# CARNEGIE MELLON UNIVERSITY
# APPLIED STOCHASTIC PROCESSES
# (COURSE 18-751)
# HOMEWORK 5

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**Q.1**

# Q.2

## (a) Compute the MAP and MLE decision boundaries

we can get the MAP decision boundary by finding such that

$$\frac{f_{r|0}p(0)}{f_{r|1}p(1)} > 1 \tag{1}$$

$$f_{r|0} = \frac{1}{\sqrt{2\pi}}e^{\frac{-(x+a)^2}{2}} \tag{2}$$

$$f_{r|1} = \frac{1}{\sqrt{2\pi}}e^{\frac{-(x-a)^2}{2}} \tag{3}$$

$\implies$

$$\frac{e^{\frac{-(r-a)^2}{2}}(1-p)}{e^{\frac{-(r+a)^2}{2}}p} > 1$$

$$e^{\frac{-(r-a)^2+(r+a)^2}{2}} > \frac{p}{1-p}$$

$$e^{-2ar} > \frac{p}{1-p}$$

$$-2ar > ln\frac{p}{1-p}$$

$$r < \frac{-1}{2a}ln\frac{p}{1-p}$$

$$r < \frac{1}{2a}ln\frac{1-p}{p}$$

That means when $r < \frac{1}{2a}ln\frac{1-p}{p}$ we interpret it as 0 else we interpret it as a 1.

$MLE = MAP$ when $p = 1 - p$

$\implies$

$$r < \frac{1}{2a}ln\frac{p}{p} \implies r < 0$$

So the MLE decision boundary is $r = 0$

(b)

