CARNEGIE MELLON UNIVERSITY
APPLIED STOCHASTIC PROCESSES
(COURSE 18-751)
HOMEWORK 1

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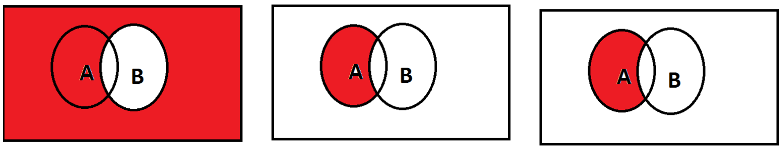
September 3, 2017

Q.1 prove with Venn diagrams

(a) $A \cap B^c = A - B$

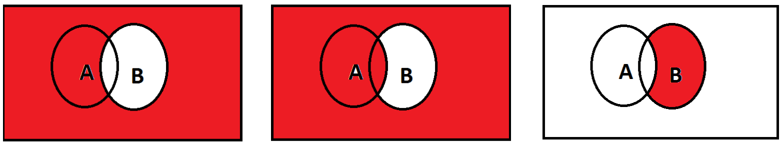


(a) B^c (b) $A \cap B^c$ (c) $A - B$

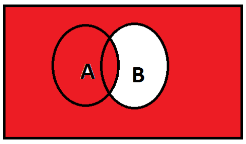
Figure 1

this implies $A \cap B^c = A - B$ is true

(b) $A \cup B^c = (A^c \cap B)^c$



(a) B^c (b) $A \cup B^c$ (c) $A^c \cap B$

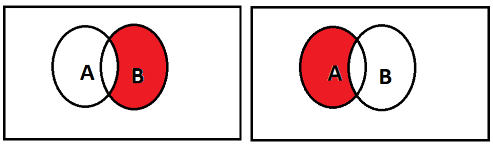


(d) $(A^c \cap B)^c$

Figure 2

hence $A \cup B^c = (A^c \cap B)^c$ is true

(c) $B - A \neq A - B$



(a) $B - A$ (b) $A - B$

Figure 3

therefor $B - A \neq A - B$ is true

Q.2

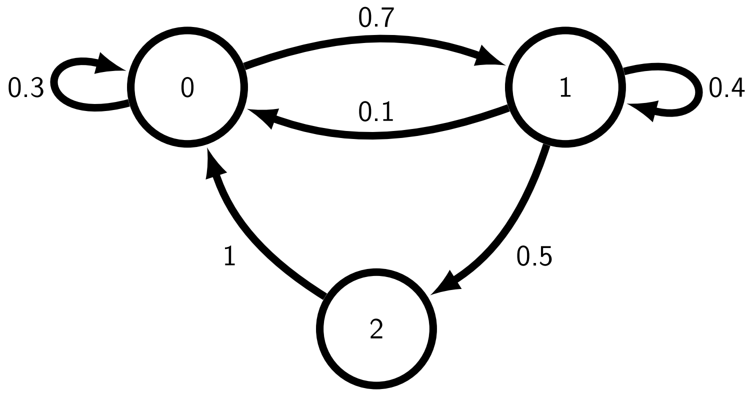


Figure 4: MC

(a) Irreduciblity and periodicity

Irreduciblity

A markov chain is said to be Irreducible if

$$\forall i, j \in S, \exists m < \infty : P(X_{n+m} = j | X_n = i) > 0 \tag{1}$$

i.e regardless the present state we can reach any other state in finit time.[1] On the Markov Chain in figure 4 we can go from any of the states to the other states in a finit number of steps hence it is irreducible.

Periodicity

for an irreducible Markov Chain the periodicity of state i is defined by

$$d(i) = g.c.d\{n \geq 1 | P^n(i, i) > 0\} \tag{2}$$

and $d(i)$ has the same value d for all i. A Markov Chain is said to be aperiodic if $d = 1$ [2]

for the Markov Chain on figure 4 at $n = 1$

$$P^1 = \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.1 & 0.4 & 0.5 \\ 1 & 0 & 0 \end{bmatrix}$$

already we can find i such that $P(i, i) > 0$. For example $P(0, 0) = 0.3 > 0$ and the $g.c.d$ of any set of integers that contains 1 is 1. hence figure 4 is aperiodic.

