

Directed containers, what are they good for?

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(based on joint work with James Chapman and Tarmo Uustalu)



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Outline

D. Ahman, J. Chapman, T. Uustalu.

When is a Container a Comonad? (FoSSaCS'12, LMCS 2014)

D. Ahman, T. Uustalu.

Distributive laws of directed containers (Progress in Inf. 2013)

D. Ahman, T. Uustalu.

Update Monads: Cointerpreting Dir. Cons. (TYPES'13)

D. Ahman, T. Uustalu.

Coalgebraic update lenses (MFPS'14)

D. Ahman, T. Uustalu.

Directed containers as categories (MSFP'16)

D. Ahman, T. Uustalu.

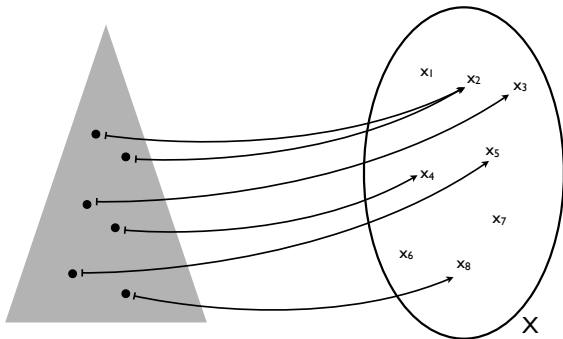
Taking Updates Seriously (BX'17)

Directed containers

(and directed polynomials)

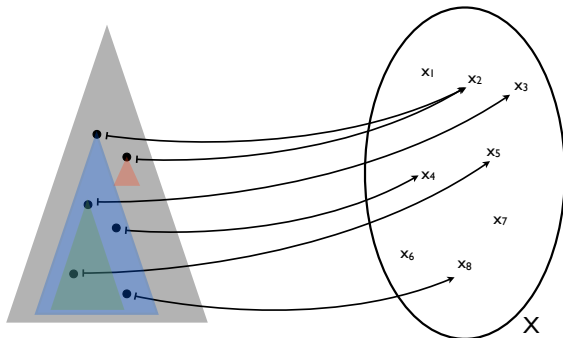
Container syntax of datatypes

- Many **datatypes** can be represented in terms of
 - **shapes**
 - **positions** in shapes
- **Containers** provide us with a handy **syntax** to analyse them
- **Examples:** lists, streams, colists, trees, zippers, etc.



Directing containers?

- Containers often exhibit a natural notion of **subshape**
- Natural questions arise:
 - What is the appropriate **specialisation** of containers?
 - Does this admit a nice **categorical** theory?
 - What else is this structure **useful** for?



Directed containers

- A directed container is given by

- $S : \mathbf{Set}$ *(shapes)*
- $P : S \rightarrow \mathbf{Set}$ *(positions)*

and

- $\downarrow : \prod s : S. P s \rightarrow S$ *(subshape)*
- $\circ : \prod \{s : S\}. P s$ *(root position)*
- $\oplus : \prod \{s : S\}. \prod p : P s. P (s \downarrow p) \rightarrow P s$ *(subshape positions)*

such that

- $s \downarrow \circ = s$
- $s \downarrow (p \oplus p') = (s \downarrow p) \downarrow p'$
- $p \oplus \{s\} \circ = p$
- $\circ \{s\} \oplus p = p$
- $(p \oplus \{s\} p') \oplus p'' = p \oplus (p' \oplus p'')$

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- $\circ : \prod \{s : S\}. P\ s$ *(root position)*
- $\oplus : \prod \{s : S\}. \prod p : P\ s. P\ (s \downarrow p) \rightarrow P\ s$ *(subshape positions)*

such that

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- $s \downarrow (p \oplus p') = (s \downarrow p) \downarrow p'$
- $p \oplus_{\{s\}} \circ = p$
- $\circ_{\{s\}} \oplus p = p$
- $(p \oplus_{\{s\}} p') \oplus p'' = p \oplus (p' \oplus p'')$

Directed **polynomials**

- A **polynomial** (in one variable) is given by

$$1 \xleftarrow{!} \overline{P} \xrightarrow{s} S \xrightarrow{!} 1$$

where

- $S : \mathbf{Set}$ (or more generally, in suitable \mathcal{C}) (*shapes*)
- $\overline{P} : \mathbf{Set}$ (or more generally, in suitable \mathcal{C}) (*total positions*)
- $\overline{P} \cong \sum_{s:S} S \cdot P_s$

