Directed Containers

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(based on joint work with James Chapman and Tarmo Uustalu)

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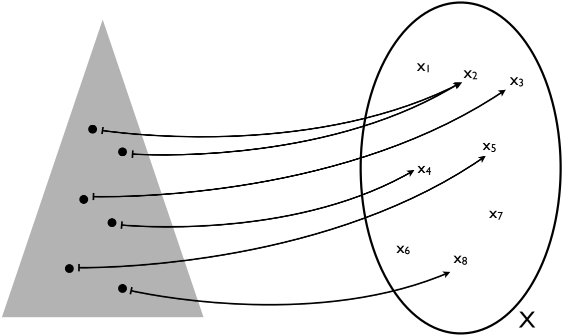
Today's plan

- Directed containers
  - type-theoretic and polynomial presentations
  - their use in functional programming
  - why are they canonical such structure?
- Some constructions on directed containers (see more in papers)
  - coproducts of directed containers
  - strict directed containers and their products
  - focussing a container
- Directed containers and computational effects
- Directed containers and BX
- Directed containers and categories

Prelude

Container syntax of datatypes

- Many datatypes can be represented in terms of
  - shapes and
  - positions in shapes



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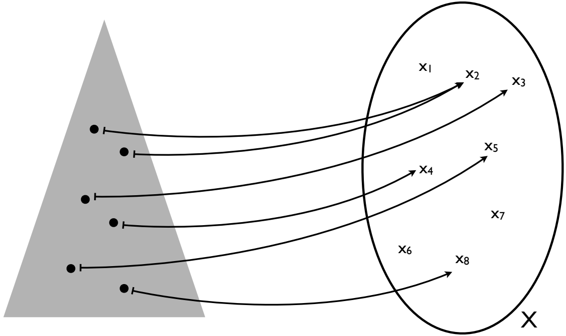
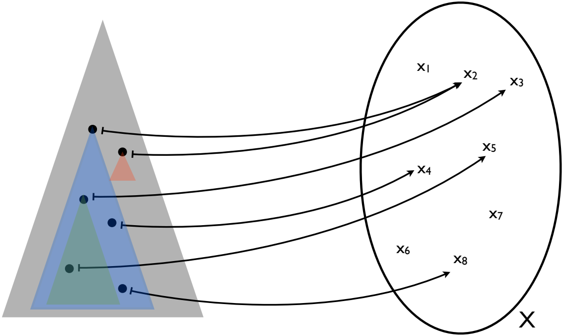


- **Examples:** lists, streams, trees, zippers, ...
- Containers provide us with a handy syntax to analyse them

Directing containers?

- Containers often exhibit a natural notion of subshape



- Natural questions arise:
  - What is the appropriate specialisation of containers?
  - Does this admit a nice mathematical theory?
  - What else is this structure useful for?

Directed containers

Directed containers

- A directed container is given by
  - S : **Set** *(shapes)*
  - $P : S \to$ **Set** *(positions)*

  and

  - $\downarrow : \Pi s : S.\, P\, s \to S$ *(subshape)*
  - $\mathrm{o} : \Pi\{s : S\}.\, P\, s$ *(root position)*
  - $\oplus : \Pi\{s : S\}.\, \Pi p : P\, s.\, P\,(s \downarrow p) \to P\, s$ *(subshape positions)*

  such that

  - $s \downarrow \mathrm{o} = s$
  - $s \downarrow (p \oplus p') = (s \downarrow p) \downarrow p'$
  - $p \oplus \{s\}\, \mathrm{o} = p$
  - $\mathrm{o}\{s\} \oplus p = p$
  - $(p \oplus \{s\}\, p') \oplus p'' = p \oplus (p' \oplus p'')$

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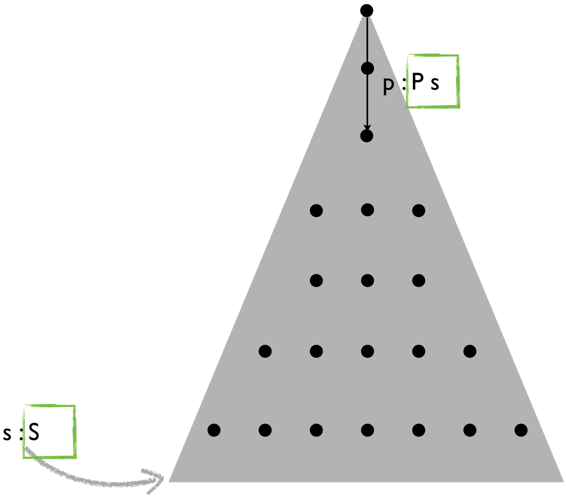
Directed containers illustrated



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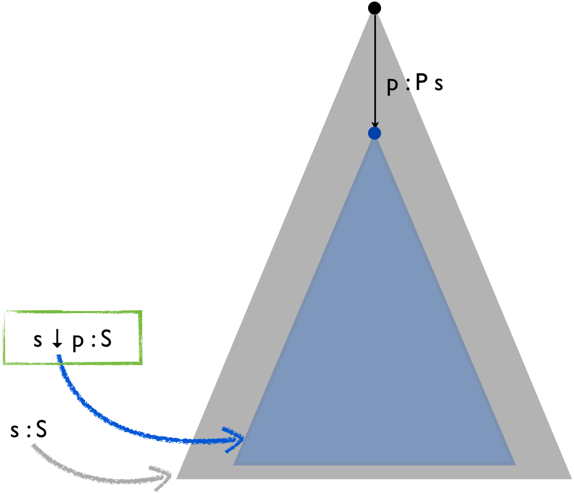

