

Interacting with the **external world** using **comodels** (aka **runners**)

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The plan

- **Computational effects** and **external resources** in PL
- **Runners** – a natural model for **top-level runtime**
- **T-runners** – for also modelling **non-top-level runtimes**
- Turning **T**-runners into a **useful programming construct**
- Some **programming examples**
- Some **implementation details**

Computational effects
and
external resources

Computational effects in PL

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- Using **monads** (as in HASKELL)

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type St a = String → (a,String)
```

```
f :: St a → St (a,a)
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f c = c >>= (\x → c >>= (\y → return (x,y)))
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- Using **alg. effects** and **handlers** (as in EFF, FRANK, KOKA)

```
effect Get : int
```

```
effect Put : int → unit
```

```
let g (c:Unit → a!{Get,Put}) =
```

```
  with st_handler handle (perform (Put 42); c ()) (* : int → a * int *)
```

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effect Put : int → unit
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```
let g (c:Unit → a!{Get,Put}) =
```

```
  with st_handler handle (perform (Put 42); c ()) (* : int → a * int *)
```

- Both are good for **faking comp. effects** in a pure language!
But what about effects that need access to the **external world**?

External resources in PL

External resources in PL

- Declare a **signature of monads** or **algebraic effects**, e.g.,

```
(* System.IO *)  
type IO a  
openFile :: FilePath → IOMode → IO Handle
```

```
(* pervasives.eff *)  
effect RandomInt : int → int  
effect RandomFloat : float → float
```

- And then **treat them specially** in the compiler, e.g.,

```
(* eff/src/backends/eval.ml *)  
let rec top_handle op =  
  match op with  
  | ...
```

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but there are some issues with that approach ...

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- Difficult to cover all possible use cases
 - **external resources hard-coded** into the top-level runtime
 - **non-trivial to change** what's available and how it's implemented

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 **Ohad** 8:35 PM
So here's the hack I added. We should do something a bit more principled

In `pervasives.eff`:

```
effect Write : (string*string) -> unit
```

in `eval.ml`, under `let rec top_handle op =` add the case:

```
| "Write" ->  
  (match v with  
  | V.Tuple vs ->  
    let (file_name :: str :: _) = List.map V.to_str vs in  
    let file_handle = open_out_gen  
                        [Open_wronly  
                        ;Open_append  
                        ;Open_creat  
                        ;Open_text  
                        ] 0o666 file_name in  
    Printf.fprintf file_handle "%s" str;  
    close_out file_handle;  
    top_handle (k V.unit_value)  
  )
```

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This talk — a principled modular (co)algebraic approach!

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- **Lack of linearity** for external resources

```
let f (s:string) =  
  let fh = fopen "foo.txt" in  
  fwrite (fh,s^s);  
  fclose fh;  
  return fh
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```
let g s =  
  let fh = f s in fread fh
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  let fh = f s in fread fh           (* fh not open ! *)
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let g s =  
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```

- We shall address these kinds of issues **indirectly**,
 - by **not** introducing a linear typing discipline
 - but instead make it convenient to **hide external resources**
(and address stronger typing disciplines in the future)

Third issue

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- **Excessive generality** of effect handlers

```
let f (s:string) =  
  let fh = fopen "foo.txt" in  
  fwrite (fh,s^s);  
  fclose fh  
  
let h = handler { fwrite (fh,s) k → return () }  
  
let f' s = handle (f "bar") with h
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where misuse of external resources can also be **purely accidental**

```
let g (s:string) =  
  let fh = fopen "foo.txt" in  
  let b = choose () in  
  if b then (fwrite (fh,s)) else (fwrite (fh,s^s));  
  fclose fh  
  