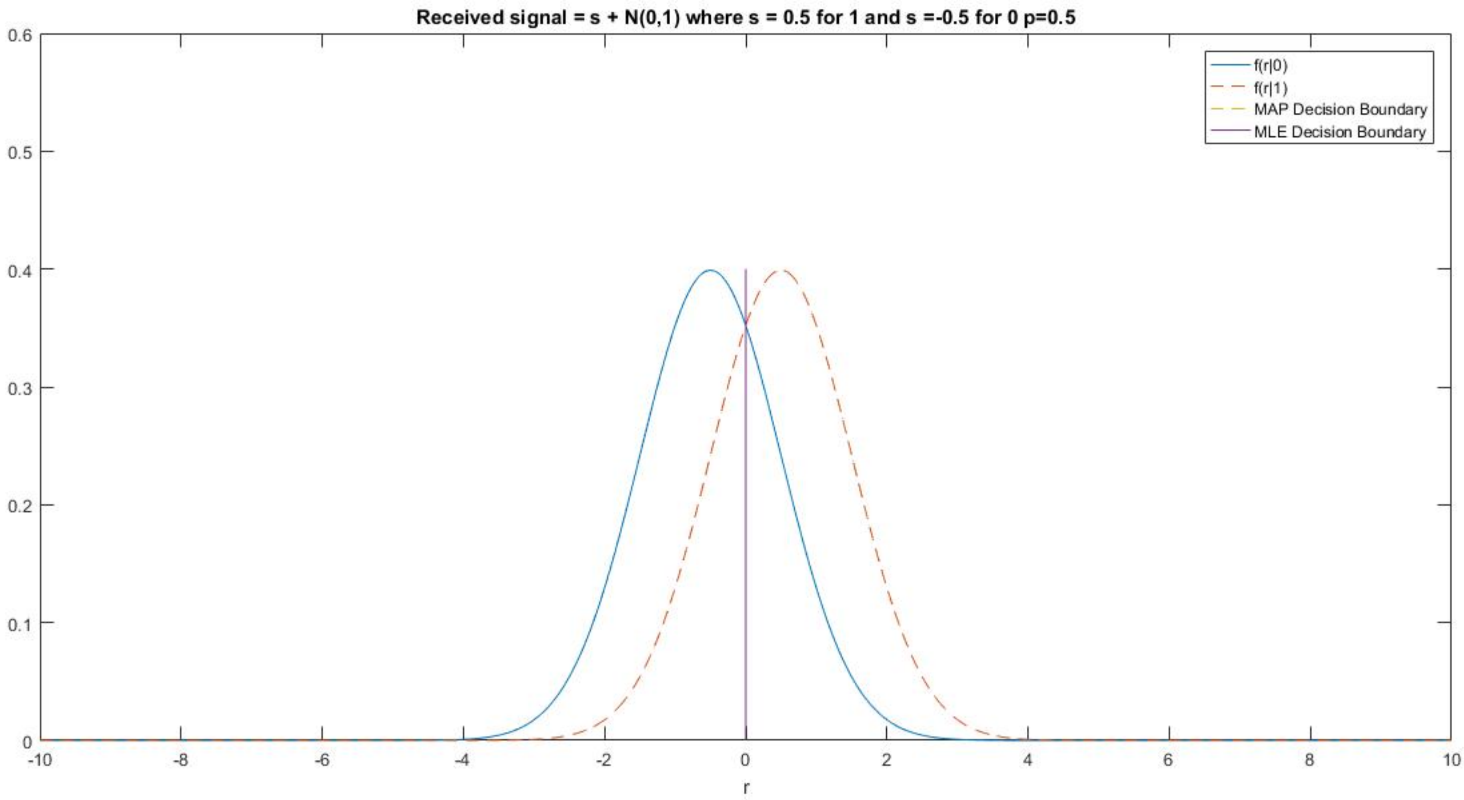


Figure 1: pdf of r , MAP and MLE decision boundaries when $p = 0.5$ and $a = 0.5$

