# A fibrational view on computational effects

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#### **Background – dependent types**

#### The Curry-Howard correspondence:

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\begin{array}{lll} \text{Simple Types} & \sim & \text{Propositional Logic} & & (\text{Nat}, \text{String}, \ldots) \\ \\ \text{Dependent Types} & \sim & \text{Predicate Logic} & & (\Sigma, \Pi, =, \ldots) \end{array}
```

A tiny example: we can use dep. types to express sorted lists

$$\ell$$
: (List Nat)  $\vdash$  Sorted( $\ell$ )  $\stackrel{\text{def}}{=}$   $\Pi i$ : Nat. ( $0 < i < \text{len } \ell$ )  $\rightarrow$  ( $\ell[i-1] \le \ell[i]$ )

which in turn could be used for typing sorting functions

```
\forall sort : \Pi \ell: (List Nat) . \Sigma \ell': (List Nat) . (Sorted(\ell') \times \dots)
```

Large examples: CompCert (Coq), miTLS and HACL\* (F\*), ...

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### **Background – computational effects**

#### **Examples:**

• state, exceptions, divergence, IO, nondeterminism, probability, . . .

#### Meta-languages and models for comp. effects: based on

• monads ( $\lambda_c$ ,  $\lambda_{ML}$ , FGCBV) (Moggi, Levy)

$$\llbracket \Gamma \vdash M : A \rrbracket_{\lambda_{\mathsf{c}}} : \llbracket \Gamma \rrbracket \longrightarrow T \llbracket A \rrbracket$$

• adjunctions (CBPV, EEC) (Levy, Egger et al.)

$$\llbracket \Gamma \vdash V : A \rrbracket_{CBPV} : \llbracket \Gamma \rrbracket \longrightarrow \llbracket A \rrbracket \qquad \llbracket \Gamma \vdash M : \underline{C} \rrbracket_{CBPV} : \llbracket \Gamma \rrbracket \longrightarrow U(\llbracket \underline{C} \rrbracket)$$

• algebraic presentations (Plotkin and Power)

get : 
$$1 \rightharpoonup S$$
 put :  $S \rightharpoonup 1$  (+ equations)

#### We investigate the combination of

```
• dependent types  (\Pi, \Sigma, V =_{\mathcal{A}} W, ...)
```

• computational effects (state, nondeterminism, IO, ...)

#### Goals

- tell a mathematically natural story
- use established math. techniques
- cover a wide range of comp. effects
- discover smth. interesting

- effectful programs in types (e.g., get and put in types)
- typing of effectful programs (e.g., sequential composition)

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- cover a wide range of comp. effects (alg. effects, continuations)
- discover smth. interesting (using handlers to reason about effects)

- effectful programs in types (e.g., get and put in types)
- typing of effectful programs (e.g., sequential composition)

(type-dependency in the presence of effects)

**Q:** Should we allow situations such as Sorted[receive(y.M)/ $\ell$ ]?

A1: In this work, we say not directly

- types should only depend on static information about effects
- allow dependency on effectful comps. via analysing thunks

**A2:** Various people are also looking at the direct case

- type-dependency needs to be "homomorphic"
- intuitively,
  - need to lift Sorted( $\ell$ ) to Sorted<sup>†</sup>(c), where c: T(List Chr)
    - $\mathsf{Sorted}^\dagger(\mathtt{receive}(y.\mathtt{return}\,y)) = \langle \mathtt{receive} \rangle (y.\mathtt{Sorted}(y))$
  - for this Sorted needs to be a T-algebra
- (cf. recent papers by Pédrot and Tabareau; and Bowman et al.)

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**Aim:** Types should only depend on static info about effects

Solution: CBPV/EEC style distinction between vals. and comps

- value types  $\Gamma \vdash A$  (MLTT + thunks + ...)
- computation types  $\Gamma \vdash \underline{C}$  (dep. typed CBPV/EEC)
- where  $\Gamma$  contains only value variables  $x_1: A_1, \ldots, x_n: A_n$

Could have also considered Moggi's  $\lambda_{\mathsf{ML}}$  or Levy's FGCBV

- building on CBPV/EEC gives a more general story
- especially for the treatment of sequential composition
- and also for (Idris-style parameterised) dependent effect-typing

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### Typing of effectful programs

(e.g., sequential composition)

The problem: The standard typing rule for seq. composition

$$\frac{\Gamma \vdash_{\overline{c}} M : F \land A \qquad \Gamma, x : A \vdash_{\overline{c}} N : \underline{C}(x)}{\Gamma \vdash_{\overline{c}} M \text{ to } x : A \text{ in } N : \underline{C}(x)}$$

is not correct any more because potentially

$$x \in FV(\underline{C})$$

in the conclusion

Aim: To fix the typing rule of sequential composition

**Option 1:** We could restrict the free variables in  $\underline{C}$ : [Levy'04]  $\underline{\Gamma \vDash M : FA \qquad \Gamma \vdash \underline{C} \qquad \Gamma, x : A \vDash N : \underline{C}}$ 

**But:** Sometimes it is useful if  $\underline{C}$  can depend on x!

sav we consider

fopen (return true, return false) to x: Bool in N

• then it would be natural to let  $\underline{C}$  depend on x, e.g.,

 $x: Bool \vdash \underline{C}(x) \stackrel{\text{def}}{=} \text{if } x \text{ then "allow fread, fwrite, and fclose"}$  else "allow fopen"

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Option 2: One could lift sequential composition to type level

$$\Gamma \vdash M \text{ to } x : A \text{ in } N : M \text{ to } x : A \text{ in } C$$

But: Then comp. types would be singleton-like!?!

**Option 3:** In the monadic metalanguage  $\lambda_{ML}$ , one could try

$$\Gamma \vdash M : TA$$
  $\Gamma, x : A \vdash N : TB(x)$   
 $\Gamma \vdash M \text{ to } x : A \text{ in } N : T(\Sigma x : A.B)$ 

But: What makes this a principled solution? Why is it correct?

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Our solution: We draw inspiration from algebraic effects
and combine this with restricting <u>C</u> in seq. comp. (Option 1)

E.g., consider the non-deterministic prog. (for  $x : \text{Nat } \vdash N : \underline{C}(x)$ )  $M \stackrel{\text{def}}{=} \text{choose (return 4. return 2) to } x : \text{Nat in } N$ 

After making the non-det. choice, this program evaluates as either  $N[4/x] : \underline{C}[4/x]$  or  $N[2/x] : \underline{C}[2/x]$ 

**Idea:** M denotes an element of the coproduct of algebras

$$\underline{C}[4/x] + \underline{C}[2/x] \stackrel{\text{def}}{=} F\left(U\left(\underline{C}[4/x]\right) + U\left(\underline{C}[2/x]\right)\right)_{=}$$

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#### Putting these ideas together

(eMLTT: a core dep.-typed calculus with comp. effects)

### eMLTT – value and comp. types

Value types: MLTT + thunks + ...

$$A, B ::=$$
Nat  $\mid 1 \mid 0 \mid \Pi x : A . B \mid \Sigma x : A . B \mid V =_A W \mid U \subseteq \mid \dots$ 

•  $U\underline{C}$  is the type of thunked (i.e., suspended) computations

Computation types: dep.-typed version of EEC's comp. types

$$\underline{C}, \underline{D} ::= FA \mid \Pi x : A \cdot \underline{C} \mid \Sigma x : A \cdot \underline{C}$$

- FA is the type of computations returning values of type A
- $\Pi x: A. C$  is the type of dependent effectful functions
  - generalises CBPV/EEC's comp. types  $A \to \underline{C}$  and  $\underline{C} \times \underline{D}$
- $\Sigma x: A \cdot C$  is the type of dep. pairs of values and effectful comps.
  - captures the intuition about seq. comp. and coprods. of algebras
  - generalises EEC's comp. types  $!A \otimes C$  and  $C \oplus D$

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### eMLTT – value and comp. terms