Ans. **Irreducible** and **Aperiodic**.

(b) invariant distribution (π)

$$\pi_j = \sum_{k=1}^{n} \pi_k p_{kj} \tag{3}$$

$$\pi_0 = \pi_0 P_{00} + \pi_1 P_{10} + \pi_2 P_{20} \tag{4}$$
$$\pi_1 = \pi_0 P_{01} + \pi_1 P_{11} + \pi_2 P_{21} \tag{5}$$
$$\pi_2 = \pi_0 P_{02} + \pi_1 P_{12} + \pi_2 P_{22} \tag{6}$$

i.e $\pi = \pi P$

$$P = \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.1 & 0.4 & 0.5 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\pi_0 = 0.3\pi_0 + 0.1\pi_1 + \pi_2 \tag{7}$$
$$\pi_1 = 0.7\pi_0 + 0.4\pi_1 \tag{8}$$
$$\pi_2 = 0.5\pi_1 \tag{9}$$
$$\pi_0 + \pi_1 + \pi_2 = 1 \tag{10}$$

lets write everything interms of π_1

$$\pi_1 = \pi_1 \tag{11}$$
$$\pi_2 = 0.5\pi_1 \tag{12}$$
$$\pi_0 = 0.3\pi_0 + 0.1\pi_1 + 0.5\pi_1 \tag{13}$$
$$0.7\pi_0 = 0.6\pi_1 \tag{14}$$
$$\pi_0 = \frac{6}{7}\pi_1 \tag{15}$$
$$\pi_0 + \pi_1 + \pi_2 = \frac{6}{7}\pi_1 + \pi_1 + 0.5\pi_1 = 1 \tag{16}$$

$$\frac{33}{14}\pi_1 = 1 \tag{17}$$

$$\pi_1 = \frac{14}{33} \tag{18}$$

$$\pi_0 = \frac{6}{7}.\frac{14}{33} = \frac{4}{11} \tag{19}$$

$$\pi_2 = 0.5.\frac{14}{33} = \frac{7}{33} \tag{20}$$

Ans.

$$\pi = \begin{bmatrix} \frac{4}{11} & \frac{14}{33} & \frac{7}{33} \end{bmatrix}$$

$$\pi \approx \begin{bmatrix} 0.3636 & 0.4242 & 0.2121 \end{bmatrix}$$

(c) Expected Time from 0 to 2

We can calculate $\beta(0)$

i.e
$\beta(0) = E$[average time to reach 2 | current state is 0]

$$\beta(2) = 0 \tag{21}$$

$$\beta(0) = 1 + 0.7\beta(1) + 0.3\beta(0) \tag{22}$$

$$\beta(1) = 1 + 0.1\beta(1) + 0.4\beta(1) + 0.5\beta(2) \tag{23}$$

since $\beta(2) = 0$

$$\beta(1) = 1 + 0.1\beta(0) + 0.4\beta(1)\beta(0) = 1 + 0.3\beta(0) + 0.7\beta(1) \tag{24}$$

$$0.7\beta(0) - 0.7\beta(1) = 1 \tag{25}$$

$$0.6\beta(1) - 0.1\beta(0) = 1 \tag{26}$$

solving the equations (25) and (26) we get $\beta(0) \approx 3.708$ and $\beta(1) \approx 2.286$

Ans. So the expected time from 0 to 2 is 3.708

(d) Probability that starting from 0, the MC has reached 2 after n-steps vs n

The n-step transition probability is defined by [3]

$$r_{ij}(n) = P(X_n = j | X_0 = i) \tag{27}$$

$r_{ij}(n)$ is the probability that the state after n time periods will be j, given that the current state is i.

