# A fibrational view on computational effects

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### Language design principles for combining

- dependent types  $(\Pi, \Sigma, V =_{\mathcal{A}} W, ...)$
- computational effects (state, I/O, probability, recursion, ...)

#### Two guiding problems

- effectful programs in types (e.g., read and write in types)
- types of effectful programs (e.g., of sequential composition)

- tell a mathematically natural story
- use established math. techniques
- cover a wide range of comp. effects
- discover smth. interesting

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- discover smth. interesting (using handlers to reason about effects)

(type-dependency in the presence of effects)

Let's assume that we have some dependent type A, e.g.:

$$\ell$$
:(List Chr)  $\vdash$   $A(\ell) \stackrel{\text{def}}{=} \Sigma \ell'$ :(List Chr).(length  $\ell =_{\mathsf{Nat}} \mathsf{length} \ \ell' \times \ldots$ ) which could be used to type the dependent function  $\mathsf{sort} : \Pi \, \ell$ :(List Chr). $A(\ell)$ 

**Q:** Should we allow A[M/x] if M is an effectful program?

A1: In this work we say no

- types should only depend on static information
- e.g., how would one compute A[receive(y.M)/x] statically?
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Aim: Types should only depend on static info about effects

**Solution:** CBPV/EEC style distinction between vals. and comps.

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• value types \Gamma \vdash A (MLTT + thunks + ....
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- computation types  $\Gamma \vdash \underline{C}$  (dep. version of CBPV/EEC
- where  $\Gamma$  contains **only** value variables  $x_1: A_1, \ldots, x_n: A_n$

**Note:** Some of the other options are  $\lambda_{ML}$  and FGCBV

- building on CBPV/EEC gives a more general story
- especially for the treatment of sequential composition
- and also for integrating dependent- and effect-typing

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(e.g., typing sequential composition)

The problem: The standard typing rule for seq. composition

$$\frac{\Gamma \vdash_{c} M : FA \qquad \Gamma, x : A \vdash_{c} N : \underline{C}}{\Gamma \vdash_{c} M \text{ to } x : A \text{ in } N : \underline{C}}$$

is not correct any more because x can appear free in the type

(

in the conclusion

Aim: To fix the typing rule of sequential composition

**Option 1:** We could restrict the free variables in  $\underline{C}$ , i.e., [Levy'04]

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But sometimes it is useful if  $\underline{C}$  can depend on x!

- if M involves opening a file and the return values of M model whether fopen succeeded,
- ullet then it would be natural to let  $\underline{C}$  to depend such values, e.g.,
- $x: Bool \vdash \underline{C}(x) = if x then "allow fread, fwrite, and fclose" else "allow fopen"$

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Option 2: One could lift sequential composition to type level

$$\Gamma \vdash M \text{ to } x : A \text{ in } N : M \text{ to } x : A \text{ in } C$$

But then all comp. types would be singleton-like:

- comp. types would contain exactly the terms we want to type!?!
- but could be useful for dependency on comps. (the yes case)

**Option 3:** In the monadic metalanguage  $\lambda_{ML}$ , one could also try

$$\Gamma \vdash M : TA$$
  $\Gamma, x : A \vdash N : TB$   
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But what makes this a principled solution? Why is it correct?

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Option 4: We draw inspiration from algebraic effects
and combine it with restricting <u>C</u> in seq. comp. (Option 1)

E.g., consider the non-det. program (for x: Nat  $\vdash N$ :  $\underline{C}(x)$ )

After tossing the coin, this program evaluates as either N[4/x] at type  $\underline{C}[4/x]$  or N[2/x] at type  $\underline{C}[2/x]$ 

**Idea:** M denotes an element of the coproduct of algebras  $\underline{C}[4/x] + \underline{C}[2/x] \quad "\stackrel{\text{def}}{=} " \quad F\left(U\left(\underline{C}[4/x]\right) + U\left(\underline{C}[2/x]\right)\right)_{/\equiv}$ 

and thus we would like to type M at the type  $\Sigma x$ : Nat.  $\underline{C}$ 

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### Putting these ideas together

(eMLTT: a core dep.-typed language with comp. effects)

### **eMLTT**

Recall: We aim to define a dependently-typed language with

- general computational effects
- a clear distinction between
  - values
  - computations
- with a principled treatment of sequential composition
  - restricting free variables in seq. composition
  - typing based on coproducts of algebras
- with a natural denotational semantics, using standard techniques
  - dep. types split closed comprehension categories
  - comp. effects adjunction models

