# A fibrational view on computational effects

Danel Ahman

Prosecco Team, Inria Paris

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#### We investigate the combination of

- dependent types  $(\Pi, \Sigma, V =_{\mathcal{A}} W, ...)$
- computational effects (state, I/O, probability, recursion, ...)

#### Two guiding problems

- effectful programs in types (e.g., read and write in types)
- types of effectful programs (e.g., of sequential composition)

- tell a mathematically natural story
- use established math. techniques
- cover a wide range of comp. effects
- discover smth. interesting

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- cover a wide range of comp. effects (alg. effects, continuations)
- discover smth. interesting (using handlers to reason about effects)

(type-dependency in the presence of effects)

Let's assume that we have some dependent type A, e.g.:

$$\ell\!:\!(\mathsf{List}\;\mathsf{Chr})\vdash A(\ell) \quad \stackrel{\scriptscriptstyle\mathsf{def}}{=} \quad \Sigma\,\ell'\!:\!(\mathsf{List}\;\mathsf{Chr})\,.\,(\mathsf{length}\;\ell =_{\mathsf{Nat}}\mathsf{length}\;\ell'\times\ldots)$$

which could be used to type the dependent function

$$\mathtt{sort}: \mathsf{\Pi}\,\ell \colon (\mathsf{List}\;\mathsf{Chr}) \cdot A(\ell)$$

- **Q:** Should we allow  $A[\text{receive}(y, M)/\ell]$ ?
  - i.e., should we be allowed to type receive(y. M): List Chr
- A1: In this work we say no
  - types should only depend on static information about effects
  - we recover dependency on effectful computations via thunks
- **A2:** We are also looking into the yes case
  - type-dependency needs to be "homomorphic" (more on this later)

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Aim: Types should only depend on static info about effects

**Solution:** CBPV/EEC style distinction between vals. and comps

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• value types \Gamma \vdash A (MLTT + thunks + ...)
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- computation types  $\Gamma \vdash \underline{C}$  (dep. CBPV/EEC)
- where  $\Gamma$  contains **only** value variables  $x_1: A_1, \ldots, x_n: A_n$

**Note:** Could have also considered  $\lambda_{ML}$  and FGCBV

- building on CBPV/EEC gives a more general story
- especially for the treatment of sequential composition
- and also for integrating dependent- and effect-typing

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(e.g., sequential composition)

The problem: The standard typing rule for seq. composition

$$\frac{\Gamma \vdash_{c} M : FA \qquad \Gamma, x : A \vdash_{c} N : \underline{C}}{\Gamma \vdash_{c} M \text{ to } x : A \text{ in } N : \underline{C}}$$

is not correct any more because x can appear free in the type

(

in the conclusion

Aim: To fix the typing rule of sequential composition

**Option 1:** We could restrict the free variables in  $\underline{C}$ : [Levy'04]

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But sometimes it is useful if  $\underline{C}$  can depend on x!

if we consider

fopen (return true, return false) to 
$$x$$
:Bool in  $N$ 

• then it would be natural to let  $\underline{C}$  depend on x, e.g.,

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x: \mathsf{Bool} \vdash \underline{C}(x) \stackrel{\mathsf{def}}{=} \mathsf{if} \ x \ \mathsf{then} \ \text{``allow fread, fwrite, and fclose''} else "allow fopen"
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Option 2: One could lift sequential composition to type level

$$\Gamma \vdash M \text{ to } x : A \text{ in } N : M \text{ to } x : A \text{ in } C$$

But then all computation types would be singleton-like!?!

**Option 3:** In the monadic metalanguage  $\lambda_{ML}$ , one could also try

$$\Gamma \vdash M : T A \qquad \Gamma, x : A \vdash N : T B(x) 
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But what makes this a principled solution? Why is it correct?

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**Option 4:** We draw inspiration from algebraic effects  $\bullet$  and combine it with restricting  $\underline{C}$  in seq. comp. (**Option 1**)

E.g., consider the non-det. program (for  $x: Nat \vdash N : \underline{C}(x)$ )  $M \stackrel{\text{def}}{=} \text{choose (return 4, return 2) to } x: Nat in N$ 

After tossing the coin, this program evaluates as either  $N[4/x] : \underline{C}[4/x]$  or  $N[2/x] : \underline{C}[2/x]$ 

