A fibrational view on computational effects

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Outline

We investigate the combination of

- dependent types $(\Pi, \Sigma, V =_A W, \ldots)$
- computational effects (state, I/O, probability, recursion, ...)

Two guiding problems

- effectful programs in types (e.g., `read` and `write` in types)
- types of effectful programs (e.g., of sequential composition)

Our goals

- tell a mathematically natural story
- use established math. techniques
- cover a wide range of comp. effects
- discover smth. interesting

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- cover a wide range of comp. effects (alg. effects, continuations)
- discover smth. interesting (using handlers to reason about effects)

Effectful programs in types

(type-dependency in the presence of effects)

Effectful programs in types

Let's assume that we have some dependent type A, e.g.:

$$\ell : (\text{List Chr}) \vdash A(\ell) \quad \stackrel{\text{def}}{=} \quad \Sigma\, \ell' : (\text{List Chr})\,.\,(\texttt{length}\ \ell =_{\text{Nat}} \texttt{length}\ \ell' \times \ldots)$$

which could be used to type the dependent function

$$\texttt{sort} : \Pi\, \ell : (\text{List Chr})\,.\,A(\ell)$$

Q: Should we allow $A[\texttt{receive}(y.\,M)/\ell]$?

A1: In this talk, we say no

- types should only depend on static information about effects
- we allow dependency on effectful computations via thunks

A2: But we are also looking into yes

- type-dependency needs to be "homomorphic"
- intuitively, need to lift $A(\ell)$ to $A^{\dagger}(c)$, where $c : T(\text{List Chr})$

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Effectful programs in types

Aim: Types should only depend on static info about effects

Solution: CBPV/EEC style distinction between vals. and comps.

- value types $\Gamma \vdash A$ (MLTT + thunks + ...)
- computation types $\Gamma \vdash \underline{C}$ (dep. CBPV/EEC)
- where Γ contains only value variables $x_1 : A_1, \ldots, x_n : A_n$

Note: Could have also considered λ_{ML} and FGCBV

- building on CBPV/EEC gives a more general story
- especially for the treatment of sequential composition
- and also for integrating dependent- and effect-typing

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Assigning types to effectful programs

(e.g., sequential composition)

Assigning types to effectful programs

The problem: The standard typing rule for seq. composition

$$\frac{\Gamma \vdash_{\mathsf{c}} M : F\,A \qquad \Gamma, x : A \vdash_{\mathsf{c}} N : \underline{C}}{\Gamma \vdash_{\mathsf{c}} M \;\mathtt{to}\; x : A \;\mathtt{in}\; N : \underline{C}}$$

is not correct any more because x can appear free in the type

$$\underline{C}$$

in the conclusion

Assigning types to effectful programs

Aim: To fix the typing rule of sequential composition

Option 1: We could restrict the free variables in $\underline{C}$: [Levy'04]

$$\frac{\Gamma \vdash_{\mathsf{c}} M : F\,A \qquad \Gamma \vdash \underline{C} \qquad \Gamma, x : A \vdash_{\mathsf{c}} N : \underline{C}}{\Gamma \vdash_{\mathsf{c}} M \;\mathtt{to}\; x : A \;\mathtt{in}\; N : \underline{C}}$$

But sometimes it is useful if $\underline{C}$ can depend on x!

- if we consider

$$\mathtt{fopen}\,(\mathtt{return}\;\mathtt{true}, \mathtt{return}\;\mathtt{false})\;\mathtt{to}\;x : \mathsf{Bool}\;\mathtt{in}\;N$$

- then it would be natural to let $\underline{C}$ depend on x, e.g.,

$$x : \mathsf{Bool} \vdash \underline{C}(x) \stackrel{\text{def}}{=} \mathtt{if}\;x\;\mathtt{then}\;\text{“allow }\mathtt{fread}\text{, }\mathtt{fwrite}\text{, and }\mathtt{fclose}\text{”}\;\mathtt{else}\;\text{“allow }\mathtt{fopen}\text{”}$$

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Option 2: One could lift sequential composition to type level

$$\Gamma \vdash_{\mathbb{C}} M \texttt{ to } x{:}A \texttt{ in } N : M \texttt{ to } x{:}A \texttt{ in } \underline{C}$$

But then comp. types would be singleton-like!?!

However, something like this is probably needed for the yes case.

Option 3: In the monadic metalanguage λ_{ML}, one could try

$$\frac{\Gamma \vdash M : T\,A \qquad \Gamma, x{:}A \vdash N : T\,B(x)}{\Gamma \vdash M \texttt{ to } x{:}A \texttt{ in } N : T\,(\Sigma x : A.B)}$$

But what makes this a principled solution? Why is it correct?