### eMLTT - types

**Value types:** MLTT + thunks + ...

$$A, B ::=$$
Nat  $\mid 1 \mid 0 \mid \Pi x : A.B \mid \Sigma x : A.B \mid V =_A W \mid U \subseteq | \dots |$ 

•  $U\underline{C}$  is the type of thunked (i.e., suspended) computations

Computation types: dep.-typed version of EEC's comp. types

$$\underline{C}, \underline{D} ::= FA \mid \Pi x : A . \underline{C} \mid \Sigma x : A . \underline{C}$$

- F A is the type of computations returning values of type A
- Πx: A. <u>C</u> is the type of dependent effectful functions
  - generalises CBPV/EEC's comp. types  $A \to \underline{C}$  and  $\underline{C} \times \underline{D}$
- $\Sigma x: A.\underline{C}$  is the type of dep. pairs of values and effectful comps.
  - captures the intuition about seq. comp. and coprods. of algebras
  - generalises EEC's comp. types  $!A \otimes C$  and  $C \oplus D$

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Value terms: MLTT + thunks + ...  $V, W ::= x \mid zero \mid succ V \mid ... \mid thunk M \mid ...$ 

• equational theory based on MLTT with intensional  $V =_A W$ 

**Comp. terms:**  $\mathsf{dep.-typed}$  version of CBPV/EEC's  $\mathsf{comp.}$   $\mathsf{terms}$ 

```
\begin{array}{lll} M,N ::= & \operatorname{force} V \\ & | & \operatorname{return} V \\ & | & M \operatorname{to} x : A \operatorname{in} N \\ & | & \lambda x : A . M \\ & | & MV \\ & | & \langle V,M \rangle & (\operatorname{comp.} \Sigma \operatorname{intro.}) \\ & | & M \operatorname{to} \langle x : A,z : \underline{C} \rangle \operatorname{in} K & (\operatorname{comp.} \Sigma \operatorname{elim.}) \end{array}
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But: Value and comp. terms alone do not suffice, as in EEC!

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**Comp. terms:** dep.-typed version of CBPV/EEC's comp. terms

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**Note:** We need to define K in such a way that the intended left-to-right evaluation order is preserved, e.g., consider

$$\Gamma \vdash_{\!\!\!\!\!c} \langle V,M\rangle \text{ to } \langle x\!:\!A, \textcolor{red}{z}\!:\!\underline{C}\rangle \text{ in } \textcolor{red}{K} = \textcolor{red}{K}[V/x,M/\textcolor{red}{z}]:\underline{D}$$

Homomorphism terms: dep.-typed version of EEC's linear terms

$$K, L := z$$
 (linear comp. vars.)  
 $\mid K \text{ to } x : A \text{ in } M$   
 $\mid \lambda x : A, K$   
 $\mid KV$   
 $\mid \langle V, K \rangle$  (comp.  $\Sigma \text{ intro.}$ )  
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#### Typing judgments:

- Γ ⋈ V : A
- Γ la M : <u>C</u>
- $\Gamma \mid z : \underline{C} \mid_{\overline{h}} K : \underline{D}$  (linear in z; comp. bound to z happens first

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$$\Gamma \vdash \langle V, M \rangle$$
 to  $\langle x : A, z : \underline{C} \rangle$  in  $K = K[V/x, M/z] : \underline{D}$ 

Homomorphism terms: dep.-typed version of EEC's linear terms

```
\begin{array}{lll} \textit{K}, \textit{L} ::= & \textit{z} & \text{(linear comp. vars.)} \\ & | & \textit{K} \text{ to } x : \textit{A} \text{ in } \textit{M} \\ & | & \lambda x : \textit{A} . \textit{K} \\ & | & \textit{K} \textit{V} \\ & | & \langle \textit{V}, \textit{K} \rangle & \text{(comp. } \Sigma \text{ intro.)} \\ & | & \textit{K} \text{ to } \langle x : \textit{A}, \textit{z} : \underline{\textit{C}} \rangle \text{ in } \textit{L} & \text{(comp. } \Sigma \text{ elim.)} \end{array}
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# eMLTT – typing sequential composition