**Idea:** M denotes an element of the coproduct of algebras  $\underline{C}[4/x] + \underline{C}[2/x] \quad "\stackrel{\text{def}}{=} " \quad F\left(U\left(\underline{C}[4/x]\right) + U\left(\underline{C}[2/x]\right)\right)_{/\equiv}$ 

and thus we would like to type M at the type  $\Sigma x$ : Nat.  $\underline{C}$ 

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### Putting these ideas together

(eMLTT: a core dep.-typed language with comp. effects)

### eMLTT - types

**Value types:** MLTT + thunks + ...

$$A, B ::=$$
Nat  $\mid 1 \mid 0 \mid \Pi x : A.B \mid \Sigma x : A.B \mid V =_A W \mid U \subseteq | \dots |$ 

•  $U\underline{C}$  is the type of thunked (i.e., suspended) computations

Computation types: dep.-typed version of EEC's comp. types

$$\underline{C}, \underline{D} ::= FA \mid \Pi x : A . \underline{C} \mid \Sigma x : A . \underline{C}$$

- F A is the type of computations returning values of type A
- Πx: A. <u>C</u> is the type of dependent effectful functions
  - generalises CBPV/EEC's comp. types  $A \to \underline{C}$  and  $\underline{C} \times \underline{D}$
- $\Sigma x: A.\underline{C}$  is the type of dep. pairs of values and effectful comps.
  - captures the intuition about seq. comp. and coprods. of algebras
  - generalises EEC's comp. types  $!A \otimes C$  and  $C \oplus D$

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Value terms: MLTT + thunks + ... V, W ::= x \mid zero \mid succ V \mid ... \mid thunk M \mid ...
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equational theory based on intensional MLTT

