Handling fibred algebraic effects

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We study algebraic computational effects and their handlers in the dependently typed setting. We describe computational effects using a generalisation of Plotkin and Pretnar's effect theories, whose dependently typed operations allow us to capture precise notions of computation, e.g., state with location-dependent store types and dependently typed update monads. Our treatment of handlers is based on an observation that their conventional term-level definition leads to unsound program equivalences being derivable in languages that include a notion of homomorphism. We solve this problem by giving handlers a novel type-based treatment via a new computation type, the user-defined algebra type, which pairs a value type (the carrier) with a family of value terms (the operations), based on Plotkin and Pretnar's insight that handlers denote algebras for a given algebraic theory. The conventional presentation of handlers can then be routinely derived from our type-based treatment. We also demonstrate that our treatment of handlers provides a useful mechanism for reasoning about effectful computations.

CCS Concepts: •**Software and its engineering** → *Functional languages;* •**Theory of computation** → *Type theory; Control primitives; Functional constructs; Type structures; Program specifications; Denotational semantics; Categorical semantics;*

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1 INTRODUCTION

An important feature of many widely-used programming languages is their support for computational effects, e.g., raising exceptions, accessing memory, performing I/O, etc., which allows programmers to write more efficient and conceptually clearer programs. Therefore, if dependently typed languages are to live up to their promise of providing a lightweight means for integrating formal verification and practical programming, we must first understand how to properly account for computational effects in such languages. While there already exists a range of work on combining these two fields (Ahman et al. 2016, 2017; Brady 2013; Casinghino 2014; Hancock and Setzer 2000; McBride 2011; Nanevski et al. 2008; Pédrot and Tabareau 2017; Pitts et al. 2015), there is still a gap between the rigorous and comprehensive understanding we have of computational effects in the simply typed setting, and what we know about them in the presence of dependent types. For example, in the above-mentioned works, either the mathematical foundations of the languages developed are not settled, the available effects are limited, or they lack a systematic treatment of (equational) effect specification. In this paper, we contribute to the intersection of these two fields by studying algebraic effects and their handlers in the dependently typed setting.

Algebraic effects form a wide class of computational effects that lend themselves to specification using operations and equations; examples include exceptions, state, input-output, nondeterminism, probability, etc. Their study originated with the pioneering work of Plotkin and Power (2001, 2002); they have since been successfully applied to, e.g., modularly combining effects (Hyland et al. 2006) and effect-dependent program optimisations (Kammar and Plotkin 2012). A key insight of Plotkin and Power was that most of Moggi's monads (Moggi 1989, 1991) are determined by algebraic effects, with the notable exception of continuations, which are not algebraic.

A significant role in the recent rise of interest in algebraic effects can be attributed to their *handlers*, as introduced by Plotkin and Pretnar (2013). These are a generalisation of exception handlers, based on the idea that handlers denote algebras for the given algebraic theory and the handling construct denotes the homomorphism

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induced by the universal property of the free algebra. From a programming languages perspective, a handler

$$\{\mathsf{op}_x(x') \mapsto N_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathsf{eff}}}$$

provides redefinitions of the algebraic operations in the signature $\mathcal{S}_{\mathsf{eff}}$ and the handling construct

$$M \;\mathtt{handled}\;\mathtt{with}\;\{\mathsf{op}_x(x') \mapsto N_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathsf{eff}}}\;\mathtt{to}\;y{:}A\;\mathtt{in}\;N_{\mathsf{ret}}$$

then recursively traverses the given program M, replacing each algebraic operation op with the corresponding user-defined term N_{op}, as illustrated by the following β-equation:

$$\begin{aligned}&\Gamma \vdash (\mathsf{op}_V^{FA}(y'.M))\;\mathtt{handled}\;\mathtt{with}\;\{\mathsf{op}_x(x') \mapsto N_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathsf{eff}}}\;\mathtt{to}\;y{:}A\;\mathtt{in}\;N_{\mathsf{ret}}\\ &\quad = N_{\mathsf{op}}[V/x][\lambda y'{:}O[V/x].\mathtt{thunk}\;H/x'] : \underline{C}\end{aligned}$$

where

$$H \stackrel{\mathsf{def}}{=} M\;\mathtt{handled}\;\mathtt{with}\;\{\mathsf{op}_x(x') \mapsto N_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathsf{eff}}}\;\mathtt{to}\;y{:}A\;\mathtt{in}\;N_{\mathsf{ret}}$$

Plotkin and Pretnar (2013, §3) also showed that handlers can be used to neatly implement timeouts, rollbacks, stream redirection, etc. More recently, handlers have gained popularity as a practical and modular programming language abstraction, allowing programmers to write their programs generically in terms of algebraic operations, and then use handlers to modularly provide different fit-for-purpose implementations for these programs. A prototypical example of this style of programming involves implementing the state operations (`get` and `put`) using the natural representation of stateful programs as functions $\mathsf{St} \to A \times \mathsf{St}$. In order to support this style of programming, existing languages have been extended with algebraic effects and their handlers (Hillerström and Lindley 2016; Kammar et al. 2013; Leijen 2017), and new languages have been built around them (Bauer and Pretnar 2015; Lindley et al. 2017). Algebraic effects and their handlers are also central to the recent extension of OCaml with shared memory multicore parallelism, available at https://github.com/ocamllabs/ocaml-multicore/.

Contributions. Our *key contribution* is an observation that the conventional term-level of definition of handlers leads to unsound program equivalences being derivable in languages that include a notion of homomorphism (§4.1). Our *other contributions* include: i) a dependently typed generalisation of Plotkin and Pretnar's effect theories (§3.1); ii) demonstrating that these theories can be used to capture precise notions of computation such as dependently typed update monads (§3.2); iii) introducing a new computation type, the user-defined algebra type, so as to give a type-based treatment of handlers and solve the problem with unsound program equivalences (§4.2); iv) showing how to derive the conventional term-level definition of handlers from our type-based treatment (§4.3); v) demonstrating that such handlers provide a useful mechanism for reasoning about effectful computations (§7); and vi) equipping the resulting dependently typed language with a sound fibrational denotational semantics (§8).

2 EMLTT: AN EFFECTFUL DEPENDENT TYPE THEORY

We begin with an overview of the language we use as a basis for studying algebraic effects and their handlers in the dependently typed setting, namely, the effectful dependently typed language proposed by Ahman et al. (2016). This language is a natural extension of Martin-Löf's intensional type theory (MLTT) with computational effects, making a clear distinction between values and computations, both at the level of types and terms, analogously to Levy's Call-By-Push-Value (CBPV) (Levy 2004) and Egger et al.'s Enriched Effect Calculus (EEC) (Egger et al. 2014). In fact, we base our work on a minor extension of Ahman et al.'s dependently typed language, as discussed later in this section; we refer to this extended language as EMLTT (for effectful Martin-Löf's type theory).

As usual for dependently typed languages, the sets of EMLTT (value and computation) types and terms are defined mutually inductively. First, one assumes two countable sets of *value variables* $x, y, \ldots$ and *computation variables* $z, \ldots$. Next, the grammar of EMLTT *value types* $A, B, \ldots$ and *computation types* $\underline{C}, \underline{D}, \ldots$ is given by

$$A ::= \mathsf{Nat} \mid 1 \mid 0 \mid A + B \mid \Sigma x{:}A.B \mid \Pi x{:}A.B \mid V =_A W \mid U\underline{C} \mid \underline{C} \multimap \underline{D} \qquad \underline{C} ::= FA \mid \Sigma x{:}A.\underline{C} \mid \Pi x{:}A.\underline{C}$$

As standard, we write $A \times B$ and $A \to B$ for $\Sigma x{:}A.B$ and $\Pi x{:}A.B$ when x is not free in B, and similarly for the computational Σ- and Π-types. As in Ahman et al. (2016), we omit general inductive types and use the type of natural numbers as a representative example. However, compared to op. cit., we include the empty type 0, the sum type $A + B$, and the homomorphic function type $\underline{C} \multimap \underline{D}$. We include the first two as to specify signatures of algebraic effects (see §3.2); and the latter as it is useful for writing libraries of effectful code without excessive thunking and forcing, and because it enables us to eliminate values into homomorphism terms, as discussed below. Finally, we note that FA is the type of possibly effectful computations that return values of type A.

Next, the grammar of EMLTT *value terms* $V, W, \ldots$ is given by

$$\begin{aligned}
V \ ::= \ & x \mid \star \\
\mid \ & \mathtt{zero} \mid \mathtt{succ}\ V \mid \mathtt{nat\text{-}elim}_{x.A}(V_z, y_1.y_2.V_s, V) \\
\mid \ & \mathtt{case}\ V\ \mathtt{of}_{x.A}\ () \\
\mid \ & \mathtt{inl}_{A+B}\ V \mid \mathtt{inr}_{A+B}\ V \mid \mathtt{case}\ V\ \mathtt{of}_{x.B}\ (\mathtt{inl}(y_1{:}A_1) \mapsto W_1, \mathtt{inr}(y_2{:}A_2) \mapsto W_2) \\
\mid \ & \langle V, W \rangle_{(x:A).B} \mid \mathtt{pm}\ V\ \mathtt{as}\ (x_1{:}A_1, x_2{:}A_2)\ \mathtt{in}_{y.B}\ W \\
\mid \ & \lambda x{:}A.V \mid V(W)_{(x:A).B} \\
\mid \ & \mathtt{refl}\ V \mid \mathtt{eq\text{-}elim}_A(x_1.x_2.x_3.B, y.W, V_1, V_2, V_p) \\
\mid \ & \mathtt{thunk}\ M \\
\mid \ & \lambda z{:}\underline{C}.K
\end{aligned}$$

Observe that in addition to the introduction and elimination forms for the types EMLTT inherits from MLTT, the value terms of EMLTT also include thunks of effectful computations and homomorphic lambda abstractions.

Regarding effectful programs, EMLTT makes a further distinction between *computation terms* $M, N, \ldots$ and *homomorphism terms* $K, L, \ldots$. The latter are necessary for correctly defining the elimination form (computational pattern-matching) for the computational Σ-type $\Sigma x{:}A.\underline{C}$. The grammar of these two kinds of terms is given by

$$\begin{aligned}
M \ ::= \ & \mathtt{return}\ V \\
\mid \ & M\ \mathtt{to}\ x{:}A\ \mathtt{in}_{\underline{C}}\ N \\
\mid \ & \langle V, M \rangle_{(x:A).\underline{C}} \mid M\ \mathtt{to}\ (x{:}A, z{:}\underline{C})\ \mathtt{in}_{\underline{D}}\ K \\
\mid \ & \lambda x{:}A.M \mid M(V)_{(x:A).\underline{C}} \\
\mid \ & V(M)_{\underline{C},\underline{D}} \\
\mid \ & \mathtt{force}_{\underline{C}}\ V
\end{aligned}
\qquad
\begin{aligned}
K \ ::= \ & z \\
\mid \ & K\ \mathtt{to}\ x{:}A\ \mathtt{in}_{\underline{C}}\ M \\
\mid \ & \langle V, K \rangle_{(x:A).\underline{C}} \mid K\ \mathtt{to}\ (x{:}A, z{:}\underline{C})\ \mathtt{in}_{\underline{D}}\ L \\
\mid \ & \lambda x{:}A.K \mid K(V)_{(x:A).\underline{C}} \\
\mid \ & V(K)_{\underline{C},\underline{D}}
\end{aligned}$$

Computation terms include standard combinators for effectful programming, such as the sequential composition of M and N, written as $M\ \mathtt{to}\ x{:}A\ \mathtt{in}_{\underline{C}}\ N$. They also include introduction and elimination forms for computational Σ- and Π-types, forcing of thunked computations, and homomorphic function applications. Homomorphism terms are similar, but also include computation variables z, which have to be used linearly and in a way that ensures that the computation bound to z "happens first" in the term containing it. This restriction on the use of z guarantees that every K denotes an algebra homomorphism in the categorical models we study in §8.

As one can use thunking and forcing (resp. the homomorphic function type) to derive elimination forms for value types into computation (resp. homomorphism) terms, these elimination forms are not included primitively. For example, one can eliminate natural numbers into computation terms as follows:

$$\mathtt{nat\text{-}elim}_{x.\underline{C}}(M_z, y_1.y_2.M_s, V) \stackrel{\text{def}}{=} \mathtt{force}_{\underline{C}[V/x]}\left(\mathtt{nat\text{-}elim}_{x.U\underline{C}}(\mathtt{thunk}\ M_z, y_1.y_2.\mathtt{thunk}\ M_s, V)\right)$$

and into homomorphism terms as follows:

$$\mathtt{nat\text{-}elim}_{\underline{C}}(K_z, y.K_s, V) \stackrel{\text{def}}{=} \left(\mathtt{nat\text{-}elim}_{x_1.\underline{C} \multimap \underline{C}}(\lambda z{:}\underline{C}.K_z, y.x_2.\lambda z{:}\underline{C}.K_s[x_2\ z/z], V)\right)\ z$$

Value types: **Computation types:**

$$\frac{\vdash \Gamma}{\Gamma \vdash \mathsf{Nat}} \quad \frac{\Gamma \vdash V : A \quad \Gamma \vdash W : A}{\Gamma \vdash V =_A W} \quad \frac{\Gamma \vdash \underline{C}}{\Gamma \vdash U\underline{C}} \quad \frac{\Gamma \vdash \underline{C} \quad \Gamma \vdash \underline{D}}{\Gamma \vdash \underline{C} \multimap \underline{D}} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash FA} \quad \frac{\Gamma, x{:}A \vdash \underline{C}}{\Gamma \vdash \Sigma x{:}A.\underline{C}} \quad \frac{\Gamma, x{:}A \vdash \underline{C}}{\Gamma \vdash \Pi x{:}A.\underline{C}}$$

Value terms:

$$\frac{\vdash \Gamma \quad x{:}A \in \Gamma}{\Gamma \vdash x : A} \quad \frac{\vdash \Gamma}{\Gamma \vdash \mathsf{zero} : \mathsf{Nat}} \quad \frac{\Gamma \vdash V : \mathsf{Nat}}{\Gamma \vdash \mathsf{succ}\ V : \mathsf{Nat}} \quad \frac{\begin{array}{c}\Gamma, x{:}\mathsf{Nat} \vdash A \quad \Gamma \vdash V : \mathsf{Nat} \quad \Gamma \vdash V_z : A[\mathsf{zero}/x] \\ \Gamma, y_1{:}\mathsf{Nat}, y_2{:}A[y_1/x] \vdash V_s : A[\mathsf{succ}\ y_1/x]\end{array}}{\Gamma \vdash \mathsf{nat\text{-}elim}_{x.A}(V_z, y_1.y_2.V_s, V) : A[V/x]}$$

$$\frac{\Gamma, x{:}A \vdash V : B}{\Gamma \vdash \lambda x{:}A.V : \Pi x{:}A.B} \quad \frac{\Gamma \vdash V : \Pi x{:}A.B \quad \Gamma, x{:}A \vdash B \quad \Gamma \vdash W : A}{\Gamma \vdash V(W)_{(x:A).B} : B[W/x]} \quad \frac{\Gamma \vdash M : \underline{C}}{\Gamma \vdash \mathsf{thunk}\ M : U\underline{C}} \quad \frac{\Gamma \mid z{:}\underline{C} \vdash K : \underline{D}}{\Gamma \vdash \lambda z{:}\underline{C}.K : \underline{C} \multimap \underline{D}}$$

Computation terms:

$$\frac{\Gamma \vdash V : A}{\Gamma \vdash \mathsf{return}\ V : FA} \quad \frac{\Gamma \vdash M : FA \quad \Gamma \vdash \underline{C} \quad \Gamma, x{:}A \vdash N : \underline{C}}{\Gamma \vdash M\ \mathsf{to}\ x{:}A\ \mathsf{in}_{\underline{C}}\ N : \underline{C}} \quad \frac{\Gamma \vdash V : U\underline{C}}{\Gamma \vdash \mathsf{force}_{\underline{C}}\ V : \underline{C}} \quad \frac{\Gamma \vdash V : \underline{C} \multimap \underline{D} \quad \Gamma \vdash M : \underline{C}}{\Gamma \vdash V(M)_{\underline{C},\underline{D}} : \underline{D}}$$

Homomorphism terms:

$$\frac{\Gamma \vdash \underline{C}}{\Gamma \mid z{:}\underline{C} \vdash z : \underline{C}} \quad \frac{\Gamma \mid z{:}\underline{C} \vdash K : FA \quad \Gamma \vdash \underline{D} \quad \Gamma, x{:}A \vdash M : \underline{D}}{\Gamma \mid z{:}\underline{C} \vdash K\ \mathsf{to}\ x{:}A\ \mathsf{in}_{\underline{D}}\ M : \underline{D}} \quad \frac{\Gamma \vdash V : A \quad \Gamma, x{:}A \vdash \underline{D} \quad \Gamma \mid z{:}\underline{C} \vdash K : \underline{D}[V/x]}{\Gamma \mid z{:}\underline{C} \vdash \langle V, K\rangle_{(x:A).\underline{D}} : \Sigma x{:}A.\underline{D}}$$

$$\frac{\Gamma \mid z_1{:}\underline{C} \vdash K : \Sigma x{:}A.\underline{D}_1 \quad \Gamma \vdash \underline{D}_2 \quad \Gamma, x{:}A \mid z_2{:}\underline{D}_1 \vdash L : \underline{D}_2}{\Gamma \mid z_1{:}\underline{C} \vdash K\ \mathsf{to}\ (x{:}A, z_2{:}\underline{D}_1)\ \mathsf{in}_{\underline{D}_2}\ L : \underline{D}_2} \quad \frac{\Gamma \vdash \underline{C} \quad \Gamma, x{:}A \mid z{:}\underline{C} \vdash K : \underline{D}}{\Gamma \mid z{:}\underline{C} \vdash \lambda x{:}A.K : \Pi x{:}A.\underline{D}}$$

$$\frac{\Gamma, x{:}A \vdash \underline{D} \quad \Gamma \mid z{:}\underline{C} \vdash K : \Pi x{:}A.\underline{D} \quad \Gamma \vdash V : A}{\Gamma \mid z{:}\underline{C} \vdash K(V)_{(x:A).\underline{D}} : \underline{D}[V/x]} \quad \frac{\Gamma \vdash V : \underline{D}_1 \multimap \underline{D}_2 \quad \Gamma \mid z{:}\underline{C} \vdash K : \underline{D}_1}{\Gamma \mid z{:}\underline{C} \vdash V(K)_{\underline{D}_1,\underline{D}_2} : \underline{D}_2}$$

Fig. 1. Selected formation and typing rules for EMLTT types and terms.

where we require that $FCV(K_z) = FCV(K_s) = z$, and where x_1 and x_2 are chosen fresh.

The *well-formed syntax* of EMLTT is defined using judgments of well-formed value contexts $\vdash \Gamma$, value types $\Gamma \vdash A$, and computation types $\Gamma \vdash \underline{C}$; and well-typed value terms $\Gamma \vdash V : A$, computation terms $\Gamma \vdash M : \underline{C}$, and homomorphism terms $\Gamma \mid z{:}\underline{C} \vdash K : \underline{D}$, where the *value context* Γ is a list of distinct value variables annotated with value types; the empty context is written $\diamond$. We present selected rules for these judgments in Fig. 1; the full set of typing rules can be found in Appendix A. It is worthwhile to note that the elimination forms for the value types that EMLTT inherits from MLTT are dependently typed, e.g., see the typing rule of `nat-elim`.

Analogously to Ahman et al. (2016), we decorate value, computation, and homomorphism terms with a number of type annotations. We use these annotations to define the denotational semantics of EMLTT on raw expressions, so as to avoid well-known coherence problems arising in the interpretation of dependently typed languages; this is a standard technique in the literature (Hofmann 1997; Streicher 1991). We omit these annotations in examples.

The well-formed syntax of EMLTT is defined mutually with an *equational theory*, consisting of a collection of mutually defined equivalence relations given by *definitional equations* between well-formed value contexts, written $\vdash \Gamma_1 = \Gamma_2$; well-formed types, written $\Gamma \vdash A = B$ and $\Gamma \vdash \underline{C} = \underline{D}$; and well-typed terms, written $\Gamma \vdash V = W : A$, $\Gamma \vdash M = N : \underline{C}$, and $\Gamma \mid z{:}\underline{C} \vdash K = L : \underline{D}$. We give a selection of these equations in Fig. 2; the full set of definitional equations can be found in Appendix B. The definitional equations interact with the well-formed syntax via

Value terms:

$$\frac{\Gamma, x : \mathsf{Nat} \vdash A \quad \Gamma \vdash V_z : A[\mathtt{zero}/x] \quad \Gamma, y_1 : \mathsf{Nat}, y_2 : A[y_1/x] \vdash V_s : A[\mathtt{succ}\ y_1/x]}{\Gamma \vdash \mathtt{nat\text{-}elim}_{x.A}(V_z, y_1.y_2.V_s, \mathtt{zero}) = V_z : A[\mathtt{zero}/x]}$$

$$\frac{\Gamma, x : \mathsf{Nat} \vdash A \quad \Gamma \vdash V : \mathsf{Nat} \quad \Gamma \vdash V_z : A[\mathtt{zero}/x] \quad \Gamma, y_1 : \mathsf{Nat}, y_2 : A[y_1/x] \vdash V_s : A[\mathtt{succ}\ y_1/x]}{\Gamma \vdash \mathtt{nat\text{-}elim}_{x.A}(V_z, y_1.y_2.V_s, \mathtt{succ}\ V) = V_s[V/y_1][\mathtt{nat\text{-}elim}_{x.A}(V_z, y_1.y_2.V_s, V)/y_2] : A[\mathtt{succ}\ V/x]}$$

$$\frac{\Gamma \vdash A \quad \Gamma, x_1 : A, x_2 : A, x_3 : x_1 =_A x_2 \vdash B \quad \Gamma \vdash V : A \quad \Gamma, y : A \vdash W : B[y/x_1][y/x_2][\mathtt{refl}\ y/x_3]}{\Gamma \vdash \mathtt{eq\text{-}elim}_A(x_1.x_2.x_3.B, y.W, V, V, \mathtt{refl}\ V) = W[V/y] : B[V/x_1][V/x_2][\mathtt{refl}\ V/x_3]}$$

$$\frac{\Gamma \vdash V : U\underline{C}}{\Gamma \vdash \mathtt{thunk}(\mathtt{force}_{\underline{C}}\ V) = V : U\underline{C}} \qquad \frac{\Gamma \vdash V : \underline{C} \multimap \underline{D}}{\Gamma \vdash V = \lambda z{:}\underline{C}.V(z)_{\underline{C},\underline{D}} : \underline{C} \multimap \underline{D}}$$

Computation terms:

$$\frac{\Gamma \vdash V : A \quad \Gamma \vdash \underline{C} \quad \Gamma, x : A \vdash M : \underline{C}}{\Gamma \vdash \mathtt{return}\ V\ \mathtt{to}\ x{:}A\ \mathtt{in}_{\underline{C}}\ M = M[V/x] : \underline{C}} \qquad \frac{\Gamma \vdash M : FA \quad \Gamma \vdash \underline{C} \quad \Gamma \mid z{:}FA \vdash K : \underline{C}}{\Gamma \vdash M\ \mathtt{to}\ x{:}A\ \mathtt{in}_{\underline{C}}\ K[\mathtt{return}\ x/z] = K[M/z] : \underline{C}}$$

$$\frac{\Gamma \vdash M : \underline{C}}{\Gamma \vdash \mathtt{force}_{\underline{C}}\,(\mathtt{thunk} M) = M : \underline{C}} \qquad \frac{\Gamma \vdash M : \underline{C} \quad \Gamma \mid z{:}\underline{C} \vdash K : \underline{D}}{\Gamma \vdash (\lambda z{:}\underline{C}.K)(M)_{\underline{C},\underline{D}} = K[M/z] : \underline{D}}$$

Homomorphism terms:

$$\frac{\Gamma \vdash V : A \quad \Gamma \mid z_1{:}\underline{C} \vdash K : \underline{D}_1[V/x] \quad \Gamma \vdash \underline{D}_2 \quad \Gamma, x{:}A \mid z_2{:}\underline{D}_1 \vdash L : \underline{D}_2}{\Gamma \mid z_1{:}\underline{C} \vdash \langle V, K\rangle_{(x:A).\underline{D}_1}\ \mathtt{to}\ (x{:}A, z_2{:}\underline{D}_1)\ \mathtt{in}_{\underline{D}_2}\ L = L[V/x][K/z_2] : \underline{D}_2}$$

$$\frac{\Gamma, x{:}A \vdash \underline{D}_1 \quad \Gamma \mid z_1{:}\underline{C} \vdash K : \Sigma x{:}A.\underline{D}_1 \quad \Gamma \vdash \underline{D}_2 \quad \Gamma \mid z_3{:}\Sigma x{:}A.\underline{D}_1 \vdash K : \underline{D}_2}{\Gamma \mid z_1{:}\underline{C} \vdash K\ \mathtt{to}\ (x{:}A, z_2{:}\underline{D}_1)\ \mathtt{in}_{\underline{D}_2}\ L[\langle x, z_2\rangle_{(x:A).\underline{D}_1}/z_3] = L[K/z_3] : \underline{D}_2}$$

$$\frac{\Gamma \vdash \underline{C} \quad \Gamma, x{:}A \mid z{:}\underline{C} \vdash K : \underline{D} \quad \Gamma \vdash V : A}{\Gamma \mid z{:}\underline{C} \vdash (\lambda x{:}A.K)(V)_{(x:A).\underline{D}} = K[V/x] : \underline{D}[V/x]}$$

$$\frac{\Gamma, x{:}A \vdash \underline{D} \quad \Gamma \mid z{:}\underline{C} \vdash K : \Pi x{:}A.\underline{D}}{\Gamma \mid z{:}\underline{C} \vdash K = \lambda x{:}A.K(x)_{(x:A).\underline{D}} : \Pi x{:}A.\underline{D}}$$

Fig. 2. Selected definitional equations from the equational theory of EMLTT.

context and type conversion rules, e.g.,

$$\frac{\vdash \Gamma_1 = \Gamma_2 \quad \Gamma_1 \vdash V : A_1 \quad \Gamma_1 \vdash A_1 = A_2}{\Gamma_2 \vdash V : A_2} \qquad \frac{\vdash \Gamma_1 = \Gamma_2 \quad \Gamma_1 \vdash M : \underline{C}_1 \quad \Gamma_1 \vdash \underline{C}_1 = \underline{C}_2}{\Gamma_2 \vdash M : \underline{C}_2}$$

We note that as EMLTT is based on Martin-Löf's intensional type theory, the elimination form for propositional equality $V =_A W$ supports only a β-equation (see Fig. 2). Similarly, the elimination form for natural numbers also supports only β-equations. In both cases, this is done so as to avoid known sources of undecidability for the equational theory—see the analysis by Hofmann (1995), and Okada and Scott (1999), respectively. We also note that the elimination forms for all other value and computation types come equipped with both β- and η-equations.