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Option 4: We draw inspiration from algebraic effects

- and combine it with restricting $\underline{C}$ in seq. comp. (**Option 1**)

E.g., consider the non-det. program $\left(\text{for } x : \mathsf{Nat} \vdash_{\mathsf{c}} N : \underline{C}(x)\right)$

$$M \stackrel{\text{def}}{=} \texttt{choose}\left(\texttt{return}\ 4, \texttt{return}\ 2\right)\ \texttt{to}\ x : \mathsf{Nat}\ \texttt{in}\ N$$

After tossing the coin, this program evaluates as either

$$N[4/x] : \underline{C}[4/x] \qquad \text{or} \qquad N[2/x] : \underline{C}[2/x]$$

Idea: M denotes an element of the coproduct of algebras

$$\underline{C}[4/x] + \underline{C}[2/x] \stackrel{\text{def}}{=} F\left(U\left(\underline{C}[4/x]\right) + U\left(\underline{C}[2/x]\right)\right)_{/\equiv}$$

and thus we would like to type M at the type $\Sigma x : \mathsf{Nat}.\underline{C}$

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Putting these ideas together

(eMLTT: a core dep.-typed language with comp. effects)

eMLTT – types

Value types: MLTT + thunks + ...

$$A, B ::= \mathsf{Nat} \mid 1 \mid 0 \mid \Pi x{:}A.B \mid \Sigma x{:}A.B \mid V =_A W \mid U\,\underline{C} \mid \ldots$$

- $U\,\underline{C}$ is the type of thunked (i.e., suspended) computations

Computation types: dep.-typed version of EEC's comp. types

$$\underline{C}, \underline{D} ::= F\,A \mid \Pi x{:}A.\underline{C} \mid \Sigma x{:}A.\underline{C}$$

- $F\,A$ is the type of computations returning values of type A
- $\Pi x{:}A.\underline{C}$ is the type of dependent effectful functions
  - generalises CBPV/EEC's comp. types $A \to \underline{C}$ and $\underline{C} \times \underline{D}$
- $\Sigma x{:}A.\underline{C}$ is the type of dep. pairs of values and effectful comps.
  - captures the intuition about seq. comp. and coprods. of algebras
  - generalises EEC's comp. types $!A \otimes \underline{C}$ and $\underline{C} \oplus \underline{D}$

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$$V, W ::= x \mid \texttt{zero} \mid \texttt{succ}\, V \mid \ldots \mid \texttt{thunk}\, M \mid \ldots$$

- equational theory based on intensional MLTT

Comp. terms: dep.-typed version of CBPV/EEC's comp. terms

$$\begin{aligned} M, N ::= {} & \texttt{force}\, V \\ \mid {} & \texttt{return}\, V \\ \mid {} & M \texttt{ to } x{:}A \texttt{ in } N \\ \mid {} & \lambda x{:}A.M \\ \mid {} & MV \\ \mid {} & \langle V, M \rangle && \text{(comp. } \Sigma \text{ intro.)} \\ \mid {} & M \texttt{ to } \langle x{:}A, z{:}\underline{C} \rangle \texttt{ in } K && \text{(comp. } \Sigma \text{ elim.)} \end{aligned}$$

But: Value and comp. terms alone do not suffice, as in EEC!

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Note: We need to define K in such a way that the intended left-to-right evaluation order is preserved, e.g., consider

$$\Gamma \vDash_{\mathsf{c}} \langle V, M \rangle \ \mathtt{to}\ \langle x{:}A, z{:}\underline{C} \rangle \ \mathtt{in}\ K = K[V/x, M/z] : \underline{D}$$

Homomorphism terms: dep.-typed version of EEC's linear terms

$$\begin{array}{rll} K, L ::= & z & \text{(linear comp. vars.)} \\ \mid & K \ \mathtt{to}\ x{:}A \ \mathtt{in}\ M & \\ \mid & \lambda x{:}A.K & \\ \mid & K\,V & \\ \mid & \langle V, K \rangle & \text{(comp. } \Sigma \text{ intro.)} \\ \mid & K \ \mathtt{to}\ \langle x{:}A, z{:}\underline{C} \rangle \ \mathtt{in}\ L & \text{(comp. } \Sigma \text{ elim.)} \end{array}$$

Typing judgments:

- $\Gamma \vDash_{\mathsf{v}} V : A$
- $\Gamma \vDash_{\mathsf{c}} M : \underline{C}$
- $\Gamma \mid z{:}\underline{C} \vDash_{\mathsf{h}} K : \underline{D}$ (linear in z; comp. bound to z happens first)

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eMLTT – typing sequential composition

We can then account for type-dependency in seq. comp. as

$$\frac{\Gamma \vDash_{\mathsf{c}} M : F\,A \qquad \Gamma \vdash \Sigma y : A.\underline{C}(y) \qquad \dfrac{\Gamma, x : A \vDash_{\mathsf{c}} N : \underline{C}(x)}{\Gamma, x : A \vDash_{\mathsf{c}} \langle x, N\rangle : \Sigma y : A.\underline{C}(y)}}{\Gamma \vDash_{\mathsf{c}} M \texttt{ to } x : A \texttt{ in } \langle x, N\rangle : \Sigma y : A.\underline{C}(y)}$$

The seq. comp. rule for λ_{ML} is justified by the type isomorphism

$$\frac{\Gamma \vdash A \qquad \Gamma, x : A \vdash B(x)}{\Gamma \vdash U(\Sigma x : A.F(B)) \cong UF(\Sigma x : A.B) = T(\Sigma x : A.B)}$$

Categorical semantics of eMLTT

(fibrations + adjunctions)

Categorical semantics – MLTT part

We define fibred adjunction models

- **Theorem:** a sound and complete class of models for eMLTT

given by: i) a split closed comprehension cat. p with Nat, ...