**Comp. terms:** dep.-typed version of CBPV/EEC's comp. terms

```
\begin{array}{lll} M,N ::= & \text{force } V \\ & \mid & \text{return } V \\ & \mid & M \text{ to } x \colon A \text{ in } N \\ & \mid & \lambda x \colon A \ldotp M \\ & \mid & MV \\ & \mid & \langle V,M \rangle & \text{(comp. $\Sigma$ intro.)} \\ & \mid & M \text{ to } \langle x \colon A,z \colon \underline{C} \rangle \text{ in } K & \text{(comp. $\Sigma$ elim.)} \end{array}
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**But:** Value and comp. terms alone do not suffice, as in EEC!

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**Note:** We need to define K in such a way that the intended left-to-right evaluation order is preserved, e.g., consider

$$\Gamma \vDash \langle V, M \rangle \text{ to } \langle x \colon A, \mathbf{z} \colon \underline{C} \rangle \text{ in } \mathbf{K} = \mathbf{K}[V/x, M/\mathbf{z}] \colon \underline{D}$$

Homomorphism terms: dep.-typed version of EEC's linear terms

$$K, L := z$$
 (linear comp. vars.)  
 $\mid K \text{ to } x : A \text{ in } M$   
 $\mid \lambda x : A . K$   
 $\mid KV$   
 $\mid \langle V, K \rangle$  (comp.  $\Sigma \text{ intro.}$ )  
 $\mid K \text{ to } \langle x : A, z : C \rangle \text{ in } L$  (comp.  $\Sigma \text{ elim.}$ )

#### Typing judgments:

- Γ ⋈ V : A
- Γ la M : C
- $\Gamma \mid z : \underline{C} \mid_{\overline{h}} K : \underline{D}$  (linear in z; comp. bound to z happens first

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#### **Typing judgments:**

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### eMLTT – typing sequential composition

We can then account for type-dependency in seq. comp. as

$$\frac{\Gamma, x : A \vdash N : \underline{C}(x)}{\Gamma \vdash R \quad \Gamma \vdash \Sigma y : A \cdot \underline{C}(y) \quad \overline{\Gamma, x : A \vdash \langle x, N \rangle : \Sigma y : A \cdot \underline{C}(y)}}{\Gamma \vdash R \quad \text{to } x : A \text{ in } \langle x, N \rangle : \Sigma y : A \cdot \underline{C}(y)}$$

The seq. comp. rule for  $\lambda_{\rm ML}$  is justified by the type isomorphism

$$\frac{\Gamma \vdash A \qquad \Gamma, x : A \vdash B(x)}{\Gamma \vdash U(\Sigma x : A . F(B)) \cong UF(\Sigma x : A . B) = T(\Sigma x : A . B)}$$

# Categorical semantics of eMLTT

(fibrations + adjunctions)

# Categorical semantics - MLTT part

We define fibred adjunction models  $(\mathcal{B}, \mathcal{V}, p, q, F \dashv U)$ 

• Theorem: a sound and complete class of models for eMLTT

given by: i) a split closed comprehension cat. p (with s. fib. 0, ...)



- ullet we define a partial interpretation fun.  $[\![-]\!]$ , that (if defined) maps:
  - a context  $\Gamma$  to and object  $\llbracket \Gamma \rrbracket$  in  $\mathcal{B}$ , with  $\llbracket \Gamma, x : A \rrbracket \stackrel{\text{def}}{=} \{ \llbracket \Gamma; A \rrbracket \}$
  - ullet a context  $\Gamma$  and a value type A to an object  $[\![\Gamma;A]\!]$  in  $\mathcal{V}_{[\![\Gamma]\!]}$
  - ullet a context  $\Gamma$  and a value term V to  $[\![\Gamma;V]\!]:1_{[\![\Gamma]\!]}\longrightarrow A$  in  $\mathcal{V}_{[\![\Gamma]\!]}$

#### **Categorical semantics – MLTT part**

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- we define a partial interpretation fun. [-], that (if defined) maps:
  - a context  $\Gamma$  to and object  $\llbracket \Gamma \rrbracket$  in  $\mathcal{B}$ , with  $\llbracket \Gamma, x : A \rrbracket \stackrel{\text{def}}{=} \{ \llbracket \Gamma; A \rrbracket \}$
  - a context  $\Gamma$  and a value type A to an object  $\llbracket \Gamma ; A 
    rbracket$  in  $\mathcal{V}_{\llbracket \Gamma 
    rbracket}$
  - a context  $\Gamma$  and a value term V to  $[\![\Gamma;V]\!]:1_{[\![\Gamma]\!]}\longrightarrow A$  in  $\mathcal{V}_{[\![\Gamma]\!]}$

#### Categorical semantics - MLTT part

We define fibred adjunction models  $(\mathcal{B}, \mathcal{V}, p, q, F \dashv U)$ 

• **Theorem:** a sound and complete class of models for eMLTT given by: i) a split closed comprehension cat. p with s. fib. 0, ...