Regarding the meta-theory of EMLTT, one can readily prove standard weakening and substitution results, the latter for both value and computation variables. For example, we write $A[V/x]$ for the substitution of V for x in A.

Analogously we write $K[M/z]$ for the substitution of M for z in K. The definitions of both kinds of substitution are straightforward: they proceed by recursion on the structure of the given term, making use of the standard convention of identifying types and terms that differ only in the names of bound variables and assuming that in any definition, etc., the bound variables of types and terms are chosen to be different from free variables.

We conclude by recalling that one of the notable features of EMLTT is the computational Σ-type $\Sigma x{:}A.\underline{C}$, see Ahman et al. (2016). In particular, the computational Σ-type provides a uniform treatment of type-dependency in sequential composition by allowing one to "close-off" the type of the the second computation with $\Sigma x{:}A.\underline{C}$ before using the typing rule for sequential composition which prohibits x to appear in the type of the second computation. A similar restriction on free variables also appears in many other computational typing rules. As a result, EMLTT lends itself to a very natural general denotational semantics based on *fibred* adjunctions, as studied in detail by Ahman et al. (2016). Thus, one says that the computational effects in EMLTT are *fibred*.

3 FIBRED ALGEBRAIC EFFECTS

In this section we develop a means for formally specifying algebraic effects in EMLTT in terms of operations and equations, using a natural dependently typed generalisation of Plotkin and Pretnar's effect theories (Plotkin and Pretnar 2013). We note that in Ahman et al. (2016), this extension of EMLTT was sketched rather informally.

3.1 Fibred effect theories

We begin by identifying the fragment of EMLTT which we use to define the types of our operation symbols.

A value type is said to be *pure* if it is built up from only Nat, 1, $\Sigma x{:}A.B$, $\Pi x{:}A.B$, 0, $A + B$, and $V =_A W$, with the type annotation A required to be pure in propositional equality $V =_A W$. This notion of pureness extends straightforwardly to contexts and terms. A value context Γ is said to be *pure* if A_i is pure for every $x_i{:}A_i \in \Gamma$. A value term is *pure* if it does not contain thunked computations and homomorphic lambda abstractions, and all its type annotations are pure. In other words, the pure fragment of EMLTT corresponds precisely to MLTT.

Assuming a countable set of *effect variables* $w, \ldots$, we now define fibred effect theories. We start with signatures of operation symbols and then add equations, so as to specify both the effects at hand and their behaviour.

Definition 3.1. A *fibred effect signature* $\mathcal{S}_{\text{eff}}$ consists of a finite set of typed operation symbols $\text{op} : (x{:}I) \longrightarrow O$, where $\diamond \vdash I$ and $x{:}I \vdash O$ are required to be pure value types, called the *input* and *output* type of op, respectively.

The *effect terms* T that one can derive from the given fibred effect signature $\mathcal{S}_{\text{eff}}$ are given by

$$\begin{array}{rcl} T & ::= & w\,(V) \\ & | & \text{op}_V(y.T) \\ & | & \text{pm } V \text{ as } (x_1{:}A_1, x_2{:}A_2) \text{ in } T \\ & | & \text{case } V \text{ of } (\text{inl}(x_1{:}A_1) \mapsto T_1, \text{inr}(x_2{:}A_2) \mapsto T_2) \end{array}$$

with the involved value types and value terms all required to be pure.

We use the convention of omitting V in $\text{op}_V(y.T)$ when the input type of op is 1, and y when the output type is 1.

An *effect context* Δ is a list of distinct effect variables annotated with pure value types. We say that Δ is *well-formed* in a pure value context Γ, written $\Gamma \vdash \Delta$, if $\vdash \Gamma$ and $\Gamma \vdash A_i$ for every $w_j{:}A_j \in \Delta$.

The *well-formed* effect terms are defined using the judgement $\Gamma \mid \Delta \vdash T$ and are given by the following rules:

$$\frac{\Gamma \vdash \Delta_1, w{:}A, \Delta_2 \quad \Gamma \vdash V : A}{\Gamma \mid \Delta_1, w{:}A, \Delta_2 \vdash w\,(V)} \qquad \frac{\Gamma \vdash V : \Sigma x_1{:}A_1.A_2 \quad \Gamma \vdash \Delta \quad \Gamma, x_1{:}A_1, x_2{:}A_2 \mid \Delta \vdash T}{\Gamma \mid \Delta \vdash \text{pm } V \text{ as } (x_1{:}A_1, x_2{:}A_2) \text{ in } T}$$

$$\frac{\Gamma \vdash V : I \quad \Gamma \vdash \Delta \quad \Gamma, y{:}O[V/x] \mid \Delta \vdash T}{\Gamma \mid \Delta \vdash \text{op}_V(y.T)} \qquad \frac{\Gamma \vdash V : A_1 + A_2 \quad \Gamma \vdash \Delta \quad \Gamma, x_1{:}A_1 \mid \Delta \vdash T_1 \quad \Gamma, x_2{:}A_2 \mid \Delta \vdash T_2}{\Gamma \mid \Delta \vdash \text{case } V \text{ of } (\text{inl}(x_1{:}A_1) \mapsto T_1, \text{inr}(x_2{:}A_2) \mapsto T_2)}$$

Definition 3.2. A *fibred effect theory* $\mathcal{T}_{\text{eff}}$ is given by a fibred effect signature $\mathcal{S}_{\text{eff}}$ and a finite set $\mathcal{E}_{\text{eff}}$ of equations $\Gamma \mid \Delta \vdash T_1 = T_2$, where the effect terms T_1 and T_2 are well-formed as $\Gamma \mid \Delta \vdash T_1$ and $\Gamma \mid \Delta \vdash T_2$.

In order to simplify the presentation of typing rules and equations involving fibred effect theories, we assume $\Gamma = x_1 : A_1, \ldots, x_n : A_n$ and $\Delta = w_1 : A'_1, \ldots, w_m : A'_m$ whenever we need to quantify over the variables of Γ and Δ.

3.2 Examples of fibred effect theories

As our fibred effect theories are a natural dependently typed generalisation of Plotkin and Pretnar's effect theories (Plotkin and Pretnar 2013), we can capture all the effects they can. For example, assuming a pure value type $\diamond \vdash \mathsf{Exc}$ of exception names, the *theory $\mathcal{T}_{EXC}$ of exceptions* is given by one operation symbol $\mathsf{raise} : \mathsf{Exc} \longrightarrow 0$ and no equations. Another standard example is the *theory $\mathcal{T}_{ND}$ of nondeterminism*, which is given by one operation symbol $\mathsf{or} : 1 \longrightarrow 1 + 1$ and three equations that make or into a semilattice. See op. cit. for more examples.

However, compared to Plotkin and Pretnar's work in the simply typed setting, our dependently typed operations further allow us to capture more precise notions of computation. We discuss two such examples: i) global state in which the store types are dependent on locations; and ii) dependently typed update monads that model state in which the store is changed not by overwriting but instead by applying (store-dependent) updates to it, examples of which include non-overflowing buffers and non-underflowing stacks—see Ahman and Uustalu (2014).

Global state. Assuming given well-formed pure value types of *memory locations* and *values* stored at them:

$$\diamond \vdash \mathsf{Loc} \qquad x : \mathsf{Loc} \vdash \mathsf{Val}$$

the fibred effect signature $\mathcal{S}_{GS}$ of global state is given by the following two operation symbols:

$$\mathsf{get} : (x : \mathsf{Loc}) \longrightarrow \mathsf{Val} \qquad \mathsf{put} : \Sigma x : \mathsf{Loc}.\mathsf{Val} \longrightarrow 1$$

The high-level idea is that get denotes an effectful command that returns the current value of the store at the given location; and put denotes a command that sets the store at the given location to the given value.

The corresponding fibred effect theory $\mathcal{T}_{\text{GS}}$ is then given by the following six equations:

$$x : \mathsf{Loc} \mid w : 1 \vdash \mathsf{get}_x(y.\mathsf{put}_{\langle x, y\rangle}(w\,(\star))) = w\,(\star)$$

$$x : \mathsf{Loc}, y : \mathsf{Val} \mid w : \mathsf{Val} \vdash \mathsf{put}_{\langle x, y\rangle}(\mathsf{get}_x(y'.w\,(y'))) = \mathsf{put}_{\langle x, y\rangle}(w\,(y))$$

$$x : \mathsf{Loc}, y_1 : \mathsf{Val}, y_2 : \mathsf{Val} \mid w : 1 \vdash \mathsf{put}_{\langle x, y_1\rangle}(\mathsf{put}_{\langle x, y_2\rangle}(w\,(\star))) = \mathsf{put}_{\langle x, y_2\rangle}(w\,(\star))$$

$$x_1 : \mathsf{Loc}, x_2 : \mathsf{Loc} \mid w : \mathsf{Val}[x_1/x] \times \mathsf{Val}[x_2/x] \vdash \mathsf{get}_{x_1}(y_1.\mathsf{get}_{x_2}(y_2.w\,(\langle y_1, y_2\rangle))) = \mathsf{get}_{x_2}(y_2.\mathsf{get}_{x_1}(y_1.w\,(\langle y_1, y_2\rangle)))$$

$$x_1 : \mathsf{Loc}, x_2 : \mathsf{Loc}, y_2 : \mathsf{Val}[x_2/x] \mid w : \mathsf{Val}[x_1/x] \vdash \mathsf{get}_{x_1}(y_1.\mathsf{put}_{\langle x_2, y_2\rangle}(w\,(y_1))) = \mathsf{put}_{\langle x_2, y_2\rangle}(\mathsf{get}_{x_1}(y_1.w\,(y_1)))$$

$$x_1 : \mathsf{Loc}, x_2 : \mathsf{Loc}, y_1 : \mathsf{Val}[x_1/x], y_2 : \mathsf{Val}[x_2/x] \mid w : 1 \vdash \mathsf{put}_{\langle x_1, y_1\rangle}(\mathsf{put}_{\langle x_2, y_2\rangle}(w\,(\star))) = \mathsf{put}_{\langle x_2, y_2\rangle}(\mathsf{put}_{\langle x_1, y_1\rangle}(w\,(\star)))$$

where the last three equations each come with a side-condition $x_1 \neq x_2$ on the locations denoted by x_1 and x_2.

Similarly to Plotkin and Pretnar (2013), the notation we use for these side-conditions is simply an informal shorthand. Formally, we require the type Loc to come with decidable equality (for simplicity, boolean-valued), given by a pure value term $\diamond \vdash \mathsf{eq} : \mathsf{Loc} \to \mathsf{Loc} \to 1 + 1$, and then write the right-hand sides of these two equations using case analysis on $\mathsf{eq}\,x_1\,x_2$, e.g., the right-hand side of the last equation would be formally written as

$$\mathsf{case}\ (\mathsf{eq}\,x_1\,x_2)\ \mathsf{of}\ \left(\mathsf{inl}(x'_1 : 1) \mapsto \mathsf{put}_{\langle x_1, y_1\rangle}(\mathsf{put}_{\langle x_2, y_2\rangle}(w\,(\star))), \mathsf{inr}(x'_2 : 1) \mapsto \mathsf{put}_{\langle x_2, y_2\rangle}(\mathsf{put}_{\langle x_1, y_1\rangle}(w\,(\star)))\right)$$

These five equations describe the expected behaviour of global state: trivial store changes are not observable (1st equation); `get` returns the most recent value the store has been set to (2nd equation); `put` overwrites the content of the store (3rd equation); and `gets` and `puts` at different locations are independent (4th and 5th equation).

Dependently typed update monads. We assume given two well-formed pure value types

$$\diamond \vdash \mathsf{St} \qquad x : \mathsf{St} \vdash \mathsf{Upd}$$

of *store values* and *store updates*, together with well-typed closed pure value terms (omitting the empty contexts)

$$\downarrow : \Pi x : \mathsf{St}.\mathsf{Upd} \to \mathsf{St} \qquad \mathsf{o} : \Pi x : \mathsf{St}.\mathsf{Upd} \qquad \oplus : \Pi x : \mathsf{St}.\Pi y : \mathsf{Upd}.\mathsf{Upd}[x \downarrow y/x] \to \mathsf{Upd}$$

satisfying the following five closed equations (in the pure fragment of the equational theory of EMLTT value terms; for better readability, we omit contexts and types, and write the first argument to $\oplus$ as a subscript):

$$V \downarrow (\mathsf{o}\, V) = V \qquad V \downarrow (W_1 \oplus_V W_2) = (V \downarrow W_1) \downarrow W_2$$

$$W \oplus_V (\mathsf{o}\, (V \downarrow W)) = W \qquad (\mathsf{o}\, V) \oplus_V W = W \qquad (W_1 \oplus_V W_2) \oplus_V W_3 = W_1 \oplus_V (W_2 \oplus_{V \downarrow W_1} W_3)$$

The signature $\mathcal{S}_{\mathrm{UPD}}$ of a dependently typed update monad is then given by the following two operation symbols:

$$\mathsf{lookup} : 1 \longrightarrow \mathsf{St} \qquad \mathsf{update} : \Pi x : \mathsf{St}.\mathsf{Upd} \longrightarrow 1$$

The idea is that $(\mathsf{Upd}, \mathsf{o}, \oplus)$ forms a dependently typed monoid of updates which can be applied to the store values via its action $\downarrow$ on St; lookup denotes an effectful command that returns the current value of the store; and update denotes a command that applies an appropriate update to the current store (from the family of updates given as its input). The dependency of Upd on St gives us fine-grain control over which updates are applicable to which store values, and allows this to be enforced statically during typechecking. It is worth noting that in the literature, the 5-tuple $(\mathsf{St}, \mathsf{Upd}, \downarrow, \mathsf{o}, \oplus)$ is also known under the name of *directed containers* (Ahman et al. 2014).

The corresponding fibred effect theory $\mathcal{T}_{\mathrm{UPD}}$ is then given by the following three equations:

$$\diamond \mid w : 1 \vdash \mathsf{lookup}(x.\mathsf{update}_{\lambda y : \mathsf{St}.\, \mathsf{o}\, y}(w\, (\star))) = w\, (\star)$$

$$\begin{aligned} x : (\Pi x' : \mathsf{St}.\mathsf{Upd}[x'/x]) \mid w : \mathsf{St} \times \mathsf{St} \vdash\ & \mathsf{lookup}(y.\mathsf{update}_x(\mathsf{lookup}(y'.w\, (\langle y, y' \rangle)))) \\ & = \mathsf{lookup}(y.\mathsf{update}_x(w\, (\langle y, y \downarrow (x\, y) \rangle)))) \end{aligned}$$

$$\begin{aligned} x : (\Pi x' : \mathsf{St}.\mathsf{Upd}[x'/x]), y : (\Pi y' : \mathsf{St}.\mathsf{Upd}[y'/x]) \mid w : 1 \vdash\ & \mathsf{update}_x(\mathsf{update}_y(w\, (\star))) \\ & = \mathsf{update}_{\lambda x''.(x\, x'') \oplus_{x''} (y\, (x'' \downarrow (x\, x')))}(w\, (\star)) \end{aligned}$$

These equations are similar to the first three equations of the global state theory $\mathcal{T}_{\mathrm{GS}}$, but instead of an overwriting behaviour, they describe how the store is changed using updates. In particular, observe how subsequent updates are combined using the $\oplus$ operation, and how o provides us with the "do nothing" updates.

3.3 Extending EMLTT with fibred algebraic effects

We now show how to extend EMLTT with fibred algebraic effects given by a fibred effect theory $\mathcal{T}_{\mathrm{eff}} = (\mathcal{S}_{\mathrm{eff}}, \mathcal{E}_{\mathrm{eff}})$.

First, we extend the grammar of computation terms with *algebraic operations*:

$$M \ ::= \ \ldots \ \mid \mathsf{op}^{\underline{C}}_{V}(y.M)$$

for all $\mathsf{op} : (x : I) \longrightarrow O \in \mathcal{S}_{\mathrm{eff}}$ and computation types $\underline{C}$.

Next, in order to extend the well-formed syntax with a corresponding typing rule and definitional equations, we first define a translation of effect terms into value terms. Observe that while it might be more intuitive and natural to translate effect terms directly into computation terms, giving the translation from effect terms into value terms allows us to later reuse it in §4 where we extend EMLTT with handlers of fibred algebraic effects.

We only translate well-formed effect terms $\Gamma \mid \Delta \vdash T$ because it makes it easier to account for substituting value terms for effect variables—various subsequent results refer to substituting value terms for all effect variables in Δ, not just for the free effect variables appearing in T. For better readability, we write $\overrightarrow{V_i}$ for the set $\{V_1, \ldots, V_n\}$.

Definition 3.3. Given $\Gamma \mid \Delta \vdash T$, a value type A, value terms V_i (for all $x_i : A_i \in \Gamma$), value terms V_j' (for all $w_j : A_j' \in \Delta$), and value terms W_{op} (for all $\mathsf{op} : (x : I) \longrightarrow O \in \mathcal{S}_{\mathrm{eff}}$), the *translation* of the effect term T into a value term $(\!| T |\!)_{A;\overrightarrow{V_i};\overrightarrow{V_j'};\overrightarrow{W_{\mathsf{op}}}}$ is defined by recursion on the structure of T as follows:

$$
\begin{aligned}
&(\!| w_j\,(V) |\!) && \stackrel{\mathrm{def}}{=} V_j'\ (V[\overrightarrow{V_i}/\overrightarrow{x_i}])\\
&(\!| \mathsf{op}_V(y.T) |\!) && \stackrel{\mathrm{def}}{=} W_{\mathsf{op}}\ \langle V[\overrightarrow{V_i}/\overrightarrow{x_i}], \lambda y : O[V[\overrightarrow{V_i}/\overrightarrow{x_i}]/x].(\!| T |\!)\rangle\\
&(\!| \mathsf{pm}\ V\ \mathsf{as}\ (y_1 : B_1, y_2 : B_2)\ \mathsf{in}\ T |\!) && \stackrel{\mathrm{def}}{=} \mathsf{pm}\ V[\overrightarrow{V_i}/\overrightarrow{x_i}]\ \mathsf{as}\ (y_1 : B_1[\overrightarrow{V_i}/\overrightarrow{x_i}], y_2 : B_2[\overrightarrow{V_i}/\overrightarrow{x_i}])\ \mathsf{in}\ (\!| T |\!)\\
&(\!| \mathsf{case}\ V\ \mathsf{of}\ (\mathsf{inl}(y_1 : B_1) \mapsto T_1, \mathsf{inr}(y_2 : B_2) \mapsto T_2) |\!) && \stackrel{\mathrm{def}}{=} \mathsf{case}\ V[\overrightarrow{V_i}/\overrightarrow{x_i}]\ \mathsf{of}\ (\mathsf{inl}(y_1 : B_1[\overrightarrow{V_i}/\overrightarrow{x_i}]) \mapsto (\!| T_1 |\!),\\
&&& \qquad\qquad \mathsf{inr}(y_2 : B_2[\overrightarrow{V_i}/\overrightarrow{x_i}]) \mapsto (\!| T_2 |\!))
\end{aligned}
$$

where we omit the subscripts on the translation so as to improve readability. However, it is still important to note that in the cases where the given effect term involves variable bindings, the set of value terms $\overrightarrow{V_i}$ is extended in the right-hand side with the corresponding bound value variables, e.g., the second case is formally defined as

$$(\!| \mathsf{op}_V(y.T) |\!)_{A;\overrightarrow{V_i};\overrightarrow{V_j'};\overrightarrow{W_{\mathsf{op}}}} \stackrel{\mathrm{def}}{=} W_{\mathsf{op}}\ \langle V[\overrightarrow{V_i}/\overrightarrow{x_i}], \lambda y : O[V[\overrightarrow{V_i}/\overrightarrow{x_i}]/x].(\!| T |\!)_{A;\overrightarrow{V_i},y;\overrightarrow{V_j'};\overrightarrow{W_{\mathsf{op}}}}\rangle$$

Using this translation, we define the typing rule and definitional equations for algebraic operations in Fig. 3. Observe that for presentational convenience, we include the equations given in $\mathcal{E}_{\mathrm{eff}}$ as definitional equations between value terms. The corresponding equations between computation terms are easily derivable, e.g.,

$$\Gamma \vdash \mathsf{get}_V^{\underline{C}}(y.\mathsf{put}_{\langle V, y\rangle}^{\underline{C}}(M)) = M : \underline{C}$$

can be easily derived from the translation of the corresponding equation in the global state theory $\mathcal{T}_{\mathrm{GS}}$.

Further, it is also worth noting that the definitional equations corresponding to the equations given in $\mathcal{E}_{\mathrm{eff}}$ can be instantiated with any suitable value context Γ'. We do so to ensure that the weakening and substitution theorems remain derivable for this extension of EMLTT. We also note that the disjointness requirement on value contexts imposed in these equations is important for the substitution theorem (Thm. 6.3) to go through. While at first sight this requirement might seem limiting on the number of definitional equations we can derive, it turns out that the corresponding definitional equation without the disjointness requirement is derivable, as shown in Prop. 6.9, which we postpone to §6.2 because its proof crucially relies on the meta-theory we establish in §6.

4 HANDLERS VIA THE USER-DEFINED ALGEBRA TYPE

4.1 Problem with adding conventional handlers to EMLTT

Before we show how to extend EMLTT with handlers of fibred algebraic effects using a novel computation type, called the *user-defined algebra type*, we first explain how extending EMLTT with the conventional term-level definition of handlers leads to unsound program equivalences becoming derivable.

Recall from §1 that Plotkin and Pretnar (and others since) include handlers in effectful languages by extending the syntax of computation terms with the following handling construct:

$$M\ \mathsf{handled}\ \mathsf{with}\ \{\mathsf{op}_x(x') \mapsto N_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathrm{eff}}}\ \mathsf{to}\ y : A\ \mathsf{in}\ N_{\mathrm{ret}}$$

whose semantics is given using the mediating homomorphism from the free algebra over A to the algebra denoted by the handler $\{\mathsf{op}_x(x') \mapsto N_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathrm{eff}}}$. Now, when extending a language such as EMLTT that includes a notion of homomorphism (homomorphism terms), this algebraic understanding of handlers suggests that one ought to also extend the given notion of homomorphism with the handling construct. Unfortunately, if one simply adds

$$K\ \mathsf{handled}\ \mathsf{with}\ \{\mathsf{op}_x(x') \mapsto N_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathrm{eff}}}\ \mathsf{to}\ y : A\ \mathsf{in}\ N_{\mathrm{ret}}$$

Typing rule for algebraic operations:

$$\frac{\Gamma \vdash V : I \quad \Gamma \vdash \underline{C} \quad \Gamma, y{:}O[V/x] \vdash M : \underline{C}}{\Gamma \vdash \mathsf{op}_V^C(y.M) : \underline{C}}$$

for all $\mathsf{op} : (x{:}I) \longrightarrow O \in \mathcal{S}_{\text{eff}}$.

