A fibrational view on computational effects

(or some things I did during my PhD)

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Outline – what I did during my PhD

2012

Refinement Types and Computational Effects

2015

Dependent Types and Computational Effects

2017

Directed Containers

Outline – dependent types

The Curry-Howard correspondence:

Simple Types	$\sim$	Propositional Logic	$(\mathsf{Nat}, \mathsf{String}, \ldots)$
Dependent Types	$\sim$	Predicate Logic	$(\Sigma, \Pi, =, \ldots)$

A tiny example: we can use dep. types to express sorted lists

$$\ell : (\mathsf{List\ Nat}) \vdash \mathsf{Sorted}(\ell) \ \stackrel{\text{def}}{=}\ \overset{\forall}{\Pi}\, i : \mathsf{Nat}\,.\,(0 < i < \mathtt{len}\ \ell) \overset{\Rightarrow}{\rightarrow} (\ell[i\text{-}1] \leq \ell[i])$$

which in turn could be used to type a sorting function

$$\mathtt{sort} : \overset{\forall}{\Pi}\, \ell : (\mathsf{List\ Nat})\,.\,\overset{\exists}{\Sigma}\, \ell' : (\mathsf{List\ Nat})\,.\,\Big(\mathsf{Sorted}(\ell) \overset{\wedge}{\times} \ldots\Big)$$

Large examples: CompCert (Coq), miTLS and HACL* (F*), ...

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Large examples: CompCert (Coq), miTLS and HACL* (F*), . . .

Outline – computational effects

Examples:

- state
- exceptions
- nondeterminism
- I/O
- ...

Meta-languages and models:

- based on monads (T, η, μ) (Moggi)
- based on adjunctions (Levy)

$$\begin{tikzcd} \mathcal{V} \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] & & \mathcal{C} \arrow[ll, bend left, "U"] \end{tikzcd}$$

- based on algebraic presentations (Plotkin and Power)

  get : $1 \rightharpoonup S$ put : $S \rightharpoonup 1$ + equations

Outline – putting the two together

We investigate the combination of

- dependent types $(\Pi, \Sigma, V =_A W, \ldots)$
- computational effects (state, nondeterminism, I/O, ...)

Two guiding problems

- effectful programs in types (e.g., `get` and `put` in types)
- types of effectful programs (e.g., of sequential composition)

Our goals

- tell a mathematically natural story
- use established math. techniques
- cover a wide range of comp. effects
- discover smth. interesting

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- use established math. techniques (fibrations and adjunctions)
- cover a wide range of comp. effects (alg. effects, continuations)
- discover smth. interesting (using handlers to reason about effects)

Effectful programs in types

(type-dependency in the presence of effects)

Effectful programs in types

Q: Should we allow situations such as Sorted[$\texttt{receive}(y.\, M)/\ell$]?

A1: In this talk, we say not directly

- types should only depend on static information about effects
- we allow dependency on effectful comps. via analysing thunks

A2: But we are also looking into the direct case

- type-dependency needs to be "homomorphic", but not only so
- intuitively, lift Sorted(ℓ) to Sorted†(c), where $c : T(\text{List Chr})$

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Aim: Types should only depend on static info about effects

Solution: CBPV/EEC style distinction between vals. and comps.

- value types $\Gamma \vdash A$ (MLTT + thunks + ...)
- computation types $\Gamma \vdash \underline{C}$ (dep. typed CBPV/EEC)
- where Γ contains only value variables $x_1 : A_1, \ldots, x_n : A_n$

Could have also considered Moggi's λ_{ML} and Levy's FGCBV

- building on CBPV/EEC gives a more general story
- especially for the treatment of sequential composition
- and also for integrating dependent- and effect-typing (ongoing)

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Assigning types to effectful programs

(e.g., sequential composition)

Assigning types to effectful programs

The problem: The standard typing rule for seq. composition

$$\frac{\Gamma \vdash_{\mathsf{c}} M : F\,A \qquad \Gamma, x\!:\!A \vdash_{\mathsf{c}} N : \underline{C}}{\Gamma \vdash_{\mathsf{c}} M \;\mathtt{to}\; x\!:\!A \;\mathtt{in}\; N : \underline{C}}$$

is not correct any more because x can appear free in the type

$$\underline{C}$$

in the conclusion

Assigning types to effectful programs

Aim: To fix the typing rule of sequential composition

Option 1: We could restrict the free variables in $\underline{C}$: [Levy'04]

$$\frac{\Gamma \vdash_{\mathsf{c}} M : F\,A \qquad \Gamma \vdash \underline{C} \qquad \Gamma, x\!:\!A \vdash_{\mathsf{c}} N : \underline{C}}{\Gamma \vdash_{\mathsf{c}} M \texttt{ to } x\!:\!A \texttt{ in } N : \underline{C}}$$

But: sometimes it is useful if $\underline{C}$ can depend on x!

- if we consider

$$\texttt{fopen}\left(\texttt{return true}, \texttt{return false}\right) \texttt{ to } x\!:\!\mathsf{Bool} \texttt{ in } N$$

- then it would be natural to let $\underline{C}$ depend on x, e.g.,

$$x\!:\!\mathsf{Bool} \vdash \underline{C}(x) \stackrel{\text{def}}{=} \begin{array}{l}\texttt{if } x \texttt{ then } \text{"allow } \texttt{fread}, \texttt{fwrite}, \text{ and } \texttt{fclose}\text{"} \\ \qquad\qquad\quad \texttt{else } \text{"allow } \texttt{fopen}\text{"}\end{array}$$

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Option 2: One could lift sequential composition to type level

$$\Gamma \vDash_{c} M \;\texttt{to}\; x\!:\!A \;\texttt{in}\; N : M \;\texttt{to}\; x\!:\!A \;\texttt{in}\; \underline{C}$$

But: then comp. types would be singleton-like!?!

