

# A fibrational view on computational effects

(or some things I did during my PhD)

Danel Ahman

Prosecco Team, Inria Paris

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# Outline – what I did during my PhD

2012

Refinement Types and  
Computational Effects

2015

Dependent Types and  
Computational Effects

2017

Directed Containers

# Outline – dependent types

## The Curry-Howard correspondence:

Simple Types  $\sim$  Propositional Logic  $(\text{Nat}, \text{String}, \dots)$

Dependent Types  $\sim$  Predicate Logic  $(\Sigma, \Pi, =, \dots)$

A tiny example: we can use dep. types to express sorted lists

$$\ell : (\text{List Nat}) \vdash \text{Sorted}(\ell) \stackrel{\text{def}}{=} \forall i : \text{Nat} . (0 < i < \text{len } \ell \rightarrow (\ell[i-1] \leq \ell[i]))$$

which in turn could be used to type a sorting function

$$\text{sort} : \forall \ell : (\text{List Nat}) . \exists \ell' : (\text{List Nat}) . \left( \text{Sorted}(\ell) \times \dots \right)$$

Large examples: CompCert (Coq), miTLS and HACL\* (F\*), ...

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# Outline – computational effects

## Examples:

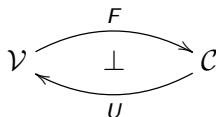
- state
- exceptions
- nondeterminism
- I/O
- ...

## Meta-languages and models:

- based on monads  $(T, \eta, \mu)$
- based on adjunctions

(Moggi)

(Levy)



- based on algebraic presentations

(Plotkin and Power)

get :  $1 \multimap S$       put :  $S \multimap 1$       + equations

# Outline – putting the two together

## We investigate the combination of

- dependent types  $(\Pi, \Sigma, V =_A W, \dots)$
- computational effects (state, nondeterminism, I/O, ...)

## Two guiding problems

- effectful programs in types (e.g., get and put in types)
- types of effectful programs (e.g., of sequential composition)

## Our goals

- tell a mathematically natural story
- use established math. techniques
- cover a wide range of comp. effects
- discover smth. interesting

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- discover smth. interesting (using handlers to reason about effects)

# **Effectful programs in types**

**(type-dependency in the presence of effects)**

# Effectful programs in types

**Q:** Should we allow situations such as  $\text{Sorted}[\text{receive}(y. M)/\ell]$ ?

**A1:** In this talk, we say **not directly**

- types should only depend on static information about effects
- we allow dependency on effectful comps. via analysing **thunks**

**A2:** But we are also looking into the **direct** case

- type-dependency needs to be “homomorphic”, but not only so
- intuitively, lift  $\text{Sorted}(\ell)$  to  $\text{Sorted}^\dagger(c)$ , where  $c: T(\text{List Chr})$

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**Aim:** Types should only depend on static info about effects

**Solution:** CBPV/EEC style distinction between vals. and comps.

- value types  $\Gamma \vdash A$  (MLTT + thunks + ...)
- computation types  $\Gamma \vdash \underline{C}$  (dep. typed CBPV/EEC)
- where  $\Gamma$  contains only value variables  $x_1 : A_1, \dots, x_n : A_n$

Could have also considered Moggi's  $\lambda_{ML}$  and Levy's FGCBV

- building on CBPV/EEC gives a more general story
- especially for the treatment of sequential composition
- and also for integrating dependent- and effect-typing (ongoing)



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# Assigning types to effectful programs

(e.g., sequential composition)

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**The problem:** The standard typing rule for seq. composition

$$\frac{\Gamma \vdash M : F A \quad \Gamma, x:A \vdash N : \underline{C}}{\Gamma \vdash M \text{ to } x:A \text{ in } N : \underline{C}}$$

is not correct any more because  $x$  can appear free in the type

$C$

in the conclusion

# Assigning types to effectful programs

**Aim:** To fix the typing rule of **sequential composition**

**Option 1:** We could restrict the free variables in  $\underline{C}$ : [Levy'04]

$$\frac{\Gamma \Vdash M : FA \quad \Gamma \vdash \underline{C} \quad \Gamma, x:A \Vdash N : \underline{C}}{\Gamma \Vdash M \text{ to } x:A \text{ in } N : \underline{C}}$$

**But:** sometimes it is useful if  $\underline{C}$  can depend on  $x$ !

- if we consider

`fopen (return true, return false) to x:Bool in N`

- then it would be natural to let  $\underline{C}$  depend on  $x$ , e.g.,

$$x:\text{Bool} \vdash \underline{C}(x) \stackrel{\text{def}}{=} \text{if } x \text{ then "allow fread, fwrite, and fclose"} \\ \text{else "allow fopen"}$$

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**But:** then comp. types would be singleton-like?!