- we define a partial interpretation fun. $[\![-]\!]$, that (if defined) maps:
  - a context Γ to and object $[\![\Gamma]\!]$ in $\mathcal{B}$, with $[\![\Gamma, x : A]\!] \stackrel{\text{def}}{=} \{[\![\Gamma; A]\!]\}$
  - a context Γ and a value type A to an object $[\![\Gamma; A]\!]$ in $\mathcal{V}_{[\![\Gamma]\!]}$
  - a context Γ and a value term V to $[\![\Gamma; V]\!] : 1_{[\![\Gamma]\!]} \longrightarrow A$ in $\mathcal{V}_{[\![\Gamma]\!]}$

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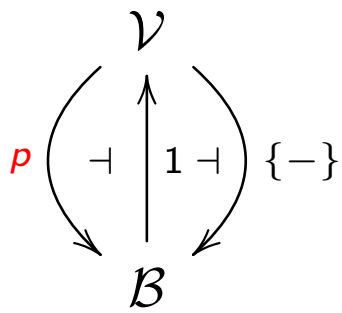

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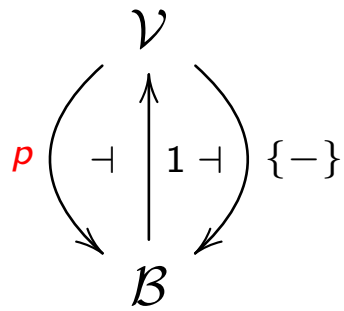

- the display maps $\pi_{[\![\Gamma;A]\!]} : [\![\Gamma, x:A]\!] \longrightarrow [\![\Gamma]\!]$ in $\mathcal{B}$

  induce the weakening functors $\pi^*_{[\![\Gamma;A]\!]} : \mathcal{V}_{[\![\Gamma]\!]} \longrightarrow \mathcal{V}_{[\![\Gamma,x:A]\!]}$, and
- the value Σ- and Π-types are interpreted as adjoints

  $\Sigma_{[\![\Gamma;A]\!]} \dashv \pi^*_{[\![\Gamma;A]\!]} : \mathcal{V}_{[\![\Gamma]\!]} \longrightarrow \mathcal{V}_{[\![\Gamma,x:A]\!]}$ (such that Σ is strong)

  $\pi^*_{[\![\Gamma;A]\!]} \dashv \Pi_{[\![\Gamma;A]\!]} : \mathcal{V}_{[\![\Gamma,x:A]\!]} \longrightarrow \mathcal{V}_{[\![\Gamma]\!]}$

Categorical semantics – effects part

We define fibred adjunction models

- **Theorem:** a sound and complete class of models for eMLTT

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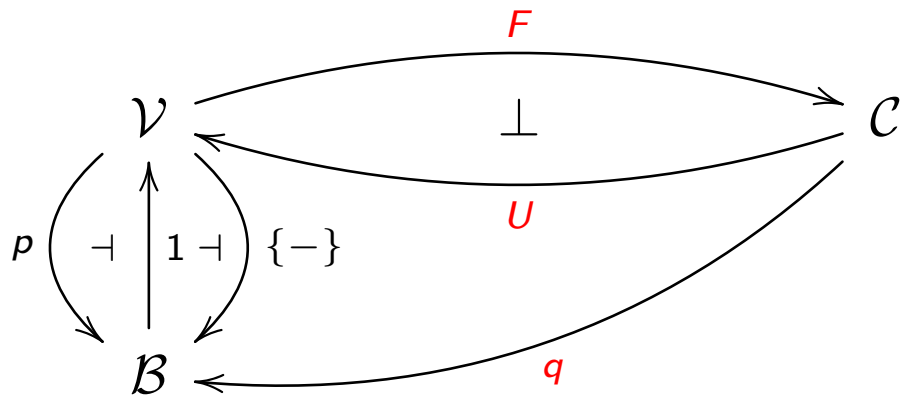

- we extend $[\![-]\!]$ so that (if defined) it maps:
  - a ctx. Γ and a comp. type $\underline{C}$ to an object $[\![\Gamma; \underline{C}]\!]$ in $\mathcal{C}_{[\![\Gamma]\!]}$
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  - a ctx. Γ, a c. var. z, a c. type $\underline{C}$, and a hom. term K to $[\![\Gamma; z : \underline{C}; K]\!] : [\![\Gamma; \underline{C}]\!] \longrightarrow \underline{D}$ in $\mathcal{C}_{[\![\Gamma]\!]}$

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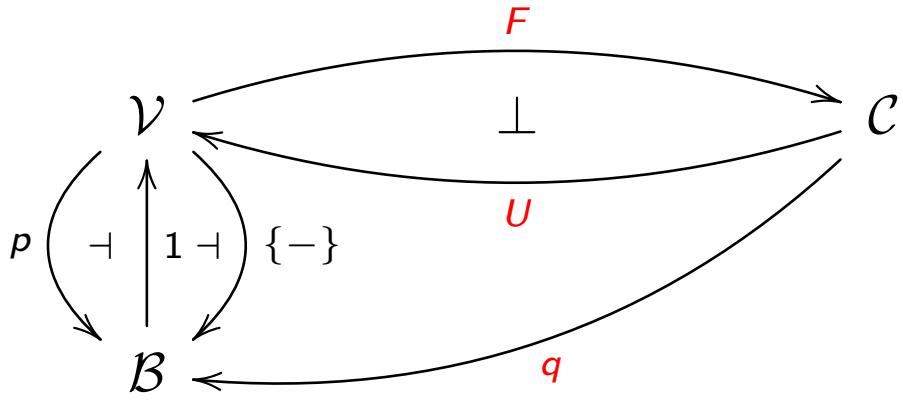

