A fibrational view on computational effects

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We investigate the combination of

- dependent types $(\Pi, \Sigma, V =_{\mathcal{A}} W, ...)$
- computational effects (state, I/O, probability, recursion, ...)

Two guiding problems

- effectful programs in types (e.g., read and write in types)
- types of effectful programs (e.g., of sequential composition)

- tell a mathematically natural story
- use established math. techniques
- cover a wide range of comp. effects
- discover smth. interesting

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- cover a wide range of comp. effects (alg. effects, continuations)
- discover smth. interesting (using handlers to reason about effects)

(type-dependency in the presence of effects)

Let's assume that we have some dependent type A, e.g.:

```
\ell: (List Chr) \vdash A(\ell) \stackrel{\text{def}}{=} \Sigma \ell': (List Chr). (length \ell = N_{\text{at}} \text{ length } \ell' \times \ldots)

which could be used to type the dependent function

sort: \Pi \ell: (List Chr). A(\ell)
```

Q: Should we allow $A[receive(y, M)/\ell]$?

A1: In this talk, we say no

- types should only depend on static information about effects
- we allow dependency on effectful computations via thunks

A2: But we are also looking into ves

- type-dependency needs to be "homomorphic", but not only so
- intuitively, need to lift $A(\ell)$ to $A^{\dagger}(c)$, where c: T(List Chr)

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Aim: Types should only depend on static info about effects

Solution: CBPV/EEC style distinction between vals. and comps

```
• value types \Gamma \vdash A (MLTT + thunks + ...)
```

- computation types $\Gamma \vdash \underline{C}$ (dep. CBPV/EEC)
- where Γ contains only value variables $x_1: A_1, \ldots, x_n: A_n$

Note: Could have also considered λ_{ML} and FGCBV

- building on CBPV/EEC gives a more general story
- especially for the treatment of sequential composition
- and also for integrating dependent- and effect-typing

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- and also for integrating dependent- and effect-typing

(e.g., sequential composition)

The problem: The standard typing rule for seq. composition

$$\frac{\Gamma \vdash_{c} M : FA \qquad \Gamma, x : A \vdash_{c} N : \underline{C}}{\Gamma \vdash_{c} M \text{ to } x : A \text{ in } N : \underline{C}}$$

is not correct any more because x can appear free in the type

(

in the conclusion

Aim: To fix the typing rule of sequential composition

Option 1: We could restrict the free variables in
$$\underline{C}$$
: [Levy'04]
$$\underline{\Gamma \vdash M : FA \qquad \Gamma \vdash \underline{C} \qquad \Gamma, x : A \vdash N : \underline{C}}$$

But sometimes it is useful if C can depend on x!