We can get $r_{ij}(n)$ by using Chapman-Kolmogorov Equation [3]

$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1) p_{kj} \tag{28}$$

for $n > 1$, and all i, j, starting with

$$r_{ij}(1) = P_{ij}$$

[CODE] = Code Appendix 2. d

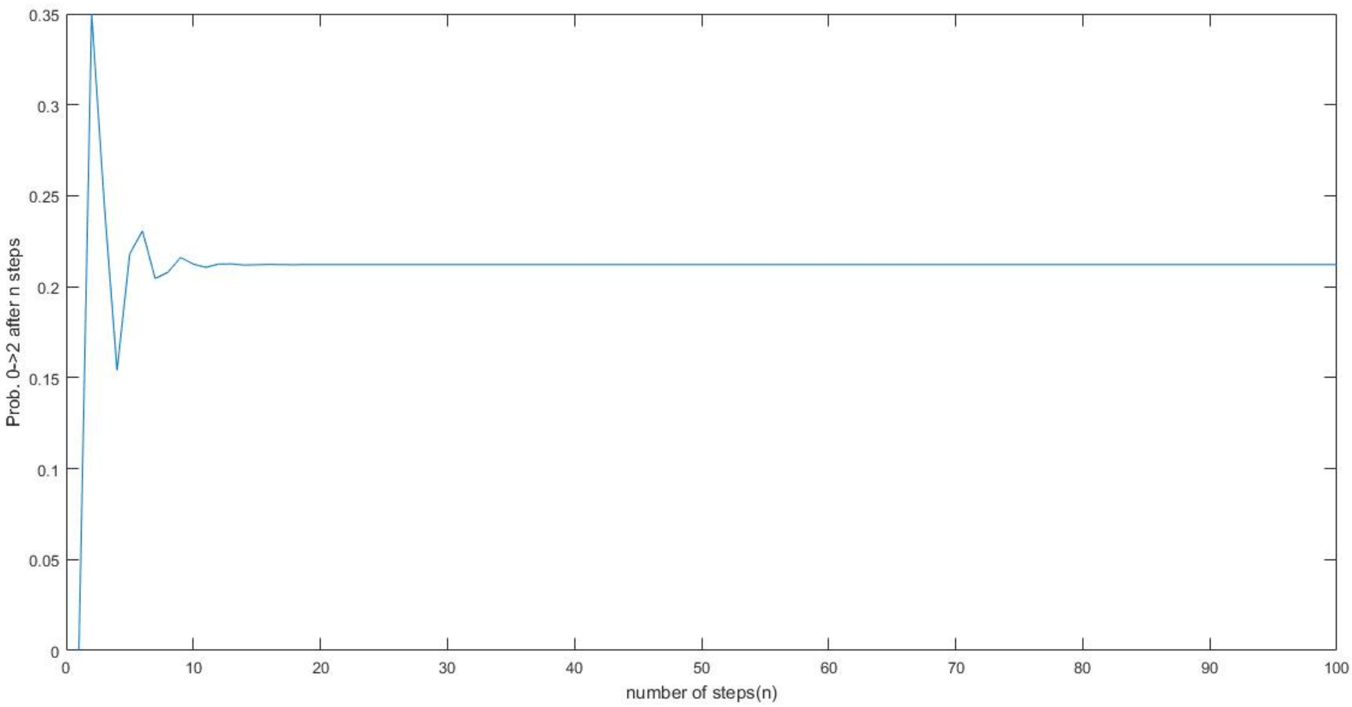


Figure 5: Probability that starting from 0, the MC has reached 2 after n-steps vs n