However, smth. like this is probably needed for the direct case.

Option 3: In the monadic metalanguage λ_{ML}, one could try

$$\frac{\Gamma \vdash M : T\,A \qquad \Gamma, x\!:\!A \vdash N : T\,B(x)}{\Gamma \vdash M \;\texttt{to}\; x\!:\!A \;\texttt{in}\; N : T\,(\Sigma x : A.B)}$$

But: what makes this a principled solution? Why is it correct?

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Option 4: We draw inspiration from algebraic effects

- and combine it with restricting $\underline{C}$ in seq. comp. (**Option 1**)

E.g., consider the non-det. program $\Big(\text{for } x : \mathsf{Nat} \vDash_c N : \underline{C}(x)\Big)$

$$M \quad \stackrel{\text{def}}{=} \quad \texttt{choose}\,\big(\texttt{return}\,4, \texttt{return}\,2\big)\ \texttt{to}\ x : \mathsf{Nat}\ \texttt{in}\ N$$

After tossing the coin, this program evaluates as either

$$N[4/x] : \underline{C}[4/x] \qquad \text{or} \qquad N[2/x] : \underline{C}[2/x]$$

Idea: M denotes an element of the coproduct of algebras

$$\underline{C}[4/x] + \underline{C}[2/x] \quad \stackrel{\text{def}}{=} \quad F\Big(U\left(\underline{C}[4/x]\right) + U\left(\underline{C}[2/x]\right)\Big)_{/\equiv}$$

and thus we would like to type M at the type $\Sigma x : \mathsf{Nat}.\underline{C}$

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Putting these ideas together

(eMLTT: a core dep.-typed language with comp. effects)

eMLTT – value and comp. types

Value types: MLTT + thunks + ...

$$A, B ::= \mathsf{Nat} \mid 1 \mid 0 \mid \Pi x\!:\!A.B \mid \Sigma x\!:\!A.B \mid V =_A W \mid U\,\underline{C} \mid \ldots$$

- $U\,\underline{C}$ is the type of thunked (i.e., suspended) computations

Computation types: dep.-typed version of EEC's comp. types

$$\underline{C}, \underline{D} ::= F\,A \mid \Pi x\!:\!A.\underline{C} \mid \Sigma x\!:\!A.\underline{C}$$

- $F\,A$ is the type of computations returning values of type A
- $\Pi x\!:\!A.\underline{C}$ is the type of dependent effectful functions
  - generalises CBPV/EEC's comp. types $A \to \underline{C}$ and $\underline{C} \times \underline{D}$
- $\Sigma x\!:\!A.\underline{C}$ is the type of dep. pairs of values and effectful comps.
  - captures the intuition about seq. comp. and coprods. of algebras
  - generalises EEC's comp. types $!A \otimes \underline{C}$ and $\underline{C} \oplus \underline{D}$

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eMLTT – value and comp. terms

Value terms: MLTT + thunks + ...

$$V, W ::= x \mid \texttt{zero} \mid \texttt{succ}\, V \mid \ldots \mid \texttt{thunk}\, M \mid \ldots$$

- equational theory based on intensional MLTT

Comp. terms: dep.-typed version of CBPV/EEC's comp. terms

$$
\begin{array}{rcll}
M, N & ::= & \texttt{force}\, V & \\
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& \mid & M \texttt{ to } x : A \texttt{ in } N & \\
& \mid & \lambda x : A.M & \\
& \mid & M V & \\
& \mid & \langle V, M \rangle & \text{(comp. } \Sigma \text{ intro.)} \\
& \mid & M \texttt{ to } \langle x : A, z : \underline{C} \rangle \texttt{ in } K & \text{(comp. } \Sigma \text{ elim.)}
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But: Value and comp. terms alone do not suffice, as in EEC!

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eMLTT – homomorphism terms

Note: We need to define K in such a way that the intended left-to-right evaluation order is preserved, e.g., consider

$$\Gamma \vDash_{\mathsf{c}} \langle V, M \rangle \ \mathtt{to}\ \langle x \mathbin{:} A, z \mathbin{:} \underline{C} \rangle \ \mathtt{in}\ K = K[V/x, M/z] : \underline{D}$$

Homomorphism terms: dep.-typed version of EEC's linear terms

$$
\begin{array}{rcll}
K, L & ::= & z & \text{(linear comp. vars.)} \\
& \mid & K \ \mathtt{to}\ x \mathbin{:} A \ \mathtt{in}\ M & \\
& \mid & \lambda x \mathbin{:} A. K & \\
& \mid & K\, V & \\
& \mid & \langle V, K \rangle & \text{(comp. } \Sigma \text{ intro.)} \\
& \mid & K \ \mathtt{to}\ \langle x \mathbin{:} A, z \mathbin{:} \underline{C} \rangle \ \mathtt{in}\ L & \text{(comp. } \Sigma \text{ elim.)}
\end{array}
$$

Typing judgments:

- $\Gamma \vDash_{\mathsf{v}} V : A$
- $\Gamma \vDash_{\mathsf{c}} M : \underline{C}$
- $\Gamma \mid z \mathbin{:} \underline{C} \vDash_{\mathsf{h}} K : \underline{D}$ (linear in z; comp. bound to z happens first)

eMLTT – homomorphism terms

Note: We need to define K in such a way that the intended left-to-right evaluation order is preserved, e.g., consider

$$\Gamma \vDash_{\mathsf{c}} \langle V, M \rangle \;\texttt{to}\; \langle x : A, z : \underline{C} \rangle \;\texttt{in}\; K = K[V/x, M/z] : \underline{D}$$

Homomorphism terms: dep.-typed version of EEC's linear terms

$$\begin{array}{rcll}
K, L & ::= & z & \text{(linear comp. vars.)} \\
& | & K \;\texttt{to}\; x : A \;\texttt{in}\; M & \\
& | & \lambda x : A . K & \\
& | & K\, V & \\
& | & \langle V, K \rangle & \text{(comp. } \Sigma \text{ intro.)} \\
& | & K \;\texttt{to}\; \langle x : A, z : \underline{C} \rangle \;\texttt{in}\; L & \text{(comp. } \Sigma \text{ elim.)}
\end{array}$$

Typing judgments:

- $\Gamma \vDash_{\mathsf{v}} V : A$
- $\Gamma \vDash_{\mathsf{c}} M : \underline{C}$
- $\Gamma \,|\, z : \underline{C} \vDash_{\mathsf{h}} K : \underline{D}$ (linear in z; comp. bound to z happens first)

eMLTT – typing sequential composition

We can then account for type-dependency in seq. comp. as

$$\frac{\Gamma \vDash_{\mathsf{c}} M : F\,A \qquad \Gamma \vdash \Sigma x{:}A.\underline{C}(x) \qquad \dfrac{\Gamma, x{:}A \vDash_{\mathsf{c}} N : \underline{C}(x)}{\Gamma, x{:}A \vDash_{\mathsf{c}} \langle x, N\rangle : \Sigma x{:}A.\underline{C}(x)}}{\Gamma \vDash_{\mathsf{c}} M \;\texttt{to}\; x{:}A \;\texttt{in}\; \langle x, N\rangle : \Sigma x{:}A.\underline{C}(x)}$$

The seq. comp. rule for λ_{ML} is justified by the type isomorphism

$$\frac{\Gamma \vdash A \qquad \Gamma, x{:}A \vdash B(x)}{\Gamma \vdash U(\Sigma x{:}A.F(B)) \cong UF(\Sigma x{:}A.B) = T(\Sigma x{:}A.B)}$$

Categorical semantics of eMLTT

(fibrations + adjunctions)

Categorical semantics – value part

Given by a split closed comprehension category p, as in

$$\begin{tikzcd}
\mathcal{V} \arrow[d, bend right=60, "p"'] \arrow[d, bend left=60, "\{-\}"] \\
\mathcal{B} \arrow[u, "1"']
\end{tikzcd}
\qquad p \dashv 1 \dashv \{-\}$$

such that

- p has split fibred strong colimits of shape $\mathbf{0}$ and $\mathbf{2}$
  - (in thesis, also Jacobs-style axiomatisation for arbitrary shapes)
  - (all one needs are cocones and fully-faithfulness of induced func.)
- p has weak split fibred strong natural numbers
- p has split intensional propositional equality

Categorical semantics – value part

Given by a split closed comprehension category p, as in

$$
\begin{tikzcd}
\mathcal{V} \arrow[d, bend right=60, "p"'] \arrow[d, bend left=60, "\{-\}"] \\
\mathcal{B} \arrow[u, "1"'] \arrow[u, phantom, shift left=4, "\dashv"] \arrow[u, phantom, shift right=4, "\dashv"]
\end{tikzcd}
$$

allowing us to define a partial interpretation fun. $[\![-]\!]$, that maps:

- a context Γ to and object $[\![\Gamma]\!]$ in $\mathcal{B}$, with
  - $[\![\diamond]\!] \stackrel{\text{def}}{=} 1$
  - $[\![\Gamma, x\!:\!A]\!] \stackrel{\text{def}}{=} \{[\![\Gamma; A]\!]\}$ (if $x \not\in \mathit{Vars}(\Gamma)$ and $[\![\Gamma; A]\!]$ is defined)
- a context Γ and a value type A to an object $[\![\Gamma; A]\!]$ in $\mathcal{V}_{[\![\Gamma]\!]}$
- a context Γ and a value term V to $[\![\Gamma; V]\!] : 1_{[\![\Gamma]\!]} \longrightarrow A$ in $\mathcal{V}_{[\![\Gamma]\!]}$

Categorical semantics – effects part

Given by a split fibration q and a split fib. adjunction $F \dashv U$, as in

$$
\begin{tikzcd}
\mathcal{V} \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] \arrow[d, bend right=60, "p"'] \arrow[d, bend left=60, "\{-\}"] & & \mathcal{C} \arrow[ll, bend left, "U"] \arrow[dll, bend left, "q"] \\
\mathcal{B} \arrow[u, "\dashv\ 1\ \dashv" description] & &
\end{tikzcd}
$$

such that

- q has split dependent p-products (comp. Π-type; r. adj. to wk.)
- q has split dependent p-coproducts (comp. Σ-type; l. adj. to wk.)