We can then account for type-dependency in seq. comp. as

$$\frac{\Gamma, x : A \vdash_{\overline{c}} N : \underline{C}(x)}{\Gamma \vdash_{\overline{c}} M : F A} \frac{\Gamma, x : A \vdash_{\overline{c}} N : \underline{C}(x)}{\Gamma, x : A \vdash_{\overline{c}} \langle x, N \rangle : \Sigma y : A . \underline{C}(y)}}{\Gamma \vdash_{\overline{c}} M \text{ to } x : A \text{ in } \langle x, N \rangle : \Sigma y : A . \underline{C}(y)}$$

The seq. comp. rule for  $\lambda_{\rm ML}$  is justified by the type isomorphism

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B \qquad \Gamma \vdash \Sigma x : A.F(B) \cong F(\Sigma x : A.B)}{\Gamma \vdash U(\Sigma x : A.F(B)) \cong UF(\Sigma x : A.B) = T(\Sigma x : A.B)}$$

# Categorical semantics of eMLTT

(fibrations + adjunctions)

# Categorical semantics - MLTT part

We define fibred adjunction models  $(\mathcal{B}, \mathcal{V}, p, q, F \dashv U)$ 

• Theorem: a sound and complete class of models for eMLTT

given by: i) a split closed comprehension cat.  $\rho$  with s. fib. 0, ...



- we define a partial interpretation fun. [-], that (if defined) maps:
  - a context  $\Gamma$  to and object  $\llbracket \Gamma \rrbracket$  in  $\mathcal{B}$ , with  $\llbracket \Gamma, x : A \rrbracket \stackrel{\text{def}}{=} \{ \llbracket \Gamma; A \rrbracket \}$
  - ullet a context  $\Gamma$  and a value type A to an object  $[\![\Gamma;A]\!]$  in  $\mathcal{V}_{[\![\Gamma]\!]}$
  - ullet a context  $\Gamma$  and a value term V to  $[\![\Gamma;V]\!]:1_{[\![\Gamma]\!]}\longrightarrow A$  in  $\mathcal{V}_{[\![\Gamma]\!]}$

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  - a context  $\Gamma$  to and object  $\llbracket \Gamma \rrbracket$  in  $\mathcal{B}$ , with  $\llbracket \Gamma, x : A \rrbracket \stackrel{\text{def}}{=} \{ \llbracket \Gamma; A \rrbracket \}$
  - a context  $\Gamma$  and a value type A to an object  $[\![\Gamma;A]\!]$  in  $\mathcal{V}_{[\![\Gamma]\!]}$
  - a context  $\Gamma$  and a value term V to  $\llbracket \Gamma; V \rrbracket : 1_{\llbracket \Gamma \rrbracket} \longrightarrow A$  in  $\mathcal{V}_{\llbracket \Gamma \rrbracket}$

### Categorical semantics - MLTT part

We define fibred adjunction models  $(\mathcal{B}, \mathcal{V}, p, q, F \dashv U)$ 

• Theorem: a sound and complete class of models for eMLTT

given by: i) a split closed comprehension cat. p with s. fib. 0, ...

$$\begin{array}{c}
V \\
\downarrow \\
P \left( \neg \uparrow \downarrow 1 \neg \downarrow \{-\} \right)
\end{array}$$

- the display maps  $\pi_{\llbracket\Gamma;A\rrbracket}:\llbracket\Gamma,x:A\rrbracket\longrightarrow \llbracket\Gamma\rrbracket$  in  $\mathcal{B}$  induce the weakening functors  $\pi^*_{\llbracket\Gamma;A\rrbracket}:\mathcal{V}_{\llbracket\Gamma\rrbracket}\longrightarrow\mathcal{V}_{\llbracket\Gamma,x:A\rrbracket}$ , and
- the value  $\Sigma$  and  $\Pi$ -types are interpreted as adjoints

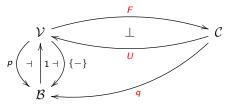
$$\begin{split} & \sum_{\llbracket \Gamma; A \rrbracket} \dashv \pi_{\llbracket \Gamma; A \rrbracket}^* : \mathcal{V}_{\llbracket \Gamma \rrbracket} \longrightarrow \mathcal{V}_{\llbracket \Gamma, x : A \rrbracket} \\ & \pi_{\llbracket \Gamma; A \rrbracket}^* \dashv \Pi_{\llbracket \Gamma; A \rrbracket} : \mathcal{V}_{\llbracket \Gamma, x : A \rrbracket} \longrightarrow \mathcal{V}_{\llbracket \Gamma \rrbracket} \end{split}$$

