A fibrational view on computational effects

(dependent types + computational effects)

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Overview – dependent types

The Curry-Howard correspondence:

```
\begin{array}{lll} \text{Simple Types} & \sim & \text{Propositional Logic} & & (\text{Nat}, \text{String}, \ldots) \\ \\ \text{Dependent Types} & \sim & \text{Predicate Logic} & & (\Sigma, \Pi, =, \ldots) \end{array}
```

A tiny example: we can use dep. types to express sorted lists

$$\ell$$
: (List Nat) \vdash Sorted(ℓ) $\stackrel{\text{def}}{=}$ Πi : Nat. ($0 < i < \mathtt{len} \ \ell$) \rightarrow ($\ell[i-1] \le \ell[i]$)

which in turn could be used to type a sorting function

```
\forall sort : \Pi \ell: (List Nat) . \Sigma \ell': (List Nat) . (Sorted(\ell') \times \dots)
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Large examples: CompCert (Coq), miTLS and HACL* (F*), ...

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Overview – computational effects

Examples:

- state
- exceptions
- nondeterminism
- interactive IO
- . . .

Meta-languages and models: based on

• monads (λ_c , λ_{ML} , FGCBV)

(Moggi)

$$T: \mathcal{V} \longrightarrow \mathcal{V}$$

adjunctions (CBPV, EEC)

(Levy, Egger et al.)

$$F: \mathcal{V} \longrightarrow \mathcal{C} \qquad U: \mathcal{C} \longrightarrow \mathcal{V}$$

• algebraic presentations

(Plotkin and Power)

get: $1 \rightarrow S$ put: $S \rightarrow 1$ + equations

We investigate the combination of

```
• dependent types  (\Pi, \Sigma, V =_A W, ...)
```

• computational effects (state, nondeterminism, IO, ...)

Two guiding problems

- effectful programs in types (e.g., get and put in types)
- types of effectful programs (e.g., of sequential composition)

- tell a mathematically natural story
- use established math. techniques
- cover a wide range of comp. effects
- discover smth. interesting

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- use established math. techniques (fibrations and adjunctions)
- cover a wide range of comp. effects (alg. effects, continuations)
- discover smth. interesting (using handlers to reason about effects)

(type-dependency in the presence of effects)

Q: Should we allow situations such as Sorted[receive(y.M)/ ℓ]?

A1: In this work, we say not directly

- types should only depend on static information about effects
- we allow dependency on effectful comps. via analysing thunks

A2: But we are also looking into the direct case

- type-dependency needs to be "homomorphic"
- intuitively, lift Sorted (ℓ) to Sorted $^\dagger(c)$, where c : $\mathcal{T}(\mathsf{List} \ \mathsf{Chr})$

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Aim: Types should only depend on static info about effects

Solution: CBPV/EEC style distinction between vals. and comps

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• value types \Gamma \vdash A (MLTT + thunks + ...]
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- computation types $\Gamma \vdash \underline{C}$ (dep. typed CBPV/EEC
- where Γ contains only value variables $x_1: A_1, \ldots, x_n: A_n$

Could have also considered Moggi's λ_{ML} and Levy's FGCBV

- building on CBPV/EEC gives a more general story
- especially for the treatment of sequential composition
- and also for integrating dependent- and effect-typing (ongoing)

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(e.g., sequential composition)

The problem: The standard typing rule for seq. composition

$$\frac{\Gamma \vdash_{\overline{c}} M : F \land A \qquad \Gamma, x : A \vdash_{\overline{c}} N : \underline{C}(x)}{\Gamma \vdash_{\overline{c}} M \text{ to } x : A \text{ in } N : \underline{C}(x)}$$

is not correct any more because x can appear free in the type

(

in the conclusion

Aim: To fix the typing rule of sequential composition

Option 1: We could restrict the free variables in \underline{C} : [Levy'04] $\underline{\Gamma \models M : FA \qquad \Gamma \vdash \underline{C} \qquad \Gamma, x : A \models N : \underline{C}}$

But: Sometimes it is useful if \underline{C} can depend on x!

sav we consider

fopen $(\mathtt{return}\ \mathtt{true},\mathtt{return}\ \mathtt{false})$ to $x\mathtt{:}\mathsf{Bool}\ \mathtt{in}\ \mathsf{N}$

• then it would be natural to let \underline{C} depend on x, e.g.,

 $x: Bool \vdash \underline{C}(x) \stackrel{\text{def}}{=} \text{if } x \text{ then "allow fread, fwrite, and fclose"}$ else "allow fopen"

needs more expressive comp. types than you see in this talk)

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Option 2: One could lift sequential composition to type level

$$\Gamma \vdash M \text{ to } x : A \text{ in } N : M \text{ to } x : A \text{ in } \underline{C}$$

But: Then comp. types would be singleton-like!?!

However, smth. like this is probably needed for the direct case.

Option 3: In the monadic metalanguage λ_{ML} , one could try

$$\frac{\Gamma \vdash M : T A \qquad \Gamma, x : A \vdash N : T B(x)}{\Gamma \vdash M \text{ to } x : A \text{ in } N : T (\Sigma x : A . B)}$$

But: What makes this a principled solution? Why is it correct?

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Option 4: We draw inspiration from algebraic effects
and combine this with restricting <u>C</u> in seq. comp. (Option 1)

E.g., consider the non-deterministic prog. (for $x : \text{Nat } \vdash N : \underline{C}(x)$)

After making the non-det. choice, this program evaluates as either N[4/x] : $\underline{C}[4/x]$ or N[2/x] : $\underline{C}[2/x]$

Idea: M denotes an element of the coproduct of algebras

$$\underline{C}[4/x] + \underline{C}[2/x] \stackrel{\text{def}}{=} F\left(U\left(\underline{C}[4/x]\right) + U\left(\underline{C}[2/x]\right)\right)_{=}$$

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After making the non-det. choice, this program evaluates as either

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Putting these ideas together

(eMLTT: a core dep.-typed language with comp. effects)

eMLTT – value and comp. types

Value types: MLTT + thunks + ...

$$A, B ::=$$
Nat $\mid 1 \mid 0 \mid \Pi x : A . B \mid \Sigma x : A . B \mid V =_A W \mid U \subseteq \mid \dots$

• $U\underline{C}$ is the type of thunked (i.e., suspended) computations

Computation types: dep.-typed version of EEC's comp. types

$$\underline{C}, \underline{D} ::= FA \mid \Pi x : A \cdot \underline{C} \mid \Sigma x : A \cdot \underline{C}$$

- FA is the type of computations returning values of type A
- Π x : A . C is the type of dependent effectful functions
 - generalises CBPV/EEC's comp. types $A \rightarrow \underline{C}$ and $\underline{C} \times \underline{D}$
- $\Sigma x: A \cdot C$ is the type of dep. pairs of values and effectful comps.
 - captures the intuition about seq. comp. and coprods. of algebras
 - generalises EEC's comp. types $!A \otimes C$ and $C \oplus D$

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eMLTT – value and comp. terms