let nd_handler =  
  handler { choose () k → return (k true ++ k false) }
```

Third issue

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  let fh = fopen "foo.txt" in  
  fwrite (fh,s^s);  
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let h = handler { fwrite (fh,s) k → return () }  
  
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- We shall address these kinds of issues **directly**,
 - by proposing a **restricted form of handlers** for resources
 - that support **controlled initialisation** and **finalisation**,
 - and **limit** how general handlers can be used

Runners enter the spotlight

A natural model of **top-level runtime**

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- Given a **signature**¹ Σ of operation symbols ($A_{\text{op}}, B_{\text{op}}$ are sets)

$$\text{op} : A_{\text{op}} \rightsquigarrow B_{\text{op}}$$

a **runner**² \mathcal{R} for Σ is given by a carrier $|\mathcal{R}|$ and co-operations

$$\left(\overline{\text{op}}_{\mathcal{R}} : A_{\text{op}} \times |\mathcal{R}| \longrightarrow B_{\text{op}} \times |\mathcal{R}| \right)_{\text{op} \in \Sigma}$$

¹We consider runners for signatures, but the work generalises to alg. theories.

²In the literature also known as **comodels** for Σ (or for an algebraic theory).

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- For example, a natural runner \mathcal{R} for **S-valued state** signature

$$\text{get} : \mathbb{1} \rightsquigarrow S \quad \text{set} : S \rightsquigarrow \mathbb{1}$$

is given by

$$|\mathcal{R}| \stackrel{\text{def}}{=} S \quad \overline{\text{get}}_{\mathcal{R}}(\star, s) \stackrel{\text{def}}{=} (s, s) \quad \overline{\text{set}}_{\mathcal{R}}(s, s) \stackrel{\text{def}}{=} (\star, s)$$

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A natural model of **top-level runtime** ctd.

- Runners/comodels have been used for
 - **operational semantics** using tensors of models and comodels
[Plotkin and Power '08]
and
 - **stateful running** of algebraic effects [Uustalu '15]
 - **linear-use state-passing translation**
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and
 - **stateful running** of algebraic effects [Uustalu '15]
 - **linear-use state-passing translation** [Møgelberg and Staton '11, '14]
- The latter explicitly rely on one-to-one correspondence between
 - **runners** \mathcal{R}
 - **monad morphisms**³ $r : \mathbf{Free}_{\Sigma}(-) \longrightarrow \mathbf{St}_{|\mathcal{R}|}$

where

$$\mathbf{St}_C X \stackrel{\text{def}}{=} C \Rightarrow X \times C$$

³ $\mathbf{Free}_{\Sigma}(X)$ is the free monad ind. defined with leaves $\text{val } x$ and nodes $\text{op}(a, \kappa)$.

A natural model of **top-level runtime** ctd.

- For our purposes, we see runners

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as describing how operations affect **runtime configurations** $|\mathcal{R}|$

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- But what if this runtime is not **the** runtime?
 - hardware vs OS
 - OS vs VMs
 - VMs vs sandboxes

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- Unfortunately, runners, as defined above, are **not readily able to**
 - use **external resources**
 - **signal failure** caused by unavoidable circumstances

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 - hardware vs OS
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- Unfortunately, runners, as defined above, are **not readily able to**
 - use **external resources**
 - **signal failure** caused by unavoidable circumstances
- But is there a **useful generalisation** that would achieve this?

Effectful runners for modular top-levels

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- Møgelberg and Staton usefully observed that a **runner** \mathcal{R} is equivalently simply a family of **generic effects** for $\mathbf{St}_{|\mathcal{R}|}$, i.e.,

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- The one-to-one correspondence with **monad morphisms**

$$r : \mathbf{Free}_{\Sigma}(-) \longrightarrow \mathbf{T}$$

now simply amounts to the **univ. property of free models**, i.e.,

$$r_X(\text{val } x) = \eta_X x \qquad r_X(\text{op}(a, \kappa)) = \underbrace{(r_X \circ \kappa)^{\dagger}(\overline{\text{op}}_{\mathcal{R}} a)}_{\text{op}_{\mathcal{M}}(a, r_X \circ \kappa)}$$

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- Observe that κ appears in a **tail call position** on the right!

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- We want a runner to be a bit like a **kernel** of an OS, i.e., to
 - (i) provide management of **(internal) resources**
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- **Algebraically** (and pragmatically), this amounts to taking
 - (i) $\text{getenv} : \mathbb{1} \rightsquigarrow C$, $\text{setenv} : C \rightsquigarrow \mathbb{1}$
 - (ii) $\text{op} : A_{\text{op}} \rightsquigarrow B_{\text{op}}$ ($\text{op} \in \Sigma'$, for some external Σ')
 - (iii) $\text{kill} : S \rightsquigarrow \mathbb{0}$s.t., (i) satisfy state equations; and (i) commute with (ii) and (iii)

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- The **induced monad** is then isomorphic to

$$\mathbf{T} X \stackrel{\text{def}}{=} C \Rightarrow \mathbf{Free}_{\Sigma'}((X \times C) + S)$$

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- The corresponding **T-runners** \mathcal{R} for Σ are then of the form

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- With this, our **T-runners** \mathcal{R} for Σ are (with “primitive” excs.)

$$\left(\overline{\text{op}}_{\mathcal{R}} : A_{\text{op}} \longrightarrow \mathbf{K}_C^{\Sigma' ! E_{\text{op}} \not\downarrow S} B_{\text{op}} \right)_{\text{op} \in \Sigma}$$

where we call $\mathbf{K}_C^{\Sigma' ! E_{\text{op}} \not\downarrow S}$ a **kernel monad**, given by

$$\mathbf{K}_C^{\Sigma' ! E_{\text{op}} \not\downarrow S} X \stackrel{\text{def}}{=} C \Rightarrow \mathbf{Free}_{\Sigma}(((X + E) \times C) + S)$$

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we accommodate **runners as values** and **co-ops. as kernel code**