```
Value terms: MLTT + thunks + ... V, W ::= x \mid zero \mid succ V \mid ... \mid thunk M \mid ...
```

equational theory based on intensional MLTT

**Comp. terms:** dep.-typed version of CBPV/EEC's comp. terms

```
\begin{array}{lll} M,N ::= & \operatorname{force} V \\ & | & \operatorname{return} V \\ & | & M \operatorname{to} x : A \operatorname{in} N \\ & | & \lambda x : A . M \\ & | & MV \\ & | & \langle V,M \rangle & (\operatorname{comp.} \Sigma \operatorname{intro.}) \\ & | & M \operatorname{to} \langle x : A,z : \underline{C} \rangle \operatorname{in} K & (\operatorname{comp.} \Sigma \operatorname{elim.}) \end{array}
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But: Value and comp. terms alone do not suffice, as in EEC!

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equational theory based on intensional MLTT

#### Comp. terms: dep.-typed version of CBPV/EEC's comp. terms

But: Value and comp. terms alone do not suffice, as in EEC!

### eMLTT - homomorphism terms

**Note:** We need to define K in such a way that the intended left-to-right evaluation order is preserved, e.g., consider

$$\Gamma \vdash_{\!\!\!\!c} \langle V, M \rangle \text{ to } \langle x \colon\! A, \mathbf{z} \colon\! \underline{C} \rangle \text{ in } \mathbf{K} = \mathbf{K}[V/x, M/\mathbf{z}] \colon\! \underline{D}$$

Homomorphism terms: dep.-typed version of EEC's linear terms

$$K, L := z$$
 (linear comp. vars.)  
 $\mid K \text{ to } x : A \text{ in } M$   
 $\mid \lambda x : A . K$   
 $\mid KV$   
 $\mid \langle V, K \rangle$  (comp.  $\Sigma \text{ intro.}$ )  
 $\mid K \text{ to } \langle x : A, z : C \rangle \text{ in } L$  (comp.  $\Sigma \text{ elim.}$ )

#### Typing judgments:

- Γ ⋈ V : A
- [ to M : C
- $\Gamma \mid z : \underline{C} \mid_{\overline{h}} K : \underline{D}$  (linear in z; comp. bound to z happens first

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$$\Gamma \vdash \langle V, M \rangle$$
 to  $\langle x : A, z : \underline{C} \rangle$  in  $K = K[V/x, M/z] : \underline{D}$ 