and to account for the full calculus presented in the thesis,

- q admits split fibred pre-enrichment in p (hom. function type $\multimap$)

Categorical semantics – effects part

Given by a split fibration q and a split fib. adjunction $F \dashv U$, as in

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathcal{V} \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] \arrow[d, bend right=60, "p"'] \arrow[d, bend left=60, "\{-\}"] & & \mathcal{C} \arrow[ll, bend left, "U"] \arrow[dll, bend left, "q"] \\
\mathcal{B} \arrow[u, "\dashv \;\; 1 \;\; \dashv" description] & &
\end{tikzcd}
$$

we extend the partial interpretation fun. $[\![-]\!]$ so that it maps:

- a ctx. Γ and a comp. type $\underline{C}$ to an object $[\![\Gamma; \underline{C}]\!]$ in $\mathcal{C}_{[\![\Gamma]\!]}$
- a ctx. Γ and a comp. term M to $[\![\Gamma; M]\!] : 1_{[\![\Gamma]\!]} \longrightarrow U(\underline{C})$ in $\mathcal{V}_{[\![\Gamma]\!]}$
- a ctx. Γ, a comp. var. z, a comp. type $\underline{C}$, and a hom. term K to
$$[\![\Gamma; z : \underline{C}; K]\!] : [\![\Gamma; \underline{C}]\!] \longrightarrow \underline{D} \text{ in } \mathcal{C}_{[\![\Gamma]\!]}$$

Categorical semantics – correctness

Theorem (Soundness):

- If $\Gamma \vdash \underline{C}$, then $[\![\Gamma; \underline{C}]\!] \in \mathcal{C}_{[\![\Gamma]\!]}$
- If $\Gamma \vdash_{\mathsf{c}} M : \underline{C}$, then $[\![\Gamma; M]\!] : 1_{[\![\Gamma]\!]} \longrightarrow U([\![\Gamma; \underline{C}]\!])$
- If $\Gamma \,|\, z : \underline{C} \vdash_{\mathsf{h}} K : \underline{D}$, then $[\![\Gamma; z : \underline{C}; K]\!] : [\![\Gamma; \underline{C}]\!] \longrightarrow [\![\Gamma; \underline{D}]\!]$
- If $\Gamma \vdash \underline{C} = \underline{D}$, then $[\![\Gamma; \underline{C}]\!] = [\![\Gamma; \underline{D}]\!] \in \mathcal{C}_{[\![\Gamma]\!]}$
- ...

Theorem (Classifying model):

- The well-formed syntax of eMLTT forms a fib. adjunction model.

Theorem (Completeness):

- If two types or terms are equal in all fibred adjunction models, then they are also equal in the equational theory of eMLTT.

Examples of fibred adjunction models

$$
\begin{tikzcd}
\mathcal{V} \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] \arrow[d, bend right, "p"'] \arrow[d, bend left, "\{-\}"] & & \mathcal{C} \arrow[ll, bend left, "U"] \arrow[lld, bend left, "q"] \\
\mathcal{B} \arrow[u, "\dashv \; 1 \; \dashv"'] & &
\end{tikzcd}
$$

Examples of fibred adjunction models

Example 1 (identity adjunctions):

$$\begin{tikzcd} \mathcal{V} \arrow[rr, bend left, "\mathsf{Id}_{\mathcal{V}}"] \arrow[rr, phantom, "\perp"] \arrow[dr, "p"'] & & \mathcal{V} \arrow[ll, bend left, "\mathsf{Id}_{\mathcal{V}}"] \arrow[dl, "p"] \\ & \mathcal{B} & \end{tikzcd}$$

Note: sound model as long as we haven't included any effects

Examples of fibred adjunction models

Example 2 (simple models from Egger et al.'s EEC):

- given an adjunction $F_{\mathsf{EEC}} \dashv U_{\mathsf{EEC}} : \mathcal{E} \longrightarrow \mathcal{D}$, such that
  - $\mathcal{D}$ is Cartesian closed (with Nat, ...), and
  - $\mathcal{E}$ and $F_{\mathsf{EEC}} \dashv U_{\mathsf{EEC}}$ are $\mathcal{D}$-enriched, and
  - $\mathcal{E}$ has all $\mathcal{D}$-tensors ($A \otimes \underline{C}$) and $\mathcal{D}$-cotensors ($A \Rightarrow \underline{C}$)

- we use simple fibration $\mathsf{s}_{\mathcal{D}}$ and simpl. $\mathcal{D}$-enrich. fibration $\mathsf{s}_{\mathcal{D},\mathcal{E}}$

$$\begin{tikzcd} \mathsf{s}(\mathcal{D}) \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] \arrow[dr, "\mathsf{s}_{\mathcal{D}}"'] & & \mathsf{s}(\mathcal{D},\mathcal{E}) \arrow[ll, bend left, "U"] \arrow[dl, "\mathsf{s}_{\mathcal{D},\mathcal{E}}"] \\ & \mathcal{D} & \end{tikzcd}$$

$$F(X, A) \stackrel{\text{def}}{=} (X, F_{\mathsf{EEC}}(A))$$

$$U(X, \underline{C}) \stackrel{\text{def}}{=} (X, U_{\mathsf{EEC}}(\underline{C}))$$

$\mathsf{s}(\mathcal{D})$: $(f, g) : (X, A) \longrightarrow (Y, B)$ where $f : X \longrightarrow Y$ $g : X \times A \longrightarrow B$

