1 Kinetics & Kinematics

- $\rightarrow \sum F_x = ma_{g_x}$
- $\rightarrow \sum F_y = ma_{g_y}$
- $\circlearrowleft \sum M_A = I_A \alpha$

General 2D Plane Motion (No Slip)

- $\overline{v}_B = \overline{v}_A + \overline{\omega} \times \overline{r}_{B/A}$
- $\overline{a}_B = \overline{a}_A + \overline{\alpha} \times \overline{r}_{B/A} + \overline{\omega} \times (\overline{\omega} \times \overline{r}_{B/A})$
- $\overline{\omega} \times (\overline{\omega} \times \overline{r}_{B/A}) = -\omega^2 \overline{r}_{B/A}$
- $v_G = \omega \times r$
- $a = \omega^2 r$
- $a = \alpha \times r$
- $s = r\theta$
- $\overline{v}_B = \overline{v}_A + \overline{v}_{B/A}$
- $\overline{a}_B = \overline{a}_A + \overline{a}_{B/A}$
- $\overline{v} = \overline{\omega} \times \overline{r}_p$
- $\alpha_A r_A = \alpha_B r_B$
- $L_{rope} = S_A + S_B$+constants
- $\dot{L} = V_A + V_B = 0, \ddot{L} = a_A + a_B = 0$
- Raising winch-$\dot{L} = -\omega r$

Rotating Coordinate Systems

- $\hat{i} = \cos\theta \hat{I} + \sin\theta \hat{J}$
- $\hat{j} = -\sin\theta \hat{I} + \cos\theta \hat{J}$
- $(\overline{v}_B) = (\overline{v}_A)_{XYZ} + (\overline{v}_{B/A})_{xyz} + \overline{\Omega}_{AB} \times (\overline{r}_{B/A})_{xyz}$
- $(\overline{a}_B)_{XYZ} = (\overline{a}_A)_{XYZ} + (\overline{a}_{B/A})_{xyz} + \dot{\overline{\Omega}}_{AB} \times (\overline{r}_{B/A})_{xyz} - \Omega_{AB}^2 (\overline{r}_{B/A})_{xyz} + 2\overline{\Omega}_{AB} \times (\overline{v}_{B/A})_{xyz}$
- Only useful for cases of constant acceleration.

Horizontal

$$\begin{aligned} v &= v_0 + at \\ \Delta x &= v_0 t + \tfrac{1}{2} a t^2 \\ v^2 &= v_0^2 + 2a\Delta x \end{aligned}$$

Vertical

$$\begin{aligned} v &= v_{0_y} - gt \\ \Delta y &= v_{0_y} t - \tfrac{1}{2} g t^2 \\ v_y^2 &= v_{0_y}^2 - 2g\Delta y \end{aligned}$$

Rotational

$$\begin{aligned} \omega &= \omega_0 + \alpha t \\ \Delta\theta &= \omega_{0_t} t + \tfrac{1}{2} \alpha t^2 \\ \omega^2 &= \omega_0^2 + 2\alpha\Delta\theta \end{aligned}$$

2 S-V-A-T Equations

- Dot notation ($\dot{s}$) indicates time derivative
- **Given function of time**
- $$\begin{aligned} s = s(t) &\rightarrow v = \tfrac{ds}{dt} \\ v = v(t) &\rightarrow a = \tfrac{dv}{dt} \\ & \quad\;\; s = s_0 + \textstyle\int_{t_0}^{t} v(t)dt \\ a = a(t) &\rightarrow v = v_0 + \textstyle\int_{t_0}^{t} a(t)dt \end{aligned}$$
- **Given function of position**
- $$\begin{aligned} t = t(s) &\rightarrow v(s) = 1/(\tfrac{dt}{ds}) \\ v = v(s) &\rightarrow a(s) = v(s)\tfrac{dv}{ds} \\ & \quad\;\; t(s) = t_0 + \textstyle\int_{s_0}^{s} \tfrac{ds}{v(s)} ds \\ a = a(s) &\rightarrow v(s) = \pm\sqrt{v_0^2 + 2\textstyle\int_{s_0}^{s} a(s)ds} \end{aligned}$$
- **Given function of speed**
- $$\begin{aligned} t = t(v) &\rightarrow a(v) = 1/(\tfrac{dt}{dv}) \\ a = a(v) &\rightarrow s(v) = s_0 + \textstyle\int_{v_0}^{v} \tfrac{v}{a(v)} dv \\ & \quad\;\; t(v) = t_0 + \textstyle\int_{v_0}^{v} \tfrac{dv}{a(v)} \\ s = s(v) &\rightarrow a(v) = v/(\tfrac{ds}{dv}) \end{aligned}$$
- $\alpha d\theta = \omega d\omega$

