

A fibrational view on computational effects

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Outline

We investigate the combination of

- dependent types $(\Pi, \Sigma, V =_A W, \dots)$
- computational effects (state, I/O, probability, recursion, ...)

Two guiding problems

- effectful programs in types (e.g., read and write in types)
- types of effectful programs (e.g., of sequential composition)

Our goals

- tell a mathematically natural story
- use established math. techniques
- cover a wide range of comp. effects
- discover smth. interesting

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- use established math. techniques (fibrations and adjunctions)
- cover a wide range of comp. effects (alg. effects, continuations)
- discover smth. interesting (using handlers to reason about effects)

Effectful programs in types

(type-dependency in the presence of effects)

Effectful programs in types

Let's consider an example **dependent type**

$$\ell : (\text{List Chr}) \vdash \text{Sorted}(\ell) \stackrel{\text{def}}{=} \Sigma \ell' : (\text{List Chr}). (\text{len } \ell =_{\text{Nat}} \text{len } \ell' \times \dots)$$

which could be used to type the **dependent sorting function**

$$\text{sort} : \Pi \ell : (\text{List Chr}). \text{Sorted}(\ell)$$

Q: Should we allow situations such as $\text{Sorted}[\text{receive}(y.M)/\ell]$?

A1: In this talk, we say **not directly**

- types should only depend on static information about effects
- we allow dependency on effectful comps. via analysing **thunks**

A2: But we are also looking into the **direct** case

- type-dependency needs to be “homomorphic”, but not only so
- intuitively, lift $\text{Sorted}(\ell)$ to $\text{Sorted}^\dagger(c)$, where $c : T(\text{List Chr})$

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Effectful programs in types

Aim: Types should only depend on static info about effects

Solution: CBPV/EEC style distinction between vals. and comps.

- value types $\Gamma \vdash A$ (MLTT + thunks + ...)
- computation types $\Gamma \vdash \underline{C}$ (dep. CBPV/EEC)
- where Γ contains only value variables $x_1 : A_1, \dots, x_n : A_n$

Note: Could have also considered λ_{ML} and FGCBV

- building on CBPV/EEC gives a more general story
- especially for the treatment of sequential composition
- and also for integrating dependent- and effect-typing

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Assigning types to effectful programs

(e.g., sequential composition)

Assigning types to effectful programs

The problem: The standard typing rule for seq. composition

$$\frac{\Gamma \vdash M : F A \quad \Gamma, x:A \vdash N : \underline{C}}{\Gamma \vdash M \text{ to } x:A \text{ in } N : \underline{C}}$$

is not correct any more because x can appear free in the type

C

in the conclusion

Assigning types to effectful programs

Aim: To fix the typing rule of **sequential composition**

Option 1: We could restrict the free variables in \underline{C} : [Levy'04]

$$\frac{\Gamma \Vdash M : FA \quad \Gamma \vdash \underline{C} \quad \Gamma, x:A \Vdash N : \underline{C}}{\Gamma \Vdash M \text{ to } x:A \text{ in } N : \underline{C}}$$

But sometimes it is useful if \underline{C} can depend on x !

- if we consider

`fopen (return true, return false) to x:Bool in N`

- then it would be natural to let \underline{C} depend on x , e.g.,

$$x:\text{Bool} \vdash \underline{C}(x) \stackrel{\text{def}}{=} \text{if } x \text{ then "allow fread, fwrite, and fclose"} \\ \text{else "allow fopen"}$$

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Option 2: One could lift sequential composition to type level

$$\Gamma \Vdash M \text{ to } x:A \text{ in } N : M \text{ to } x:A \text{ in } \underline{C}$$

But then comp. types would be singleton-like?!

However, smth. like this is probably needed for the **direct** case.

Option 3: In the monadic metalanguage λ_{ML} , one could try

$$\frac{\Gamma \vdash M : T A \quad \Gamma, x:A \vdash N : T B(x)}{\Gamma \vdash M \text{ to } x:A \text{ in } N : T (\Sigma x:A. B)}$$

But what makes this a principled solution? Why is it correct?