```
Value terms: MLTT + thunks + ... V, W ::= x \mid zero \mid succ V \mid ... \mid thunk M \mid ...
```

equational theory based on intensional MLTT

Comp. terms: dep.-typed version of CBPV/EEC's comp. terms

```
\begin{array}{lll} M,N ::= & \operatorname{force} V \\ & | & \operatorname{return} V \\ & | & M \operatorname{to} x : A \operatorname{in} N \\ & | & \lambda x : A . M \\ & | & MV \\ & | & \langle V,M \rangle & (\operatorname{comp.} \Sigma \operatorname{intro.}) \\ & | & M \operatorname{to} \langle x : A,z : \underline{C} \rangle \operatorname{in} K & (\operatorname{comp.} \Sigma \operatorname{elim.}) \end{array}
```

But: Value and comp. terms alone do not suffice, as in EEC!

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\begin{array}{lll} \textit{M}, \textit{N} ::= & \text{force } \textit{V} \\ & | & \text{return } \textit{V} \\ & | & \textit{M} \text{ to } \textit{x} \colon \textit{A} \text{ in } \textit{N} \\ & | & \lambda \textit{x} \colon \textit{A} \colon \textit{M} \\ & | & \textit{MV} \\ & | & \langle \textit{V}, \textit{M} \rangle & \text{(comp. } \Sigma \text{ intro.)} \\ & | & \textit{M} \text{ to } \langle \textit{x} \colon \textit{A}, \textit{z} \colon \underline{\textit{C}} \rangle \text{ in } \textit{K} \end{array}
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But: Value and comp. terms alone do not suffice, as in EEC!

eMLTT - homomorphism terms

Note: We need to define K in such a way that the intended left-to-right evaluation order is preserved, e.g., consider

$$\Gamma \vdash_{\!\!\!\!c} \langle V, M \rangle \text{ to } \langle x \colon\! A, \mathbf{z} \colon\! \underline{C} \rangle \text{ in } \mathbf{K} = \mathbf{K}[V/x, M/\mathbf{z}] \colon\! \underline{D}$$

Homomorphism terms: dep.-typed version of EEC's linear terms

$$K, L := z$$
 (linear comp. vars.)
 $\mid K \text{ to } x : A \text{ in } M$
 $\mid \lambda x : A . K$
 $\mid KV$
 $\mid \langle V, K \rangle$ (comp. $\Sigma \text{ intro.}$)
 $\mid K \text{ to } \langle x : A, z : C \rangle \text{ in } L$ (comp. $\Sigma \text{ elim.}$)

Typing judgments:

- Γ ⋈ V : A
- [to M : C
- $\Gamma \mid z : \underline{C} \mid_{\overline{h}} K : \underline{D}$ (linear in z; comp. bound to z happens first

eMLTT - homomorphism terms

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$$\Gamma \vdash \langle V, M \rangle$$
 to $\langle x : A, z : \underline{C} \rangle$ in $K = K[V/x, M/z] : \underline{D}$

Homomorphism terms: dep.-typed version of EEC's linear terms