$$\begin{array}{c|c}
V \\
\uparrow \\
\uparrow \\
\downarrow \\
P
\end{array}$$

- the display maps  $\pi_{\llbracket\Gamma;A\rrbracket}:\llbracket\Gamma,x:A\rrbracket\longrightarrow \llbracket\Gamma\rrbracket$  in  $\mathcal B$  induce the weakening functors  $\pi_{\llbracket\Gamma;A\rrbracket}^*:\mathcal V_{\llbracket\Gamma\rrbracket}\longrightarrow \mathcal V_{\llbracket\Gamma,x:A\rrbracket}$ , and
- the value  $\Sigma$  and  $\Pi$ -types are interpreted as adjoints

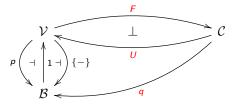
$$\begin{array}{l} \Sigma_{\llbracket\Gamma;A\rrbracket} \dashv \pi_{\llbracket\Gamma;A\rrbracket}^* : \mathcal{V}_{\llbracket\Gamma\rrbracket} \longrightarrow \mathcal{V}_{\llbracket\Gamma,x:A\rrbracket} \qquad \text{(such that $\Sigma$ is strong)} \\ \pi_{\llbracket\Gamma;A\rrbracket}^* \dashv \Pi_{\llbracket\Gamma;A\rrbracket} : \mathcal{V}_{\llbracket\Gamma,x:A\rrbracket} \longrightarrow \mathcal{V}_{\llbracket\Gamma\rrbracket} \end{array}$$

### Categorical semantics - effects part

We define fibred adjunction models  $(\mathcal{B}, \mathcal{V}, p, q, F \dashv U)$ 

• Theorem: a sound and complete class of models for eMLTT

given by: ii) a split fibration q (with ...) and a s. fib. adj.  $F \dashv U$ 



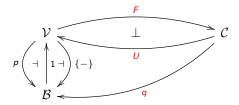
- - a ctx.  $\Gamma$  and a comp. type  $\underline{C}$  to an object  $\llbracket \Gamma ; \underline{C} \rrbracket$  in  $\mathcal{C}_{\llbracket \Gamma \rrbracket}$
  - a ctx.  $\Gamma$  and a comp. term M to  $\llbracket \Gamma; M \rrbracket : 1_{\llbracket \Gamma \rrbracket} \longrightarrow U(\underline{C})$  in  $\mathcal{V}_{\llbracket \Gamma \rrbracket}$
  - a ctx.  $\Gamma$ , a c. var. z, a c. type  $\underline{C}$ , and a hom. term K to  $\llbracket \Gamma; z \colon \underline{C}; K \rrbracket \colon \llbracket \Gamma; \underline{C} \rrbracket \longrightarrow \underline{D} \text{ in } \mathcal{C}_{\llbracket \Gamma \rrbracket}$

# Categorical semantics - effects part

We define fibred adjunction models  $(\mathcal{B}, \mathcal{V}, p, q, F \dashv U)$ 

• Theorem: a sound and complete class of models for eMLTT

given by: ii) a split fibration q (with ...) and a s. fib. adj.  $F \dashv U$ 



- we again have weakening functors  $\pi_{\llbracket\Gamma:A\rrbracket}^*:\mathcal{C}_{\llbracket\Gamma\rrbracket}\longrightarrow\mathcal{C}_{\llbracket\Gamma,x:A\rrbracket}$ , and
- the comp.  $\Sigma$  and  $\Pi$ -types are interpreted again as adjoints

$$\begin{split} & \Sigma_{\llbracket \Gamma; A \rrbracket} \dashv \pi_{\llbracket \Gamma; A \rrbracket}^* : \mathcal{C}_{\llbracket \Gamma \rrbracket} \longrightarrow \mathcal{C}_{\llbracket \Gamma, \mathbf{x} : A \rrbracket} \\ & \pi_{\llbracket \Gamma; A \rrbracket}^* \dashv \Pi_{\llbracket \Gamma; A \rrbracket} : \mathcal{C}_{\llbracket \Gamma, \mathbf{x} : A \rrbracket} \longrightarrow \mathcal{C}_{\llbracket \Gamma \rrbracket} \end{split}$$

### **Digression:** dep. elimination of 0 and +

The coproduct type A + B:

[Jacobs'99]

- require  $p: \mathcal{V} \longrightarrow \mathcal{B}$  to have split fibred coproducts  $A +_X B$ , and
- $\langle \{ \mathsf{inl}_A \}^*, \{ \mathsf{inr}_B \}^* \rangle : \mathcal{V}_{\{A+_X B\}} \longrightarrow \mathcal{V}_{\{A\}} \times \mathcal{V}_{\{B\}}$  to be fully-faith.
- allows one to interpret dependent case analysis, i.e.,

$$\mathcal{V}_{\{A\}}\left(1_{\{A\}}, \{\operatorname{inl}_A\}^*(C)\right) \times \mathcal{V}_{\{B\}}\left(1_{\{B\}}, \{\operatorname{inr}_B\}^*(C)\right) \\ \cong \\ \mathcal{V}_{\{A+_{X}B\}}\left(1_{\{A+_{X}B\}}, C\right)$$

provides semantics for

$$\frac{\Gamma, y_1 : A \vdash W_1 : C[\operatorname{inl}_A y_1/x] \quad \Gamma, y_2 : B \vdash W_2 : C[\operatorname{inr}_B y_2/x]}{\Gamma, x : A + B \vdash \operatorname{case} x \text{ of } (\operatorname{inl}(y_1) \mapsto W_1, \operatorname{inr}(y_2) \mapsto W_2) : C[\operatorname{inr}_B y_2/x]}$$

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### Digression: dep. elimination of colimits

#### A generalisation:

[Ahman'17]