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Option 4: We draw inspiration from algebraic effects

- and combine it with restricting \underline{C} in seq. comp. (**Option 1**)

E.g., consider the non-det. program $(\text{for } x:\text{Nat} \models N : \underline{C}(x))$

$$M \stackrel{\text{def}}{=} \text{choose}(\text{return } 4, \text{return } 2) \text{ to } x:\text{Nat} \text{ in } N$$

After tossing the coin, this program evaluates as either

$$N[4/x] : \underline{C}[4/x] \quad \text{or} \quad N[2/x] : \underline{C}[2/x]$$

Idea: M denotes an element of the coproduct of algebras

$$\underline{C}[4/x] + \underline{C}[2/x] \stackrel{\text{def}}{=} F\left(U(\underline{C}[4/x]) + U(\underline{C}[2/x])\right)_{/\equiv}$$

and thus we would like to type M at the type $\sum x:\text{Nat}. \underline{C}$

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Putting these ideas together

(eMLTT: a core dep.-typed language with comp. effects)

eMLTT – types

Value types: MLTT + **thunks** + ...

$A, B ::= \text{Nat} \mid 1 \mid 0 \mid \Pi x:A. B \mid \Sigma x:A. B \mid V =_A W \mid \underline{U} \underline{C} \mid \dots$

- $\underline{U} \underline{C}$ is the type of **thunked** (i.e., suspended) **computations**

Computation types: dep.-typed version of EEC's comp. types

$\underline{C}, \underline{D} ::= F A \mid \Pi x:A. \underline{C} \mid \Sigma x:A. \underline{C}$

- $F A$ is the type of computations returning values of type A
- $\Pi x:A. \underline{C}$ is the type of dependent effectful functions
 - generalises CBPV/EEC's comp. types $A \rightarrow \underline{C}$ and $\underline{C} \times \underline{D}$
- $\Sigma x:A. \underline{C}$ is the type of dep. pairs of values and effectful comps.
 - captures the intuition about seq. comp. and coprods. of algebras
 - generalises EEC's comp. types $!A \otimes \underline{C}$ and $\underline{C} \oplus \underline{D}$

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$$V, W ::= x \mid \text{zero} \mid \text{succ } V \mid \dots \mid \text{thunk } M \mid \dots$$

- equational theory based on *intensional* MLTT

Comp. terms: dep.-typed version of CBPV/EEC's comp. terms

$$\begin{array}{lcl} M, N ::= & \text{force } V & \\ & \text{return } V & \\ & M \text{ to } x:A \text{ in } N & \\ & \lambda x:A. M & \\ & MV & \\ & \langle V, M \rangle & \text{(comp. } \Sigma \text{ intro.)} \\ & M \text{ to } \langle x:A, z:\underline{C} \rangle \text{ in } K & \text{(comp. } \Sigma \text{ elim.)} \end{array}$$

But: Value and comp. terms alone do not suffice, as in EEC!

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eMLTT – terms

Note: We need to define K in such a way that the intended left-to-right evaluation order is preserved, e.g., consider

$$\Gamma \Vdash \langle V, M \rangle \text{ to } \langle x:A, z:\underline{C} \rangle \text{ in } K = K[V/x, M/z] : \underline{D}$$

Homomorphism terms: dep.-typed version of EEC's linear terms

$$\begin{array}{ll} K, L ::= & z \quad \text{(linear comp. vars.)} \\ & K \text{ to } x:A \text{ in } M \\ & \lambda x:A. K \\ & KV \\ & \langle V, K \rangle \quad \text{(comp. } \Sigma \text{ intro.)} \\ & K \text{ to } \langle x:A, z:\underline{C} \rangle \text{ in } L \quad \text{(comp. } \Sigma \text{ elim.)} \end{array}$$

Typing judgments:

- $\Gamma \Vdash V : A$
- $\Gamma \Vdash M : \underline{C}$
- $\Gamma \mid z:\underline{C} \Vdash K : \underline{D}$ (linear in z ; comp. bound to z happens first)

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eMLTT – typing sequential composition

We can then account for **type-dependency in seq. comp.** as

$$\frac{\Gamma \Vdash M : F A \quad \Gamma \vdash \Sigma y:A. \underline{C}(y) \quad \frac{\Gamma, x:A \Vdash N : \underline{C}(x)}{\Gamma, x:A \Vdash \langle x, N \rangle : \Sigma y:A. \underline{C}(y)}}{\Gamma \Vdash M \text{ to } x:A \text{ in } \langle x, N \rangle : \Sigma y:A. \underline{C}(y)}$$