```
\begin{array}{lll} \textit{K}, \textit{L} ::= & \textit{z} & \text{(linear comp. vars.)} \\ & \mid & \textit{K} \text{ to } x : \textit{A} \text{ in } \textit{M} \\ & \mid & \lambda x : \textit{A} . \textit{K} \\ & \mid & \textit{KV} \\ & \mid & \langle \textit{V}, \textit{K} \rangle & \text{(comp. } \Sigma \text{ intro.)} \\ & \mid & \textit{K} \text{ to } \langle x : \textit{A}, \textit{z} : \underline{\textit{C}} \rangle \text{ in } \textit{L} & \text{(comp. } \Sigma \text{ elim.)} \end{array}
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Typing judgments:

- Γ ⋈ V : A
- Γ |_c M : C
- $\Gamma \mid z : \underline{C} \mid_{\overline{h}} K : \underline{D}$ (linear in z; comp. bound to z happens first)

eMLTT – typing sequential composition

We can then account for type-dependency in seq. comp. as

$$\frac{\Gamma, x : A \vDash N : \underline{C}(x)}{\Gamma \vDash M : FA} \frac{\Gamma, x : A \vDash N : \underline{C}(x)}{\Gamma, x : A \vDash \langle x, N \rangle : \Sigma x : A \cdot \underline{C}(x)}$$
$$\frac{\Gamma \vDash M \text{ to } x : A \text{ in } \langle x, N \rangle : \Sigma x : A \cdot \underline{C}(x)}{\Gamma \vDash M \text{ to } x : A \text{ in } \langle x, N \rangle : \Sigma x : A \cdot \underline{C}(x)}$$

The seq. comp. rule for $\lambda_{\rm ML}$ is justified by the type isomorphism

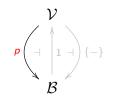
$$\frac{\Gamma \vdash A \qquad \Gamma, x : A \vdash B(x)}{\Gamma \vdash U(\Sigma x : A \cdot FB(x)) \cong UF(\Sigma x : A \cdot B(x)) = T(\Sigma x : A \cdot B(x))}$$

Categorical semantics of eMLTT

(fibrations + adjunctions)

Fibred adjunction models – value part

Given by a split closed comprehension category p, as in



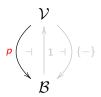
allowing us to define a partial interpretation fun. [-], that maps:

- a context Γ to and object $\llbracket \Gamma \rrbracket$ in \mathcal{B} , with

 - $\llbracket \Gamma, x : A \rrbracket \stackrel{\mathsf{def}}{=} \{ \llbracket \Gamma; A \rrbracket \}$ (if $x \notin \mathit{Vars}(\Gamma)$ and $\llbracket \Gamma; A \rrbracket$ is defined)
- a context Γ and a value type A to an object $\llbracket \Gamma; A \rrbracket$ in $\mathcal{V}_{\llbracket \Gamma \rrbracket}$
- a context Γ and a value term V to $\llbracket \Gamma; V \rrbracket : 1_{\llbracket \Gamma \rrbracket} \longrightarrow A$ in $\mathcal{V}_{\llbracket \Gamma \rrbracket}$