However, smth. like this is probably needed for the **direct** case.

**Option 3:** In the monadic metalanguage  $\lambda_{\text{ML}}$ , one could try

$$\frac{\Gamma \vdash M : T A \quad \Gamma, x:A \vdash N : T B(x)}{\Gamma \vdash M \text{ to } x:A \text{ in } N : T (\Sigma x:A. B)}$$

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- and combine it with restricting  $\underline{C}$  in seq. comp. (**Option 1**)

E.g., consider the non-det. program  $(\text{for } x:\text{Nat} \models N : \underline{C}(x))$

$$M \stackrel{\text{def}}{=} \text{choose}(\text{return } 4, \text{return } 2) \text{ to } x:\text{Nat} \text{ in } N$$

After tossing the coin, this program evaluates as either

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**Idea:**  $M$  denotes an element of the coproduct of algebras

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## Putting these ideas together

(eMLTT: a core dep.-typed language with comp. effects)



# eMLTT – value and comp. types

**Value types:** MLTT + **thunks** + ...

$A, B ::= \text{Nat} \mid 1 \mid 0 \mid \Pi x:A. B \mid \Sigma x:A. B \mid V =_A W \mid \underline{U} \underline{C} \mid \dots$

- $\underline{U} \underline{C}$  is the type of **thunked** (i.e., suspended) **computations**

**Computation types:** dep.-typed version of EEC's comp. types

$\underline{C}, \underline{D} ::= F A \mid \Pi x:A. \underline{C} \mid \Sigma x:A. \underline{C}$

- $F A$  is the type of computations returning values of type  $A$
- $\Pi x:A. \underline{C}$  is the type of dependent effectful functions
  - generalises CBPV/EEC's comp. types  $A \rightarrow \underline{C}$  and  $\underline{C} \times \underline{D}$
- $\Sigma x:A. \underline{C}$  is the type of dep. pairs of values and effectful comps.
  - captures the intuition about seq. comp. and coprods. of algebras
  - generalises EEC's comp. types  $!A \otimes \underline{C}$  and  $\underline{C} \oplus \underline{D}$

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$$V, W ::= x \mid \text{zero} \mid \text{succ } V \mid \dots \mid \text{thunk } M \mid \dots$$

- equational theory based on *intensional* MLTT

**Comp. terms:** dep.-typed version of CBPV/EEC's comp. terms

$$\begin{array}{lcl} M, N ::= & \text{force } V & \\ & \text{return } V & \\ & M \text{ to } x:A \text{ in } N & \\ & \lambda x:A. M & \\ & MV & \\ & \langle V, M \rangle & \text{(comp. } \Sigma \text{ intro.)} \\ & M \text{ to } \langle x:A, z:\underline{C} \rangle \text{ in } K & \text{(comp. } \Sigma \text{ elim.)} \end{array}$$

**But:** Value and comp. terms alone do not suffice, as in EEC!

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**Note:** We need to define  $K$  in such a way that the intended left-to-right evaluation order is preserved, e.g., consider

$$\Gamma \Vdash \langle V, M \rangle \text{ to } \langle x:A, z:\underline{C} \rangle \text{ in } K = K[V/x, M/z] : \underline{D}$$

**Homomorphism terms:** dep.-typed version of EEC's linear terms

$$\begin{array}{ll} K, L ::= & z \quad \text{(linear comp. vars.)} \\ & K \text{ to } x:A \text{ in } M \\ & \lambda x:A. K \\ & KV \\ & \langle V, K \rangle \quad \text{(comp. } \Sigma \text{ intro.)} \\ & K \text{ to } \langle x:A, z:\underline{C} \rangle \text{ in } L \quad \text{(comp. } \Sigma \text{ elim.)} \end{array}$$

**Typing judgments:**

- $\Gamma \Vdash V : A$
- $\Gamma \Vdash M : \underline{C}$
- $\Gamma \mid z:\underline{C} \Vdash K : \underline{D}$  (linear in  $z$ ; comp. bound to  $z$  happens first)

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**Note:** We need to define  $K$  in such a way that the intended left-to-right evaluation order is preserved, e.g., consider

$$\Gamma \Vdash \langle V, M \rangle \text{ to } \langle x:A, z:\underline{C} \rangle \text{ in } K = K[V/x, M/z] : \underline{D}$$

