Handling Fibred Computational Effects

Effect Handlers in a Dependently Typed Setting

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Outline

- Setting the scene
  - Algebraic effects and their handlers
  - A core dependently typed effectful calculus (FoSSaCS'16)
- Why handlers + dependent types?
  - Programming with handlers + expressiveness of dep. types
  - Useful for defining predicates/types depending on computations
- Extending the FoSSaCS'16 calculus with handlers
  - Take 1: The common term-level def. of handlers (unsound)
  - Take 2: A type-level treatment of handlers

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Algebraic effects and their handlers

- Moggi taught us to model comp. effects using **monads** $(T, \eta, (-)^\dagger)$

$$\eta_A : A \to TA \qquad (f : A \to TB)^\dagger_{A,B} : TA \to TB$$

- Plotkin and Power showed that most of these monads arise from
  - **operations** - representing sources of effects

$$\mathsf{raise} : \mathsf{Exc} \longrightarrow 0 \qquad \mathsf{read} : \mathsf{Loc} \longrightarrow \mathsf{Val} \qquad \mathsf{write} : \mathsf{Loc} \times \mathsf{Val} \longrightarrow 1$$

  - **equations** - describing the computational behaviour

$$\ell{:}\mathsf{Loc} \mid w{:}1 \vdash \mathtt{read}_\ell(x.\mathtt{write}_{\langle \ell, x\rangle}(w(\star))) = w(\star)$$

- The algebraic approach significantly simplifies
  - **choosing** a monad/adjunction to model a given language
  - modelling **combinations** of two or more comp. effects
  - **reasoning** about effects in terms of computation trees
  - **generic programming** with effects (via handlers)

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Algebraic effects and their handlers ctd.

- Plotkin and Pretnar's **handlers** of algebraic effects
  - generalise exception handlers
  - given by redefining the given operations (they denote **algebras**)
  - example uses - rollbacks, stream redirection, concurrency, ...

- Usually included in languages using the **handling** construct

$$M \; \texttt{handled with} \; \{\mathsf{op}_x(x') \mapsto N_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathsf{eff}}} \; \texttt{to} \; y : A \; \texttt{in}_{\underline{C}} \; N_{\mathsf{ret}}$$

denoting the **homomorphism** $FA \longrightarrow \{\mathsf{op}_x(x') \mapsto N_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathsf{eff}}}$

$$(\mathtt{op}_V(y.M)) \; \texttt{handled with} \; \{\ldots\}_{\mathsf{op} \in \mathcal{S}_{\mathsf{eff}}} \; \texttt{to} \; y : A \; \texttt{in}_{\underline{C}} \; N_{\mathsf{ret}}$$
$$=$$
$$N_{\mathsf{op}}[V/x][\lambda \, y : O \, . \, \texttt{thunk}\,(M \; \texttt{handled with} \; \ldots)/x']$$

and

$$(\texttt{return}\; V) \; \texttt{handled with} \; \{\ldots\}_{\mathsf{op} \in \mathcal{S}_{\mathsf{eff}}} \; \texttt{to} \; y : A \; \texttt{in}_{\underline{C}} \; N_{\mathsf{ret}} \; = \; N_{\mathsf{ret}}[V/y]$$

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A core dependently typed effectful calculus

- (Model-theoretically) natural extension of type theory
  - clear distinction between **values** and **computations** (CBPV, EEC)

- **Value types** ($\Gamma \vdash A$) and **computation types** ($\Gamma \vdash \underline{C}$)

$$A, B ::= \ \dots \ \mid \ U\underline{C} \qquad \underline{C}, \underline{D} ::= FA \mid \Pi\, x{:}A\,.\,\underline{C} \ \mid \ \Sigma\, x{:}A\,.\,\underline{C}$$

- **Value terms** ($\Gamma \vdash V : A$)

$$V, W ::= x \ \mid \ \dots \ \mid \ \mathtt{thunk}\, M$$

- **Computation terms** ($\Gamma \vdash M : \underline{C}$)

$$\begin{aligned} M, N ::= {} & \mathtt{return}\, V \ \mid \ M \ \mathtt{to}\ x{:}A\ \mathtt{in}_{\underline{C}}\ N \ \mid \ \lambda\, x{:}A\,.\,M \ \mid \ M\,V \\ & \mid \ \langle V, M \rangle \ \mid \ M \ \mathtt{to}\ (x{:}A, z{:}\underline{C})\ \mathtt{in}_{\underline{D}}\ K \ \mid \ \mathtt{force}_{\underline{C}}\, V \end{aligned}$$

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Defining predicates on effectful comps.