if we consider

fopen (return true, return false) to
$$x$$
:Bool in N

• then it would be natural to let \underline{C} depend on x, e.g.,

```
x: \mathsf{Bool} \vdash \underline{C}(x) \stackrel{\mathsf{def}}{=} \mathsf{if} \ x \ \mathsf{then} \ "\mathsf{allow} \ \mathsf{fread}, \ \mathsf{fwrite}, \ \mathsf{and} \ \mathsf{fclose}" else "allow fopen"
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(needs more expressive comp. types than we consider here)

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Option 2: One could lift sequential composition to type level

$$\Gamma \vdash M \text{ to } x : A \text{ in } N : M \text{ to } x : A \text{ in } \underline{C}$$

But then comp. types would be singleton-like!?!

However, something like this is probably needed for the yes case

Option 3: In the monadic metalanguage λ_{ML} , one could try

$$\frac{\Gamma \vdash M : T A \qquad \Gamma, x : A \vdash N : T B(x)}{\Gamma \vdash M \text{ to } x : A \text{ in } N : T (\Sigma x : A . B)}$$

But what makes this a principled solution? Why is it correct?

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Option 4: We draw inspiration from algebraic effects
and combine it with restricting <u>C</u> in seq. comp. (Option 1)

E.g., consider the non-det. program (for $x: Nat \vdash N : \underline{C}(x)$) $M \stackrel{\text{def}}{=} \text{choose (return 4, return 2) to } x: Nat \text{ in } N$

After tossing the coin, this program evaluates as either N[4/x]: C[4/x] or N[2/x]: C[2/x]

Idea: M denotes an element of the coproduct of algebras $\underline{C}[4/x] + \underline{C}[2/x] \stackrel{\text{def}}{=} F\left(U\left(\underline{C}[4/x]\right) + U\left(\underline{C}[2/x]\right)\right)_{/=}$

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Putting these ideas together

(eMLTT: a core dep.-typed language with comp. effects)

eMLTT – types

Value types: MLTT + thunks + ...

$$A, B ::=$$
Nat $\mid 1 \mid 0 \mid \Pi x : A.B \mid \Sigma x : A.B \mid V =_A W \mid U \subseteq | \dots |$

• $U \subseteq C$ is the type of thunked (i.e., suspended) computations

Computation types: dep.-typed version of EEC's comp. types

$$\underline{C}, \underline{D} ::= FA \mid \Pi x : A . \underline{C} \mid \Sigma x : A . \underline{C}$$

- F A is the type of computations returning values of type A
- Πx: A. <u>C</u> is the type of dependent effectful functions
 - generalises CBPV/EEC's comp. types $A \to \underline{C}$ and $\underline{C} \times \underline{D}$
- Σx: A.C is the type of dep. pairs of values and effectful comps.
 - captures the intuition about seq. comp. and coprods. of algebras
 - generalises EEC's comp. types $!A \otimes C$ and $C \oplus D$

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```
Value terms: MLTT + thunks + ... V, W ::= x \mid zero \mid succ V \mid ... \mid thunk M \mid ...
```

equational theory based on intensional MLTT

Comp. terms: dep.-typed version of CBPV/EEC's comp. terms

```
\begin{array}{lll} M,N ::= & \text{force } V \\ & \mid & \text{return } V \\ & \mid & M \text{ to } x \colon A \text{ in } N \\ & \mid & \lambda x \colon A \ldotp M \\ & \mid & MV \\ & \mid & \langle V,M \rangle & \text{(comp. $\Sigma$ intro.)} \\ & \mid & M \text{ to } \langle x \colon A,z \colon \underline{C} \rangle \text{ in } K & \text{(comp. $\Sigma$ elim.)} \end{array}
```

But: Value and comp. terms alone do not suffice, as in EEC!

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But: Value and comp. terms alone do not suffice, as in EEC!

Note: We need to define K in such a way that the intended left-to-right evaluation order is preserved, e.g., consider

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Homomorphism terms: dep.-typed version of EEC's linear terms

$$K, L := z$$
 (linear comp. vars.)
 $\mid K \text{ to } x : A \text{ in } M$
 $\mid \lambda x : A, K$
 $\mid KV$
 $\mid \langle V, K \rangle$ (comp. $\Sigma \text{ intro.}$)
 $\mid K \text{ to } \langle x : A, z : C \rangle \text{ in } L$ (comp. $\Sigma \text{ elim.}$)

Typing judgments:

- Γ ∀ V : A
- Γ la M : C
- $\Gamma \mid z : \underline{C} \mid_{\overline{h}} K : \underline{D}$ (linear in z; comp. bound to z happens first

Note: We need to define K in such a way that the intended left-to-right evaluation order is preserved, e.g., consider

$$\Gamma \vdash \langle V, M \rangle$$
 to $\langle x : A, z : \underline{C} \rangle$ in $K = K[V/x, M/z] : \underline{D}$

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```
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eMLTT – typing sequential composition

We can then account for type-dependency in seq. comp. as

$$\frac{\Gamma, x : A \vdash N : \underline{C}(x)}{\Gamma \vdash R \quad \Gamma \vdash \Sigma y : A \cdot \underline{C}(y) \quad \overline{\Gamma, x : A \vdash \langle x, N \rangle : \Sigma y : A \cdot \underline{C}(y)}}{\Gamma \vdash R \quad \text{to } x : A \text{ in } \langle x, N \rangle : \Sigma y : A \cdot \underline{C}(y)}$$

The seq. comp. rule for $\lambda_{\rm ML}$ is justified by the type isomorphism

$$\frac{\Gamma \vdash A \qquad \Gamma, x : A \vdash B(x)}{\Gamma \vdash U(\Sigma x : A . F(B)) \cong UF(\Sigma x : A . B) = T(\Sigma x : A . B)}$$

Categorical semantics of eMLTT

(fibrations + adjunctions)

Categorical semantics – MLTT part

We define fibred adjunction models

• Theorem: a sound and complete class of models for eMLTT

given by: i) a split closed comprehension cat. p with Nat, ...