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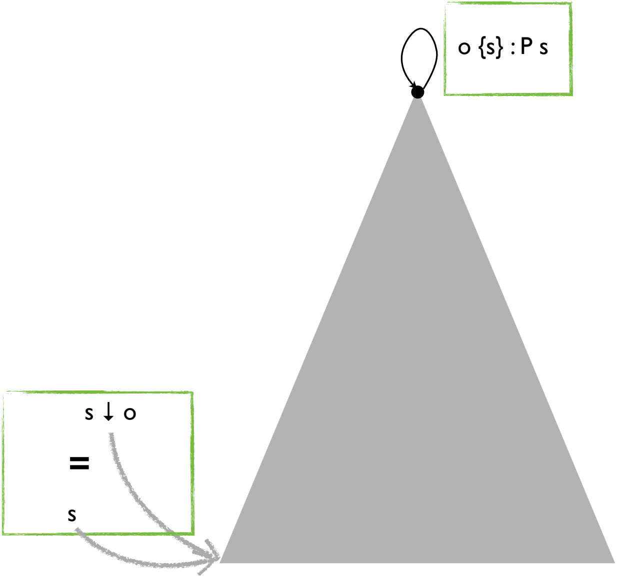

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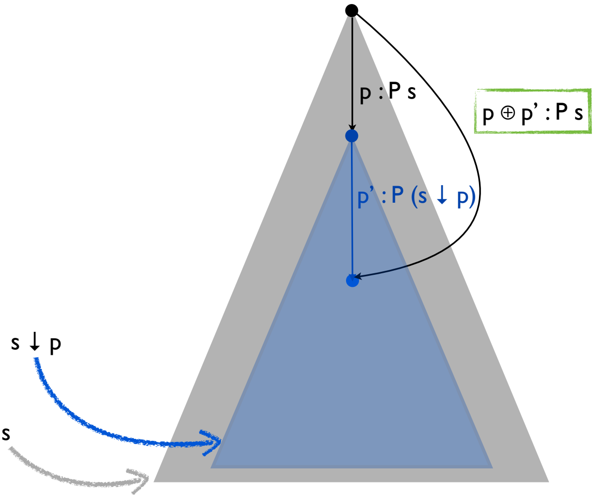

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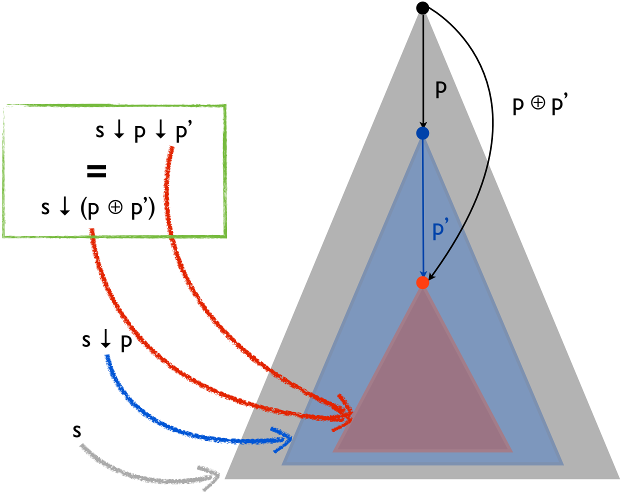

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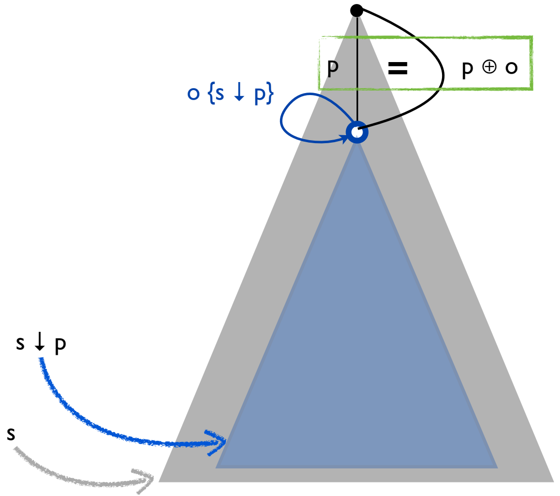

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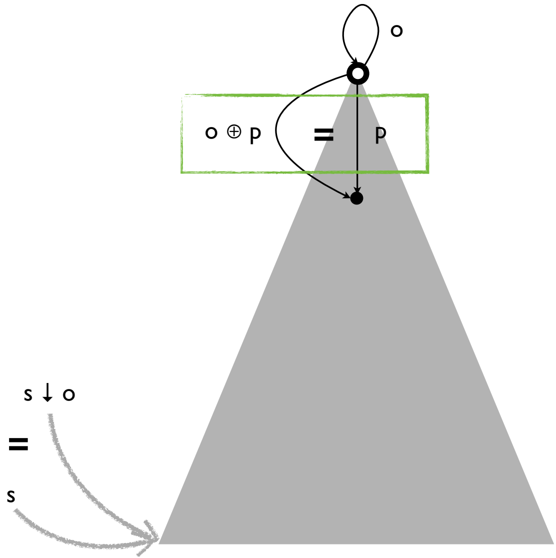

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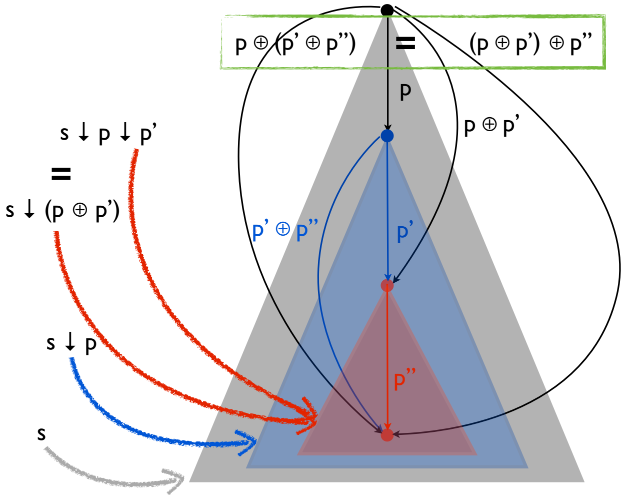