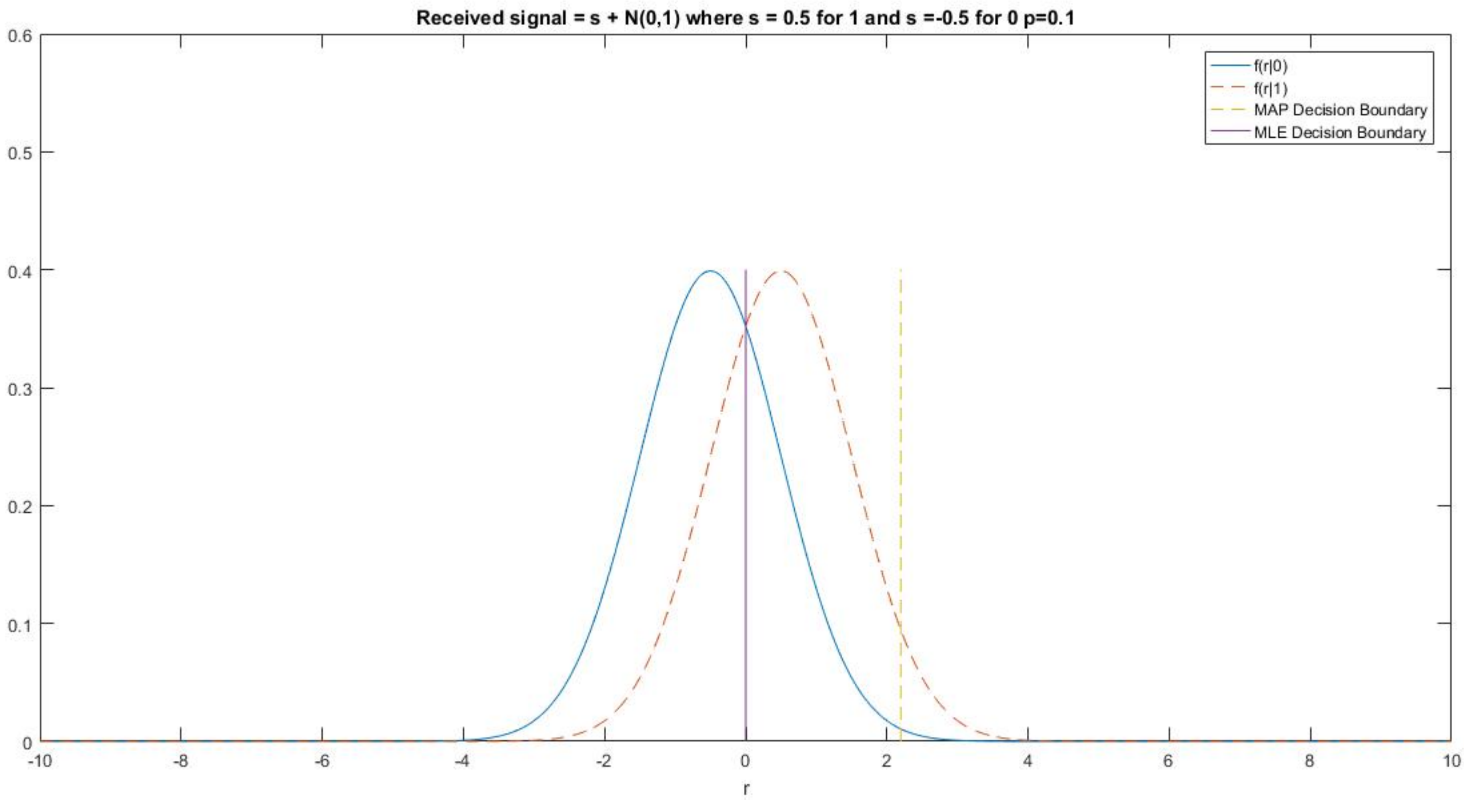


Figure 2: pdf of r , MAP and MLE decision boundaries when $p = 0.1$ and $a = 0.5$

### (c) Compute the expected bit error rate

$$P(biterror) = P(biterror|1)P(1) + P(biterror|0)P(0)$$

$$P(biterror|0) = \int_{\beta}^{\infty} f_{r|0}(r)dr$$

$$P(biterror|1) = \int_{-\infty}^{\beta} f_{r|1}(r)dr$$

where $\beta$ is the decision boundary.
If we have a normal variable $X \sim N(\mu, \sigma^2)$, the probability that $X > x$ is

$$Pr\{X > c\} = Q(\frac{x-\mu}{\sigma}) \tag{4}$$

where Q is the Q-function
$\Longrightarrow$

$$P(biterror|0) = \int_{\beta}^{\infty} f_{r|0}(r)dr = Q\left(\frac{\beta-(-a)}{1}\right) = Q\big(\beta + a\big)$$

similarly,

$$P(biterror|1) = \int_{-\infty}^{\beta} f_{r|1}(r)dr = 1 - \int_{\beta}^{\infty} f_{r|1}(r)dr = 1 - Q\left(\frac{\beta - a}{1}\right) = 1 - Q\big(\beta - a\big)$$

$$P(biterror) = P(0)Q\big(\beta + a\big) + P(1)(1 - Q\big(\beta - a\big))$$

MAP decision boundary and $p = 0.5 \implies \beta = 0$

$$P(biterror) = P(0)Q\big(0+a\big) + P(1)(1-Q\big(0-a\big)) = 0.5(Q(a) + (1 - Q(-a)))$$

$$Q(-a) = 1 - Q(a)$$

$$P(biterror) = 0.5(Q\big(a\big) + (1 - (1 - Q(a)))) = Q(a)$$

for a $= 0.5$ bit error rate $= P(biterror) = Q(a) = Q(0.5) = 0.3085$

MLE decision boundary $\implies \beta = 0$ the same as MAP above for a $=$ $0.5$ bit error rate $= P(biterror) = Q(a) = Q(0.5) = 0.3085$