The **seq. comp. rule for λ_{ML}** is justified by the type isomorphism

$$\frac{\Gamma \vdash A \quad \Gamma, x:A \vdash B(x)}{\Gamma \vdash U(\Sigma x:A. F(B)) \cong UF(\Sigma x:A. B) = T(\Sigma x:A. B)}$$

Categorical semantics of eMLTT

(fibrations + adjunctions)

Categorical semantics – MLTT part

We define **fibred adjunction models**

- **Theorem:** a sound and complete class of models for eMLTT given by: i) a split closed comprehension cat. \mathcal{P} with Nat , ...



- we define a partial interpretation fun. $\llbracket - \rrbracket$, that (if defined) maps:
 - a context Γ to an object $\llbracket \Gamma \rrbracket$ in \mathcal{B} , with $\llbracket \Gamma, x:A \rrbracket \stackrel{\text{def}}{=} \{\llbracket \Gamma; A \rrbracket\}$
 - a context Γ and a value type A to an object $\llbracket \Gamma; A \rrbracket$ in $\mathcal{V}_{\llbracket \Gamma \rrbracket}$
 - a context Γ and a value term V to $\llbracket \Gamma; V \rrbracket : 1_{\llbracket \Gamma \rrbracket} \longrightarrow \llbracket \Gamma; A \rrbracket$ in $\mathcal{V}_{\llbracket \Gamma \rrbracket}$

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- **Theorem:** a sound and complete class of models for eMLTT given by: i) a **split closed comprehension cat.** \mathcal{P} with Nat , ...

$$\begin{array}{c} \mathcal{V} \\ \left(\begin{array}{c} \swarrow \quad \uparrow \quad \searrow \\ \vdash \quad 1 \vdash \end{array} \right) \{-\} \\ \mathcal{B} \end{array}$$

- we define a **partial interpretation fun.** $\llbracket - \rrbracket$, that (if defined) maps:
 - a **context** Γ to an object $\llbracket \Gamma \rrbracket$ in \mathcal{B} , with $\llbracket \Gamma, x:A \rrbracket \stackrel{\text{def}}{=} \{\llbracket \Gamma; A \rrbracket\}$
 - a context Γ and a **value type** A to an object $\llbracket \Gamma; A \rrbracket$ in $\mathcal{V}_{\llbracket \Gamma \rrbracket}$
 - a context Γ and a **value term** V to $\llbracket \Gamma; V \rrbracket : 1_{\llbracket \Gamma \rrbracket} \longrightarrow A$ in $\mathcal{V}_{\llbracket \Gamma \rrbracket}$

Categorical semantics – MLTT part

We define **fibred adjunction models**

- **Theorem:** a sound and complete class of models for eMLTT given by: i) a **split closed comprehension cat.** \mathcal{P} with Nat, \dots

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 \left(\begin{array}{c} \lrcorner \uparrow \lrcorner \\ \downarrow 1 \downarrow \end{array} \right) \{-\} \\
 \mathcal{B}
 \end{array}$$

- the **display maps** $\pi_{[\Gamma;A]} : [\Gamma, x:A] \longrightarrow [\Gamma]$ in \mathcal{B} induce the **weakening functors** $\pi_{[\Gamma;A]}^* : \mathcal{V}_{[\Gamma]} \longrightarrow \mathcal{V}_{[\Gamma, x:A]}$, and
- the value Σ - and Π -types are interpreted as **adjoints**

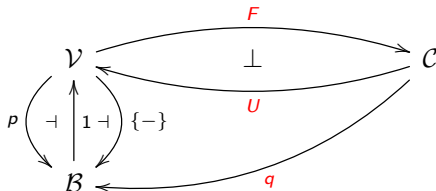
$$\Sigma_{[\Gamma;A]} \dashv \pi_{[\Gamma;A]}^* : \mathcal{V}_{[\Gamma]} \longrightarrow \mathcal{V}_{[\Gamma, x:A]} \quad (\text{such that } \Sigma \text{ is strong})$$

$$\pi_{[\Gamma;A]}^* \dashv \Pi_{[\Gamma;A]} : \mathcal{V}_{[\Gamma, x:A]} \longrightarrow \mathcal{V}_{[\Gamma]}$$