3 Friction

- $F_s \leq F_{f(max)} = \mu_s F_n \quad F_k = \mu_k F_n$
- $F_{static} = -F_{applied}$
- Friction forces are not conservative
- Neglect rolling resistance does not mean neglect friction
- Assumption of no slip requires $F_f \leq \mu_s F_N$
- If $F_f > \mu_s F_N$, slipping occurs

4 Moments & Moment of Inertia

- Counterclockwise is generally positive
- Couple is two equal moments acting in opposite directions that causes rotation
- $\overline{M} = \overline{r} \times \overline{F} = rF\sin\theta$
- $M = |F|d$
- $\hat{\lambda} \cdot \overline{M_o} = M_o \cos\theta$
- Moment about an axis- $M_a = u_a \cdot (r \times F)$
- u_A = unit vector along the axis
- Resulting crossproduct is $|u_x u_y u_z|$ in first row, $|u_a r_x F_x|$ in first row
- $\sum M_{O_{Bar}} = \frac{1}{3} m l^2 \alpha_{Bar}$
- $\sum M_{G_{Bar}} = \frac{1}{12} m l^2 \alpha_{Bar}$
- $\sum M_{G_{Disk}} = \frac{1}{2} m r^2 \alpha_{Disk}$
- $\sum \overline{M}_p = I_G \overline{\alpha} + \overline{r}_{G/P} \times m\overline{a}_G$
- $\sum \overline{M}_O = I_O \overline{\alpha}$
- $I_G = k_G^2 m$
- $I_{Ball} = \frac{2}{5} m r^2$
- $I_{Bar} = \frac{1}{12} m l^2$
- $I_{Bar\ end} = \frac{1}{3} m l^2$
- $\sum \overline{M}_p = I_G \overline{\alpha} + \overline{r}_{G/P} \times m\overline{a}_G$
- $I_O = I_G + (\text{mass})(\text{dist O}\rightarrow\text{CG})^2$
- $m = \frac{w}{g} = \rho V = \rho \int V = \rho \int y t dx$
- $dm = \rho t y dx$
- $I_y = \int x_{el}^2 dA$
- $I_{yG} = I_y - A x_G^2$
- $I_x = \int \frac{1}{12} y^3 dx + \int y_{el}^2 dA$
- $CG_{semicircle} = \frac{4r}{3\pi}$

- $I_{G_{rect}} = \frac{m}{12}(a^2 + b^2)$
- $I_{x_{rect}} = \frac{1}{12}bh^3 + A(distO \to CG)^2$
- $I_{x_{sphere}} = \frac{1}{4}\pi r^4 + A(distO \to CG)^2$
- $I_{x_{rect}} = m[\frac{1}{12}(z^2 + y^2) + (CG_z^2 + CG_y^2)]$
- $I_{x_{cyl}} = m[(\frac{1}{4}r^2 + \frac{1}{12}h^2) + (CG_z^2 + CG_y^2)]$
- $I_x = I_{x_G} + m(y_G^2 + z_G^2)$

5 Work, Energy & Power

- T- Kinetic Energy, V- Potential Energy, U- Work
- U = all work done by external forces
- $T_1 + U_{1\to 2} = T_2 = \int_1^2 \sum \overline{F} \cdot d\overline{r}$
- $T = \frac{1}{2}mv^2 + \frac{1}{2}I_G\omega^2$
- $T = \frac{1}{2}I_O\omega^2$(for translation & rotation about fixed point)
- $V_1 + T_1 = V_2 + T_2$
- $V_{Spring} = \frac{k_s}{2}s^2 = \frac{k_\theta}{2}\theta^2$
- $U_{1\to 2} = Fd = M_G\theta$
- $\int_1^2 \sum \overline{M}_G \cdot d\overline{\theta} = \int_1^2 I_G\overline{\alpha} \cdot d\overline{\theta}$
- $U_{On\ spring} = \frac{k_s}{2}(s_2^2 - s_1^2)$
- $U_{By\ spring} = -\frac{k_s}{2}(s_2^2 - s_1^2)$
- $U_{By\ torsional\ spring} = -\frac{k_\theta}{2}(\Delta\theta_2^2 - \Delta\theta_1^2)$
- $\sum_{i=1}^{n}(U_{1\to 2})_i = \sum_{i=1}^{n}(T_2)_i - \sum_{i=1}^{n}(T_1)_i$
- $P = Fv = M\omega = Fv\cos\theta$
- $P = \overline{F} \cdot \overline{v} + \frac{d\overline{F}}{dt} \cdot d\overline{r} = \overline{M} \cdot \overline{\omega} + \frac{d\overline{M}}{dt} \cdot d\overline{\theta}$
  $= \frac{U_{1\to 2}}{t_2 - t_1}$
- $\epsilon = \frac{P_{out}}{P_{in}}$