MAP decision boundary a=0.5 and $p = 0.1$
$\Longrightarrow$

$$\beta = \frac{1}{2a} ln \frac{1-p}{p} = ln \frac{0.9}{0.1} = 2.1972$$

$$
\begin{aligned}
P(biterror) &= 0.9Q\big(2.1972 + a\big) + 0.1(1 - Q\big(2.1972 - a\big)) \\
&= 0.9Q\big(2.1972 + 0.5\big) + 0.1(1 - Q\big(2.1972 - 0.5\big)) \\
&= 0.9Q\big(2.6972\big) + 0.1(1 - Q\big(1.6972\big)) \\
&= 0.9 * 0.0035 + 0.1(1 - 0.0448) = 0.0987
\end{aligned}
$$

for $a = 0.5$ and $p = 0.1$ bit error rate $= P(biterror) = 0.0987$ MAP decision boundary

MLE decision boundary $\implies \beta = 0$

$$
\begin{aligned}
P(biterror) &= 0.9Q\big(a\big) + 0.1(1 - Q\big(-a\big)) \\
P(biterror) &= 0.9Q(a) + 0.1(1 - (1 - Q(a))) \\
&= 0.9Q(0.5) + 0.1(1 - (1 - Q(0.5))) = 0.3085
\end{aligned}
$$

for $a = 0.5$ and $p = 0.1$ bit error rate $= P(biterror) = 0.3085$ with MLE decision boundary

## (d) ROC

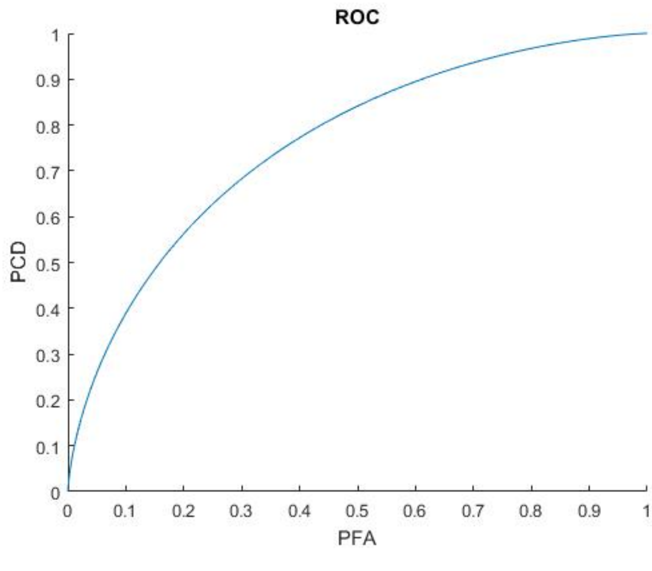


Figure 3: ROC for $a = 0.5$

# Q.3

## (a) Code [1,1,0]

With MAP decision boundary when $p = 0.5$
bit error rate = $P(biterror) = Q(a)$ from 2.c

$\implies$

$$Q(a) = 10^{-6}$$
$$a = Q^{-1}(10^{-6})$$
$$a = 4.7534$$

Average energy per bit = $\beta a^2 = 22.5950\beta$

## (b) Code [7,4,1]

With MAP decision boundary when $p = 0.5$
bit error rate = $P(biterror) = Q(a)$ from 2.c

$\implies$

$$Q(a) = 10^{-6}$$
$$a = Q^{-1}(10^{-6})$$
$$a = 4.7534$$

For every 4 bit we send 7 bit long code word. That means for every bit we will be using $\frac{7}{4}$x(the energy to send a single bit).

Average energy per bit = $\frac{7}{4}\beta a^2 = 39.5413\beta$

## (c) Which of the two codes is the most efficient and by what factor

from energy consumption perspective (a) is more efficient.
$factor$ = Average energy per bit (a)/ Average energy per bit (b) = $\frac{\beta a^2}{\frac{7}{4}\beta a^2}$
$factor = \frac{4}{7}$
i.e with code (a) we will need only $\frac{4}{7}$th of the energy will need with code (b)