- For time being, **assume** that we have handlers in the calculus
- In particular, assume that we can also **handle into values**

$$M \texttt{ handled with } \{\mathtt{op}_x(x') \mapsto V_{\mathtt{op}}\}_{\mathtt{op} \in \mathcal{S}_{\mathrm{eff}}} \texttt{ to } y \!:\! A \texttt{ in}_B \; V_{\mathrm{ret}}$$

- Also assume that we have a Tarski-style **value universe** $\mathcal{U}$
- Then we can define **predicates** $V : UFA \to \mathcal{U}$ by
  - equipping $\mathcal{U}$ with an **algebra** structure
  - **handling** the given computation using that algebra
  - essentially, each such V computes a **proof obligation**
- Examples
  - **lifting predicates** from return values to (I/O)-computations
  - Dijkstra's **weakest precondition semantics** of state
  - specifying **allowed patterns** of (I/O)-computations

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Lifting predicates to effectful comps.

- Given a predicate $V_P : A \to \mathcal{U}$ on **return values**,

  we define a predicate $V_{\widehat{P}} : UFA \to \mathcal{U}$ on **(I/O)-comps.** by

  $\lambda\, y : UFA \,.\, (\texttt{force}\, y)\ \texttt{handled with}\ \{\ldots\}_{\mathsf{op} \in \mathcal{S}_{\mathsf{IO}}}\ \texttt{to}\ y' : A\ \texttt{in}_{\mathcal{U}}\ V_P\, y'$

  using the **handler** given by

  $$V_{\mathsf{read}} \stackrel{\text{def}}{=} \lambda\, y : (\Sigma\, x : 1 \,.\, \mathsf{Chr} \to \mathcal{U}) \,.\, \texttt{v-pi-code}(\texttt{chr-code}\,, y' .\, (\texttt{snd}\, y)\, y')$$

  $$V_{\mathsf{write}} \stackrel{\text{def}}{=} \lambda\, y : (\Sigma\, x : \mathsf{Chr} \,.\, 1 \to \mathcal{U}) \,.\, (\texttt{snd}\, y) \star$$

- $V_{\widehat{P}}$ is similar to the **necessity modality** from Evaluation Logic

  $$\Gamma \vdash \mathsf{El}(V_{\widehat{P}}\ (\texttt{thunk}\,(\texttt{read}^{FA}(x \,.\, \texttt{return}\, W)))) = \Pi\, x : \mathsf{Chr} \,.\, V_P\, W$$

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Dijkstra's weakest precondition semantics

- Given a postcondition on **return values** and **final states**

$$V_Q : A \to \mathsf{St} \to \mathcal{U}$$

we define a precondition for **stateful comps.** on **initial states**

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by handling the given term using

$$V_{\mathrm{get}}, V_{\mathrm{put}} \qquad \text{on} \qquad \mathsf{St} \to (\mathcal{U} \times \mathsf{St})$$

- Then the following equations hold

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  $$\Gamma \vdash V_{\widehat{Q}}\ (\mathtt{thunk}\,(\mathtt{return}\,V)) \quad = \quad \lambda\, x_S \,{:}\, \mathsf{St}\,.\, V_Q\ V\ x_S$$

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Dijkstra's weakest precondition semantics

- Given a postcondition on **return values** and **final states**

$$V_Q : A \to \mathrm{St} \to \mathcal{U}$$

we define a precondition for **stateful comps.** on **initial states**

$$V_{\widehat{Q}} : UFA \to \mathrm{St} \to \mathcal{U}$$

by handling the given term using

$$V_{\mathrm{get}}, V_{\mathrm{put}} \qquad \text{on} \qquad \mathrm{St} \to (\mathcal{U} \times \mathrm{St})$$

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Specifying allowed patterns of I/O-effects

- We assume an **inductive type** Protocol, given by

$$\mathtt{e} : \text{Protocol} \qquad \mathtt{r} : (\text{Chr} \to \text{Protocol}) \to \text{Protocol}$$

