Applied Stochastic Processes, 18-751, TX Brown, Fall 2017
Homework #10

Due 8pm Monday November 13.

Read ch 11 in JW. On your own do Quizzes 6.8, 7.2, 10.1, and 10.2

1. In a game with two dice, the event snake eyes refers to both dice showing one spot. Let R denote the number of dice rolls needed to observe the third occurrence of snake eyes. Find
   (a) the upper bound to $P[R \geq 250]$ based on the Markov inequality,
   (b) the upper bound to $P[R \geq 250]$ based on the Chebyshev inequality,
   (c) the upper bound to $P[R \geq 250]$ based on the Chernoff bound,
   (d) the upper bound to the exact value of $P[R \geq 250]$.
2. Let $g(x)$ be an arbitrary deterministic function. If $X(t)$ is a stationary random process, is $Y(t) = g(X(t))$ a stationary process? Prove your answer one way or the other.
3. In class we discussed two ways to think about discrete time random processes.
   (a) As a collection of sequences (i.e. discrete time functions): From this perspective the experiment being performed selects a member, ω, of the sample space Ω, and associated with that ω is a countable sequence (i.e. a real-valued function of $n \in \mathbb{N}$) which we denote by $X(\omega, n)$. In class we used the example of a data source where a number between 0 and 1 is randomly selected as the message ω_0, and the bits of ω_0's binary expansion are then output as its associated sequence (i.e. the message's encoding). In other words, for this perspective, we emphasize "fixing" ω to the value ω_0 and think about n changing. For each ω we get a function.
   (b) As a collection of RVs: From this perspective we have, of course, the same underlying probability space $(\Omega, \mathcal{F}, P)$. However, we now emphasize what happens at a particular time index n_0. Associated with time n_0 we have the Random Variable $X(\omega, n_0)$. In other words, for this perspective, we emphasize "fixing" n to the value n_0 and think about ω varying. At each index n we get an RV.

   Of course for any of this to make sense, and for us to investigate the statistical properties of a Random Process (e.g. correlations between RVs in the process, the RV means at time n, etc.) we need to have a probability space on which all the RVs in the RP are defined. Although we reserve the term Random Process for the case where we have an *infinite* number of RVs in our collection, RP's are a natural extension of the case where we have a *finite* number of RVs defined on the same probability space. This problem will show how a probability space can be easily constructed for 2 RVs (and by extension for any finite number of RVs). In the next class lecture we'll discuss how to create a probability space for a countable collection of independent RVs all of which have the same marginal distribution. The RVs in such RPs are said to be *independent identically distributed (i.i.d).* i.i.d RPs are very useful in practice as they model repeated trial experiments where each trial (like repeated coin flipping) is independent of all the other trials. Furthermore, i.i.d processes are often filtered or otherwise processed to model more complex phenomena where the RV's in the RP are correlated with each other.

   Problem We plan to conduct an experiment where we select 10,000 twenty year old men at random and measure their height and their weight. We will round off their height to one of three categories: $\{$*Short, Average, Tall*$\}$ and their weight to one of four categories $\{$*Light, Medium, Heavy, Obese*$\}$. Note that these category names where cleverly selected to have unique first letters! Our goal is to construct a probability space $(\Omega, \mathcal{F}, P)$ for this experiment.

(a) We will make two measurements on each men. List an appropriate sample space (think ordered pairs). In other words, what are the ω?

(b) How many members are in the sample space?

(c) What should we choose as the event space $\mathcal{F}$ (you don't need to list it)? How many elements are in the event space?

(d) Once we have our experimental results, we will compute $P(\{\omega\})$ for each event $\{\omega\}$ based on its relative frequency. How can we extend $P()$ given these core probabilities so that every $A \in \mathcal{F}$ has a probability assigned to it?

(e) Assume your experimental results show that $P(\{\omega\}) = \frac{1}{N}$ for every ω in your sample space where N is the number of elements in your sample space.

   i. Compute $P(S), P(A), P(T), P(L), P(M), P(H), P(O)$ where each event is defined in this manner: $S = \{\omega : \text{the height associated with } \omega \text{ is "Short"}\}$

   ii. Are the events S and O independent?

   iii. Define a RV X on your Ω such that $X(\omega) = 1$ if $\omega \in S$, 2 if $\omega \in A$, and 3 if $\omega \in T$. Define a second RV Y on your Omega such that $Y(\omega) = 5$ if $\omega \in L$, 6 if $\omega \in M$, 7 if $\omega \in H$, and 8 if $\omega \in O$. What are EX, EX^2, EY, EY^2, σ_X^2, and σ_Y^2? Are X and Y uncorrelated (justify your answer)?

   iv. Divide your Ω space into two equal-sized groups. You can do so in any arbitrary manner. List the members of each of your two groups. Now let $P(\{\omega\}) = \frac{1}{4N}$ for each ω in your first group and $P(\{\omega\}) = \frac{7}{4N}$ for each ω in your second group. Repeat parts a.), b.), and c.) of this problem.