GENERALIZED RULE OF COUNTING

$n_1 \times n_2 \times \cdots \times n_r = \prod_{i=1}^{r} n_i$

PERMUTATIONS

$n \times (n-1) \times \cdots \times 2 \times 1 = n!$

If n objects are alike,

$\frac{n!}{n!n_2!\ldots n_r!}$

NUMBER OF PERMUTATIONS

$n \times (n-1) \times \cdots \times (n-r+1) = \frac{n!}{(n-r)!}$

COMBINATIONS

If we have n items and want to select r of them,

$\binom{n}{r} = \frac{n!}{(n-r)!r!}$

$\binom{n}{0} = 1$

$\binom{n}{1} = n$

$\binom{n}{r} = \binom{n}{n-r}$

$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$

BINOMIAL THEOREM

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

Any event E is a subset of the sample space S.

$E \subset S$

Intersection- $A \cap B$ (both occur)

Union- $A \cup B$ (at least one occurs)

Complement- A^c (A does not occur)

Two events A and B are mutually exclusive/disjoint if they have no outcomes in common, i.e. $A \cap B = \varnothing$

Rules for these:

$A \cup B = B \cup A, A \cap B = B \cap A$

$(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

DEMORGAN's LAWS

$(A \cup B)^c = A^c \cap B^c, (A \cap B)^c = A^c \cup B^c$

$\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} A_i^c$

AXIOMS OF PROBABILITY

Let P be a function that assigns a nonnegative real number to each event E of a sample space S. We call P a probability if:

Axiom 1: non-negative

$0 \leq P(E) \leq 1$

Axiom 2: total one

$P(S) = 1$

Axiom 3: countable addition

$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$ if $E_i \cap E_j = \varnothing$ for $i \neq j$

For k disjoint events $E_1 \ldots E_k$,

$$P\left(\bigcup_{i=1}^{k} E_i\right) = \sum_{i=1}^{k} P(E_i)$$

COMPLEMENT RULE

$P(A^c) = 1 - P(A)$

DIFFERENCE RULE

$P(B \cap A^c) = P(B) - P(A)$, if $A \subseteq B$

INCLUSION-EXCLUSION

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

SAMPLE SPACES WITH EQUALLY LIKELY OUTCOMES

For event E in a sample space S with equally likely outcomes,

$P(E) = \frac{\#(E)}{\#(S)}$

CONDITIONAL PROBABILITY

Given two events A and B with $P(B) > 0$, the conditional probability of A given B has occurred is:

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

$P(A|A) = 1$

$P(A^c|A) = 0$

$P(A^c|B) = 1 - P(A|B)$

MULTIPLICATION RULE

$P(A \cap B) = P(A|B)P(B)$

$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1)P(A_2|A_1)P(A_3|A_1, A_2)\ldots P(A_n|A_1, \ldots A_{n-1})$

LAW OF TOTAL PROBABILITY

For events $A_1, \ldots, A_n$ are disjoint, and $\bigcup_{i=1}^{n} A_i = S$, then for any event B,

$P(B) = P(B|A_1)P(A_1) + \cdots + P(B|A_n)P(A_n)$

BAYES THEOREM

Suppose events $A_1, \ldots, A_n$ are disjoint, and $\bigcup_{i=1}^{n} A_i = S$, with $P(A_i) > 0, i = 1, 2, \ldots, n$. Then for any event B with $P(B) > 0$,

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^{n} P(B|A_j)P(A_j)}, i = 1, \ldots, n$$

$$= \frac{P(B|A_i)P(A_i)}{P(B|A_1)P(A_1) + \cdots + P(B|A_n)P(A_n)}$$

$P(A_i)$ is the prior probability. $P(A_i|B)$ is called posterior probability.

INDEPENDENCE

Events A and B are independent if:

$P(A \cap B) = P(A) \times P(B)$

If A and B are independent, then:

$P(A|B) = \frac{P(A)P(B)}{P(B)} = P(A)$

If A and B are independent, then B and A are independent. A and B^c are also independent, as are A^c and B^c. Independent $\neq$ disjoint.