Congruence equations:

$$\frac{\Gamma \vdash V = W : I \quad \Gamma \vdash \underline{C} = \underline{D} \quad \Gamma, y{:}O[V/x] \vdash M = N : \underline{C}}{\Gamma \vdash \mathsf{op}_V^C(y.M) = \mathsf{op}_W^D(y.N) : \underline{C}}$$

for all $\mathsf{op} : (x{:}I) \longrightarrow O \in \mathcal{S}_{\text{eff}}$.

General algebraicity equations:

$$\frac{\Gamma \vdash V : I \quad \Gamma, y{:}O[V/x] \vdash M : \underline{C} \quad \Gamma \mid z{:}\underline{C} \vdash K : \underline{D}}{\Gamma \vdash K[\mathsf{op}_V^C(y.M)/z] = \mathsf{op}_V^D(y.K[M/z]) : \underline{D}}$$

for all $\mathsf{op} : (x{:}I) \longrightarrow O \in \mathcal{S}_{\text{eff}}$.

Equations of the given fibred effect theory:

$$\frac{\begin{array}{c} Vars(\Gamma') \cap Vars(\Gamma) = \emptyset \\ \Gamma' \vdash V_i : A_i[V_1/x_1, \ldots, V_{i-1}/x_{i-1}] \qquad (1 \leq i \leq n) \\ \Gamma' \vdash \underline{C} \quad \Gamma' \vdash V_j' : A_j'[\overrightarrow{V_i}/\overrightarrow{x_i}] \to U\underline{C} \qquad (1 \leq j \leq m) \end{array}}{\Gamma' \vdash (\!|\, T_1 \,|\!)_{U\underline{C};\overrightarrow{V_i};\overrightarrow{V_j'};\overrightarrow{W_{\mathsf{op}}}} = (\!|\, T_2 \,|\!)_{U\underline{C};\overrightarrow{V_i};\overrightarrow{V_j'};\overrightarrow{W_{\mathsf{op}}}} : U\underline{C}}$$

for all $\Gamma \mid \Delta \vdash T_1 = T_2 \in \mathcal{E}_{\text{eff}}$, with the well-typed value terms $\Gamma' \vdash W_{\mathsf{op}} : (\Sigma x{:}I.O \to U\underline{C}) \to U\underline{C}$ given by

$$W_{\mathsf{op}} \stackrel{\text{def}}{=} \lambda x' {:} (\Sigma x{:}I.O \to U\underline{C}).\mathtt{pm}\ x'\ \mathtt{as}\ (x{:}I, y{:}O \to U\underline{C})\ \mathtt{in}\ \mathtt{thunk}(\mathsf{op}_x^C(y'.\mathtt{force}_{\underline{C}}\ (y\ y')))$$

for all $\mathsf{op} : (x{:}I) \longrightarrow O \in \mathcal{S}_{\text{eff}}$.

Fig. 3. Rules for extending EMLTT with fibred algebraic effects.

to EMLTT, the combination of the general algebraicity equation (see Fig. 3) and the definitional β-equations associated with the handling construct (see §1) now give rise to unsound definitional equations.

To explain this problem in more detail, let us consider the theory $\mathcal{T}_{\text{I/O}}$ of *interactive input-output of bits*, given by two operation symbols $\mathsf{read} : 1 \longrightarrow 1 + 1$ and $\mathsf{write} : 1 + 1 \longrightarrow 1$, and no equations. Next, recall from §1 that a handler redefines the operations of a given effect theory, e.g., by flipping the bits written to the output, as in

$$\begin{array}{ll} \{\ \mathsf{read}(x') & \mapsto \mathsf{read}^{F1}(y.\mathtt{force}\ (x'\ y)), \\ \ \mathsf{write}_x(x') & \mapsto \mathsf{write}^{F1}_{\neg x}(\mathtt{force}\ (x'\ \star))\ \} \end{array}$$

where $\neg : 1 + 1 \to 1 + 1$ is a pure value term that swaps the left and right injections, denoting a logical negation.

Now, let us consider handling a simple program, given by $\mathsf{write}^{F1}_{\mathsf{inl}\,\star}(\mathtt{return}\ \star)$, using the handler we defined above. On the one hand, using the β-equations for handling (see §1), we can prove the following equation:

$$\begin{array}{l} \Gamma \vdash (\mathsf{write}^{F1}_{\mathsf{inl}\,\star}(\mathtt{return}\ \star))\ \mathtt{handled\ with}\ \{\mathsf{op}_x(x') \mapsto N_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\text{I/O}}}\ \mathtt{to}\ y{:}1\ \mathtt{in}\ \mathtt{return}\ \star \\ \quad = \mathsf{write}^{F1}_{\mathsf{inr}\,\star}(\mathtt{return}\ \star) : F1 \end{array}$$

On the other hand, using the general algebraicity equation given in Fig. 3, which ensures that homomorphism terms indeed behave as if they were algebra homomorphisms, we can prove the following equation:

$$\Gamma \vdash (\mathtt{write}^{F1}_{\mathtt{inl}\,\star}(\mathtt{return}\,\star))\ \mathtt{handled\ with}\ \{\mathsf{op}_x(x') \mapsto N_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathrm{I/O}}}\ \mathtt{to}\ y{:}1\ \mathtt{in}\ \mathtt{return}\,\star$$
$$= \mathtt{write}^{F1}_{\mathtt{inl}\,\star}(\mathtt{return}\,\star) : F1$$

Clearly, if we want to equip EMLTT with a sound denotational semantics based on the models of the fibred effect theory $\mathcal{T}_{\mathrm{I/O}}$, these two equations had better not be derivable at the same time because $\mathsf{inl}\,\star \neq \mathsf{inr}\,\star$.

The reason for this discrepancy lies in the term-level definition of handlers in their conventional presentation. In particular, while the homomorphic behaviour of homomorphism terms is determined exclusively by the computation types involved (via the general algebraicity equation given in Fig. 3), the type of the above handling construct contains no trace of the algebra denoted by the user-defined handler $\{\mathsf{op}_x(x') \mapsto N_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathrm{I/O}}}$.

It is worth noting that this problem is not inherent to EMLTT but would also arise in the simply typed setting, when combining handlers with CBPV and its stack terms, or with EEC and its linear (computation) terms. The reason why Plotkin and Pretnar (2013) were able to give a sound denotational semantics to their language was exactly due to their choice of using CBPV *without* stack terms, i.e., with only value and computation terms.

4.2 Extending EMLTT with the user-defined algebra type

In this section we solve the problem discussed in the previous section by giving handlers a novel type-based treatment, internalising Plotkin and Pretnar's idea that they denote algebras for the given algebraic theory.

First, given a fibred effect theory $\mathcal{T}_{\mathrm{eff}} = (\mathcal{S}_{\mathrm{eff}}, \mathcal{E}_{\mathrm{eff}})$, we extend EMLTT with the *user-defined algebra type*:

$$\underline{C} \ ::= \ \ldots \ \mid \langle A, \{V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathrm{eff}}}\rangle$$

which pairs a value type A (the carrier) with a family of value terms $\{V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathrm{eff}}}$ (the operations).

In addition, we extend the computation and homomorphism terms of EMLTT with two *composition operations*:

$$M \ ::= \ \ldots \ \mid M\ \mathtt{as}\ x{:}U\underline{C}\ \mathtt{in}_{\underline{D}}\ N \qquad\qquad K \ ::= \ \ldots \ \mid K\ \mathtt{as}\ x{:}U\underline{C}\ \mathtt{in}_{\underline{D}}\ N$$

which provide elimination forms for the user-defined algebra type when we choose $\underline{C}$ to be $\langle A, \{V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathrm{eff}}}\rangle$.

It is worth noting that in principle we could have restricted these composition operations to only the user-defined algebra type, but then we would not have been able to derive a useful type isomorphism to coerce computations between a general $\underline{C}$ and its canonical representation as a user-defined algebra type (see Prop. 4.1).

Conceptually, these two composition operations are a form of explicit substitution of thunked computations for value variables, e.g., we will be able to show that the computation term $M\ \mathtt{as}\ x{:}U\underline{C}\ \mathtt{in}_{\underline{D}}\ N$ is definitionally equal to $N[\mathtt{thunk}\ M/x]$. As such, the value variable x refers to the whole of (the thunk of) M, compared to sequential composition $M\ \mathtt{to}\ x{:}A\ \mathtt{in}\ N$, where x refers to the value computed by M. Thus, regarding our notation, we use as (running M *as* if it was x in N), compared to to (running M *to* produce a value to be bound to x in N).

However, importantly, the typing rules for $M\ \mathtt{as}\ x{:}U\underline{C}\ \mathtt{in}_{\underline{D}}\ N$ and $K\ \mathtt{as}\ x{:}U\underline{C}\ \mathtt{in}_{\underline{D}}\ N$ require that x is to be used in N as if it was a computation variable, in that x must not be duplicated or discarded arbitrarily. We do so as to ensure that N behaves as if it was a homomorphism term, meaning that in $M\ \mathtt{as}\ x{:}U\underline{C}\ \mathtt{in}_{\underline{D}}\ N$ the effects of M are guaranteed to "happen before" those of N. However, rather than trying to extend EMLTT further with some form of linearity for such value variables, we impose these requirements via equational proof obligations, requiring that N commutes with algebraic operations (when substituted for x using thunking).

We make the above discussion formal in Fig. 4 where we give the rules for extending the well-formed syntax and equational theory of EMLTT with the user-defined algebra type and the two composition operations.

In the rules concerning the user-defined algebra type, we use the following auxiliary judgment:

$$\Gamma' \vdash \{V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathrm{eff}}}\ \mathsf{on}\ A$$

Formation rule for the user-defined algebra type:

$$\frac{\Gamma \vdash \{V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathsf{eff}}} \text{ on } A}{\Gamma \vdash \langle A, \{V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathsf{eff}}}\rangle}$$

Typing rules for the composition operations:

$$\frac{\Gamma \vdash M : \underline{C} \quad \Gamma \vdash \underline{D} \quad \Gamma, x{:}U\underline{C} \vdash_{\mathsf{hom}} N : \underline{D}}{\Gamma \vdash M \,\mathsf{as}\, x{:}U\underline{C}\, \mathtt{in}_{\underline{D}}\, N : \underline{D}} \qquad \frac{\Gamma \mid z{:}\underline{C} \vdash K : \underline{D}_1 \quad \Gamma \vdash \underline{D}_2 \quad \Gamma, x{:}U\underline{D}_1 \vdash_{\mathsf{hom}} M : \underline{D}_2}{\Gamma \mid z{:}\underline{C} \vdash K \,\mathsf{as}\, x : U\underline{D}_1\, \mathtt{in}_{\underline{D}_2}\, M : \underline{D}_2}$$

Congruence equations:

$$\frac{\begin{array}{c}\Gamma \vdash \langle A, \{V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathsf{eff}}}\rangle \quad \Gamma \vdash \langle B, \{W_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathsf{eff}}}\rangle \quad \Gamma \vdash A = B \\ \Gamma \vdash V_{\mathsf{op}} = W_{\mathsf{op}} : (\Sigma x{:}I.O \to A) \to A \quad (\mathsf{op} : (x{:}I) \longrightarrow O)\end{array}}{\Gamma \vdash \langle A, \{V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathsf{eff}}}\rangle = \langle B, \{W_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathsf{eff}}}\rangle}$$

plus similar two equations for composition operations.

β-equation for the user-defined algebra type:

$$\frac{\Gamma \vdash \langle A, \{V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathsf{eff}}}\rangle}{\Gamma \vdash U\langle A, \{V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathsf{eff}}}\rangle = A}$$

β-equation for the composition operations:

$$\frac{\Gamma \vdash V : U\underline{C} \quad \Gamma \vdash \underline{D} \quad \Gamma, x{:}U\underline{C} \vdash_{\mathsf{hom}} M : \underline{D}}{\Gamma \vdash (\mathtt{force}_{\underline{C}}\, V) \,\mathsf{as}\, x{:}U\underline{C}\, \mathtt{in}_{\underline{D}}\, M = M[V/x] : \underline{D}}$$

η-equations for the composition operations:

$$\frac{\Gamma \vdash M : \underline{C} \quad \Gamma \mid z{:}\underline{C} \vdash K : \underline{D}}{\Gamma \vdash M \,\mathsf{as}\, x{:}U\underline{C}\, \mathtt{in}_{\underline{D}}\, K[\mathtt{force}_{\underline{C}}\, x/z] = K[M/z] : \underline{D}}$$

$$\frac{\Gamma \mid z_1{:}\underline{C} \vdash K : \underline{D}_1 \quad \Gamma \mid z_2{:}\underline{D}_1 \vdash L : \underline{D}_2}{\Gamma \mid z_1{:}\underline{C} \vdash K \,\mathsf{as}\, x{:}U\underline{D}_1\, \mathtt{in}_{\underline{D}_2}\, L[\mathtt{force}_{\underline{D}_1}\, x/z_2] = L[K/z_2] : \underline{D}_2}$$

η-equation for algebraic operations:

$$\frac{\Gamma \vdash V : I \quad \Gamma \vdash \langle A, \{V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathsf{eff}}}\rangle \quad \Gamma, y{:}O[V/x] \vdash M : \langle A, \{V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathsf{eff}}}\rangle}{\Gamma \vdash \mathsf{op}_V^{\langle A, \{V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathsf{eff}}}\rangle}(y.M) = \mathtt{force}_{\langle A, \{V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathsf{eff}}}\rangle}\,(V_{\mathsf{op}}\,\langle V, \lambda y{:}O[V/x].\mathtt{thunk}\, M\rangle) : \langle A, \{V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathsf{eff}}}\rangle}$$

Fig. 4. Rules for extending EMLTT with the user-defined algebra type.

which holds iff the value terms V_{op} form an algebra *on* the value type A, i.e., iff $\Gamma' \vdash A, \Gamma' \vdash V_{\mathsf{op}} : (\Sigma x{:}I.O \to A) \to A$ (for all $\mathsf{op} : (x{:}I) \longrightarrow O$ in $\mathcal{S}_{\mathsf{eff}}$), and we can prove for all $\Gamma \mid \Delta \vdash T_1 = T_2 \in \mathcal{E}_{\mathsf{eff}}$ that the following equation holds:

$$\Gamma', \Gamma \vdash \overrightarrow{\lambda x_{w_j} : A'_j \to A.}(\!|\, T_1 \,|\!)_{A;\overrightarrow{x_i};\overrightarrow{x_{w_j}};\overrightarrow{V_{\mathsf{op}}}} = \overrightarrow{\lambda x_{w_j} : A'_j \to A.}(\!|\, T_2 \,|\!)_{A;\overrightarrow{x_i};\overrightarrow{x_{w_j}};\overrightarrow{V_{\mathsf{op}}}} : \overrightarrow{A'_j \to A} \to A$$

where we write $\overrightarrow{\lambda x_{w_j} : A'_j \to A.}$ for the sequence of lambda abstractions $\lambda x_{w_1} : A'_1 \to A. \ldots \lambda x_{w_m} : A'_m \to A.$, and $\overrightarrow{A'_j \to A}$ for the corresponding sequence of function types $(A'_1 \to A) \to \ldots \to (A'_m \to A)$.

It is worth noting that if one works exclusively with equation-free fibred effect theories, e.g., as used in simply typed languages (Bauer and Pretnar 2015; Hillerström and Lindley 2016) and discussed in §1, then the equational proof obligations in $\Gamma' \vdash \{V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathsf{eff}}}$ on A hold vacuously, and thus do not put any additional burden on the programmer. However, the possibility of also being able to specify effects using equations ensures that the fit-for-purpose handler implementations of given notion of computation (say, global state) are correct.

In the rules concerning the composition operations, we use the following auxiliary judgment:

$$\Gamma, y{:}U\underline{C} \vdash_{\mathsf{hom}} N : \underline{D}$$

which holds iff N behaves like a homomorphism from the algebra denoted by $\underline{C}$ to the algebra denoted by $\underline{D}$, i.e., iff $\Gamma, y{:}U\underline{C} \vdash N : \underline{D}$ and we can prove for all $\mathsf{op} : (x{:}I) \longrightarrow O \in \mathcal{S}_{\mathsf{eff}}$ that the following equation holds:

$$\Gamma \vdash \lambda x.\lambda x'.N[\mathtt{thunk}\,(\mathsf{op}^{\underline{C}}_{x}(y'.\mathtt{force}_{\underline{C}}\,(x'\ y')))/y] = \lambda x.\lambda x'.\mathsf{op}^{\underline{D}}_{x}(y'.N[x'\ y'/y]) : \Pi x{:}I.(O \to U\underline{C}) \to \underline{D}$$

where the omitted type annotations on x and x' are given by the value types I and $O \to U\underline{C}$, respectively.

It is worth noting that the equational proof obligations in the judgement $\Gamma, y{:}U\underline{C} \vdash_{\mathsf{hom}} N : \underline{D}$ are reminiscent of the equational conditions used by Pédrot and Tabareau (2017) to require linearity in type-dependence so as to ensure the correctness of their monadic translation of type theory. This connection will become more evident in §7 where we use the composition operations to define type-theoretic predicates on (thunks of) computations.

Regarding the definitional equations given in Fig. 4, observe that the β-equation for the user-defined algebra type captures the intuition that the value type A denotes the carrier of the algebra denoted by $\langle A, \{V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathsf{eff}}}\rangle$. Analogously, the η-equation for algebraic operations captures the intuition that the value terms V_{op} denote the operations associated with the algebra denoted by the user-defined algebra type $\langle A, \{V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathsf{eff}}}\rangle$.

It is also worth noting that we have not included an η-equation for the user defined algebra type. We do so because it does not hold in the natural denotational semantics we develop for this extension of EMLTT in §8. Instead, as promised earlier, we can construct a corresponding computation type isomorphism, as given below.

Proposition 4.1. *Given a computation type $\Gamma \vdash \underline{C}$, we can construct the following computation type isomorphism:*

$$\Gamma \vdash \underline{C} \cong \langle U\underline{C}, \{V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathit{eff}}}\rangle$$

where each well-formed value term $\Gamma \vdash V_{\mathsf{op}} : (\Sigma x{:}I.O \to U\underline{C}) \to U\underline{C}$ is defined as follows:

$$V_{\mathsf{op}} \stackrel{\mathit{def}}{=} \lambda y{:}(\Sigma x{:}I.O \to U\underline{C}).\mathtt{pm}\ y\ \mathtt{as}\ (x{:}I, x'{:}O \to U\underline{C})\ \mathtt{in}\ \mathtt{thunk}\,(\mathsf{op}^{\underline{C}}_{x}(y'.\mathtt{force}_{\underline{C}}\,(x'\ y')))$$

Proof sketch. This computation type isomorphism is witnessed by the following two homomorphic functions:

$$\lambda z{:}\underline{C}.z\ \mathtt{as}\ x{:}U\underline{C}\ \mathtt{in}\ \mathtt{force}_{\langle U\underline{C}, \{V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathsf{eff}}}\rangle}\,x \qquad \lambda z{:}\langle U\underline{C}, \{V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathsf{eff}}}\rangle.z\ \mathtt{as}\ x{:}U\langle U\underline{C}, \{V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathsf{eff}}}\rangle\ \mathtt{in}\ \mathtt{force}_{\underline{C}}\,x$$

□

4.3 Deriving the term-level definition of handlers

In this section we show how to derive the conventional term-level definition of handlers from our type-based treatment. In particular, we define the handling construct using sequential composition:

$$M\ \mathtt{handled}\ \mathtt{with}\ \{\mathsf{op}_x(x') \mapsto N_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathsf{eff}}}\ \mathtt{to}\ y{:}A\ \mathtt{in}_{\underline{C}}\ N_{\mathsf{ret}}$$
$$\stackrel{\mathsf{def}}{=}$$
$$\mathtt{force}_{\underline{C}}\,(\mathtt{thunk}\,(M\ \mathtt{to}\ y{:}A\ \mathtt{in}\ \mathtt{force}_{\langle U\underline{C}, \{V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathsf{eff}}}\rangle}\,(\mathtt{thunk}\,N_{\mathsf{ret}})))$$

where each well-formed value term $\Gamma \vdash V_{\mathsf{op}} : (\Sigma x{:}I.O \to U\underline{C}) \to U\underline{C}$ is defined as follows:

$$V_{\mathsf{op}} \stackrel{\mathsf{def}}{=} \lambda y'{:}(\Sigma x{:}I.O \to U\underline{C}).\mathtt{pm}\ y'\ \mathtt{as}\ (x{:}I, x'{:}O \to U\underline{C})\ \mathtt{in}\ \mathtt{thunk}\,N_{\mathsf{op}}$$

Next, we show that the expected typing rule and the two β-equations are then derivable.

Proposition 4.2. *The following typing rule is derivable:*

$$\frac{\begin{array}{c}\Gamma, x{:}I, x'{:}O \to U\underline{C} \vdash N_{\mathsf{op}} : \underline{C} \quad (\mathsf{op} : (x : I) \longrightarrow O \in \mathcal{S}_{\mathit{eff}}) \\ \Gamma \vdash M : FA \quad \Gamma \vdash \underline{C} \quad \Gamma, y{:}A \vdash N_{\mathsf{ret}} : \underline{C} \quad \Gamma \vdash \{V_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathit{eff}}} \text{ on } U\underline{C}\end{array}}{\Gamma \vdash M \text{ handled with } \{\mathsf{op}_x(x') \mapsto N_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathit{eff}}} \text{ to } y{:}A \text{ in}_{\underline{C}}\ N_{\mathsf{ret}} : \underline{C}}$$

where each V_{op} is derived from the corresponding N_{op} as defined above.