- Idea: fully-faith. for cocones  $A \longrightarrow A \circledast_X B \longleftarrow B$  is enough, and we can generalise this to all split fibred colimits
- Theorem:
  - if for every object  $X \in \mathcal{B}$  and diagram  $J : \mathcal{D} \longrightarrow \mathcal{V}_X$ there exists a cocone  $\underline{\operatorname{in}}^J : J \longrightarrow \Delta(\underline{\operatorname{colim}}(J))$  in  $\mathcal{V}_X$ ,
  - such that f\*(in<sup>J</sup><sub>D</sub>) = in<sup>f\*oJ</sup><sub>D</sub>, for any f : X → Y, and such that the unique mediating functor

$$\begin{split} & \langle \{\underline{\operatorname{in}}_D^J\}_{D\in\mathcal{D}}^* \rangle : \mathcal{V}_{\{\underline{\operatorname{colim}}(J)\}} \longrightarrow \operatorname{lim}(\widehat{J}) \\ & \text{s fully-faithful (for } \widehat{J} : \mathcal{D}^{op} \longrightarrow \operatorname{Cat, where } \widehat{J}(D) = \mathcal{V}_{\{J(D)\}}) \end{split}$$

then p has split fibred colimits of shape D, and
 p supports dependent elimination for them (analogously to +x)

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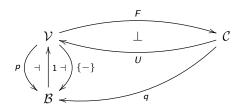
• Idea: fully-faith. for cocones  $A \longrightarrow A \circledast_X B \longleftarrow B$  is enough, and we can generalise this to all split fibred colimits

#### • Theorem:

- if for every object X ∈ B and diagram J : D → V<sub>X</sub>
   there exists a cocone in J : J → Δ(colim(J)) in V<sub>X</sub>,
- such that  $f^*(\underline{\operatorname{in}}_D^J) = \underline{\operatorname{in}}_D^{f^*\circ J}$ , for any  $f: X \longrightarrow Y$ , and such that the unique mediating functor

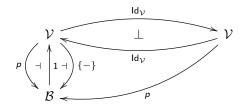
$$\langle \{\underline{\mathsf{in}}_D^J\}_{D\in\mathcal{D}}^* \rangle : \mathcal{V}_{\{\underline{\mathsf{colim}}(J)\}} \longrightarrow \mathsf{lim}(\widehat{J})$$
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then p has split fibred colimits of shape D, and
 p supports dependent elimination for them (analogously to +x)



**Example 1** (identity adjunctions):  $Id_{\mathcal{V}} \dashv Id_{\mathcal{V}} : \mathcal{V} \longrightarrow \mathcal{V}$ 

- $\mathcal{C} \stackrel{\mathsf{def}}{=} \mathcal{V}$
- $q \stackrel{\text{def}}{=} p$
- $F \stackrel{\text{def}}{=} \operatorname{Id}_{\mathcal{V}}$
- $U \stackrel{\text{def}}{=} \operatorname{Id}_{\mathcal{V}}$



Note: sound as long as we haven't included any effects

#### **Example 2** (models of Egger et al.'s EEC):

- given an adjunction  $F_{\mathsf{EEC}} \dashv U_{\mathsf{EEC}} : \mathcal{E} \longrightarrow \mathcal{D}$ , such that
  - $\mathcal{D}$  is a CCC (with 0, ...), and
  - $F_{\mathsf{EEC}} \dashv U_{\mathsf{EEC}}$  and  $\mathcal{E}$  are  $\mathcal{D}$ -enriched, and
  - $\mathcal E$  has all  $\mathcal D$ -tensors  $(A \otimes \underline{\mathcal C})$  and  $\mathcal D$ -cotensors  $(A \Rightarrow \underline{\mathcal C})$
- we use the simple fibration  $p: s(\mathcal{D}) \longrightarrow \mathcal{D}$ , where

$$p(X,A) \stackrel{\text{def}}{=} X$$
  $p(f,g) \stackrel{\text{def}}{=} f$  where  $f:X \longrightarrow Y$   $g:X \times A \longrightarrow B$ 

• then, we define the simpl.  $\mathcal{D}$ -enrich. fib.  $q: \mathbf{s}(\mathcal{D}, \mathcal{E}) \longrightarrow \mathcal{D}$ , where

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• finally, we define  $F\dashv U$  as the lifting of  $F_{\sf EEC}\dashv U_{\sf EEC}$ 

$$F(X,A) \stackrel{\text{def}}{=} (X, F_{\text{EEC}}(A))$$
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#### **Example 3** (families fibrations):

- given an adjunction  $F_{\mathcal{D}} \dashv U_{\mathcal{D}} : \mathcal{D} \longrightarrow \mathsf{Set}$ , such that
  - ullet  $\mathcal D$  has set-indexed products and coproducts
- we use the families of sets fibration  $p : Fam(Set) \longrightarrow Set$

$$p(X,A) \stackrel{\text{def}}{=} X$$
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where

$$X, Y \in \mathsf{Set}$$
  $A: X \longrightarrow \mathsf{Set}$   $B: Y \longrightarrow \mathsf{Set}$ 

