CARNEGIE MELLON UNIVERSITY
APPLIED STOCHASTIC PROCESSES
(COURSE 18-751)
HOMEWORK 5

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Q.1

(a)

The codeword is in error when there are more than e errors
$\Longrightarrow$

$$P_{codeerror} = \sum_{j=e+1}^{n} \binom{n}{j} p^j (1-p)^{n-j}$$

(b)

The most significant term in $P_{codeerror}$ is the first one.i.e when $j = e+1$

$\Longrightarrow$

$$P_{codeerror} \approx \binom{n}{e+1} p^{e+1} (1-p)^{n-e-1}$$

For small p and $N < 100 \quad 1-p \approx 1$

$\Longrightarrow$

$$P_{codeerror} \approx \binom{n}{e+1} p^{e+1}$$

(c) Bit Error Rate

$$P(biterror|codeerror) \approx \frac{1}{2}$$

$\Longrightarrow P(biterror) = P(biterror|codeerror)P_{codeerror} + P(biterror|Nocodeerror)P(Nocodeerror)$

Since $P(biterror|Nocodeerror) = 0$

$$P_{biterror} = P(biterror|codeerror)P_{codeerror}$$

$$P_{biterror} \approx \frac{1}{2} P_{codeerror}$$

$$P_{biterror} \approx \frac{1}{2} \sum_{j=e+1}^{n} \binom{n}{j} p^j (1-p)^{n-j}$$

(d)

BER for [1,1,0]

It is the same as sending a single bit across the channel hence, BER is the same as the channel bit rate
$\implies P_{biterror} = p$
$p = 0.01 \implies P_{biterror} = 0.01$
$p = 0.001 \implies P_{biterror} = 0.001$
$p = 10^{-6} \implies P_{biterror} = 10^{-6}$

BER for [7,4,1]

$$P_{biterror} = \frac{1}{2} P_{codeerror}$$

$$P_{biterror} = \frac{1}{2} \sum_{j=1+1}^{7} \binom{7}{j} p^j (1-p)^{7-j}$$

$$= \frac{1}{2}(1 - \sum_{j=0}^{1} \binom{7}{j} p^j (1-p)^{7-j})$$

$$= \frac{1}{2}(1 - \binom{7}{0}(1-p)^7 - \binom{7}{1}(1-p)^6)$$

$$= \frac{1}{2}(1 - (1-p)^7 - 7(1-p)^6)$$

$p = 0.01 \implies P_{biterror} = \frac{1}{2}(1 - (1-0.01)^7 - 7(1-0.01)^6) = 0.001$
$p = 0.001 \implies P_{biterror} = \frac{1}{2}(1 - (1-0.001)^7 - 7(1-0.001)^6) = 1.04x10^{-5}$
$p = 10^{-6} \implies P_{biterror} = \frac{1}{2}(1 - (1-10^{-6})^7 - 7(1-10^{-6})^6) = 1.05x10^{-11}$

(e)

[1,1,0] has the same bit error rate as the channel
$\implies p = 10^{-6}$

For [7,4,1]