(e) Fraction of Time

[CODE] = Code Appendix 2. e

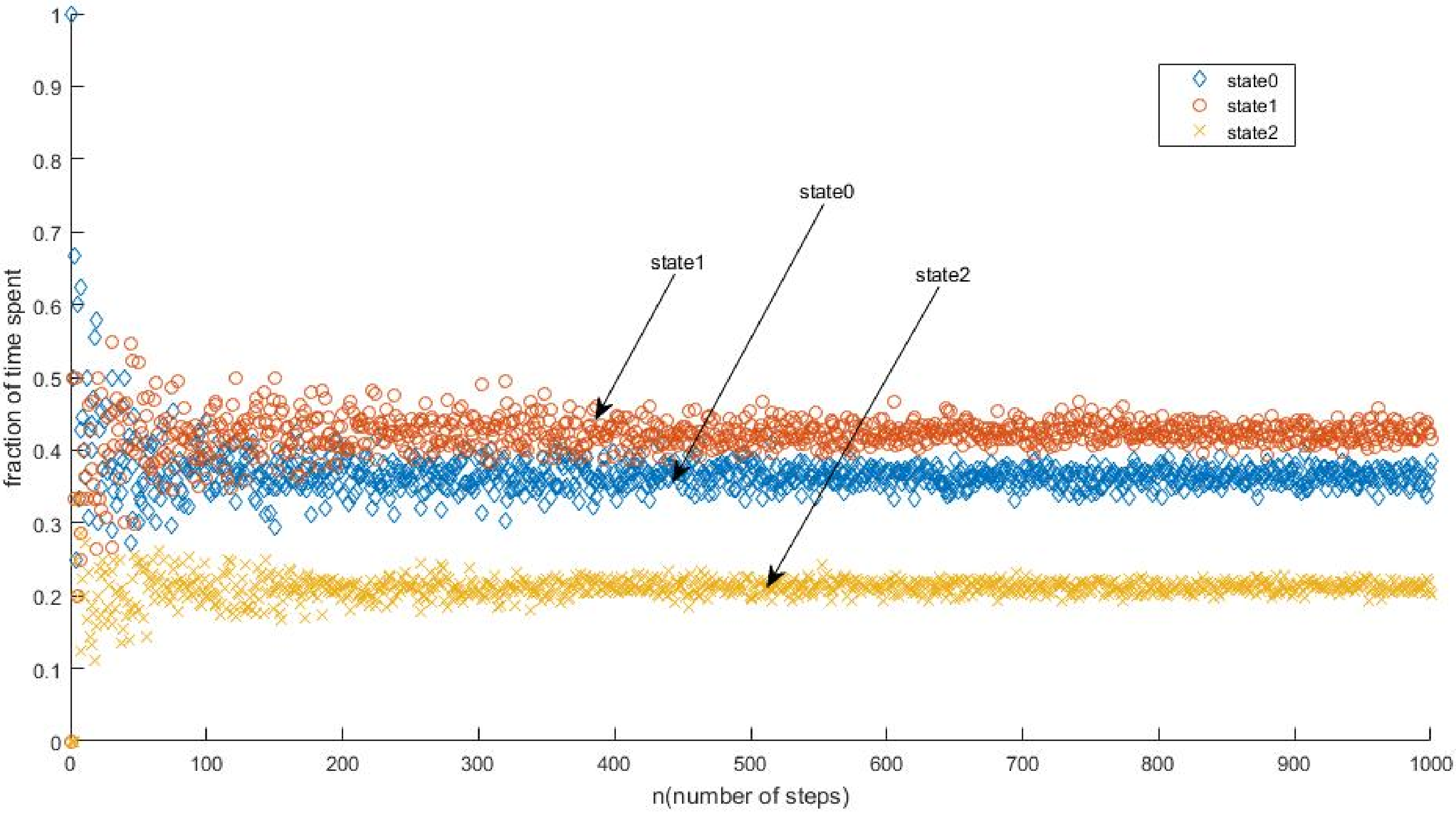


Figure 6: Fraction of time it spends in each of the states vs n

Remark:
On every run we get slightly different outcome just because of the inherent randomness of the Markov chain. But for large n all of the states will converge to the values in the invariant distribution.

