CARNEGIE MELLON UNIVERSITY
APPLIED STOCHASTIC PROCESSES
(COURSE 18-751)
HOMEWORK 6

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Q.1

$$g(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } 0 < x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

Domain if Y is $= [0,1] \implies F_Y(y) = 1$ when $y > 1$ and $F_Y(y) = 0$ when $y < 0$

$y = g(x) = x$ for $0 < x \leq 1$

$$F_Y(y) = P[Y \leq y] = P[X \leq y] = F_X(y) = \Phi(y)$$

$$F_Y(y) = \begin{cases} 0 & \text{if } y \leq 0 \\ \Phi(y) & \text{if } 0 < y \leq 1 \\ 1 & \text{if } y > 1 \end{cases}$$

Threre are jumps at 0 and 1. At 0 from 0 to $\Phi(0) = \frac{1}{2}$
at 1 from $\Phi(1) = 0.8413$ to 1

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} \frac{1}{2}\delta(y) & y = 0 \\ \frac{1}{\sqrt{2\pi}}e^{\frac{-y^2}{2}} & 0 < y \leq 1 \\ 0.1587\delta(y-1) & y = 1 \\ 0 & \text{otherwise} \end{cases}$$

Q.2

(a)

$$Z = min\{X_1, X_2, \ldots, X_n\}$$

$$\begin{aligned} F_Z(z) = P[Z \leq z] &= P[min(X_1, X_2, \ldots, X_n) \leq z] \\ &= 1 - P[min(X1, X2, \ldots, Xn) \geq z] \end{aligned}$$

If $min(\{X_1, X_2, \ldots, X_n\}) > z$ then all the values in $\{X_1, X_2, \ldots, X_n\} > z$

$\implies F_Z(z) = 1 - P[X1 \geq z, X2 \geq z, \ldots, X_n \geq z]$

$$\begin{aligned} F_Z(z) &= 1 - \prod_i^n P[X_i \geq z] \\ &= 1 - \prod_i^n (1 - P[X_i \leq z]) \\ F_Z(z) &= 1 - (1 - F_X(z))^n \end{aligned}$$

$$\begin{aligned} f_Z(z) = \frac{dF_Z(z)}{dz} &= \frac{d(1 - (1 - F_X(z))^n)}{dz} \\ &= -n(1 - F_X(z))^{n-1} * \frac{d(1 - F_X(z))}{dz} \\ &= -n(1 - F_X(z))^{n-1} * (-f_X(z)) \\ &= n(1 - F_X(z))^{n-1} * f_X(z) \end{aligned}$$

(b)

$F_X(x) = 1 - e^{-x}$, $x > 0$, $n = 3$, $f_X(x) = e^{-x}$

$$\begin{aligned} f_Z(z) &= 3(1 - (1 - e^{-z}))^2 e^{-z} \\ &= 3(e^{-z})^2 e^{-z} \\ &= 3(e^{-2z})e^{-z} \\ &= 3e^{-3z} \end{aligned}$$

$$f_Y(z) = \begin{cases} 3e^{-3z} & z > 0 \\ 0 & \text{otherwise} \end{cases}$$

(c)

For $\alpha = 0.05$ the rejection region is $R = \{z > r\}$

$$\alpha = P[z > r] = 1 - P[z \leq r]$$
$$\alpha = P[z > r] = 1 - F_Z(r)$$
$$F_Z(r) = 1 - (1 - F_X(r))^3 = 1 - (1 - (1 - e^{-r}))^3$$
$$F_Z(r) = 1 - e^{-3r}$$
$$\alpha = 1 - F_Z(r) = e^{-3r}$$
$$r = -\frac{ln\alpha}{3} = -\frac{ln0.05}{3} = 0.9986$$

$\implies$ we reject the hypothesis if $z > 0.9986$

(d)

$F_Y(y) = 1 - e^{-\lambda y}$ and $\lambda < 1$

$\implies$ For rejection region $R = \{z > r\}$

$$\alpha = 1 - P[z \leq r] = 1 - F_Z(r)$$
$$F_Z(r) = 1 - (1 - F_X(r))^3 = 1 - (1 - (1 - e^{-\lambda r}))^3$$
$$F_Z(r) = 1 - e^{-3\lambda r}$$
$$\alpha = 1 - F_Z(r) = e^{-3\lambda r}$$
$$e^{-3\lambda r} = \alpha$$
$$r = -\frac{ln\alpha}{3\lambda} = -\frac{ln0.05}{3\lambda} = \frac{0.9986}{\lambda}$$