$$P_{biterror} = \frac{1}{2} P_{codeerror}$$

$$P_{biterror} = \frac{1}{2} \binom{n}{e+1} p^{e+1} = 10^{-6}$$

$$\implies p^{e+1} = \frac{2x10^{-6}}{\binom{n}{e+1}}$$

$$p^2 = \frac{2x10^{-6}}{\binom{7}{2}}$$

$$p = \sqrt{\frac{2x10^{-6}}{\binom{7}{2}}}$$

$$p = 3.08x10^{-4}$$

Q.2

(a) Compute the MAP and MLE decision boundaries

we can get the MAP decision boundary by finding such that

$$\frac{f_{r|0}p(0)}{f_{r|1}p(1)} > 1 \tag{1}$$

$$f_{r|0} = \frac{1}{\sqrt{2\pi}}e^{\frac{-(x+a)^2}{2}} \tag{2}$$

$$f_{r|1} = \frac{1}{\sqrt{2\pi}}e^{\frac{-(x-a)^2}{2}} \tag{3}$$

$\implies$

$$\frac{e^{\frac{-(r-a)^2}{2}}(1-p)}{e^{\frac{-(r+a)^2}{2}}p} > 1$$

$$e^{\frac{-(r-a)^2+(r+a)^2}{2}} > \frac{p}{1-p}$$

$$e^{-2ar} > \frac{p}{1-p}$$

$$-2ar > ln\frac{p}{1-p}$$

$$r < \frac{-1}{2a}ln\frac{p}{1-p}$$

$$r < \frac{1}{2a}ln\frac{1-p}{p}$$

That means when $r < \frac{1}{2a}ln\frac{1-p}{p}$ we interpret it as 0 else we interpret it as a 1.

$MLE = MAP$ when $p = 1-p$

$\implies$

$$r < \frac{1}{2a}ln\frac{p}{p} \implies r < 0$$

So the MLE decision boundary is $r = 0$

(b)