$$f:X\longrightarrow Y \qquad g_{x}:A(x)\longrightarrow (B\circ f)(x)$$

- ullet analogously, we use the  ${\mathcal D}$ -valued fam. fib. q :  $\mathsf{Fam}({\mathcal D}) \longrightarrow \mathsf{Set}$
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- Ex.: EM-cats.  $(\mathcal{D} \stackrel{\text{def}}{=} \operatorname{Set}^{\mathsf{T}})$  and Lawere ths.  $(\mathcal{D} \stackrel{\text{def}}{=} \operatorname{Mod}(\mathcal{L}, \operatorname{Set}))$

**Example 4** (continuous families of cpos for  $\mu x : U\underline{C}.M$ ):

- given a CPO-enriched monad T on CPO, such that
  - **T** supports least zero-ary alg. op.  $(\bot_A : 1 \longrightarrow TA)$ , and
  - CPO<sup>T</sup> has reflexive coequalizers
- we use the continuous fam. fib.  $p : CFam(CPO) \longrightarrow CPO$

$$p(X, A) \stackrel{\text{def}}{=} X$$
  $p(f, \{g_X\}_{X \in X}) \stackrel{\text{def}}{=} f$ 

$$X \in \mathsf{CPO}$$
  $A: X \longrightarrow \mathsf{CPO}^{\mathsf{EP}}$  an  $\omega$ -continuous functor

$$f:X\longrightarrow Y$$
  $g_{x}:A(x)\longrightarrow (B\circ f)(x)$  s.t. idx. is  $\omega$ -continuous

- ullet analogously, we use the cont. fam. fib.  $q:\mathsf{CFam}(\mathsf{CPO}^\mathsf{T})\longrightarrow \mathsf{CPO}$
- finally, we define  $F \dashv U$  as the lifting of  $F^{\mathsf{T}} \dashv U^{\mathsf{T}}$
- Ex.: monads from discrete CPO-enriched countable Lawere ths.

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- Why not use  $p: \mathsf{CPO}^{\to} \longrightarrow \mathsf{CPO}$ ?
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  - Idea: Not all functors  $f^*$ : CPO/Y  $\rightarrow$  CPO/X are left adjoints
  - consider the epimorphism  $e \stackrel{\mathsf{def}}{=} n \mapsto n : \mathbb{N}_{=} \longrightarrow \mathbb{N}_{\omega}$  in CPO, and
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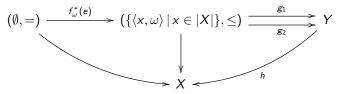
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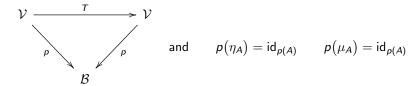
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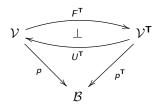


#### **Example 5** (EM-resolutions of split fibred monads):

• given a split fibred monad  $\mathbf{T} = (T, \eta, \mu)$  on p, i.e.,



• we consider models based on the EM-resolution of T



and show that three familiar results hold for this situation

**Example 5** (EM-resolutions of split fibred monads):

• **Theorem 1:** if p supports  $\Pi$ -types, then  $p^{\mathsf{T}}$  also supports  $\Pi$ -types

• **Theorem 2:** if p supports  $\Sigma$ -types and the dependent strength

$$\sigma_A:\Sigma_A\circ T\longrightarrow T\circ \Sigma_A$$

is a natural isomorphism, then  $p^{\mathsf{T}}$  also supports  $\Sigma$ -types

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### **Examples of fibred adjunction models**

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#### Fibred effect theories $\mathcal{T}_{\text{eff}}$ :

signatures of dep. typed operation symbols

$$\frac{\cdot \vdash I \qquad x_i \colon I \vdash O \qquad I \text{ and } O \text{ are pure value types}}{\mathsf{op} \colon (x_i \colon I) \longrightarrow O}$$

equipped with equations on derivable effect terms

#### In eMLTT:

$$M ::= \ldots \mid \operatorname{op}_{V}^{C}(x.M)$$

**General algebraicity equations** (in addition to eff. th. eqs.):

$$\frac{\Gamma \trianglerighteq V : I \quad \Gamma, x : O[V/x_i] \trianglerighteq M : \underline{C} \quad \Gamma \mid z : \underline{C} \mid_{\overline{h}} K : \underline{D}}{\Gamma \trianglerighteq K[\operatorname{op}_{\overline{V}}^{\underline{C}}(x.M)/z] = \operatorname{op}_{\overline{V}}^{\underline{D}}(x.K[M/z]) : \underline{D}} \text{ (op : } (x_i : I) \longrightarrow O)$$

Sound semantics: based on

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#### **Example 1** (interactive I/O):

- ullet read :  $1 \longrightarrow \mathsf{Chr}$   $(\mathsf{Chr} \stackrel{\mathsf{def}}{=} 1 + \ldots + 1)$  write :  $\mathsf{Chr} \longrightarrow 1$
- no equations

Example 2 (global state with location-dependent store type):