**Homomorphism terms:** dep.-typed version of EEC's linear terms

$$\begin{array}{ll} K, L ::= & z \quad \text{(linear comp. vars.)} \\ & | \quad K \text{ to } x:A \text{ in } M \\ & | \quad \lambda x:A. K \\ & | \quad KV \\ & | \quad \langle V, K \rangle \quad \text{(comp. } \Sigma \text{ intro.)} \\ & | \quad K \text{ to } \langle x:A, z:\underline{C} \rangle \text{ in } L \quad \text{(comp. } \Sigma \text{ elim.)} \end{array}$$

**Typing judgments:**

- $\Gamma \Vdash V : A$
- $\Gamma \Vdash M : \underline{C}$
- $\Gamma \mid z:\underline{C} \Vdash K : \underline{D}$  (linear in  $z$ ; comp. bound to  $z$  happens first)

# eMLTT – typing sequential composition

We can then account for **type-dependency in seq. comp.** as

$$\frac{\Gamma \Vdash M : F A \quad \Gamma \vdash \Sigma_{x:A} \underline{C}(x) \quad \frac{\Gamma, x:A \Vdash N : \underline{C}(x)}{\Gamma, x:A \Vdash \langle x, N \rangle : \Sigma_{x:A} \underline{C}(x)}}{\Gamma \Vdash M \text{ to } x:A \text{ in } \langle x, N \rangle : \Sigma_{x:A} \underline{C}(x)}$$

The **seq. comp. rule for  $\lambda_{ML}$**  is justified by the type isomorphism

$$\frac{\Gamma \vdash A \quad \Gamma, x:A \vdash B(x)}{\Gamma \vdash U(\Sigma_{x:A} F(B)) \cong UF(\Sigma_{x:A} B) = T(\Sigma_{x:A} B)}$$



# Categorical semantics of eMLTT

(fibrations + adjunctions)

# Categorical semantics – value part

Given by a **split closed comprehension category**  $p$ , as in

$$\begin{array}{c} \mathcal{V} \\ \begin{array}{c} \curvearrowleft \quad \uparrow \quad \curvearrowright \\ \textcolor{red}{p} \quad \dashv \quad 1 \quad \dashv \quad \{-\} \\ \downarrow \\ \mathcal{B} \end{array} \end{array}$$

such that

- $p$  has split fibred strong **colimits** of shape **0** and **2**
  - (in thesis, also Jacobs-style axiomatisation for arbitrary shapes)
  - (all one needs are cocones and fully-faithfulness of induced func.)
- $p$  has weak split fibred strong **natural numbers**
- $p$  has split intensional **propositional equality**

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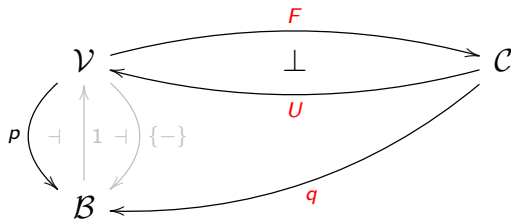
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allowing us to define a **partial interpretation fun.**  $\llbracket - \rrbracket$ , that maps:

- a **context**  $\Gamma$  to an object  $\llbracket \Gamma \rrbracket$  in  $\mathcal{B}$ , with
  - $\llbracket \diamond \rrbracket \stackrel{\text{def}}{=} 1$
  - $\llbracket \Gamma, x:A \rrbracket \stackrel{\text{def}}{=} \{ \llbracket \Gamma; A \rrbracket \}$  (if  $x \notin \text{Vars}(\Gamma)$  and  $\llbracket \Gamma; A \rrbracket$  is defined)
- a context  $\Gamma$  and a **value type**  $A$  to an object  $\llbracket \Gamma; A \rrbracket$  in  $\mathcal{V}_{\llbracket \Gamma \rrbracket}$
- a context  $\Gamma$  and a **value term**  $V$  to  $\llbracket \Gamma; V \rrbracket : 1_{\llbracket \Gamma \rrbracket} \longrightarrow A$  in  $\mathcal{V}_{\llbracket \Gamma \rrbracket}$

# Categorical semantics – effects part

Given by a **split fibration**  $q$  and a split fib. adjunction  $F \dashv U$ , as in



such that

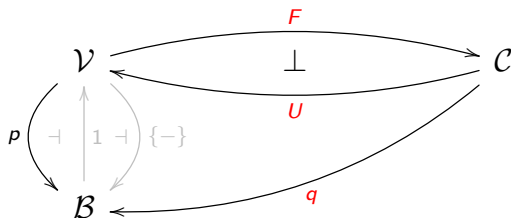
- $q$  has split dependent  $p$ -products (comp.  $\Pi$ -type; r. adj. to wk.)
- $q$  has split dependent  $p$ -coproducts (comp.  $\Sigma$ -type; l. adj. to wk.)