- we again have weakening functors $\pi^*_{[\![\Gamma;A]\!]} : \mathcal{C}_{[\![\Gamma]\!]} \longrightarrow \mathcal{C}_{[\![\Gamma,x:A]\!]}$, and
- the comp. Σ- and Π-types are interpreted again as adjoints

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Digression: dep. elimination of 0 and $+$

The coproduct type $A + B$: [Jacobs'99]

- require $p : \mathcal{V} \longrightarrow \mathcal{B}$ to have split fibred coproducts $A +_X B$, and
- $\langle \{\mathsf{inl}_A\}^*, \{\mathsf{inr}_B\}^* \rangle : \mathcal{V}_{\{A +_X B\}} \longrightarrow \mathcal{V}_{\{A\}} \times \mathcal{V}_{\{B\}}$ to be fully-faith.

- allows one to interpret dependent case analysis, i.e.,

$$\mathcal{V}_{\{A\}}\Big(1_{\{A\}}, \{\mathsf{inl}_A\}^*(C)\Big) \times \mathcal{V}_{\{B\}}\Big(1_{\{B\}}, \{\mathsf{inr}_B\}^*(C)\Big) \cong \mathcal{V}_{\{A +_X B\}}\Big(1_{\{A +_X B\}}, C\Big)$$

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A generalisation: [Ahman'17]

- **Idea:** fully-faith. for cocones $A \longrightarrow A \circledast_X B \longleftarrow B$ is enough, and we can generalise this to all split fibred colimits

- **Theorem:**
  - if for every object $X \in \mathcal{B}$ and diagram $J : \mathcal{D} \longrightarrow \mathcal{V}_X$ there exists a cocone $\underline{\mathsf{in}}^J : J \longrightarrow \Delta(\underline{\mathsf{colim}}(J))$ in $\mathcal{V}_X$,
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    is fully-faithful (for $\widehat{J} : \mathcal{D}^{op} \longrightarrow \mathsf{Cat}$, where $\widehat{J}(D) = \mathcal{V}_{\{J(D)\}}$),
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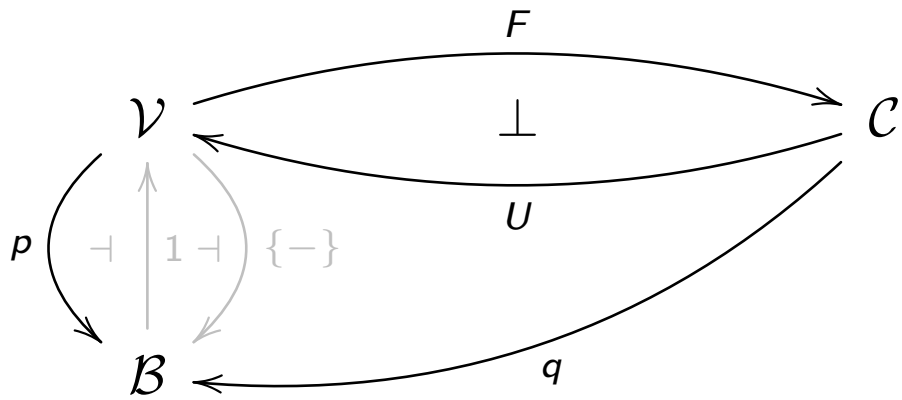
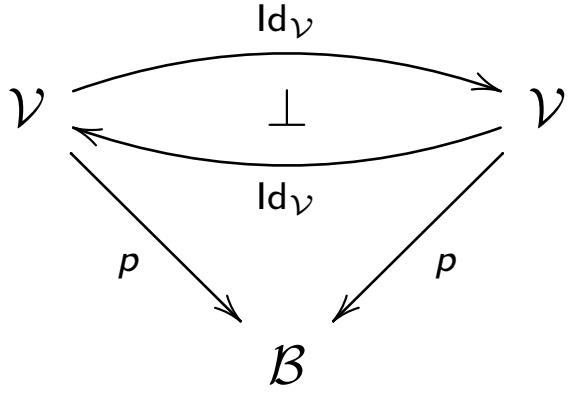
Examples of fibred adjunction models



Examples of fibred adjunction models

Example 1 (identity adjunctions):



Note: sound model as long as we haven't included any effects

Examples of fibred adjunction models

Example 2 (simple models from Egger et al.'s EEC):

- given an adjunction $F_{\mathsf{EEC}} \dashv U_{\mathsf{EEC}} : \mathcal{E} \longrightarrow \mathcal{D}$, such that
  - $\mathcal{D}$ is Cartesian closed (with Nat, ...), and
  - $\mathcal{E}$ and $F_{\mathsf{EEC}} \dashv U_{\mathsf{EEC}}$ are $\mathcal{D}$-enriched, and
  - $\mathcal{E}$ has all $\mathcal{D}$-tensors $(A \otimes \underline{C})$ and $\mathcal{D}$-cotensors $(A \Rightarrow \underline{C})$
- we use simple fibration $\mathsf{s}_{\mathcal{D}}$ and simpl. $\mathcal{D}$-enrich. fibration $\mathsf{s}_{\mathcal{D},\mathcal{E}}$

$$F(X,A) \stackrel{\text{def}}{=} (X, F_{\mathsf{EEC}}(A))$$

$$U(X,\underline{C}) \stackrel{\text{def}}{=} (X, U_{\mathsf{EEC}}(\underline{C}))$$

$$\mathsf{s}(\mathcal{D}): \quad (f,g) : (X,A) \longrightarrow (Y,B) \quad \text{where} \quad f : X \longrightarrow Y \quad g : X \times A \longrightarrow B$$