- we define a partial interpretation fun. [-], that (if defined) maps:
 - a context Γ to and object $\llbracket \Gamma \rrbracket$ in \mathcal{B} , with $\llbracket \Gamma, x : A \rrbracket \stackrel{\text{def}}{=} \{ \llbracket \Gamma; A \rrbracket \}$
 - ullet a context Γ and a value type A to an object $[\![\Gamma;A]\!]$ in $\mathcal{V}_{[\![\Gamma]\!]}$
 - a context Γ and a value term V to $[\![\Gamma;V]\!]:1_{[\![\Gamma]\!]}\longrightarrow A$ in $\mathcal{V}_{[\![\Gamma]\!]}$

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 rbracket$ in $\mathcal{V}_{\llbracket \Gamma
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 - a context Γ and a value term V to $\llbracket \Gamma; V \rrbracket : 1_{\llbracket \Gamma \rrbracket} \longrightarrow A$ in $\mathcal{V}_{\llbracket \Gamma \rrbracket}$

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• **Theorem:** a sound and complete class of models for eMLTT given by: i) a split closed comprehension cat. p with Nat, ...

$$\begin{array}{c|c}
V \\
\uparrow \\
\uparrow \\
B
\end{array}$$

- the display maps $\pi_{\llbracket\Gamma;A\rrbracket}:\llbracket\Gamma,x:A\rrbracket\longrightarrow\llbracket\Gamma\rrbracket$ in \mathcal{B} induce the weakening functors $\pi^*_{\llbracket\Gamma;A\rrbracket}:\mathcal{V}_{\llbracket\Gamma\rrbracket}\longrightarrow\mathcal{V}_{\llbracket\Gamma,x:A\rrbracket}$, and
- the value Σ and Π -types are interpreted as adjoints

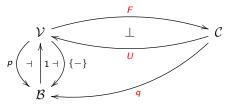
$$\begin{array}{l} \sum_{\llbracket \Gamma;A\rrbracket} \dashv \pi_{\llbracket \Gamma;A\rrbracket}^* : \mathcal{V}_{\llbracket \Gamma\rrbracket} \longrightarrow \mathcal{V}_{\llbracket \Gamma,x:A\rrbracket} \qquad \text{(such that Σ is strong)} \\ \pi_{\llbracket \Gamma;A\rrbracket}^* \dashv \Pi_{\llbracket \Gamma;A\rrbracket} : \mathcal{V}_{\llbracket \Gamma,x:A\rrbracket} \longrightarrow \mathcal{V}_{\llbracket \Gamma\rrbracket} \end{array}$$

Categorical semantics – effects part

We define fibred adjunction models

Theorem: a sound and complete class of models for eMLTT

given by: ii) a split fibration q (with ...) and a s. fib. adj. $F \dashv U$



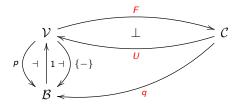
- - a ctx. Γ and a comp. type \underline{C} to an object $\llbracket \Gamma ; \underline{C} \rrbracket$ in $\mathcal{C}_{\llbracket \Gamma \rrbracket}$
 - a ctx. Γ and a comp. term M to $\llbracket \Gamma; M \rrbracket : 1_{\llbracket \Gamma \rrbracket} \longrightarrow U(\underline{C})$ in $\mathcal{V}_{\llbracket \Gamma \rrbracket}$
 - a ctx. Γ , a c. var. z, a c. type \underline{C} , and a hom. term K to $[\![\Gamma;z\colon\underline{C};K]\!]:[\![\Gamma;\underline{C}]\!]\longrightarrow\underline{D}$ in $\mathcal{C}_{[\![\Gamma]\!]}$

