Handling Fibred Algebraic Effects

Danel Ahman

INRIA Paris

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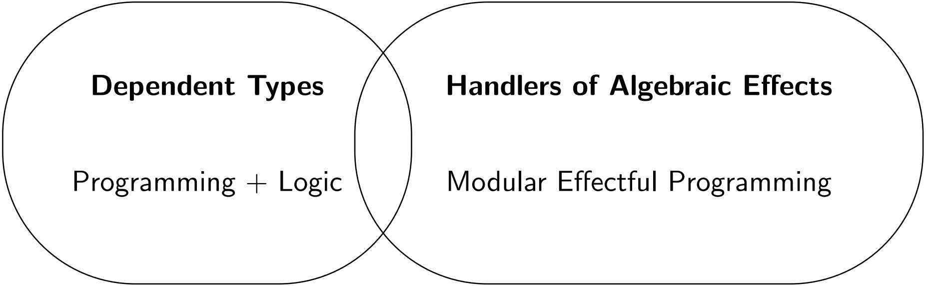
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Programming + Logic
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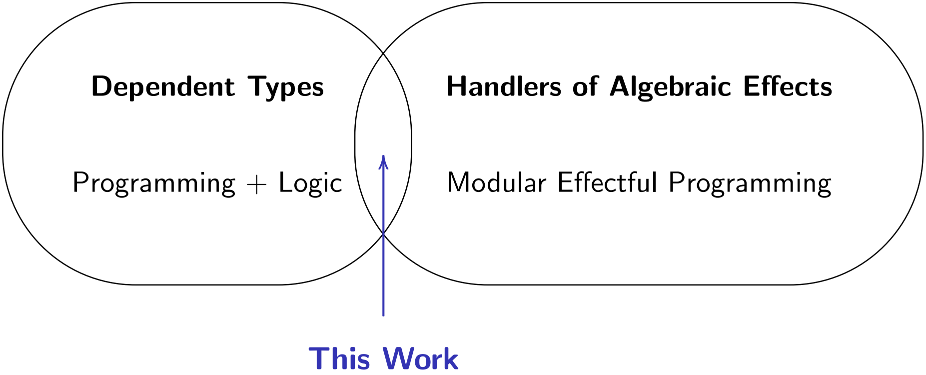
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This Work

Outline

- Setting the scene
    - Algebraic effects and their handlers
    - A core effectful dependently typed calculus (FoSSaCS'16)

      [A., Ghani, Plotkin'16]
- What can we gain from handlers + dependent types?
    - Modular programming with handlers + expressiveness of d. types
    - Reasoning about effectful computations
- Extending the FoSSaCS'16 calculus with alg. effects and handlers
    - Take 1: The common term-level def. of handlers (unsound)
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Algebraic effects and their handlers

- Moggi taught us to model comp. effects using **monads** $(T, \eta, (-)^{\dagger})$

$$\eta_A : A \to TA \qquad (f : A \to TB)^{\dagger}_{A,B} : TA \to TB$$

- Plotkin and Power showed that most of these monads arise from
  - **operation symbols** – representing the **sources** of effects

$$\mathsf{raise} : \mathsf{Exc} \longrightarrow 0 \qquad \mathsf{get} : \mathsf{Loc} \longrightarrow \mathsf{Val} \qquad \mathsf{put} : \mathsf{Loc} \times \mathsf{Val} \longrightarrow 1$$

  - **equations** – describing the computational **behaviour**

$$\ell : \mathsf{Loc} \mid w : 1 \vdash \mathtt{get}_{\ell}(x.\mathtt{put}_{\langle \ell, x \rangle}(w(\star))) = w(\star)$$

- The algebraic approach significantly simplifies
  - **choosing** a monad/adjunction to model a given language
  - modelling **combinations** of two or more comp. effects
  - **generic** effectful programming (via **handlers**)

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Algebraic effects and their handlers ctd.