and to account for the full calculus presented in the thesis,

- $q$  admits split fibred pre-enrichment in  $p$  (hom. function type  $\multimap$ )

## Categorical semantics – effects part

Given by a **split fibration**  $q$  and a split fib. adjunction  $F \dashv U$ , as in



we extend the **partial interpretation fun.**  $\llbracket - \rrbracket$  so that it maps:

- a ctx.  $\Gamma$  and a **comp. type**  $\underline{C}$  to an object  $\llbracket \Gamma; \underline{C} \rrbracket$  in  $\mathcal{C}_{\llbracket \Gamma \rrbracket}$
- a ctx.  $\Gamma$  and a **comp. term**  $M$  to  $\llbracket \Gamma; M \rrbracket : 1_{\llbracket \Gamma \rrbracket} \longrightarrow U(\underline{C})$  in  $\mathcal{V}_{\llbracket \Gamma \rrbracket}$
- a ctx.  $\Gamma$ , a comp. var.  $z$ , a comp. type  $\underline{C}$ , and a **hom. term**  $K$  to  $\llbracket \Gamma; z : \underline{C}; K \rrbracket : \llbracket \Gamma; \underline{C} \rrbracket \longrightarrow \underline{D}$  in  $\mathcal{C}_{\llbracket \Gamma \rrbracket}$

# Categorical semantics – correctness

**Theorem** (Soundness):

- If  $\Gamma \vdash \underline{C}$ , then  $\llbracket \Gamma; \underline{C} \rrbracket \in \mathcal{C}_{\llbracket \Gamma \rrbracket}$
- If  $\Gamma \Vdash M : \underline{C}$ , then  $\llbracket \Gamma; M \rrbracket : 1_{\llbracket \Gamma \rrbracket} \longrightarrow U(\llbracket \Gamma; \underline{C} \rrbracket)$
- If  $\Gamma \mid z : \underline{C} \Vdash K : \underline{D}$ , then  $\llbracket \Gamma; z : \underline{C}; K \rrbracket : \llbracket \Gamma; \underline{C} \rrbracket \longrightarrow \llbracket \Gamma; \underline{D} \rrbracket$
- If  $\Gamma \vdash \underline{C} = \underline{D}$ , then  $\llbracket \Gamma; \underline{C} \rrbracket = \llbracket \Gamma; \underline{D} \rrbracket \in \mathcal{C}_{\llbracket \Gamma \rrbracket}$
- ...

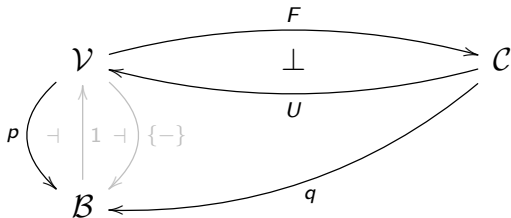
**Theorem** (Classifying model):

- The well-formed syntax of eMLTT forms a fib. adjunction model.

**Theorem** (Completeness):

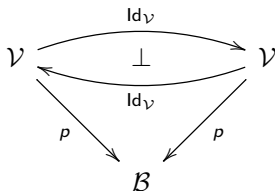
- If two types or terms are equal in all fibred adjunction models, then they are also equal in the equational theory of eMLTT.

## Examples of fibred adjunction models



# Examples of fibred adjunction models

**Example 1** (identity adjunctions):



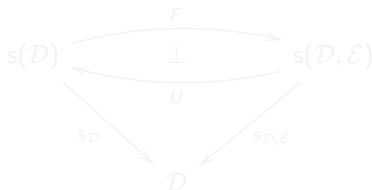
**Note:** sound model as long as we haven't included any effects



# Examples of fibred adjunction models

**Example 2** (simple models from Egger et al.'s EEC):

- given an adjunction  $F_{\text{EEC}} \dashv U_{\text{EEC}} : \mathcal{E} \longrightarrow \mathcal{D}$ , such that
  - $\mathcal{D}$  is Cartesian closed (with  $\text{Nat}, \dots$ ), and
  - $\mathcal{E}$  and  $F_{\text{EEC}} \dashv U_{\text{EEC}}$  are  $\mathcal{D}$ -enriched, and
  - $\mathcal{E}$  has all  $\mathcal{D}$ -tensors ( $A \otimes \underline{C}$ ) and  $\mathcal{D}$ -cotensors ( $A \rightrightarrows \underline{C}$ )
- we use simple fibration  $s_{\mathcal{D}}$  and simpl.  $\mathcal{D}$ -enrich. fibration  $s_{\mathcal{D}, \mathcal{E}}$