Directed polynomials

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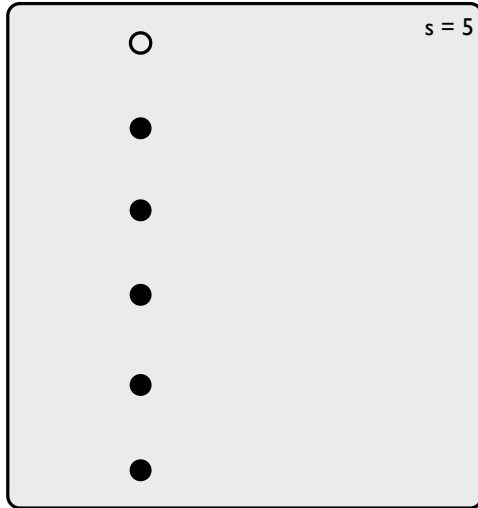
where

- $S : \mathbf{Set}$ (or more generally, in suitable \mathcal{C}) *(shapes)*
- $\overline{P} : \mathbf{Set}$ (or more generally, in suitable \mathcal{C}) *(total positions)*
- $\overline{P} \cong \sum s : S. P_s$
- A **directed polynomial** is given by
 - $s : \overline{P} \rightarrow S$ *(a polynomial)*
 - $\downarrow : \overline{P} \rightarrow S$
 - $o : S \rightarrow \overline{P}$ s.t. $s \circ o = \text{id}_S$ and $\downarrow \circ o = \text{id}_S$
 - ...
 - def. is remarkably **symmetric in s and \downarrow** (more on this later)

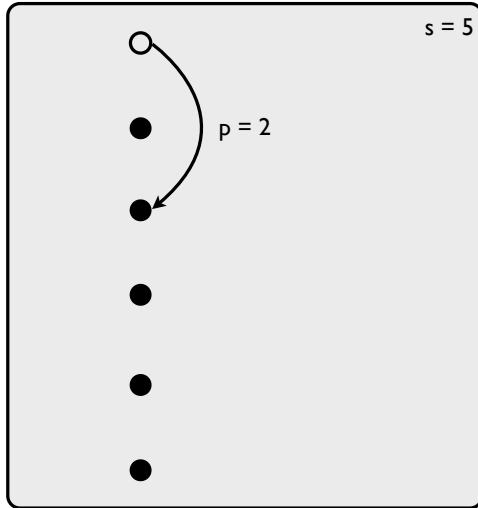
Non-empty lists and streams

- Non-empty lists are represented as
 - $S \stackrel{\text{def}}{=} \text{Nat}$ *(shapes)*
 - $P\ s \stackrel{\text{def}}{=} [0..s]$ *(positions)*
 - $s \downarrow p \stackrel{\text{def}}{=} s - p$ *(subshapes)*
 - $o_{\{s\}} \stackrel{\text{def}}{=} 0$ *(root position)*
 - $p \oplus_{\{s\}} p' \stackrel{\text{def}}{=} p + p'$ *(subshape positions)*
- Streams are represented similarly
 - $S \stackrel{\text{def}}{=} 1$ *(shapes)*
 - $P\ * \stackrel{\text{def}}{=} \text{Nat}$ *(positions)*
 - ...
- Another example is lists with **cyclic shifts** as “sublists”