Directed containers (recap)

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Examples: non-empty lists and streams

- Non-empty lists are represented as
  - $S \stackrel{\text{def}}{=} \mathsf{Nat}$ *(shapes)*
  - $P\,n \stackrel{\text{def}}{=} \mathsf{Fin}\,(n+1) = \{0, ..., n\}$ *(positions)*
  - $n \downarrow m \stackrel{\text{def}}{=} n - m$ *(subshapes)*
  - $\mathsf{o}_{\{n\}} \stackrel{\text{def}}{=} 0$ *(root position)*
  - $m \oplus_{\{n\}} m' \stackrel{\text{def}}{=} m + m'$ *(subshape positions)*

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- Another example is non-empty lists with cyclic shifts
- Streams are represented similarly
  - $S \stackrel{\text{def}}{=} 1$ *(shapes)*
  - $P\, * \stackrel{\text{def}}{=} \text{Nat}$ *(positions)*
  - ...

Examples: non-empty lists with a focus

- Zippers – tree-like data-structures consisting of
  - a context and a focal subtree

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- Zippers – tree-like data-structures consisting of
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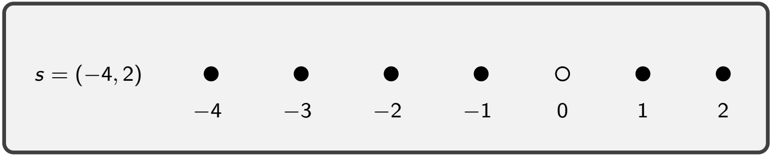
- Non-empty lists with a focus
  - $S \stackrel{\text{def}}{=} \mathrm{Nat} \times \mathrm{Nat}$ *(shapes)*
  - $P\,(n_0, n_1) \stackrel{\text{def}}{=} \{-n_0, \ldots, n_1\} = \{-n_0, \ldots, -1\} \cup \{0, \ldots, n_1\}$ *(pos.)*



  - $(n_0, n_1) \downarrow m \stackrel{\text{def}}{=} (n_0 + m, n_1 - m)$ *(subshapes)*
  - $\mathsf{o}_{\{n_0,n_1\}} \stackrel{\text{def}}{=} 0$ *(root)*
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Directed container morphisms

- A directed container morphism

$$t \lhd q : (S \lhd P, \downarrow, \mathsf{o}, \oplus) \longrightarrow (S' \lhd P', \downarrow', \mathsf{o}', \oplus')$$

  is given by

  - $t : S \to S'$
  - $q : \Pi\{s : S\}.\, P'\,(t\,s) \to P\,s$ (note the direction!)

  such that

  - $t\,(s \downarrow q\,p) = t\,s \downarrow' p$
  - $\mathsf{o}_{\{s\}} = q\,(\mathsf{o}'_{\{t\,s\}})$
  - $q\,p \oplus_{\{s\}} q\,p' = q\,(p \oplus'_{\{t\,s\}} p')$

- Identities and composition are defined component-wise
- Directed containers form a category **DCont**

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- Identities and composition are defined component-wise

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Digression: directed polynomials

Directed polynomials

- A polynomial (in one variable) is given by

$$1 \xleftarrow{\;!\;} \overline{P} \xrightarrow{\;\mathsf{s}\;} S \xrightarrow{\;!\;} 1$$

  where

  - S : **Set** (or some other suitable $\mathcal{C}$) *(shapes)*
  - $\overline{P}$: **Set** (or some other suitable $\mathcal{C}$) *(total positions)*

- Polynomials correspond to containers via $\overline{P} \cong \Sigma s : S.\, P\, s$

Directed polynomials

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$$\begin{tikzcd} 1 & \overline{P} \arrow[l, "!"'] \arrow[r, "s"] & S \arrow[r, "!"] & 1 \end{tikzcd}$$

  where
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- Polynomials correspond to containers via $\overline{P} \cong \Sigma s\!:\!S.\, P\, s$

- They interpret into polynomial functors as

$$\begin{tikzcd}
 & \bullet \arrow[r] \arrow[d] & {[\![s]\!]\, X} \arrow[dd] \arrow[ddr, "!"] & \\
X \arrow[d, "!"'] \arrow[dr, phantom, "\text{pb.}"] & X \times \overline{P} \arrow[l] \arrow[d] \arrow[r, phantom, "\text{dpb.}"] & & \\
1 & \overline{P} \arrow[l, "!"] \arrow[r, "s"'] & S \arrow[r, "!"'] & 1
\end{tikzcd}$$

Directed polynomials

Are given by the following data

$$
\begin{tikzcd}
\overline{P} \arrow[r, "s"'] & S
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{P} \arrow[d, "\downarrow"] \\
S
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{P} \arrow[r, "s"] & S \\
 & S \arrow[ul, "\circ"] \arrow[u, "\mathrm{id}"'] \arrow[ul, phantom, "\text{s.t.}" near start, shift left=4]
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{P} \arrow[rrrd, bend left, "s"] & & & \\
 & \bullet \arrow[ul, "\oplus"] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \overline{P} \arrow[r, "s"'] \arrow[d, "\downarrow"] & S \\
 & \overline{P} \arrow[r, "s"'] & S &
\end{tikzcd}
\quad \text{s.t.}
$$

such that five diagrams commute

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\begin{tikzcd}
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 & \overline{P} \arrow[r, "s"'] \arrow[d, "\downarrow"] & S \\
 & S &
\end{tikzcd}
\qquad
\cdots
$$