Fibred adjunction models – value part

Given by a split closed comprehension category p, as in

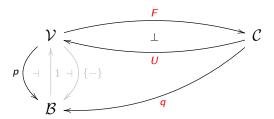


such that

- p has split fibred strong colimits of shape **0** and **2** [Jacobs'99]
 - (in thesis, also Jacobs-style axiomatisation for arbitrary shapes)
- p has weak split fibred strong natural numbers
 - (axiomatisation is given in the style of fibrational induction)
- p has split intensional propositional equality
 - (currently very synthetic ax., would like a weak form of adjoints)

Fibred adjunction models - effects part

Given by a split fibration q and a split fib. adjunction $F \dashv U$, as in

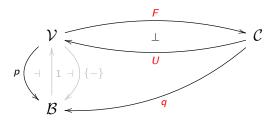


we extend the partial interpretation fun. [-] so that it maps:

- a ctx. Γ and a comp. type \underline{C} to an object $[\![\Gamma;\underline{C}]\!]$ in $\mathcal{C}_{[\![\Gamma]\!]}$
- a ctx. Γ and a comp. term M to $[\![\Gamma;M]\!]:1_{[\![\Gamma]\!]}\longrightarrow U(\underline{C})$ in $\mathcal{V}_{[\![\Gamma]\!]}$
- a ctx. Γ , a comp. var. z, a comp. type \underline{C} , and a hom. term K to $\llbracket \Gamma; z : \underline{C}; K \rrbracket : \llbracket \Gamma; \underline{C} \rrbracket \longrightarrow \underline{D}$ in $\mathcal{C}_{\llbracket \Gamma \rrbracket}$

Fibred adjunction models – effects part

Given by a split fibration q and a split fib. adjunction $F \dashv U$, as in



such that

- q has split dependent p-products (comp. Π-type; r. adj. to wk.)
- q has split dependent p-coproducts (comp. Σ-type; l. adj. to wk.)

and to account for the full calculus presented in the thesis,

• q admits split fibred pre-enrichment in p (hom. function type $-\circ$)

Fibred adjunction models – correctness

Theorem (Soundness):

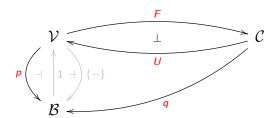
- If $\Gamma \vdash \underline{C}$, then $[\![\Gamma;\underline{C}]\!] \in \mathcal{C}_{[\![\Gamma]\!]}$
- $\bullet \ \, \text{If} \,\, \Gamma \models M : \underline{C}, \,\, \text{then} \,\, \llbracket \Gamma; M \rrbracket : 1_{\llbracket \Gamma \rrbracket} \longrightarrow \textit{U}(\llbracket \Gamma; \underline{C} \rrbracket)$
- $\bullet \ \ \mathsf{lf} \ \Gamma \vdash \underline{\mathcal{C}} = \underline{\mathcal{D}}, \ \mathsf{then} \ \llbracket \Gamma ; \underline{\mathcal{C}} \rrbracket = \llbracket \Gamma ; \underline{\mathcal{D}} \rrbracket \in \mathcal{C}_{\llbracket \Gamma \rrbracket}$
- ...

Theorem (Classifying model):

• The well-formed syntax of eMLTT forms a fib. adjunction model.

Theorem (Completeness):

• If two types or terms are equal in all fibred adjunction models, then they are also equal in the equational theory of eMLTT.



Example 1 (identity adjunctions):

• sound as long as we haven't included any actual comp. effects

Example 2 (simple fibrations from enriched adj. models of EEC):

• doesn't support any real type dependency (constant families

Example 3 (families fibrations and lifting of adjunctions):

- $\bullet \ (\llbracket \Gamma \rrbracket, \llbracket A \rrbracket) \in \mathsf{Fam}(\mathsf{Set}) \qquad \qquad (\mathsf{where} \ \llbracket A \rrbracket \in \llbracket \Gamma \rrbracket \longrightarrow \mathsf{Set}$
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Example 4 (continuous families and CPO-enriched monads)

- $(\llbracket \Gamma \rrbracket, \llbracket A \rrbracket) \in \mathsf{CFam}(\mathsf{CPO})$ (where $\llbracket A \rrbracket \in \llbracket \Gamma \rrbracket \longrightarrow \mathsf{CPO}^{\mathit{EP}} \rrbracket$
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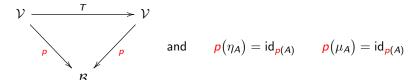
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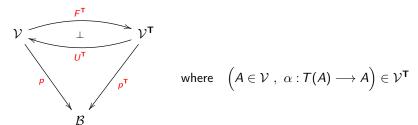
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Example 5 (EM-resolutions of split fibred monads):