Categorical semantics – effects part

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• Theorem: a sound and complete class of models for eMLTT

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- we again have weakening functors $\pi^*_{\llbracket\Gamma;A\rrbracket}:\mathcal{C}_{\llbracket\Gamma\rrbracket}\longrightarrow\mathcal{C}_{\llbracket\Gamma,x:A\rrbracket}$, and
- the comp. Σ and Π -types are interpreted again as adjoints

$$\begin{split} & \Sigma_{\llbracket \Gamma; A \rrbracket} \dashv \pi_{\llbracket \Gamma; A \rrbracket}^* : \mathcal{C}_{\llbracket \Gamma \rrbracket} \longrightarrow \mathcal{C}_{\llbracket \Gamma, \mathbf{x} : A \rrbracket} \\ & \pi_{\llbracket \Gamma; A \rrbracket}^* \dashv \Pi_{\llbracket \Gamma; A \rrbracket} : \mathcal{C}_{\llbracket \Gamma, \mathbf{x} : A \rrbracket} \longrightarrow \mathcal{C}_{\llbracket \Gamma \rrbracket} \end{split}$$

Digression: dep. elimination of 0 and +

The coproduct type A + B:

[Jacobs'99]

- require $p: \mathcal{V} \longrightarrow \mathcal{B}$ to have split fibred coproducts $A +_X B$, and
- ullet $\langle \{ \mathsf{inl}_A \}^*, \{ \mathsf{inr}_B \}^* \rangle : \mathcal{V}_{\{A+_XB\}} \longrightarrow \mathcal{V}_{\{A\}} \times \mathcal{V}_{\{B\}}$ to be fully-faith.
- allows one to interpret dependent case analysis, i.e.,

$$\mathcal{V}_{\{A\}}\Big(1_{\{A\}}, \{\operatorname{inl}_A\}^*(C)\Big) \times \mathcal{V}_{\{B\}}\Big(1_{\{B\}}, \{\operatorname{inr}_B\}^*(C)\Big) \\ \cong \\ \mathcal{V}_{\{A+_{X}B\}}\Big(1_{\{A+_{X}B\}}, C\Big)$$

provides semantics for

$$\frac{\Gamma, y_1 : A \trianglerighteq W_1 : C[\operatorname{inl}_A y_1/x] \quad \Gamma, y_2 : B \trianglerighteq W_2 : C[\operatorname{inr}_B y_2/x]}{\Gamma, x : A + B \trianglerighteq \operatorname{case} x \text{ of } (\operatorname{inl}(y_1) \mapsto W_1, \operatorname{inr}(y_2) \mapsto W_2) : C[\operatorname{inr}_B y_2/x]}$$

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Digression: dep. elimination of colimits

A generalisation:

[Ahman'17]

- Idea: fully-faith. for cocones $A \longrightarrow A \circledast_X B \longleftarrow B$ is enough, and we can generalise this to all split fibred colimits
- Theorem:
 - if for every object $X \in \mathcal{B}$ and diagram $J : \mathcal{D} \longrightarrow \mathcal{V}_X$ there exists a cocone $\underline{\operatorname{in}}^J : J \longrightarrow \Delta(\underline{\operatorname{colim}}(J))$ in \mathcal{V}_X ,
 - such that f*(in^J_D) = in^{f*oJ}_D, for any f : X → Y, and such that the unique mediating functor

$$\begin{split} & \langle \{\underline{\operatorname{in}}_D^J\}_{D\in\mathcal{D}}^* \rangle : \mathcal{V}_{\{\underline{\operatorname{colim}}(J)\}} \longrightarrow \operatorname{lim}(\widehat{J}) \\ & \text{s fully-faithful (for } \widehat{J} : \mathcal{D}^{op} \longrightarrow \operatorname{Cat, where } \widehat{J}(D) = \mathcal{V}_{\{J(D)\}}) \end{split}$$

then p has split fibred colimits of shape D, and
 p supports dependent elimination for them (analogously to +x)