```
let R = runner { op1 x1 → K1 , ... , opn xn → Kn } @ C
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- For instance, we can implement a **write-only file handle** as

```
let RFH = runner {  
  write s → if (length s > max)  
    then (raise WriteSizeExceeded)  
    else (let fh = getenv () in  
          if (isValid fh) then (fwrite (fh,s)) else (kill IOError))  
} @ FileHandle
```

where

$$(\text{fwrite} : \text{FileHandle} \times \text{String} \rightsquigarrow 1 ! E) \in \Sigma'$$

$$\Sigma \stackrel{\text{def}}{=} \{ \text{write} : \text{String} \rightsquigarrow 1 ! E \cup \{\text{WriteSizeExceeded}\} \} \quad \text{IOError} \in S$$

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using R @ Minit  
run M  
finally {return x @ c → Mret , ... raise e @ c → Me ... , ... kill s → Ms ...}
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(a **user monad**)

- M_s are **user code**, modelled using $\mathbf{U}^{\Sigma!E} X \stackrel{\text{def}}{=} \mathbf{Free}_\Sigma(X + E)$

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- M_{ret} , M_e , M_s **finalise** for return values, exceptions, and signals

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- M is the user code being **run using the runner** R
- M_{ret} , M_e , M_s **finalise** for return values, exceptions, and signals
- M_{ret} and M_e **depend on the final state** c , but M_s **does not**

Controlled **initialisation** and **finalisation** ctd.

- For instance, we can define a PYTHON-esque **with construct**

```
with fileName do M
=
using R_FH @ (fopen fileName)
run M
finally {
  return x @ fh → fclose fh; return x ,
  raise e @ fh → fclose fh; raise e ,
  kill s → return () }
```

- Importantly, here
 - the **file handle is hidden** from `M`
 - `M` can only use `write` but not `fopen` and `fclose`
 - `write` : $\text{String} \rightsquigarrow 1 ! E \cup \{\text{WriteSizeExceeded}\}$
 - `fopen` and `fclose` are limited to **initialisation-finalisation**

**A core calculus for
programming with runners**

Core calculus (syntax)

Core calculus (syntax)

- **Ground types** (types of ops. and kernel state)

$$A, B, C ::= B \mid 1 \mid 0 \mid A \times B \mid A + B$$

- **Types**

$$\begin{aligned} X, Y &::= B \mid 1 \mid 0 \mid X \times Y \mid X + Y \\ &\mid X \xrightarrow{\Sigma} Y ! E \\ &\mid X \xrightarrow{\Sigma} Y ! E \Downarrow S @ C \\ &\mid \Sigma \Rightarrow \Sigma' \Downarrow S @ C \end{aligned}$$

- **Values**

$$\Gamma \vdash V : X$$

- **User computations**

$$\Gamma \vDash M : X ! E$$

- **Kernel computations**

$$\Gamma \vDash K : X ! E \Downarrow S @ C$$

Core calculus (user computations)

$$\begin{aligned} M, N ::= & \text{ return } V \\ & | \text{ try } M \text{ with } \{ \text{ return } x \mapsto M_{\text{return}}, \\ & \quad (\text{ raise } e \mapsto N_e)_{e \in E} \} \\ & | V \ W \\ & | \text{ match } V \text{ with } \{ \langle x, y \rangle \mapsto M \} \\ & | \text{ match } V \text{ with } \{ \} _X \\ & | \text{ match } V \text{ with } \{ \text{ inl } x \mapsto M, \text{ inr } y \mapsto N \} \\ & | \text{ op}_X(V, (x . M), (N_e)_{e \in E}) \\ & | \text{ raise}_X e \\ & | \text{ using } V @ W \text{ run } M \text{ finally } \{ \text{ return } x @ c \mapsto M, \\ & \quad (\text{ raise } e @ c \mapsto N_e)_{e \in E} \\ & \quad (\text{ kill } s \mapsto N_s)_{s \in S} \} \\ & | \text{ kernel } K @ V \text{ finally } \{ \text{ return } x @ c \mapsto M, \\ & \quad (\text{ raise } e @ c \mapsto N_e)_{e \in E} \\ & \quad (\text{ kill } s \mapsto N_s)_{s \in S} \} \end{aligned}$$