$$F(X, A) \stackrel{\text{def}}{=} (X, F_{\text{EEC}}(A))$$

$$U(X, \underline{C}) \stackrel{\text{def}}{=} (X, U_{\text{EEC}}(\underline{C}))$$

$$s(\mathcal{D}): \quad (f, g) : (X, A) \longrightarrow (Y, B) \quad \text{where} \quad f : X \longrightarrow Y \quad g : X \times A \longrightarrow B$$

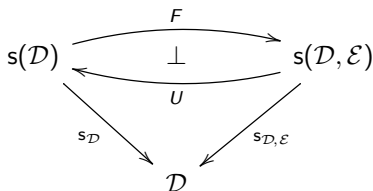
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**Note:** this model doesn't support any real type-dependency

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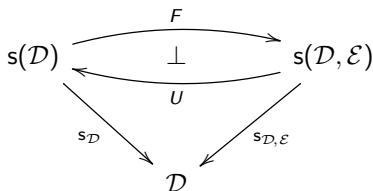
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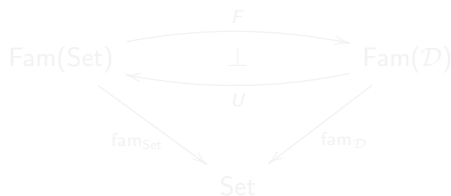
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# Examples of fibred adjunction models

## Example 3 (families fibrations):

- given an adjunction  $F_{\mathcal{D}} \dashv U_{\mathcal{D}} : \mathcal{D} \longrightarrow \mathbf{Set}$ , such that
  - $\mathcal{D}$  has set-indexed products and set-indexed coproducts
- such adjunctions arise from
  - EM-cats. ( $\mathcal{D} \stackrel{\text{def}}{=} \mathbf{Set}^{\mathbf{T}}$ ) and Law. ths. ( $\mathcal{D} \stackrel{\text{def}}{=} \mathbf{Mod}(\mathcal{L}, \mathbf{Set})$ )
  - resolutions of  $S \Rightarrow (-) \times S$  and  $((-) \Rightarrow R) \Rightarrow R$
- we use families fibrations  $\text{fam}_{\mathbf{Set}}$  and  $\text{fam}_{\mathcal{D}}$



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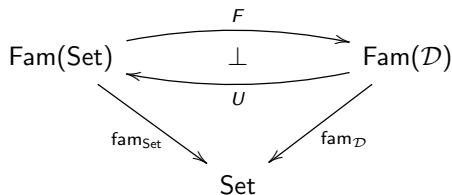
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# Examples of fibred adjunction models

**Example 4** (continuous families for  $\mu x: U\underline{C}. M$ ):

- given a **CPO-enriched monad**  $\mathbf{T}$  on CPO, such that
  - $\mathbf{T}$  supports least zero-ary alg. op. ( $\perp_A : 1 \rightarrow TA$ ), and
  - $\text{CPO}^{\mathbf{T}}$  has reflexive coequalizers
- such  $\mathbf{T}$  arise from **discrete CPO-enriched countable Law. ths.**
- we use continuous families fibrations  $\text{cfam}_{\text{CPO}}$  and  $\text{cfam}_{\text{CPO}^{\mathbf{T}}}$



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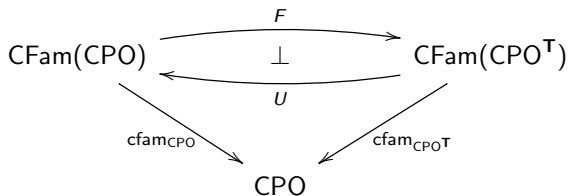
$\text{CFam}(\text{CPO})$ :  $(X, A)$  where  $X \in \text{CPO}$   $A : X \rightarrow \text{CPO}^{\text{EP}}$  an  $\omega$ -cont. fun.