$\mathsf{s}(\mathcal{D},\mathcal{E})$: $(f, h) : (X, \underline{C}) \longrightarrow (Y, \underline{D})$ where $f : X \longrightarrow Y$ $h : X \otimes \underline{C} \longrightarrow \underline{D}$

Note: this model doesn't support any real type-dependency

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& \mathcal{D} &
\end{tikzcd}$$

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Examples of fibred adjunction models

Example 3 (families fibrations):

- given an adjunction $F_{\mathcal{D}} \dashv U_{\mathcal{D}} : \mathcal{D} \longrightarrow \mathsf{Set}$, such that
  - $\mathcal{D}$ has set-indexed products and set-indexed coproducts
- such adjunctions arise from
  - EM-cats. ($\mathcal{D} \stackrel{\text{def}}{=} \mathsf{Set}^{\mathbf{T}}$) and Law. ths. ($\mathcal{D} \stackrel{\text{def}}{=} \mathsf{Mod}(\mathcal{L}, \mathsf{Set})$)
  - resolutions of $S \Rightarrow (-) \times S$ and $((-) \Rightarrow R) \Rightarrow R$

- we use families fibrations $\mathsf{fam}_{\mathsf{Set}}$ and $\mathsf{fam}_{\mathcal{D}}$

$$\begin{tikzcd}
\mathsf{Fam}(\mathsf{Set}) \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] \arrow[dr, "\mathsf{fam}_{\mathsf{Set}}"'] & & \mathsf{Fam}(\mathcal{D}) \arrow[ll, bend left, "U"] \arrow[dl, "\mathsf{fam}_{\mathcal{D}}"] \\
& \mathsf{Set} &
\end{tikzcd}$$

$$F(X, A) \stackrel{\text{def}}{=} (X, F_{\mathcal{D}} \circ A)$$

$$U(X, \underline{C}) \stackrel{\text{def}}{=} (X, U_{\mathcal{D}} \circ \underline{C})$$

$\mathsf{Fam}(\mathsf{Set})$: (X, A) where $X \in \mathsf{Set}$ $A : X \longrightarrow \mathsf{Set}$

$(f, \{g_x\}_{x \in X}) : (X, A) \longrightarrow (Y, B)$ where $g_x : A(x) \longrightarrow (B \circ f)(x)$

Examples of fibred adjunction models

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Examples of fibred adjunction models

Example 4 (continuous families for $\mu\, x : U\underline{C} \,.\, M$):

- given a CPO-enriched monad $\mathbf{T}$ on CPO, such that
  - $\mathbf{T}$ supports least zero-ary alg. op. ($\perp_A : 1 \longrightarrow TA$), and
  - $\mathrm{CPO}^{\mathbf{T}}$ has reflexive coequalizers
- such $\mathbf{T}$ arise from discrete CPO-enriched countable Law. ths.
- we use continuous families fibrations $\mathrm{cfam}_{\mathrm{CPO}}$ and $\mathrm{cfam}_{\mathrm{CPO}^{\mathbf{T}}}$

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\mathrm{CFam}(\mathrm{CPO}) \arrow[rr, bend left, "F"] \arrow[dr, "\mathrm{cfam}_{\mathrm{CPO}}"'] \arrow[rr, phantom, "\perp"] & & \mathrm{CFam}(\mathrm{CPO}^{\mathbf{T}}) \arrow[ll, bend left, "U"] \arrow[dl, "\mathrm{cfam}_{\mathrm{CPO}^{\mathbf{T}}}"] \\
& \mathrm{CPO} &
\end{tikzcd}$$

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$\mathrm{CFam}(\mathrm{CPO})$: (X, A) where $X \in \mathrm{CPO}$ $A : X \longrightarrow \mathrm{CPO}^{\mathrm{EP}}$ an ω-cont. fun.

Thm.: we don't use $\mathrm{cod} : \mathrm{CPO}^{\rightarrow} \rightarrow \mathrm{CPO}$ because CPO isn't LCCC

Examples of fibred adjunction models

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Examples of fibred adjunction models

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Examples of fibred adjunction models

Example 5 (EM-resolutions of split fibred monads):

- given a split fibred monad $\mathbf{T} = (T, \eta, \mu)$ on p, i.e.,

$$\begin{tikzcd} \mathcal{V} \arrow[rr, "T"] \arrow[dr, "p"'] & & \mathcal{V} \arrow[dl, "p"] \\ & \mathcal{B} & \end{tikzcd}$$

and $\qquad p(\eta_A) = \mathrm{id}_{p(A)} \qquad p(\mu_A) = \mathrm{id}_{p(A)}$

- we consider models based on the EM-resolution of $\mathbf{T}$

$$\begin{tikzcd} \mathcal{V} \arrow[rr, bend left, "F^{\mathbf{T}}"] \arrow[rr, phantom, "\perp"] \arrow[dr, "p"'] & & \mathcal{V}^{\mathbf{T}} \arrow[ll, bend left, "U^{\mathbf{T}}"] \arrow[dl, "p^{\mathbf{T}}"] \\ & \mathcal{B} & \end{tikzcd}$$

- and show that three familiar results hold for this situation

Examples of fibred adjunction models

Example 5 (EM-resolutions of split fibred monads):

- **Theorem 1:** if p supports Π-types, then $p^{\mathbf{T}}$ also supports Π-types

$$\Pi^{\mathbf{T}}_A(B, \beta) \overset{\text{def}}{=} (\Pi_A(B), \beta_{\Pi^{\mathbf{T}}_A})$$