Categorical semantics – effects part

We define **fibred adjunction models**

- **Theorem:** a sound and complete class of models for eMLTT given by: ii) a **split fibration** q (with ...) and a **s. fib. adj.** $F \dashv U$



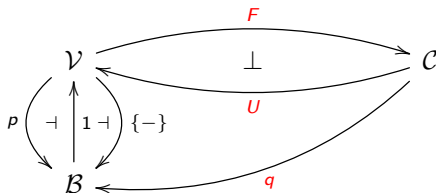
- we **extend** $\llbracket - \rrbracket$ so that (if defined) it maps:
 - a ctx. Γ and a **comp. type** \underline{C} to an object $\llbracket \Gamma; \underline{C} \rrbracket$ in $\mathcal{C}_{\llbracket \Gamma \rrbracket}$
 - a ctx. Γ and a **comp. term** M to $\llbracket \Gamma; M \rrbracket : 1_{\llbracket \Gamma \rrbracket} \longrightarrow U(\underline{C})$ in $\mathcal{V}_{\llbracket \Gamma \rrbracket}$
 - a ctx. Γ , a c. var. z , a c. type \underline{C} , and a **hom. term** K to

$$\llbracket \Gamma; z : \underline{C}; K \rrbracket : \llbracket \Gamma; \underline{C} \rrbracket \longrightarrow \underline{D} \text{ in } \mathcal{C}_{\llbracket \Gamma \rrbracket}$$

Categorical semantics – effects part

We define **fibred adjunction models**

- **Theorem:** a sound and complete class of models for eMLTT given by: ii) a **split fibration** q (with ...) and a **s. fib. adj.** $F \dashv U$



- we again have **weakening functors** $\pi_{[\Gamma;A]}^*: \mathcal{C}_{[\Gamma]} \longrightarrow \mathcal{C}_{[\Gamma, x:A]}$, and
- the comp. Σ - and Π -types are interpreted again as **adjoints**

$$\Sigma_{[\Gamma;A]} \dashv \pi_{[\Gamma;A]}^*: \mathcal{C}_{[\Gamma]} \longrightarrow \mathcal{C}_{[\Gamma, x:A]}$$

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Digression: dep. elimination of 0 and +

The coproduct type $A + B$:

[Jacobs'99]

- require $p : \mathcal{V} \longrightarrow \mathcal{B}$ to have split fibred coproducts $A +_X B$, and
- $\langle \{\text{inl}_A\}^*, \{\text{inr}_B\}^* \rangle : \mathcal{V}_{\{A +_X B\}} \longrightarrow \mathcal{V}_{\{A\}} \times \mathcal{V}_{\{B\}}$ to be fully-faith.
- allows one to interpret dependent case analysis, i.e.,

$$\begin{aligned} \mathcal{V}_{\{A\}} \left(1_{\{A\}}, \{\text{inl}_A\}^*(C) \right) \times \mathcal{V}_{\{B\}} \left(1_{\{B\}}, \{\text{inr}_B\}^*(C) \right) \\ \cong \\ \mathcal{V}_{\{A +_X B\}} \left(1_{\{A +_X B\}}, C \right) \end{aligned}$$

provides semantics for

$$\frac{\Gamma, y_1 : A \Vdash W_1 : C[\text{inl}_A y_1/x] \quad \Gamma, y_2 : B \Vdash W_2 : C[\text{inr}_B y_2/x]}{\Gamma, x : A + B \Vdash \text{case } x \text{ of } (\text{inl}(y_1) \mapsto W_1, \text{inr}(y_2) \mapsto W_2) : C}$$

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A generalisation:

[Ahman'17]

- **Idea:** **fully-faith.** for cocones $A \longrightarrow A \otimes_X B \longleftarrow B$ is enough, and we can generalise this to all **split fibred colimits**

- **Theorem:**

- if for every object $X \in \mathcal{B}$ and diagram $J : \mathcal{D} \longrightarrow \mathcal{V}_X$ there exists a cocone $\underline{\text{in}}^J : J \longrightarrow \Delta(\underline{\text{colim}}(J))$ in \mathcal{V}_X ,
- such that $f^*(\underline{\text{in}}_D^J) = \underline{\text{in}}_D^{f^* \circ J}$, for any $f : X \longrightarrow Y$, and such that the unique mediating functor

$$\langle \{\underline{\text{in}}_D^J\}_{D \in \mathcal{D}}^* \rangle : \mathcal{V}_{\{\underline{\text{colim}}(J)\}} \longrightarrow \lim(\hat{J})$$

is fully-faithful (for $\hat{J} : \mathcal{D}^{op} \longrightarrow \text{Cat}$, where $\hat{J}(D) = \mathcal{V}_{\{J(D)\}}$),