- Plotkin and Pretnar's **handlers** of algebraic effects
  - generalisation of exception handlers
  - given by **redefining** the given ops. (handlers denote **algebras**)
  - many uses – rollbacks, stream redirection, concurrency, ...
- Usually included in languages using the **handling** construct

$$M \;\texttt{handled with}\; \{\mathrm{op}_{x_v}(x_k) \mapsto N_{\mathrm{op}}\}_{\mathrm{op} \in \mathcal{S}_{\mathrm{eff}}} \;\texttt{to}\; y{:}A \;\texttt{in}_{\underline{C}}\; N_{\mathrm{ret}}$$

interpreted using the **homomorphism** $FA \longrightarrow \langle U\underline{C}, \overrightarrow{N_{\mathrm{op}}}\rangle$

$$(\mathrm{op}_V(y.M)) \;\texttt{handled with}\; \{\ldots\}_{\mathrm{op} \in \mathcal{S}_{\mathrm{eff}}} \;\texttt{to}\; y{:}A \;\texttt{in}_{\underline{C}}\; N_{\mathrm{ret}}$$
$$=$$
$$N_{\mathrm{op}}[V/x_v][\lambda\, y{:}O\,.\,\texttt{thunk}\,(M \;\texttt{handled with}\; \ldots)/x_k]$$

and

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A core dependently typed effectful calculus

- (Model-theoretically) natural extension of type theory
  - clear distinction between **values** and **computations** (CBPV, EEC)
- **Value types** ($\Gamma \vdash A$) and **computation types** ($\Gamma \vdash \underline{C}$)

$$A, B \;::=\; \ldots \;\mid\; U\underline{C} \qquad \underline{C}, \underline{D} \;::=\; FA \mid \Pi x{:}A.\underline{C} \mid \boxed{\Sigma x{:}A.\underline{C}}$$

- **Value terms** ($\Gamma \vdash V : A$)

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$$M, N \;::=\; \texttt{return}\, V \mid M \texttt{ to } x{:}A \texttt{ in}_{\underline{C}}\, N \mid \lambda x{:}A.M \mid M\,V$$
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The calculus we work in

- We work in an extension to the FoSSaCS'16 calculus, with
  - a Tarski-style **value universe** $\mathcal{U}$
    - with **codes** written as $\widehat{\Pi}$, $\widehat{\Sigma}$, $\widehat{0}$, $\widehat{1}$, ...
    - but thinking of them as $\forall$, $\exists$, $\bot$, $\top$, ...
  - fibred **algebraic effects**
    - dep. typed **operation symbols** $\mathsf{op} : (x_v : I) \longrightarrow O(x_v)$
    - ops. determine **comp. terms** $\mathsf{op}^{\underline{C}}_{V}(y : O[V/x_v] \,.\, M)$
    - effect eqs. determine **definitional eqs.**
  - a **derivable** "into-comps." variant of **handlers** and **handling**

    M $\texttt{handled with}$ $\{\mathsf{op}_{x_v}(x_k) \mapsto N_{\mathsf{op}} ; \overrightarrow{W_{\mathsf{eq}}}\}_{\mathsf{op} \in \mathcal{S}_{\mathsf{eff}}}$ $\texttt{to}$ $y : A$ $\texttt{in}_{\underline{C}}$ N_{ret}
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Reasoning about effectful computations

- Handlers are **useful** in various ways!

- They enable **extrinsic reasoning** about computations $M : FA$
  - Can be used to define **predicates** $P : UFA \to \mathcal{U}$ by
    1) equipping $\mathcal{U}$ (or a resp. type) with an **algebra** structure
    2) **handling** the given computation using that algebra
  - Intuitively, $P\ (\mathtt{thunk}\ M)$ computes a **proof obligation** for M
  - We discuss **three examples** of such predicates

- Also, an alternative to mon. reification for **rel. reasoning**
  - E.g., relating **stateful comps.** $M : FA$ as **functions** $S \to A \times S$
  - Not investigated in this paper
  - See [Grimm et al.'18] for **reification-based** rel. reasoning

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Ex1: Lifting predicates to effectful comps.