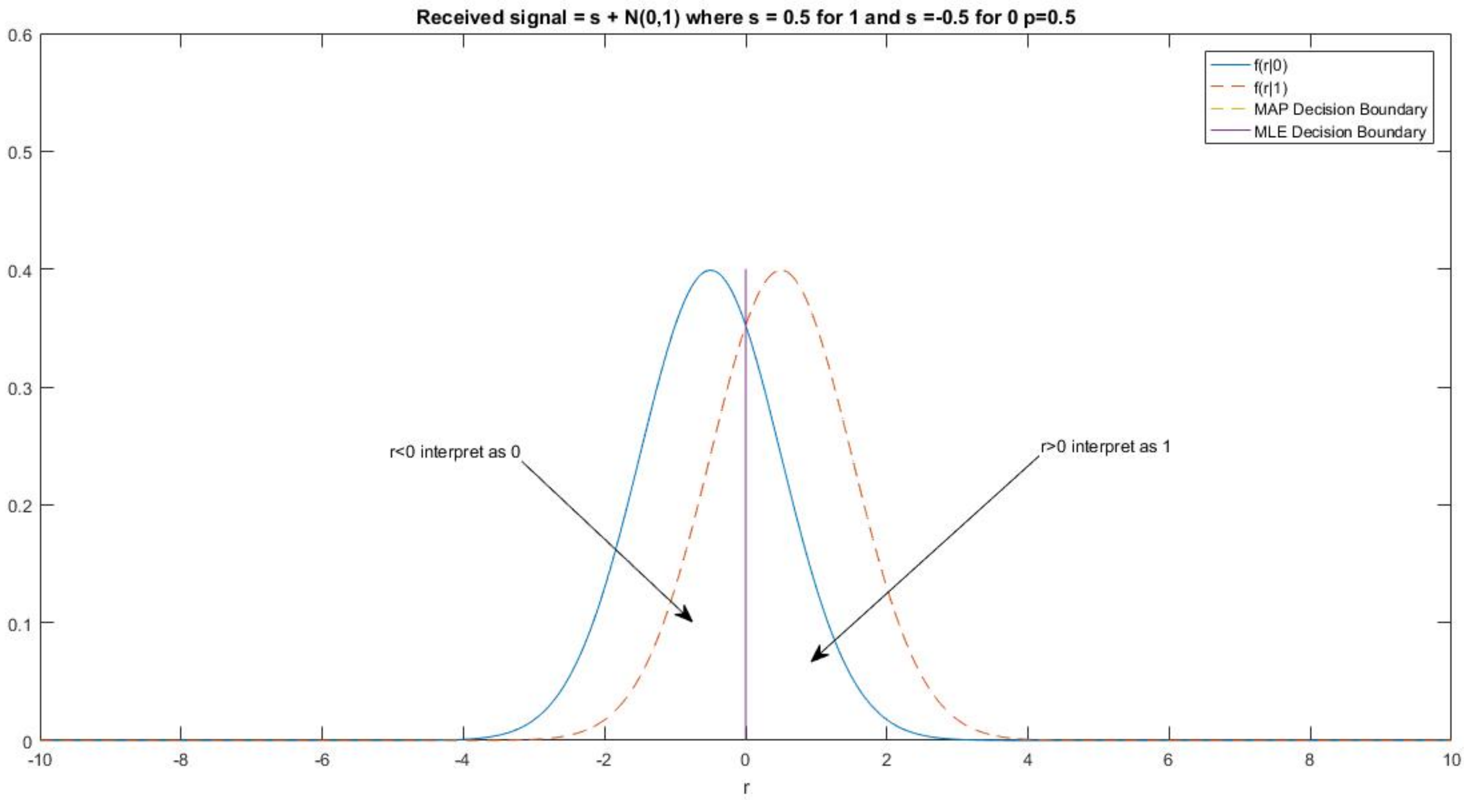

Figure 1: pdf of r , MAP and MLE decision boundaries when $p = 0.5$ and $a = 0.5$