6 Impulse & Momentum

- $\frac{\Delta V}{\Delta T}$during collision
- Impulsive- magnitude is a function of the time length of the impulse (i.e. reaction forces, connections)
- Non-impulsive- magnitude does not change as the time length of the impulse changes (i.e. weight, soft spring, constants)
- Linear (can be applied in all three directions)
- $m(\overline{v}_G)_1 + \int_{t_1}^{t_2} \sum \overline{F} dt = m(\overline{v}_G)_2$
- **Angular**
- $I_G\overline{\omega}_1 + \int_{t_1}^{t_2} \sum \overline{M}_G dt = I_G\overline{\omega}_2$
- **Translation with a fixed point**
- $(\overline{H}_p)_1 + \int_{t_1}^{t_2} \sum \overline{M}_p dt = (\overline{H}_p)_2$
- $(\overline{H}_p)_1 = I_G\overline{\omega}_1 + r_{G/p_1} \times m\overline{v}_{G_1}$
- $(\overline{H}_p)_2 = I_G\overline{\omega}_2 + r_{G/p_2} \times m\overline{v}_{G_2}$
- **Special Case**
- $H_O = [I_G + m(r_{G/O})^2]\overline{\omega} = I_O\overline{\omega}$ (fixed axis)
- $F_{avg}\Delta t = \int_{t_1}^{t_2} F dt$

7 Coordinates

N-T Coordinates (normal-tangential)

- $\overline{v} = v\hat{u}_t = \dot{s}\hat{u}_t$
- $\overline{a} = a_t\hat{u}_t + a_n\hat{u}_n = \dot{v}\hat{u}_t + v\dot{\theta}\hat{u}_n \qquad (\dot{\theta} = \frac{v^2}{\rho})$
- $a_t = \frac{dv}{dt} = \dot{v} = \ddot{s} = \alpha r$
- $a_n = \frac{v^2}{\rho} = \omega^2 r$
- $\rho = \frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2}$ (radius of curvature)

Polar Coordinates

- $\overline{r} = r\hat{u}_r$
- $\overline{v} = \dot{r}\hat{u}_r + r\frac{d\hat{u}_r}{dt} = \dot{r}\hat{u}_r + r\dot{\theta}\hat{u}_\theta$
- $v_r = \dot{r}$
- $v_\theta = r\dot{\theta}$
- $\overline{a} = (\ddot{r} - r\dot{\theta}^2)\hat{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{u}_\theta$
- $a_r = (\ddot{r} - r\dot{\theta}^2)$
- $a_\theta = (r\ddot{\theta} + 2\dot{r}\dot{\theta})$
- $\dot{r} = \dot{\theta}\frac{dr}{d\theta}$
- $\ddot{r} = \dot{\theta}^2\frac{d^2r}{d\theta^2} + \ddot{\theta}\frac{dr}{d\theta}$
- For 3D position, add z$\hat{k}$; take time derivatives for v & a

Relating The Systems

- $\tan\psi = \frac{rd\theta}{dr} = \frac{r}{dr/d\theta}$
- ψ =angle between $\hat{u}_t$ and $\hat{u}_r$(or a_r and a_t)
- $\hat{u_r} = (\cos\theta)\hat{\imath} + (\sin\theta)\hat{\jmath}$
- $\hat{u_\theta} = -(\sin\theta)\hat{\imath} + (\cos\theta)\hat{\jmath}$
- $\overline{F} = F_r\hat{u}_r + F_\theta\hat{u}_\theta$

8 Disks

- $\frac{T_2}{T_1} = e^{\mu\beta}$
- β = angle between ropes in radians
- $\cdot$ T_2 is in direction of motion
- $\Sigma M_{pin} = T_2 r + T_1 r + M = 0$

9 Rollling Resistance

- $F_{RR} = mg\cos\beta\frac{\sin\theta}{\cos\theta}$
- $mgdsin\beta - mg\cos\beta\tan\theta d = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$