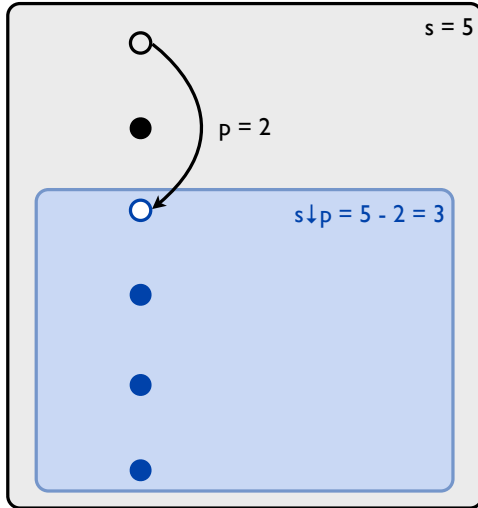
Non-empty lists illustrated



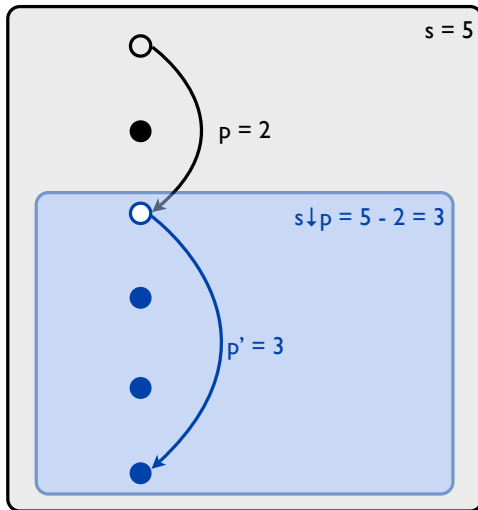
Non-empty lists illustrated



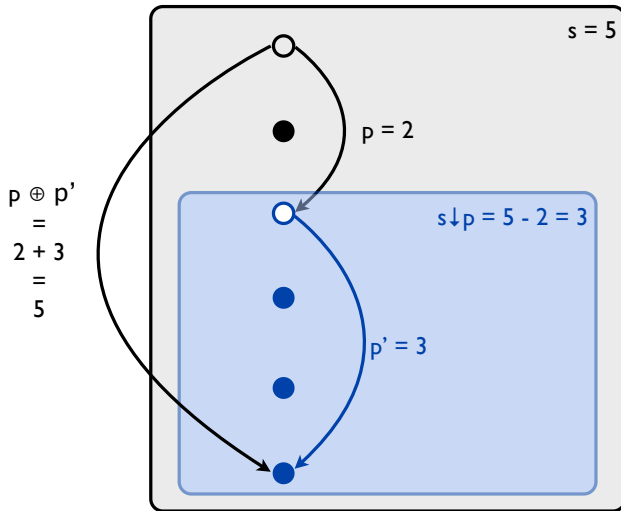
Non-empty lists illustrated



Non-empty lists illustrated



Non-empty lists illustrated



Non-empty lists with a focus

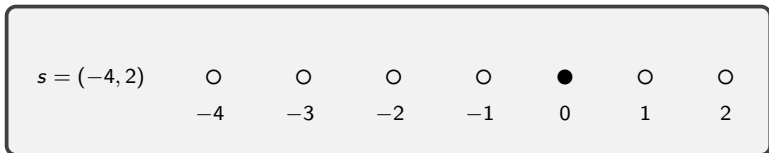
- **Zipper** – tree-like data-structures consisting of

- a **context** and a **focal subtree**

- Non-empty lists with a **focus**

- $S \stackrel{\text{def}}{=} \text{Nat} \times \text{Nat}$ *(shapes)*

- $P(s_0, s_1) \stackrel{\text{def}}{=} [-s_0..s_1] = [-s_0..-1] \cup [0..s_1]$ *(positions)*



- $(s_0, s_1) \downarrow p \stackrel{\text{def}}{=} (s_0 + p, s_1 - p)$ *(subshapes)*

- $o_{\{s_0, s_1\}} \stackrel{\text{def}}{=} 0$ *(root)*

- $p \oplus_{\{s_0, s_1\}} p' \stackrel{\text{def}}{=} p + p'$ *(subshape positions)*

Directed container morphisms

- A directed container morphism

$$t \triangleleft q : (S \triangleleft P, \downarrow, o, \oplus) \longrightarrow (S' \triangleleft P', \downarrow', o', \oplus')$$

is given by

- $t : S \rightarrow S'$
- $q : \Pi\{s : S\}. P'(ts) \rightarrow P s$

such that

- $t(s \downarrow q p) = t s \downarrow' p$
- $o_{\{s\}} = q(o'_{\{ts\}})$
- $q p \oplus_{\{s\}} q p' = q(p \oplus'_{\{ts\}} p')$
- Identities and composition are defined component-wise
- Directed containers form a category **DCont**

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 - $qp \oplus_{\{s\}} qp' = q(p \oplus'_{\{ts\}} p')$
- **Identities** and **composition** are defined component-wise
 - Directed containers form a **category** **DCont**