- Given a predicate $P : A \to \mathcal{U}$ on **return values**,

  we define a predicate $\Box P : UFA \to \mathcal{U}$ on **(I/O)-comps.** as

$$\Box P \stackrel{\text{def}}{=} \lambda\, y \,{:}\, UFA \,.\, (\mathtt{force}\, y)\ \mathtt{handled\ with}\ \{\ldots\}_{\mathsf{op} \in \mathcal{S}_{\mathsf{I/O}}}\ \mathtt{to}\ y' \,{:}\, A\ \mathtt{in}_{\mathcal{U}}\ P\, y'$$

  using the **handler** given by

$$\begin{array}{lll} \mathsf{read}(x_k) & \mapsto & \widehat{\Pi}\, y \,{:}\, \mathsf{El}(\widehat{\mathsf{Chr}}) \,.\, x_k\ y \qquad\qquad (\text{where } x_k \,{:}\, \mathsf{Chr} \to \mathcal{U}) \\ \mathsf{write}_{x_v}(x_k) & \mapsto & x_k\ \star \qquad\qquad (\text{where } x_v \,{:}\, \mathsf{Chr},\ x_k \,{:}\, 1 \to \mathcal{U}) \end{array}$$

- $\Box P$ is similar to the **necessity modality** from Evaluation Logic

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$$\Gamma \vdash \Box P\, \big(\mathtt{thunk}\, (\mathtt{read}(x\, .\, \mathtt{write}_{\prime e'}(\mathtt{return}\; V)))\big) = \widehat{\Pi}\, x : \mathsf{El}(\widehat{\mathsf{Chr}})\, .\, P\;\; V$$

- To get $\Diamond P$, we only have to replace $\widehat{\Pi}$ with $\widehat{\Sigma}$ in the handler

Ex2: Dijkstra's weakest precondition sem.

- Given a postcondition on **return values** and **final states**

$$Q : A \to S \to \mathcal{U} \qquad\qquad (S \stackrel{\text{def}}{=} \Pi\, \ell\!:\!\mathrm{Loc}\,.\mathrm{Val}(\ell))$$

  we define a precondition for **stateful comps.** on **initial states**

$$\mathrm{wp}_Q : UFA \to S \to \mathcal{U}$$

  by

  1) handling the given comp. into a **state-passing function** using

$$V_{\mathrm{get}}\,, V_{\mathrm{put}} \quad \text{on} \quad S \to \mathcal{U} \times S \qquad\qquad \text{and} \qquad\qquad V_{\mathrm{ret}} \text{ “}=\text{” } Q$$

  2) feeding in the **initial state**; and **3)** projecting out the **value of** $\mathcal{U}$

- Then, wp_Q satisfies the **expected properties**, such as

$$\Gamma \vdash \mathrm{wp}_Q\,(\mathtt{thunk}\,(\mathtt{return}\,V)) \;=\; \lambda\, x_S\!:\!S\,.\,Q\;V\;x_S$$

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Ex3: Allowed patterns of (I/O)-effects

- Assuming an inductive type of **I/O-protocols**, given by

$$\texttt{e} : \mathsf{Protocol} \qquad \texttt{r} : (\mathsf{Chr} \to \mathsf{Protocol}) \to \mathsf{Protocol}$$

$$\texttt{w} : (\mathsf{Chr} \to \mathcal{U}) \times \mathsf{Protocol} \to \mathsf{Protocol}$$

- We can define a **relation** between **comps.** and **protocols**

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given by (using pattern-matching lambda notation)

$$\begin{array}{lll} \mathsf{read}(x_k) & \mapsto & \lambda \left\{ (\texttt{r}\ x_{pr}) \to \widehat{\Pi}\, y \,{:}\, \mathsf{El}(\widehat{\mathsf{Chr}})\,.\, x_k\ y\ (x_{pr}\ y)\ ; \right. \\ & & \left. \quad \_ \to \widehat{0} \right\} \\ \mathsf{write}_{x_v}(x_k) & \mapsto & \lambda \left\{ (\texttt{w}\ P\ x_{pr}) \to \widehat{\Sigma}\, y \,{:}\, \mathsf{El}(P\, x_v)\,.\, x_k\ \star\ x_{pr}\ ; \right. \\ & & \left. \quad \_ \to \widehat{0} \right\} \end{array}$$