Proposition 4.3. *The following definitional β-equations are derivable:*

$$\frac{\begin{array}{c}\Gamma, x{:}I, x'{:}O \to U\underline{C} \vdash N_{\mathsf{op}} : \underline{C} \qquad (\mathsf{op} : (x : I) \longrightarrow O \in \mathcal{S}_{\mathit{eff}}) \\ \Gamma \vdash V : I \quad \Gamma, y'{:}O[V/x] \vdash M : FA \quad \Gamma \vdash \underline{C} \quad \Gamma, y{:}A \vdash N_{\mathsf{ret}} : \underline{C} \quad \Gamma \vdash \{V_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathit{eff}}} \text{ on } U\underline{C}\end{array}}{\begin{array}{l}\Gamma \vdash (\mathsf{op}_V^{FA}(y'.M)) \text{ handled with } \{\mathsf{op}_x(x') \mapsto N_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathit{eff}}} \text{ to } y{:}A \text{ in}_{\underline{C}}\ N_{\mathsf{ret}} \\ \quad = N_{\mathsf{op}}[V/x][\lambda y'{:}O[V/x].\mathsf{thunk}\ H/x'] : \underline{C}\end{array}}$$

$$\frac{\begin{array}{c}\Gamma, x{:}I, x'{:}O \to U\underline{C} \vdash N_{\mathsf{op}} : \underline{C} \qquad (\mathsf{op} : (x : I) \longrightarrow O \in \mathcal{S}_{\mathit{eff}}) \\ \Gamma \vdash V : A \quad \Gamma \vdash \underline{C} \quad \Gamma, y{:}A \vdash N_{\mathsf{ret}} : \underline{C} \quad \Gamma \vdash \{V_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathit{eff}}} \text{ on } U\underline{C}\end{array}}{\begin{array}{l}\Gamma \vdash (\mathsf{return}\ V) \text{ handled with } \{\mathsf{op}_x(x') \mapsto N_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathit{eff}}} \text{ to } y{:}A \text{ in}_{\underline{C}}\ N_{\mathsf{ret}} \\ \quad = N_{\mathsf{ret}}[V/y] : \underline{C}\end{array}}$$

where

$$H \stackrel{def}{=} M \text{ handled with } \{\mathsf{op}_x(x') \mapsto N_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathit{eff}}} \text{ to } y{:}A \text{ in}_{\underline{C}}\ N_{\mathsf{ret}}$$

Proof. The first equation is proved as follows:

$$\begin{array}{l}
\Gamma \vdash (\mathsf{op}_V^{FA}(y'.M)) \text{ handled with } \{\mathsf{op}_x(x') \mapsto N_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathrm{eff}}} \text{ to } y{:}A \text{ in}_{\underline{C}}\ N_{\mathsf{ret}} \\
\quad = \mathsf{force}_{\underline{C}}\left(\mathsf{thunk}\left((\mathsf{op}_V^{FA}(y'.M)) \text{ to } y{:}A \text{ in } \mathsf{force}_{\langle U\underline{C}, \{V_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathrm{eff}}}\rangle} \mathsf{thunk}\ (N_{\mathsf{ret}})\right)\right) \\
\quad = \mathsf{force}_{\underline{C}}\left(\mathsf{thunk}\left(\mathsf{op}_V^{\langle U\underline{C}, \{V_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathrm{eff}}}\rangle}(y'.M \text{ to } y{:}A \text{ in } \mathsf{force}_{\langle U\underline{C}, \{V_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathrm{eff}}}\rangle}\ (\mathsf{thunk}\ N_{\mathsf{ret}}))\right)\right) \\
\quad = \mathsf{force}_{\underline{C}}\left(\mathsf{thunk}\left(\mathsf{force}_{\langle U\underline{C}, \{V_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathrm{eff}}}\rangle}\left(V_{\mathsf{op}}\left\langle V, \lambda y'{:}O[V/x].\mathsf{thunk}\ (\right.\right.\right.\right. \\
\qquad\qquad\qquad\qquad M \text{ to } y{:}A \text{ in } \mathsf{force}_{\langle U\underline{C}, \{V_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathrm{eff}}}\rangle}\ (\mathsf{thunk}\ N_{\mathsf{ret}}))\left.\left.\left.\left.\right\rangle\right)\right)\right)\right) \\
\quad = \mathsf{force}_{\underline{C}}\left(V_{\mathsf{op}}\left\langle V, \lambda y'{:}O[V/x].\mathsf{thunk}\ (M \text{ to } y{:}A \text{ in } \mathsf{force}_{\langle U\underline{C}, \{V_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathrm{eff}}}\rangle}\ (\mathsf{thunk}\ N_{\mathsf{ret}}))\right\rangle\right) \\
\quad = \mathsf{force}_{\underline{C}}\left(\mathsf{thunk}\left(N_{\mathsf{op}}[V/x]\left[\lambda y'{:}O[V/x].\mathsf{thunk}\ (M \text{ to } y{:}A \text{ in } \mathsf{force}_{\langle U\underline{C}, \{V_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathrm{eff}}}\rangle}\ (\mathsf{thunk}\ N_{\mathsf{ret}}))/x'\right]\right)\right) \\
\quad = N_{\mathsf{op}}[V/x]\left[\lambda y'{:}O[V/x].\mathsf{thunk}\ (M \text{ to } y{:}A \text{ in } \mathsf{force}_{\langle U\underline{C}, \{V_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathrm{eff}}}\rangle}\ (\mathsf{thunk}\ N_{\mathsf{ret}}))/x'\right] \\
\quad = N_{\mathsf{op}}[V/x]\left[\lambda y'{:}O[V/x].\mathsf{thunk}\ (\mathsf{force}_{\underline{C}}\ (\mathsf{thunk}\ (M \text{ to } y{:}A \text{ in } \mathsf{force}_{\langle U\underline{C}, \{V_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathrm{eff}}}\rangle}\ (\mathsf{thunk}\ N_{\mathsf{ret}}))))/x'\right] \\
\quad = N_{\mathsf{op}}[V/x]\left[\lambda y'{:}O[V/x].\mathsf{thunk}\ (M \text{ handled with } \{\mathsf{op}_x(x') \mapsto N_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathrm{eff}}} \text{ to } y{:}A \text{ in}_{\underline{C}}\ N_{\mathsf{ret}})/x'\right] : \underline{C}
\end{array}$$

The second equation is proved as follows:

$$\begin{aligned}
&\Gamma \vdash (\mathtt{return}\ V)\ \mathtt{handled\ with}\ \{\mathsf{op}_x(x') \mapsto N_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\mathsf{eff}}}\ \mathtt{to}\ y{:}A\ \mathtt{in}_{\underline{C}}\ N_{\mathsf{ret}} \\
&\quad = \mathtt{force}_{\underline{C}}\left(\mathtt{thunk}\left((\mathtt{return}\ V)\ \mathtt{to}\ y{:}A\ \mathtt{in}\ \mathtt{force}_{\langle U\underline{C},\{V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathsf{eff}}}\rangle}\ (\mathtt{thunk}\ N_{\mathsf{ret}})\right)\right) \\
&\quad = \mathtt{force}_{\underline{C}}\left(\mathtt{thunk}\left(\mathtt{force}_{\langle U\underline{C},\{V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathsf{eff}}}\rangle}\ (\mathtt{thunk}\ N_{\mathsf{ret}}[V/y])\right)\right) \\
&\quad = \mathtt{force}_{\underline{C}}\ (\mathtt{thunk}\ N_{\mathsf{ret}}[V/y]) \\
&\quad = N_{\mathsf{ret}}[V/y] : \underline{C}
\end{aligned}$$

□

It is worth recalling that Plotkin and Pretnar do not enforce the correctness of their handlers during typechecking as it is in general undecidable (Plotkin and Pretnar 2013, §6). In particular, they do not require the family of user-defined terms N_{op} to satisfy the equations given in $\mathcal{E}_{\mathsf{eff}}$. We will address decidable typechecking in future extensions of this work. For example, one could develop a normaliser that is optimised for important fibred effect theories (e.g., for state, as studied in the simply typed setting by Ahman and Staton (2013, §5.2)) and require programmers to manually prove equations that can not be established automatically. To enable the latter, we could change EMLTT to use propositional equalities in proof obligations instead of definitional equations.

4.4 Handling computations into values

We conclude this section by noting that in addition to the standard (computation term variant of the) handling construct, as discussed in §4.3, we can also use the user-defined algebra type and the composition operations to define a handling construct that handles computations directly into values, e.g., as discussed by Ahman and Staton (2013, §6) in the context of Levy's fine-grain call-by-value language (Levy 2004, Appendix A.3.2). We use this value term variant of the handling construct in §7 to define type-theoretic predicates on computations.

Proposition 4.4. *The following typing rule for well-formed value terms is derivable:*

$$\frac{\begin{array}{c}\Gamma, x{:}I, x'{:}O \to B \vdash V_{\mathsf{op}} : B \quad (\mathsf{op} : (x : I) \longrightarrow O \in \mathcal{S}_{\mathit{eff}}) \\ \Gamma \vdash M : FA \quad \Gamma \vdash B \quad \Gamma, y{:}A \vdash V_{\mathsf{ret}} : B \\ \Gamma \vdash \{\lambda x''{:}(\Sigma x{:}I.O \to B).\mathtt{pm}\ x''\ \mathtt{as}\ (x{:}I, x'{:}O \to B)\ \mathtt{in}\ V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathit{eff}}}\ \mathtt{on}\ B\end{array}}{\Gamma \vdash M\ \mathtt{handled\ with}\ \{\mathsf{op}_x(x') \mapsto V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathit{eff}}}\ \mathtt{to}\ y{:}A\ \mathtt{in}_B\ V_{\mathsf{ret}} : B}$$

where

$$\begin{aligned}
&M\ \mathtt{handled\ with}\ \{\mathsf{op}_x(x') \mapsto V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathit{eff}}}\ \mathtt{to}\ y{:}A\ \mathtt{in}_B\ V_{\mathsf{ret}} \\
&\quad \stackrel{\mathit{def}}{=} \\
&\mathtt{thunk}\ (M\ \mathtt{to}\ y{:}A\ \mathtt{in}\ \mathtt{force}_{\langle \mathrm{B},\{\lambda x''.\mathtt{pm}\ x''\ \mathtt{as}\ (x,x')\ \mathtt{in}\ V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathit{eff}}}\rangle}\ V_{\mathsf{ret}})
\end{aligned}$$

satisfying two β-equations analogous to the definitional equations considered in Prop. 4.3.

5 AN ALTERNATIVE PRESENTATION OF EMLTT WITH FIBRED ALGEBRAIC EFFECTS

It is worth noting that as we only consider EMLTT with fibred algebraic effects in this paper, it would be possible to give EMLTT a different presentation, namely, by omitting computation variables z and homomorphism terms K, and instead use value variables and the auxiliary judgement $\Gamma, x{:}U\underline{C} \vdash_{\mathsf{hom}} N : \underline{D}$ to define and type the elimination form for the computational Σ-type $\Sigma x{:}A.\underline{C}$, analogously to the composition operations introduced in §4.2 . In more detail, this alternative presentation of EMLTT would involve the following elimination form for $\Sigma x{:}A.\underline{C}$:

$$\frac{\Gamma \vdash M : \Sigma x{:}A.\underline{C} \quad \Gamma \vdash \underline{D} \quad \Gamma, x{:}A, y{:}U\underline{C} \vdash_{\mathsf{hom}} N : \underline{D}}{\Gamma \vdash M\ \mathtt{to}\ (x{:}A, y{:}U\underline{C})\ \mathtt{in}_{\underline{D}}\ N : \underline{D}}$$

Observe that while the computation term M is now eliminated into a pair of values, the auxiliary judgement $\Gamma, x{:}A, y{:}U\underline{C} \vdash_{\mathsf{hom}} N : \underline{D}$ ensures that the value variable y is used in N as if it was a computation variable.

We note that in this paper we choose to include both homomorphism terms and the auxiliary judgement $\Gamma, x:A, y:U\underline{C} \vdash_{\mathsf{hom}} N : \underline{D}$ for two main reasons. First, as one of the main aims of this paper is to show how to formally extend the language of Ahman et al. (2016) with handlers of fibred algebraic effects, we wanted to keep the underlying language as close to op. cit. as possible. Second, aesthetically, using homomorphism terms provides a cleaner presentation of the elimination form for $\Sigma x:A.\underline{C}$, compared to equational proof obligations.

6 META-THEORY

We now discuss some meta-theoretic properties of the extension of EMLTT with fibred algebraic effects and their handlers, given by $\mathcal{T}_{\mathrm{eff}} = (\mathcal{S}_{\mathrm{eff}}, \mathcal{E}_{\mathrm{eff}})$. As various typing rules and definitional equations now include (translations of) effect terms, we also prove corresponding properties for effect terms, which the results for EMLTT then use.

6.1 Weakening and substitution

First, we note that weakening is admissible for value variables. Due to the disjointness requirement on value contexts used in Fig. 3, we also require the given variable to be fresh with respect to the equations given in $\mathcal{E}_{\mathrm{eff}}$.

Theorem 6.1 (Weakening). *Given $\Gamma_1, \Gamma_2 \vdash B$, $\Gamma_1 \vdash A$, and x such that $x \notin Vars(\Gamma_1, \Gamma_2)$ and $x \notin Vars(\Gamma)$ (for all equations $\Gamma \mid \Delta \vdash T_1 = T_2 \in \mathcal{E}_{\mathit{eff}}$), then $\Gamma_1, x:A, \Gamma_2 \vdash B$, and similarly for other types, terms, and definitional equations.*

Next, we note that substitution is admissible for both value and computation variables.

Proposition 6.2. *Given $\Gamma \mid \Delta \vdash T$, y such that $y \notin Vars(\Gamma)$, and W such that $FVV(W) \cap Vars(\Gamma) = \emptyset$, then*

$$(\!|\, T \,|\!)_{A;\overrightarrow{V_i};\overrightarrow{V'_j};\overrightarrow{W_{\mathrm{op}}}}[W/y] = (\!|\, T \,|\!)_{A[W/y];\overrightarrow{V_i[W/y]};\overrightarrow{V'_j[W/y]};\overrightarrow{W_{\mathrm{op}}[W/y]}}$$

Theorem 6.3 (Substitution). *Given $\Gamma_1, x:A, \Gamma_2 \vdash B$, and $\Gamma_1 \vdash V : A$, then we have $\Gamma_1, \Gamma_2[V/x] \vdash B[V/x]$, and similarly for other judgments of types, terms, and definitional equations.*

Theorem 6.4. *Given $\Gamma \mid z:\underline{C} \vdash K : \underline{D}$ and $\Gamma \vdash M : \underline{C}$, then we have $\Gamma \vdash K[M/z] : \underline{D}$.*

Theorem 6.5. *Given $\Gamma \mid z_2:\underline{D}_1 \vdash L : \underline{D}_2$ and $\Gamma \mid z_1:\underline{C} \vdash K : \underline{D}_1$, then we have $\Gamma \mid z_1:\underline{C} \vdash L[K/z_2] : \underline{D}_2$.*

Proof. We first prove Prop. 6.2 and then Thm. 6.3, 6.4, and 6.5, all by induction on the given derivations. □

Finally, we note that judgments of well-formed types, etc. only refer to well-formed contexts, etc.

Proposition 6.6. *Given $\Gamma \mid \Delta \vdash T$ and Γ' such that $Vars(\Gamma') \cap Vars(\Gamma) = \emptyset$, $\Gamma' \vdash A$, and the value terms in the subscripts are well-typed in Γ' (as in Fig. 3), then we have $\Gamma' \vdash (\!|\, T \,|\!)_{A;\overrightarrow{V_i};\overrightarrow{V'_j};\overrightarrow{W_{\mathrm{op}}}} : A$.*

Proposition 6.7. *Given $\Gamma \mid \Delta \vdash T$ and Γ' such that $Vars(\Gamma') \cap Vars(\Gamma) = \emptyset$, $\Gamma' \vdash A = B$, and the corresponding value terms in the subscripts are definitionally equal in Γ', then we have $\Gamma' \vdash (\!|\, T \,|\!)_{A;\overrightarrow{V_i};\overrightarrow{V'_j};\overrightarrow{V_{\mathrm{op}}}} = (\!|\, T \,|\!)_{B;\overrightarrow{W_i};\overrightarrow{W'_j};\overrightarrow{W_{\mathrm{op}}}} : A$.*

Theorem 6.8. *Given $\Gamma \vdash V : A$, then $\vdash \Gamma$ and $\Gamma \vdash A$, and similarly for other judgments of types, terms and equations.*

Proof. We first prove Prop. 6.6, and then Prop. 6.7 and Thm. 6.8, all by induction on the given derivations. □

6.2 Derivable definitional equations

We begin by showing that the disjointness requirement on value contexts that we used in Fig. 3 can be omitted.

Proposition 6.9. *Given $\Gamma \mid \Delta \vdash T_1 = T_2 \in \mathcal{E}_{\mathit{eff}}$, then*

$$\frac{\begin{array}{ll}\Gamma' \vdash V_i : A_i[V_1/x_1, \ldots, V_{i-1}/x_{i-1}] & (1 \leq i \leq n) \\ \Gamma' \vdash \underline{C} \quad \Gamma' \vdash V'_j : A'_j[\overrightarrow{V_i}/\overrightarrow{x_i}] \to U\underline{C} & (1 \leq j \leq m)\end{array}}{\Gamma' \vdash (\!|\, T_1 \,|\!)_{U\underline{C};\overrightarrow{V_i};\overrightarrow{V'_j};\overrightarrow{W_{\mathrm{op}}}} = (\!|\, T_2 \,|\!)_{U\underline{C};\overrightarrow{V_i};\overrightarrow{V'_j};\overrightarrow{W_{\mathrm{op}}}} : U\underline{C}}$$

Proof. There are two cases to consider. If Γ' and Γ are disjoint, we use the corresponding restricted rule from Fig. 3. If Γ' and Γ are non-disjoint, we first systematically replace the overlapping variables with fresh ones using the weakening and substitution theorems from §6, and then use the corresponding restricted rule from Fig. 3. □

Next, we note that one can derive specialised versions of the general algebraicity equation from Fig. 3.

Proposition 6.10. *We can derive the following specialised algebraicity equation for sequential composition:*

$$\frac{\Gamma \vdash V : I \quad \Gamma, y{:}O[V/x] \vdash M : FA \quad \Gamma \vdash \underline{C} \quad \Gamma, y'{:}A \vdash N : \underline{C}}{\Gamma \vdash \mathsf{op}_V^{FA}(y.M)\ \mathtt{to}\ y'{:}A\ \mathtt{in}\ N = \mathsf{op}_V^{\underline{C}}(y.M\ \mathtt{to}\ y'{:}A\ \mathtt{in}\ N) : \underline{C}}$$

and similarly for other computation term formers.

Finally, we present some useful derivable equations for composition operations.

Proposition 6.11. *The following unit and associativity equations are derivable for the composition operations:*

$$\frac{\Gamma \vdash M : \underline{C}}{\Gamma \vdash M\ \mathtt{as}\ x{:}U\underline{C}\ \mathtt{in}\ \mathtt{force}_{\underline{C}}\, x = M : \underline{C}}$$

$$\frac{\Gamma \vdash M : \underline{C}_1 \quad \Gamma \vdash \underline{C}_2 \quad \Gamma \vdash \underline{D} \quad \Gamma, x_1{:}U\underline{C}_1 \vdash_{\mathsf{hom}} N_1 : \underline{C}_2 \quad \Gamma, x_2{:}U\underline{C}_2 \vdash_{\mathsf{hom}} N_2 : \underline{D}}{\Gamma \vdash M\ \mathtt{as}\ x_1{:}U\underline{C}_1\ \mathtt{in}\ (N_1\ \mathtt{as}\ x_2{:}U\underline{C}_2\ \mathtt{in}\ N_2) = (M\ \mathtt{as}\ x_1{:}U\underline{C}_1\ \mathtt{in}\ N_1)\ \mathtt{as}\ x_2{:}U\underline{C}_2\ \mathtt{in}\ N_2 : \underline{D}}$$

and similarly for the composition operation for homomorphism terms.

Proposition 6.12. *The following definitional equations are derivable:*

$$\frac{\Gamma \vdash M : FA \quad \Gamma \vdash \underline{C} \quad \Gamma \vdash \underline{D} \quad \Gamma, x_1{:}A \vdash N_1 : \underline{C} \quad \Gamma, x_2{:}U\underline{C} \vdash_{\mathsf{hom}} N_2 : \underline{D}}{\Gamma \vdash M\ \mathtt{to}\ x_1{:}A\ \mathtt{in}\ (N_1\ \mathtt{as}\ x_2{:}U\underline{C}\ \mathtt{in}\ N_2) = (M\ \mathtt{to}\ x_1{:}A\ \mathtt{in}\ N_1)\ \mathtt{as}\ x_2{:}U\underline{C}\ \mathtt{in}\ N_2 : \underline{D}}$$

$$\frac{\Gamma \vdash M : \underline{C} \quad \Gamma, x_1{:}U\underline{C} \vdash_{\mathsf{hom}} N_1 : FA \quad \Gamma \vdash \underline{D} \quad \Gamma, x_2{:}A \vdash N_2 : \underline{D}}{\Gamma \vdash M\ \mathtt{as}\ x_1{:}U\underline{C}\ \mathtt{in}\ (N_1\ \mathtt{to}\ x_2{:}A\ \mathtt{in}\ N_2) = (M\ \mathtt{as}\ x_1{:}U\underline{C}\ \mathtt{in}\ N_1)\ \mathtt{to}\ x_2{:}A\ \mathtt{in}\ N_2 : \underline{D}}$$

and similarly for computational pattern-matching, and the corresponding homomorphism terms.

Proposition 6.13. *The composition operations commute with computational pairing, computational lambda abstraction, and computational and homomorphic function applications, e.g., we have*

$$\frac{\Gamma \vdash M : \underline{C} \quad \Gamma \vdash V : A \qquad \Gamma, y{:}A \vdash \underline{D} \quad \Gamma, x{:}U\underline{C} \vdash_{\mathsf{hom}} N : \underline{D}[V/y]}{\begin{array}{l}\Gamma \vdash M\ \mathtt{as}\ x{:}U\underline{C}\ \mathtt{in}\ \langle V, N\rangle \\ \quad = \langle V, M\ \mathtt{as}\ x{:}U\underline{C}\ \mathtt{in}\ N\rangle : \Sigma y{:}A.\underline{D}\end{array}}$$

$$\frac{\Gamma \vdash M : \underline{C} \quad \Gamma \vdash V : \underline{D}_1 \multimap \underline{D}_2 \qquad \Gamma, y_1{:}U\underline{C} \vdash_{\mathsf{hom}} N : \underline{D}_1}{\begin{array}{l}\Gamma \vdash M\ \mathtt{as}\ y_1{:}U\underline{C}\ \mathtt{in}\ V\ N \\ \quad = V\ (M\ \mathtt{as}\ y_1{:}U\underline{C}\ \mathtt{in}\ N) : \underline{D}_2\end{array}}$$

7 USING HANDLERS TO REASON ABOUT EFFECTFUL COMPUTATIONS

We now demonstrate that our type-based treatment of handlers provides a useful mechanism for reasoning about effectful computations, giving us an alternative to defining predicates using propositional equality on thunks.

In order to facilitate such reasoning, we introduce a *universe* à la Tarski (Martin-Löf 1984), by extending eMLTT with a *value universe* VU, the corresponding *decoding function* $\mathsf{El}(V)$, and the corresponding *codes of value types*:

$$A ::= \ldots \mid \mathsf{VU} \mid \mathsf{El}(V) \qquad V ::= \ldots \mid \mathtt{unit\text{-}c} \mid \mathtt{empty\text{-}c} \mid \mathtt{coprod\text{-}c}(\mathtt{V}, \mathtt{W}) \mid \mathtt{sig\text{-}c}(V, x.W) \mid \mathtt{pi\text{-}c}(V, x.W)$$

We also extend eMLTT with corresponding typing rules and definitional equations, e.g., for `pi-c` we have

$$\frac{\Gamma \vdash V : \mathsf{VU} \quad \Gamma, x{:}\mathsf{El}(V) \vdash W : \mathsf{VU}}{\Gamma \vdash \mathtt{pi\text{-}c}(V, x.W) : \mathsf{VU}} \qquad \frac{\Gamma \vdash V : \mathsf{VU} \quad \Gamma, x{:}\mathsf{El}(V) \vdash W : \mathsf{VU}}{\Gamma \vdash \mathsf{El}(\mathtt{pi\text{-}c}(V, x.W)) = \Pi x{:}\mathsf{El}(V).\mathsf{El}(W)}$$

Using this universe, we can define our type-theoretic predicates on computations (of type FA) as value terms of the form $\Gamma \vdash V : UFA \to \mathsf{VU}$, with the aim of using these predicates to refine (thunks of) computations using value Σ-types, i.e., as $\Sigma x{:}UFA.\mathsf{El}(Vx)$. In particular, we define the predicates $\Gamma \vdash V : UFA \to \mathsf{VU}$ by equipping VU with an algebra for the given notion of computation, and by using the handling construct from Prop. 4.4.

It is worth noting that our approach of defining predicates on computations (essentially, defining types that depend on effectful computations in a "well-behaved" manner) by equipping the universe with an algebra structure is reminiscent of the recent work by Pédrot and Tabareau (2017) we mentioned earlier. In particular, their monadic translation of type theory crucially relies on equipping types with an algebra structure for the given monad.

Below, we give examples of two kinds of predicates on effectful computations: i) lifting predicates given on return values to effectful computations (§7.1); and ii) specifying patterns of allowed effects in computations (§7.2).

7.1 Lifting predicates from return values to effectful computations

Lifting predicates given on return values to effectful computations is easiest when the given fibred effect theory does not contain any equations. Thus, let us consider the theory $\mathcal{T}_{\mathrm{I/O}}$ of *input-output of bits* from §4.1 for our first example; other equation-free fibred algebraic effects can be reasoned about similarly, e.g., exceptions.

Then, assuming given a predicate $\Gamma \vdash V_P : A \to \mathsf{VU}$ on return values, we lift V_P to a predicate $V_{\widehat{P}}$ on UFA by

$$V_{\widehat{P}} \stackrel{\text{def}}{=} \lambda y{:}UFA\,.\,(\mathtt{force}_{FA}\,y)\ \mathtt{handled\ with}\ \{\mathsf{op}_x(x') \mapsto V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathrm{I/O}}}\ \mathtt{to}\ y'{:}A\ \mathtt{in}_{\mathsf{VU}}\ (V_P\,y')$$

where we let $\mathtt{bit\text{-}c} \stackrel{\text{def}}{=} \mathtt{coprod\text{-}c}(\mathtt{unit\text{-}c}, \mathtt{unit\text{-}c})$ and define

$$x{:}1, x'{:}1+1 \to \mathsf{VU} \vdash V_{\mathsf{read}} \stackrel{\text{def}}{=} \mathtt{sig\text{-}c}(\mathtt{bit\text{-}c}, y'.\,x'\,y') : \mathsf{VU}$$
$$x{:}1+1, x'{:}1 \to \mathsf{VU} \vdash V_{\mathsf{write}} \stackrel{\text{def}}{=} x' \star$$

On closer inspection, we see that $V_{\widehat{P}}$ agrees with the possibility modality from Evaluation Logic (Pitts 1991), in that a computation satisfies $V_{\widehat{P}}$ if there *exists* a return value that satisfies V_P. If we replace `sig-c` (code for value Σ-type) with `pi-c` (code for value Π-type), we get a predicate that holds if *all* the return values satisfy V_P.