- then p has split fibred colimits of shape \mathcal{D} , and p supports dependent elimination for them (analogously to $+_X$)

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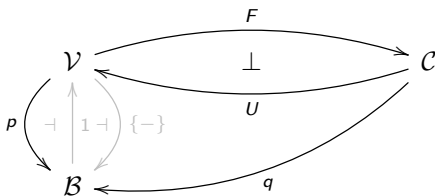
- if for every object $X \in \mathcal{B}$ and diagram $J : \mathcal{D} \longrightarrow \mathcal{V}_X$ there exists a **cocone** $\underline{\text{in}}^J : J \longrightarrow \Delta(\underline{\text{colim}}(J))$ in \mathcal{V}_X ,
- such that $f^*(\underline{\text{in}}^J_D) = \underline{\text{in}}^{f^* \circ J}_D$, for any $f : X \longrightarrow Y$, and such that the **unique mediating functor**

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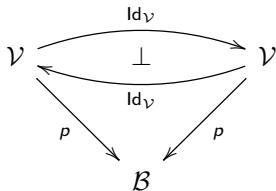
- then p has **split fibred colimits** of shape \mathcal{D} , and p supports **dependent elimination** for them (analogously to $+_X$)

Examples of fibred adjunction models



Examples of fibred adjunction models

Example 1 (identity adjunctions):

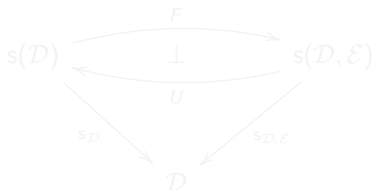


Note: sound model as long as we haven't included any effects

Examples of fibred adjunction models

Example 2 (simple models from Egger et al.'s EEC):

- given an **adjunction** $F_{\text{EEC}} \dashv U_{\text{EEC}} : \mathcal{E} \longrightarrow \mathcal{D}$, such that
 - \mathcal{D} is **Cartesian closed** (with Nat, \dots), and
 - \mathcal{E} and $F_{\text{EEC}} \dashv U_{\text{EEC}}$ are **\mathcal{D} -enriched**, and
 - \mathcal{E} has all **\mathcal{D} -tensors** ($A \otimes \underline{C}$) and **\mathcal{D} -cotensors** ($A \rightrightarrows \underline{C}$)
- we use simple fibration $s_{\mathcal{D}}$ and simpl. \mathcal{D} -enrich. fibration $s_{\mathcal{D}, \mathcal{E}}$



$$F(X, A) \stackrel{\text{def}}{=} (X, F_{\text{EEC}}(A))$$

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$$s(\mathcal{D}): \quad (f, g) : (X, A) \longrightarrow (Y, B) \quad \text{where} \quad f : X \longrightarrow Y \quad g : X \times A \longrightarrow B$$

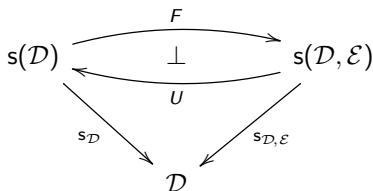
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Note: this model doesn't support any real type-dependency

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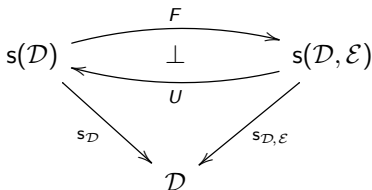
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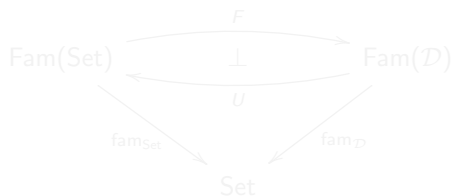
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Examples of fibred adjunction models

Example 3 (families fibrations):