### Categorical semantics - effects part

We define fibred adjunction models  $(\mathcal{B}, \mathcal{V}, p, q, F \dashv U)$ 

• Theorem: a sound and complete class of models for eMLTT

given by: ii) a split fibration q (with ...) and a s. fib. adj.  $F \dashv U$ 



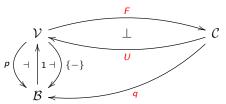
- we extend [-] so that (if defined) it maps:
  - a ctx.  $\Gamma$  and a comp. type  $\underline{C}$  to an object  $\llbracket \Gamma ; \underline{C} \rrbracket$  in  $\mathcal{C}_{\llbracket \Gamma \rrbracket}$
  - a ctx.  $\Gamma$  and a comp. term M to  $\llbracket \Gamma; M \rrbracket : 1_{\llbracket \Gamma \rrbracket} \longrightarrow U(\underline{C})$  in  $\mathcal{V}_{\llbracket \Gamma \rrbracket}$
  - a ctx.  $\Gamma$ , a c. var. z, a c. type  $\underline{C}$ , and a hom. term K to  $\llbracket \Gamma; z \colon \underline{C}; K \rrbracket \colon \llbracket \Gamma; \underline{C} \rrbracket \longrightarrow \underline{D} \text{ in } \mathcal{C}_{\llbracket \Gamma \rrbracket}$

### Categorical semantics - effects part

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$$\Sigma_{\llbracket\Gamma;A\rrbracket} \dashv \pi_{\llbracket\Gamma;A\rrbracket}^* : \mathcal{C}_{\llbracket\Gamma\rrbracket} \longrightarrow \mathcal{C}_{\llbracket\Gamma,x:A\rrbracket}$$
$$\pi_{\llbracket\Gamma;A\rrbracket}^* \dashv \Pi_{\llbracket\Gamma;A\rrbracket} : \mathcal{C}_{\llbracket\Gamma,x:A\rrbracket} \longrightarrow \mathcal{C}_{\llbracket\Gamma\rrbracket}$$

The coproduct type A + B:

[Jacobs'99]

- require  $p: \mathcal{V} \longrightarrow \mathcal{B}$  to have split fibred coproducts  $A +_X B$ , and
- $\langle \{ \mathsf{inl}_A \}^*, \{ \mathsf{inr}_B \}^* \rangle : \mathcal{V}_{\{A+_X B\}} \longrightarrow \mathcal{V}_{\{A\}} \times \mathcal{V}_{\{B\}}$  to be fully-faith.
- allows one to interpret dependent case analysis, i.e.,

$$\mathcal{V}_{\{A\}}\left(1_{\{A\}}, \{\operatorname{inl}_A\}^*(C)\right) \times \mathcal{V}_{\{B\}}\left(1_{\{B\}}, \{\operatorname{inr}_B\}^*(C)\right) \\ \cong \\ \mathcal{V}_{\{A+_{X}B\}}\left(1_{\{A+_{X}B\}}, C\right)$$

provides semantics for

$$\frac{\Gamma, y_1 : A \vdash W_1 : C[\operatorname{inl}_A y_1/x] \quad \Gamma, y_2 : B \vdash W_2 : C[\operatorname{inr}_B y_2/x]}{\Gamma, x : A + B \vdash \operatorname{case} x \text{ of } (\operatorname{inl}(y_1) \mapsto W_1, \operatorname{inr}(y_2) \mapsto W_2) : C[\operatorname{inr}_B y_2/x]}$$

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### A generalisation:

[Ahman'17]

- **Idea:** fully-faith. for cocones  $A \longrightarrow A \circledast_X B \longleftarrow B$  is enough, and we can generalise this to all split fibred colimits
- Theorem:
  - if for every object  $X \in \mathcal{B}$  and diagram  $J : \mathcal{D} \longrightarrow \mathcal{V}_X$ there exists a cocone  $\underline{\operatorname{in}}^J : J \longrightarrow \Delta(\underline{\operatorname{colim}}(J))$  in  $\mathcal{V}_X$ ,
  - such that f\*(in<sup>J</sup><sub>D</sub>) = in<sup>f\*oJ</sup><sub>D</sub>, for any f : X → Y, and such that the unique mediating functor