For our second example of lifting predicates given on return values to effectful computations, let us consider a fibred effect theory that also includes equations, namely, the theory $\mathcal{T}_{\mathrm{GS}}$ of *global state* from §3.2.

Then, assuming given a predicate $\Gamma \vdash V_Q : A \to \mathsf{S} \to \mathsf{VU}$ on return values and *final states*, where $\mathsf{S} \stackrel{\text{def}}{=} \Pi x{:}\mathsf{Loc}.\mathsf{Val}$, we can define the predicate

$$V_{\widehat{Q}} \stackrel{\text{def}}{=} \lambda y{:}UFA\,.\,\lambda x_S{:}\mathsf{S}\,.\ \mathtt{fst}\left(\left(\mathtt{thunk}\ \left((\mathtt{force}_{FA}\,y)\ \mathtt{handled\ with}\ \{\mathsf{op}_x(x') \mapsto V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathrm{GS}}}\ \mathtt{to}\ y'{:}A\ \mathtt{in}_{\mathsf{S}\to\mathsf{VU}\times\mathsf{S}}\ V_{\mathsf{ret}}\right)\right)\,x_S\right)$$

on (thunks of) computations and *initial states*, with V_{get} and V_{put} defined using the natural representation of stateful programs as functions $\mathsf{S} \to \mathsf{VU} \times \mathsf{S}$. In other words, V_{get} and V_{put} are defined as if they were operations of the free algebra on VU for an equational theory of state corresponding to the fibred effect theory $\mathcal{T}_{\mathrm{GS}}$. Further,

$$\Gamma, y{:}UFA, x_S{:}\mathsf{S}, y'{:}A \vdash V_{\mathsf{ret}} \stackrel{\text{def}}{=} \lambda x'_S{:}\mathsf{S}\,.\,\langle V_Q\,y'\,x'_S, x'_S\rangle : \mathsf{S} \to \mathsf{VU}\times\mathsf{S}$$

On closer inspection, $V_{\widehat{Q}}$ corresponds to Dijkstra's weakest precondition semantics of stateful programs (Dijkstra 1975), e.g., taking $\mathsf{Loc} \stackrel{\text{def}}{=} 1$ and omitting location arguments, we can prove the following definitional equations:

$$\Gamma \vdash V_{\widehat{Q}}\ (\mathtt{thunk}(\mathtt{return}\ V))\ V_S = V_Q\,V\,V_S : \mathsf{VU}$$
$$\Gamma \vdash V_{\widehat{Q}}\ (\mathtt{thunk}(\mathtt{get}^{FA}(y.M)))\,V_S = V_{\widehat{Q}}\ (\mathtt{thunk}\,M[V_S/y])\,V_S : \mathsf{VU}$$
$$\Gamma \vdash V_{\widehat{Q}}\ (\mathtt{thunk}(\mathtt{put}^{FA}_{V'_S}(M)))\,V_S = V_{\widehat{Q}}\ (\mathtt{thunk}\,M)\,V'_S : \mathsf{VU}$$

7.2 Specifying patterns of allowed effects in computations

Analogously to lifting predicates from return values to effectful computations, specifying patterns of allowed effects is easiest when the given fibred effect theory does not contain any equations. Thus, for simplicity, we again consider the theory $\mathcal{T}_{\text{I/O}}$ of *input-output* for our examples; other equation-free effects are treated similarly.

As a first example, we consider a coarse grained specification on I/O-effects, namely, disallowing all writes:

$$V_{\text{no-w}} \stackrel{\text{def}}{=} \lambda y{:}UFA\,.\,(\mathtt{force}_{FA}\,y)\ \mathtt{handled\ with}\ \{\mathtt{op}_x(x') \mapsto V_{\mathtt{op}}\}_{\mathtt{op}\in\mathcal{S}_{\text{I/O}}}\ \mathtt{to}\ y'{:}A\ \mathtt{in}_{\mathsf{VU}}\ \mathtt{unit\text{-}c}$$

where

$$x{:}1, x'{:}1+1 \to \mathsf{VU} \vdash V_{\mathsf{read}} \stackrel{\text{def}}{=} \mathtt{pi\text{-}c}(\mathtt{bit\text{-}c}, y'.\,x'\,y')$$
$$x{:}1+1, x'{:}1 \to \mathsf{VU} \vdash V_{\mathsf{write}} \stackrel{\text{def}}{=} \mathtt{empty\text{-}c}$$

For example, we can then show that $\mathtt{read}^{FA}(x.\mathtt{write}^{FA}_{V}(M))$ does not satisfy $V_{\text{no-w}}$ because we have

$$\Gamma \vdash \mathsf{El}(V_{\text{no-w}}(\mathtt{thunk}\,(\mathtt{read}^{FA}(y.\mathtt{write}^{FA}_{V}(M))))) = \Pi y{:}1{+}1.0 \cong 0$$

As a more involved example, we consider specifications on I/O-effects in the style of session types (Honda et al. 1998). In particular, we first assume given an inductive value type $\diamond \vdash \mathsf{Protocol}$, given by three constructors:

$$\mathtt{r} : (1{+}1 \to \mathsf{Protocol}) \to \mathsf{Protocol} \qquad \mathtt{w} : (1{+}1 \to \mathsf{VU}) \times \mathsf{Protocol} \to \mathsf{Protocol} \qquad \mathtt{e} : \mathsf{Protocol}$$

describing *patterns of allowed I/O-effects*. Intuitively, `r` specifies that the first allowed I/O-effect is reading; `w` specifies that the first allowed I/O-effect is writing, with the value written required to satisfy the predicate given as an argument to `w`; and `e` specifies that no further communication must happen (end of communication). The Protocol-valued arguments of `r` and `w` specify how the computation is allowed to evolve after reading and writing.

Then, given some particular protocol $\Gamma \vdash V_{\mathsf{pr}} : \mathsf{Protocol}$, we can define a predicate

$$V_{\widehat{\mathsf{pr}}} \stackrel{\text{def}}{=} \lambda y{:}UFA\,.\,\Big(\mathtt{thunk}\,\big((\mathtt{force}_{FA}\,y)\ \mathtt{handled\ with}\ \{\mathtt{op}_x(x') \mapsto V_{\mathtt{op}}\}_{\mathtt{op}\in\mathcal{S}_{\text{I/O}}}\ \mathtt{to}\ y'{:}A\ \mathtt{in}_{\mathsf{Protocol}\to\mathsf{VU}}\ V_{\mathsf{ret}}\big)\Big)\,V_{\mathsf{pr}}$$

where the value terms V_{read}, V_{write}, and V_{ret} are defined as follows (for simplicity and better readability, we give their structural-recursive definitions by pattern-matching on their respective arguments of type Protocol):

$$\Gamma, y{:}UFA, y'{:}A \vdash V_{\mathsf{ret}}\quad \mathtt{e} \qquad \stackrel{\text{def}}{=} \mathtt{unit\text{-}c}$$
$$\Gamma, y{:}UFA, x{:}1, x'{:}1+1 \to \mathsf{Protocol} \to \mathsf{VU} \vdash V_{\mathsf{read}}\ (\mathtt{r}\ V'_{\mathsf{pr}}) \qquad \stackrel{\text{def}}{=} \mathtt{pi\text{-}c}(\mathtt{bit\text{-}c}, y.(x'\,y)\,(V'_{\mathsf{pr}}\,y))$$
$$\Gamma, y{:}UFA, x{:}1+1, x'{:}1 \to \mathsf{Protocol} \to \mathsf{VU} \vdash V_{\mathsf{write}}\ (\mathtt{w}\,\langle V_P, V'_{\mathsf{pr}}\rangle) \stackrel{\text{def}}{=} \mathtt{sig\text{-}c}(V_P\,x, y.\,x' \star V'_{\mathsf{pr}})$$

with all other cases defined to be equal to `empty-c`. As a result, a computation satisfies the predicate $V_{\widehat{\mathsf{pr}}}$ only if its I/O-effects precisely follow the specific pattern of I/O-effects specified by the given protocol V_{pr}.

It is worth noting that this example can be easily extended to account for *sets* of patterns of allowed I/O-effects. For example, we could extend Protocol with a fourth constructor $\mathtt{or} : \mathsf{Protocol} \times \mathsf{Protocol} \longrightarrow \mathsf{Protocol}$, and the above definitions of value terms V_{read}, V_{write}, and V_{ret} with the case $V\ (V'_{\mathsf{pr}}\ \mathtt{or}\ V''_{\mathsf{pr}}) \stackrel{\text{def}}{=} \mathtt{coprod\text{-}c}(V\,V'_{\mathsf{pr}}, V\,V''_{\mathsf{pr}})$.

Finally, we highlight that one can also easily combine these specifications of I/O-effects with those discussed in §7.1, namely, by replacing `unit-c` in the definition of V_{ret} with a suitable predicate V_P on return values.

8 SEMANTICS

We conclude this paper by describing how to give a denotational semantics to our extension of EMLTT with fibred algebraic effects and their handlers, given by $\mathcal{T}_{\text{eff}} = (\mathcal{S}_{\text{eff}}, \mathcal{E}_{\text{eff}})$. The semantics we develop is an instance of a more general class of models of EMLTT, based on *fibrations* (functors with extra structure for modelling substitution, Σ- and Π-types, etc.) and *adjunctions* between them, as investigated in detail by Ahman et al. (2016).

We proceed in three steps. First, we recall the interpretation of the pure fragment of EMLTT from the work of Ahman et al. (2016). Next, we derive a countable Lawvere theory $\mathcal{L}_{\mathcal{T}_{\text{eff}}}$ from the given fibred effect theory $\mathcal{T}_{\text{eff}}$. Finally, we define the interpretation of the rest of EMLTT using the models of the countable Lawvere theory $\mathcal{L}_{\mathcal{T}_{\text{eff}}}$.

We leave studying the denotational semantics of the extension of EMLTT with universes (see §7) for future work, expecting it to closely follow the standard fibrational treatment of induction-recursion (Ghani et al. 2013).

8.1 Families fibrations

We begin with an overview of the particular fibrations we use for defining our denotational semantics. For a general treatment of fibrations and their use in modelling dependent types, we suggest the book by Jacobs (1999).

Given a category C, it is well-known that one can define a new category $\mathsf{Fam}(C)$ whose objects are pairs (X, A) of a set X and a functor $A : X \longrightarrow C$ (treating X as a discrete category); the morphisms $(X, A) \to (Y, B)$ are pairs of a function $f : X \longrightarrow Y$ and a natural transformation $g : A \longrightarrow B \circ f$. The corresponding *$C$-valued families fibration* $\mathsf{fam}_C : \mathsf{Fam}(C) \longrightarrow \mathsf{Set}$ is defined on objects as $\mathsf{fam}_C(X, A) \stackrel{\text{def}}{=} X$ and on morphisms as $\mathsf{fam}_C(f, g) \stackrel{\text{def}}{=} f$.

For any set X, the category $\mathsf{Fam}_X(C)$ is called the *fibre* over X. It is a subcategory of $\mathsf{Fam}(C)$ whose objects and morphisms are of the form (X, A) and (id_X, g). Given a function $f : X \longrightarrow Y$, the corresponding *reindexing functor* $f^* : \mathsf{Fam}_Y(C) \longrightarrow \mathsf{Fam}_X(C)$ is given by $f^*(Y, A) \stackrel{\text{def}}{=} (X, A \circ f)$ and analogously on morphisms. As standard, we write $\overline{f}(Y, A) \stackrel{\text{def}}{=} (f, (\mathsf{id}_{A(f(x))})_x) : f^*(Y, A) \longrightarrow (Y, A)$ for the *Cartesian morphism* over $f : X \longrightarrow Y$.

It is worth noting that we get a prototypical model of dependent types when we take $C \stackrel{\text{def}}{=} \mathsf{Set}$. In this case, there also exists a pair of adjunctions $\mathsf{fam}_{\mathsf{Set}} \dashv 1 \dashv \{-\}$, where the *terminal object functor* $1 : \mathsf{Set} \longrightarrow \mathsf{Fam}(\mathsf{Set})$ is given by $1(X) \stackrel{\text{def}}{=} (X, x \mapsto \{\star\})$ and the *comprehension functor* $\{-\} : \mathsf{Fam}(\mathsf{Set}) \longrightarrow \mathsf{Set}$ by $\{(X, A)\} \stackrel{\text{def}}{=} \coprod_{x \in X} A(x)$. The latter provides semantics to context extensions $\Gamma, x : A$; it also gives us canonical *projection maps* $\pi_{(X,A)} : \{(X, A)\} \longrightarrow X$.

8.2 Interpretation of the pure fragment of EMLTT

As a first step, we recall from the work of Ahman et al. (2016) how the pure fragment of EMLTT is interpreted in the families of sets fibration $\mathsf{fam}_{\mathsf{Set}} : \mathsf{Fam}(\mathsf{Set}) \longrightarrow \mathsf{Set}$ we described in the previous section.

In detail, the interpretation of the pure fragment of EMLTT is defined as a *partial interpretation function* $[\![-]\!]$, which, if defined, maps a context Γ to a set $[\![\Gamma]\!]$, a context Γ and value type A to an object $[\![\Gamma; A]\!]$ in $\mathsf{Fam}_{[\![\Gamma]\!]}(\mathsf{Set})$, and a context Γ and value term V to $[\![\Gamma; V]\!] : 1([\![\Gamma]\!]) \longrightarrow ([\![\Gamma]\!], A)$ in $\mathsf{Fam}_{[\![\Gamma]\!]}(\mathsf{Set})$, for some functor $A : [\![\Gamma]\!] \longrightarrow \mathsf{Set}$. For better readability, we denote the first and second components of $[\![\Gamma; A]\!]$ and $[\![\Gamma; V]\!]$ using subscripts 1, 2.

First, the types Nat, 1, 0, and $A + B$ are interpreted by using the corresponding categorical structure in the fibres of $\mathsf{Fam}(\mathsf{Set})$, e.g., $[\![\Gamma; \mathsf{Nat}]\!] \stackrel{\text{def}}{=} ([\![\Gamma]\!], \gamma \mapsto \mathbb{N})$ and $[\![\Gamma; A + B]\!] \stackrel{\text{def}}{=} [\![\Gamma; A]\!] + [\![\Gamma; B]\!] \stackrel{\text{def}}{=} ([\![\Gamma]\!], \gamma \mapsto [\![\Gamma; A]\!]_2(\gamma) + [\![\Gamma; B]\!]_2(\gamma))$.

Next, assuming that $[\![\Gamma; A]\!]$ and $[\![\Gamma, x : A; B]\!]$ are defined, and further that $[\![\Gamma, x : A; B]\!]_1 = \coprod_{\gamma \in [\![\Gamma]\!]} [\![\Gamma; A]\!]_2(\gamma)$, then

$$[\![\Gamma; \Sigma x : A.B]\!] \stackrel{\text{def}}{=} ([\![\Gamma]\!], \gamma \mapsto \coprod_{a \in [\![\Gamma; A]\!]_2(\gamma)} [\![\Gamma, x : A; B]\!]_2(\langle \gamma, a \rangle))$$
$$[\![\Gamma; \Pi x : A.B]\!] \stackrel{\text{def}}{=} ([\![\Gamma]\!], \gamma \mapsto \prod_{a \in [\![\Gamma; A]\!]_2(\gamma)} [\![\Gamma, x : A; B]\!]_2(\langle \gamma, a \rangle))$$

Finally, propositional equality $V =_A W$ is interpreted in terms of equality of the denotations of V and W, i.e.,

$$[\![\Gamma; V =_A W]\!] \stackrel{\text{def}}{=} ([\![\Gamma]\!], \gamma \mapsto \{ \star \mid ([\![\Gamma; V]\!]_2)_\gamma(\star) = ([\![\Gamma; W]\!]_2)_\gamma(\star) \})$$

For value terms, one then defines, e.g., $[\![\Gamma; \mathtt{inl}_{A+B}\ V]\!]_1 \stackrel{\text{def}}{=} \mathsf{id}_{[\![\Gamma]\!]}$ and $([\![\Gamma; \mathtt{inl}_{A+B}\ V]\!]_2)_\gamma \stackrel{\text{def}}{=} \star \mapsto \mathsf{inl}\,(([\![\Gamma; V]\!]_2)_\gamma(\star))$, assuming that $[\![\Gamma; V]\!] : 1([\![\Gamma]\!]) \longrightarrow [\![\Gamma; A]\!]$ and $[\![\Gamma; B]\!]$ are defined—see Ahman et al. (2016) for further details.

From the soundness theorem for EMLTT (Ahman et al. 2016, Thm. 1), we get that $[\![-]\!]$ is in fact defined on all well-formed pure value types and well-typed pure value terms, and that it validates pure definitional equations.

8.3 Deriving a countable Lawvere theory from a fibred effect theory

As a next step, we now show how to derive a countable Lawvere theory from the given fibred effect theory $\mathcal{T}_{\text{eff}}$.

We begin by recalling some basic definitions and results about countable Lawvere theories from Power (2006).

A *countable Lawvere theory* consists of a small category $\mathcal{L}$ with countable products and a strict countable-product preserving identity-on-objects functor $I : \aleph_1^{\text{op}} \longrightarrow \mathcal{L}$, where $\aleph_1$ is the skeleton of the category of countable sets. A *model* of a countable Lawvere theory $\mathcal{L}$ in a category C with countable products is given by a countable-product preserving functor $\mathcal{M} : \mathcal{L} \longrightarrow C$. A *morphism* of models from $\mathcal{M}_1 : \mathcal{L} \longrightarrow C$ to $\mathcal{M}_2 : \mathcal{L} \longrightarrow C$ is given by a natural transformation $\mathcal{M}_1 \longrightarrow \mathcal{M}_2$. Models and morphisms between them form the category $\mathsf{Mod}(\mathcal{L}, C)$.

Conceptually, a countable Lawvere theory is nothing but an abstract, category-theoretic description of the clone of countable equational theories (Grätzer 1979). In particular, one should think of morphisms $n \longrightarrow 1$ in $\mathcal{L}$ as terms in n free variables, and of morphisms $n \longrightarrow m$ as m-tuples of terms in n free variables. Analogously, the models of a countable Lawvere theory correspond to the models of the corresponding equational theories.

We also recall from Power (2006) that there exists a canonical forgetful functor $U_{\mathcal{L}} : \mathsf{Mod}(\mathcal{L}, C) \longrightarrow C$, given on objects by $U_{\mathcal{L}}(\mathcal{M}) \stackrel{\text{def}}{=} \mathcal{M}(1)$. A well known result then states that if C is locally countably presentable (Adamek and Rosicky 1994), the functor $U_{\mathcal{L}}$ has a left adjoint $F_{\mathcal{L}} : C \longrightarrow \mathsf{Mod}(\mathcal{L}, C)$. Importantly for the purposes of this paper, Set is locally countably presentable. A further useful property of $\mathsf{Mod}(\mathcal{L}, \mathsf{Set})$ is that it is both complete and cocomplete. For better readability, we write Mod for $\mathsf{Mod}(\mathcal{L}_{\mathcal{T}_{\text{eff}}}, \mathsf{Set})$ in the rest of this paper.

To be able to derive a countable Lawvere theory $\mathcal{L}_{\mathcal{T}_{\text{eff}}}$ from the given fibred effect theory $\mathcal{T}_{\text{eff}}$, we require $\mathcal{T}_{\text{eff}}$ to also be *countable*, i.e., for all $\mathsf{op} : (x : I) \longrightarrow O \in \mathcal{S}_{\text{eff}}$, we require the interpretations $[\![x : I; O]\!]_2$ and $[\![\Gamma; A'_j]\!]_2$ to be given by families of countable sets (for all equations $\Gamma \mid \Delta \vdash T_1 = T_2 \in \mathcal{E}_{\text{eff}}$ and effect variables $w_j : A'_j \in \Delta$).

We then construct $\mathcal{L}_{\mathcal{T}_{\text{eff}}}$ by expanding $\mathcal{T}_{\text{eff}}$ into a countable equational theory (Grätzer 1979), analogously to how Plotkin and Pretnar (2013) expanded their effect theories. More specifically, $\mathcal{S}_{\text{eff}}$ determines a countable signature consisting of operation symbols $\mathsf{op}_i : |[\![x : I; O]\!]_2\ (\langle\star, i\rangle)|$, for all $\mathsf{op} : (x : I) \longrightarrow O \in \mathcal{S}_{\text{eff}}$ and $i \in [\![\diamond; I]\!]_2\ (\star)$. Every effect term $\Gamma \mid \Delta \vdash T$ then naturally determines a family of terms $\Delta^\gamma \vdash T^\gamma$ derivable from this countable signature (for all $\gamma \in [\![\Gamma]\!]$), where Δ^γ consists of variables $x^a_{w_j}$ for all $w_j : A_j \in \Delta$ and $a \in [\![\Gamma; A'_j]\!]_2(\gamma)$. In detail, we have

$$
\begin{aligned}
&(w_j(V))^\gamma && \stackrel{\text{def}}{=} x^{([\![\Gamma;V]\!]_2)_\gamma(\star)}_{w_j} && \\
&(\mathsf{op}_V(y.T))^\gamma && \stackrel{\text{def}}{=} \mathsf{op}_{([\![\Gamma;V]\!]_2)_\gamma(\star)}(T^{\langle\gamma,o\rangle})_{1\le o\le|[\![x:I;O]\!]_2\ (\langle\star,([\![\Gamma;V]\!]_2)_\gamma(\star)\rangle)|} && \\
&(\mathsf{pm}\ V\ \mathsf{as}\ (y_1 : B_1, y_2 : B_2)\ \mathsf{in}\ T)^\gamma && \stackrel{\text{def}}{=} T^{\langle\langle\gamma,b_1\rangle,b_2\rangle} && (\text{when } ([\![\Gamma;V]\!]_2)_\gamma(\star) = \langle b_1, b_2\rangle)\\
&(\mathsf{case}\ V\ \mathsf{of}\ (\mathsf{inl}(y_1 : B_1) \mapsto T_1, \mathsf{inr}(y_2 : B_2) \mapsto T_2))^\gamma && \stackrel{\text{def}}{=} T_1^{\langle\gamma,b\rangle} && (\text{when } ([\![\Gamma;V]\!]_2)_\gamma(\star) = \mathsf{inl}\, b)\\
&(\mathsf{case}\ V\ \mathsf{of}\ (\mathsf{inl}(y_1 : B_1) \mapsto T_1, \mathsf{inr}(y_2 : B_2) \mapsto T_2))^\gamma && \stackrel{\text{def}}{=} T_2^{\langle\gamma,b\rangle} && (\text{when } ([\![\Gamma;V]\!]_2)_\gamma(\star) = \mathsf{inr}\, b)
\end{aligned}
$$

We get a countable equational theory by taking equations $\Delta^\gamma \vdash T_1^\gamma = T_2^\gamma$, for all $\Gamma \mid \Delta \vdash T_1 = T_2 \in \mathcal{E}_{\text{eff}}$ and $\gamma \in [\![\Gamma]\!]$, and by closing these equations under rules of reflexivity, symmetry, transitivity, replacement, and substitution.

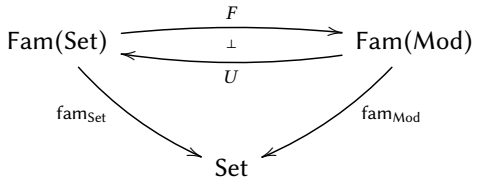
The countable Lawvere theory $\mathcal{L}_{\mathcal{T}_{\text{eff}}}$ is then given by taking the morphisms $n \longrightarrow m$ in $\mathcal{L}_{\mathcal{T}_{\text{eff}}}$ to be m-tuples $(\overrightarrow{x_i} \vdash t_j)_{1\le j\le m}$ of equivalence classes of terms in n variables (in the countable equational theory defined above). The identity morphisms are given by tuples of variables, while the composition of morphisms is given by substitution. We define the functor $I_{\mathcal{T}_{\text{eff}}} : \aleph_1^{\text{op}} \longrightarrow \mathcal{L}_{\mathcal{T}_{\text{eff}}}$ by $I_{\mathcal{T}_{\text{eff}}}(n) \stackrel{\text{def}}{=} n$ and $I_{\mathcal{T}_{\text{eff}}}(f) \stackrel{\text{def}}{=} (\overrightarrow{x_i} \vdash x_{f(j)})_{1\,\le\, j\,\le\, m} : n \to m$. It is easy to verify that $\mathcal{L}_{\mathcal{T}_{\text{eff}}}$ has countable products (given using the cardinal sums in $\aleph_1$) and $I_{\mathcal{T}_{\text{eff}}}$ strictly preserves them.