**Thm.:** we don't use  $\text{cod} : \text{CPO}^{\rightarrow} \rightarrow \text{CPO}$  because CPO isn't LCCC

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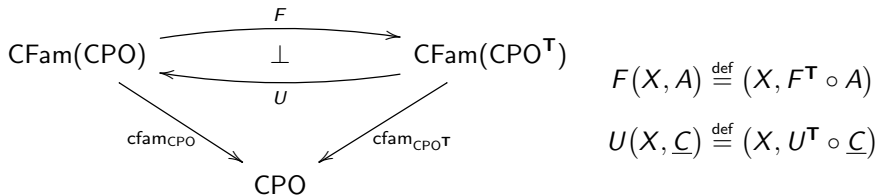
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# Examples of fibred adjunction models

**Example 5** (EM-resolutions of split fibred monads):

- given a **split fibred monad**  $\mathbf{T} = (T, \eta, \mu)$  on  $p$ , i.e.,

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{T} & \mathcal{V} \\ & \searrow p & \swarrow p \\ & \mathcal{B} & \end{array} \quad \text{and} \quad p(\eta_A) = \text{id}_{p(A)} \quad p(\mu_A) = \text{id}_{p(A)}$$

- we consider models based on the **EM-resolution** of  $\mathbf{T}$

$$\begin{array}{ccc} \mathcal{V} & \begin{array}{c} \xrightarrow{F^T} \\ \perp \\ \xleftarrow{U^T} \end{array} & \mathcal{V}^T \\ & \searrow p & \swarrow p^T \\ & \mathcal{B} & \end{array}$$

- and show that **three familiar results** hold for this situation

# Examples of fibred adjunction models

**Example 5** (EM-resolutions of split fibred monads):

- **Theorem 1:** if  $p$  supports  $\Pi$ -types, then  $p^{\mathbf{T}}$  also supports  $\Pi$ -types

$$\Pi_A^{\mathbf{T}}(B, \beta) \stackrel{\text{def}}{=} (\Pi_A(B), \beta_{\Pi_A^{\mathbf{T}}})$$

- **Prop.:** every  $\mathbf{T}$  on a split closed comp. cat. has a dep. strength

$$\sigma_A : \Sigma_A \circ T \longrightarrow T \circ \Sigma_A \quad (A \in \mathcal{V})$$

- **Theorem 2:** if  $p$  supports  $\Sigma$ -types and  $\sigma_A$  are natural isos., then  $p^{\mathbf{T}}$  also supports  $\Sigma$ -types

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# **Algebraic effects**

# Algebraic effects – ops. and eqs.

**Fibred effect theories**  $\mathcal{T}_{\text{eff}}$ :

- signatures of **dep. typed operation symbols**

$$\frac{\cdot \vdash I \quad x_i : I \vdash O \quad I \text{ and } O \text{ are pure value types}}{\text{op} : (x_i : I) \longrightarrow O}$$

- equipped with **equations** on derivable effect terms

**In eMLTT:**

$$M ::= \dots \mid \text{op}_V^C(x.M)$$

**General algebraicity equations** (in addition to eff. th. eqs.):

$$\frac{\Gamma \Vdash V : I \quad \Gamma, x : O[V/x_i] \Vdash M : \underline{C} \quad \Gamma \mid z : \underline{C} \Vdash K : \underline{D}}{\Gamma \Vdash K[\text{op}_V^C(x.M)/z] = \text{op}_V^D(x.K[M/z]) : \underline{D}} \quad (\text{op} : (x_i : I) \longrightarrow O)$$

**Sound semantics:** based on

- $p : \text{Fam}(\text{Set}) \longrightarrow \text{Set}$  and  $q : \text{Fam}(\text{Mod}(\mathcal{L}_{\mathcal{T}_{\text{eff}}}, \text{Set})) \longrightarrow \text{Set}$

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$$\frac{\Gamma \Vdash V : I \quad \Gamma, x : O[V/x_i] \Vdash M : \underline{C} \quad \Gamma \mid z : \underline{C} \Vdash K : \underline{D}}{\Gamma \Vdash K[\text{op}_V^C(x.M)/z] = \text{op}_V^D(x.K[M/z]) : \underline{D}} \quad (\text{op} : (x_i : I) \longrightarrow O)$$

**Sound semantics:** based on

- $p : \text{Fam}(\text{Set}) \longrightarrow \text{Set}$  and  $q : \text{Fam}(\text{Mod}(\mathcal{L}_{\mathcal{T}_{\text{eff}}}, \text{Set})) \longrightarrow \text{Set}$

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**Fibred effect theories**  $\mathcal{T}_{\text{eff}}$ :

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# Algebraic effects – examples