Homomorphism terms: dep.-typed version of EEC's linear terms

```
\begin{array}{lll} \textit{K}, \textit{L} ::= & \textit{z} & \text{(linear comp. vars.)} \\ & \mid & \textit{K} \text{ to } x : \textit{A} \text{ in } \textit{M} \\ & \mid & \lambda x : \textit{A} . \textit{K} \\ & \mid & \textit{KV} \\ & \mid & \langle \textit{V}, \textit{K} \rangle & \text{(comp. } \Sigma \text{ intro.)} \\ & \mid & \textit{K} \text{ to } \langle x : \textit{A}, \textit{z} : \underline{\textit{C}} \rangle \text{ in } \textit{L} & \text{(comp. } \Sigma \text{ elim.)} \end{array}
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- Γ |<sub>c</sub> M : C
- $\Gamma \mid z : \underline{C} \mid_{\overline{h}} K : \underline{D}$  (linear in z; comp. bound to z happens first)

## eMLTT – typing sequential composition

We can then account for type-dependency in seq. comp. as

$$\frac{\Gamma, x : A \vdash_{\overline{c}} N : \underline{C}(x)}{\Gamma \vdash_{\overline{c}} M : FA \qquad \Gamma \vdash_{\overline{c}} \Sigma x : A \cdot \underline{C}(x) \qquad \overline{\Gamma, x : A \vdash_{\overline{c}} \langle x, N \rangle : \Sigma x : A \cdot \underline{C}(x)}}{\Gamma \vdash_{\overline{c}} M \text{ to } x : A \text{ in } \langle x, N \rangle : \Sigma x : A \cdot \underline{C}(x)}$$

ullet As a bonus, the comp.  $\Sigma$ -type can also be used to explain Idris's

$$\begin{array}{c|c} \Gamma \vdash \varepsilon_1 : \mathsf{Effect} & \Gamma \vdash A & \Gamma \vdash \varepsilon_2 : A \to \mathsf{Effect} \\ \hline \qquad \qquad \Gamma \vdash T \varepsilon_1 A \varepsilon_2 \end{array}$$

in terms of standard parameterised effect-typing as

$$T \varepsilon_1 A \varepsilon_2 \stackrel{\text{def}}{=} U_{\varepsilon_1}(\Sigma \times : A \cdot F_{\varepsilon_2 \times} 1)$$

and thus naturally accommodate examples like

fopen (return true, return false) to x: Bool in  $\Lambda$ 

### eMLTT – typing sequential composition

We can then account for type-dependency in seq. comp. as

$$\frac{\Gamma, x : A \vdash N : \underline{C}(x)}{\Gamma \vdash E M : F A \qquad \Gamma \vdash \Sigma x : A \cdot \underline{C}(x) \qquad \overline{\Gamma, x : A \vdash C}(x, N) : \Sigma x : A \cdot \underline{C}(x)}{\Gamma \vdash E M \text{ to } x : A \text{ in } \langle x, N \rangle : \Sigma x : A \cdot \underline{C}(x)}$$

ullet As a bonus, the comp.  $\Sigma$ -type can also be used to explain Idris's

$$\frac{\Gamma \vdash \varepsilon_1 : \mathsf{Effect} \quad \Gamma \vdash A \quad \Gamma \vdash \varepsilon_2 : A \to \mathsf{Effect}}{\Gamma \vdash T \varepsilon_1 A \varepsilon_2}$$

in terms of standard parameterised effect-typing as

$$T \varepsilon_1 A \varepsilon_2 \stackrel{\text{def}}{=} U_{\varepsilon_1}(\Sigma \times : A \cdot F_{\varepsilon_2 \times} 1)$$

and thus naturally accommodate examples like

fopen (return true, return false) to x: Bool in N

## eMLTT – typing sequential composition

We can then account for type-dependency in seq. comp. as

$$\frac{\Gamma, x : A \vdash R \quad : \underline{C}(x)}{\Gamma \vdash R \quad \Gamma \vdash \Sigma x : A \cdot \underline{C}(x)} \frac{\Gamma, x : A \vdash R \quad : \underline{C}(x)}{\Gamma, x : A \vdash R \quad \langle x, N \rangle : \Sigma x : A \cdot \underline{C}(x)}$$

$$\Gamma \vdash R \quad \text{to } x : A \text{ in } \langle x, N \rangle : \Sigma x : A \cdot \underline{C}(x)$$

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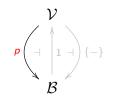
and thus naturally accommodate examples like

### Fibred adjunction models

(categorical semantics of eMLTT)

## Fibred adjunction models – value part

Given by a split closed comprehension category p, as in



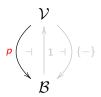
allowing us to define a partial interpretation fun. [-], that maps:

- a context  $\Gamma$  to and object  $\llbracket \Gamma \rrbracket$  in  $\mathcal{B}$ , with

  - $\llbracket \Gamma, x : A \rrbracket \stackrel{\mathsf{def}}{=} \{ \llbracket \Gamma; A \rrbracket \}$  (if  $x \notin \mathit{Vars}(\Gamma)$  and  $\llbracket \Gamma; A \rrbracket$  is defined)
- a context  $\Gamma$  and a value type A to an object  $\llbracket \Gamma; A \rrbracket$  in  $\mathcal{V}_{\llbracket \Gamma \rrbracket}$
- a context  $\Gamma$  and a value term V to  $\llbracket \Gamma; V \rrbracket : 1_{\llbracket \Gamma \rrbracket} \longrightarrow A$  in  $\mathcal{V}_{\llbracket \Gamma \rrbracket}$

### Fibred adjunction models – value part

Given by a split closed comprehension category p, as in

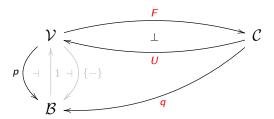


#### such that

- p has split fibred strong colimits of shape **0** and **2** [Jacobs'99]
  - (in thesis, also Jacobs-style characterisation for arbitrary shapes)
- p has weak split fibred strong natural numbers
  - (axiomatisation is given in the style of fibrational induction)
- p has split intensional propositional equality