• given a split fibred monad $\mathbf{T} = (T, \eta, \mu)$ on \mathbf{p} , i.e.,



we consider models based on the EM-resolution of T



and show that three familiar results hold for this situation

Example 5 (EM-resolutions of split fibred monads):

• **Theorem 1:** If p supports Π -types, then p^{T} also supports Π -types

$$\Pi_A^{\mathsf{T}}(B,\beta) \ \stackrel{\scriptscriptstyle\mathsf{def}}{=} \ \left(\Pi_A(B),\beta_{\Pi_A^{\mathsf{T}}}\right)$$

• **Prop.:** If p supports Σ -types, then T has a dependent strength

$$\sigma_A: \Sigma_A \circ T \longrightarrow T \circ \Sigma_A \qquad (A \in \mathcal{V})$$

• Theorem 2: If σ_A are natural isos., then p^T supports Σ -types

$$\Sigma_A^{\mathsf{T}}(B,\beta) \stackrel{\text{def}}{=} (\Sigma_A(B), \beta_{\Sigma_A^{\mathsf{T}}})$$

 Theorem 3: If p supports Σ-types and p^T has split fibred reflexive coequalizers, then p^T also supports Σ-types

$$\Sigma_A^{\mathsf{T}}(B,\beta) \stackrel{\text{def}}{=} F^{\mathsf{T}}(\Sigma_A(B))_{/\equiv}$$

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Algebraic effects

(operations and equations)

Fibred effect theories \mathcal{T}_{eff} :

signatures of dependently typed operation symbols

$$\frac{\cdot \vdash I \qquad x_i : I \vdash O \qquad I \text{ and } O \text{ are pure value types}}{\text{op} : (x_i : I) \rightharpoonup O}$$

equipped with equations on derivable effect terms

In eMLTT:

$$M ::= \ldots \mid \operatorname{op}_{V}^{\mathcal{C}}(x.M)$$

General algebraicity equations (in addition to eff. th. eqs.):

$$\frac{\Gamma \trianglerighteq V : I \quad \Gamma, x : O[V/x_i] \trianglerighteq M : \underline{C} \quad \Gamma \mid z : \underline{C} \trianglerighteq_{\overline{h}} K : \underline{D}}{\Gamma \trianglerighteq K[\operatorname{op}_{\overline{V}}^{\underline{C}}(x.M)/z] = \operatorname{op}_{\overline{V}}^{\underline{D}}(x.K[M/z]) : \underline{D}} \text{ (op : } (x_i : I) \to O)$$

•
$$p : \mathsf{Fam}(\mathsf{Set}) \longrightarrow \mathsf{Set}$$
 and $g : \mathsf{Fam}(\mathsf{Mod}(\mathcal{L}_{\mathcal{T}_{\mathsf{eff}}}, \mathsf{Set})) \longrightarrow \mathsf{Set}$

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Algebraic effects – examples

Example 1 (interactive IO):

- read : $1
 ightharpoonup \mathsf{Chr} = 1 + \ldots + 1)$ write : $\mathsf{Chr} \rightharpoonup 1$
- no equations

Example 2 (global state with location-dependent store type):

- \diamond \vdash Loc ℓ :Loc \vdash Val \diamond \forall isDec_{Loc}: $\Pi \ell$:Loc. $\Pi \ell'$:Loc. $(\ell =_{Loc} \ell') + (\ell =_{Loc} \ell' \to 0)$
 - get: $(\ell:\mathsf{Loc})
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- five equations (two of them branching on isDec_{Loc})

Example 3 (dep. typed update monads $TX \stackrel{\text{def}}{=} \Pi_{s:S}$. $Ps \times X$)

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Handlers of algebraic effects (for programming and extrinsic reasoning)

Usual term-level presentation:

 $\Gamma \models M \text{ handled with } \{\operatorname{op}_{\mathsf{x}_v}(\mathsf{x}_k) \mapsto \mathsf{N}_{\operatorname{op}}\}_{\operatorname{op} \in \mathcal{T}_{\operatorname{eff}}} \text{ to } y : A \text{ in}_{\underline{C}} \ \mathsf{N}_{\operatorname{ret}} : \underline{C}$ satisfying

```
 (\operatorname{return} V) \text{ handled with } \{...\}_{\operatorname{op} \in \mathcal{T}_{\operatorname{eff}}} \text{ to } y \colon A \text{ in } N_{\operatorname{ret}} = N_{\operatorname{ret}}[V/x]   (\operatorname{op}_V^{\underline{C}}(x.M)) \text{ handled with } \{...\}_{\operatorname{op} \in \mathcal{T}_{\operatorname{eff}}} \text{ to } y \colon A \text{ in } N_{\operatorname{ret}} = N_{\operatorname{op}}[V/x_v][.../x_k]
```