$$\begin{split} & \langle \{\underline{\operatorname{in}}_D^J\}_{D\in\mathcal{D}}^* \rangle : \mathcal{V}_{\{\underline{\operatorname{colim}}(J)\}} \longrightarrow \operatorname{lim}(\widehat{J}) \\ & \text{s fully-faithful (for } \widehat{J} : \mathcal{D}^{op} \longrightarrow \operatorname{Cat, where } \widehat{J}(D) = \mathcal{V}_{\{J(D)\}}) \end{split}$$

then p has split fibred colimits of shape D, and
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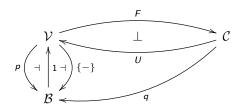
• Idea: fully-faith. for cocones  $A \longrightarrow A \circledast_X B \longleftarrow B$  is enough, and we can generalise this to all split fibred colimits

#### • Theorem:

- if for every object X ∈ B and diagram J : D → V<sub>X</sub>
   there exists a cocone in J : J → Δ(colim(J)) in V<sub>X</sub>,
- such that  $f^*(\underline{\operatorname{in}}_D^J) = \underline{\operatorname{in}}_D^{f^*\circ J}$ , for any  $f: X \longrightarrow Y$ , and such that the unique mediating functor

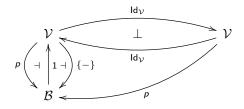
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then p has split fibred colimits of shape D, and
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**Example 1** (identity adjunctions):  $Id_{\mathcal{V}} \dashv Id_{\mathcal{V}} : \mathcal{V} \longrightarrow \mathcal{V}$ 

- $m{\mathcal{C}}\stackrel{\mathsf{def}}{=} \mathcal{V}$
- $q \stackrel{\text{def}}{=} p$
- $\bullet \ \ F \stackrel{\scriptscriptstyle\mathsf{def}}{=} \mathsf{Id}_{\mathcal{V}}$
- $U \stackrel{\text{def}}{=} \operatorname{Id}_{\mathcal{V}}$



sound as long as we haven't included any effects

### **Example 2** (models of Egger et al.'s EEC):

- given an adjunction  $F_{\mathsf{EEC}} \dashv U_{\mathsf{EEC}} : \mathcal{E} \longrightarrow \mathcal{D}$ , such that
  - $\mathcal{D}$  is a CCC (with 0, ...), and
  - $F_{\mathsf{EEC}} \dashv U_{\mathsf{EEC}}$  and  $\mathcal{E}$  are  $\mathcal{D}$ -enriched, and
  - $\mathcal{E}$  has all  $\mathcal{D}$ -tensors  $(A \otimes \underline{C})$  and  $\mathcal{D}$ -cotensors  $(A \Rightarrow \underline{C})$
- we use the simple fibration  $p: s(\mathcal{D}) \longrightarrow \mathcal{D}$ , where

$$p(f: X \longrightarrow Y, g: X \times A \longrightarrow B) \stackrel{\text{def}}{=} f: p(X, A) \longrightarrow p(Y, B)$$

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• finally, we define  $F \dashv U$  as the lifting of  $F_{\text{EEC}} \dashv U_{\text{EEC}}$  as follows:

$$F(A, B) \stackrel{\text{def}}{=} (A, F_{\text{EEC}}(B))$$
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### Example 3 (families fibrations):

- given an adjunction  $F_{\mathcal{D}} \dashv U_{\mathcal{D}} : \mathcal{D} \longrightarrow \mathsf{Set}$ , such that
  - ullet D has set-indexed products and coproducts
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• ex: EM-cats.  $(\mathcal{D} \stackrel{\text{def}}{=} \mathsf{Set}^\mathsf{T})$  and Lawere ths.  $(\mathcal{D} \stackrel{\text{def}}{=} \mathsf{Mod}(\mathcal{L}, \mathsf{Set}))$ 

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### **Example 4** (continuous families for $\mu x : U\underline{C}.M$ ):

- given a CPO-enriched monad T on CPO, such that
  - **T** supports least zero-ary alg. op.  $(\bot_A : 1 \longrightarrow TA)$ , and
  - CPO<sup>T</sup> has reflexive coequalizers
- ullet we use the continuous families fibration  $p:\mathsf{CFam}(\mathsf{CPO})\longrightarrow\mathsf{CPC}$

