Applied Stochastic Processes, 18-751, TX Brown, Fall 2017
Homework #4

Due 5pm Monday Sep. 25.

Read Chapters 2 and 3 of Goodman and Yates

1. A computer broadcast protocol attempts to get a message from a wireless router to N listening computers. Time is divided into slots of length $\tau = 100msec$. In one slot the router sends the message and then the computers respond with an acknowledgment (aka "ACK"). The probability any one computer can receive the message and successfully ACK it is p. Analyze the following two strategies.

   (a) All Strategy: The router requires all N of the ACKs to be received in response to a given transmission attempt for the packet transmission to be declared successful. Let T_A be the time to a successful transmission. Compute the PMF, $P_{T_A}(t)$, mean, μ_{T_A}, and CDF $F_A(m) = P(T_A \leq m\tau)$.

   (b) Cumulative Strategy: The router only requires ACKs to be received from those computers that have not ACK'd already. Let T_C be the time to a successful transmission. Compute the PMF, $P_{T_C}(t)$, mean, μ_{T_C}, and CDF $F_C(m) = P(T_C \leq m\tau)$. Hint: first compute it for $N = 1$ and then generalize for larger N.

   (c) For $p = 0.9$ and for $N = 5$ and $N = 20$ compute the mean time and the probability that it takes longer than 1 sec to achieve success for both strategies.

2. A local restaurant is known for its unusual service. The time X, in minutes, that a customer has to wait before she captures the attention of a waitress is specified by the following distribution function:

$$F_X(x) = \begin{cases} \frac{x^2}{4} & \text{for } 0 \leq x \leq 1 \\ \frac{x}{4} & \text{for } 1 \leq x \leq 2 \\ \frac{1}{2} & \text{for } 2 \leq x \leq 10 \\ \frac{x}{20} & \text{for } 10 \leq x \leq 20 \\ 1 & \text{for } x \geq 20 \end{cases}$$

   (a) Sketch $F_X(x)$.

   (b) Compute and sketch $f_X(x)$. Verify that the area under the pdf is indeed unity.

   (c) What is the probability that the customer will have to wait:

      i. At least 10 minutes.

      ii. Less than 5 minutes.

      iii. Between 5 and 10 minutes.

      iv. Exactly 1 minute.

3. Let $f_{XY}(x,y) = A(x^2 + y^2)$ if $0 \leq x \leq 10$ and $0 \leq y \leq 5$ for some constant A.

   (a) Compute A.

   (b) Compute $F_{XY}(x,y)$ for $0 \leq x \leq 10$ and $0 \leq y \leq 5$.

   (c) Let $B = \{(x,y)|0 \leq x \leq 1, \quad 0 \leq y \leq 1)$. Compute the P[B].

   (d) Compute the marginal densities $f_X(x)$ and $f_Y(y)$.

   (e) Are X and Y independent? Why or why not?

4. In Bitcoin, so-called miners compete for rewards by cryptographically flipping a biased coin. The coin has very small success probability, p. Miners are rewarded each time they get a success. Currently miners are flipping at a rate of 8×10^{18} flips per second.

   (a) Calculate p so that there is one success every 10 minutes.

   (b) A success has just been announced and so miners start competing for the next reward. Calculate the probability that the time to the next reward is more than 1 minute, more than 10 minutes, and more than one hour.

   (c) No success is announced after 20 minutes. Given there is no success after 20 minutes, what is the probability the time to the next reward is more than 21 minutes, more than 30 minutes and more than 70 minutes?

   (d) Mining equipment uses a lot of energy. Suppose that 20 minutes has passed since the last block and you have just finished setting up your equipment so it is ready to turn on. Would your energy be better spent if you started mining right away or if you waited until the next success before mining? Explain why.

   (e) You decide to use your fancy new laptop to be a miner (anybody, including you, can be a miner). It can flip 40 million coins per second. Given the p you calculated in (a), calculate the expected time to your first success. Put the answer in appropriate units such as hours or days or weeks or $\ldots$ as appropriate.