- write your programs using alg. ops. (e.g., get and put)
- use handlers to provide fit-for-purpose impl. (e.g., $S \to X \times S$)

Usual term-level presentation:

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(return V) handled with $\{...\}_{\mathsf{op}\in\mathcal{T}_{\mathsf{eff}}}$ to y:A in $N_{\mathsf{ret}}=N_{\mathsf{ret}}[V/x]$ ($\mathsf{op}_V^{\underline{C}}(x.M)$) handled with $\{...\}_{\mathsf{op}\in\mathcal{T}_{\mathsf{eff}}}$ to y:A in $N_{\mathsf{ret}}=N_{\mathsf{op}}[V/x_v][.../x_k]$

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 $\begin{tabular}{ll} \textbf{Idea:} & Generalisation of exception handlers} & & [Plotkin,Pretnar'09] \\ & & Handler \sim Algebra & and & Handling \sim Homomorphism \\ \end{tabular}$

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```
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```

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Idea: Using a derived handle-into-values handling construct

$$M$$
 handled with $\{\operatorname{op}_{x_v}(x_k)\mapsto V_{\operatorname{op}}\}_{\operatorname{op}\in\mathcal{T}_{\operatorname{eff}}}$ to $y\colon A$ in B V_{ret} we can define natural predicates (essentially, dependent types

$$\sqcap \vdash P : UFA \rightarrow U$$

by

- ullet equipping a universe ${\cal U}$ with an algebra for $\mathcal{T}_{\mathsf{eff}}$, and
- using the above handle-into-values construct to define P

Note 1: P(thunk M) computes a proof obligation for M

- a universe $\mathcal U$ closed under Nat, 1, 0, +, Σ , and Π
- a type-based treatment of handlers $\underline{C} ::= \ldots \mid \langle A; \overrightarrow{V_{\mathsf{op}}}; \overrightarrow{W_{\mathsf{eq}}} \rangle$
- function extensionality (actually, it's a bit more extensional)

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- function extensionality (actually, it's a bit more extensional)

Example 1 (Evaluation Logic style modalities):

- Given a predicate $P:A \to \mathcal{U}$ on return values, we define a predicate $\Diamond P:UFA \to \mathcal{U}$ on IO-computations as
- $\Diamond P \stackrel{\text{def}}{=} \lambda x : UFA . (\text{force } x) \text{ handled with } \{...\}_{\text{op} \in \mathcal{T}_{10}} \text{ to } y : A \text{ in}_{\mathcal{U}} P y$ using the handler given by

$$\begin{array}{ll} V_{\mathsf{read}} & \stackrel{\mathsf{def}}{=} & \lambda \, x \colon \! \big(\Sigma \, x_{\!\scriptscriptstyle V} \colon \! 1 \cdot \mathsf{Chr} \to \mathcal{U} \big) \cdot \widehat{\Sigma} \, y \colon \! \mathsf{El}(\widehat{\mathsf{Chr}}) \cdot \big(\mathsf{snd} \, x \big) \, y \\ \\ V_{\mathsf{write}} & \stackrel{\mathsf{def}}{=} & \lambda \, x \colon \! \big(\Sigma \, x_{\!\scriptscriptstyle V} \colon \! \mathsf{Chr} \cdot 1 \to \mathcal{U} \big) \cdot \big(\mathsf{snd} \, x \big) \, \star \end{array}$$

• $\Diamond P$ corresponds to Evaluation Logic's possibility modality

$$\lozenge P\left(exttt{thunk}\left(exttt{read}(x. exttt{write}_{e'}(exttt{return}\ V)
ight)
ight) = \widehat{\Sigma}x: \widehat{\mathsf{El}}(\widehat{\mathsf{Chr}}).P\ V$$

• To get the necessity modality $\Box P$, we use $\widehat{\Pi} x$: El $(\widehat{\mathsf{Chr}})$ in V_{read}

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 - $\Diamond P\left(\operatorname{thunk}\left(\operatorname{read}(x.\operatorname{write}_{e'}(\operatorname{return}V)\right)\right)\right) = \widehat{\Sigma}x:\operatorname{El}(\widehat{\operatorname{Chr}}).PV$
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Example 1 (Evaluation Logic style modalities):