Core calculus (kernel computations)

$$\begin{aligned} K, L ::= & \text{return}_C V \\ & | \text{try } K \text{ with } \{ \text{return } x \mapsto K_{\text{return}}, \\ & \quad (\text{raise } e \mapsto L_e)_{e \in E} \} \\ & | V W \\ & | \text{match } V \text{ with } \{ \langle x, y \rangle \mapsto K \} \\ & | \text{match } V \text{ with } \{ \}_{X@C} \\ & | \text{match } V \text{ with } \{ \text{inl } x \mapsto K, \text{inr } y \mapsto L \} \\ & | \text{op}_{X@C}(V, (x . K), (L_e)_{e \in E}) \\ & | \text{raise}_{X@C} e \\ & | \text{kill}_{X@C} s \\ & | \text{getenv}_C(c . K) \\ & | \text{setenv}(V, K) \\ & | \text{user } M \text{ with } \{ \text{return } x \mapsto K_{\text{return}}, \\ & \quad (\text{raise } e \mapsto L_e)_{e \in E} \} \end{aligned}$$

Core calculus (type system and eq. theory)

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- For example, the **typing rule for running user comps.** is

$$\begin{array}{c}
 \Gamma \vdash V : \Sigma \Rightarrow \Sigma' \not\downarrow S @ C \quad \Gamma \vdash W : C \\
 \Gamma \Vdash M : X ! E \quad \Gamma, x : X, c : C \Vdash' N_{ret} : Y ! E' \\
 \frac{(\Gamma, c : C \Vdash' N_e : Y ! E')_{e \in E} \quad (\Gamma \Vdash' N_s : Y ! E')_{s \in S}}{\Gamma \Vdash' \text{using } V @ W \text{ run } M \text{ finally } \{ \text{return } x @ c \mapsto N_{ret} , \\
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 \text{(kill } s \mapsto N_s)_{s \in S} \} : Y ! E'}
 \end{array}$$

- and the **main β -equation for running user comps.** is

$$\begin{aligned}
 &\Gamma \preceq' \text{using } R_C @ W \text{ run } (\text{op}_X(V, (x.M), (M_e)_{e \in E_{\text{op}}})) \text{ finally } F \\
 &\equiv \text{kernel } R_{\text{op}}[V] @ W \text{ finally } \{ \\
 &\quad \text{return } x @ c' \mapsto \text{using } R_C @ c' \text{ run } M \text{ finally } F , \\
 &\quad (\text{raise } e @ c' \mapsto \text{using } R_C @ c' \text{ run } M_e \text{ finally } F)_{e \in E_{\text{op}}} , \\
 &\quad (\text{kill } s \mapsto N_s)_{s \in S} \} : Y ! E'
 \end{aligned}$$

Core calculus (type system and eq. theory)

- The calculus also includes **subtyping**, and **subsumption rules**

$$\frac{\Gamma \vdash V : A \quad A <: B}{\Gamma \vdash V : B}$$

$$\frac{\Gamma \Vdash M : A ! E \quad \Sigma \subseteq \Sigma' \quad A <: B \quad E \subseteq E'}{\Gamma \Vdash' M : B ! E'}$$

$$\frac{\begin{array}{cccc} \Gamma \Vdash K : A ! E \downarrow S @ C & \Sigma \subseteq \Sigma' & & \\ A <: B & E \subseteq E' & S \subseteq S' & C = C' \end{array}}{\Gamma \Vdash' K : B ! E' \downarrow S' @ C'}$$

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- We use $C = C'$ to have (standard) **proof-irrelevant subtyping**
- Otherwise, instead of just $C <: C'$, we would need a **lens** $C' \leftrightarrow C$

Core calculus (semantics)