- **Prop.:** every $\mathbf{T}$ on a split closed comp. cat. has a dep. strength

$$\sigma_A : \Sigma_A \circ T \longrightarrow T \circ \Sigma_A \qquad (A \in \mathcal{V})$$

- **Theorem 2:** if p supports Σ-types and σ_A are natural isos., then $p^{\mathbf{T}}$ also supports Σ-types

$$\Sigma^{\mathbf{T}}_A(B, \beta) \overset{\text{def}}{=} (\Sigma_A(B), \beta_{\Sigma^{\mathbf{T}}_A})$$

- **Theorem 3:** if p supports Σ-types and $p^{\mathbf{T}}$ has split fibred reflexive coequalizers, then $p^{\mathbf{T}}$ also supports Σ-types

$$\Sigma^{\mathbf{T}}_A(B, \beta) \overset{\text{def}}{=} F^{\mathbf{T}}(\Sigma_A(B))_{/\equiv}$$

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Algebraic effects

Algebraic effects – ops. and eqs.

Fibred effect theories $\mathcal{T}_{\mathsf{eff}}$:

- signatures of dep. typed operation symbols

$$\frac{\cdot \vdash I \qquad x_i : I \vdash O \qquad I \text{ and } O \text{ are pure value types}}{\mathsf{op} : (x_i : I) \longrightarrow O}$$

- equipped with equations on derivable effect terms

In eMLTT:

$$M ::= \ldots \mid \mathsf{op}^{\underline{C}}_{V}(x.M)$$

General algebraicity equations (in addition to eff. th. eqs.):

$$\frac{\Gamma \vDash_{\mathsf{v}} V : I \quad \Gamma, x : O[V/x_i] \vDash_{\mathsf{c}} M : \underline{C} \quad \Gamma \mid z : \underline{C} \vDash_{\mathsf{h}} K : \underline{D}}{\Gamma \vDash_{\mathsf{c}} K[\mathsf{op}^{\underline{C}}_{V}(x.M)/z] = \mathsf{op}^{\underline{D}}_{V}(x.K[M/z]) : \underline{D}} \ (\mathsf{op} : (x_i : I) \longrightarrow O)$$

Sound semantics: based on

- $p : \mathsf{Fam}(\mathsf{Set}) \longrightarrow \mathsf{Set}$ and $q : \mathsf{Fam}(\mathsf{Mod}(\mathcal{L}_{\mathcal{T}_{\mathsf{eff}}}, \mathsf{Set})) \longrightarrow \mathsf{Set}$

Algebraic effects – ops. and eqs.

Fibred effect theories $\mathcal{T}_{\mathsf{eff}}$:

- signatures of dep. typed operation symbols

$$\frac{\cdot \vdash I \qquad x_i : I \vdash O \qquad I \text{ and } O \text{ are pure value types}}{\mathsf{op} : (x_i : I) \longrightarrow O}$$

- equipped with equations on derivable effect terms

In eMLTT:

$$M ::= \ldots \mid \mathsf{op}^{\underline{C}}_{V}(x.M)$$

General algebraicity equations (in addition to eff. th. eqs.):

$$\frac{\Gamma \vDash_{\mathsf{v}} V : I \qquad \Gamma, x : O[V/x_i] \vDash_{\mathsf{c}} M : \underline{C} \qquad \Gamma \,|\, z : \underline{C} \vDash_{\mathsf{h}} K : \underline{D}}{\Gamma \vDash_{\mathsf{c}} K[\mathsf{op}^{\underline{C}}_{V}(x.M)/z] = \mathsf{op}^{\underline{D}}_{V}(x.K[M/z]) : \underline{D}} \; (\mathsf{op} : (x_i : I) \longrightarrow O)$$

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Algebraic effects – examples

Example 1 (interactive I/O):

- $\mathsf{read} : 1 \longrightarrow \mathsf{Chr}$ $\qquad\qquad (\mathsf{Chr} \stackrel{\text{def}}{=} 1 + \ldots + 1)$

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- no equations

Example 2 (global state with location-dependent store type):

- $\diamond \vdash \mathsf{Loc}$

  $\ell : \mathsf{Loc} \vdash \mathsf{Val}$

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- five equations (two of them branching on $\mathsf{isDec}_{\mathsf{Loc}}$)

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Handlers of algebraic effects

(for programming and extrinsic reasoning)

Handlers of alg. effects – for programming

Idea: Generalisation of exception handlers [Plotkin,Pretnar'09]

Handler = Algebra and Handling = Homomorphism

Usual term-level presentation:

$$\Gamma \vdash_{\mathsf{c}} M \ \texttt{handled with} \ \{\mathrm{op}_{x_v}(x_k) \mapsto N_{\mathrm{op}}\}_{\mathrm{op} \in \mathcal{T}_{\mathrm{eff}}} \ \texttt{to} \ y \mathbin{:} A \ \texttt{in}_{\underline{C}} \ N_{\mathrm{ret}} : \underline{C}$$

satisfying

$$(\texttt{return}\, V) \ \texttt{handled with} \ \{\ldots\}_{\mathrm{op} \in \mathcal{T}_{\mathrm{eff}}} \ \texttt{to} \ y \mathbin{:} A \ \texttt{in} \ N_{\mathrm{ret}} = N_{\mathrm{ret}}[V/x]$$