Directed polynomials

Are given by the following data

$$\overline{P} \xrightarrow{s} S \qquad
\begin{tikzcd}
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S
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{P} \arrow[r, "s"] & S \\
& S \arrow[u, "id"'] \arrow[ul, "\circ"] \arrow[ul, phantom, "\text{s.t.}"', shift right=3ex]
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{P} \arrow[drrr, bend left, "s"] & & & \\
& \bullet \arrow[ul, "\oplus"] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[ul, phantom, "\text{s.t.}"', shift right=6ex] & \overline{P} \arrow[r, "s"'] \arrow[d, "\downarrow"] & S \\
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\end{tikzcd}$$

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& \overline{P} \arrow[r, "s"'] \arrow[d, "\downarrow"] & S \\
& S &
\end{tikzcd}
\qquad \cdots$$

Note the symmetry in s and ↓ !

Directed containers

$$=$$

containers $\cap$ comonads

Interpretation of directed containers

- Any directed container

$$(S \lhd P, \downarrow, \mathrm{o}, \oplus)$$

defines a functor/~~comonad~~

$$[\![S \lhd P, \downarrow, \mathrm{o}, \oplus]\!]^{\mathrm{dc}} \stackrel{\mathrm{def}}{=} (D, \varepsilon, \delta)$$

where

  - $D : \mathbf{Set} \longrightarrow \mathbf{Set}$

    $D\,X \stackrel{\mathrm{def}}{=} \Sigma s : S.\,(P\,s \to X)$

  - $\varepsilon_X : D\,X \longrightarrow X$

    $\varepsilon_X\,(s, v) \stackrel{\mathrm{def}}{=} v\,(\mathrm{o}_{\{s\}})$

  - $\delta_X : D\,X \longrightarrow D\,D\,X$

    $\delta_X\,(s, v) \stackrel{\mathrm{def}}{=} (s, \lambda p.\,(s \downarrow p, \lambda p'.\,v\,(p \oplus_{\{s\}} p')))$

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Interpretation of dcon. morphisms

- Any directed container morphism

  $$t \lhd q : (S \lhd P, \downarrow, \mathsf{o}, \oplus) \longrightarrow (S' \lhd P', \downarrow', \mathsf{o}', \oplus')$$

  defines a natural transformation/~~comonad morphism~~

  $$[\![t \lhd q]\!]^{\mathrm{dc}} : [\![S \lhd P, \downarrow, \mathsf{o}, \oplus]\!]^{\mathrm{dc}} \longrightarrow [\![S' \lhd P', \downarrow', \mathsf{o}', \oplus']\!]^{\mathrm{dc}}$$

  by

  - $[\![t \lhd q]\!]^{\mathrm{dc}}_X : \Sigma s : S.\, (P\, s \to X) \longrightarrow \Sigma s' : S'.\, (P'\, s' \to X)$

    $[\![t \lhd q]\!]^{\mathrm{dc}}_X\, (s, v) \stackrel{\text{def}}{=} (t\, s, v \circ q_{\{s\}})$

- $[\![-]\!]^{\mathrm{dc}}$ preserves the identities and composition

- $[\![-]\!]^{\mathrm{dc}}$ is a functor from **DCont** to **[Set, Set]**/~~$\mathit{Comonads}(\mathit{Set})$~~

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Interpretation is fully faithful

- Every natural transformation / ~~comonad morphism~~

$$\tau : [\![S \lhd P, \downarrow, \mathrm{o}, \oplus]\!]^{\mathrm{dc}} \longrightarrow [\![S' \lhd P', \downarrow', \mathrm{o}', \oplus']\!]^{\mathrm{dc}}$$

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  satisfying

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Directed containers = cons. ∩ cmnds.

- Any comonad (D, ε, δ), such that $D = [\![S \lhd P]\!]^{\mathrm{c}}$, determines

$$\lceil (D, \varepsilon, \delta), S \lhd P \rceil \stackrel{\mathsf{def}}{=} (S \lhd P, \downarrow, \mathsf{o}, \oplus)$$

- $\lceil - \rceil$ satisfies

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$$\lceil [\![S \lhd P, \downarrow, \mathsf{o}, \oplus]\!]^{\mathrm{dc}}, S \lhd P \rceil = (S \lhd P, \downarrow, \mathsf{o}, \oplus)$$

- The following is a pullback in **CAT**:

$$\begin{tikzcd}
\mathbf{DCont} \arrow[r, "U"] \arrow[d, "{[\![-]\!]^{\mathrm{dc}}}"', "\text{f.f.}"] & \mathbf{Cont} \arrow[d, "{[\![-]\!]^{\mathrm{c}}}", "\text{f.f.}"'] \\
\mathbf{Comonads}(\mathbf{Set}) \arrow[r, "U"] & {[\mathbf{Set}, \mathbf{Set}]}
\end{tikzcd}$$