(f) π_n vs n

[CODE] = Code Appendix 2. f

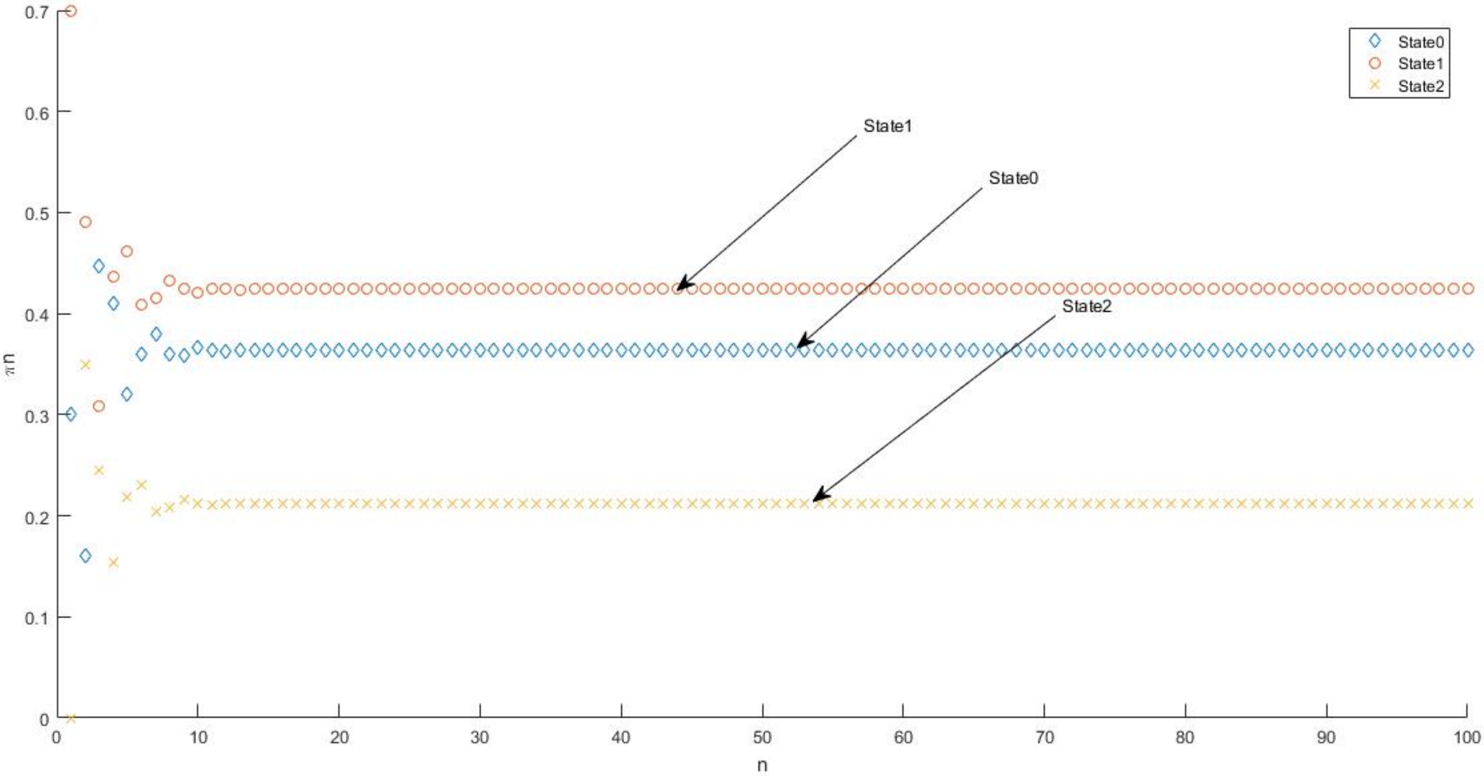


Figure 7: π_n vs n

Q3

(a) Page Rank

We can rank the states by their value in the invariant distribution(π)

1. state(page) 1
2. state(page) 0
3. state(page) 2

(b) Removing self connections and Normalizing

After removing the self connections, there are an infinit number of ways of normalizing the Markov Chain. One way is