Proposition 8.1. *$\mathcal{L}_{\mathcal{T}_{\text{eff}}}$ is a countable Lawvere theory.*

We conclude our discussion about the countable Lawvere theory $\mathcal{L}_{\mathcal{T}_{\text{eff}}}$ by highlighting that for any set A, one can intuitively view the *free* model $F_{\mathcal{L}_{\mathcal{T}_{\text{eff}}}}(A)$ as given by the set of equivalence classes of terms in the countable equational theory we derived from $\mathcal{T}_{\text{eff}}$, with the variables of these terms replaced by elements of the given set A.

8.4 Interpretation of the non-pure fragment of EMLTT

We now describe how to extend the interpretation of the pure fragment of EMLTT to the rest of our extension of EMLTT with fibred algebraic effects and their handlers, based on the following fibred adjunction:



where the two fibred functors are defined (on objects) as $F(X,A) \stackrel{\text{def}}{=} (X, F_{\mathcal{L}_{\mathcal{T}_{\text{eff}}}} \circ A)$ and $U(X,\underline{C}) \stackrel{\text{def}}{=} (X, U_{\mathcal{L}_{\mathcal{T}_{\text{eff}}}} \circ \underline{C})$.

The fact that the functors F and U indeed form a fibred adjunction, and that this adjunction is suitable for modelling EMLTT from §2, is an instance of a general result about models of EMLTT (Ahman et al. 2016, Thm. 3).

Using this fibred adjunction $F \dashv U$, one can now extend the definition of $[\![-]\!]$ given in §8.2 so that, if defined, it maps a context Γ and computation type $\underline{C}$ to an object $[\![\Gamma;\underline{C}]\!]$ in $\mathsf{Fam}_{[\![\Gamma]\!]}(\mathsf{Mod})$; a context Γ and computation term M to $[\![\Gamma;M]\!] : 1([\![\Gamma]\!]) \longrightarrow U([\![\Gamma]\!],\underline{C})$ in $\mathsf{Fam}_{[\![\Gamma]\!]}(\mathsf{Set})$, for some $\underline{C} : [\![\Gamma]\!] \longrightarrow \mathsf{Set}$; and a context Γ, variable z, computation type $\underline{C}$ and homomorphism term K to $[\![\Gamma;z{:}\underline{C};K]\!] : [\![\Gamma;\underline{C}]\!] \longrightarrow ([\![\Gamma]\!],\underline{D})$ in $\mathsf{Fam}_{[\![\Gamma]\!]}(\mathsf{Mod})$, for some $\underline{D}$.

In particular, the interpretation of the types $\underline{C} \multimap \underline{D}$, FA, and $U\underline{C}$ is defined as follows:

$$[\![\Gamma;\underline{C} \multimap \underline{D}]\!] \stackrel{\text{def}}{=} \Big([\![\Gamma]\!], \gamma \mapsto \mathsf{Hom}_{\mathsf{Mod}}\big([\![\Gamma;\underline{C}]\!]_2(\gamma), [\![\Gamma;\underline{D}]\!]_2(\gamma)\big)\Big) \qquad [\![\Gamma;FA]\!] \stackrel{\text{def}}{=} F([\![\Gamma;A]\!]) \qquad [\![\Gamma;U\underline{C}]\!] \stackrel{\text{def}}{=} U([\![\Gamma;\underline{C}]\!])$$

assuming that the objects $[\![\Gamma;A]\!]$, $[\![\Gamma;\underline{C}]\!]$, and $[\![\Gamma;\underline{D}]\!]$ are defined. In the rest of this section, we omit such routine assumptions. We also note that the computational Σ- and Π-types are interpreted similarly to their value counterparts, using the set-indexed coproducts and products in Mod—see Ahman et al. (2016) for further details.

We omit the definition of $[\![-]\!]$ for the cases (of computation and homomorphism terms) that are already covered in great detail by Ahman et al. (2016), and instead concentrate on demonstrating how to define the interpretation for algebraic operations, the user-defined algebra type, and the two composition operations. We further note that diagrammatic and more detailed definitions of the cases of $[\![-]\!]$ that we discuss below can be found in Appendix C.

First, we define $[\![-]\!]$ on the algebraic operation $\mathsf{op}_V^C(y.M)$ as follows:

$$[\![\Gamma;\mathsf{op}_V^C(y.M)]\!]_1 \quad \stackrel{\text{def}}{=} \mathsf{id}_{[\![\Gamma]\!]}$$
$$([\![\Gamma;\mathsf{op}_V^C(y.M)]\!]_2)_\gamma \stackrel{\text{def}}{=} \mathsf{op}^\gamma \circ \iota \circ \textstyle\prod_o \Big(([\![\Gamma, y{:}O[V/x];M]\!]_2)_{\langle\gamma,o\rangle}\Big) \circ \langle \mathsf{id}_1 \rangle_{o \in [\![\Gamma;O[V/x]]\!]_2(\gamma)} : 1 \longrightarrow U_{\mathcal{L}_{\mathcal{T}_{\text{eff}}}}([\![\Gamma;\underline{C}]\!]_2(\gamma))$$

where op^γ is the corresponding *operation* of $[\![\Gamma;\underline{C}]\!]_2(\gamma)$, i.e.,

$$([\![\Gamma;\underline{C}]\!]_2(\gamma))(\overrightarrow{x_o} \vdash \mathsf{op}_{([\![\Gamma;V]\!]_2)_\gamma(\star)}(x_o)_{1 \le o \le |[\![\Gamma;O[V/x]]\!]_2(\gamma)|}) : ([\![\Gamma;\underline{C}]\!]_2(\gamma))(|[\![\Gamma;O[V/x]]\!]_2(\gamma)|) \longrightarrow ([\![\Gamma;\underline{C}]\!]_2(\gamma))(1)$$

and where ι is the following countable-product preservation isomorphism:

$$\textstyle\prod_o([\![\Gamma;\underline{C}]\!]_2(\gamma))(1) \cong ([\![\Gamma;\underline{C}]\!]_2(\gamma))(|[\![\Gamma;O[V/x]]\!]_2(\gamma)|)$$

Next, we define $[\![-]\!]$ on the user-defined algebra type $\langle A, \{V_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\text{eff}}}\rangle$ as follows:

$$[\![\Gamma;\langle A, \{V_{\mathsf{op}}\}_{\mathsf{op} \in \mathcal{S}_{\text{eff}}}\rangle]\!] \stackrel{\text{def}}{=} ([\![\Gamma]\!], \gamma \mapsto \mathcal{M}^\gamma)$$

where the functor $\mathcal{M}^\gamma : \mathcal{L}_{\mathcal{T}_{\text{eff}}} \longrightarrow \mathsf{Set}$ is defined on morphisms of the form $n \longrightarrow 1$ as follows:

$$\mathcal{M}^\gamma(n) \stackrel{\text{def}}{=} \textstyle\prod_{1 \le j \le n} [\![\Gamma;A]\!]_2(\gamma) \qquad \mathcal{M}^\gamma(\overrightarrow{x_j} \vdash x_j) \stackrel{\text{def}}{=} \mathsf{proj}_j$$
$$\mathcal{M}^\gamma(\Delta \vdash \mathsf{op}_i(t_o)_{1 \le o \le |[\![x:I;O]\!]_2(\langle\star,i\rangle)|}) \stackrel{\text{def}}{=} f^\gamma_{\mathsf{op}_i} \circ \langle \mathcal{M}^\gamma(\Delta \vdash t_o)\rangle_{o \in [\![x:I;O]\!]_2(\langle\star,i\rangle)}$$

where the function $f^{\gamma}_{\mathsf{op}_i}$ is derived from $[\![\Gamma; V_{\mathsf{op}}]\!]$ as follows:

$$f^{\gamma}_{\mathsf{op}_i} \stackrel{\text{def}}{=} f \mapsto \mathsf{proj}_{\langle i,f\rangle}\Big(([\![\Gamma; V_{\mathsf{op}}]\!]_2)_{\gamma}(\star)\Big) : \textstyle\prod_{o\in[\![x:I;O]\!]_2(\langle\star,i\rangle)}[\![\Gamma; A]\!]_2(\gamma) \longrightarrow [\![\Gamma; A]\!]_2(\gamma)$$

We also note that $\mathcal{M}^{\gamma}$ extends straightforwardly to m-tuples of terms, so as to account for all morphisms in $\mathcal{L}_{\mathcal{T}_{\text{eff}}}$:

$$\mathcal{M}^{\gamma}((\Delta \vdash t)_{1\le j\le m}) \stackrel{\text{def}}{=} \langle \mathcal{M}^{\gamma}(\Delta \vdash t)\rangle_{1\le j\le m}$$

We note that for this case of $[\![-]\!]$ to be defined, we additionally require that $\mathcal{M}^{\gamma}$ validates the equations given in $\mathcal{E}_{\text{eff}}$. In detail, we require that $\mathcal{M}^{\gamma}(\Delta^{\gamma'} \vdash T_1^{\gamma'}) = \mathcal{M}^{\gamma}(\Delta^{\gamma'} \vdash T_2^{\gamma'})$ for all $\Gamma' \mid \Delta \vdash T_1 = T_2 \in \mathcal{E}_{\text{eff}}$ and $\gamma' \in [\![\Gamma']\!]$.

Finally, we define $[\![-]\!]$ on the two composition operations as follows:

$$\begin{aligned}
&[\![\Gamma; M \text{ as } x{:}U\underline{C} \text{ in}_{\underline{D}} N]\!]_1 && \stackrel{\text{def}}{=} \mathsf{id}_{[\![\Gamma]\!]}\\
&([\![\Gamma; M \text{ as } x{:}U\underline{C} \text{ in}_{\underline{D}} N]\!]_2)_{\gamma} && \stackrel{\text{def}}{=} f^{\gamma} \circ ([\![\Gamma; M]\!]_2)_{\gamma} : 1 \longrightarrow U_{\mathcal{L}_{\mathcal{T}_{\text{eff}}}}([\![\Gamma; \underline{D}]\!]_2(\gamma))\\
&[\![\Gamma; z{:}\underline{C}'; K \text{ as } x{:}U\underline{C} \text{ in}_{\underline{D}} N]\!]_1 && \stackrel{\text{def}}{=} \mathsf{id}_{[\![\Gamma]\!]}\\
&([\![\Gamma; z{:}\underline{C}'; K \text{ as } x{:}U\underline{C} \text{ in}_{\underline{D}} N]\!]_2)_{\gamma} && \stackrel{\text{def}}{=} \mathsf{hom}(f^{\gamma}) \circ ([\![\Gamma; z{:}\underline{C}'; K]\!]_2)_{\gamma} : [\![\Gamma; \underline{C}]\!]_2(\gamma) \longrightarrow [\![\Gamma; \underline{D}]\!]_2(\gamma)
\end{aligned}$$

where the function f^{γ} is derived from $[\![\Gamma, x{:}U\underline{C}; N]\!]$ as follows:

$$f^{\gamma} \stackrel{\text{def}}{=} c \mapsto ([\![\Gamma, x{:}U\underline{C}; N]\!]_2)_{\langle\gamma,c\rangle}(\star) : U_{\mathcal{L}_{\mathcal{T}_{\text{eff}}}}([\![\Gamma; \underline{C}]\!]_2(\gamma)) \longrightarrow U_{\mathcal{L}_{\mathcal{T}_{\text{eff}}}}([\![\Gamma; \underline{D}]\!]_2(\gamma))$$

and where $\mathsf{hom}(f^{\gamma})$ is a morphism of models of $\mathcal{L}_{\mathcal{T}_{\text{eff}}}$, given by components

$$(\mathsf{hom}(f^{\gamma}))_n \stackrel{\text{def}}{=} \iota_{[\![\Gamma;\underline{D}]\!]} \circ \textstyle\prod_{1\le j\le n}(f^{\gamma}) \circ \iota^{-1}_{[\![\Gamma;\underline{C}]\!]} : ([\![\Gamma; \underline{C}]\!]_2(\gamma))(n) \longrightarrow ([\![\Gamma; \underline{D}]\!]_2(\gamma))(n)$$

where $\iota_{[\![\Gamma;\underline{C}]\!]}$ and $\iota_{[\![\Gamma;\underline{D}]\!]}$ are respectively the following countable-product preservation isomorphisms:

$$\textstyle\prod_{1\le j\le n}([\![\Gamma; \underline{C}]\!]_2(\gamma))(1) \cong ([\![\Gamma; \underline{C}]\!]_2(\gamma))(n) \qquad \prod_{1\le j\le n}([\![\Gamma; \underline{D}]\!]_2(\gamma))(1) \cong ([\![\Gamma; \underline{D}]\!]_2(\gamma))(n)$$

We note that for these two cases of $[\![-]\!]$ to be defined, we additionally require that the function f^{γ} commutes with the operations of $[\![\Gamma; \underline{C}]\!]_2(\gamma)$ and $[\![\Gamma; \underline{D}]\!]_2(\gamma)$, as depicted in the following diagram:

$$\begin{array}{ccc}
\prod_o([\![\Gamma; \underline{C}]\!]_2(\gamma))(1) & \xrightarrow{\prod_{o\in[\![x:I;O]\!]_2(\langle\star,i\rangle)}(f^{\gamma})} & \prod_o([\![\Gamma; \underline{D}]\!]_2(\gamma))(1)\\
\cong\downarrow & & \downarrow\cong\\
([\![\Gamma; \underline{C}]\!]_2(\gamma))(|[\![x:I;O]\!]_2(\langle\star,i\rangle)|) & & ([\![\Gamma; \underline{D}]\!]_2(\gamma))(|[\![x:I;O]\!]_2(\langle\star,i\rangle)|)\\
{\scriptstyle([\![\Gamma;\underline{C}]\!]_2(\gamma))(\overrightarrow{x_o} \vdash \mathsf{op}_i(x_o)_o)}\downarrow & & \downarrow{\scriptstyle([\![\Gamma;\underline{D}]\!]_2(\gamma))(\overrightarrow{x_o} \vdash \mathsf{op}_i(x_o)_o)}\\
([\![\Gamma; \underline{C}]\!]_2(\gamma))(1) & \xrightarrow[f^{\gamma}]{} & ([\![\Gamma; \underline{D}]\!]_2(\gamma))(1)
\end{array}$$

8.5 Soundness

To establish the soundness of this interpretation, we first prove standard semantic weakening and substitution lemmas. In detail, we begin by defining *a priori* partial *semantic projection* and *substitution* morphisms

$$\mathsf{proj}_{\Gamma_1;x:A;\Gamma_2} : [\![\Gamma_1, x{:}A, \Gamma_2]\!] \longrightarrow [\![\Gamma_1, \Gamma_2]\!] \qquad \mathsf{subst}_{\Gamma_1;x:A;\Gamma_2;V} : [\![\Gamma_1, \Gamma_2[V/x]]\!] \longrightarrow [\![\Gamma_1, x{:}A, \Gamma_2]\!]$$

by induction on the size of Γ_2 as follows:

$$\begin{aligned}
&\mathsf{proj}_{\Gamma_1;x:A;\diamond} && \stackrel{\text{def}}{=} \pi_{[\![\Gamma_1;A]\!]} && \mathsf{proj}_{\Gamma_1;x:A;\Gamma_2,y:B} && \stackrel{\text{def}}{=} \{\overline{\mathsf{proj}_{\Gamma_1;x:A;\Gamma_2}}([\![\Gamma_1, \Gamma_2; B]\!])\}\\
&\mathsf{subst}_{\Gamma_1;x:A;\diamond;V} && \stackrel{\text{def}}{=} [\![\Gamma; V]\!] && \mathsf{subst}_{\Gamma_1;x:A;\Gamma_2,y:B;V} && \stackrel{\text{def}}{=} \{\overline{\mathsf{subst}_{\Gamma_1;x:A;\Gamma_2;V}}([\![\Gamma_1, x{:}A, \Gamma_2; B]\!])\}
\end{aligned}$$

and then show that both morphisms are in fact defined if the interpretations of the involved value contexts and terms are defined, and that reindexing along these morphisms models syntactic weakening and substitution.

Proposition 8.2. *Given value contexts* Γ_1 *and* Γ_2*, a value type* A*, and a value variable* x *such that* $x \notin Vars(\Gamma_1)$*,* $x \notin Vars(\Gamma_2)$*,* $[\![\Gamma_1, \Gamma_2]\!] \in \mathcal{B}$*, and* $[\![\Gamma_1, x{:}A, \Gamma_2]\!] \in \mathcal{B}$*, then i) the semantic projection morphism* $\mathsf{proj}_{\Gamma_1;x:A;\Gamma_2}$ *is defined; and ii) given a value type* B *such that* $[\![\Gamma_1, \Gamma_2; B]\!] \in \mathcal{V}_{[\![\Gamma_1,\Gamma_2]\!]}$*, then* $[\![\Gamma_1, x{:}A, \Gamma_2; B]\!] = \mathsf{proj}^*_{\Gamma_1;x:A;\Gamma_2}([\![\Gamma_1, \Gamma_2; B]\!])$ *as objects of* $\mathsf{Fam}_{[\![\Gamma_1, x:A, \Gamma_2]\!]}(\mathsf{Set})$*, and similarly for computation types, and value, computation, and homomorphism terms.*

Proposition 8.3. *Given value contexts* Γ_1 *and* Γ_2*, a value type* A*, a value variable* x*, and a value term* V *such that* $x \notin Vars(\Gamma_1)$*,* $x \notin Vars(\Gamma_2)$*,* $[\![\Gamma_1, x{:}A, \Gamma_2]\!] \in \mathcal{B}$*,* $[\![\Gamma_1, \Gamma_2[V/x]]\!] \in \mathcal{B}$*, and* $[\![\Gamma_1; V]\!] : 1_{[\![\Gamma_1]\!]} \longrightarrow [\![\Gamma_1; A]\!]$*, then i) the semantic substitution morphism* $\mathsf{subst}_{\Gamma_1;x:A;\Gamma_2;V}$ *is defined; and ii) given a value type* B *such that* $[\![\Gamma_1, x{:}A, \Gamma_2; B]\!] \in \mathcal{V}_{[\![\Gamma_1, x:A, \Gamma_2]\!]}$*, then* $[\![\Gamma_1, \Gamma_2[V/x]; B[V/x]]\!] = \mathsf{subst}^*_{\Gamma_1;x:A;\Gamma_2;V}([\![\Gamma_1, x : A, \Gamma_2; B]\!])$ *as objects of* $\mathsf{Fam}_{[\![\Gamma_1,\Gamma_2[V/x]]\!]}(\mathsf{Set})$*, and similarly for computation types, and value, computation, and homomorphism terms.*

In addition, we note that substituting computation and homomorphism terms for computation variables corresponds to composition of the morphisms that the given terms denote.

Proposition 8.4. *Given a value context* Γ*, a computation variable* z*, a computation type* $\underline{C}$*, a computation term* M*, and a homomorphism term* K *such that* $[\![\Gamma; M]\!] : 1_{[\![\Gamma]\!]} \longrightarrow U([\![\Gamma; \underline{C}]\!])$ *and* $[\![\Gamma; z{:}\underline{C}; K]\!] : [\![\Gamma; \underline{C}]\!] \longrightarrow \underline{D}$*, then we have* $[\![\Gamma; K[M/z]]\!] = U([\![\Gamma; z : \underline{C}; K]\!]) \circ [\![\Gamma; M]\!] : 1_{[\![\Gamma]\!]} \longrightarrow U(\underline{D})$ *in* $\mathsf{Fam}_{[\![\Gamma]\!]}(\mathsf{Mod})$*, and similarly for homomorphism terms.*

Finally, we prove the soundness of this interpretation.

Theorem 8.5 (Soundness). *The interpretation function* $[\![-]\!]$ *is defined on all well-formed contexts, well-formed types, and well-typed terms, and it further identifies all definitionally equal contexts, types, and terms.*

Proof. We prove this theorem by induction on the given derivations. In particular, for the definitional equations that correspond to the equations given in $\mathcal{E}_{\text{eff}}$ (see Fig. 3), we recall that these equations hold in $\mathcal{L}_{\mathcal{T}_{\text{eff}}}$ by construction. Therefore, all models of $\mathcal{L}_{\mathcal{T}_{\text{eff}}}$ validate them, including those modelling our computation types. For computation types, we prove that $[\![\Gamma; \langle A, \{V_{\text{op}}\}_{\text{op}\in\mathcal{S}_{\text{eff}}}\rangle]\!]$ is defined by observing that the equational proof obligations included in $\Gamma \vdash \{V_{\text{op}}\}_{\text{op}\in\mathcal{S}_{\text{eff}}}$ on A ensure that each $\mathcal{M}^\gamma$ validates the equations given in $\mathcal{E}_{\text{eff}}$. For computation terms, we prove that $[\![\Gamma; M \;\mathtt{as}\; x{:}U\underline{C} \;\mathtt{in}_{\underline{D}}\; N]\!]$ is defined by observing that the assumption $\Gamma, x{:}U\underline{C} \vdash_{\mathsf{hom}} N : \underline{D}$ ensures that the functions f^γ we derive from $[\![\Gamma, x{:}U\underline{C}; N]\!]$ commute with the operations of $[\![\Gamma; \underline{C}]\!]_2(\gamma)$ and $[\![\Gamma; \underline{D}]\!]_2(\gamma)$; the case for the other composition operation, namely, $K \;\mathtt{as}\; x{:}U\underline{C} \;\mathtt{in}_{\underline{D}}\; N$, is proved analogously. □

9 CONCLUSIONS AND FUTURE WORK

In this paper we have given a comprehensive account of algebraic effects and their handlers in the dependently typed setting. In detail, we gave handlers a novel type-based treatment and demonstrated that they provide a useful mechanism for reasoning about effectful computations. We also showed how to equip the resulting language with a denotational semantics, based on families fibrations and models of countable Lawvere theories.

In future, we plan to combine our treatment of handlers with effect-typing (Kammar et al. 2013) and multi-handlers (Lindley et al. 2017). We also plan to compare our handler-based definition of Dijkstra's predicate transformers with their CPS-translation based definition used in the F* language (Ahman et al. 2017). Further, we plan to extend computation terms with recursion following the analyses of Ahman et al. (2016); Plotkin and Pretnar (2013). Specifically, we plan to generalise from an equational presentation of effects to an inequational presentation, and develop the corresponding denotational semantics using fibrations of continuous families of ω-cpos and models of countable discrete CPO-enriched Lawvere theories (Hyland and Power 2006). More generally, we plan to extend our work from families fibrations to more general fibrational models of dependent types, where definitional and propositional proof obligations (see discussion in §4.3) might not coincide.

REFERENCES

J. Adamek and J. Rosicky. 1994. *Locally Presentable and Accessible Categories.* Number 189 in London Mathematical Society Lecture Note Series. Cambridge Univ. Press.

Danel Ahman, James Chapman, and Tarmo Uustalu. 2014. When is a container a comonad? *Logical Methods in Computer Science* 10, 3 (2014).

Danel Ahman, Neil Ghani, and Gordon D. Plotkin. 2016. Dependent Types and Fibred Computational Effects. In *Proc. of 19th Int. Conf. on Foundations of Software Science and Computation Structures, FoSSaCS 2016 (LNCS)*, Vol. 9634. Springer, 1–19.

Danel Ahman, Cătălin Hriţcu, Kenji Maillard, Guido Martínez, Gordon Plotkin, Jonathan Protzenko, Aseem Rastogi, and Nikhil Swamy. 2017. Dijkstra Monads for Free. In *Proc. of 44th ACM SIGPLAN Symp. on Principles of Programming Languages, POPL 2017.* ACM, 515–529.

Danel Ahman and Sam Staton. 2013. Normalization by Evaluation and Algebraic Effects. In *Proc. of 29th Conf. on the Mathematical Foundations of Programming Semantics, MFPS XXIX (ENTCS)*, Vol. 298. Elsevier, 51–69.

Danel Ahman and Tarmo Uustalu. 2014. Update Monads: Cointerpreting Directed Containers. In *Post-proc. of the 19th Meeting "Types for Proofs and Programs", TYPES 2013 (LIPIcs)*, Vol. 26. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, 1–23.

Andrej Bauer and Matija Pretnar. 2015. Programming with algebraic effects and handlers. *J. Log. Algebr. Meth. Program.* 84, 1 (2015), 108–123.

Edwin Brady. 2013. Programming and reasoning with algebraic effects and dependent types. In *Proc. of 18th ACM SIGPLAN Int. Conf. on Functional Programming, ICFP 2013.* ACM, 133–144.

Chris Casinghino. 2014. *Combining Proofs and Programs.* Ph.D. Dissertation. University of Pennsylvania.

Edsger W. Dijkstra. 1975. Guarded Commands, Nondeterminacy and Formal Derivation of Programs. *CACM* 18, 8 (1975), 5.

Jeff Egger, Rasmus Ejlers Møgelberg, and Alex Simpson. 2014. The enriched effect calculus: syntax and semantics. *J. Log. Comput.* 24, 3 (2014), 615–654.