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where

$$X \in \mathsf{CPO}$$
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  - Idea: Not all functors  $f^*: CPO/Y \to CPO/X$  are left adjoints
  - consider the epimorphism  $e \stackrel{\mathsf{def}}{=} n \mapsto n : \mathbb{N}_{=} \longrightarrow \mathbb{N}_{\omega}$  in CPO, and
  - assume given a non-empty cpo X, and
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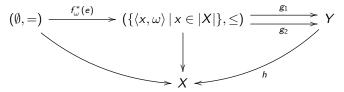
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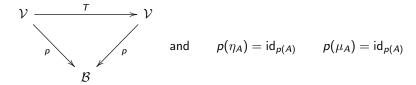
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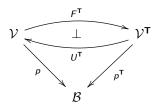


### **Example 5** (EM-resolutions of split fibred monads):

• given a split fibred monad  $\mathbf{T} = (T, \eta, \mu)$  on p, i.e.,



we consider models based on the EM-resolution of T



and show that three familiar results hold for this situation

### **Example 5** (EM-resolutions of split fibred monads):

• Theorem 1: if p supports  $\Pi$ -types, then  $p^T$  also supports  $\Pi$ -types

• **Theorem 2:** if p supports  $\Sigma$ -types and the dependent strength

$$\sigma_A:\Sigma_A\circ T\longrightarrow T\circ \Sigma_A$$

is a natural isomorphism, then  $p^{\mathsf{T}}$  also supports  $\Sigma$ -types

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(for corresponding simply typed results, see [Borceux'94] and [Linton'69])

#### Fibred effect theories $\mathcal{T}_{\text{eff}}$ :

• we consider signatures of dep. typed operation symbols

$$\frac{\cdot \vdash I \qquad x_i : I \vdash O \qquad I \text{ and } O \text{ are pure value types}}{\text{op} : (x_i : I) \longrightarrow O}$$

equipped with equations on derivable effect terms

Typing rule:

$$\frac{\Gamma \uplus V: I \quad \Gamma \vdash \underline{C} \quad \Gamma, x: O[V/x_i] \uplus M: \underline{C}}{\Gamma \vDash \operatorname{op}_{V}^{\underline{C}}(x.M): \underline{C}} \text{ (op:} (x_i:I) \longrightarrow O)$$

**General algebraicity equations** (in addition to eff. th. eqs.)

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- read :  $1 \longrightarrow \mathsf{Chr}$   $(\mathsf{Chr} \stackrel{\mathsf{def}}{=} 1 + \ldots + 1)$  write :  $\mathsf{Chr} \longrightarrow 1$
- no equations

Example 2 (global state with location-dependent store type):

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x : \mathsf{Loc} \vdash \mathsf{Val}

\diamond \vdash \mathsf{isDec}_{\mathsf{Loc}} : \Pi x : \mathsf{Loc} . \Pi y : \mathsf{Loc} . (x =_{\mathsf{Loc}} y) + (x =_{\mathsf{Loc}} y \to 0)
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# Handlers of algebraic effects

(for programming and reasoning)

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**Idea:** Generalisation of exception handlers to all algebraic effects [Plotkin, Pretnar'09]

 ${\sf Handler} = {\sf Algebra} \quad {\sf and} \quad {\sf Handling} = {\sf Homomorphism}$ 

Usual term-level presentation:

$$\frac{\Gamma \vdash R M : FA \quad \Gamma, x_v : I, x_k : O[x_v/x_i] \to U\underline{C} \vdash R_{op} : \underline{C} \quad \Gamma, y : A \vdash R_{ret} : \underline{C}}{\Gamma \vdash R M \text{ handled with } \{op_{x_v}(x_k) \mapsto N_{op}\}_{op \in \mathcal{T}_{eff}} \text{ to } y : A \text{ in}_{\underline{C}} N_{ret} : \underline{C}}$$
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**Note 1:** P(thunk M) computes a proof obligation for M

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- a type-based treatment of handlers  $\underline{C} ::= \ldots \mid \langle A; \overrightarrow{V_{\text{op}}}; \overrightarrow{W_{\text{eq}}} \rangle$
- function extensionality (actually, a it's a bit more extensional)

Idea: Assuming we were able to handle into values

$$M$$
 handled with  $\{\operatorname{op}_{\mathsf{x}_{\mathsf{v}}}(\mathsf{x}_{\mathsf{k}})\mapsto V_{\operatorname{op}}\}_{\operatorname{op}\in\mathcal{T}_{\operatorname{eff}}}$  to  $y\!:\!A$  in  $_{\!B}$   $V_{\operatorname{ret}}$ 

we could define predicates (value terms)

$$\Gamma \vdash P : UFA \rightarrow \mathcal{U}$$

by

- ullet equipping a universe  ${\cal U}$  with an algebra for  ${\cal T}_{\sf eff}$ , and
- using the above handle-into-values construct to define P