  - (currently very synthetic ax., would like a weak form of adjoints)

### Fibred adjunction models - effects part

Given by a split fibration q and a split fib. adjunction  $F \dashv U$ , as in

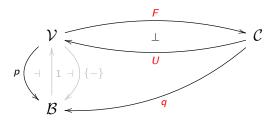


we extend the partial interpretation fun. [-] so that it maps:

- a ctx.  $\Gamma$  and a comp. type  $\underline{C}$  to an object  $[\![\Gamma;\underline{C}]\!]$  in  $\mathcal{C}_{[\![\Gamma]\!]}$
- a ctx.  $\Gamma$  and a comp. term M to  $[\![\Gamma;M]\!]:1_{[\![\Gamma]\!]}\longrightarrow U(\underline{C})$  in  $\mathcal{V}_{[\![\Gamma]\!]}$
- a ctx.  $\Gamma$ , a comp. var. z, a comp. type  $\underline{C}$ , and a hom. term K to  $\llbracket \Gamma; z : \underline{C}; K \rrbracket : \llbracket \Gamma; \underline{C} \rrbracket \longrightarrow \underline{D}$  in  $\mathcal{C}_{\llbracket \Gamma \rrbracket}$

### Fibred adjunction models – effects part

Given by a split fibration q and a split fib. adjunction  $F \dashv U$ , as in



#### such that

- q has split dependent p-products (comp. Π-type; r. adj. to wk.)
- q has split dependent p-coproducts (comp. Σ-type; l. adj. to wk.)

and to account for the full calculus presented in the thesis,

• q admits split fibred pre-enrichment in p (hom. function type  $-\circ$ )

### Fibred adjunction models – correctness

#### **Theorem** (Soundness):

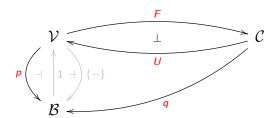
- If  $\Gamma \vdash \underline{\mathcal{C}}$ , then  $[\![\Gamma;\underline{\mathcal{C}}]\!] \in \mathcal{C}_{[\![\Gamma]\!]}$
- If  $\Gamma \vDash M : \underline{C}$ , then  $\llbracket \Gamma ; M \rrbracket : 1_{\llbracket \Gamma \rrbracket} \longrightarrow U(\llbracket \Gamma ; \underline{C} \rrbracket)$
- If  $\Gamma \mid z : \underline{C} \models K : \underline{D}$ , then  $\llbracket \Gamma; z : \underline{C}; K \rrbracket : \llbracket \Gamma; \underline{C} \rrbracket \longrightarrow \llbracket \Gamma; \underline{D} \rrbracket$
- $\bullet \ \ \mathsf{If} \ \Gamma \vdash \underline{C} = \underline{D}, \ \mathsf{then} \ [\![\Gamma;\underline{C}]\!] = [\![\Gamma;\underline{D}]\!] \in \mathcal{C}_{[\![\Gamma]\!]}$
- ...

#### Theorem (Classifying model):

• The well-formed syntax of eMLTT forms a fib. adjunction model.

#### Theorem (Completeness):

• If two types or terms are equal in all fibred adjunction models, then they are also equal in the equational theory of eMLTT.



#### **Example 1** (identity adjunctions):

• sound as long as no actual comp. effects in the calculus

Example 2 (simple fibrations from enriched adj. models of EEC):

• given an adj. model of EEC  $F\dashv U:\mathcal{C}\longrightarrow\mathcal{V}$   $(\mathcal{V}\text{ a CCC},\dots)$  we can lift it to simple fibrations  $\widehat{F}\dashv\widehat{U}:\mathsf{s}(\mathcal{V},\mathcal{C})\longrightarrow\mathsf{s}(\mathcal{V})$  where

$$\mathsf{s}_{\mathcal{V}\!,\mathcal{C}}:\mathsf{s}(\mathcal{V},\mathcal{C})\longrightarrow\mathcal{V}$$

is defined as

$$\mathsf{s}_{\mathcal{V},\mathcal{C}} \Big( X \in \mathcal{V} \,,\, \underline{C} \in \mathcal{C} \Big) \stackrel{\mathsf{def}}{=} X$$

$$\mathsf{s}_{\mathcal{V}\!,\mathcal{C}}\!\left(f:X\longrightarrow Y\,,\,h:X\otimes\underline{C}\longrightarrow\underline{D}\right)\stackrel{\mathsf{def}}{=}f\qquad:\mathsf{s}_{\mathcal{V}\!,\mathcal{C}}(X,\underline{C})\longrightarrow\mathsf{s}_{\mathcal{V}\!,\mathcal{C}}(Y,\underline{D})$$

• doesn't support any real type dependency (constant families)

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• doesn't support any real type dependency (constant families)

**Example 3** (families fibrations and lifting of adjunctions):

• given a suitable adjunction  $F_{\mathcal{D}}\dashv U_{\mathcal{D}}:\mathcal{D}\longrightarrow \mathsf{Set},$  we can lift it to  $\widehat{F_{\mathcal{D}}}\dashv \widehat{U_{\mathcal{D}}}:\mathsf{Fam}(\mathcal{D})\longrightarrow \mathsf{Fam}(\mathsf{Set})$  between  $\mathsf{fam}_{\mathsf{Set}}:\mathsf{Fam}(\mathsf{Set})\longrightarrow \mathsf{Set}$ 

$$\mathsf{fam}_{\mathsf{Set}} : \mathsf{Fam}(\mathsf{Set}) \longrightarrow \mathsf{Set}$$
 $\mathsf{fam}_{\mathcal{D}} : \mathsf{Fam}(\mathcal{D}) \longrightarrow \mathsf{Set}$ 

- resulting in
  - $(\llbracket \Gamma \rrbracket, \llbracket A \rrbracket) \in \mathsf{Fam}(\mathsf{Set})$
  - $(\llbracket \Gamma \rrbracket, \llbracket \underline{C} \rrbracket) \in \mathsf{Fam}(\mathcal{D})$

(where 
$$[\![A]\!]\in [\![\Gamma]\!]\longrightarrow \mathsf{Set}$$
)

(where  $[\![\underline{C}]\!] \in [\![\Gamma]\!] \longrightarrow \mathcal{D}$ )

- examples
  - $F^{\mathsf{T}} \dashv U^{\mathsf{T}} : \mathsf{Set}^{\mathsf{T}} \longrightarrow \mathsf{Set}$
  - $(-) \times S \dashv (-)^S : \mathsf{Set} \longrightarrow \mathsf{Set}$
  - $R^{(-)} \dashv R^{(-)} : \mathsf{Set}^{op} \longrightarrow \mathsf{Set}$

**Example 4** (continuous families and CPO-enriched monads):