Neil Ghani, Lorenzo Malatesta, Fredrik Nordvall Forsberg, and Anton Setzer. 2013. Fibred Data Types. In *Proc. of 28th Ann. Symp on Logic in Computer Science, LICS 2013.* IEEE Computer Society, 243–252.

George A. Grätzer. 1979. *Universal Algebra* (2nd ed.). Springer.

Peter Hancock and Anton Setzer. 2000. Interactive programs in dependent type theory. In *Proc. of 14th Ann. Conf. of the EACSL on Computer Science Logic, CSL 2000 (LNCS)*, Vol. 1862. Springer, 317–331.

Daniel Hillerström and Sam Lindley. 2016. Liberating Effects with Rows and Handlers. In *Proc. of 1st Wksh. on Type-Driven Development, TyDe 2016.* ACM, 15–27.

Martin Hofmann. 1995. *Extensional concepts in intensional type theory.* Ph.D. Dissertation. Laboratory for Foundations in Computer Science, University of Edinburgh.

Martin Hofmann. 1997. Syntax and Semantics of Dependent Types. In *Semantics and Logics of Computation*, Andrew M. Pitts and P. Dybjer (Eds.). Cambridge Univ. Press, 79–130.

Kohei Honda, Vasco Thudichum Vasconcelos, and Makoto Kubo. 1998. Language Primitives and Type Discipline for Structured Communication-Based Programming. In *Proc. of 7th European Symp. on Programming, ESOP 1998 (LNCS)*, Vol. 1381. Springer, 122–138.

Martin Hyland, Gordon Plotkin, and John Power. 2006. Combining effects: Sum and tensor. *Theor. Comput. Sci.* 357, 1–3 (2006), 70–99.

Martin Hyland and John Power. 2006. Discrete Lawvere theories and computational effects. *Theor. Comput. Sci.* 366, 1-2 (2006), 144–162.

Bart Jacobs. 1999. *Categorical Logic and Type Theory.* Number 141 in Studies in Logic and the Foundations of Mathematics. North Holland, Elsevier.

Ohad Kammar, Sam Lindley, and Nicolas Oury. 2013. Handlers in Action. In *Proc. of 18th ACM SIGPLAN Int. Conf. on Functional Programming, ICFP 2013.* ACM, 145–158.

Ohad Kammar and Gordon D. Plotkin. 2012. Algebraic Foundations for Effect-dependent Optimisations. In *Proc. of 39th ACM SIGPLAN-SIGACT Symp. on Principles of Programming Languages, POPL 2012.* ACM, 349–360.

Daan Leijen. 2017. Type Directed Compilation of Row-Typed Algebraic Effects. In *Proc. of 44th ACM SIGPLAN Symp. on Principles of Programming Languages, POPL 2017.* ACM, 486–499.

Paul Blain Levy. 2004. *Call-By-Push-Value: A Functional/Imperative Synthesis.* Semantics Structures in Computation, Vol. 2. Springer.

Sam Lindley, Conor McBride, and Craig McLaughlin. 2017. Do Be Do Be Do. In *Proc. of 44th ACM SIGPLAN Symp. on Principles of Programming Languages, POPL 2017.* ACM, 500–514.

Per Martin-Löf. 1984. *Intuitionistic Type Theory.* Bibliopolis.

Conor McBride. 2011. Functional Pearl: Kleisli arrows of outrageous fortune. *J. Funct. Program.* (2011). (To appear).

Eugenio Moggi. 1989. Computational Lambda-Calculus and Monads. In *Proc. of 4th Ann. Symp. on Logic in Computer Science, LICS 1989*, Rohit Parikh (Ed.). IEEE, 14–23.

Eugenio Moggi. 1991. Notions of Computation and Monads. *Inf. Comput.* 93, 1 (1991), 55–92.

Aleksandar Nanevski, Greg Morrisett, and Lars Birkedal. 2008. Hoare Type Theory, polymorphism and separation. *J. Funct. Program.* 18, 5-6 (2008), 865–911.

M. Okada and P. J. Scott. 1999. A Note on Rewriting Theory for Uniqueness of Iteration. *Theory Appl. Categ.* 6, 4 (1999), 47–64.

Pierre-Marie Pédrot and Nicolas Tabareau. 2017. An Effectful Way to Eliminate Addiction to Dependence. In *Proc. of 32nd Ann. Symp on Logic in Computer Science, LICS 2017*. To appear.

A. M. Pitts. 1991. Evaluation Logic. In *Proc. IVth Higher Order Workshop (Workshops in Computing)*. Springer, 162–189.

A. M. Pitts, J. Matthiesen, and J. Derikx. 2015. A Dependent Type Theory with Abstractable Names. In *Proc. of 9th Wksh. on Logical and Semantic Frameworks, with Applications, LSFA 2014 (ENTCS)*, Vol. 312. Elsevier, 19–50.

Gordon Plotkin and John Power. 2001. Semantics for Algebraic Operations. In *Proc. of 17th Conf. on the Mathematical Foundations of Programming Semantics, MFPS XVII (ENTCS)*, Vol. 45. Elsevier, 332–345.

Gordon D. Plotkin and John Power. 2002. Notions of Computation Determine Monads. In *Proc. of 5th Int. Conf. on Foundations of Software Science and Computation Structures, FOSSACS 2002 (LNCS)*, Vol. 2303. Springer, 342–356.

Gordon D. Plotkin and Matija Pretnar. 2013. Handling Algebraic Effects. *Logical Methods in Computer Science* 9, 4:23 (2013).

John Power. 2006. Countable Lawvere Theories and Computational Effects. In *Proc. of 3rd Irish Conf. on the Mathematical Foundations of Computer Science and Information Technology, MFCSIT 2004 (ENTCS)*, Vol. 161. Elsevier, 59–71.

Thomas Streicher. 1991. *Semantics of Type Theory. Correctness, Completeness and Independence Results*. Birkhäuser Boston.

A TYPING RULES FOR THE CORE OF EMLTT

Value contexts:

$$\frac{}{\vdash \diamond} \qquad \frac{\vdash \Gamma \quad \Gamma \vdash A \quad x \notin \mathit{Vars}(\Gamma)}{\vdash \Gamma, x{:}A}$$

Value and computation variables:

$$\frac{\vdash \Gamma_1, x{:}A, \Gamma_2}{\Gamma_1, x{:}A, \Gamma_2 \vdash x : A} \qquad \frac{\Gamma \vdash \underline{C}}{\Gamma \mid z{:}\underline{C} \vdash z : \underline{C}}$$

Type of natural numbers:

$$\frac{\vdash \Gamma}{\Gamma \vdash \mathsf{zero} : \mathsf{Nat}} \qquad \frac{\Gamma \vdash V : \mathsf{Nat}}{\Gamma \vdash \mathsf{succ}\ V : \mathsf{Nat}}$$

$$\frac{\Gamma, x{:}\mathsf{Nat} \vdash A \quad \Gamma \vdash V : \mathsf{Nat} \quad \Gamma \vdash V_z : A[\mathsf{zero}/x] \quad \Gamma, y_1{:}\mathsf{Nat}, y_2{:}A[y_1/x] \vdash V_s : A[\mathsf{succ}\ y_1/x]}{\Gamma \vdash \mathsf{nat\text{-}elim}_{x.A}(V_z, y_1.y_2.V_s, V) : A[V/x]}$$

Unit type:

$$\frac{\vdash \Gamma}{\Gamma \vdash \star : 1}$$

Empty type:

$$\frac{\Gamma, x{:}0 \vdash A \quad \Gamma \vdash V : 0}{\Gamma \vdash \mathsf{case}\ V\ \mathsf{of}_{x.A}\ () : A[V/x]}$$

Coproduct type:

$$\frac{\Gamma \vdash V : A}{\Gamma \vdash \mathsf{inl}_{A+B}\ V : A + B} \qquad \frac{\Gamma \vdash V : B}{\Gamma \vdash \mathsf{inr}_{A+B}\ V : A + B}$$

$$\frac{\Gamma, x{:}A_1 + A_2 \vdash B \quad \Gamma \vdash V : A_1 + A_2 \quad \Gamma, y_1{:}A_1 \vdash W_1 : B[\mathsf{inl}_{A_1+A_2}\ y_1/x] \quad \Gamma, y_2{:}A_2 \vdash W_2 : B[\mathsf{inr}_{A_1+A_2}\ y_2/x]}{\Gamma \vdash \mathsf{case}\ V\ \mathsf{of}_{x.B}\ (\mathsf{inl}(y_1{:}A_1) \mapsto W_1, \mathsf{inr}(y_2{:}A_2) \mapsto W_2) : B[V/x]}$$

Value Σ-type:

$$\frac{\Gamma \vdash V : A \quad \Gamma, x{:}A \vdash B \quad \Gamma \vdash W : B[V/x]}{\Gamma \vdash \langle V, W \rangle_{(x:A).B} : \Sigma x{:}A.B}$$

$$\frac{\Gamma, y{:}\Sigma x_1{:}A_1.A_2 \vdash B \quad \Gamma \vdash V : \Sigma x_1{:}A_1.A_2 \quad \Gamma, x_1{:}A_1, x_2{:}A_2 \vdash W : B[\langle x_1, x_2 \rangle_{(x_1:A_1).A_2}/y]}{\Gamma \vdash \mathsf{pm}\ V\ \mathsf{as}\ (x_1{:}A_1, x_2{:}A_2)\ \mathsf{in}_{y.B}\ W : B[V/y]}$$

Value Π-type:

$$\frac{\Gamma, x{:}A \vdash V : B}{\Gamma \vdash \lambda x{:}A.V : \Pi x{:}A.B} \qquad \frac{\Gamma, x{:}A \vdash B \quad \Gamma \vdash V : \Pi x{:}A.B \quad \Gamma \vdash W : A}{\Gamma \vdash V(W)_{(x:A).B} : B[W/x]}$$

Propositional equality:

$$\frac{\Gamma \vdash V : A}{\Gamma \vdash \mathsf{refl}\ V : V =_A V} \qquad \frac{\begin{array}{c}\Gamma \vdash A \quad \Gamma, x_1{:}A, x_2{:}A, x_3{:}x_1 =_A x_2 \vdash B \quad \Gamma \vdash V_1 : A \quad \Gamma \vdash V_2 : A \\ \Gamma \vdash V_p : V_1 =_A V_2 \quad \Gamma, y{:}A \vdash W : B[y/x_1][y/x_2][\mathsf{refl}\ y/x_3]\end{array}}{\Gamma \vdash \mathsf{eq\text{-}elim}_A(x_1.x_2.x_3.B, y.W, V_1, V_2, V_p) : B[V_1/x_1][V_2/x_2][V_p/x_3]}$$

Thunking and forcing:

$$\frac{\Gamma \vdash M : \underline{C}}{\Gamma \vdash \mathsf{thunk}\ M : U\underline{C}} \qquad \frac{\Gamma \vdash V : U\underline{C}}{\Gamma \vdash \mathsf{force}_{\underline{C}}\ V : \underline{C}}$$

Homomorphic function type:

$$\frac{\Gamma \mid z{:}\underline{C} \vdash K : \underline{D}}{\Gamma \vdash \lambda z{:}\underline{C}.K : \underline{C} \multimap \underline{D}} \qquad \frac{\Gamma \vdash V : \underline{C} \multimap \underline{D} \quad \Gamma \vdash M : \underline{C}}{\Gamma \vdash V(M)_{\underline{C},\underline{D}} : \underline{D}} \qquad \frac{\Gamma \vdash V : \underline{D}_1 \multimap \underline{D}_2 \quad \Gamma \mid z{:}\underline{C} \vdash K : \underline{D}_1}{\Gamma \mid z{:}\underline{C} \vdash V(K)_{\underline{D}_1,\underline{D}_2} : \underline{D}_2}$$

Sequential composition:

$$\frac{\Gamma \vdash V : A}{\Gamma \vdash \mathtt{return}\ V : FA} \qquad \frac{\Gamma \vdash M : FA \quad \Gamma \vdash \underline{C} \quad \Gamma, x{:}A \vdash N : \underline{C}}{\Gamma \vdash M\ \mathtt{to}\ x{:}A\ \mathtt{in}_{\underline{C}}\ N : \underline{C}} \qquad \frac{\Gamma \mid z{:}\underline{C} \vdash K : FA \quad \Gamma \vdash \underline{D} \quad \Gamma, x{:}A \vdash M : \underline{D}}{\Gamma \mid z{:}\underline{C} \vdash K\ \mathtt{to}\ x{:}A\ \mathtt{in}_{\underline{D}}\ M : \underline{D}}$$

Computational Σ-type:

$$\frac{\Gamma \vdash V : A \quad \Gamma, x{:}A \vdash \underline{C} \quad \Gamma \vdash M : \underline{C}[V/x]}{\Gamma \vdash \langle V, M\rangle_{(x:A).\underline{C}} : \Sigma x{:}A.\underline{C}} \qquad \frac{\Gamma \vdash M : \Sigma x{:}A.\underline{C} \quad \Gamma \vdash \underline{D} \quad \Gamma, x{:}A \mid z{:}\underline{C} \vdash K : \underline{D}}{\Gamma \vdash M\ \mathtt{to}\ (x{:}A, z{:}\underline{C})\ \mathtt{in}_{\underline{D}}\ K : \underline{D}}$$

$$\frac{\Gamma \vdash V : A \quad \Gamma, x{:}A \vdash \underline{D} \quad \Gamma \mid z{:}\underline{C} \vdash K : \underline{D}[V/x]}{\Gamma \mid z{:}\underline{C} \vdash \langle V, K\rangle_{(x:A).\underline{D}} : \Sigma x{:}A.\underline{D}} \qquad \frac{\Gamma \mid z_1{:}\underline{C} \vdash K : \Sigma x{:}A.\underline{D}_1 \quad \Gamma \vdash \underline{D}_2 \quad \Gamma, x{:}A \mid z_2{:}\underline{D}_1 \vdash L : \underline{D}_2}{\Gamma \mid z_1{:}\underline{C} \vdash K\ \mathtt{to}\ (x{:}A, z_2{:}\underline{D}_1)\ \mathtt{in}_{\underline{D}_2}\ L : \underline{D}_2}$$

Computational Π-type:

$$\frac{\Gamma, x{:}A \vdash M : \underline{C}}{\Gamma \vdash \lambda x{:}A.M : \Pi x{:}A.\underline{C}} \qquad \frac{\Gamma, x{:}A \vdash \underline{C} \quad \Gamma \vdash M : \Pi x{:}A.\underline{C} \quad \Gamma \vdash V : A}{\Gamma \vdash M(V)_{(x:A).\underline{C}} : \underline{C}[V/x]}$$

$$\frac{\Gamma \vdash \underline{C} \quad \Gamma, x{:}A \mid z{:}\underline{C} \vdash K : \underline{D}}{\Gamma \mid z{:}\underline{C} \vdash \lambda x{:}A.K : \Pi x{:}A.\underline{D}} \qquad \frac{\Gamma, x{:}A \vdash \underline{D} \quad \Gamma \mid z{:}\underline{C} \vdash K : \Pi x{:}A.\underline{D} \quad \Gamma \vdash V : A}{\Gamma \mid z{:}\underline{C} \vdash K(V)_{(x:A).\underline{D}} : \underline{D}[V/x]}$$

Context and type conversion rules:

$$\frac{\Gamma_1 \vdash A \quad \vdash \Gamma_1 = \Gamma_2}{\Gamma_2 \vdash A} \qquad \frac{\Gamma_1 \vdash \underline{C} \quad \vdash \Gamma_1 = \Gamma_2}{\Gamma_2 \vdash \underline{C}}$$

$$\frac{\Gamma_1 \vdash V : A \quad \vdash \Gamma_1 = \Gamma_2 \quad \Gamma_1 \vdash A = B}{\Gamma_2 \vdash V : B} \qquad \frac{\Gamma_1 \vdash M : \underline{C} \quad \vdash \Gamma_1 = \Gamma_2 \quad \Gamma_1 \vdash \underline{C} = \underline{D}}{\Gamma_2 \vdash M : \underline{D}}$$

$$\frac{\Gamma_1 \mid z{:}\underline{C}_1 \vdash K : \underline{D}_1 \quad \vdash \Gamma_1 = \Gamma_2 \quad \Gamma_1 \vdash \underline{C}_1 = \underline{C}_2 \quad \Gamma_1 \vdash \underline{D}_1 = \underline{D}_2}{\Gamma_2 \mid z{:}\underline{C}_2 \vdash K : \underline{D}_2}$$

B DEFINITIONAL EQUATIONS FOR THE CORE OF EMLTT

We omit the rules for the standard equations of reflexivity, symmetry, transitivity, replacement, and congruence.

Value contexts:

$$\frac{}{\vdash \diamond = \diamond} \qquad \frac{\vdash \Gamma_1 = \Gamma_2 \quad \Gamma_1 \vdash A = B \quad x \notin \mathit{Vars}(\Gamma_1) \quad x \notin \mathit{Vars}(\Gamma_2)}{\vdash \Gamma_1, x{:}A = \Gamma_2, x{:}B}$$

Type of natural numbers:

$$\frac{\Gamma, x{:}\mathsf{Nat} \vdash A \quad \Gamma \vdash V_z : A[\mathsf{zero}/x] \quad \Gamma, y_1{:}\mathsf{Nat}, y_2{:}A[y_1/x] \vdash V_s : A[\mathsf{succ}\ y_1/x]}{\Gamma \vdash \mathsf{nat\text{-}elim}_{x.A}(V_z, y_1.y_2.V_s, \mathsf{zero}) = V_z : A[\mathsf{zero}/x]}$$

$$\frac{\Gamma, x{:}\mathsf{Nat} \vdash A \quad \Gamma \vdash V : \mathsf{Nat} \quad \Gamma \vdash V_z : A[\mathsf{zero}/x] \quad \Gamma, y_1{:}\mathsf{Nat}, y_2{:}A[y_1/x] \vdash V_s : A[\mathsf{succ}\ y_1/x]}{\Gamma \vdash \mathsf{nat\text{-}elim}_{x.A}(V_z, y_1.y_2.V_s, \mathsf{succ}\ V) = V_s[V/y_1][\mathsf{nat\text{-}elim}_{x.A}(V_z, y_1.y_2.V_s, V)/y_2] : A[\mathsf{succ}\ V/x]}$$

Unit type:

$$\frac{\Gamma \vdash V : 1}{\Gamma \vdash V = \star : 1}$$

Empty type:

$$\frac{\Gamma, x{:}0 \vdash A \quad \Gamma \vdash V : 0 \quad \Gamma, x{:}0 \vdash W : A}{\Gamma \vdash \mathsf{case}\ V\ \mathsf{of}_{x.A}\ () = W[V/x] : A[V/x]}$$

Coproduct type:

$$\frac{\Gamma, x{:}A_1 + A_2 \vdash B \quad \Gamma \vdash V : A_1 \quad \Gamma, y_1{:}A_1 \vdash W_1 : B[\mathsf{inl}_{A_1+A_2}\ y_1/x] \quad \Gamma, y_2{:}A_2 \vdash W_2 : B[\mathsf{inr}_{A_1+A_2}\ y_2/x]}{\Gamma \vdash \mathsf{case}\ (\mathsf{inl}_{A_1+A_2}\ V)\ \mathsf{of}_{x.B}\ (\mathsf{inl}(y_1{:}A_1) \mapsto W_1, \mathsf{inr}(y_2{:}A_2) \mapsto W_2) = W_1[V/y_1] : B[\mathsf{inl}_{A_1+A_2}\ V/x]}$$

$$\frac{\Gamma, x{:}A_1 + A_2 \vdash B \quad \Gamma \vdash V : A_2 \quad \Gamma, y_1{:}A_1 \vdash W_1 : B[\mathsf{inl}_{A_1+A_2}\ y_1/x] \quad \Gamma, y_2{:}A_2 \vdash W_2 : B[\mathsf{inr}_{A_1+A_2}\ y_2/x]}{\Gamma \vdash \mathsf{case}\ (\mathsf{inr}_{A_1+A_2}\ V)\ \mathsf{of}_{x.B}\ (\mathsf{inl}(y_1{:}A_1) \mapsto W_1, \mathsf{inr}(y_2{:}A_2) \mapsto W_2) = W_2[V/y_2] : B[\mathsf{inr}_{A_1+A_2}\ V/x]}$$

$$\frac{\begin{array}{c} y_1 \notin \mathit{Vars}(\Gamma) \cup \{x_1\} \quad y_2 \notin \mathit{Vars}(\Gamma) \cup \{x_2\} \quad x_2 \notin \{y_1, y_2\} \\ \Gamma, x_1{:}A_1 + A_2 \vdash B \quad \Gamma \vdash V : A_1 + A_2 \quad \Gamma, x_2{:}A_1 + A_2 \vdash W : B \end{array}}{\begin{array}{l} \Gamma \vdash \mathsf{case}\ V\ \mathsf{of}_{x_1.B}\ (\mathsf{inl}(y_1{:}A_1) \mapsto W[\mathsf{inl}_{A_1+A_2}\ y_1/x_2], \\ \qquad\qquad \mathsf{inr}(y_2{:}A_2) \mapsto W[\mathsf{inr}_{A_1+A_2}\ y_2/x_2]) = W[V/x_2] : B[V/x_1] \end{array}}$$

Value Σ-type:

$$\frac{\Gamma, y{:}\Sigma x_1{:}A_1.A_2 \vdash B \quad \Gamma \vdash V_1 : A_1 \quad \Gamma \vdash V_2 : A_2[V_1/x_1] \quad \Gamma, x_1{:}A_1, x_2{:}A_2 \vdash W : B[\langle x_1, x_2\rangle_{(x_1:A_1).A_2}/y]}{\Gamma \vdash \mathsf{pm}\ \langle V_1, V_2\rangle_{(x_1:A_1).A_2}\ \mathsf{as}\ (x_1{:}A_1, x_2{:}A_2)\ \mathsf{in}_{y.B}\ W = W[V_1/x_1][V_2/x_2] : B[\langle V_1, V_2\rangle_{(x_1:A_1).A_2}/y]}$$

$$\frac{\begin{array}{c} \Gamma \vdash A_1 \quad \Gamma, x_1{:}A_1 \vdash A_2 \quad x_2 \notin \mathit{Vars}(\Gamma) \cup \{x_1\} \quad y_2 \notin \{x_1, x_2\} \\ \Gamma \vdash V : \Sigma x_1{:}A_1.A_2 \quad \Gamma, y_1{:}\Sigma x_1{:}A_1.A_2 \vdash B \quad \Gamma, y_2{:}\Sigma x_1{:}A_1.A_2 \vdash W : B[y_2/y_1] \end{array}}{\Gamma \vdash \mathsf{pm}\ V\ \mathsf{as}\ (x_1{:}A_1, x_2{:}A_2)\ \mathsf{in}_{y_1.B}\ W[\langle x_1, x_2\rangle_{(x_1:A_1).A_2}/y_2] = W[V/y_2] : B[V/y_1]}$$

Value Π-type:

$$\frac{\Gamma, x{:}A \vdash V : B \quad \Gamma \vdash W : A}{\Gamma \vdash (\lambda x{:}A.V)(W)_{(x:A).B} = V[W/x] : B[W/x]} \qquad \frac{\Gamma, x{:}A \vdash B \quad \Gamma \vdash V : \Pi x{:}A.B}{\Gamma \vdash V = \lambda x{:}A.V(x)_{(x:A).B} : \Pi x{:}A.B}$$

Propositional equality:

$$\frac{\Gamma \vdash A \quad \Gamma, x_1{:}A, x_2{:}A, x_3{:}x_1 =_A x_2 \vdash B \quad \Gamma \vdash V : A \quad \Gamma, y{:}A \vdash W : B[y/x_1][y/x_2][\mathtt{refl}\ y/x_3]}{\Gamma \vdash \mathtt{eq\text{-}elim}_A(x_1.x_2.x_3.B, y.W, V, V, \mathtt{refl}\ V) = W[V/y] : B[V/x_1][V/x_2][\mathtt{refl}\ V/x_3]}$$

Thunking and forcing:

$$\frac{\Gamma \vdash V : U\underline{C}}{\Gamma \vdash \mathtt{thunk}\ (\mathtt{force}_{\underline{C}}\ V) = V : U\underline{C}} \qquad \frac{\Gamma \vdash M : \underline{C}}{\Gamma \vdash \mathtt{force}_{\underline{C}}\ (\mathtt{thunk}\ M) = M : \underline{C}}$$