- given an adjunction $F_{\mathcal{D}} \dashv U_{\mathcal{D}} : \mathcal{D} \longrightarrow \mathbf{Set}$, such that
 - \mathcal{D} has set-indexed products and set-indexed coproducts
- such adjunctions arise from
 - EM-cats. ($\mathcal{D} \stackrel{\text{def}}{=} \mathbf{Set}^{\mathbf{T}}$) and Law. ths. ($\mathcal{D} \stackrel{\text{def}}{=} \mathbf{Mod}(\mathcal{L}, \mathbf{Set})$)
 - resolutions of $S \Rightarrow (-) \times S$ and $((-) \Rightarrow R) \Rightarrow R$
- we use families fibrations $\text{fam}_{\mathbf{Set}}$ and $\text{fam}_{\mathcal{D}}$



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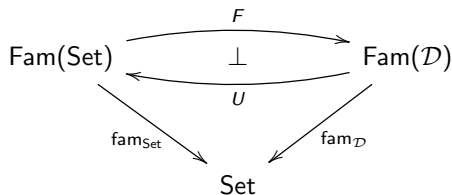
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Examples of fibred adjunction models

Example 4 (continuous families for $\mu x: U\underline{C}. M$):

- given a **CPO-enriched monad** \mathbf{T} on CPO, such that
 - \mathbf{T} supports least zero-ary alg. op. ($\perp_A : 1 \rightarrow TA$), and
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- such \mathbf{T} arise from **discrete CPO-enriched countable Law. ths.**
- we use continuous families fibrations cfam_{CPO} and $\text{cfam}_{\text{CPO}^{\mathbf{T}}}$



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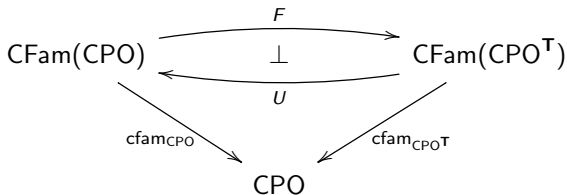
$\text{CFam}(\text{CPO})$: (X, A) where $X \in \text{CPO}$ $A : X \rightarrow \text{CPO}^{\text{EP}}$ an ω -cont. fun.

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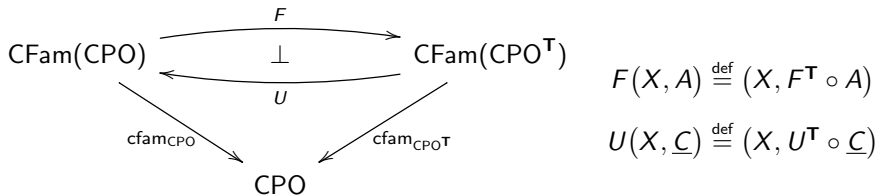
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Examples of fibred adjunction models

Example 5 (EM-resolutions of split fibred monads):

- given a **split fibred monad** $\mathbf{T} = (T, \eta, \mu)$ on p , i.e.,

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{T} & \mathcal{V} \\ & \searrow p & \swarrow p \\ & \mathcal{B} & \end{array} \quad \text{and} \quad p(\eta_A) = \text{id}_{p(A)} \quad p(\mu_A) = \text{id}_{p(A)}$$

- we consider models based on the **EM-resolution** of \mathbf{T}

$$\begin{array}{ccc} \mathcal{V} & \begin{array}{c} \xrightarrow{F^T} \\ \perp \\ \xleftarrow{U^T} \end{array} & \mathcal{V}^T \\ & \searrow p & \swarrow p^T \\ & \mathcal{B} & \end{array}$$

- and show that **three familiar results** hold for this situation

Examples of fibred adjunction models

Example 5 (EM-resolutions of split fibred monads):

- **Theorem 1:** if p supports Π -types, then $p^{\mathbf{T}}$ also supports Π -types

$$\Pi_A^{\mathbf{T}}(B, \beta) \stackrel{\text{def}}{=} (\Pi_A(B), \beta_{\Pi_A^{\mathbf{T}}})$$

- **Prop.:** every \mathbf{T} on a split closed comp. cat. has a dep. strength

$$\sigma_A : \Sigma_A \circ T \longrightarrow T \circ \Sigma_A \quad (A \in \mathcal{V})$$

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Algebraic effects

Algebraic effects – ops. and eqs.