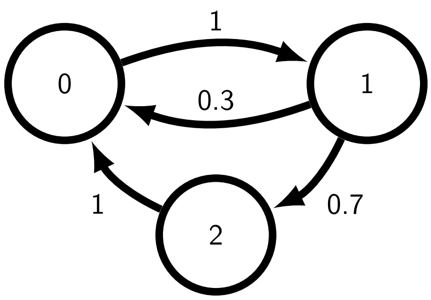


Figure 8: Updated MC

(c) Adding States to Trick the MC

After adding 1a and 0a we get a Markov Chain that looks like figure 9

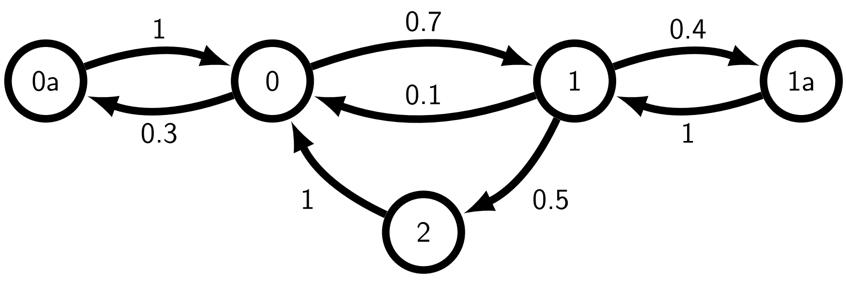


Figure 9: Updated MC

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 & 0 \\ 0 & 0.1 & 0 & 0.5 & 0.4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{align} \pi_{0a} &= 0.3\pi_0 \tag{29} \\ \pi_0 &= \pi_{0a} + 0.1\pi_1 + \pi_2 \tag{30} \\ \pi_1 &= 0.7\pi_0 + \pi_{1a} \tag{31} \\ \pi_2 &= 0.5\pi_1 \tag{32} \\ \pi_{1a} &= 0.4\pi_1 \tag{33} \\ \pi_{0a} + \pi_0 + \pi_1 + \pi_2 + \pi_{1a} &= 1 \tag{34} \end{align}$$

lets write everything interms of π_1

$$\begin{align} \pi_1 &= \pi_1 \tag{35} \\ \pi_2 &= 0.5\pi_1 \tag{36} \\ \pi_{1a} &= 0.4\pi_1 \tag{37} \\ \pi_0 &= 0.3\pi_0 + 0.1\pi_1 + 0.5\pi_1 \tag{38} \\ 0.7\pi_0 &= 0.6\pi_1 \tag{39} \\ \pi_0 &= \frac{6}{7}\pi_1 \tag{40} \\ \pi_{0a} &= 0.3.\frac{6}{7}\pi_1 = \frac{18}{70}\pi_1 \tag{41} \\ \frac{18}{70}\pi_1 + \frac{6}{7}\pi_1 + \pi_1 + \frac{1}{2}\pi_1 + \frac{4}{10}\pi_1 &= 1 \tag{42} \\ \pi_1 &= \frac{70}{211} \approx 0.331 \tag{43} \\ & \tag{44} \end{align}$$

similarly $\pi_{oa} = \frac{18}{70}\pi_1 \approx 0.085$, $\pi_0 = \frac{6}{7}\pi_1 \approx 0.284$, $\pi_2 = 0.5\pi_1 \approx 0.165$, $\pi_{1a} = 0.4\pi_1 \approx 0.132$

$$\pi = \begin{bmatrix} 0.085 & 0.284 & 0.331 & 0.165 & 0.132 \end{bmatrix}$$

We can rank the states(0, 1 and 2) by their value in the invariant distribution(π) form highest to lowest

1. state(page) 1
2. state(page) 0
3. state(page) 2

for the MC in (b)

$$\pi_0 = 0.3\pi_1 + \pi_2 \tag{45}$$

$$\pi_1 = \pi_0 \tag{46}$$

$$\pi_2 = 0.7\pi_1 \tag{47}$$

$$\pi_0 + \pi_1 + \pi_2 = 1 \tag{48}$$

solving the above equations we get

$$\pi = \begin{bmatrix} \frac{10}{27} & \frac{10}{27} & \frac{7}{27} \end{bmatrix}$$