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Typical use case for programming:

- write your programs using alg. ops. (e.g., `get` and `put`)
- use handlers to provide fit-for-purpose impl. (e.g., $S \to X \times S$)

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Handlers of alg. effects – for reasoning

Idea: Using a derived handle-into-values handling construct

$$M \texttt{ handled with } \{\mathtt{op}_{x_v}(x_k) \mapsto V_{\mathtt{op}}\}_{\mathtt{op} \in \mathcal{T}_{\mathrm{eff}}} \texttt{ to } y : A \texttt{ in}_{\underline{B}} \; V_{\mathrm{ret}}$$

we can define natural predicates (essentially, dependent types)

$$\Gamma \vDash_{\mathrm{v}} P : UFA \to \mathcal{U}$$

by

- equipping a universe $\mathcal{U}$ with an algebra for $\mathcal{T}_{\mathrm{eff}}$, and
- using the above handle-into-values construct to define P

Note 1: $P\,(\mathtt{thunk}\,M)$ computes a proof obligation for M

Note 2: Formally, we work in an extension of eMLTT with

- a universe $\mathcal{U}$ closed under Nat, 1, 0, $+$, Σ, and Π
- a type-based treatment of handlers $\quad \underline{C} ::= \ldots \mid \langle A; \overrightarrow{V_{\mathrm{op}}}; \overrightarrow{W_{\mathrm{eq}}} \rangle$
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Handlers of alg. effects – for reasoning

Example 1 (Evaluation Logic style modalities):

- Given a predicate $P : A \to \mathcal{U}$ on return values,

  we define a predicate $\Diamond P : UFA \to \mathcal{U}$ on I/O-computations as

$$\Diamond P \stackrel{\text{def}}{=} \lambda\, x\!:\!UFA\,.\,(\texttt{force}\,x)\ \texttt{handled with}\ \{\ldots\}_{\mathsf{op}\in\mathcal{T}_{\mathsf{IO}}}\ \texttt{to}\ y\!:\!A\ \texttt{in}_{\mathcal{U}}\ P\,y$$

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- $\Diamond P$ corresponds to Evaluation Logic's possibility modality

$$\Diamond P\,(\texttt{thunk}\,(\texttt{read}(x\,.\,\texttt{write}_{'e'}(\texttt{return}\,V)))) = \widehat{\Sigma}\, x\!:\!\mathsf{El}(\widehat{\mathsf{Chr}})\,.\,P\ V$$

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Handlers of alg. effects – for reasoning

Example 2 (Dijkstra's weakest precondition semantics):

- Given a postcondition on return values and final states

$$Q : A \to S \to \mathcal{U} \qquad\qquad (S \stackrel{\text{def}}{=} \Pi x : \mathsf{Loc} \,.\, \mathsf{Val})$$

  we define a precondition for stateful comps. on initial states

$$\mathsf{wp}_Q : UFA \to S \to \mathcal{U}$$

  by

  i) handling the given comp. into a state-passing function using

$$V_{\mathsf{get}}, V_{\mathsf{put}} \quad \text{on} \quad S \to (\mathcal{U} \times S) \qquad \text{and} \qquad V_{\mathsf{ret}} \ \text{“} = \text{”} \ Q$$

  ii) feeding in the initial state; and *iii*) projecting out $\mathcal{U}$

- **Theorem:** wp_Q satisfies expected properties of WPs, e.g.,

$$\mathsf{wp}_Q\,(\mathtt{thunk}\,(\mathtt{return}\,V)) \quad = \quad \lambda\, x_S : S \,.\, Q\ V\ x_S$$

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- Assuming an inductive type Protocol, given by

$$\mathtt{e} : \text{Protocol} \qquad \mathtt{r} : (\text{Chr} \to \text{Protocol}) \to \text{Protocol}$$

$$\mathtt{w} : (\text{Chr} \to \mathcal{U}) \to \text{Protocol} \to \text{Protocol}$$

  and potentially also by $\wedge$, $\vee$, ...

- Then, we define the predicate (rel. between comps. and protocols)

$$\text{Allowed} : UFA \to \text{Protocol} \to \mathcal{U}$$

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$$V_{\text{read}}\,,\, V_{\text{write}} \qquad \text{on} \qquad \text{Protocol} \to \mathcal{U}$$

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$$\begin{array}{lcl} V_{\mathsf{read}}\ \langle \_ , V_{\mathsf{rk}} \rangle\ \ (\mathtt{r}\ \mathsf{Pr}') & \stackrel{\mathrm{def}}{=} & \widehat{\Pi}\, x : \mathsf{El}(\widehat{\mathsf{Chr}})\,.\,(V_{\mathsf{rk}}\ x)\ (\mathsf{Pr}'\ x) \\ V_{\mathsf{write}}\ \langle V , V_{\mathsf{wk}} \rangle\ (\mathtt{w}\ P\ \mathsf{Pr}') & \stackrel{\mathrm{def}}{=} & \widehat{\Sigma}\, x : \mathsf{El}(P\ V)\,.\,V_{\mathsf{wk}} \star \mathsf{Pr}' \\ \_ & \stackrel{\mathrm{def}}{=} & \widehat{0} \end{array}$$

Handlers of alg. effects – for reasoning

Example 3 (Patterns of allowed I/O-effects):

- Assuming an inductive type Protocol, given by

$$\mathtt{e} : \mathsf{Protocol} \qquad \mathtt{r} : (\mathsf{Chr} \to \mathsf{Protocol}) \to \mathsf{Protocol}$$

$$\mathtt{w} : (\mathsf{Chr} \to \mathcal{U}) \to \mathsf{Protocol} \to \mathsf{Protocol}$$

and potentially also by $\wedge$, $\vee$, ...