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- **Monadic semantics**, for simplicity in **Set**, using
 - **user monads** $\mathbf{U}^{\Sigma!E} X \stackrel{\text{def}}{=} \mathbf{Free}_{\Sigma}(X + E)$
 - **kernel monads** $\mathbf{K}_C^{\Sigma!E \not\vdash S} X \stackrel{\text{def}}{=} C \Rightarrow \mathbf{Free}_{\Sigma}(((X + E) \times C) + S)$

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 - **kernel monads** $\mathbf{K}_C^{\Sigma!E \not\vdash S} X \stackrel{\text{def}}{=} C \Rightarrow \mathbf{Free}_{\Sigma}(((X + E) \times C) + S)$
- (At a high level) the **judgements are interpreted** as

$$\llbracket \Gamma \vdash V : X \rrbracket : \llbracket \Gamma \rrbracket \longrightarrow \llbracket X \rrbracket$$

$$\llbracket \Gamma \models M : X ! E \rrbracket : \llbracket \Gamma \rrbracket \longrightarrow \mathbf{U}^{\Sigma!E} \llbracket X \rrbracket$$

$$\llbracket \Gamma \models K : X ! E \not\vdash S @ C \rrbracket : \llbracket \Gamma \rrbracket \longrightarrow \mathbf{K}_{\llbracket C \rrbracket}^{\Sigma!E \not\vdash S} \llbracket X \rrbracket$$

Core calculus (semantics ctd.)

- However, to prove **coherence** of the semantics (**subtyping!**), we actually give the semantics in the **subset fibration**

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- For instance, **kernel computations** are interpreted as

$$\begin{array}{ccc}
 \llbracket \Gamma \rrbracket & \xrightarrow{\llbracket \Gamma \models K : X ! E \not\leq S @ C \rrbracket} & \mathbf{K}_{\llbracket C \rrbracket}^{\Sigma ! E \not\leq S} \llbracket X \rrbracket \\
 \downarrow \subseteq & & \downarrow \subseteq \\
 \llbracket \Gamma^s \rrbracket & \xrightarrow{\llbracket \Gamma^s \vdash K : X^s @ C \rrbracket} & \mathbf{K}_{\llbracket C \rrbracket}^{\mathcal{O} ! E \not\leq S + \{\perp\}} \llbracket X^s \rrbracket
 \end{array}$$

where $\Gamma^s \vdash K : X^s @ C$ is a **skeletal kernel typing judgement**

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 \end{array}$$

where $\Gamma^s \vdash K : X^s @ C$ is a **skeletal kernel typing judgement**

- No essential obstacles to extending to **Sub(Cpo)** and beyond
- **Ground type restriction** on C needed to stay within **Sub(—)**
 - Otherwise, analogously to subtyping, we'd need **lenses** instead

Runners in action

Runners can be **vertically nested**

Runners can be **vertically nested**

```
• using RFH @ (fopen fileName)
  run (
    using RFC @ (return "")
    run M
    finally {
      return x @ str → write str; return x ,
      raise e @ str → raise e }
  )
  finally {
    return x @ fh → fclose fh; return x ,
    raise e @ fh → fclose fh; raise e , kill IOError → return ()}
```

where the **file contents runner** (with $\Sigma' = \emptyset$) is defined as

```
let RFC = runner {
  write str → let str' = getenv () in
    if (length (str^str') > max) then (raise WriteSizeExceeded)
    else (setenv (str^str'))
} @ String
```

Vertical nesting for instrumentation

Vertical nesting for instrumentation

- ```
using RSniffer @ (return 0)
run M
finally {
 return x @ c →
 let fh = fopen "nsa.txt" in fwrite (fh,toStr c); fclose fh }
```

where the **instrumenting runner** is defined as

```
let RSniffer = runner {
 ... ,
 op a → let c = getenv () in
 setenv (c + 1);
 op a ,
 ...
} @ Nat
```

(\* forwards op outwards \*)

- The runner  $R_{\text{Sniffer}}$  implements the same sig.  $\Sigma$  that  $M$  is using
- As a result, the runner  $R_{\text{Sniffer}}$  is **invisible** from  $M$ 's viewpoint