## Example 1 (interactive I/O):

- $\text{read} : 1 \longrightarrow \text{Chr}$   
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- no equations

$$(\text{Chr} \stackrel{\text{def}}{=} 1 + \dots + 1)$$

## Example 2 (global state with location-dependent store type):

- $\diamond \vdash \text{Loc}$   
 $\ell : \text{Loc} \vdash \text{Val}$   
 $\diamond \Vdash \text{isDec}_{\text{Loc}} : \prod \ell : \text{Loc} . \prod \ell' : \text{Loc} . (\ell =_{\text{Loc}} \ell') + (\ell =_{\text{Loc}} \ell' \rightarrow 0)$
- $\text{get} : (\ell : \text{Loc}) \longrightarrow \text{Val}$   
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# **Handlers of algebraic effects**

**(for programming and extrinsic reasoning)**

# Handlers of alg. effects – for programming

**Idea:** Generalisation of exception handlers [Plotkin, Pretnar'09]

Handler = Algebra and Handling = Homomorphism

Usual term-level presentation:

$\Gamma \models M$  handled with  $\{\text{op}_{x_v}(x_k) \mapsto N_{\text{op}}\}_{\text{op} \in \mathcal{T}_{\text{eff}}}$  to  $y:A$  in  $\underline{C}$   $N_{\text{ret}} : \underline{C}$

satisfying

$(\text{return } V)$  handled with  $\{\dots\}_{\text{op} \in \mathcal{T}_{\text{eff}}}$  to  $y:A$  in  $N_{\text{ret}} = N_{\text{ret}}[V/x]$

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Typical use case for programming:

- write your programs using alg. ops. (e.g., get and put)
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**Idea:** Using a derived handle-into-values handling construct

$M$  handled with  $\{\text{op}_{x_v}(x_k) \mapsto V_{\text{op}}\}_{\text{op} \in \mathcal{T}_{\text{eff}}}$  to  $y:A \text{ in}_B V_{\text{ret}}$

we can define natural predicates (essentially, dependent types)

$$\Gamma \Vdash P : UFA \rightarrow \mathcal{U}$$

by

- equipping a universe  $\mathcal{U}$  with an algebra for  $\mathcal{T}_{\text{eff}}$ , and
- using the above handle-into-values construct to define  $P$

**Note 1:**  $P(\text{thunk } M)$  computes a proof obligation for  $M$

**Note 2:** Formally, we work in an extension of eMLTT with

- a universe  $\mathcal{U}$  closed under  $\text{Nat}$ ,  $1$ ,  $0$ ,  $+$ ,  $\Sigma$ , and  $\Pi$
- a type-based treatment of handlers  $\underline{C} ::= \dots \mid \langle A; \overrightarrow{V_{\text{op}}}; \overrightarrow{W_{\text{eq}}} \rangle$
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## Example 1 (Evaluation Logic style modalities):

- Given a predicate  $P : A \rightarrow \mathcal{U}$  on return values,  
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$$\Diamond P \stackrel{\text{def}}{=} \lambda x : UFA. (\text{force } x) \text{ handled with } \{\dots\}_{\text{op} \in \mathcal{T}_{\text{IO}}} \text{ to } y : A \text{ in } P y$$

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$$V_{\text{read}} \stackrel{\text{def}}{=} \lambda x : (\Sigma x_v : 1. \text{Chr} \rightarrow \mathcal{U}). \widehat{\Sigma} y : \text{El}(\widehat{\text{Chr}}). (\text{snd } x) y$$

$$V_{\text{write}} \stackrel{\text{def}}{=} \lambda x : (\Sigma x_v : \text{Chr}. 1 \rightarrow \mathcal{U}). (\text{snd } x) \star$$

- $\Diamond P$  corresponds to Evaluation Logic's possibility modality

$$\Diamond P (\text{think}(\text{read}(x.\text{write}_{e'}(\text{return } V)))) = \widehat{\Sigma} x : \text{El}(\widehat{\text{Chr}}). P V$$

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**Example 2** (Dijkstra's weakest precondition semantics):

- Given a postcondition on return values and final states

$$Q : A \rightarrow S \rightarrow \mathcal{U} \quad (S \stackrel{\text{def}}{=} \prod x:\text{Loc}. \text{Val})$$

we define a precondition for stateful comps. on initial states

$$\text{wp}_Q : \text{UFA} \rightarrow S \rightarrow \mathcal{U}$$

by

- i) handling the given comp. into a state-passing function using

$$V_{\text{get}}, V_{\text{put}} \text{ on } S \rightarrow (\mathcal{U} \times S) \quad \text{and} \quad V_{\text{ret}} \text{ " = " } Q$$