10 Impacts

- $e = \frac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1}$
- $m_A(v_A)_1 + m_B(v_B)_1 = m_A(v_A)_2 + m_B(v_B)_2$ (when no external impulses)
- Tangent (no friction) - $(v_{BT})_1 = (v_{BT})_2, (v_{AT})_1 = (v_{AT})_2$

11 Center of Gravity

- For uniform thickness/density, centroid = center of gravity
- $\overline{r}_{G/P} = \frac{1}{m}\sum_{i=1}^{n}(r_{i/P}\Delta m_i) = \frac{1}{m}\int \overline{r}_p dm$
- Other forms: $\frac{1}{V}\int \overline{r}_p dV \quad \frac{1}{A}\int \overline{r}_p dA \quad \frac{1}{L}\int \overline{r}_p dL$
- **Common Center of Gravity Formulas:**
- Bar- $dA = hdx$
- $A = \int dA = \int \text{height } dx$
- $V = \int dv = \int Adx$
- Wedge of circle- $dA = \frac{1}{2}r^2 d\theta$
- Rectangular prism- $dV = bhdz$
- Cylinder- $dV = \pi(R_{out}^2 - R_{in}^2)dy$
- Triangle- $x_{cm} = \frac{b}{3}, y_{cm} = \frac{h}{3}$
- Semicircle- $x_{cm} = \frac{4r}{3\pi}$
- Thin semicirclular wire = $\frac{2r}{\pi}$
- Thin wire $\quad dL = \sqrt{\frac{dy}{dx}^2 + 1dx}$
- Volume for a 3D object- $dm = \rho dV, m = \rho V$
- Area for a 2D object (flat plate)- $dm = \rho t dA, m = \rho t A$
- Length for a 1D object (wire)- $dm = \rho A_c dL, m = \rho A_c L$
- $dA = (\text{top curve}-\text{bottom curve})dx \quad x_{el} = x$
- y_{el} =top$-\frac{top-bottom}{2} = \frac{top+bottom}{2}$
- $y_G = \frac{\int \frac{y}{2}dA}{A}$

12 Pulleys

- $L_{rope} = S_A + S_B$+constants
- $\dot{L} = V_A + V_B = 0, \ddot{L} = a_A + a_B = 0$
- Raising winch-$\dot{L} = -\omega r$

13 3D Vector Systems

- $\overline{F} = F\hat{\lambda} = F_x\hat{i} + F_y\hat{j} = F[\frac{F_x}{F}\hat{i} + \frac{F_y}{F}\hat{j}] = F[\lambda_x\hat{i} + \lambda_y\hat{j}]$
- Method 1: $\overline{F} = F\cos\theta_x\hat{i} + F\cos\theta_y\hat{j} + F\cos\theta_z\hat{k}$
- Method 2: $\overline{F} = F\sin\theta_y\hat{i} + F\cos\theta_y\hat{j} + F\sin\theta_y\sin\theta_{xz}\hat{k}$
- Method 3: $\overline{F} = F[\frac{d\hat{i}+h\hat{j}}{\sqrt{d^2+h^2}}]$
- $\lambda_x = \frac{F_x}{F} = \cos\theta_x$, same for y and z

14 Axial, Shear, Bending Moment

- $\frac{dM}{dx} = V, \frac{dV}{dx} = -w$
- Single arrow is resultant force, double arrow is resultant couple

15 General

- 1 hp = $550\frac{ft\cdot lb}{s} = 746W$
- Write directions and draw diagrams for all energy, momentum, etc.
- $F = ks$
- Only one $\overline{\omega}$ and $\overline{\alpha}$ for a rigid body
- Imperial unit of mass- slugs
- Tires on ground generally have x & y forces for friction
- Smooth = no friction
- Couple is written as one object on FBD (units of $N \cdot m$)
- For constant speed, $a_t = 0$
- λ is actually 2 angles- the one you solve for and one that adds 180 degrees
- Banking angle, no slip, no friction- $\theta = tan^{-1}\left(\frac{v^2}{g\rho}\right)$
- $a \cdot ds = v \cdot dv$
- Impending slip- $\mu_s = \mu_{s_{max}}$
- $\overline{A} \cdot \overline{B} = AB\cos\theta$
- Statically determine = same number of unknowns and equations
- Statically indeterminate = more unknowns than equations
- Unstable = no solution
- List whether forces are in tension or compression (for FBD, assume tension)
- Weight of distributed load acts at center
- Always draw free-body diagram and kinetic diagram
- Shear acts in opposite direction of moment
- Constant slip rate- use μ_k