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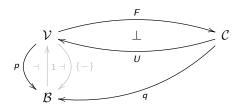
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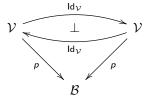
- if for every object X ∈ B and diagram J : D → V_X
 there exists a cocone in J : J → Δ(colim(J)) in V_X,
- such that $f^*(\underline{\operatorname{in}}_D^J) = \underline{\operatorname{in}}_D^{f^*\circ J}$, for any $f: X \longrightarrow Y$, and such that the unique mediating functor

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then p has split fibred colimits of shape D, and
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Example 1 (identity adjunctions):



Note: sound model as long as we haven't included any effects

Example 2 (simple models from Egger et al.'s EEC):

- given an adjunction $F_{\mathsf{EEC}} \dashv U_{\mathsf{EEC}} : \mathcal{E} \longrightarrow \mathcal{D}$, such that
 - \mathcal{D} is Cartesian closed (with Nat, ...), and
 - \mathcal{E} and $F_{\text{EEC}} \dashv U_{\text{EEC}}$ are \mathcal{D} -enriched, and
 - $\mathcal E$ has all $\mathcal D$ -tensors $(A \otimes \underline{\mathcal C})$ and $\mathcal D$ -cotensors $(A \Rightarrow \underline{\mathcal C})$
- ullet we use simple fibration $\mathbf{s}_{\mathcal{D}}$ and simpl. \mathcal{D} -enrich. fibration $\mathbf{s}_{\mathcal{D},\mathcal{E}}$

$$s(\mathcal{D}) \xrightarrow{f} s(\mathcal{D}, \mathcal{E})$$

$$v \qquad F(X, A) \stackrel{\text{def}}{=} (X, F_{\text{EEC}}(A))$$

$$U(X, \underline{C}) \stackrel{\text{def}}{=} (X, U_{\text{EEC}}(\underline{C}))$$

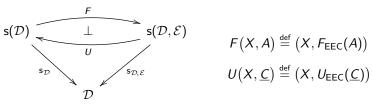
$$(D): (f,g): (X,A) \longrightarrow (Y,B) \quad \text{where} \quad f: X \longrightarrow Y \quad g: X \times A \longrightarrow E$$

$$(D,S): (f,h): (X,C) \longrightarrow (Y,D) \quad \text{where} \quad f: X \longrightarrow Y \quad h: X \otimes C \longrightarrow Y$$

Note: this model doesn't support any real type-dependency

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 - \mathcal{E} has all \mathcal{D} -tensors $(A \otimes \underline{C})$ and \mathcal{D} -cotensors $(A \Rightarrow \underline{C})$
- we use simple fibration $s_{\mathcal{D}}$ and simpl. \mathcal{D} -enrich. fibration $s_{\mathcal{D},\mathcal{E}}$



$$s(\mathcal{D})$$
: $(f,g):(X,A)\longrightarrow (Y,B)$ where $f:X\longrightarrow Y$ $g:X\times A\longrightarrow B$ $s(\mathcal{D},\mathcal{E})$: $(f,h):(X,C)\longrightarrow (Y,D)$ where $f:X\longrightarrow Y$ $h:X\otimes C\longrightarrow D$

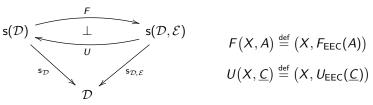
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where $f: X \longrightarrow Y$ $g: X \times A \longrightarrow B$

 $s(\mathcal{D},\mathcal{E})$: $(f,h):(X,\underline{C})\longrightarrow (Y,\underline{D})$ where $f:X\longrightarrow Y$ $h:X\otimes\underline{C}\longrightarrow\underline{D}$

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Example 3 (families fibrations):