```
• given the EM-adjunction F^{\mathsf{T}}\dashv U^{\mathsf{T}}:\mathsf{CPO}^{\mathsf{T}}\longrightarrow\mathsf{CPO},
we can lift it to \widehat{F_{\mathcal{D}}}\dashv\widehat{U_{\mathcal{D}}}:\mathsf{CFam}(\mathsf{CPO}^{\mathsf{T}})\longrightarrow\mathsf{CFam}(\mathsf{CPO})
between \mathsf{cfam}_{\mathsf{CPO}}:\mathsf{CFam}(\mathsf{CPO})\longrightarrow\mathsf{CPO}\mathsf{cfam}_{\mathsf{CPO}^{\mathsf{T}}}:\mathsf{CFam}(\mathsf{CPO}^{\mathsf{T}})\longrightarrow\mathsf{CPO}
```

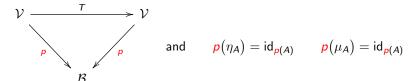
resulting in

```
• (\llbracket \Gamma \rrbracket, \llbracket A \rrbracket) \in CFam(CPO) (where \llbracket A \rrbracket \in \llbracket \Gamma \rrbracket \longrightarrow \mathsf{CPO}^{EP})
• (\llbracket \Gamma \rrbracket, \llbracket C \rrbracket) \in CFam(CPO^{\mathsf{T}}) (where \llbracket C \rrbracket \in \llbracket \Gamma \rrbracket \longrightarrow (\mathsf{CPO}^{\mathsf{T}})^{EP})
```

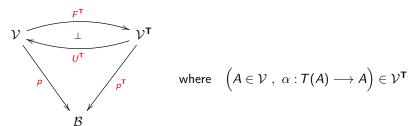
• Note: cod<sub>CPO</sub> is not suitable because CPO is not an LCCC.

**Example 5** (EM-resolutions of split fibred monads):

• given a split fibred monad  $\mathbf{T} = (T, \eta, \mu)$  on  $\mathbf{p}$ , i.e.,



we consider models based on the EM-resolution of T



and show that three familiar results hold for this situation

**Example 5** (EM-resolutions of split fibred monads):

• **Theorem 1:** If p supports  $\Pi$ -types, then  $p^{\mathsf{T}}$  also supports  $\Pi$ -types

$$\Pi_A^{\mathsf{T}}(B,\beta) \ \stackrel{\scriptscriptstyle\mathsf{def}}{=} \ \left(\Pi_A(B),\beta_{\Pi_A^{\mathsf{T}}}\right)$$

• **Prop.:** If p supports  $\Sigma$ -types, then T has a dependent strength

$$\sigma_A: \Sigma_A \circ T \longrightarrow T \circ \Sigma_A \qquad (A \in \mathcal{V})$$

• Theorem 2: If  $\sigma_A$  are natural isos., then  $p^T$  supports  $\Sigma$ -types

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 Theorem 3: If p supports Σ-types and p<sup>T</sup> has split fibred reflexive coequalizers, then p<sup>T</sup> also supports Σ-types

$$\Sigma_A^{\mathsf{T}}(B,\beta) \stackrel{\text{def}}{=} F^{\mathsf{T}}(\Sigma_A(B))_{/\equiv}$$

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# **Algebraic effects**

(operations and equations)

#### Fibred effect theories $\mathcal{T}_{\text{eff}}$ :

signatures of dependently typed operation symbols

$$\frac{\cdot \vdash I \qquad x_i : I \vdash O \qquad I \text{ and } O \text{ are pure value types}}{\text{op} : (x_i : I) \rightharpoonup O}$$

equipped with equations on derivable effect terms

#### In eMLTT:

$$M ::= \ldots \mid \operatorname{op}_{V}^{\mathcal{C}}(x.M)$$

**General algebraicity equations** (in addition to eff. th. eqs.):

$$\frac{\Gamma \trianglerighteq V : I \quad \Gamma, x : O[V/x_i] \trianglerighteq M : \underline{C} \quad \Gamma \mid z : \underline{C} \trianglerighteq_{\overline{h}} K : \underline{D}}{\Gamma \trianglerighteq K[\operatorname{op}_{\overline{V}}^{\underline{C}}(x.M)/z] = \operatorname{op}_{\overline{V}}^{\underline{D}}(x.K[M/z]) : \underline{D}} \text{ (op : } (x_i : I) \to O)$$

• 
$$p : \mathsf{Fam}(\mathsf{Set}) \longrightarrow \mathsf{Set}$$
 and  $q : \mathsf{Fam}(\mathsf{Mod}(\mathcal{L}_{\mathcal{T}_{\mathsf{eff}}}, \mathsf{Set})) \longrightarrow \mathsf{Set}$ 

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• 
$$p : \mathsf{Fam}(\mathsf{Set}) \longrightarrow \mathsf{Set}$$
 and  $g : \mathsf{Fam}(\mathsf{Mod}(\mathcal{L}_{\mathcal{T}_{\mathsf{eff}}}, \mathsf{Set})) \longrightarrow \mathsf{Set}$ 

#### Fibred effect theories $\mathcal{T}_{eff}$ :

signatures of dependently typed operation symbols

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equipped with equations on derivable effect terms

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• signatures of dependently typed operation symbols

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## Algebraic effects – examples

#### **Example 1** (interactive IO):

- read :  $1 
  ightharpoonup \mathsf{Chr} = 1 + \ldots + 1)$  write :  $\mathsf{Chr} \rightharpoonup 1$
- no equations

#### **Example 2** (global state with location-dependent store type):

- $\diamond$   $\vdash$  Loc  $\ell$ : Loc  $\vdash$  Val  $\diamond$   $\forall$  isDec<sub>Loc</sub> :  $\Pi \ell$ : Loc .  $\Pi \ell'$ : Loc .  $(\ell =_{Loc} \ell') + (\ell =_{Loc} \ell' \to 0)$ 
  - $\begin{array}{l} \bullet \;\; \mathsf{get} : (\ell \colon \mathsf{Loc}) \, \rightharpoonup \, \mathsf{Val} \\ \\ \mathsf{put} : (\Sigma \, \ell \colon \mathsf{Loc} \, . \mathsf{Val}) \, \rightharpoonup \, 1 \end{array}$
- five equations (two of them branching on isDecLoc

**Example 3** (dep. typed update monads  $TX \stackrel{\text{def}}{=} \Pi_{s:S}$ .  $Ps \times X$ )

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- get :  $(\ell : \mathsf{Loc}) \rightharpoonup \mathsf{Val}$ put :  $(\Sigma \ell : \mathsf{Loc} . \mathsf{Val}) \rightharpoonup 1$
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#### Handlers of algebraic effects (for programming and extrinsic reasoning)

Usual term-level presentation:

 $\Gamma \models M \text{ handled with } \{\operatorname{op}_{\mathsf{x}_v}(\mathsf{x}_k) \mapsto \mathsf{N}_{\operatorname{op}}\}_{\operatorname{op} \in \mathcal{T}_{\operatorname{eff}}} \text{ to } y : A \text{ in}_{\underline{C}} \ \mathsf{N}_{\operatorname{ret}} : \underline{C}$  satisfying