Homomorphic function type:

$$\frac{\Gamma \vdash M : \underline{C} \quad \Gamma \mid z{:}\underline{C} \vdash K : \underline{D}}{\Gamma \vdash (\lambda z{:}\underline{C}.K)(M)_{\underline{C},\underline{D}} = K[M/z] : \underline{D}} \qquad \frac{\Gamma \mid z_1{:}\underline{C} \vdash K : \underline{D}_1 \quad \Gamma \mid z_2{:}\underline{D}_1 \vdash L : \underline{D}_2}{\Gamma \mid z_1{:}\underline{C} \vdash (\lambda z_2{:}\underline{D}_1.L)(K)_{\underline{D}_1,\underline{D}_2} = L[K/z_2] : \underline{D}_2}$$

$$\frac{\Gamma \vdash V : \underline{C} \multimap \underline{D}}{\Gamma \vdash V = \lambda z{:}\underline{C}.V(z)_{\underline{C},\underline{D}} : \underline{C} \multimap \underline{D}}$$

Sequential composition:

$$\frac{\Gamma \vdash V : A \quad \Gamma \vdash \underline{C} \quad \Gamma, x{:}A \vdash M : \underline{C}}{\Gamma \vdash \mathtt{return}\ V\ \mathtt{to}\ x{:}A\ \mathtt{in}_{\underline{C}}\ M = M[V/x] : \underline{C}}$$

$$\frac{\Gamma \vdash M : FA \quad \Gamma \vdash \underline{C} \quad \Gamma \mid z{:}FA \vdash K : \underline{C}}{\Gamma \vdash M\ \mathtt{to}\ x{:}A\ \mathtt{in}_{\underline{C}}\ K[\mathtt{return}\ x/z] = K[M/z] : \underline{C}} \qquad \frac{\Gamma \mid z_1{:}\underline{C} \vdash K : FA \quad \Gamma \vdash \underline{D} \quad \Gamma \mid z_2{:}FA \vdash L : \underline{D}}{\Gamma \mid z_1{:}\underline{C} \vdash K\ \mathtt{to}\ x{:}A\ \mathtt{in}_{\underline{D}}\ L[\mathtt{return}\ x/z_2] = L[K/z_2] : \underline{D}}$$

Computational Σ-type:

$$\frac{\Gamma \vdash V : A \quad \Gamma \vdash M : \underline{C}[V/x] \quad \Gamma \vdash \underline{D} \quad \Gamma, x{:}A \mid z{:}\underline{C} \vdash K : \underline{D}}{\Gamma \vdash \langle V, M\rangle_{(x:A).\underline{C}}\ \mathtt{to}\ (x{:}A, z{:}\underline{C})\ \mathtt{in}_{\underline{D}}\ K = K[V/x][M/z] : \underline{D}}$$

$$\frac{\Gamma, x{:}A \vdash \underline{C} \quad \Gamma \vdash M : \Sigma x{:}A.\underline{C} \quad \Gamma \vdash \underline{D} \quad \Gamma \mid z_2{:}\Sigma x{:}A.\underline{C} \vdash K : \underline{D}}{\Gamma \vdash M\ \mathtt{to}\ (x{:}A, z_1{:}\underline{C})\ \mathtt{in}_{\underline{D}}\ K[\langle x, z_1\rangle_{(x:A).\underline{C}}/z_2] = K[M/z_2] : \underline{D}}$$

$$\frac{\Gamma \vdash V : A \quad \Gamma \mid z_1{:}\underline{C} \vdash K : \underline{D}_1[V/x] \quad \Gamma \vdash \underline{D}_2 \quad \Gamma, x{:}A \mid z_2{:}\underline{D}_1 \vdash L : \underline{D}_2}{\Gamma \mid z_1{:}\underline{C} \vdash \langle V, K\rangle_{(x:A).\underline{D}_1}\ \mathtt{to}\ (x{:}A, z_2{:}\underline{D}_1)\ \mathtt{in}_{\underline{D}_2}\ L = L[V/x][K/z_2] : \underline{D}_2}$$

$$\frac{\Gamma, x{:}A \vdash \underline{D}_1 \quad \Gamma \mid z_1{:}\underline{C} \vdash K : \Sigma x{:}A.\underline{D}_1 \quad \Gamma \vdash \underline{D}_2 \quad \Gamma \mid z_3{:}\Sigma x{:}A.\underline{D}_1 \vdash K : \underline{D}_2}{\Gamma \mid z_1{:}\underline{C} \vdash K\ \mathtt{to}\ (x{:}A, z_2{:}\underline{D}_1)\ \mathtt{in}_{\underline{D}_2}\ L[\langle x, z_2\rangle_{(x:A).\underline{D}_1}/z_3] = L[K/z_3] : \underline{D}_2}$$

Computational Π-type:

$$\frac{\Gamma, x{:}A \vdash M : \underline{C} \quad \Gamma \vdash V : A}{\Gamma \vdash (\lambda x{:}A.M)(V)_{(x:A).\underline{C}} = M[V/x] : \underline{C}[V/x]} \qquad \frac{\Gamma, x{:}A \vdash \underline{C} \quad \Gamma \vdash M : \Pi x{:}A.\underline{C}}{\Gamma \vdash M = \lambda x{:}A.M(x)_{(x:A).\underline{C}} : \Pi x{:}A.\underline{C}}$$

$$\frac{\Gamma \vdash \underline{C} \quad \Gamma, x{:}A \mid z{:}\underline{C} \vdash K : \underline{D} \quad \Gamma \vdash V : A}{\Gamma \mid z{:}\underline{C} \vdash (\lambda x{:}A.K)(V)_{(x:A).\underline{D}} = K[V/x] : \underline{D}[V/x]} \qquad \frac{\Gamma, x{:}A \vdash \underline{D} \quad \Gamma \mid z{:}\underline{C} \vdash K : \Pi x{:}A.\underline{D}}{\Gamma \mid z{:}\underline{C} \vdash K = \lambda x{:}A.K(x)_{(x:A).\underline{D}} : \Pi x{:}A.\underline{D}}$$

Context and type conversion rules:

$$\frac{\Gamma_1 \vdash A = B \quad \vdash \Gamma_1 = \Gamma_2}{\Gamma_2 \vdash A = B} \qquad \frac{\Gamma_1 \vdash \underline{C} = \underline{D} \quad \vdash \Gamma_1 = \Gamma_2}{\Gamma_2 \vdash \underline{C} = \underline{D}}$$

$$\frac{\Gamma_1 \vdash V = W : A \quad \vdash \Gamma_1 = \Gamma_2 \quad \Gamma_1 \vdash A = B}{\Gamma_2 \vdash V = W : B} \qquad \frac{\Gamma_1 \vdash M = N : \underline{C} \quad \vdash \Gamma_1 = \Gamma_2 \quad \Gamma_1 \vdash \underline{C} = \underline{D}}{\Gamma_2 \vdash M = N : \underline{D}}$$

$$\frac{\Gamma_1 \mid z{:}\underline{C}_1 \vdash K = L : \underline{D}_1 \quad \vdash \Gamma_1 = \Gamma_2 \quad \Gamma_1 \vdash \underline{C}_1 = \underline{C}_2 \quad \Gamma_1 \vdash \underline{D}_1 = \underline{D}_2}{\Gamma_2 \mid z{:}\underline{C}_2 \vdash K = L : \underline{D}_2}$$

C DIAGRAMMATIC AND MORE DETAILED DEFINITION OF $[\![-]\!]$

In this appendix we give diagrammatic and more detailed definition of $[\![-]\!]$ for algebraic operations, the user-defined algebra type, and the two composition operations. We present the partial definition in a natural deduction style, where the premises of the rule are assumed to hold for the conclusion to be defined. Further, we write

$$[\![\Gamma; A]\!]_1 = [\![\Gamma]\!] \in \mathrm{Set} \qquad [\![\Gamma; A]\!]_2 : [\![\Gamma]\!] \longrightarrow \mathrm{Set}$$

to mean that: i) $[\![\Gamma; A]\!]$ is defined; ii) its first component is equal to the set $[\![\Gamma]\!]$; and iii) its second component is given by a functor $[\![\Gamma]\!] \longrightarrow \mathrm{Set}$. The notation we use for the interpretation of terms is analogous.

C.1 Algebraic operations:

Given $\mathsf{op} : (x{:}I) \longrightarrow O \in \mathcal{S}_{\mathrm{eff}}$, we give the definition of $[\![\Gamma; \mathsf{op}_V^C(y.M)]\!]$ in Fig. 5, where the morphism

$$\mathsf{op}^{[\![\Gamma;\underline{C}]\!]_2(\gamma)}_{([\![\Gamma;V]\!]_2)_\gamma(\star)}$$

is defined using the countable-product preservation property of $[\![\Gamma; \underline{C}]\!]_2(\gamma)$, namely, as the following composite:

$$\begin{array}{c}
\prod_{o \in [\![\Gamma; O[V/x]]\!]_2(\gamma)} U_{\mathcal{L}_{\mathcal{T}_{\mathrm{eff}}}}([\![\Gamma; \underline{C}]\!]_2(\gamma)) \\
\Big\downarrow{\scriptstyle =} \\
\prod_{o \in [\![\Gamma; O[V/x]]\!]_2(\gamma)} ([\![\Gamma; \underline{C}]\!]_2(\gamma))(1) \\
\Big\downarrow{\scriptstyle \cong} \\
([\![\Gamma; \underline{C}]\!]_2(\gamma))(+_{1 \le o \le |[\![\Gamma; O[V/x]]\!]_2(\gamma)|} 1) \\
\Big\downarrow{\scriptstyle =} \\
([\![\Gamma; \underline{C}]\!]_2(\gamma))(|[\![\Gamma; O[V/x]]\!]_2(\gamma)|) \\
\Big\downarrow{\scriptstyle (*)} \\
([\![\Gamma; \underline{C}]\!]_2(\gamma))(1) \\
\Big\downarrow{\scriptstyle =} \\
U_{\mathcal{L}_{\mathcal{T}_{\mathrm{eff}}}}([\![\Gamma; \underline{C}]\!]_2(\gamma))
\end{array}$$

$$[\![\Gamma; V]\!]_1 = \mathsf{id}_{[\![\Gamma]\!]} : [\![\Gamma]\!] \longrightarrow [\![\Gamma]\!] \qquad ([\![\Gamma; V]\!]_2)_\gamma : 1 \longrightarrow [\![\diamond; I]\!]_2(\star)$$
$$[\![\Gamma; O[V/x]]\!]_1 = [\![\Gamma]\!] \in \mathsf{Set} \qquad [\![\Gamma; O[V/x]]\!]_2(\gamma) = [\![\Gamma, x{:}I; O]\!]_2(\langle \gamma, ([\![\Gamma; V]\!]_2)_\gamma(\star)\rangle) \in \mathsf{Set}$$
$$[\![\Gamma, y{:}O[V/x]; M]\!]_1 = \mathsf{id}_{\coprod_{\gamma \in [\![\Gamma]\!]} [\![\Gamma; O[V/x]]\!]_2(\gamma)}$$
$$([\![\Gamma, y{:}O[V/x]; M]\!]_2)_{\langle \gamma, o\rangle} : 1 \longrightarrow U_{\mathcal{L}_{\mathcal{T}_{\mathrm{eff}}}}([\![\Gamma; \underline{C}]\!]_2(\gamma))$$

$$[\![\Gamma; \mathsf{op}^C_V(y.M)]\!]_1 \overset{\mathrm{def}}{=} [\![\Gamma]\!] \xrightarrow{\mathsf{id}_{[\![\Gamma]\!]}} [\![\Gamma]\!]$$

$$([\![\Gamma; \mathsf{op}^C_V(y.M)]\!]_2)_\gamma \overset{\mathrm{def}}{=} 1 \xrightarrow{\langle \mathsf{id}_1 \rangle_{o \in [\![\Gamma; O[V/x]]\!]_2(\gamma)}} \prod_{o \in [\![\Gamma; O[V/x]]\!]_2(\gamma)} 1 \xrightarrow{\prod_{o \in [\![\Gamma; O[V/x]]\!]_2(\gamma)} ([\![\Gamma, y{:}O[V/x]; M]\!]_2)_{\langle \gamma, o\rangle}} \prod_{o \in [\![\Gamma; O[V/x]]\!]_2(\gamma)} U_{\mathcal{L}_{\mathcal{T}_{\mathrm{eff}}}}([\![\Gamma; \underline{C}]\!]_2(\gamma)) \xrightarrow{\mathsf{op}^{[\![\Gamma; \underline{C}]\!]_2(\gamma)}_{([\![\Gamma; V]\!]_2)_\gamma(\star)}} U_{\mathcal{L}_{\mathcal{T}_{\mathrm{eff}}}}([\![\Gamma; \underline{C}]\!]_2(\gamma))$$

Fig. 5. Appendix: Interpretation of algebraic operations.

where $(*)$ denotes the following function:

$$([\![\Gamma; \underline{C}]\!]_2(\gamma))(\overrightarrow{x_o} \vdash \mathsf{op}_{([\![\Gamma; V]\!]_2)_\gamma(\star)}(x_o)_{1 \le o \le |[\![\Gamma; O[V/x]]\!]_2(\gamma)|})$$

C.2 User-defined algebra type:

We give the definition of $[\![\Gamma'; \langle A, \{V_{\mathrm{op}}\}_{\mathrm{op} \in \mathcal{S}_{\mathrm{eff}}}\rangle]\!]$ in Fig. 6. The models $\mathcal{M}^{\gamma'}$ of $\mathcal{L}_{\mathcal{T}_{\mathrm{eff}}}$ are defined as in §8.4.

C.3 Composition operations:

We define $[\![-]\!]$ for the two composition operations in Fig. 7 and Fig. 8, respectively.

For better readability, we present the assumptions about the functions f^γ diagrammatically in Fig. 7 and Fig. 8.

$$[\![\Gamma';A]\!]_1 = [\![\Gamma']\!] \in \mathsf{Set} \qquad [\![\Gamma';A]\!]_2 : [\![\Gamma']\!] \longrightarrow \mathsf{Set}$$
$$[\![\Gamma';V_{\mathsf{op}}]\!]_1 = \mathsf{id}_{[\![\Gamma']\!]} : [\![\Gamma']\!] \longrightarrow [\![\Gamma']\!]$$
$$([\![\Gamma';V_{\mathsf{op}}]\!]_2)_{\gamma'} : 1 \longrightarrow \textstyle\prod_{\langle i,f\rangle \in \coprod_{i \in [\![\diamond;I]\!]_2(\star)} \prod_{o \in [\![x:I;O]\!]_2(\langle\star,i\rangle)} [\![\Gamma';A]\!]_2(\gamma')} [\![\Gamma';A]\!]_2(\gamma')$$
$$\mathcal{M}^{\gamma'}(\Delta^{\gamma} \vdash T_1^{\gamma}) = \mathcal{M}^{\gamma'}(\Delta^{\gamma} \vdash T_2^{\gamma})$$
$$(\text{for all } \Gamma \mid \Delta \vdash T_1 = T_2 \text{ in } \mathcal{E}_{\mathsf{eff}} \text{ and } \gamma \text{ in } [\![\Gamma]\!])$$

$$[\![\Gamma';\langle A,\{V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathsf{eff}}}\rangle]\!]_1 \stackrel{\text{def}}{=} [\![\Gamma']\!]$$
$$[\![\Gamma';\langle A,\{V_{\mathsf{op}}\}_{\mathsf{op}\in\mathcal{S}_{\mathsf{eff}}}\rangle]\!]_2(\gamma') \stackrel{\text{def}}{=} \mathcal{M}^{\gamma'}$$

Fig. 6. Appendix: Interpretation of the user-defined algebra type.

For Fig. 8, we define the morphism $\mathsf{hom}(f^{\gamma})$ of models of $\mathcal{L}_{\mathcal{T}_{\mathsf{eff}}}$ from $[\![\Gamma;\underline{D}_1]\!]_2(\gamma)$ to $[\![\Gamma;\underline{D}_2]\!]_2(\gamma)$ (i.e., a natural transformation) using components $(\mathsf{hom}(f^{\gamma}))_n$ given by

$$\begin{array}{c}
([\![\Gamma;\underline{D}_1]\!]_2(\gamma))(n) \\
\downarrow \cong \\
\textstyle\prod_{1 \le j \le n}([\![\Gamma;\underline{D}_1]\!]_2(\gamma))(1) \\
\downarrow \textstyle\prod_j(f^{\gamma}) \\
\textstyle\prod_{1 \le j \le n}([\![\Gamma;\underline{D}_2]\!]_2(\gamma))(1) \\
\downarrow \cong \\
([\![\Gamma;\underline{D}_2]\!]_2(\gamma))(n)
\end{array}$$

$$[\![\Gamma; M]\!]_1 = \mathsf{id}_{[\![\Gamma]\!]} : [\![\Gamma]\!] \longrightarrow [\![\Gamma]\!] \qquad ([\![\Gamma; M]\!]_2)_\gamma : 1 \longrightarrow U_{\mathcal{L}_{\mathcal{T}_{\mathrm{eff}}}}([\![\Gamma; \underline{C}]\!]_2(\gamma))$$

$$[\![\Gamma, y{:}U\underline{C}; N]\!]_1 = \mathsf{id}_{\coprod_{\gamma \in [\![\Gamma]\!]} U_{\mathcal{L}_{\mathcal{T}_{\mathrm{eff}}}}([\![\Gamma; \underline{C}]\!]_2(\gamma))} : [\![\Gamma, y{:}U\underline{C}]\!] \longrightarrow [\![\Gamma, y{:}U\underline{C}]\!]$$

$$([\![\Gamma, y{:}U\underline{C}; N]\!]_2)_{\langle \gamma, c\rangle} : 1 \longrightarrow U_{\mathcal{L}_{\mathcal{T}_{\mathrm{eff}}}}([\![\Gamma; \underline{D}]\!]_2(\gamma))$$

$$\begin{array}{ccc}
\prod_o \mathcal{M}_1^\gamma(1) & \xrightarrow{\prod_{o \in [\![x:I;O]\!]_2(\langle\star, i\rangle)}(f^\gamma)} & \prod_o \mathcal{M}_2^\gamma(1) \\
\cong \downarrow & & \downarrow \cong \\
\mathcal{M}_1^\gamma(|[\![x:I;O]\!]_2(\langle\star, i\rangle)|) & & \mathcal{M}_2^\gamma(|[\![x:I;O]\!]_2(\langle\star, i\rangle)|) \\
{\scriptstyle \mathcal{M}_1^\gamma(\overrightarrow{x_o} \vdash \mathrm{op}_i(x_o)_o)} \downarrow & & \downarrow {\scriptstyle \mathcal{M}_2^\gamma(\overrightarrow{x_o} \vdash \mathrm{op}_i(x_o)_o)} \\
\mathcal{M}_1^\gamma(1) & \xrightarrow[f^\gamma]{} & \mathcal{M}_2^\gamma(1)
\end{array}$$

for all $\mathrm{op} : (x{:}I) \longrightarrow O \in \mathcal{S}_{\mathrm{eff}}$, $\gamma \in [\![\Gamma]\!]$ and $i \in [\![\diamond; I]\!]_2(\star)$, and where

$$\mathcal{M}_1^\gamma \stackrel{\mathrm{def}}{=} [\![\Gamma; \underline{C}]\!]_2(\gamma) \qquad \mathcal{M}_2^\gamma \stackrel{\mathrm{def}}{=} [\![\Gamma; \underline{D}]\!]_2(\gamma) \qquad f^\gamma \stackrel{\mathrm{def}}{=} c \mapsto ([\![\Gamma, y{:}U\underline{C}; N]\!]_2)_{\langle \gamma, c\rangle}(\star)$$

$$\begin{array}{ccc}
[\![\Gamma; M \text{ as } y{:}U\underline{C} \text{ in}_{\underline{D}} N]\!]_1 & & ([\![\Gamma; M \text{ as } y{:}U\underline{C} \text{ in}_{\underline{D}} N]\!]_2)_\gamma \\
\stackrel{\mathrm{def}}{=} & & \stackrel{\mathrm{def}}{=} \\
[\![\Gamma]\!] & & 1 \\
\downarrow & & \downarrow {\scriptstyle ([\![\Gamma; M]\!]_2)_\gamma} \\
{\scriptstyle \mathsf{id}_{[\![\Gamma]\!]}} \downarrow & & U_{\mathcal{L}_{\mathcal{T}_{\mathrm{eff}}}}([\![\Gamma; \underline{C}]\!]_2(\gamma)) \\
 & & \downarrow {\scriptstyle f^\gamma} \\
[\![\Gamma]\!] & & U_{\mathcal{L}_{\mathcal{T}_{\mathrm{eff}}}}([\![\Gamma; \underline{D}]\!]_2(\gamma))
\end{array}$$

Fig. 7. Appendix: Interpretation of the composition operation for computation terms.

$$[\![\Gamma; z{:}\underline{C}; K]\!]_1 = \mathsf{id}_{[\![\Gamma]\!]} : [\![\Gamma]\!] \longrightarrow [\![\Gamma]\!] \qquad ([\![\Gamma; z{:}\underline{C}; K]\!]_2)_\gamma : [\![\Gamma; \underline{C}]\!]_2(\gamma) \longrightarrow [\![\Gamma; \underline{D}_1]\!]_2(\gamma)$$

$$[\![\Gamma, y{:}U\underline{D}_1; M]\!]_1 = \mathsf{id}_{\coprod_{\gamma \in [\![\Gamma]\!]} U_{\mathcal{L}_{\mathcal{T}_{\mathrm{eff}}}}([\![\Gamma; \underline{D}_1]\!]_2(\gamma))} : [\![\Gamma, y{:}U\underline{D}_1]\!] \longrightarrow [\![\Gamma, y{:}U\underline{D}_1]\!]$$

$$([\![\Gamma, y{:}U\underline{D}_1; M]\!]_2)_{\langle \gamma, d \rangle} : 1 \longrightarrow U_{\mathcal{L}_{\mathcal{T}_{\mathrm{eff}}}}([\![\Gamma; \underline{D}_2]\!]_2(\gamma))$$

$$\begin{array}{ccc}
\prod_o \mathcal{M}_1^\gamma(1) & \xrightarrow{\prod_{o \in [\![x:I;O]\!]_2(\langle \star, i \rangle)}(f^\gamma)} & \prod_o \mathcal{M}_2^\gamma(1) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
\mathcal{M}_1^\gamma(|[\![x:I;O]\!]_2(\langle \star, i \rangle)|) & & \mathcal{M}_2^\gamma(|[\![x:I;O]\!]_2(\langle \star, i \rangle)|) \\
\downarrow{\scriptstyle \mathcal{M}_1^\gamma(\overrightarrow{x_o} \vdash \mathsf{op}_i(x_o)_o)} & & \downarrow{\scriptstyle \mathcal{M}_2^\gamma(\overrightarrow{x_o} \vdash \mathsf{op}_i(x_o)_o)} \\
\mathcal{M}_1^\gamma(1) & \xrightarrow[f^\gamma]{} & \mathcal{M}_2^\gamma(1)
\end{array}$$

for all $\mathsf{op} : (x{:}I) \longrightarrow O \in \mathcal{S}_{\mathrm{eff}}$, $\gamma \in [\![\Gamma]\!]$ and $i \in [\![\diamond; I]\!]_2(\star)$, and where

$$\mathcal{M}_1^\gamma \stackrel{\mathrm{def}}{=} [\![\Gamma; \underline{D}_1]\!]_2(\gamma) \qquad \mathcal{M}_2^\gamma \stackrel{\mathrm{def}}{=} [\![\Gamma; \underline{D}_2]\!]_2(\gamma) \qquad f^\gamma \stackrel{\mathrm{def}}{=} d \mapsto ([\![\Gamma, y{:}U\underline{D}_1; M]\!]_2)_{\langle \gamma, d \rangle}(\star)$$

$$\begin{array}{ccc}
[\![\Gamma; z{:}\underline{C}; K \ \mathsf{as}\ y{:}U\underline{D}_1\ \mathsf{in}_{\underline{D}_2}\ M]\!]_1 & & ([\![\Gamma; z{:}\underline{C}; K \ \mathsf{as}\ y{:}U\underline{D}_1\ \mathsf{in}_{\underline{D}_2}\ M]\!]_2)_\gamma \\
\stackrel{\mathrm{def}}{=} & & \stackrel{\mathrm{def}}{=} \\
[\![\Gamma]\!] & & [\![\Gamma; \underline{C}]\!]_2(\gamma) \\
\downarrow{\scriptstyle \mathsf{id}_{[\![\Gamma]\!]}} & & \downarrow{\scriptstyle ([\![\Gamma; z{:}\underline{C}; K]\!]_2)_\gamma} \\
[\![\Gamma]\!] & & [\![\Gamma; \underline{D}_1]\!]_2(\gamma) \\
 & & \downarrow{\scriptstyle \mathsf{hom}(f^\gamma)} \\
 & & [\![\Gamma; \underline{D}_2]\!]_2(\gamma)
\end{array}$$

Fig. 8. Appendix: Interpretation of the composition operation for homomorphism terms.