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Outline

- Setting the scene
  - Algebraic effects and their handlers
  - A core effectful dependently typed calculus (FoSSaCS'16)

    [A., Ghani, Plotkin'16]
- What can we gain from handlers + dependent types?
  - Modular programming with handlers + expressiveness of d. types
  - Reasoning about effectful computations
- Extending the FoSSaCS'16 calculus with alg. effects and handlers
  - Take 1: The common term-level def. of handlers (unsound)
  - Take 2: A new type-level treatment of handlers

Extending the FoSSaCS'16 calculus

- We assume given a **fibred effect theory** $\mathcal{T} = (\mathcal{S}, \mathcal{E})$

- First, we extend the calculus with **algebraic effects** as follows:

  - we extend the **computation terms** with

$$M, N \;::=\; \ldots \;\mid\; \mathrm{op}^{\underline{C}}_{V}(y \,{:}\, O[V/x_v] \,.\, M) \qquad (\mathrm{op} : (x_v : I) \longrightarrow O \in \mathcal{S})$$

  - we extend the **equational theory** with equations given in $\mathcal{E}$

  - we capture the **interaction** of comp. terms and ops. with the eq.

$$\frac{\Gamma \vdash V : I \quad \Gamma, x \,{:}\, O[V/x_v] \vdash M : \underline{C} \quad \Gamma \mid z \,{:}\, \underline{C} \vdash K : \underline{D}}{\Gamma \vdash K[\mathrm{op}^{\underline{C}}_{V}(x.M)/z] = \mathrm{op}^{\underline{D}}_{V}(x.K[M/z]) : \underline{D}} \; (\mathrm{op} : (x_v : I) \longrightarrow O \in \mathcal{S})$$

- Second, we extend the calculus with a support for **handlers** . . .

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$$\Gamma \vdash \mathtt{write}_{\mathsf{a}}(\mathtt{return}\,\star) = \mathtt{write}_{\mathsf{z}}(\mathtt{return}\,\star) : F1$$

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How to proceed?

- Possible ways to solve this unsoundness problem
  - **Option 1:** Change the FoSSaCS'16 calculus
    - change the equational theory of homomorphism terms
    - hom. terms would not denote homomorphisms any more
    - investigated for exceptions in CBPV with stacks by [Levy'06]
  - **Option 2:** Keep the FoSSaCS'16 calculus unchanged
    - extend it so that handling for comp. terms is derivable
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Take 2: A type-level treatment of handlers

- Instead, we extend the FoSSaCS'16 **computation types** with
  - a **user-defined algebra type**

$$\underline{C}\,,\underline{D} \;::=\; \ldots \;\mid\; \langle A; \overrightarrow{V_{\mathrm{op}}}; \overrightarrow{W_{\mathrm{eq}}} \rangle$$

    where

    - A is the **carrier** value type
    - $\overrightarrow{V_{\mathrm{op}}}$ is a set of user-defined **operations**
    - $\overrightarrow{W_{\mathrm{eq}}}$ is a set of **witnesses** of equational proof obligations

- As a result, we can derive the **handing construct** as

$$M \;\mathtt{handled\ with}\; \{\mathrm{op}_{x_v}(x_k) \mapsto N_{\mathrm{op}}; \overrightarrow{W_{\mathrm{eq}}}\}_{\mathrm{op} \in \mathcal{S}_{\mathrm{eff}}} \;\mathtt{to}\; y \mathbin{:} A \;\mathtt{in}_{\underline{C}}\; N_{\mathrm{ret}}$$

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Conclusion

- In conclusion
  - handlers are natural for defining **predicates on computations**
    - lifting predicates from return values to computations
    - Dijkstra's weakest precondition semantics of state
    - specifying patterns of allowed (I/O)-effects
  - they admit a principled **type-based treatment**

- See the paper for
  - **formal details** of what I have shown you today
  - families fibrations based **denotational semantics** of the calculus
  - discussion about the calculus's inherent **extensional nature**
  - **Agda code** for the example predicates $P : UFA \to \mathcal{U}$

Thank you!

Questions?