$\lambda < 1 \implies r > 0.9986$ hence it is more likely that $\lambda = 1$ will reject the hypothesis than $\lambda < 1$

Q.3

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left[-\left(\frac{x^2 - 2\rho xy + y^2}{2\sigma^2(1-\rho^2)}\right)\right] \tag{1}$$

We can rewrite eqn 1 as a product of two Gaussians

$$f_{XY}(x,y) = \frac{1}{\sqrt{2\pi}\sigma\sqrt{1-\rho^2}} e^{\frac{-(y-\rho x)^2}{2\sigma^2(1-\rho^2)}} . \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-x^2}{2\sigma^2}}$$

because it is symmetric we can also rewrite as

$$f_{XY}(x,y) = \frac{1}{\sqrt{2\pi}\sigma\sqrt{1-\rho^2}} e^{\frac{-(x-\rho y)^2}{2\sigma^2(1-\rho^2)}} . \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-y^2}{2\sigma^2}}$$

The first Gaussian has mean $\mu_1 = 0$ and variance $\sigma_1^2 = \sigma^2$. On the other hand the second Gaussian has mean $\mu 2 = \rho y$ and variance $\sigma_2^2 = \sigma^2(1-\rho^2)$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx$$

$$f_Y(y) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-y^2}{2\sigma^2}} . \frac{1}{\sqrt{2\pi}\sigma\sqrt{1-\rho^2}} e^{\frac{-(x-\rho y)^2}{2\sigma^2(1-\rho^2)}} dx$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-y^2}{2\sigma^2}} . \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma\sqrt{1-\rho^2}} e^{\frac{-(x-\rho y)^2}{2\sigma^2(1-\rho^2)}} dx$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma\sqrt{1-\rho^2}} e^{\frac{-(x-\rho y)^2}{2\sigma^2(1-\rho^2)}} dx = 1$$

$\Longrightarrow$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-y^2}{2\sigma^2}}$$

because it is symmetric

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-x^2}{2\sigma^2}}$$

i.e both $f_Y(y)$ and $f_Y(y)$ are Gaussians with mean of $\mu = 0$ and variance of $\sigma^2 = \sigma^2$
$f_Y(y) \sim N(0,\sigma^2)$ and $f_X(x) \sim N(0,\sigma^2)$

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)dy$$

since $f_Y(y)$ is a gaussian the expected value $E[Y] = \mu = 0$

$$f_{Y|X}(x,y) = \frac{f_{XY}(x,y)}{f_X(x)}$$

$$f_{Y|X}(x,y) = \frac{\frac{1}{\sqrt{2\pi}\sigma\sqrt{1-\rho^2}} e^{\frac{-(y-\rho x)^2}{2\sigma^2(1-\rho^2)}} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-x^2}{2\sigma^2}}}{\frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-x^2}{2\sigma^2}}}$$

$$f_{Y|X}(x,y) = \frac{1}{\sqrt{2\pi}\sigma\sqrt{1-\rho^2}} e^{\frac{-(y-\rho x)^2}{2\sigma^2(1-\rho^2)}}$$

$\implies f_{Y|X}(x,y) \sim N(\rho x, \sigma^2(1-\rho^2)$
$\implies E[Y|X=x] = \mu = \rho x$

(b)

$$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2}}$$

$\implies f_Y(y) \sim N(0, \sigma^2)$
$\implies \sigma_Y = \sigma^2$

$$f_{Y|X}(x,y) = \frac{1}{\sqrt{2\pi}\sigma\sqrt{1-\rho^2}} e^{\frac{-(y-\rho x)^2}{2\sigma^2(1-\rho^2)}}$$

$\implies f_{Y|X}(x,y) \sim N(\rho x, \sigma^2(1-\rho^2)$
$\implies \sigma^2_{Y|X=x} = \sigma^2(1-\rho^2)$

(c)

Does knowing **X** provide any information about **Y**?

It depends. When $\rho = 0$ $f_{XY}(x,y) = f_X(x)f_Y(y)$ i.e X and Y are independent and knowing X does not provide any information about Y. But when $\rho \neq 0$ $f_{XY}(x,y) \neq f_X(x)f_Y(y)$ i.e X and Y are not independent hence knowing X provides information about Y and vice versa.