Applied Stochastic Processes, 18-751, TX Brown, Fall 2017
Homework #3

Due 5pm Monday Sep. 18.

1. Multiplexing: You are designing a link that is shared by many users. Your base design is to have $N_b = 100$ users share a $R_b = 1$ Mbps link and the activity factor is $p_b = 0.3$ across users. You want to analyze your base design and consider two modified designs as described below.

   (a) Base Design: Compute $\mu_b$, the mean number of active users and the range of data rates that will be experienced by users 95% of the time.

   (b) New Design 1: Here you use the same link but you aggregate many users who have very low probability of being active. In this design, you let $p_1 \to 0$ but choose $N_1$ so that $\mu_1 = \mu_b$. Compute the mean number of active users and the range of data rates that will be experienced by users 95% of the time.

   (c) New Design 2: Here you aggregate more of base users (i.e. $p_1 = p_b$) onto bigger capacity links. Let $c$ be the factor by which you increase the capacity. The number of users you multiplex also increases by the factor $c$ (i.e. there are $N_2 = 100c$ users). If $c = 10$, compute the range of data rates that will be experienced by users 95% of the time.

   (d) New Design 2': Repeat the question for Design 2 but let $c \to \infty$.

   (e) Organize all your results into a table. Include one row for each design and columns for the lower and upper limits of the data rates that will be experienced by users 95% of the time. Which design is best and why? What would you recommend to link designers?

2. Central Limit Theorem: The CLT applies even when the random variables are not independent and identically distributed (i.i.d.). Let $X_k$ correspond to independent random variables such that $E[X_k] = \mu_k$ and $\text{Var}[X_k] = \sigma_k^2$. Let $m_n = \mu_1 + \mu_2 + \ldots + \mu_n$. Let $s_n^2 = \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_n^2$. Let $b_n = \max_{\{1 \le k \le n\}}\{\sigma_k\}$. Let $Y_n = (X_1 + X_2 + \ldots + X_n - m_n)/s_n$.

   **Generalized CLT:** If $\lim_{n\to\infty} \frac{b_n}{s_n} = 0$ then $\lim_{n\to\infty} Y_n \Rightarrow Y \sim \mathcal{N}(0, 1)$.

   (a) Show that the GCLT reduces to the usual i.i.d. version of the CLT (equation (3.3) in the text) if the $\{X_i\}$ are in fact i.i.d.

   (b) Let $X_k$ for all $k$ be a random integer chosen uniformly from 1 to 5. Generate at least 100 instances of $X_1, X_2, \ldots X_{10}$ and plot a normalized histogram of $Y_{10}$ against the pdf of a the standard normal distribution. Plot a q-q plot of your data against the standard normal.

   (c) Determine whether the GCLT applies to the case where $X_k$ is a random integer chosen uniformly from 1 to $k$. Generate at least 100 instances of $X_1, X_2, \ldots X_{10}$ and plot a normalized histogram of $Y_{10}$ against the pdf of a the standard normal distribution. Plot a q-q plot of your data against the standard normal.

   (d) Determine whether the GCLT applies to the case where $X_k = B_k/2^k$ and $B_k$ is a Bernoulli trial with $p = 0.5$. Generate at least 100 instances of $X_1, X_2, \ldots X_{10}$ and plot a normalized histogram of $Y_{10}$ against the pdf of a the standard normal distribution. Plot a q-q plot of your data against the standard normal.

   (e) What can you say about how well the sum of the first ten elements is approximated by a normal distribution in each of these three cases?

3. Buffers: Write a Matlab simulation of a buffer of size $N$ where packets arrive as a Bernoulli process with rate $\lambda$ and geometric service times with rate $\mu$, where $0 < \lambda, \mu < 1$. What this means is that a new packet is generated with probability $\lambda$ and a packet (even a newly generated packet) departs with probability $\mu$. If there is a net departure, then the number of packets decreases by one to a minimum of zero. If there is a net arrival then a packet is added to the buffer if it is not already full. If the buffer is already full, the new packet is dropped. Hint: use your generic Markov chain simulator.

   (a) Let $\lambda = 0.1$. Choose an $N$. Choose a $\mu$ that corresponds to a low load. Plot the number of packets in the buffer over time. Based on the simulation compute the average number of packets in the buffer, the fraction of time the buffer is empty (in state 0) and the fraction of packet arrivals that are blocked (arrive when the buffer is full and are dropped). Low load means that less than 1% of the packets are lost.

   (b) Repeat (a) but with a $\mu$ that corresponds to a high load. High load means that more than 10% of the packets are lost.

   (c) In both cases, compute the average delay per packet using Little's law.