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Note: this model doesn't support any real type-dependency

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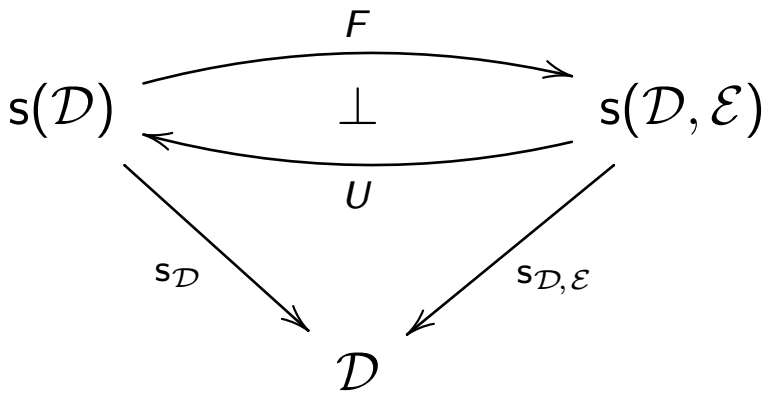

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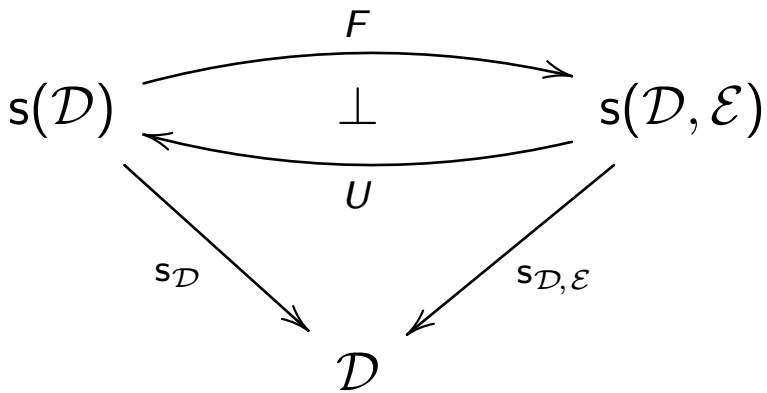

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Examples of fibred adjunction models

Example 3 (families fibrations):

- given an adjunction $F_{\mathcal{D}} \dashv U_{\mathcal{D}} : \mathcal{D} \longrightarrow \mathsf{Set}$, such that
  - $\mathcal{D}$ has set-indexed products and set-indexed coproducts
- such adjunctions arise from
  - EM-cats. ($\mathcal{D} \stackrel{\text{def}}{=} \mathsf{Set}^{\mathbf{T}}$) and Law. ths. ($\mathcal{D} \stackrel{\text{def}}{=} \mathsf{Mod}(\mathcal{L}, \mathsf{Set})$)
  - resolutions of $S \Rightarrow (-) \times S$ and $((-) \Rightarrow R) \Rightarrow R$
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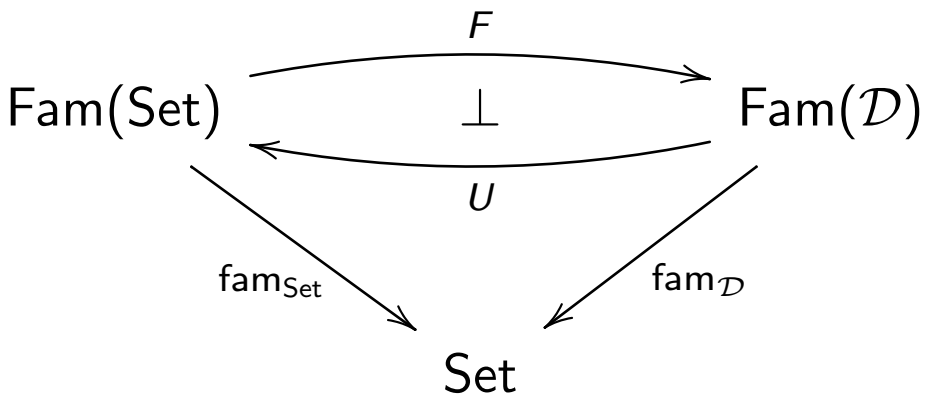

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Example 4 (continuous families for $\mu x : U\underline{C} \,.\, M$):

- given a CPO-enriched monad $\mathbf{T}$ on CPO, such that
  - $\mathbf{T}$ supports least zero-ary alg. op. ($\bot_A : 1 \longrightarrow TA$), and
  - $\mathrm{CPO}^{\mathbf{T}}$ has reflexive coequalizers
- such $\mathbf{T}$ arise from discrete CPO-enriched countable Law. ths.
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$\mathrm{CFam}(\mathrm{CPO})$: (X,A) where $X \in \mathrm{CPO}$ $A : X \longrightarrow \mathrm{CPO}^{\mathrm{EP}}$ an ω-cont. fun.