```
 (\operatorname{return} V) \text{ handled with } \{...\}_{\operatorname{op} \in \mathcal{T}_{\operatorname{eff}}} \text{ to } y \colon A \text{ in } N_{\operatorname{ret}} = N_{\operatorname{ret}}[V/x]   (\operatorname{op}_V^{\underline{C}}(x.M)) \text{ handled with } \{...\}_{\operatorname{op} \in \mathcal{T}_{\operatorname{eff}}} \text{ to } y \colon A \text{ in } N_{\operatorname{ret}} = N_{\operatorname{op}}[V/x_v][.../x_k]
```

- write your programs using alg. ops. (e.g., get and put)
- use handlers to provide fit-for-purpose impl. (e.g.,  $S \to X \times S$ )

#### Usual term-level presentation:

 $\Gamma \vdash_{\mathsf{c}} M \text{ handled with } \{ \mathsf{op}_{\mathsf{x}_\mathsf{v}}(\mathsf{x}_k) \mapsto \mathsf{N}_\mathsf{op} \}_{\mathsf{op} \in \mathcal{T}_\mathsf{eff}} \text{ to } y : A \text{ in}_{\underline{C}} \mathsf{N}_\mathsf{ret} : \underline{C}$ 

(return V) handled with  $\{...\}_{\mathsf{op}\in\mathcal{T}_{\mathsf{eff}}}$  to y:A in  $N_{\mathsf{ret}}=N_{\mathsf{ret}}[V/x]$  ( $\mathsf{op}_V^{\underline{C}}(x.M)$ ) handled with  $\{...\}_{\mathsf{op}\in\mathcal{T}_{\mathsf{eff}}}$  to y:A in  $N_{\mathsf{ret}}=N_{\mathsf{op}}[V/x_v][.../x_k]$ 

- write your programs using alg. ops. (e.g., get and put)
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#### Usual term-level presentation:

```
\Gamma \vDash M \text{ handled with } \{ \operatorname{op}_{\mathsf{X}_\mathsf{V}}(\mathsf{X}_k) \mapsto \mathsf{N}_{\operatorname{op}} \}_{\operatorname{op} \in \mathcal{T}_{\operatorname{eff}}} \text{ to } y \colon A \text{ in}_{\underline{C}} \text{ } \mathsf{N}_{\operatorname{ret}} \colon \underline{C} satisfying
```

- write your programs using alg. ops. (e.g., get and put)
- use handlers to provide fit-for-purpose impl. (e.g.,  $S \to X \times S$ )

 $\begin{tabular}{ll} \textbf{Idea:} & Generalisation of exception handlers} & & [Plotkin,Pretnar'09] \\ & & Handler \sim Algebra & and & Handling \sim Homomorphism \\ \end{tabular}$ 

#### Usual term-level presentation:

```
satisfying  (\text{return } V) \text{ handled with } \{...\}_{\texttt{op} \in \mathcal{T}_{\texttt{eff}}} \text{ to } y \colon A \text{ in } N_{\texttt{ret}} = N_{\texttt{ret}}[V/x]
```

 $\Gamma \vdash M$  handled with  $\{ op_{X_k}(x_k) \mapsto N_{op} \}_{op \in \mathcal{T}_{eff}}$  to  $y : A \text{ in }_C N_{ret} : \underline{C}$ 

 $(op_V^C(x.M))$  handled with  $\{...\}_{op \in \mathcal{T}_{eff}}$  to y:A in  $N_{ret} = N_{op}[V/x_v][.../x_k]$ 

- write your programs using alg. ops. (e.g., get and put)
- use handlers to provide fit-for-purpose impl. (e.g.,  $S \to X \times S$ )

## Handlers of alg. effects – for reasoning

Idea: Using a derived handle-into-values handling construct

$$M$$
 handled with  $\{\operatorname{op}_{x_v}(x_k)\mapsto V_{\operatorname{op}}\}_{\operatorname{op}\in\mathcal{T}_{\operatorname{eff}}}$  to  $y\!:\!A$  in  $B$   $V_{\operatorname{ret}}$  ve can define natural predicates (essentially, dependent types)

$$\Gamma \Vdash P : \mathit{UFA} o \mathcal{U}$$

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- ullet equipping a universe  ${\cal U}$  with an algebra for  $\mathcal{T}_{\mathsf{eff}}$ , and
- using the above handle-into-values construct to define P

**Note 1:** P(thunk M) computes a proof obligation for M

Note 2: Formally, this is done in an extension of eMLTT with

- a universe  $\mathcal U$  closed under Nat, 1, 0, +,  $\Sigma$ , and  $\Pi$
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#### **Example 1** (Evaluation Logic style modalities):

- Given a predicate  $P:A \to \mathcal{U}$  on return values, we define a predicate  $\Diamond P:UFA \to \mathcal{U}$  on IO-computations as
- $\Diamond P \stackrel{\text{def}}{=} \lambda x : UFA . (\text{force} x) \text{ handled with } \{...\}_{\text{op} \in \mathcal{T}_{10}} \text{ to } y : A \text{ in}_{\mathcal{U}} P y$  using the handler given by

$$\begin{split} V_{\text{read}} & \stackrel{\text{def}}{=} \lambda \, x \colon \! \left( \Sigma \, x_{v} \colon \! 1 \cdot \mathsf{Chr} \to \mathcal{U} \right) \cdot \widehat{\Sigma} \, y \colon \! \mathsf{El}(\widehat{\mathsf{Chr}}) \cdot \left( \mathsf{snd} \, x \right) \, y \\ V_{\text{write}} & \stackrel{\text{def}}{=} \lambda \, x \colon \! \left( \Sigma \, x_{v} \colon \mathsf{Chr} \cdot 1 \to \mathcal{U} \right) \cdot \left( \mathsf{snd} \, x \right) \, \star \end{split}$$

ullet  $\Diamond P$  corresponds to Evaluation Logic's possibility modality

$$\Diamond P\left(\texttt{thunk}\left(\texttt{read}(x\,.\,\texttt{write}_{\texttt{e}'}(\texttt{return}\,V))\right)\right) = \widehat{\Sigma}\,x\,: \mathsf{El}(\widehat{\mathsf{Chr}})\,.\,P\,\,V$$

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#### **Example 2** (Dijkstra's weakest precondition semantics for state):

Given a postcondition on return values and final states

$$Q: A \to S \to \mathcal{U}$$
 ( $S \stackrel{\text{def}}{=} \Pi \ell$ : Loc .Val

we define a precondition for stateful comps. on initial states

$$\operatorname{wp}_Q: \mathit{UFA} \to \mathit{S} \to \mathit{U}$$

by

1) handling the given comp. into a state-passing function using

$$V_{
m get}, V_{