Some constructions on directed containers

Coproducts of directed containers

- Given $(S_0 \lhd P_0, \downarrow_0, \mathsf{o}_0, \oplus_0)$ and $(S_1 \lhd P_1, \downarrow_1, \mathsf{o}_1, \oplus_1)$, their coproduct is $(S \lhd P, \downarrow, \mathsf{o}, \oplus)$ where
  - $S \lhd P \stackrel{\text{def}}{=} (S_0 \lhd P_0) + (S_1 \lhd P_1) = \left(S_0 + S_1 \lhd \left[\lambda s.\, P_0\, s, \lambda s.\, P_1\, s\right]\right)$
  - $\mathsf{inl}\, s \downarrow p \stackrel{\text{def}}{=} \mathsf{inl}\,(s \downarrow_0 p)$
    $\mathsf{inr}\, s \downarrow p \stackrel{\text{def}}{=} \mathsf{inr}\,(s \downarrow_1 p)$
  - $\mathsf{o}_{\{\mathsf{inl}\, s\}} \stackrel{\text{def}}{=} \mathsf{o}_{0\{s\}}$
    $\mathsf{o}_{\{\mathsf{inr}\, s\}} \stackrel{\text{def}}{=} \mathsf{o}_{1\{s\}}$
  - $p \oplus_{\{\mathsf{inl}\, s\}} p' \stackrel{\text{def}}{=} p \oplus_{0\,\{s\}} p'$
    $p \oplus_{\{\mathsf{inr}\, s\}} p' \stackrel{\text{def}}{=} p \oplus_{1\,\{s\}} p'$

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  - $\mathsf{inl}\, s \downarrow p \stackrel{\text{def}}{=} \mathsf{inl}\,(s \downarrow_0 p)$
    $\mathsf{inr}\, s \downarrow p \stackrel{\text{def}}{=} \mathsf{inr}\,(s \downarrow_1 p)$
  - $\mathsf{o}_{\{\mathsf{inl}\, s\}} \stackrel{\text{def}}{=} \mathsf{o}_{0\{s\}}$
    $\mathsf{o}_{\{\mathsf{inr}\, s\}} \stackrel{\text{def}}{=} \mathsf{o}_{1\{s\}}$
  - $p \oplus_{\{\mathsf{inl}\, s\}} p' \stackrel{\text{def}}{=} p \oplus_{0\,\{s\}} p'$
    $p \oplus_{\{\mathsf{inr}\, s\}} p' \stackrel{\text{def}}{=} p \oplus_{1\,\{s\}} p'$
- It interprets as $[\![S_0 \lhd P_0, \downarrow_0, \mathsf{o}_0, \oplus_0]\!]^{\mathrm{dc}} + [\![S_1 \lhd P_1, \downarrow_1, \mathsf{o}_1, \oplus_1]\!]^{\mathrm{dc}}$

Products of <u>strict</u> directed containers

- Given $(S_0 \lhd P_0, \downarrow_0, \mathsf{o}_0, \oplus_0)$ and $(S_1 \lhd P_1, \downarrow_1, \mathsf{o}_1, \oplus_1)$, there is no general way to endow $(S_0 \lhd P_0) \times (S_1 \lhd P_1)$ with dcon. struct.

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- But analogously to the coprod. of (ideal) monads, the product exists for strict directed containers/coideal comonads:
  - $S : \mathbf{Set}$
  - $P^+ : S \to \mathbf{Set}$
  - $\downarrow^+ : \Pi\, s : S.\, P^+\, s \to S$
  - $\oplus^+ : \Pi\, \{s : S\}.\, \Pi\, p : P^+\, s.\, P^+\, (s \downarrow^+ p) \to P^+\, s$
  - satisfying two laws (omitted)

- The directed container determined by a strict dcon. has
  - $P\, s \stackrel{\text{def}}{=} 1 + P^+\, s$
  - ...

Products of <u>strict</u> directed containers ctd.

- Now, given $(S_0 \lhd P_0^+, \downarrow_0^+, \oplus_0^+)$ and $(S_1 \lhd P_1^+, \downarrow_1^+, \oplus_1^+)$, we can define $(S \lhd P^+, \downarrow^+, \oplus^+)$ where
  - $S \stackrel{\text{def}}{=} \overline{S_0} \times \overline{S_1}$

    with

    $(\overline{S_0}, \overline{S_1}) \stackrel{\text{def}}{=} \nu(Z_0, Z_1).\, \left(\Sigma s_0 : S_0.\, P_0^+\, s_0 \to Z_1\, ,\, \Sigma s_1 : S_1.\, P_1^+\, s_1 \to Z_0\right)$
  - $P^+\,(s_0, s_1) \stackrel{\text{def}}{=} \overline{P_0^+\, s_0} + \overline{P_1^+\, s_1}$

    with

    $$\begin{aligned}(\overline{P_0^+}, \overline{P_1^+}) \stackrel{\text{def}}{=} \mu(Z_0, Z_1).\, \big(&\lambda(s_0, v_0).\, \Sigma p_0 : P_0^+\, s_0.(1 + Z_1\,(v_0\, p_0))\,,\\ &\lambda(s_1, v_1).\, \Sigma p_1 : P_1^+\, s_1.(1 + Z_0\,(v_1\, p_1))\big)\end{aligned}$$
  - ...

Products of strict directed containers ctd.

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  - ...
- This gives the product of the given strict dcons. in **DCont**
- It interprets as the product of the corresponding coideal cmnds.