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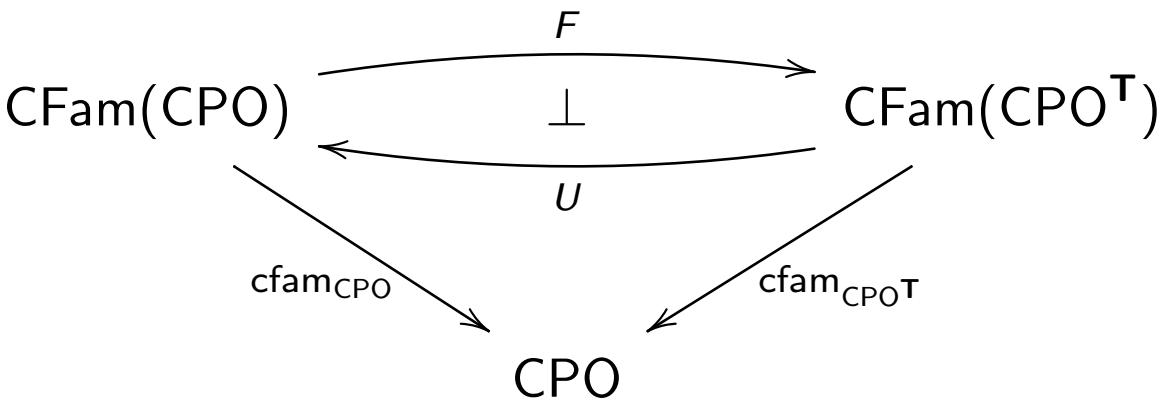

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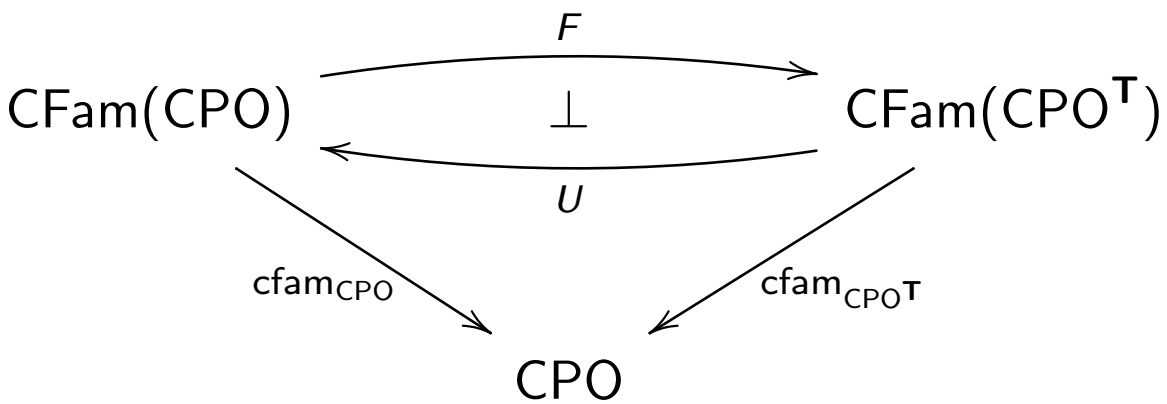

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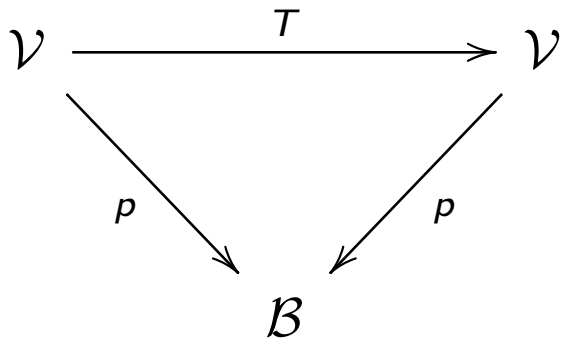
Examples of fibred adjunction models

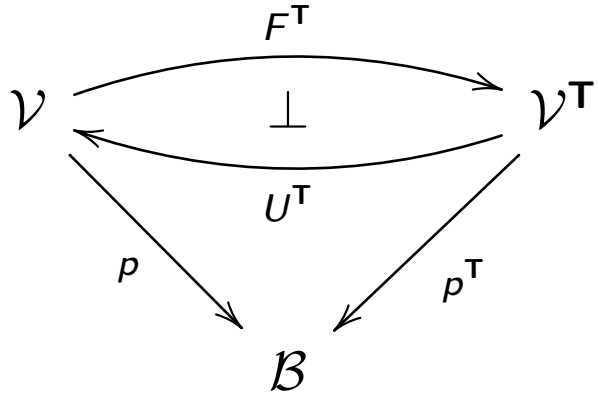
Example 5 (EM-resolutions of split fibred monads):

- given a split fibred monad $\mathbf{T} = (T, \eta, \mu)$ on p, i.e.,



and $\quad p(\eta_A) = \mathrm{id}_{p(A)} \qquad p(\mu_A) = \mathrm{id}_{p(A)}$

- we consider models based on the EM-resolution of $\mathbf{T}$



- and show that three familiar results hold for this situation

Examples of fibred adjunction models

Example 5 (EM-resolutions of split fibred monads):

- **Theorem 1:** if p supports Π-types, then p^{T} also supports Π-types

$$\Pi_A^{\mathsf{T}}(B,\beta) \stackrel{\text{def}}{=} (\Pi_A(B), \beta_{\Pi_A^{\mathsf{T}}})$$

- **Prop.:** every $\mathbf{T}$ on a split closed comp. cat. has a dep. strength

$$\sigma_A : \Sigma_A \circ T \longrightarrow T \circ \Sigma_A \qquad (A \in \mathcal{V})$$

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$$\Sigma_A^{\mathsf{T}}(B,\beta) \stackrel{\text{def}}{=} (\Sigma_A(B), \beta_{\Sigma_A^{\mathsf{T}}})$$

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Algebraic effects

Algebraic effects – ops. and eqs.