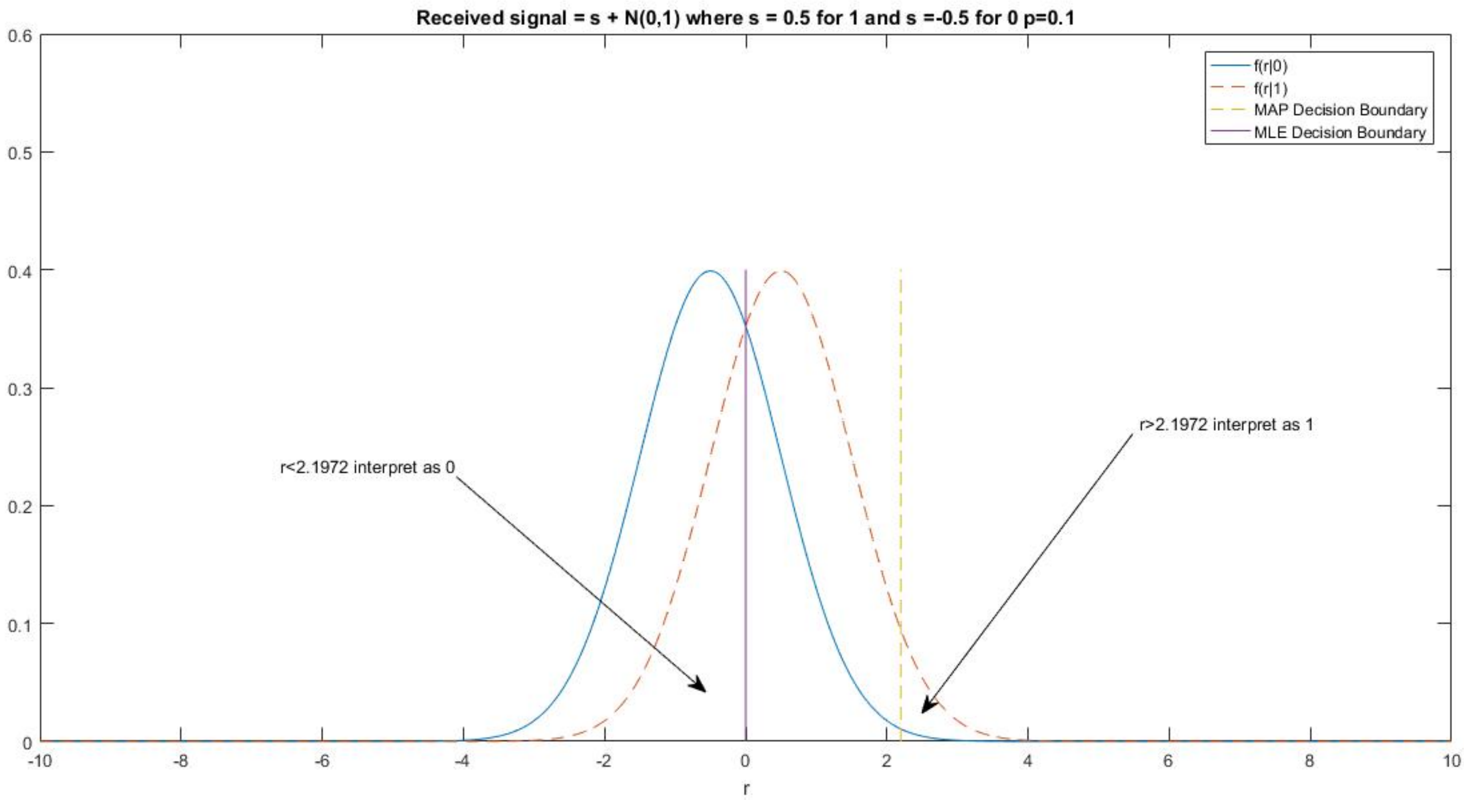

Figure 2: pdf of r , MAP and MLE decision boundaries when $p = 0.1$ and $a = 0.5$

(c) Compute the expected bit error rate

$$P(biterror) = P(biterror|1)P(1) + P(biterror|0)P(0)$$

$$P(biterror|0) = \int_{\beta}^{\infty} f_{r|0}(r)dr$$

$$P(biterror|1) = \int_{-\infty}^{\beta} f_{r|1}(r)dr$$

where β is the decision boundary.
If we have a normal variable $X \sim N(\mu, \sigma^2)$, the probability that $X > x$ is

$$Pr\{X > c\} = Q(\frac{x-\mu}{\sigma}) \quad (4)$$

where Q is the Q-function
$\Longrightarrow$

$$P(biterror|0) = \int_{\beta}^{\infty} f_{r|0}(r)dr = Q\bigg(\frac{\beta-(-a)}{1}\bigg) = Q\big(\beta+a\big)$$

similarly,

$$P(biterror|1) = \int_{-\infty}^{\beta} f_{r|1}(r)dr = 1-\int_{\beta}^{\infty} f_{r|1}(r)dr = 1-Q\bigg(\frac{\beta-a}{1}\bigg) = 1-Q\big(\beta-a\big)$$

$$P(biterror) = P(0)Q\big(\beta+a\big) + P(1)(1-Q\big(\beta-a\big))$$

MAP decision boundary and $p = 0.5 \implies \beta = 0$

$$P(biterror) = P(0)Q\big(0+a\big)+P(1)(1-Q\big(0-a\big)) = 0.5(Q(a)+(1-Q(-a)))$$

$$Q(-a) = 1-Q(a)$$

$$P(biterror) = 0.5(Q\big(a\big) + (1-(1-Q(a)))) = Q(a)$$

for a $= 0.5$ bit error rate $= P(biterror) = Q(a) = Q(0.5) = 0.3085$

MLE decision boundary $\implies$ $\beta = 0$ the same as MAP above for a $=$ 0.5 bit error rate $= P(biterror) = Q(a) = Q(0.5) = 0.3085$

MAP decision boundary a=0.5 and $p = 0.1$
$\Longrightarrow$

$$\beta = \frac{1}{2a} ln\frac{1-p}{p} = ln\frac{0.9}{0.1} = 2.1972$$

$$
\begin{aligned}
P(biterror) &= 0.9Q\big(2.1972 + a\big) + 0.1(1 - Q\big(2.1972 - a\big)) \\
&= 0.9Q\big(2.1972 + 0.5\big) + 0.1(1 - Q\big(2.1972 - 0.5\big)) \\
&= 0.9Q\big(2.6972\big) + 0.1(1 - Q\big(1.6972\big)) \\
&= 0.9 * 0.0035 + 0.1(1 - 0.0448) = 0.0987
\end{aligned}
$$

for $a = 0.5$ and $p = 0.1$ bit error rate $= P(biterror) = 0.0987$ MAP decision boundary

MLE decision boundary $\implies \beta = 0$

$$
\begin{aligned}
P(biterror) &= 0.9Q\big(a\big) + 0.1(1 - Q\big(-a\big)) \\
P(biterror) &= 0.9Q(a) + 0.1(1 - (1 - Q(a))) \\
&= 0.9Q(0.5) + 0.1(1 - (1 - Q(0.5))) = 0.3085
\end{aligned}
$$

for $a = 0.5$ and $p = 0.1$ bit error rate $= P(biterror) = 0.3085$ with MLE decision boundary