Directed containers
=
containers \cap comonads

Interpretation of directed containers

- Any directed container

$$(S \triangleleft P, \downarrow, o, \oplus)$$

defines a functor/comonad

$$[[S \triangleleft P, \downarrow, o, \oplus]]^{\text{c}} \stackrel{\text{def}}{=} (D, \varepsilon, \delta)$$

where

- $D : \mathbf{Set} \rightarrow \mathbf{Set}$ (or in any cat. with enough pullbacks)

$$DX \stackrel{\text{def}}{=} \Sigma s : S. (P s \rightarrow X)$$

- $\varepsilon_X : DX \rightarrow X$

$$\varepsilon_X(s, v) \stackrel{\text{def}}{=} v(o_{\{s\}})$$

- $\delta_X : DX \rightarrow DDX$

$$\delta_X(s, v) \stackrel{\text{def}}{=} (s, \lambda p. (s \downarrow p, \lambda p'. v(p \oplus_{\{s\}} p'))))$$

Interpretation of directed containers

- Any directed container

$$(S \triangleleft P, \downarrow, \circ, \oplus)$$

defines a ~~functor~~/comonad

$$\llbracket S \triangleleft P, \downarrow, \circ, \oplus \rrbracket^{\text{dc}} \stackrel{\text{def}}{=} (D, \varepsilon, \delta)$$

where

- $D : \mathbf{Set} \rightarrow \mathbf{Set}$ (or in any cat. with enough pullbacks)

$$D X \stackrel{\text{def}}{=} \Sigma s : S. (P s \rightarrow X)$$

- $\varepsilon_X : D X \rightarrow X$

$$\varepsilon_X(s, v) \stackrel{\text{def}}{=} v(\circ_{\{s\}})$$

- $\delta_X : D X \rightarrow D D X$

$$\delta_X(s, v) \stackrel{\text{def}}{=} (s, \lambda p. (s \downarrow p, \lambda p'. v(p \oplus_{\{s\}} p'))))$$

Interpretation of **dir. cont. morphisms**

- Any **directed container morphism**

$$t \triangleleft q : (S \triangleleft P, \downarrow, o, \oplus) \longrightarrow (S' \triangleleft P', \downarrow', o', \oplus')$$

defines a **natural transformation** / **comonad morphism**

$$\llbracket t \triangleleft q \rrbracket^c : \llbracket S \triangleleft P, \downarrow, o, \oplus \rrbracket^c \longrightarrow \llbracket S' \triangleleft P', \downarrow', o', \oplus' \rrbracket^c$$

by

- $\llbracket t \triangleleft q \rrbracket_X^c : (\Sigma s : S. P\ s \rightarrow X) \rightarrow (\Sigma s' : S'. P'\ s' \rightarrow X)$

$$\llbracket t \triangleleft q \rrbracket_X^c(s, v) \stackrel{\text{def}}{=} (t\ s, v \circ q_{\{s\}})$$

- $\llbracket - \rrbracket^c$ preserves the identities and composition
- $\llbracket - \rrbracket^c$ is a **functor** from **DCont** to **[Set, Set]** / **Comonads(Set)**

Interpretation of dir. cont. morphisms

- Any directed container morphism

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by

- $\llbracket t \triangleleft q \rrbracket_X^{\text{dc}} : (\Sigma s : S. P\ s \rightarrow X) \rightarrow (\Sigma s' : S'. P'\ s' \rightarrow X)$

$$\llbracket t \triangleleft q \rrbracket_X^{\text{dc}}(s, v) \stackrel{\text{def}}{=} (t\ s, v \circ q_{\{s\}})$$

- $\llbracket - \rrbracket^{\text{dc}}$ preserves the identities and composition
- $\llbracket - \rrbracket^{\text{dc}}$ is a functor from **DCont** to ~~[Set, Set]~~/Comonads(**Set**)

Interpretation is fully faithful

- Every natural transformation / comonad morphism

$$\tau : \llbracket S \triangleleft P, \downarrow, \circ, \oplus \rrbracket^{\text{dc}} \longrightarrow \llbracket S' \triangleleft P', \downarrow', \circ', \oplus' \rrbracket^{\text{c}}$$

defines a directed container morphism

$$\lceil \tau \rceil^{\text{dc}} : (S \triangleleft P, \downarrow, \circ, \oplus) \longrightarrow (S' \triangleleft P', \downarrow', \circ', \oplus')$$

satisfying

- $\lceil \llbracket t \triangleleft q \rrbracket^{\text{c}} \rceil^{\text{dc}} = t \triangleleft q$
- $\lceil \tau \rceil^{\text{c}} = \lceil \tau' \rceil^{\text{c}}$ implies $\tau = \tau'$
- $\llbracket - \rrbracket^{\text{dc}}$ is a fully faithful functor