- Then, we define the predicate (rel. between comps. and protocols)

$$\mathsf{Allowed} : UFA \to \mathsf{Protocol} \to \mathcal{U}$$

by handling the given comp. using

$$V_{\mathsf{read}}, V_{\mathsf{write}} \qquad \text{on} \qquad \mathsf{Protocol} \to \mathcal{U}$$

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$$\begin{array}{lcl}
V_{\mathsf{read}} \ \langle _ , V_{\mathsf{rk}} \rangle \ \ (\mathtt{r}\ \mathsf{Pr}') & \stackrel{\mathsf{def}}{=} & \widehat{\Pi}\, x : \mathsf{El}(\widehat{\mathsf{Chr}}) \,.\, (V_{\mathsf{rk}}\ x)\ (\mathsf{Pr}'\ x) \\
V_{\mathsf{write}} \ \langle V , V_{\mathsf{wk}} \rangle \ (\mathtt{w}\ P\ \mathsf{Pr}') & \stackrel{\mathsf{def}}{=} & \widehat{\Sigma}\, x : \mathsf{El}(P\ V) \,.\, V_{\mathsf{wk}} \ \star \ \mathsf{Pr}' \\
_ & \stackrel{\mathsf{def}}{=} & \widehat{0}
\end{array}$$

Conclusion

In work we told a mathematically natural story of combining

- dependent types and computational effects

In particular, we saw

- a clean core language of dependent types and comp. effects
- a natural category-theoretic semantics
- alg. effects and handlers, in particular, for reasoning using
    - Evaluation Logic style modalities
    - Dijkstra's weakest precondition semantics of state
    - patterns of allowed (I/O)-effects

Things to look at:

- type-dependency on computations (e.g., in seq. composition)
- more expressive comp. types (par. adjunctions, Dijkstra monads)

Thank you!

D. Ahman.
Fibred Computational Effects. (PhD Thesis, 2017)

D. Ahman, N. Ghani, G. Plotkin.
Dependent Types and Fibred Computational Effects. (FoSSaCS'16)

D. Ahman.
Handling Fibred Computational Effects. (POPL'18)

Digression: dep. elimination of 0 and $+$

The coproduct type $A + B$: [Jacobs'99]

- require $p : \mathcal{V} \longrightarrow \mathcal{B}$ to have split fibred coproducts $A +_X B$, and
- $\langle \{\mathsf{inl}_A\}^*, \{\mathsf{inr}_B\}^* \rangle : \mathcal{V}_{\{A +_X B\}} \longrightarrow \mathcal{V}_{\{A\}} \times \mathcal{V}_{\{B\}}$ to be fully-faith.
- allows one to interpret dependent case analysis, i.e.,

$$\mathcal{V}_{\{A\}}\Big(1_{\{A\}}, \{\mathsf{inl}_A\}^*(C)\Big) \times \mathcal{V}_{\{B\}}\Big(1_{\{B\}}, \{\mathsf{inr}_B\}^*(C)\Big)$$
$$\cong$$
$$\mathcal{V}_{\{A +_X B\}}\Big(1_{\{A +_X B\}}, C\Big)$$

provides semantics for

$$\frac{\Gamma, y_1 \!:\! A \vdash_{\mathsf{v}} W_1 : C[\mathtt{inl}_A\ y_1/x] \quad \Gamma, y_2 \!:\! B \vdash_{\mathsf{v}} W_2 : C[\mathtt{inr}_B\ y_2/x]}{\Gamma, x \!:\! A + B \vdash_{\mathsf{v}} \mathtt{case}\ x\ \mathtt{of}\ \big(\mathtt{inl}(y_1) \mapsto W_1, \mathtt{inr}(y_2) \mapsto W_2\big) : C}$$

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Digression: dep. elimination of colimits

A generalisation: [Ahman'17]

- **Idea:** fully-faith. for cocones $A \longrightarrow A \circledast_X B \longleftarrow B$ is enough, and we can generalise this to all split fibred colimits

- **Theorem:**
  - if for every object $X \in \mathcal{B}$ and diagram $J : \mathcal{D} \longrightarrow \mathcal{V}_X$ there exists a cocone $\underline{\mathsf{in}}^J : J \longrightarrow \Delta(\underline{\mathsf{colim}}(J))$ in $\mathcal{V}_X$,
  - such that $f^*(\underline{\mathsf{in}}^J_D) = \underline{\mathsf{in}}^{f^* \circ J}_D$, for any $f : X \longrightarrow Y$, and such that the unique mediating functor
  $$\langle \{\underline{\mathsf{in}}^J_D\}^*_{D \in \mathcal{D}} \rangle : \mathcal{V}_{\{\underline{\mathsf{colim}}(J)\}} \longrightarrow \lim(\widehat{J})$$
  is fully-faithful (for $\widehat{J} : \mathcal{D}^{op} \longrightarrow \mathsf{Cat}$, where $\widehat{J}(D) = \mathcal{V}_{\{J(D)\}}$),
  - then p has split fibred colimits of shape $\mathcal{D}$, and p supports dependent elimination for them (analogously to $+_X$)

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