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- Given a **protocol** V_{pr} : Protocol, we define

$$V_{\widehat{\text{pr}}} : \mathit{UFA} \to \mathcal{U}$$

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$$V_{\text{read}}, V_{\text{write}} \qquad \text{on} \qquad \text{Protocol} \to \mathcal{U}$$

where

$$V_{\text{read}}\ \langle V, V_{\text{rk}}\rangle\ (\mathtt{r}\ V'_{\text{pr}}) \stackrel{\text{def}}{=} \texttt{v-pi-code}(\texttt{chr-code}, y\,.\,(V_{\text{rk}}\,y)\,(V'_{\text{pr}}\,y))$$

$$V_{\text{write}}\ \langle V, V_{\text{wk}}\rangle\ (\mathtt{w}\,\langle V_P, V'_{\text{pr}},\rangle) \stackrel{\text{def}}{=} \texttt{v-sigma-code}(V_P\,V, y\,.\,V_{\text{wk}} \star V'_{\text{pr}})$$

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Outline

- Setting the scene
  - Algebraic effects and their handlers
  - A core dependently typed effectful calculus (FoSSaCS'16)
- Why handlers + dependent types?
  - Programming with handlers + expressiveness of dep. types
  - Useful for defining predicates/types depending on computations
- Extending the FoSSaCS'16 calculus with handlers
  - Take 1: The common term-level def. of handlers (unsound)
  - Take 2: A type-level treatment of handlers

Fibred algebraic effects

- To include fib. alg. effects $(\mathcal{S}_{\mathrm{eff}}, \mathcal{E}_{\mathrm{eff}})$ in our calculus, we
  - extend its computation terms with **algebraic operations**

$$\frac{\Gamma \vdash V : I \quad \Gamma \vdash \underline{C} \quad \Gamma, y \,{:}\, O[V/x] \vdash M : \underline{C}}{\Gamma \vdash \mathrm{op}^{\underline{C}}_{V}(y \,{:}\, O[V/x].M) : \underline{C}}$$

  - include **equations** $\Gamma \mid \Delta \vdash T_1 = T_2$ in $\mathcal{E}_{\mathrm{eff}}$

  - include a general **algebraicity equation**

$$\frac{\Gamma \mid z \,{:}\, \underline{C} \vdash K : \underline{D} \quad \Gamma \vdash V : I \quad \Gamma, y \,{:}\, O[V/x] \vdash M : \underline{C}}{\Gamma \vdash K[\mathrm{op}^{\underline{C}}_{V}(y \,{:}\, O[V/x].M)/z] = \mathrm{op}^{\underline{D}}_{V}(y \,{:}\, O[V/x].K[M/z]) : \underline{D}}$$

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Handlers for fibred algebraic effects

- **Take 1:** Let's use their conventional term-level definition
  - include the handling construct for **computation terms**

    $$M \ \mathtt{handled\ with}\ \{\mathtt{op}_{\mathsf{x}}(x') \mapsto N_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathsf{eff}}}\ \mathtt{to}\ y \mathbin{:} A\ \mathtt{in}_{\underline{C}}\ N_{\mathsf{ret}}$$

  - as handling denotes a homomorphism, also for **hom. terms**

    $$K \ \mathtt{handled\ with}\ \{\mathtt{op}_{\mathsf{x}}(x') \mapsto N_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathsf{eff}}}\ \mathtt{to}\ y \mathbin{:} A\ \mathtt{in}_{\underline{C}}\ N_{\mathsf{ret}}$$

  - but then we can prove the **unsound equation**

    $$\Gamma \vdash \mathtt{write}^{F1}_{\mathsf{a}}(\mathtt{return}\,\star) = \mathtt{write}^{F1}_{\mathsf{z}}(\mathtt{return}\,\star) : F1$$

    by **handling**

    $$\mathtt{write}^{F1}_{\mathsf{a}}(\mathtt{return}\,\star)$$

    with

    $$\mathtt{write}_{\mathsf{x}}(x') \mapsto \mathtt{write}_{\mathsf{z}}(\mathtt{force}\,(x'\,\star))$$

    and exploiting a non-convergent **critical pair** in the eq. theory

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Handlers for fibred algebraic effects ctd.