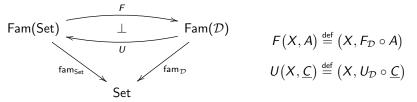
- given an adjunction $F_{\mathcal{D}} \dashv U_{\mathcal{D}} : \mathcal{D} \longrightarrow \mathsf{Set}$, such that
 - ullet ${\cal D}$ has set-indexed products and set-indexed coproducts
- such adjunctions arise from
 - EM-cats. $(\mathcal{D} \stackrel{\text{def}}{=} \operatorname{Set}^{\mathsf{T}})$ and Law. ths. $(\mathcal{D} \stackrel{\text{def}}{=} \operatorname{\mathsf{Mod}}(\mathcal{L}, \operatorname{\mathsf{Set}}))$
 - resolutions of $S \Rightarrow (-) \times S$ and $((-) \Rightarrow R) \Rightarrow R$
- ullet we use families fibrations fam $_{\mathsf{Set}}$ and fam $_{\mathcal{D}}$



Fam(Set):
$$(X,A)$$
 where $X \in \mathsf{Set}$ $A:X \longrightarrow \mathsf{Set}$ $(f,\{g_x\}_{x \in X}): (X,A) \longrightarrow (Y,B)$ where $g_x:A(x) \longrightarrow (B \circ f)(x)$

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$$\mathsf{Fam}(\mathsf{Set}) \colon \ (X,A) \qquad \text{where} \quad X \in \mathsf{Set} \quad A \colon X \longrightarrow \mathsf{Set}$$

$$(f,\{g_x\}_{x \in X}) \colon (X,A) \longrightarrow (Y,B) \qquad \text{where} \quad g_x \colon A(x) \longrightarrow (B \circ f)(x)$$

Example 4 (continuous families for $\mu x : U\underline{C} . M$):

- given a CPO-enriched monad T on CPO, such that
 - **T** supports least zero-ary alg. op. $(\bot_A : 1 \longrightarrow TA)$, and
 - ullet CPO $^{\mathsf{T}}$ has reflexive coequalizers
- such T arise from discrete CPO-enriched countable Law. ths.
- we use continuous families fibrations cfam_{CPO} and cfam_{CPO}T

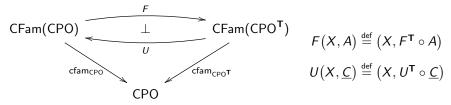
CFam(CPO)
$$\begin{array}{ccc}
& F \\
& \bot & CFam(CPO^T) \\
& U & F(X,A) \stackrel{\text{def}}{=} (X,F^T \circ A) \\
& & U(X,\underline{C}) \stackrel{\text{def}}{=} (X,U^T \circ \underline{C})
\end{array}$$
CPO

CFam(CPO): (X, A) where $X \in CPO$ $A: X \longrightarrow CPO^{EP}$ an ω -cont. fun

Thm.: we don't use $cod : CPO \rightarrow CPO$ because CPO isn't LCCC

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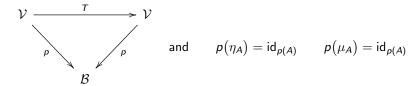
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F \\
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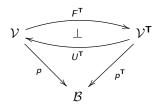
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Example 5 (EM-resolutions of split fibred monads):

• given a split fibred monad $\mathbf{T} = (T, \eta, \mu)$ on p, i.e.,



• we consider models based on the EM-resolution of T



and show that three familiar results hold for this situation

Example 5 (EM-resolutions of split fibred monads):

• **Theorem 1:** if p supports Π -types, then p^{T} also supports Π -types

$$\Pi_A^{\mathsf{T}}(B,\beta) \stackrel{\mathsf{def}}{=} (\Pi_A(B),\beta_{\Pi_A^{\mathsf{T}}})$$

- **Prop.:** every **T** on a split closed comp. cat. has a dep. strength $\sigma_A: \Sigma_A \circ \mathcal{T} \longrightarrow \mathcal{T} \circ \Sigma_A \qquad (A \in \mathcal{V})$
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Algebraic effects

Fibred effect theories \mathcal{T}_{eff} :

signatures of dep. typed operation symbols

$$\frac{\cdot \vdash I \qquad x_i \colon I \vdash O \qquad I \text{ and } O \text{ are pure value types}}{\text{op} \colon (x_i \colon I) \longrightarrow O}$$

equipped with equations on derivable effect terms

In eMLTT:

$$M ::= \dots \mid \operatorname{op}_{V}^{C}(x.M)$$

General algebraicity equations (in addition to eff. th. eqs.):

$$\frac{\Gamma \trianglerighteq V: I - \Gamma, x: O[V/x_i] \trianglerighteq M: \underline{C} - \Gamma | z: \underline{C} \trianglerighteq_{\mathbb{F}} K: \underline{D}}{\Gamma \trianglerighteq K[\operatorname{op}_V^{\underline{C}}(x.M)/z] = \operatorname{op}_V^{\underline{D}}(x.K[M/z]): \underline{D}} (\operatorname{op}: (x_i: I) \longrightarrow O)$$