Fibred effect theories \mathcal{T}_{eff} :

- signatures of **dep. typed operation symbols**

$$\frac{\cdot \vdash I \quad x_i : I \vdash O \quad I \text{ and } O \text{ are pure value types}}{\text{op} : (x_i : I) \longrightarrow O}$$

- equipped with **equations** on derivable effect terms

In eMLTT:

$$M ::= \dots \mid \text{op}_{\underline{V}}^{\underline{C}}(x.M)$$

General algebraicity equations (in addition to eff. th. eqs.):

$$\frac{\Gamma \Vdash V : I \quad \Gamma, x : O[V/x_i] \Vdash M : \underline{C} \quad \Gamma \mid z : \underline{C} \Vdash K : \underline{D}}{\Gamma \Vdash K[\text{op}_{\underline{V}}^{\underline{C}}(x.M)/z] = \text{op}_{\underline{V}}^{\underline{D}}(x.K[M/z]) : \underline{D}} \quad (\text{op} : (x_i : I) \longrightarrow O)$$

Sound semantics: based on

- $p : \text{Fam}(\text{Set}) \longrightarrow \text{Set}$ and $q : \text{Fam}(\text{Mod}(\mathcal{L}_{\mathcal{T}_{\text{eff}}}, \text{Set})) \longrightarrow \text{Set}$

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Algebraic effects – examples

Example 1 (interactive I/O):

- $\text{read} : 1 \longrightarrow \text{Chr}$
 $\text{write} : \text{Chr} \longrightarrow 1$
- no equations

$$(\text{Chr} \stackrel{\text{def}}{=} 1 + \dots + 1)$$

Example 2 (global state with location-dependent store type):

- $\diamond \vdash \text{Loc}$
 $\ell : \text{Loc} \vdash \text{Val}$
 $\diamond \Vdash \text{isDec}_{\text{Loc}} : \prod \ell : \text{Loc} . \prod \ell' : \text{Loc} . (\ell =_{\text{Loc}} \ell') + (\ell =_{\text{Loc}} \ell' \rightarrow 0)$
- $\text{get} : (\ell : \text{Loc}) \longrightarrow \text{Val}$
 $\text{put} : (\sum \ell : \text{Loc} . \text{Val}) \longrightarrow 1$
- five equations (two of them branching on $\text{isDec}_{\text{Loc}}$)

Example 3 (dep. typed update monads $T X \stackrel{\text{def}}{=} \prod_{s:S} . P s \times X$)

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Handlers of algebraic effects

(for programming and extrinsic reasoning)

Handlers of alg. effects – for programming

Idea: Generalisation of exception handlers [Plotkin, Pretnar'09]

Handler = Algebra and Handling = Homomorphism

Usual term-level presentation:

$\Gamma \models M$ handled with $\{\text{op}_{x_v}(x_k) \mapsto N_{\text{op}}\}_{\text{op} \in \mathcal{T}_{\text{eff}}}$ to $y:A$ in \underline{C} $N_{\text{ret}} : \underline{C}$

satisfying

$(\text{return } V)$ handled with $\{\dots\}_{\text{op} \in \mathcal{T}_{\text{eff}}}$ to $y:A$ in $N_{\text{ret}} = N_{\text{ret}}[V/x]$

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Typical use case for programming:

- write your programs using alg. ops. (e.g., get and put)
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Handlers of alg. effects – for reasoning

Idea: Using a derived handle-into-values handling construct

M handled with $\{\text{op}_{x_v}(x_k) \mapsto V_{\text{op}}\}_{\text{op} \in \mathcal{T}_{\text{eff}}}$ to $y:A \text{ in}_B V_{\text{ret}}$

we can define natural predicates (essentially, dependent types)

$$\Gamma \Vdash P : UFA \rightarrow \mathcal{U}$$

by

- equipping a universe \mathcal{U} with an algebra for \mathcal{T}_{eff} , and
- using the above handle-into-values construct to define P

Note 1: $P(\text{thunk } M)$ computes a proof obligation for M

Note 2: Formally, we work in an extension of eMLTT with

- a universe \mathcal{U} closed under Nat , 1 , 0 , $+$, Σ , and Π
- a type-based treatment of handlers $\underline{C} ::= \dots \mid \langle A; \overrightarrow{V_{\text{op}}}; \overrightarrow{W_{\text{eq}}} \rangle$
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Example 1 (Evaluation Logic style modalities):