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Example 1 (Evaluation Logic style modalities):

- Given a predicate P: A → U on return values,
 we define a predicate ◊P: UFA → U on IO-computations as
- $\Diamond P \stackrel{\text{def}}{=} \lambda x : UFA. \text{(force } x) \text{ handled with } \{...\}_{\text{op} \in \mathcal{T}_{\text{IO}}} \text{ to } y : A \text{ in}_{\mathcal{U}} P y$ using the handler given by

$$\begin{array}{ll} V_{\mathsf{read}} & \stackrel{\mathsf{def}}{=} & \lambda \, x \colon (\Sigma \, x_{\mathsf{v}} \colon 1 \cdot \mathsf{Chr} \to \mathcal{U}) \cdot \widehat{\Sigma} \, y \colon \mathsf{El}(\widehat{\mathsf{Chr}}) \cdot (\mathsf{snd} \, x) \, y \\ \\ V_{\mathsf{write}} & \stackrel{\mathsf{def}}{=} & \lambda \, x \colon (\Sigma \, x_{\mathsf{v}} \colon \mathsf{Chr} \cdot 1 \to \mathcal{U}) \cdot (\mathsf{snd} \, x) \, \star \end{array}$$

ullet $\Diamond P$ corresponds to Evaluation Logic's possibility modality

$$\Diamond P \left(\text{thunk} \left(\text{read}(x . \text{write}_{e'}(\text{return } V)) \right) \right) = \widehat{\Sigma} x : \widehat{\text{El}(\widehat{\text{Chr}})} . P V$$

• To get the necessity modality $\Box P$, we use $\widehat{\Pi} x : El(\widehat{Chr})$ in V_{read}

Example 2 (Dijkstra's weakest precondition semantics for state):

Given a postcondition on return values and final states

$$Q: A \to S \to \mathcal{U}$$
 ($S \stackrel{\text{def}}{=} \Pi \ell$: Loc .Val

we define a precondition for stateful comps. on initial states

$$\operatorname{wp}_{\mathcal{Q}}: \mathit{UFA} \to \mathit{S} \to \mathcal{U}$$

by

1) handling the given comp. into a state-passing function using

$$V_{
m get}, V_{
m put}$$
 on $S o (\mathcal{U} imes S)$ and $V_{
m ret}$ "=" Q

- 2) feeding in the initial state; and 3) projecting out \mathcal{U}
- Theorem: wp_Q satisfies expected properties of WPs, e.g., $\operatorname{wp}_Q\left(\operatorname{thunk}\left(\operatorname{return}V\right)\right) = \lambda x_S : S \cdot Q \cdot V \cdot x_S$

Example 2 (Dijkstra's weakest precondition semantics for state):

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- **Theorem:** wp_Q satisfies expected properties of WPs, e.g., $\operatorname{wp}_Q\left(\operatorname{thunk}\left(\operatorname{return}V\right)\right) = \lambda x_S : S \cdot Q \cdot V \cdot x_S$

Example 2 (Dijkstra's weakest precondition semantics for state):

• Given a postcondition on return values and final states

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- 2) feeding in the initial state; and 3) projecting out \mathcal{U}
- ullet Theorem: wp_Q satisfies expected properties of WPs, e.g.,

$$wp_Q (thunk (return V)) = \lambda x_S : S . Q V x_S$$

$$wp_Q (thunk (put_{(\ell,V)}(M))) = \lambda x_S : S . wp_Q (thunk M) (x_S[\ell \mapsto V])$$

Example 3 (Patterns of allowed (IO-)effects):

- Assuming an inductive type of IO-protocols, given by $\mathbf{e}:\mathsf{Protocol} \qquad \mathbf{r}:(\mathsf{Chr}\to\mathsf{Protocol})\to\mathsf{Protocol} \\ \qquad \mathbf{w}:(\mathsf{Chr}\to\mathcal{U})\to\mathsf{Protocol}\to\mathsf{Protocol}$
- Then, we define the predicate (rel. between comps. and protocols)

Allowed :
$$\mathit{UFA} o \mathsf{Protocol} o \mathcal{U}$$

by handling the given computation using

here
$$V_{\text{read}} \ \langle -, V_{\text{rk}} \rangle \ (\textbf{r} \ \textbf{Pr'}) \ \stackrel{\text{def}}{=} \ \widehat{\Pi} \, x \colon \text{El}(\widehat{\mathsf{Chr}}) \, . (V_{\text{rk}} \, x) \ (\textbf{Pr'} \, x)$$

$$V_{\text{write}} \ \langle V \, , V_{\text{wk}} \rangle \ (\textbf{w} \, P \, \textbf{Pr'}) \ \stackrel{\text{def}}{=} \ \widehat{\Sigma} \, x \colon \text{El}(P \, V) \, . \, V_{\text{wk}} \, \star \, \text{Pr'}$$