# Vertical nesting for **active monitoring**

# Vertical nesting for active monitoring

- First, we define a runner for **integer-valued ML-style state** as

**type** IntHeap = (Nat  $\rightarrow$  (Int + 1))  $\times$  Nat

**type** Ref = Nat

```
let RIntState = runner {
 alloc x \rightarrow let h = getenv () in (* alloc : Int \rightsquigarrow Ref ! 0 *)
 let (r,h') = heapAlloc h x in
 setenv h';
 return r ,

 deref r \rightarrow let h = getenv () in (* deref : Ref \rightsquigarrow Int ! 0 *)
 match (heapSel h r) with
 | inl x \rightarrow return x
 | inr () \rightarrow kill ReferenceDoesNotExist ,

 assign r y \rightarrow let h = getenv () in (* assign : Ref \times Int \rightsquigarrow 1 ! 0 *)
 match (heapUpd h r y) with
 | inl h' \rightarrow setenv h'
 | inr () \rightarrow kill ReferenceDoesNotExist
}
```

@ IntHeap



## Vertical nesting for **active monitoring** ctd.

- Next we define a runner for **monotonicity layer** on top of  $R_{\text{IntState}}$

# Vertical nesting for **active monitoring** ctd.

- Next we define a runner for **monotonicity layer** on top of  $R_{\text{IntState}}$

```
type MonMemory = Ref \rightarrow ((Int \rightarrow Int \rightarrow Bool) + 1)
```

```
let RMonState = runner {
 monAlloc x rel \rightarrow let r = alloc x in (* : Int \times Ord \rightsquigarrow Ref ! 0 *)
 let m = getenv () in
 setenv (memAdd m r rel);
 return r,

 monDeref r \rightarrow deref r , (* monDeref : Ref \rightsquigarrow Int ! 0 *)

 monAssign r y \rightarrow let x = deref r in (* : Ref \times Int \rightsquigarrow 1 ! {MV} *)
 let m = getenv () in
 match (memSel m r) with
 | inl rel \rightarrow if (rel x y)
 then (assign r y)
 else (raise MonotonicityViolation)
 | inr \rightarrow kill PreorderDoesNotExist
}

@ IntHeap
```

## Vertical nesting for **active monitoring** ctd.

- We can then perform **runtime monotonicity verification** as

# Vertical nesting for active monitoring ctd.

- We can then perform **runtime monotonicity verification** as

```
using RIntState @ ((fun _ → inr ()), 0) (* init empty ML-style heap *)
run (
 using RMonState @ (fun _ → inr ()) (* init empty preorders memory *)
 run (
 let r = monAlloc 0 (≤) in
 monAssign r 1;
 try (monAssign r 0) with { (* RMonState raises exception *)
 return _ → monAssign r 3 ,
 raise MonotonicityViolation → return ();
 monAssign r 2
)
 finally {return x @ m → return x ,
 raise MonotonicityViolation @ m → ... ,
 kill PreorderDoesNotExist → ... }
)
finally {return x @ h → print h; return x ,
 kill ReferenceDoesNotExist → ... }
```

Runners can also be **horizontally paired**

# Runners can also be horizontally paired

- Given a runner for  $\Sigma$

```
let R1 = runner { ... , op1i x → k1i , ... } @ C1
```

and a runner for  $\Sigma'$

```
let R2 = runner { ... , op2j x → k2j , ... } @ C2
```

we can **pair them** to get a runner for  $\Sigma \cup \Sigma'$

```
let R = runner {
 ... ,
 op1i x → let (c,c') = getenv () in
 let (x,c'') = k1i x in
 setenv (c'',c');
 return x ,
 ... ,
 op2j x → ... (* analogously to above *) ,
 ...
} @ C1 × C2
```

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```
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 ... ,
 op2j x → ... (* analogously to above *) ,
 ...
} @ C1 × C2
```

- For instance, this way we can build a **runner for IO and state**

## Other examples



# Other examples

- More general forms of **(ML-style) state** (for general Ref A)
  - if the host language allows it, we use GADTs, etc for safety
  - some examples extract a footprint from a larger memory
- **Combinations** of different effects and runners
  - in particular the combination of IO and state
  - good use case for both vertical and horizontal composition
- KOKA-style **ambient values** and **ambient functions**
  - **ambient values** are essentially **mutable variables/parameters**
  - **ambient functions** are **applied in their lexical context**
  - a runner that treats **amb. fun. application as a co-operation**
  - amb. funs. are stored in a context-depth-sensitive heap
  - the appl. co-operation restores the heap to the lexical context

## Implementing runners

Experimenting with the **theory in practice**

# Experimenting with the **theory in practice**

- A **small experimental language** COOP<sup>4</sup>
  - Implements the core calculus with few extras
  - The interpreter is directly based on the denotational semantics
  - Top-level containers for running external (OCaml) code

---

<sup>4</sup>coop [/ku:p/] – a cage where small animals are kept, especially chickens

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- A **HASKELL library** HASKELL-COOP
  - A shallow-embedding of the core calculus in HASKELL
  - Uses one of the Freer monad implementations underneath
  - Again, the operational aspects implement the denot. semantics
  - Top-level containers for arbitrary HASKELL monads
  - Examples make use of HASKELL's features (GADTs, ...)

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  - Again, the operational aspects implement the denot. semantics
  - Top-level containers for arbitrary HASKELL monads
  - Examples make use of HASKELL's features (GADTs, ...)
- Both still need some finishing touches, but will be public soon

---

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# Experimenting with the theory in practice