m put}$$
 on  $S o (\mathcal{U} imes S)$  and  $V_{
m ret}$  "="  $\mathcal{Q}$ 

- 2) feeding in the initial state; and 3) projecting out  $\mathcal U$
- Theorem:  $\operatorname{wp}_Q$  satisfies expected properties of WPs, e.g.,  $\operatorname{wp}_Q\left(\operatorname{thunk}\left(\operatorname{return}V\right)\right) = \lambda x_S : S \cdot Q \cdot V \cdot x_S$

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- ullet Theorem: wp $_Q$  satisfies expected properties of WPs, e.g.,

$$wp_Q (thunk (return V)) = \lambda x_S : S . Q V x_S$$

$$wp_Q (thunk (put_{(\ell,V)}(M))) = \lambda x_S : S . wp_Q (thunk M) (x_S[\ell \mapsto V])$$

#### **Example 3** (Patterns of allowed (IO-)effects):

- Assuming an inductive type of IO-protocols, given by
   e : Protocol
   w : (Chr → Protocol) → Protocol
   and potentially also by A. V.
- We can define a rel. between comps. and protocols as follows:

Allowed : 
$$\mathit{UFA} o \mathsf{Protocol} o \mathcal{U}$$

by handling the given computation using

$$V_{
m read}, V_{
m write}$$
 on  ${
m Protocol} 
ightarrow {\cal U}$  ere  $V_{
m read} \ \langle -\ , V_{
m rk} 
angle \ ({
m r}\ {
m Pr}') \stackrel{
m def}{=} \widehat{\Pi} \, x \colon {
m El}(\widehat{
m Chr}) \cdot (V_{
m rk} \, x) \ ({
m Pr}' \, x) \ V_{
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**Example 3** (Patterns of allowed (IO-)effects):

Assuming an inductive type of IO-protocols, given by

$$\begin{array}{c} \textbf{e}: \mathsf{Protocol} & \textbf{r}: (\mathsf{Chr} \to \mathsf{Protocol}) \to \mathsf{Protocol} \\ \\ \textbf{w}: (\mathsf{Chr} \to \mathcal{U}) \to \mathsf{Protocol} \to \mathsf{Protocol} \\ \\ \mathsf{and} \ \mathsf{potentially} \ \mathsf{also} \ \mathsf{by} \ \land, \ \lor, \dots \end{array}$$

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 on  $\mathsf{Protocol} o \mathcal{U}$   $V_{\mathsf{rk}}$   $(\mathbf{r} \; \mathsf{Pr'}) \stackrel{\mathsf{def}}{=} \widehat{\Pi} \, x \colon \mathsf{El}(\widehat{\mathsf{Chr}}) \, . \, (V_{\mathsf{rk}} \, x)$ 

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and potentially also by  $\wedge$ ,  $\vee$ , ...

• We can define a rel. between comps. and protocols as follows:

Allowed : 
$$UFA \rightarrow Protocol \rightarrow \mathcal{U}$$

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$$V_{\mathsf{read}}, V_{\mathsf{write}}$$
 on  $\mathsf{Protocol} \to \mathcal{U}$ 

where

$$\begin{array}{cccc} V_{\mathsf{read}} & \langle -, V_{\mathsf{rk}} \rangle & (\mathtt{r} \ \mathsf{Pr'}) & \stackrel{\mathsf{def}}{=} & \widehat{\Pi} \, x \colon \mathsf{El}(\widehat{\mathsf{Chr}}) \, . \, (V_{\mathsf{rk}} \, x) \, (\mathsf{Pr'} \, x) \\ V_{\mathsf{write}} & \langle V \, , V_{\mathsf{wk}} \rangle \, (\mathtt{w} \, P \, \mathsf{Pr'}) & \stackrel{\mathsf{def}}{=} & \widehat{\Sigma} \, x \colon \mathsf{El}(P \, V) \, . \, V_{\mathsf{wk}} \, \star \, \mathsf{Pr'} \\ & \stackrel{\mathsf{def}}{=} & \widehat{0} \end{array}$$

#### **Conclusion**

At a high-level, the presented work was about combining dependent types and computational effects

#### In particular, you saw

- a clean core calculus of dependent types and comp. effects
- a natural category-theoretic semantics
- · alg. effects and handlers, in particular, for reasoning using
  - Evaluation Logic style modalities
  - Dijkstra's weakest precondition semantics for state
  - patterns of allowed (IO-)effects

#### Some items of future work:

- uniform account of the various handler-defined predicates
- more expressive comp. types (par. adjunctions, Dijkstra monads)

# Thank you!

D. Ahman.

Fibred Computational Effects. (PhD Thesis, 2017)

D. Ahman, N. Ghani, G. Plotkin.

 $\textbf{Dependent Types and Fibred Computational Effects.} \ (\texttt{FoSSaCS'}16)$ 

D. Ahman.

Handling Fibred Computational Effects. (POPL'18)