- Given a predicate $P : A \rightarrow \mathcal{U}$ on return values,
we define a predicate $\Diamond P : UFA \rightarrow \mathcal{U}$ on I/O-computations as

$$\Diamond P \stackrel{\text{def}}{=} \lambda x : UFA. (\text{force } x) \text{ handled with } \{\dots\}_{\text{op} \in \mathcal{T}_{\text{IO}}} \text{ to } y : A \text{ in } P y$$

using the handler given by

$$V_{\text{read}} \stackrel{\text{def}}{=} \lambda x : (\Sigma x_v : 1. \text{Chr} \rightarrow \mathcal{U}). \widehat{\Sigma} y : \text{El}(\widehat{\text{Chr}}). (\text{snd } x) y$$

$$V_{\text{write}} \stackrel{\text{def}}{=} \lambda x : (\Sigma x_v : \text{Chr}. 1 \rightarrow \mathcal{U}). (\text{snd } x) \star$$

- $\Diamond P$ corresponds to Evaluation Logic's possibility modality

$$\Diamond P (\text{think}(\text{read}(x.\text{write}_{e'}(\text{return } V)))) = \widehat{\Sigma} x : \text{El}(\widehat{\text{Chr}}). P V$$

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Example 2 (Dijkstra's weakest precondition semantics):

- Given a postcondition on return values and final states

$$Q : A \rightarrow S \rightarrow \mathcal{U} \quad (S \stackrel{\text{def}}{=} \prod x:\text{Loc}. \text{Val})$$

we define a precondition for stateful comps. on initial states

$$\text{wp}_Q : \text{UFA} \rightarrow S \rightarrow \mathcal{U}$$

by

- i) handling the given comp. into a state-passing function using

$$V_{\text{get}}, V_{\text{put}} \text{ on } S \rightarrow (\mathcal{U} \times S) \quad \text{and} \quad V_{\text{ret}} \text{ " = " } Q$$

- ii) feeding in the initial state; and iii) projecting out \mathcal{U}

- Theorem:** wp_Q satisfies expected properties of WPs, e.g.,

$$\text{wp}_Q (\text{thunk}(\text{return } V)) = \lambda x_S : S. Q \ V \ x_S$$

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Example 3 (Patterns of allowed I/O-effects):

- Assuming an inductive type **Protocol**, given by

$$e : \text{Protocol} \quad r : (\text{Chr} \rightarrow \text{Protocol}) \rightarrow \text{Protocol}$$

$$w : (\text{Chr} \rightarrow \mathcal{U}) \rightarrow \text{Protocol} \rightarrow \text{Protocol}$$

and potentially also by \wedge, \vee, \dots

- Then, we define the predicate (rel. between comps. and protocols)

$$\text{Allowed} : \text{UFA} \rightarrow \text{Protocol} \rightarrow \mathcal{U}$$

by handling the given comp. using

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where

$$V_{\text{read}} \langle -, V_{\text{rk}} \rangle (\mathbf{r} \text{ Pr}') \stackrel{\text{def}}{=} \widehat{\Pi} x : \text{El}(\widehat{\text{Chr}}) . (V_{\text{rk}} x) (\text{Pr}' x)$$

$$V_{\text{write}} \langle V, V_{\text{wk}} \rangle (\mathbf{w} \text{ } P \text{ Pr}') \stackrel{\text{def}}{=} \widehat{\Sigma} x : \text{El}(P V) . V_{\text{wk}} \star \text{Pr}'$$

$$\text{—} \stackrel{\text{def}}{=} \widehat{0}$$

Conclusion

In work we told a mathematically natural story of combining

- **dependent types** and **computational effects**

In particular, we saw

- a clean **core language** of dependent types and comp. effects
- a natural category-theoretic **semantics**
- **alg. effects** and **handlers**, in particular, for **reasoning** using
 - Evaluation Logic style modalities
 - Dijkstra's weakest precondition semantics of state
 - patterns of allowed (I/O)-effects

Things to look at:

- type-dependency on computations (e.g., in seq. composition)
- more expressive comp. types (par. adjunctions, Dijkstra monads)

Other work: directed containers, F^* and monotonic state, ...

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Thank you!

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