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### **Example 1** (Evaluation logic style modalities):

- Given a predicate  $P:A\to \mathcal{U}$  on return values, we define a predicate  $\Diamond P:UFA\to \mathcal{U}$  on I/O-computations as
- $\Diamond P \stackrel{\text{def}}{=} \lambda x : UFA . (\text{force } x) \text{ handled with } \{...\}_{\text{op} \in \mathcal{T}_{\text{IO}}} \text{ to } y : A \text{ in}_{\mathcal{U}} P y$  using the handler given by

$$\begin{array}{ll} V_{\mathsf{read}} & \stackrel{\mathsf{def}}{=} & \lambda \, x \colon \! \left( \Sigma \, x_{\!\scriptscriptstyle V} \colon \! 1 \cdot \mathsf{Chr} \to \mathcal{U} \right) \cdot \widehat{\Sigma} \, y \colon \! \mathsf{El}(\widehat{\mathsf{Chr}}) \cdot \left( \mathsf{snd} \, x \right) \, y \\ V_{\mathsf{write}} & \stackrel{\mathsf{def}}{=} & \lambda \, x \colon \! \left( \Sigma \, x_{\!\scriptscriptstyle V} \colon \mathsf{Chr} \cdot 1 \to \mathcal{U} \right) \cdot \left( \mathsf{snd} \, x \right) \, \star \end{array}$$

ullet  $\Diamond P$  is the possibility modality

$$\Diamond P\left(\operatorname{thunk}\left(\operatorname{read}(x.\operatorname{write}_{e'}(\operatorname{return}V)\right)\right)\right) = \widehat{\Sigma}x:\operatorname{El}(\widehat{\operatorname{Chr}}).PV$$

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### **Example 2** (Dijkstra's weakest precondition semantics):

• Given a postcondition on return values and final states

$$Q: A \to S \to \mathcal{U}$$
  $(S \stackrel{\text{def}}{=} \Pi x: \text{Loc. Val})$ 

we define a precondition for stateful comps. on initial states

$$\mathsf{wp}_\mathcal{Q}: \mathit{UFA} o \mathit{S} o \mathcal{U}$$

by

- ) handling the given comp. Into a state-passing function using  $V_{
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  m ret}$  " = "  $({\cal U} imes S)$
- ii) feeding in the initial state; and iii) projecting out  ${\cal U}$
- **Theorem:** wp<sub>Q</sub> satisfies expected properties of WPs, e.g., wp<sub>Q</sub> (thunk (return V)) =  $\lambda x_S : S : Q : V : x_S$ wp<sub>Q</sub> (thunk (put = (M))) =  $\lambda x_S : S : x_S : (thunk <math>M$ ) ( $x_S : (f : S) : x_S : (f : S) :$

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### **Example 3** (Patterns of allowed effects):

Assuming an inductive type Protocol, given by

$$e:\mathsf{Protocol} \qquad \qquad \mathbf{r}:(\mathsf{Chr}\to\mathsf{Protocol})\to\mathsf{Protocol}$$

$$\mathtt{w}: (\mathsf{Chr} o \mathcal{U}) o \mathsf{Protocol} o \mathsf{Protocol}$$

and potentially also by ∧, ∨, ...

• Then, given a protocol Pr : Protocol, we define

$$\underline{\mathsf{Pr}}: \mathit{UFA} \to \mathcal{U}$$

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### **Conclusion**

In work we told a mathematically natural story of combining

dependent types and computational effects

### In particular, we saw

- a clean core language of dependent types and comp. effects
- a natural category-theoretic semantics
- algebraic effects and their handlers (also for reasoning)

**Future work:** type-dep. on computations, i.e.,  $\underline{D}(\text{read}(x.M))$ 

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- M to x:A in  $N: \underline{C}[\operatorname{thunk} M/y]$  (where y:UFA) [Vákár'17]
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# Thank you!

D. Ahman.

Fibred Computational Effects. (PhD Thesis, 2017)

D. Ahman, N. Ghani, G. Plotkin.

Dependent Types and Fibred Computational Effects. (FoSSaCS'16)

D. Ahman.

Handling Fibred Computational Effects. (POPL'18)