(d) ROC

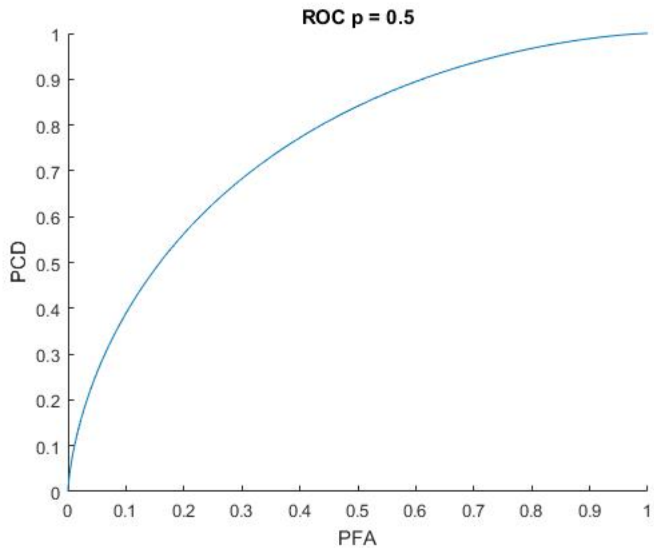

Figure 3: ROC for $a = 0.5$,$p = 0.5$

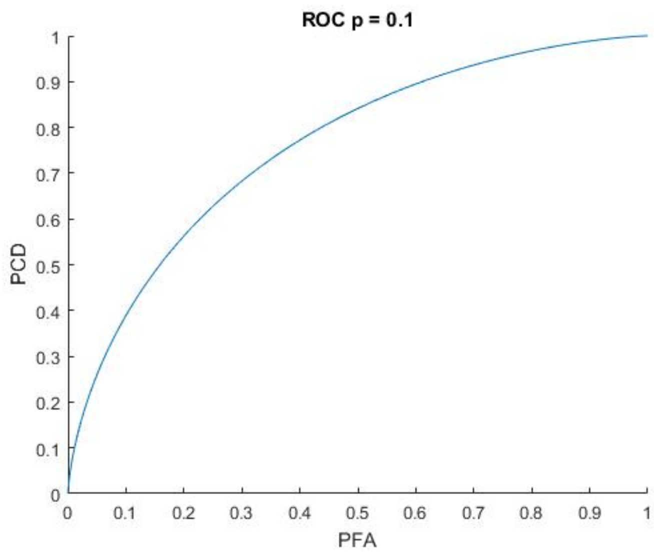

Figure 4: ROC for $a = 0.5$ $p = 0.1$

Q.3

(a) Code [1,1,0]

With MAP decision boundary when $p = 0.5$
bit error rate = $P(biterror) = P(codeerror) = Q(a)$ from 2.c

$\Longrightarrow$

$$Q(a) = 10^{-6}$$
$$a = Q^{-1}(10^{-6})$$
$$a = 4.7534$$

Average energy per bit = $\beta a^2 = 22.5950\beta$

(b) Code [7,4,1]

With MAP decision boundary when $p = 0.5$
$P(codeerror) = Q(a)$ from 2.c and $P(biterror) = \frac{1}{2}P(codeerror)$ from 1.c

$$P(codeerror) = 2P(biterror) = 2 * 10^{-6}$$

$\Longrightarrow$

$$Q(a) = 2 * 10^{-6}$$
$$a = Q^{-1}(2 * 10^{-6})$$
$$a = 4.6114$$

For every 4 bit we send 7 bit long codeword. That means for every bit we will need $\frac{7}{4}$x(the energy to send a single bit) energy.

Average energy per bit = $\frac{7}{4}\beta a^2 = 37.2135\beta$

(c) Which of the two codes is the most efficient and by what factor

From energy consumption perspective (a) is more efficient.
$factor$ = Average energy per bit (b)/ Average energy per bit (a) = $\frac{37.2135\beta}{22.5950\beta}$
$factor = 1.6469$
i.e with the second code [7,4,1] we will need 1.6469 times more energy than [1,1,0].