Example 3 (Patterns of allowed (IO-)effects):

Assuming an inductive type of IO-protocols, given by

$$\begin{tabular}{ll} \bf e: Protocol & \bf r: (Chr \rightarrow Protocol) \rightarrow Protocol \\ \hline \bf w: (Chr \rightarrow {\cal U}) \rightarrow Protocol \rightarrow Protocol \\ \hline \end{tabular}$$
 and potentially also by \land , \lor , \ldots

Then, we define the predicate (rel. between comps. and protocols

Allowed :
$$\mathit{UFA}
ightarrow \mathsf{Protocol}
ightarrow \mathcal{U}$$

by handling the given computation using

$$V_{
m read},\,V_{
m write}$$
 on Protocol $ightarrow \, {\cal U}$ were

$$\begin{array}{cccc} V_{\mathsf{read}} & \langle -, V_{\mathsf{rk}} \rangle & (\mathbf{r} \; \mathsf{Pr'}) & \stackrel{\mathsf{def}}{=} & \widehat{\Pi} \, x \colon \mathsf{El}(\widehat{\mathsf{Chr}}) \, . \, (V_{\mathsf{rk}} \, x) \, (\mathsf{Pr'} \, x) \\ V_{\mathsf{write}} & \langle V \, , V_{\mathsf{wk}} \rangle \, (\mathsf{w} \; P \; \mathsf{Pr'}) & \stackrel{\mathsf{def}}{=} & \widehat{\Sigma} \, x \colon \mathsf{El}(P \, V) \, . \, V_{\mathsf{wk}} \, \star \, \mathsf{Pr'} \\ & & \stackrel{\mathsf{def}}{=} & \widehat{0} \end{array}$$

Example 3 (Patterns of allowed (IO-)effects):

• Assuming an inductive type of IO-protocols, given by

 $w : (\mathsf{Chr} \to \mathcal{U}) \to \mathsf{Protocol} \to \mathsf{Protocol}$

and potentially also by \land , \lor , . . .

Then, we define the predicate (rel. between comps. and protocols)

Allowed :
$$UFA \rightarrow Protocol \rightarrow \mathcal{U}$$

by handling the given computation using

$$V_{\mathsf{read}}, V_{\mathsf{write}}$$
 on $\mathsf{Protocol} o \mathcal{U}$

where

$$\begin{array}{cccc} V_{\mathsf{read}} & \langle - , V_{\mathsf{rk}} \rangle & (\mathtt{r} \; \mathsf{Pr'}) & \stackrel{\mathsf{def}}{=} & \widehat{\Pi} \, x \colon \mathsf{El}(\widehat{\mathsf{Chr}}) \, . \, (V_{\mathsf{rk}} \, x) \, (\mathsf{Pr'} \, x) \\ V_{\mathsf{write}} & \langle V \, , V_{\mathsf{wk}} \rangle \, (\mathtt{w} \; P \; \mathsf{Pr'}) & \stackrel{\mathsf{def}}{=} & \widehat{\Sigma} \, x \colon \mathsf{El}(P \, V) \, . \, V_{\mathsf{wk}} \, \star \, \mathsf{Pr'} \\ - & \stackrel{\mathsf{def}}{=} & \widehat{0} \end{array}$$

Conclusion

At a high-level, the presented work was about combining dependent types and computational effects

In particular, you saw

- a clean core language of dependent types and comp. effects
- a natural category-theoretic semantics
- alg. effects and handlers, in particular, for reasoning using
 - Evaluation Logic style modalities
 - Dijkstra's weakest precondition semantics for state
 - patterns of allowed (IO-)effects

Ongoing and future work:

- uniform account of the various handler-defined predicates
- more expressive comp. types (par. adjunctions, Dijkstra monads)
- type-dependency on computations (e.g., in seq. composition)

Thank you!

D. Ahman.

Fibred Computational Effects. (PhD Thesis, 2017)

D. Ahman, N. Ghani, G. Plotkin.

 $\textbf{Dependent Types and Fibred Computational Effects.} \ (FoSSaCS'16)$

D. Ahman.

Handling Fibred Computational Effects. (POPL'18)