```
• \diamond \vdash \mathsf{Loc}

x : \mathsf{Loc} \vdash \mathsf{Val}

\diamond \vdash \mathsf{isDec}_{\mathsf{Loc}} : \Pi x : \mathsf{Loc} . \Pi y : \mathsf{Loc} . (x =_{\mathsf{Loc}} y) + (x =_{\mathsf{Loc}} y \to 0)
```

- $\begin{array}{c} \texttt{get}: (x : \mathsf{Loc}) \longrightarrow \mathsf{Val} \\ \\ \mathsf{put}: (\Sigma x : \mathsf{Loc}.\mathsf{Val}) \longrightarrow \end{array}$
- $\bullet$  five equations (two of them branching on  $is \mathsf{Dec}_\mathsf{Loc}$

**Example 3** (dep. typed update monads  $TX \stackrel{\text{def}}{=} \Pi_{s:S}$ .  $Ps \times X$ )

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### Handlers of algebraic effects (for programming and extrinsic reasoning)

### Handlers of alg. effects (for programming)

**Idea:** Generalisation of exception handlers [Plotkin, Pretnar'09]

 ${\sf Handler} = {\sf Algebra} \quad {\sf and} \quad {\sf Handling} = {\sf Homomorphism}$ 

Usual term-level presentation:

$$\underline{\Gamma} \vDash M : FA \qquad \Gamma, x_{\nu} : I, x_{k} : O[x_{\nu}/x_{i}] \to U\underline{C} \vDash N_{\text{op}} : \underline{C} \qquad \Gamma, y : A \vDash N_{\text{ret}} : \underline{C}$$

satisfying

(return 
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#### Typical use case for programming:

- write your programs using alg. ops. (e.g., get and put)
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 satisfying

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$$\sqcap \vdash P : UFA \rightarrow \mathcal{U}$$

by

- ullet equipping a universe  ${\cal U}$  with an algebra for  $\mathcal{T}_{\sf eff}$ , and
- using the above handle-into-values construct to define P

**Note 1:** P(thunk M) computes a proof obligation for M

Note 2: Formally, we work in an extension of eMLTT with

- a universe  $\mathcal{U}$  closed under Nat,  $1, 0, +, \Sigma, \Pi$
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- a universe  $\mathcal{U}$  closed under Nat,  $1, 0, +, \Sigma, \Pi$
- a type-based treatment of handlers  $\underline{C} ::= \ldots \mid \langle A; \overrightarrow{V_{op}}; \overrightarrow{W_{eq}} \rangle$
- function extensionality (actually, a it's a bit more extensional)

### **Example 1** (Evaluation Logic style modalities):

- Given a predicate  $P:A\to \mathcal{U}$  on return values, we define a predicate  $\Diamond P:UFA\to \mathcal{U}$  on I/O-computations as
- $\Diamond P \stackrel{\text{def}}{=} \lambda x : UFA . (\text{force } x) \text{ handled with } \{...\}_{\text{op} \in \mathcal{T}_{\text{IO}}} \text{ to } y : A \text{ in}_{\mathcal{U}} P y$  using the handler given by

$$\begin{array}{ll} V_{\mathsf{read}} & \stackrel{\scriptscriptstyle\mathsf{def}}{=} & \lambda \, x \colon \! \big( \Sigma \, x_{\!\scriptscriptstyle V} \colon \! 1 \cdot \mathsf{Chr} \to \mathcal{U} \big) \cdot \widehat{\Sigma} \, y \colon \! \mathsf{El}(\widehat{\mathsf{Chr}}) \cdot \big( \mathsf{snd} \, \, x \big) \, y \\ V_{\mathsf{write}} & \stackrel{\scriptscriptstyle\mathsf{def}}{=} & \lambda \, x \colon \! \big( \Sigma \, x_{\!\scriptscriptstyle V} \colon \mathsf{Chr} \cdot 1 \to \mathcal{U} \big) \cdot \big( \mathsf{snd} \, \, x \big) \, \star \end{array}$$

•  $\Diamond P$  is the possibility modality

$$\Diamond P\left(\operatorname{thunk}\left(\operatorname{read}(x.\operatorname{write}_{e'}(\operatorname{return}V)\right)\right)\right) = \widehat{\Sigma}x:\operatorname{El}(\widehat{\operatorname{Chr}}).PV$$

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#### **Example 2** (Dijkstra's weakest precondition semantics):

• Given a postcondition on return values and final states

$$Q: A \to S \to \mathcal{U}$$