Fibred effect theories $\mathcal{T}_{\text{eff}}$:

- signatures of dep. typed operation symbols

$$\frac{\cdot \vdash I \qquad x_i : I \vdash O \qquad I \text{ and } O \text{ are pure value types}}{\mathrm{op} : (x_i : I) \longrightarrow O}$$

- equipped with equations on derivable effect terms

In eMLTT:

$$M ::= \ldots \mid \mathrm{op}_V^C(x.M)$$

General algebraicity equations (in addition to eff. th. eqs.):

$$\frac{\Gamma \vdash_{\mathsf{v}} V : I \quad \Gamma, x : O[V/x_i] \vdash_{\mathsf{c}} M : \underline{C} \quad \Gamma \mid z : \underline{C} \vdash_{\mathsf{h}} K : \underline{D}}{\Gamma \vdash_{\mathsf{c}} K[\mathrm{op}_V^C(x.M)/z] = \mathrm{op}_V^D(x.K[M/z]) : \underline{D}} \; (\mathrm{op} : (x_i : I) \longrightarrow O)$$

Sound semantics: based on

- $p : \mathrm{Fam}(\mathrm{Set}) \longrightarrow \mathrm{Set}$ and $q : \mathrm{Fam}(\mathrm{Mod}(\mathcal{L}_{\mathcal{T}_{\text{eff}}}, \mathrm{Set})) \longrightarrow \mathrm{Set}$

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Example 1 (interactive I/O):

- read : $1 \longrightarrow \mathsf{Chr}$ $\qquad (\mathsf{Chr} \stackrel{\mathsf{def}}{=} 1 + \ldots + 1)$

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- $\diamond \vdash \mathsf{Loc}$

  $\ell : \mathsf{Loc} \vdash \mathsf{Val}$

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- get : $(\ell : \mathsf{Loc}) \longrightarrow \mathsf{Val}$

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- five equations (two of them branching on $\mathsf{isDec}_{\mathsf{Loc}}$)

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Handlers of algebraic effects

(for programming and extrinsic reasoning)

Handlers of alg. effects – for programming

Idea: Generalisation of exception handlers [Plotkin,Pretnar'09]

Handler = Algebra and Handling = Homomorphism

Usual term-level presentation:

$$\Gamma \vdash_c M \texttt{ handled with } \{\mathrm{op}_{x_v}(x_k) \mapsto N_{\mathrm{op}}\}_{\mathrm{op}\in\mathcal{T}_{\mathrm{eff}}} \texttt{ to } y : A \texttt{ in}_{\underline{C}} N_{\mathrm{ret}} : \underline{C}$$

satisfying

$$(\texttt{return } V) \texttt{ handled with } \{\ldots\}_{\mathrm{op}\in\mathcal{T}_{\mathrm{eff}}} \texttt{ to } y : A \texttt{ in } N_{\mathrm{ret}} = N_{\mathrm{ret}}[V/x]$$

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Typical use case for programming:

- write your programs using alg. ops. (e.g., `get` and `put`)
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Idea: Using a derived handle-into-values handling construct

$$M \texttt{ handled with } \{\mathrm{op}_{x_v}(x_k) \mapsto V_{\mathrm{op}}\}_{\mathrm{op} \in \mathcal{T}_{\mathrm{eff}}} \texttt{ to } y : A \texttt{ in}_B \; V_{\mathrm{ret}}$$

we can define natural predicates (essentially, dependent types)

$$\Gamma \vDash P : UFA \to \mathcal{U}$$

by

- equipping a universe $\mathcal{U}$ with an algebra for $\mathcal{T}_{\mathrm{eff}}$, and
- using the above handle-into-values construct to define P

Note 1: $P\,(\texttt{thunk}\,M)$ computes a proof obligation for M

Note 2: Formally, we work in an extension of eMLTT with

- a universe $\mathcal{U}$ closed under Nat, 1, 0, $+$, Σ, and Π
- a type-based treatment of handlers $\quad \underline{C} ::= \ldots \mid \langle A; \overrightarrow{V_{\mathrm{op}}}; \overrightarrow{W_{\mathrm{eq}}} \rangle$
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Example 1 (Evaluation Logic style modalities):

- Given a predicate $P : A \to \mathcal{U}$ on return values,

  we define a predicate $\Diamond P : UFA \to \mathcal{U}$ on I/O-computations as

$$\Diamond P \stackrel{\text{def}}{=} \lambda x{:}UFA\,.\,(\mathtt{force}\,x)\ \mathtt{handled\ with}\ \{\ldots\}_{\mathsf{op}\in\mathcal{T}_{\mathsf{IO}}}\ \mathtt{to}\ y{:}A\ \mathtt{in}_{\mathcal{U}}\ P\,y$$

  using the handler given by

$$V_{\mathsf{read}} \stackrel{\text{def}}{=} \lambda x{:}(\Sigma\, x_v{:}1\,.\,\mathsf{Chr}\to\mathcal{U})\,.\,\widehat{\Sigma}\, y{:}\mathsf{El}(\widehat{\mathsf{Chr}})\,.\,(\mathtt{snd}\ x)\ y$$

