $$\mathrm{wp}_Q : UFA \to S \to \mathcal{U}$$

  by

  1) handling the given comp. into a state-passing function using

  $$V_{\mathrm{get}}, V_{\mathrm{put}} \quad \text{on} \quad S \to (\mathcal{U} \times S) \qquad \text{and} \qquad V_{\mathrm{ret}} \ \text{“}=\text{”}\ Q$$

  2) feeding in the initial state; and 3) projecting out $\mathcal{U}$

- **Theorem:** wp_Q satisfies expected properties of WPs, e.g.,

  $$\mathrm{wp}_Q\,(\mathtt{thunk}\,(\mathtt{return}\,V)) \quad = \quad \lambda\, x_S : S \,.\, Q\ V\ x_S$$

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Example 3 (Patterns of allowed (IO-)effects):

- Assuming an inductive type of IO-protocols, given by

$$\mathtt{e} : \text{Protocol} \qquad \mathtt{r} : (\text{Chr} \to \text{Protocol}) \to \text{Protocol}$$

$$\mathtt{w} : (\text{Chr} \to \mathcal{U}) \to \text{Protocol} \to \text{Protocol}$$

  and potentially also by $\wedge$, $\vee$, ...

- We can define a rel. between comps. and protocols as follows:

$$\text{Allowed} : UFA \to \text{Protocol} \to \mathcal{U}$$

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\end{array}$$

Conclusion

At a high-level, the presented work was about combining

dependent types and computational effects

In particular, you saw

- a clean core calculus of dependent types and comp. effects
- a natural category-theoretic semantics
- alg. effects and handlers, in particular, for reasoning using
    - Evaluation Logic style modalities
    - Dijkstra's weakest precondition semantics for state
    - patterns of allowed (IO-)effects

Some items of future work:

- uniform account of the various handler-defined predicates
- more expressive comp. types (par. adjunctions, Dijkstra monads)

Thank you!

D. Ahman.
Fibred Computational Effects. (PhD Thesis, 2017)

D. Ahman, N. Ghani, G. Plotkin.
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