Interpretation is fully faithful

- Every ~~natural transformation~~/comonad morphism

$$\tau : \llbracket S \triangleleft P, \downarrow, \mathbf{o}, \oplus \rrbracket^{\text{dc}} \longrightarrow \llbracket S' \triangleleft P', \downarrow', \mathbf{o}', \oplus' \rrbracket^{\text{dc}}$$

defines a directed container morphism

$$\ulcorner \tau \urcorner^{\text{dc}} : (S \triangleleft P, \downarrow, \mathbf{o}, \oplus) \longrightarrow (S' \triangleleft P', \downarrow', \mathbf{o}', \oplus')$$

satisfying

- $\ulcorner \llbracket t \triangleleft q \rrbracket^{\text{dc}} \urcorner^{\text{dc}} = t \triangleleft q$
- $\ulcorner \tau \urcorner^{\text{dc}} = \ulcorner \tau' \urcorner^{\text{dc}}$ implies $\tau = \tau'$
- $\llbracket - \rrbracket^{\text{dc}}$ is a fully faithful functor

Directed containers = cons. \cap cmnds.

- Any **comonad** (D, ε, δ) , such that $D = \llbracket S \triangleleft P \rrbracket^c$, determines

$$\llbracket (D, \varepsilon, \delta), S \triangleleft P \rrbracket \stackrel{\text{def}}{=} (S \triangleleft P, \downarrow, \circ, \oplus)$$

where

$$s \downarrow p \stackrel{\text{def}}{=} \text{snd}(t^\delta s) p \quad \circ_{\{s\}} \stackrel{\text{def}}{=} q_{\{s\}}^\varepsilon * \quad p \oplus_{\{s\}} p' \stackrel{\text{def}}{=} q_{\{s\}}^\delta(p, p')$$

- $\llbracket - \rrbracket$ satisfies

$$\llbracket \llbracket (D, \varepsilon, \delta), S \triangleleft P \rrbracket \rrbracket^{\text{dc}} = (D, \varepsilon, \delta)$$

$$\llbracket \llbracket S \triangleleft P, \downarrow, \circ, \oplus \rrbracket^{\text{dc}}, S \triangleleft P \rrbracket = (S \triangleleft P, \downarrow, \circ, \oplus)$$

- The following is a **pullback** in **CAT**:

$$\begin{array}{ccc} \mathbf{DCont} & \xrightarrow{U} & \mathbf{Cont} \\ \downarrow \llbracket - \rrbracket^{\text{dc}} \text{ f.f.} & & \downarrow \llbracket - \rrbracket^c \text{ f.f.} \\ \mathbf{Comonads}(\mathbf{Set}) & \xrightarrow{U} & [\mathbf{Set}, \mathbf{Set}] \end{array}$$

Constructions on directed containers

Constructions on directed containers

- Coproduct of directed containers
- Cofree directed containers
- Focussing of a container
- Bidirected containers (**dependently typed group structure**)
 - $(-)^{-1} : \prod\{s : S\}. \prod p : P s. P(s \downarrow p)$ + two equations
 - Which comonads do these correspond to? Hopf algebra like?
- Strict directed containers and their categorical product
- Distributive laws between directed containers
- Composition of directed containers

Update monads

(update your state instead of overwriting it!)

Cointerpretation of directed containers

- In addition to the **interpretation** functor

$$\llbracket - \rrbracket^c : \mathbf{Cont} \longrightarrow [\mathbf{Set}, \mathbf{Set}]$$

one can also define a **cointerpretation** functor

$$\langle\langle - \rangle\rangle^c : \mathbf{Cont}^{\mathrm{op}} \longrightarrow [\mathbf{Set}, \mathbf{Set}]$$

given by

$$\langle\langle S \triangleleft P \rangle\rangle^c \stackrel{\mathrm{def}}{=} \prod_{s : S} (P_s \times X)$$

which lifts to $\langle\langle - \rangle\rangle^{\mathrm{dc}}$ making the following a **pullback** in **CAT**

$$\begin{array}{ccc} \mathbf{DCont}^{\mathrm{op}} & \xrightarrow{U} & \mathbf{Cont} \\ \langle\langle - \rangle\rangle^{\mathrm{dc}} \downarrow & & \downarrow \langle\langle - \rangle\rangle^c \\ \mathbf{Monads}(\mathbf{Set}) & \xrightarrow{U} & [\mathbf{Set}, \mathbf{Set}] \end{array}$$