Focussing a container

- Given any container $S_0 \lhd P_0$, we get $(S \lhd P, \downarrow, o, \oplus)$ where
  - $S \stackrel{\text{def}}{=} \Sigma s : S_0 . P_0\, s$
  - $P\,(s, p) \stackrel{\text{def}}{=} P_0\, s$
  - $(s, p) \downarrow p' \stackrel{\text{def}}{=} (s, p')$
  - $o_{\{s,p\}} \stackrel{\text{def}}{=} p$
  - $p' \oplus_{\{s,p\}} p'' \stackrel{\text{def}}{=} p''$

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- When positions in P_0 are decidable, then $[\![S \lhd P, \downarrow, \mathrm{o}, \oplus]\!]^{\mathrm{dc}}$ is isomorphic to the comonad structure on $\partial [\![S_0 \lhd P_0]\!]^{\mathrm{c}} \times \mathsf{Id}$

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- Focussing forms a functor from $\mathbf{Con}_{\mathrm{cart}}$ to $\mathbf{DCon}$

Other constructions on directed containers

- Cofree and cofree recursive directed containers

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  - A dep. typed version of the Zappa–Szép product, i.e., of:
    - Given monoid actions $\alpha : N \times M \to M$ and $\beta : N \times M \to N$ satisfying two compat. laws, we get a monoid on $M \times N$ with $(m_0, n_0) \oplus (m_1, n_1) \stackrel{\mathrm{def}}{=} \big(m_0 \oplus_M \alpha(n_0, m_1)\,,\, \beta(n_0, m_1) \oplus_N n_1\big)$

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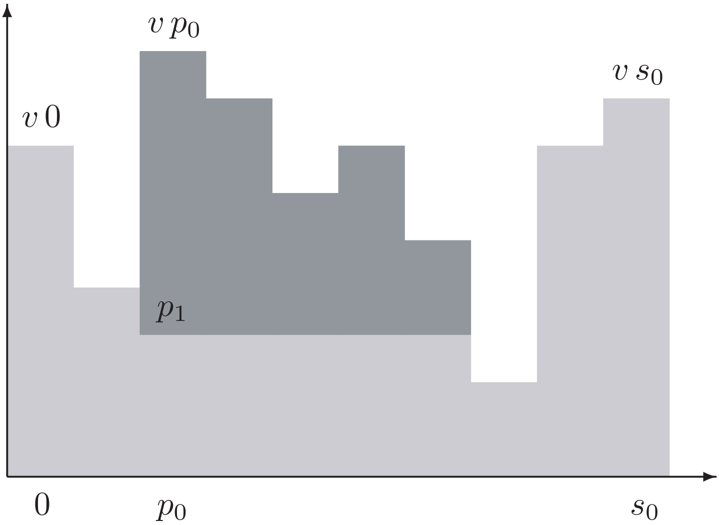
"This should be called an aqueduct" —A.M.Pitts

Non-empty lists over non-empty lists



Directed containers and effects

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  - it looks a lot like the costate comonad on $S \times (S \to -)$
  - what happens if instead we take $T\,X = \Pi s : S.\,(P\,s \times X)$?
  - it looks suspiciously like the state monad $S \to (S \times -)$

Cointerpretation of (directed) containers

- In addition to the interpretation functor

  $$[\![-]\!]^{\mathrm{c}} : \mathbf{Cont} \longrightarrow [\mathbf{Set}, \mathbf{Set}]$$

  one can also define a cointerpretation functor

  $$\langle\!\langle - \rangle\!\rangle^{\mathrm{c}} : \mathbf{Cont}^{\mathrm{op}} \longrightarrow [\mathbf{Set}, \mathbf{Set}]$$

  given by

  $$\langle\!\langle S \lhd P \rangle\!\rangle^{\mathrm{c}}\, X \;\stackrel{\mathsf{def}}{=}\; \Pi s : S.\, (P\, s \times X)$$

  which lifts to $\langle\!\langle - \rangle\!\rangle^{\mathrm{dc}}$, making the following a pullback in **CAT**

$$\begin{tikzcd}
\mathbf{DCont}^{\mathrm{op}} \arrow[r, "U"] \arrow[d, "\langle\!\langle - \rangle\!\rangle^{\mathrm{dc}}"'] & \mathbf{Cont}^{\mathrm{op}} \arrow[d, "\langle\!\langle - \rangle\!\rangle^{\mathrm{c}}"] \\
\mathbf{Monads}(\mathbf{Set}) \arrow[r, "U"] & [\mathbf{Set}, \mathbf{Set}]
\end{tikzcd}$$

Dependently typed update monads

- In more detail, given a directed container $(S \lhd P, \downarrow, \mathrm{o}, \oplus)$, the corresponding dependently typed update monad is given by
  - $T : \mathbf{Set} \longrightarrow \mathbf{Set}$
    $T\,X \stackrel{\text{def}}{=} \langle\!\langle S \lhd P \rangle\!\rangle^{\mathrm{c}}\,X = \Pi s : S.\,(P\,s \times X)$
  - $\eta_X : X \longrightarrow T\,X$
    $\eta_X\,x \stackrel{\text{def}}{=} \lambda s.\,(\mathrm{o}_{\{s\}}, x)$
  - $\mu_X : T\,T\,X \longrightarrow T\,X$
    $\mu_X\,f \stackrel{\text{def}}{=} \lambda s.\,\mathbf{let}\ (p, g) = f\,s\ \mathbf{in}$
    $\qquad \mathbf{let}\ (p', x) = g\,(s \downarrow p)\ \mathbf{in}\ (p \oplus_{\{s\}} p', x)$
- Intuitively
  - S – set/type of states
  - $(P, \mathrm{o}, \oplus)$ – dependently typed monoid of state updates

Dependently typed update monads ctd.