  $(S \stackrel{\text{def}}{=} \Pi x : \text{Loc. Val})$ 

we define a precondition for stateful comps. on initial states

$$\mathsf{wp}_{\mathcal{Q}}: \mathit{UFA} o \mathit{S} o \mathcal{U}$$

by

i) handling the given comp. into a state-passing function using

$$V_{
m get}, V_{
m put}$$
 on  $S o (\mathcal{U} imes S)$  and  $V_{
m ret}$  "  $=$  "  $Q$ 

- ii) feeding in the initial state; and iii) projecting out  ${\cal U}$
- Theorem:  $\operatorname{wp}_Q$  satisfies expected properties of WPs, e.g.,  $\operatorname{wp}_Q\left(\operatorname{thunk}\left(\operatorname{return}V\right)\right) = \lambda x_S : S . Q V x_S$

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$$\mathsf{wp}_{O}: \mathit{UFA} \to \mathit{S} \to \mathit{U}$$

by

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$$wp_Q (thunk (return V)) = \lambda x_S : S . Q V x_S$$

$$wp_Q (thunk (put_{\langle \ell, V \rangle}(M))) = \lambda x_S : S . wp_Q (thunk M) (x_S[\ell \mapsto V])$$

#### **Example 3** (Patterns of allowed effects):

Assuming an inductive type Protocol, given by

e : Protocol 
$$\mathbf{r}: (\mathsf{Chr} \to \mathsf{Protocol}) \to \mathsf{Protocol}$$

and potentially also by  $\wedge$ ,  $\vee$ , ...

• Then, given a protocol Pr : Protocol, we define

$$\underline{\mathsf{Pr}}: \mathit{UFA} \to \mathcal{U}$$

by handling the given comp. using

$$V_{\mathsf{read}}, V_{\mathsf{write}}$$
 on  $\mathsf{Protocol} o \mathcal{U}$ 

where

$$\begin{array}{lll} V_{\mathsf{read}} & \langle -, V_{\mathsf{rk}} \rangle & (\mathtt{r} \; \mathsf{Pr'}) & \stackrel{\mathsf{def}}{=} & \widehat{\Pi} \, x \colon \mathsf{El}(\widehat{\mathsf{Chr}}) \, . \, (V_{\mathsf{rk}} \, x) \, (\mathsf{Pr'} \, x) \\ V_{\mathsf{write}} & \langle V \, , V_{\mathsf{wk}} \rangle \, (\mathtt{w} \; P \; \mathsf{Pr'}) & \stackrel{\mathsf{def}}{=} & \widehat{\Sigma} \, x \colon \mathsf{El}(P \, V) \, . \, V_{\mathsf{wk}} \, \star \, \mathsf{Pr'} \\ & \stackrel{\mathsf{def}}{=} & \widehat{\Omega} \end{array}$$

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 and potentially also by  $\land$ ,  $\lor$ ,  $\ldots$ 

Then, given a protocol Pr : Protocol, we define

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### **Conclusion**

In work we told a mathematically natural story of combining

dependent types and computational effects

#### In particular, we saw

- a clean core language of dependent types and comp. effects
- a natural category-theoretic semantics
- alg. effects and handlers, in particular, for reasoning using
  - Evaluation Logic style modalities
  - Dijkstra's weakest precondition semantics
  - patterns of allowed (I/O)-effects

#### Future work involves

- type-dependency on computations
- local effects
- more expressive computation types

# Thank you!

D. Ahman.

Fibred Computational Effects. (PhD Thesis, 2017)

D. Ahman, N. Ghani, G. Plotkin.

Dependent Types and Fibred Computational Effects. (FoSSaCS'16)

D. Ahman.

Handling Fibred Computational Effects. (POPL'18)

### Future work (type-dependency on comps.)

- How to accommodate  $\underline{D}(\text{read}(x.M))$
- That is, how to avoid restricting the typing of seq. comp.?
- M to x:A in N: C[thunk M/y] (where y:UFA) [Vákár'17]
- $\alpha: \widehat{T}(\mathcal{U}_{\mathsf{comp}}) \longrightarrow \mathcal{U}_{\mathsf{comp}}$  [Pédrot, Tabareau'17]
- for eMLTT, one possible way forward
  - i) build on Vákár's proposal
  - ii) but force type-dep. to be homomorphic
    - $\underline{D}[\text{thunk}(M \text{ to } x:A \text{ in } N)/y] = M \text{ to } x:A \text{ in } \underline{D}[\text{thunk}(N/y)]$
    - D[M to x:A in N/z] = M to x:A in D[N/z]