Applied Stochastic Processes, 18-751, TX Brown, Fall 2017
Homework #2

Due 5pm Monday Sep. 11.

1. Using Figure 1.2, consider that state D earns a reward of 1 and state E earns a reward of 2 and all other states earn a reward of zero. Let there be a discount factor of 0.8. Compute the expected discounted future reward, $\gamma(i)$ for each $i \in \{A, B, C, D, E\}$.

2. Prove that for arbitrary sets $E_1, E_2, \ldots,$

$$\bigcap_{i=1}^{\infty} E_i \subset \left(\bigcap_{i=1}^{\infty} E_i^c \right)^c$$

   Hint: consider an arbitrary element of the LHS of the inequality.

   Can the left and right side ever be equal?

3. Consider the experiment of rolling a six-sided die. Consider the set of events:

$$\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3, 4, 5, 6\}\}.$$

   (a) Show that the following statements are consistent with the three axioms of probability: $P[\{1\}] = 1, P[\{2\}] = 1, P[\{1, 2\}] = 2.$

   (b) Is $\mathcal{F}$ a sigma field? Why or why not? If not, what are the minimum number of additional events required to make it a sigma field. If these additional events are added, show (a) is not true.

4. Given an ordinary deck of 52 playing cards, you are dealt four cards and order does not matter. (circle all that apply)

   (a) Compute the number of ways we can draw no Kings.

   (b) Compute the number of ways we can draw 2 Kings and 2 Queens.

   (c) Let n_h, n_d, n_s, n_c be the number of hearts, diamonds, spades, and clubs. Compute the number of possible combinations of (n_h, n_d, n_s, n_c).

   (d) Prove or disprove the event "no Kings" is independent of the event "no Queen's'.

   (e) Prove or disprove the event "no Kings" is independent of the event "no red card".

5. You take a medical test for a disease. You test positive if you have the disease with probability 0.99; and you test positive if you don't have the disease with probability 0.01. The disease is known to affect 1 in 10,000 people. If you test positive, what is the probability that you have the disease?

6. The Monty Hall problem. Monty Hall was a game show host where the climax of the show consisted of the contestant picking their prize. The show would have 3 doors and the contestant would pick a door. Behind one of the doors was a real prize, and behind the other doors would be a donkey. To make the show interesting, after a contestant had chosen a door (call it door A) but before the prize was revealed, Monty would reveal a donkey behind another door (call it door B), and allow the contestant to change their choice to the last door (call it door C). Monty always knew what was behind each door.

   (a) If Monty always followed this procedure, which strategy (sticking to door A or switching to door C) would most likely yield a good prize for the contestant? You have to prove this to me by computing the probability of a good prize with each strategy.

   (b) Monty secretly knew which door had the donkey. What if he only offered to switch when Door A was a good prize. Now, which strategy is best?

   (c) Suppose Monty applied the following rule,

   $$\text{Offer a switch with probability} = \begin{cases} p_G & \text{if door A is a good prize} \\ p_D & \text{if door A is a donkey} \end{cases}$$

   Sketch the space of possible p_G and p_D combinations and show where it is better to stay, where it is better to switch, and where either strategy is equivalent.

In fact, Monty Hall had a college degree (in pre-Med) and was well aware of the mathematical aspects of the problem. His strategy was closer to the last part of the problem. Estimating p_G and p_D (by carefully observing the show over time) would help. However, he would try to psych contestants out. He would go so far as offering them cash ($100) to change their decision (to stay or switch). In this "real-world" environment the contestant is not an abstract observer but an integral part of a social system. For a given contestant the outcome has more than theoretical interest and Monty was good at working each contestant individually. In other words, what matters are not the abstract probabilities but the probabilities given the contestant. This indicates some of the differences and difficulties of social science research (observer is part of observed system) relative to physical science research (observer is assumed outside the observed system).