- The dependently typed update monad

$$T\,X \stackrel{\text{def}}{=} \Pi s : S.\,(P\,s \times X)$$

arises as the free-model monad for a (large) Lawvere theory, whose models are given by a carrier M : **Set** and two operations

$$\mathsf{lkp} : (S \to M) \longrightarrow M \qquad\qquad \mathsf{upd} : (\Pi s : S.\,P\,s) \times M \longrightarrow M$$

subject to three natural equations

- $\mathsf{lkp}\,(\lambda s.\,\mathsf{upd}_{\lambda s.\,o_{\{s\}}}(m)) = m$
- $\mathsf{lkp}\,(\lambda s.\,\mathsf{upd}_f\,(\mathsf{lkp}\,(\lambda s'.\,m\,s'))) = \mathsf{lkp}(\lambda s.\,\mathsf{upd}_f\,(m\,(s \downarrow (f\,s))))$
- $\mathsf{upd}_f\,(\mathsf{upd}_g\,(m)) = \mathsf{upd}_{\lambda s.\,(f\,s) \oplus (g\,(s \downarrow f\,s))}\,(m)$

Examples of dep. typed update monads

- Global state
  - S : **Set**
  - $P\,s \stackrel{\text{def}}{=} S$
  - $s \downarrow s' \stackrel{\text{def}}{=} s'$
  - $\mathrm{o}_{\{s\}} \stackrel{\text{def}}{=} s$
  - $s' \oplus_{\{s\}} s'' \stackrel{\text{def}}{=} s''$
  - $T\,X \stackrel{\text{def}}{=} S \to (S \times X)$

Examples of dep. typed update monads ctd.

- Monotonic state as in F*
  - S : **Set**
  - $P\,s \stackrel{\text{def}}{=} \{s' : S \mid s\ \mathcal{R}\ s'\}$

    where $\mathcal{R}$ is some fixed preorder on S, e.g.,
    - $\leq$ when $S \stackrel{\text{def}}{=}$ Nat and modelling monotonic counters
    - transition relation of some state machine (with states in S)
    - subset relation for references when $S \stackrel{\text{def}}{=}$ heap
  - $s \downarrow s' \stackrel{\text{def}}{=} s'$
  - $o_{\{s\}} \stackrel{\text{def}}{=} s$
  - $s' \oplus_{\{s\}} s'' \stackrel{\text{def}}{=} s''$
  - $T\,X \stackrel{\text{def}}{=} \Pi s : S.\, (\{s' : S \mid s\ \mathcal{R}\ s'\} \times X)$
  - In F* it is combined with a modal logic based Hoare logic

Examples of dep. typed update monads ctd.

- A non-overflowing (non-removal) buffer
  - fixed size buffer of length n
  - storing values of some type A

  - $S \stackrel{\text{def}}{=} A^{\leq n}$
  - $P\, as \stackrel{\text{def}}{=} A^{\leq n - \mathsf{len}\, as}$

  - $as \downarrow as' \stackrel{\text{def}}{=} as \mathbin{+\!\!+} as'$
  - $\mathsf{o}_{\{as\}} \stackrel{\text{def}}{=} []$
  - $as' \oplus_{\{as\}} as'' \stackrel{\text{def}}{=} as' \mathbin{+\!\!+} as''$

  - $T\, X \stackrel{\text{def}}{=} \Pi as : A^{\leq n}. \, (A^{\leq n - \mathsf{len}\, as} \times X)$

Examples of dep. typed update monads ctd.

- A non-underflowing (unbounded) stack
  - $S = A^*$
  - $P\, as = \{ps : (1 + A)^* \mid \mathsf{removes}\, ps \leq \mathsf{len}\, as\}$

    where

    $$\mathsf{removes}\, [] = 0$$
    $$\mathsf{removes}\, (\mathsf{inl} * :: ps) = \mathsf{removes}\, ps + 1$$
    $$\mathsf{removes}\, (\mathsf{inr}\, a :: ps) = \mathsf{removes}\, ps \mathbin{\dot{-}} 1$$

  - $as \downarrow [] = as$

    $as \downarrow (\mathsf{inl} * :: ps) = as/1 \downarrow ps$

    $as \downarrow (\mathsf{inr}\, a :: ps) = (as + [a]) \downarrow ps$

  - $\mathsf{o}_{\{as\}} = []$
  - $as' \oplus_{\{as\}} as'' = as' + as''$

Simply typed update monads

- If P constant, then we get a simply typed update monad

$$T\,X \stackrel{\text{def}}{=} S \to (P \times X)$$

- In this case,
  - $(P, \mathsf{o}, \oplus)$ is a monoid in the standard sense
  - $\downarrow : S \times P \longrightarrow S$ is an action of $(P, \mathsf{o}, \oplus)$ on S

- This monad is the compatible composition of the monads

$$T_{\text{reader}}\,X \stackrel{\text{def}}{=} S \to X \qquad\qquad T_{\text{writer}}\,X \stackrel{\text{def}}{=} P \times X$$

- There is a one-to-one correspondence between
  - monoid actions $\downarrow : S \times P \longrightarrow S$
  - distributive laws $\theta : T_{\text{writer}} \circ T_{\text{reader}} \longrightarrow T_{\text{reader}} \circ T_{\text{writer}}$

Directed containers and BX

Directed containers and BX

- An asymmetric lens is a comodel for the th. of global state, i.e.,
  - X : **Set** (the database)
  - get : $X \longrightarrow S$ (computing the view)
  - put : $X \times S \longrightarrow X$ (updating the database)
  - satisfying natural laws relating get and put
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- Equivalently a coalgebra for the costate comonad $S \times (S \rightarrow -)$
- Given a simply typed dcon. $(S \lhd P, \downarrow, \mathrm{o}, \oplus)$, i.e., where P : **Set**, we define a simply typed update lens to be given by
  - X : **Set**
  - lkp : $X \longrightarrow S$
  - upd : $X \times P \longrightarrow X$
  - satisfying natural laws relating lkp and upd
- Equivalently a coalgebra for $[\![S \lhd P, \downarrow, \mathrm{o}, \oplus]\!]^{\mathrm{dc}}$

Directed containers and BX ctd.