```
module AmbientsTests where

import Control.Runner
import Control.Runner.Ambients

ambFun :: AmbVal Int -> Int -> AmbEff Int
ambFun x y =
 do x <- getVal x;
 return (x + y)

test1 :: AmbEff Int
test1 =
 withAmbVal
 (4 :: Int)
 (\ x ->
 withAmbFun
 (ambFun x)
 (\ f ->
 do rebindVal x 2;
 applyFun f 1))

test2 = ambToplevel test1
```

# Wrapping up

- **Runners** are a natural model of **top-level runtime**
- We propose **T-runners** to also model **non-top-level runtimes**
- We have turned **T-runners** into a **(practical ?) programming construct**, that supports controlled initialisation and finalisation
- I showed you some **combinators** and **programming examples**
- Two **implementations** in the works, COOP & HASKELL-COOP
- **Future:** lenses in subtyping and semantics, category of runners, handlers, larger case studies, refinement typing, compilation, ...

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**Thank you!**





# Core calculus (semantics ctd.)

$$\begin{aligned} \llbracket \Gamma \models' \text{using } V @ W \text{ run } M \text{ finally } \{ & \text{return } x @ c \mapsto N_{ret} , \\ & (\text{raise } e @ c \mapsto N_e)_{e \in E} , \\ & (\text{kill } s \mapsto N_s)_{s \in S} \} : Y ! E' \rrbracket_\gamma \stackrel{\text{def}}{=} \dots \end{aligned}$$

- $\llbracket V \rrbracket_\gamma = \mathcal{R} = \left( \overline{\text{op}}_{\mathcal{R}} : \llbracket A_{\text{op}} \rrbracket \longrightarrow \mathbf{K}_{\llbracket C \rrbracket}^{\Sigma' ! E_{\text{op}} \not\vdash S} \llbracket B_{\text{op}} \rrbracket \right)_{\text{op} \in \Sigma}$
- $\llbracket W \rrbracket_\gamma \in \llbracket C \rrbracket$
- $\llbracket M \rrbracket_\gamma \in \mathbf{U}^{\Sigma ! E} \llbracket A \rrbracket$
- $\llbracket \text{return } x @ c \mapsto N_{ret} \rrbracket_\gamma \in \llbracket A \rrbracket \times \llbracket C \rrbracket \longrightarrow \mathbf{U}^{\Sigma' ! E'} \llbracket B \rrbracket$
- $\llbracket (\text{raise } e @ c \mapsto N_e)_{e \in E} \rrbracket_\gamma \in E \times \llbracket C \rrbracket \longrightarrow \mathbf{U}^{\Sigma' ! E'} \llbracket B \rrbracket$
- $\llbracket (\text{kill } s \mapsto N_s)_{s \in S} \rrbracket_\gamma \in S \longrightarrow \mathbf{U}^{\Sigma' ! E'} \llbracket B \rrbracket$
- allowing us to use the **free model property** to get

$$\mathbf{U}^{\Sigma ! E} \llbracket A \rrbracket \xrightarrow{r_{\llbracket A \rrbracket} + E} \mathbf{K}_{\llbracket C \rrbracket}^{\Sigma' ! E \not\vdash S} \llbracket A \rrbracket \xrightarrow{(\lambda \llbracket N_{ret} \rrbracket_\gamma)^\dagger} \llbracket C \rrbracket \Rightarrow \mathbf{U}^{\Sigma' ! E'} \llbracket B \rrbracket$$

and then apply the resulting composite to  $\llbracket M \rrbracket_\gamma$  and  $\llbracket W \rrbracket_\gamma$