CARNEGIE MELLON UNIVERSITY APPLIED STOCHASTIC PROCESSES (COURSE 18-751) HOMEWORK 2

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Q.1 Compute the expected discounted future reward($\gamma(i)$)

we can compute the expected discounted future reward by using the following formula

$$\gamma(i) = h(i) + \beta \sum_j P(i,j)\gamma(j) \tag{1}$$

where β is the discount factor and $h(i)$ is the reward on state i.

$$\gamma(A) = 0 + 0.8 * (0.5 * \gamma(B) + 0.5 * \gamma(D)) \tag{2}$$

$$\gamma(B) = 0 + 0.8 * \gamma(C) \tag{3}$$

$$\gamma(C) = 0 + 0.8 * \gamma(A) \tag{4}$$

$$\gamma(D) = 1 + 0.8 * (\frac{1}{3} * \gamma(B) + \frac{1}{3} * \gamma(E) + \frac{1}{3} * \gamma(A)) \tag{5}$$

$$\gamma(E) = 2 + 0.8 * (0.5 * \gamma(B) + 0.5 * \gamma(C)) \tag{6}$$

Lets write all them in terms of $\gamma(A)$

$$\gamma(A) = \gamma(A) \tag{7}$$

$$\gamma(B) = 0.64 * \gamma(A) \tag{8}$$

$$\gamma(C) = 0.8 * \gamma(A) \tag{9}$$

$$\gamma(D) = 1.5333 + 0.5909 * \gamma(A) \tag{10}$$

$$\gamma(E) = 2 + 0.576 * \gamma(A) \tag{11}$$

writing eqn (2) interms of $\gamma(A)$s we get

$$\gamma(A) = 0.64\gamma(A) + 0.4 * 0.5909 * \gamma(A) + 0.4 * 1.5333 \tag{12}$$

$$0.50764 * \gamma(A) = 0.4 * 1.5333 \tag{13}$$

$$\tag{14}$$

so $\gamma(A) = 1.2082$,$\gamma(A) = 1.2082$, $\gamma(B) = 0.64 * \gamma(A) = 0.7732$, $\gamma(C) = 0.8 * \gamma(A) = 0.9666$, $\gamma(D) = 1.5333 + 0.5909 * \gamma(A) = 2.2472$, $\gamma(E) = 2 + 0.576 * \gamma(A) = 2.6959$

Ans.
$\gamma(A) = 1.2082$, $\gamma(B)) = 0.7732$, $\gamma(C)) = 0.9666$, $\gamma(D)) = 2.2472$, $\gamma(E)) = 2.6959$

Q.2 Prove the following

$$\bigcap_{i=1}^{\infty} E_i \subset \left(\bigcap_{i=1}^{\infty} E_i^c\right)^c \tag{15}$$

soln.
We can rewrite the RHS as (16) using De Morgan's law

$$\left(\bigcap_{i=1}^{\infty} E_i^c\right)^c = \bigcup_{i=1}^{\infty} E_i \tag{16}$$

To prove that the LHS is a subset of RHS we can show that any arbitrary element of LHS is also an element if the RHS.
lets say e_i is an arbitrary element of LHS, i.e $e_i \in \bigcap\limits_{i=1}^{\infty} E_i$ .This implies e_i is an element of any set E_i. Since all the elements of arbitrary set E_i are also elements of the union of the sets, $e_i \in \bigcup\limits_{i=1}^{\infty} E_i$.
Hence LHS is a subset of RHS. i.e $\bigcap\limits_{i=1}^{\infty} E_i \subset \bigcup\limits_{i=1}^{\infty} E_i$ which is the same as saying,

$$\bigcap_{i=1}^{\infty} E_i \subset \left(\bigcap_{i=1}^{\infty} E_i^c\right)^c \tag{17}$$

Can the left and right side ever be equal?
YES. When for example $E_1 = E_2 = E_3 ... = E_i = ...$, that is when all the E_i s are the same. The intersection and union of those sets would be the same as anyone of the sets E_i.