- Analogously, given a general dcon. $(S \lhd P, \downarrow, \mathsf{o}, \oplus)$,
  we can define a dependently typed update lens to be given by
  - X : **Set**
  - $\mathsf{lkp} : X \longrightarrow S$
  - $\mathsf{upd} : \left(\Sigma x : X . P\,(\mathsf{lkp}\,x)\right) \longrightarrow X$
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- Previous examples were about asymmetric update lenses, but it is also possible to do a more symmetric variant with dcons.:
  - $$\begin{aligned}\mathsf{fwd} \lhd \mathsf{bwd} : {} & (S_{\mathsf{db}} \lhd P_{\mathsf{db}}, \downarrow_{\mathsf{db}}, \mathsf{o}_{\mathsf{db}}, \oplus_{\mathsf{db}}) \\ & \longrightarrow \\ & (S_{\mathsf{view}} \lhd P_{\mathsf{view}}, \downarrow_{\mathsf{view}}, \mathsf{o}_{\mathsf{view}}, \oplus_{\mathsf{view}})\end{aligned}$$
  - now both the database and the view have their own updates

Directed containers and (small) categories

Directed containers and (small) categories

- Given a directed container $(S \lhd P, \downarrow, \mathsf{o}, \oplus)$ we get a corresponding small category $\mathcal{C}_{(S \lhd P, \downarrow, \mathsf{o}, \oplus)}$ as follows
  - $\mathrm{ob}(\mathcal{C}) \stackrel{\text{def}}{=} S$
  - $\mathcal{C}(s, s') \stackrel{\text{def}}{=} \Sigma p : P\, s.\, (s \downarrow p = s')$
  - identities are given using o
  - composition is given using $\oplus$
- And vice versa, every small category $\mathcal{C}$ gives us a corresponding directed container $(S_{\mathcal{C}} \lhd P_{\mathcal{C}}, \downarrow_{\mathcal{C}}, \mathsf{o}_{\mathcal{C}}, \oplus_{\mathcal{C}})$
- But then, is it simply the case that **Cat** $\cong$ **DCont**?

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- But then, is it simply the case that **Cat** $\cong$ **DCont**? NO!

Directed container morphisms as cofunctors

- Given a directed container morphism

$$t \lhd q : (S \lhd P, \downarrow, \mathrm{o}, \oplus) \longrightarrow (S' \lhd P', \downarrow', \mathrm{o}', \oplus')$$

we do not get a functor, but instead a cofunctor [Aguiar'97]

$$F_{t \lhd q} : \mathcal{C}_{(S \lhd P, \downarrow, \mathrm{o}, \oplus)} \longrightarrow \mathcal{D}_{(S' \lhd P', \downarrow', \mathrm{o}', \oplus')}$$

given by a mapping of objects

$$(F_{t \lhd q})_0 \stackrel{\text{def}}{=} t : \mathrm{ob}(\mathcal{C}) \longrightarrow \mathrm{ob}(\mathcal{D})$$

and a lifting operation on morphisms (pre-opcleavage)

$$\begin{tikzcd}
s \arrow[r, "(F_{t \lhd q})_1(s,p) \stackrel{\text{def}}{=} q_{\{s\}}\, p"] & \circledast & \text{in } \mathcal{C} \\
(F_{t \lhd q})_0(s) \arrow[r, "p"'] \arrow[u, phantom, "\uparrow"] & s' & \text{in } \mathcal{D}
\end{tikzcd}$$

Constructions on dcons. revisited

- On the one hand, we can relate existing constructions on directed containers to constructions (small) categories, e.g.,
  - the symmetry of the definition of directed polynomials in

    $$\mathsf{s} : \overline{P} \longrightarrow S \qquad \text{and} \qquad \downarrow\, : \overline{P} \longrightarrow S$$

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  - bidirected containers (ongoing, a bit of a conundrum) with

    $$(-)^{-1} : \Pi\{s : S\}. \Pi p : P\,s. P\,(s \downarrow p)$$

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- On the other hand, the (small) categories view also provides new insights into directed containers and comonads, e.g.,
  - factorisation of directed container/comonad morphisms

Factorisation of morphisms

- Given a directed container morphism

$$t \lhd q : (S \lhd P, \downarrow, \mathsf{o}, \oplus) \longrightarrow (S' \lhd P', \downarrow', \mathsf{o}', \oplus')$$

we can factorise $(t \lhd q)$ as $(t \lhd \lambda s.\, \mathsf{id}_{P'(t\,s)}) \circ (\mathsf{id}_S \lhd q)$ where

in **Cont**

$$\begin{tikzcd}
(S \lhd P) \arrow[r, dotted, "\mathsf{id}_S \lhd q"] \arrow[rr, bend left, "t \lhd q"] \arrow[dr, bend right, "\mathsf{id}_S \lhd \lambda_.?"'] & (S \lhd P' \circ t) \arrow[r, "t \lhd \lambda s.\mathsf{id}_{P'(t\,s)}"] \arrow[d, "\mathsf{id}_S \lhd \lambda_.?"'] \arrow[dr, phantom, "\text{pb.}"] & (S' \lhd P') \arrow[d, "\mathsf{id}_{S'} \lhd \lambda_.?"] \\
 & (S \lhd \lambda_.\,0) \arrow[r, "t \lhd \lambda_.\mathsf{id}_0"'] & (S' \lhd \lambda_.\,0)
\end{tikzcd}$$

inspired by the full image factorisation of ordinary functors

- Notably, this works for all pullback-preserving comonads

Conclusions

- Directed containers
  - type-theoretic and polynomial presentations
  - their use in functional programming
  - why are they canonical such structure?
- Some constructions on directed containers
  - coproducts of directed containers
  - strict directed containers and their products
  - focussing a container
  - ...
- Directed containers and computational effects
- Directed containers and BX
- Directed containers and categories