Dependently typed update monads

- In more detail, given a **directed container** $(S \triangleleft P, \downarrow, \circ, \oplus)$ the corresponding **dependently typed update monad** is given by
 - $T : \mathbf{Set} \longrightarrow \mathbf{Set}$
$$T X \stackrel{\text{def}}{=} \llbracket S \triangleleft P, \downarrow, \circ, \oplus \rrbracket^{\text{dc}} X = \prod s : S. (P s \times X)$$
 - $\eta_X : X \longrightarrow T X$
$$\eta_X x \stackrel{\text{def}}{=} \lambda s. (\circ, x)$$
 - $\mu_X : T T X \longrightarrow T X$
$$\mu_X f \stackrel{\text{def}}{=} \lambda s. \mathbf{let} (p, g) = f s \mathbf{in}$$
$$\mathbf{let} (p', x) = g (s \downarrow p) \mathbf{in} (p \oplus p', x)$$
- Intuitively
 - S – set of **states**
 - (P, \circ, \oplus) – dependently typed monoid of **updates**
- Use **cases**: non-overflowing buffers, non-underflowing stacks

Dependently typed update monads

- The dependently typed update monad

$$T X \stackrel{\text{def}}{=} \prod s : S. (P s \times X)$$

arises as the free model monad for a Lawvere theory whose models are given by a carrier $M : \mathbf{Set}$ and two operations

$$\text{lkp} : (S \rightarrow M) \longrightarrow M \qquad \text{upd} : (\prod s : S. P s) \times M \longrightarrow M$$

subject to three natural equations

- $\text{lkp} (\lambda s. \text{upd}_{\lambda s. o_{\{s\}}} (m)) = m$
- $\text{lkp} (\lambda s. \text{upd}_f (\text{lkp} (\lambda s'. m(s')))) = \text{lkp} (\lambda s. \text{upd}_f (m(s \downarrow (f s))))$
- $\text{upd}_f (\text{upd}_g (m)) = \text{upd}_{\lambda s. (f s) \oplus (g(s \downarrow f s))} (m)$

Simply typed update monads

- If $P : \mathbf{Set}$, then we get a simply typed update monad

$$T X \stackrel{\text{def}}{=} S \rightarrow (P \times X)$$

- In this case,
 - (P, o, \oplus) is a monoid in the standard sense
 - $\downarrow : S \times P \longrightarrow S$ is an action of (P, o, \oplus) on S
- This monad is the compatible composition of the monads

$$T_{\text{reader}} X \stackrel{\text{def}}{=} S \rightarrow X \qquad T_{\text{writer}} X \stackrel{\text{def}}{=} P \times X$$

- There is a one-to-one correspondence between
 - monoid actions $\downarrow : S \times P \longrightarrow S$
 - distributive laws $\theta : T_{\text{writer}} \circ T_{\text{reader}} \longrightarrow T_{\text{reader}} \circ T_{\text{writer}}$

Update lenses

(the dual of update monads)

Update lenses

- A **dependently typed update lens** is a coalgebra for the comonad

$$DX \stackrel{\text{def}}{=} \llbracket S \triangleleft P, \downarrow, \circ, \oplus \rrbracket^{\text{dc}} X = \Sigma s : S. (P s \rightarrow X)$$

that is, a **carrier** $M : \mathbf{Set}$ and **operations**

$$\text{lkp} : M \longrightarrow S \quad \text{upd} : (\prod s : S. P s) \times M \longrightarrow M$$

satisfying natural **equations** relating lkp and upd

- Equivalently, they are **comodels** for the Law. th. shown earlier
- Intuitively
 - M – set of **sources**, i.e., the **database**
 - S – set of **views**
 - (P, \circ, \oplus) – dependently typed monoid of source **updates**