- Possible ways to solve this unsoundness problem
  - **Option 1:** Change the FoSSaCS'16 calculus
    - change the equational theory of homomorphism terms
    - hom. terms wouldn't denote homomorphisms any more
    - investigated for exceptions in CBPV with stacks in [Levy'06]
  - **Option 2:** Keep the FoSSaCS'16 calculus unchanged
    - extend it so that handling for comp. terms is derivable
    - while making sure that the calculus remains sound
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  - extend comp. types with the **user-defined algebra type**

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Handlers for fibred algebraic effects ctd.

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  - extend the equational theory of **value types** with

$$\Gamma \vdash U \langle A, \{V_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathsf{eff}}} \rangle = A$$

(what about the corresponding η-equation for comp. types?)

  - extend the equational theory of **comp.** and **hom. terms** with

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Handlers for fibred algebraic effects ctd.

- **Take 2:** A type-based treatment of handlers

  - we can then routinely derive the **handling construct**

    $$M \; \texttt{handled with} \; \{\mathtt{op}_x(x') \mapsto N_{\mathtt{op}}\}_{\mathtt{op} \in \mathcal{S}_{\mathrm{eff}}} \; \texttt{to} \; y \,{:}\, A \; \texttt{in}_{\underline{C}} \; N_{\mathrm{ret}}$$

    using **sequential composition**, **thunking**, and **forcing**

    $$\texttt{force}_{\underline{C}} \left(\texttt{thunk} \left(\underbrace{M \; \texttt{to} \; y \,{:}\, A \; \texttt{in} \; \left(\texttt{force}_{\langle U\underline{C}, \{V_{\mathrm{op}}\}_{\mathrm{op} \in \mathcal{S}_{\mathrm{eff}}} \rangle} \left(\texttt{thunk}\, N_{\mathrm{ret}}\right)\right)}_{\text{has type } \langle U\underline{C}, \{V_{\mathrm{op}}\}_{\mathrm{op} \in \mathcal{S}_{\mathrm{eff}}} \rangle} \right)\right)$$

    where the value terms V_{op} are derived from the comp. terms N_{op}

  - satisfies the standard β-equations for handling

  - **handling into values** can be derived analogously

Handlers for fibred algebraic effects ctd.

- **Take 2:** A type-based treatment of handlers
  - we can then routinely derive the **handling construct**

$$M \ \texttt{handled with} \ \{\mathsf{op}_x(x') \mapsto N_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathsf{eff}}} \ \texttt{to} \ y \,{:}\, A \ \texttt{in}_{\underline{C}} \ N_{\mathsf{ret}}$$

using **sequential composition**, **thunking**, and **forcing**

$$\texttt{force}_{\underline{C}} \left(\texttt{thunk} \left(\underbrace{M \ \texttt{to} \ y \,{:}\, A \ \texttt{in} \ \left(\texttt{force}_{\langle U\underline{C}, \{V_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathsf{eff}}}\rangle} \left(\texttt{thunk}\, N_{\mathsf{ret}}\right)\right)}_{\text{has type } \langle U\underline{C}, \{V_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathsf{eff}}}\rangle} \right)\right)$$

where the value terms V_{op} are derived from the comp. terms N_{op}

  - satisfies the standard β-equations for handling
  - **handling into values** can be derived analogously

Handlers for fibred algebraic effects ctd.

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    $$\texttt{force}_{\underline{C}}\left(\texttt{thunk}\left(\underbrace{M \;\texttt{to}\; y\!:\!A \;\texttt{in}\; \left(\texttt{force}_{\langle U\underline{C},\{V_{\mathtt{op}}\}_{\mathtt{op}\in\mathcal{S}_{\mathrm{eff}}}\rangle}\left(\texttt{thunk}\, N_{\mathrm{ret}}\right)\right)}_{\text{has type } \langle U\underline{C},\{V_{\mathtt{op}}\}_{\mathtt{op}\in\mathcal{S}_{\mathrm{eff}}}\rangle}\right)\right)$$

    where the value terms $V_{\mathtt{op}}$ are derived from the comp. terms $N_{\mathtt{op}}$
  - satisfies the standard β-equations for handling
  - **handling into values** can be derived analogously

Conclusion

- In this talk, we saw
  - using (value) handlers to define predicates on computations
  - unsoundness problems when accommodating handlers
  - a principled type-based treatment of the handlers
- Future work
  - general account of defining predicates on alg. effects
  - operational semantics (complex values + eq. for ops.)
  - presentations of the calculus without hom. terms (eq. proof obl.)

Thank you!

Questions?