Sound semantics: based on

•
$$p : \mathsf{Fam}(\mathsf{Set}) \longrightarrow \mathsf{Set}$$
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Algebraic effects – examples

Example 1 (interactive I/O):

- ullet read : $1 \longrightarrow \mathsf{Chr}$ $(\mathsf{Chr} \stackrel{\mathsf{def}}{=} 1 + \ldots + 1)$ write : $\mathsf{Chr} \longrightarrow 1$
- no equations

Example 2 (global state with location-dependent store type):

```
• ♦ ⊢ Loc \ell : Loc ⊢ Val \bullet is Dec_{Loc} : \Pi\ell : Loc . \Pi\ell' : Loc . (\ell =_{Loc} \ell') + (\ell =_{Loc} \ell' \to 0)
```

- $\begin{array}{c} \bullet \ \ \mathsf{get} : (\ell \colon \mathsf{Loc}) \longrightarrow \mathsf{Val} \\ \\ \mathsf{put} : (\Sigma \ell \colon \mathsf{Loc}.\mathsf{Val}) \longrightarrow \end{array}$
- five equations (two of them branching on isDec_{Loc}

Example 3 (dep. typed update monads $TX \stackrel{\text{def}}{=} \Pi_{s:S}$. $Ps \times X$)

Algebraic effects – examples

Example 1 (interactive I/O):

- ullet read : $1 \longrightarrow \mathsf{Chr}$ $(\mathsf{Chr} \stackrel{\mathsf{def}}{=} 1 + \ldots + 1)$ write : $\mathsf{Chr} \longrightarrow 1$
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Handlers of algebraic effects (for programming and extrinsic reasoning)

Handlers of alg. effects – for programming

Idea: Generalisation of exception handlers [Plotkin,Pretnar'09] Handler = Algebra and Handling = Homomorphism

Usual term-level presentation:

 $\sqcap_{\mathsf{E}} M \text{ handled with } \{ \mathrm{op}_{\mathsf{x}_{\mathsf{v}}}(\mathsf{x}_{k}) \mapsto \mathsf{N}_{\mathsf{op}} \}_{\mathsf{op} \in \mathcal{T}_{\mathsf{eff}}} \text{ to } y \colon A \text{ in}_{\underline{C}} \ \mathsf{N}_{\mathsf{ret}} \colon \underline{C}$ satisfying

(return V) handled with $\{...\}_{\mathsf{op}\in\mathcal{T}_{\mathsf{eff}}}$ to $y\!:\!A$ in $N_{\mathsf{ret}} = N_{\mathsf{ret}}[V/x]$ (op $\frac{C}{V}(x.M)$) handled with $\{...\}_{\mathsf{op}\in\mathcal{T}_{\mathsf{eff}}}$ to $y\!:\!A$ in $N_{\mathsf{ret}} = N_{\mathsf{op}}[V/x_V][.../x_k]$

Typical use case for programming:

- write your programs using alg. ops. (e.g., get and put)
- use handlers to provide fit-for-purpose impl. (e.g., $S \to X \times S$)