Q.3

(a) Show that the following statements are consistant with the three axioms of probability

The three axioms of probability are :
(1)$P(A) \geq 0$ for all $A \subset S$
(2)$P(S) = 1$
(3) if $A \cap B = \emptyset$, then $P(A \cup B = P(A) + P(B)$

Assigning $P[\{1\}] = 1$, $P[\{2\}] = 1$, $P[\{1,2\}] = 2$ is consistant with the first axiom because all the elements of $\mathcal{F}$ have $P \geq 0$, we can also assign $P(S) = P(\{1,2,3,4,5,6\}) = 1$ so it satisfies the second axiom. When it comes to the third axiom, the only elements of $\mathcal{F}$ that are pair wise disjoint are $\{1\}$ and $\{2\}$.

$P(\{1\} \cup \{2\}) = P(\{1,2\}) = 2 = P(\{1\}) + P(\{2\})$ hence it is also consitstant with third axiom.
Therefore, $P[\{1\}] = 1$, $P[\{2\}] = 1$, $P[\{1,2\}] = 2$ is consistant with the three axioms of probability.

(b)

Is $\mathcal{F}$ a sigma field?

NO.
To be a sigma field $\mathcal{F}$ has to be closed under complementation.And we can clearly see that $\mathcal{F}$ doesn't satisfy this property.$\{1\}^c$,$\{2\}^c$ and $\{1,2\}^c$ are all missing.

Minimum number of additional events required to make $\mathcal{F}$ a sigma field.

3.
we need to add $\{1\}^c$ which is $\{2,3,4,5,6\}^c$,$\{2\}^c$ which is $\{1,3,4,5,6\}^c$ and finally $\{1,2\}^c$ which is $\{3,4,5,6\}^c$.

$$\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{3,4,5,6\}, \{1,3,4,5,6\}, \{2,3,4,5,6\}, \{1,2,3,4,5,6\}\} \tag{18}$$

Show that (a) is not true

for $\mathcal{F}$ in (18) if we assign $P[\{1\}] = 1$, $P[\{2\}] = 1$, $P[\{1,2\}] = 2$
$\implies P(\{1,2\} \cup \{3,4,5,6\}) = P(\{1,2,3,4,5,6\}) = P(S) = 1$

lets assume it is consistant with the third axiom,
that means $P(\{1,2\} \cup \{3,4,5,6\}) = P(\{1,2\}) + P(\{3,4,5,6\})$

$$P(\{1,2\}) + P(\{3,4,5,6\}) = 1$$
$$2 + P(\{3,4,5,6\}) = 1$$
$$P(\{3,4,5,6\}) = -1$$

i.e $P(\{3,4,5,6\}) \leq 0$, thus violating the first axiom of probability.
So $P[\{1\}] = 1$, $P[\{2\}] = 1$ and $P[\{1,2\}] = 2$ will not be consistant with the axioms of probability if $\mathcal{F}$ is a sigma field.

Q4

(a) No Kings

We have 48 cards that are not King cards, hence we can select four cards that are not king cards in $\binom{48}{4} = 194580$ different ways.

Ans. 194580

(b) 2 Kings and 2 Queens

We have four kings so we can select two of them in $\binom{4}{2}$ ways similarly we can draw two Queens in $\binom{4}{2}$ ways. Hence we can draw two Kings and two Queens in $\binom{4}{2} * \binom{4}{2} = 36$ ways.

Ans. 36

(c) Number of Possible combinations of (n_h, n_d, n_s, n_c)

Q5

$P(+|D) = 0.99$, $P(+|D^c) = 0.01$, $P(D) = \frac{1}{10000} = 0.0001$ and since $P(D) + P(D^c) = 1$ $P(D^c) = 0.9999$
Where D = you have the disease and + = you tested positive .

We are asked to compute, given that we tested positive for it, the probability that we actually have the disease (i.e $P(D|+)$).

$$P(D|+) = \frac{P(+|D)P(D)}{P(+)} \tag{19}$$

$$P(+) = P(+|D)P(D) + P(+|D^c)P(D^c) \tag{20}$$

$$P(D|+) = \frac{P(+|D)P(D)}{P(+|D)P(D) + P(+|D^c)P(D^c)} \tag{21}$$

hence

$$P(D|+) = \frac{0.99 * 0.0001}{0.99 * 0.0001 + 0.01 * 0.9999)}$$

$$P(D|+) = 0.009803$$

So we have a $< 1\%$ chance of actually having the disease eventhough we tested positive for it.

Q6