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Example 2 (Dijkstra's weakest precondition semantics):

- Given a postcondition on return values and final states

$$Q : A \to S \to \mathcal{U} \qquad (S \stackrel{\text{def}}{=} \Pi x : \mathrm{Loc}\,.\,\mathrm{Val})$$

we define a precondition for stateful comps. on initial states

$$\mathrm{wp}_Q : \mathit{UFA} \to S \to \mathcal{U}$$

by

i) handling the given comp. into a state-passing function using

$$V_{\mathrm{get}}, V_{\mathrm{put}} \quad \text{on} \quad S \to (\mathcal{U} \times S) \qquad \text{and} \qquad V_{\mathrm{ret}} \text{ ``}=\text{''} \; Q$$

ii) feeding in the initial state; and *iii*) projecting out $\mathcal{U}$

- **Theorem:** wp_Q satisfies expected properties of WPs, e.g.,

$$\mathrm{wp}_Q\,(\mathtt{thunk}\,(\mathtt{return}\,V)) \;=\; \lambda x_S : S\,.\,Q\;V\;x_S$$

$$\mathrm{wp}_Q\,(\mathtt{thunk}\,(\mathtt{put}_{\langle \ell, V\rangle}(M))) \;=\; \lambda x_S : S\,.\,\mathrm{wp}_Q\;(\mathtt{thunk}\,M)\;(x_S[\ell \mapsto V])$$

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$$\mathsf{wp}_Q : \mathit{UFA} \to S \to \mathcal{U}$$

by

i) handling the given comp. into a state-passing function using

$$V_{\mathsf{get}}, V_{\mathsf{put}} \quad \text{on} \quad S \to (\mathcal{U} \times S) \qquad \text{and} \qquad V_{\mathsf{ret}} \text{ “}=\text{”} \ Q$$

ii) feeding in the initial state; and *iii*) projecting out $\mathcal{U}$

- **Theorem:** wp_Q satisfies expected properties of WPs, e.g.,

$$\mathsf{wp}_Q\,(\mathtt{thunk}\,(\mathtt{return}\,V)) \quad = \quad \lambda x_S : S\,.\,Q\ V\ x_S$$

$$\mathsf{wp}_Q\,(\mathtt{thunk}\,(\mathtt{put}_{\langle \ell, V \rangle}(M))) \quad = \quad \lambda x_S : S\,.\,\mathsf{wp}_Q\ (\mathtt{thunk}\,M)\ (x_S[\ell \mapsto V])$$

Handlers of alg. effects – for reasoning

Example 2 (Dijkstra's weakest precondition semantics):

- Given a postcondition on return values and final states

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Handlers of alg. effects – for reasoning

Example 3 (Patterns of allowed I/O-effects):

- Assuming an inductive type Protocol, given by

$$\mathtt{e} : \mathsf{Protocol} \qquad \mathtt{r} : (\mathsf{Chr} \to \mathsf{Protocol}) \to \mathsf{Protocol}$$

$$\mathtt{w} : (\mathsf{Chr} \to \mathcal{U}) \to \mathsf{Protocol} \to \mathsf{Protocol}$$

and potentially also by $\wedge$, $\vee$, ...

- Then, given a protocol $\mathsf{Pr} : \mathsf{Protocol}$, we define

$$\underline{\mathsf{Pr}} : \mathit{UFA} \to \mathcal{U}$$

by handling the given comp. using

$$V_{\mathsf{read}}, V_{\mathsf{write}} \qquad \text{on} \qquad \mathsf{Protocol} \to \mathcal{U}$$

where

$$V_{\mathsf{read}}\ \langle \_ , V_{\mathsf{rk}} \rangle\ (\mathtt{r}\ \mathsf{Pr}') \stackrel{\text{def}}{=} \widehat{\Pi}\, x : \mathsf{El}(\widehat{\mathsf{Chr}})\,.\,(V_{\mathsf{rk}}\ x)\ (\mathsf{Pr}'\ x)$$

$$V_{\mathsf{write}}\ \langle V\,, V_{\mathsf{wk}} \rangle\ (\mathtt{w}\ P\ \mathsf{Pr}') \stackrel{\text{def}}{=} \widehat{\Sigma}\, x : \mathsf{El}(P\ V)\,.\,V_{\mathsf{wk}}\ \star\ \mathsf{Pr}'$$

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Conclusion

In work we told a mathematically natural story of combining

- dependent types and computational effects

In particular, we saw

- a clean core language of dependent types and comp. effects
- a natural category-theoretic semantics
- alg. effects and handlers, in particular, for reasoning using
  - Evaluation Logic style modalities
  - Dijkstra's weakest precondition semantics of state
  - patterns of allowed (I/O)-effects

Things to look at:

- type-dependency on computations (e.g., in seq. composition)
- more expressive comp. types (par. adjunctions, Dijkstra monads)

Other work: directed containers, F* and monotonic state, ...

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Thank you!

D. Ahman.
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