Directed containers as (small) categories

Directed containers as (small) categories

- Given a directed container $(S \triangleleft P, \downarrow, o, \oplus)$ we get a corresponding small category $\mathcal{C}_{(S \triangleleft P, \downarrow, o, \oplus)}$ as follows
 - $\text{ob}(\mathcal{C}_{(S \triangleleft P, \downarrow, o, \oplus)}) \stackrel{\text{def}}{=} S$
 - $\mathcal{C}_{(S \triangleleft P, \downarrow, o, \oplus)}(s, s') \stackrel{\text{def}}{=} \Sigma p : P. (s \downarrow p = s')$
 - identities are given using o
 - composition is given using \oplus
- Vice versa, every small category \mathcal{C} gives us a corresponding directed container $(S_{\mathcal{C}} \triangleleft P_{\mathcal{C}}, \downarrow_{\mathcal{C}}, o_{\mathcal{C}}, \oplus_{\mathcal{C}})$
- But then, is it simply the case that **Set** \cong **DCont**?

Directed containers as (small) categories

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 - identities are given using o
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- Vice versa, every small category \mathcal{C} gives us a corresponding directed container $(S_{\mathcal{C}} \triangleleft P_{\mathcal{C}}, \downarrow_{\mathcal{C}}, o_{\mathcal{C}}, \oplus_{\mathcal{C}})$
- But then, is it simply the case that **Set** \cong **DCont**? **NO!**

Directed container morphisms as cofunctors

- Given a directed container morphism

$$t \triangleleft q : (S \triangleleft P, \downarrow, o, \oplus) \longrightarrow (S' \triangleleft P', \downarrow', o', \oplus')$$

we do not get a functor, but instead a cofunctor [Aguiar'97]

$$F_{t \triangleleft q} : \mathcal{C}_{(S \triangleleft P, \downarrow, o, \oplus)} \longrightarrow \mathcal{D}_{(S' \triangleleft P', \downarrow', o', \oplus')}$$

given by a mapping on objects

$$(F_{t \triangleleft q})_0 : \text{ob}(\mathcal{C}) \longrightarrow \text{ob}(\mathcal{D})$$

and a lifting operation on morphisms

$$s \xrightarrow{(F_{t \triangleleft q})_1(s, p)} o \quad \text{in } \mathcal{C}$$

$$(F_{t \triangleleft q})_0(s) \xrightarrow{p} s' \quad \text{in } \mathcal{D}$$

Constructions on directed containers

- On the one hand, we can relate **existing constructions** on directed containers to constructions (small) categories
- On the other hand, the (small) categories view also provides **new constructions** on directed containers and comonads
- For example, **factorisation** of dcontainer/comonad morphisms

Factorisation of morphisms

- Given a directed container morphism

$$t \triangleleft q : (S \triangleleft P, \downarrow, o, \oplus) \longrightarrow (S' \triangleleft P', \downarrow', o', \oplus')$$

we can factorise it in **DCont** as

$$(S \triangleleft P, \downarrow, o, \oplus) \xrightarrow{\text{id}_S \triangleleft q} (S \triangleleft P' \circ t, \downarrow'', o'', \oplus'') \xrightarrow{t \triangleleft \lambda s. \text{id}_{P'(t s)}} (S' \triangleleft P', \downarrow', o', \oplus')$$

which can be characterised as the following pullback in **Cont**

$$\begin{array}{ccccc} (S \triangleleft P) & \xrightarrow{\text{id}_S \triangleleft q} & (S \triangleleft P' \circ t) & \xrightarrow{t \triangleleft \lambda s. \text{id}_{P'(t s)}} & (S' \triangleleft P') \\ & & \downarrow \text{id}_S \triangleleft \lambda _ . ? & \text{pb.} & \downarrow \text{id}_{S'} \triangleleft \lambda _ . ? \\ & & (S \triangleleft \lambda _ . 0) & \xrightarrow{t \triangleleft \lambda _ . \text{id}_0} & (S' \triangleleft \lambda _ . 0) \end{array}$$

which corresponds to the full image factorisation of functors

- Notably, this works for any comonads that preserve pullbacks!

Conclusion

- So, directed containers, what are they good for?
- They
 - describe datastructures with a notion of **subshape**
 - characterise containers that carry a **comonad** structure
 - admit a variety of natural **constructions**
 - give rise to a natural refinement of the **state** monad
 - give rise to a natural refinement of asymmetric **lenses**
 - correspond to (small) **categories**