So the ranking for (b) would be
1. state(page) 1 = state(page) 0   i.e page1 and page 2 have the same rank
2. state(page) 2

Remark:
Compared to (a) adding the new states 0a and 1a do not change the ranking. On the other hand in (b), after removing and normalizing it, Pages 0 and 1 will have the same ranking because the exit probabilty from 0 to 1 is 1(i.e if we are at Page 0 the next Page we will look at is guaranteed to be Page 1). In all cases page 2 is ranked last.

Code Appendix

2. d

```
function [R,chap_kol] = chapmanKolmogorov(P,N,start,ending)
len_p = size(P);
R = zeros(len_p(1),len_p(1),N);
R(:,:,1)=P;
for n=2:N
    for i=1:len_p(1)
     for j=1:len_p(1)
        accum=0;
        for k=1:len_p(1)
         accum = accum+R(i,k,n-1)*P(k,j);
        end
        R(i,j,n) = accum;
     end
    end
end
chap_kol = R(start,ending,:);
end
```

```
N = 100;
P = [0.3 0.7 0;0.1 0.4 0.5;1 0 0];
start  = 1;%state 0
ending = 3;%state 2
[R,ch] = chapmanKolmogorov(P,N,start,ending);
ch = ch(:);
plot(ch);
xlabel('number of steps(n)');
ylabel('Prob. 0->2 after n steps')
```

2. e [2] Appendix C.3

```
function T = discrete(P)
Pnorm = [0 P]/sum(P);
Pcum  = cumsum(Pnorm);
R = rand(1);
[~,T] = histc(R,Pcum);
end
```

```
function X = simMC(M,A,P)
X = zeros(1,M);
X(1) = A;
for m=1:M-1
      X(m+1) = discrete(P(X(m),:));
end
end
```

```
function frac_dist = get_frac_dist(state_tans,number_of_states)
frac_dist = zeros(1,number_of_states);
N = size(state_tans);
N = N(2);

for i=1:number_of_states
 frac_dist(i)= sum(state_tans==i);
end
frac_dist = frac_dist/N;
end
```

```
N =1000;
Number_of_states = 3;
start_state = 1;
P = [0.3 0.7 0;0.1 0.4 0.5;1 0 0];
Y = zeros(N,Number_of_states);
simMC(10,start_state,P)
for i=1:N
    Y(i,:)=get_frac_dist(simMC(i,start_state,P),Number_of_states
    );
end

x =(1:N);
scatter(x,Y(:,1),'d');
hold on;
scatter(x,Y(:,2),'o');
scatter(x,Y(:,3),'x');
hold off;
legend('state0','state1','state2');
xlabel('n(number of steps)');
ylabel('fraction of time spent')
```

2. f

```
function pin = PI(pi0,P,n)
    pin = pi0*(P^n);
end
```

```
function mat = PIN(pi0,P,n)
len_pi = size(pi0);

mat  = zeros(n,len_pi(2));
    for i=1:n
        mat(i,:) = PI(pi0,P,i);
    end
end
```

```
N = 100;
P = [0.3 0.7 0;0.1 0.4 0.5;1 0 0];
pi0 = [1 0 0];
pin = PIN(pi0,P,N);
x = [1:N];
scatter(x,pin(:,1),'d');
hold on;
scatter(x,pin(:,2),'o');
scatter(x,pin(:,3),'x');
hold off;
legend('State0','State1','State2');
xlabel('n');
ylabel('\pin')
```

Bibliography

[1] https://pages.dataiku.com/hubfs/Dataiku

[2] Jean Walrand. *Probability in Electrical Engineering and Computer science*. Jean Walrand, 2014.

[3] Dimitri P. Bertsekas and John N. Tsitsiklis. *Introduction to Probability.* LECTURE NOTES Course 6.041-6.431 M.I.T. FALL 2000