- ii) feeding in the initial state; and iii) projecting out  $\mathcal{U}$

- Theorem:**  $\text{wp}_Q$  satisfies expected properties of WPs, e.g.,

$$\text{wp}_Q (\text{thunk}(\text{return } V)) = \lambda x_S : S. Q \ V \ x_S$$

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- Assuming an inductive type **Protocol**, given by

$$e : \text{Protocol} \quad r : (\text{Chr} \rightarrow \text{Protocol}) \rightarrow \text{Protocol}$$

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and potentially also by  $\wedge, \vee, \dots$

- Then, we define the predicate (rel. between comps. and protocols)

$$\text{Allowed} : \text{UFA} \rightarrow \text{Protocol} \rightarrow \mathcal{U}$$

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$$V_{\text{write}} \langle V, V_{\text{wk}} \rangle (\mathbf{w} \text{ } P \text{ Pr}') \stackrel{\text{def}}{=} \widehat{\Sigma} x : \text{El}(P V) . V_{\text{wk}} \star \text{Pr}'$$

$$\text{—} \stackrel{\text{def}}{=} \widehat{0}$$

# Conclusion

In work we told a mathematically natural story of combining

- dependent types and computational effects

In particular, we saw

- a clean core language of dependent types and comp. effects
- a natural category-theoretic semantics
- alg. effects and handlers, in particular, for reasoning using
  - Evaluation Logic style modalities
  - Dijkstra's weakest precondition semantics of state
  - patterns of allowed (I/O)-effects

## Things to look at:

- type-dependency on computations (e.g., in seq. composition)
- more expressive comp. types (par. adjunctions, Dijkstra monads)

# Thank you!

D. Ahman.

**Fibred Computational Effects.** (PhD Thesis, 2017)

D. Ahman, N. Ghani, G. Plotkin.

**Dependent Types and Fibred Computational Effects.** (FoSSaCS'16)

D. Ahman.

**Handling Fibred Computational Effects.** (POPL'18)





# Digression: dep. elimination of 0 and +

The coproduct type  $A + B$ :

[Jacobs'99]

- require  $p : \mathcal{V} \longrightarrow \mathcal{B}$  to have split fibred coproducts  $A +_X B$ , and
- $\langle \{\text{inl}_A\}^*, \{\text{inr}_B\}^* \rangle : \mathcal{V}_{\{A +_X B\}} \longrightarrow \mathcal{V}_{\{A\}} \times \mathcal{V}_{\{B\}}$  to be fully-faith.
- allows one to interpret dependent case analysis, i.e.,

$$\begin{aligned} \mathcal{V}_{\{A\}} \left( 1_{\{A\}}, \{\text{inl}_A\}^*(C) \right) \times \mathcal{V}_{\{B\}} \left( 1_{\{B\}}, \{\text{inr}_B\}^*(C) \right) \\ \cong \\ \mathcal{V}_{\{A +_X B\}} \left( 1_{\{A +_X B\}}, C \right) \end{aligned}$$

provides semantics for

$$\frac{\Gamma, y_1 : A \Vdash W_1 : C[\text{inl}_A y_1/x] \quad \Gamma, y_2 : B \Vdash W_2 : C[\text{inr}_B y_2/x]}{\Gamma, x : A + B \Vdash \text{case } x \text{ of } (\text{inl}(y_1) \mapsto W_1, \text{inr}(y_2) \mapsto W_2) : C}$$

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# Digression: dep. elimination of colimits

## A generalisation:

[Ahman'17]

- **Idea:** **fully-faith.** for cocones  $A \longrightarrow A \otimes_X B \longleftarrow B$  is enough, and we can generalise this to all **split fibred colimits**

- **Theorem:**

- if for every object  $X \in \mathcal{B}$  and diagram  $J : \mathcal{D} \longrightarrow \mathcal{V}_X$  there exists a cocone  $\underline{\text{in}}^J : J \longrightarrow \Delta(\underline{\text{colim}}(J))$  in  $\mathcal{V}_X$ ,
- such that  $f^*(\underline{\text{in}}_D^J) = \underline{\text{in}}_D^{f^* \circ J}$ , for any  $f : X \longrightarrow Y$ , and such that the unique mediating functor

$$\langle \{\underline{\text{in}}_D^J\}_{D \in \mathcal{D}}^* \rangle : \mathcal{V}_{\{\underline{\text{colim}}(J)\}} \longrightarrow \lim(\hat{J})$$

is fully-faithful (for  $\hat{J} : \mathcal{D}^{op} \longrightarrow \text{Cat}$ , where  $\hat{J}(D) = \mathcal{V}_{\{J(D)\}}$ ),

- then  $p$  has split fibred colimits of shape  $\mathcal{D}$ , and  $p$  supports dependent elimination for them (analogously to  $+_X$ )

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