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```

```
\begin{split} & (\text{return } V) \text{ handled with } \{...\}_{\mathsf{op} \in \mathcal{T}_{\mathsf{eff}}} \text{ to } y \colon\! A \text{ in } N_{\mathsf{ret}} = N_{\mathsf{ret}}[V/x] \\ & (\mathsf{op}_V^{\underline{C}}(x.M)) \text{ handled with } \{...\}_{\mathsf{op} \in \mathcal{T}_{\mathsf{eff}}} \text{ to } y \colon\! A \text{ in } N_{\mathsf{ret}} = N_{\mathsf{op}}[V/x_v][.../x_k] \end{split}
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 $\Gamma \vdash M$ handled with $\{ op_{x_v}(x_k) \mapsto N_{op} \}_{op \in \mathcal{T}_{eff}}$ to $y : A \text{ in}_C N_{ret} : \underline{C}$

 $(\operatorname{op}_{V}^{\underline{C}}(x.M)) \text{ handled with } \{...\}_{\operatorname{op}\in\mathcal{T}_{\operatorname{eff}}} \text{ to } y:A \text{ in } N_{\operatorname{ret}} = N_{\operatorname{op}}[V/x_{v}][.../x_{k}]$

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$$M$$
 handled with $\{\operatorname{op}_{x_v}(x_k)\mapsto V_{\operatorname{op}}\}_{\operatorname{op}\in\mathcal{T}_{\operatorname{eff}}}$ to $y\colon A$ in B V_{ret} can define natural predicates (essentially, dependent types

$$\Gamma \vdash P : UFA \rightarrow U$$

by

- ullet equipping a universe ${\cal U}$ with an algebra for $\mathcal{T}_{\sf eff}$, and
- using the above handle-into-values construct to define P

Note 1: P(thunk M) computes a proof obligation for M

- a universe \mathcal{U} closed under Nat, 1, 0, +, Σ , and Π
- a type-based treatment of handlers $\underline{C} ::= \ldots \mid \langle A; \overrightarrow{V_{\sf op}}; \overrightarrow{W_{\sf eq}} \rangle$
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Example 1 (Evaluation Logic style modalities):

- Given a predicate $P:A\to \mathcal{U}$ on return values, we define a predicate $\Diamond P:UFA\to \mathcal{U}$ on I/O-computations as
- $\Diamond P \stackrel{\text{def}}{=} \lambda x : UFA . (\text{force } x) \text{ handled with } \{...\}_{\text{op} \in \mathcal{T}_{\text{lo}}} \text{ to } y : A \text{ in}_{\mathcal{U}} P y$ using the handler given by

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ullet $\Diamond P$ corresponds to Evaluation Logic's possibility modality

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Example 2 (Dijkstra's weakest precondition semantics):

Given a postcondition on return values and final states

$$Q: A \to S \to \mathcal{U}$$
 $(S \stackrel{\text{def}}{=} \Pi x: \text{Loc. Val})$

we define a precondition for stateful comps. on initial states

$$\mathsf{wp}_{\mathcal{Q}}: \mathit{UFA} o \mathit{S} o \mathcal{U}$$

by

i) handling the given comp. into a state-passing function using

$$V_{
m get},\,V_{
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 on $S o (\mathcal{U} imes S)$ and $V_{
m ret}$ " $=$ " Q

- ii) feeding in the initial state; and iii) projecting out ${\cal U}$
- Theorem: wp_Q satisfies expected properties of WPs, e.g., $\operatorname{wp}_Q\left(\operatorname{thunk}\left(\operatorname{return}V\right)\right) = \lambda x_S : S \cdot Q \cdot V \cdot x_S$

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$$wp_Q (thunk (return V)) = \lambda x_S : S . Q V x_S$$

$$wp_Q (thunk (put_{\langle \ell, V \rangle}(M))) = \lambda x_S : S . wp_Q (thunk M) (x_S[\ell \mapsto V])$$

Example 3 (Patterns of allowed I/O-effects):

Assuming an inductive type Protocol, given by

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$$\mathbf{r}: (\mathsf{Chr} \to \mathsf{Protocol}) \to \mathsf{Protocol}$$

and potentially also by ∧, ∨, ...

Then, given a protocol Pr : Protocol, we define

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Conclusion

In work we told a mathematically natural story of combining

dependent types and computational effects

In particular, we saw

- a clean core language of dependent types and comp. effects
- a natural category-theoretic semantics
- alg. effects and handlers, in particular, for reasoning using
 - Evaluation Logic style modalities
 - Dijkstra's weakest precondition semantics of state
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Things to look at:

- type-dependency on computations (e.g., in seq. composition)
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Thank you!

D. Ahman.

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