MUTUAL INDEPENDENCE

Three events A, B, C are mutually independent if:

$P(A \cap B) = P(A)P(B)$

$P(B \cap C) = P(B)P(C)$

$P(A \cap C) = P(A)P(C)$

$P(A \cap B \cap C) = P(A)P(B)P(C)$

If the first three hold true but not the last, then A, B, C are pairwise independent.

RANDOM VARIABLES

A random variable X is a real-valued function on the sample space S.

CUMULATIVE DISTRIBUTION FUNCTION

$F(x) = P(X \leq x), -\infty < x < \infty$

X- random variable

x- real valued number

$F(n) = P(X \leq n) = 1 - (1-p)^n$

$P(a < X \leq b) = F(b) - F(a)$

$P(X < b) \neq P(X \leq b)$

DISCRETE RANDOM VARIABLES

A random variable X that can take on at most a countable number of possible values is a discrete random variable.

$p(x) = P(X = x)$

For a discrete random variable X, there exists a countable sequence $x_1, x_2, \ldots$, so that

P(x$_i$ > 0) for $i = 1, 2, /dots$

$p(x) = 0$ for all other values of x

$\sum\limits_{i=1}^{\infty} p(x_i) = 1$

$F(a) = \sum\limits_{all x \leq a} p(x)$

EXPECTED VALUE

$E[x] = \sum\limits_{x:p(x)>0} x \cdot P(X = x) = \sum\limits_{x:p(x)>0} xp(x)$

$E[aX + b] = aE[x] + b$

$E[aX] = aE[x]$

$E[b] = b$

VARIANCE

$\sigma^2 = \text{Var}[x] = E[(x - E(x))^2] = E[(X - \mu)^2] = E(x^2) - \mu^2 = E(x^2) - (E(x))^2$

$\sigma = \text{SD}[x] = \sqrt{\text{Var}[x]}$

$\text{Var}[x] \geq 0$

$\text{Var}[aX + b] = a^2 \text{Var}[x]$

$\text{Var}[X] + b = \text{Var}[x]$

$\text{Var}[b] = 0$

$\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y]$

DIFFERENCE RULE

$P(B \cap A^c) = P(B) - P(A)$

MISC. INFO / EXTRAS

-X people shaking hands- $\binom{x}{2}$

-Solving for $P(A)$ when disjoint vs. independent:

Disjoint- $P(A) + P(B) = P(A \cup B)$

Independent- $P(A \cap B) = P(A) \cdot P(B) = P(A) + P(B) - P(A \cup B)$

-$E(Y^2) = \text{Var}(Y) + [E(Y)]^2$

-PMF (example of Bernoulli):

$p_x(1) = p, p_x(0) = 1 - p$

$p_y(Y(1)) = p, p_y(Y(0)) = 1 - p$

-CDF (example of Bernoulli):

$F_y(Y) =$

$$\begin{cases} 0 & \text{if} \quad \text{y is less than Y(0)} \\ \text{y(smaller val)} & \text{if} \quad \text{Y(0)} \leq y < Y(1) \\ 1 & \text{if} \quad y \geq Y(1) \end{cases}$$

-Number of different circles formed by N people is $(n-1)!$

-Disjoint = mutually exclusive

-$P(A \cap B^c) = P(A) - P(A \cap B)$

−For every two sets A and B, $A \cap B$ and $A \cap B^c$ are disjoint and $A = (A \cap B) \cup (A \cap B^c)$. In addition, B and $A \cap B^c$ are disjoint and $A \cup B = B \cup (A \cap B^c)$.

-Birthday shared probability- $\frac{365!}{(365-k)!365^k}$

-Other multiplication rule value- $P(A \cap B) = P(A)P(B|A)$

Joint Distributions and Expectations

Joint CDF

$F_{X,Y}(x, y) = P[X \leq x, Y \leq y]$

$F_X(x) = F_{X,Y}(x, \infty)$

$F_Y(y) = F_{X,Y}(\infty, y)$

$P(X > x, Y > y) = 1 - F_X(x) - F_Y(y) + F_{X,Y}(x, y)$

$P(X = x) = \sum\limits_y P(X = x, Y = y) = \sum\limits_y P(X = x|Y = y)P(Y = y)$

$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$

$f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$

Marginal PDF

$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy$

$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx$

Independent Random Variables

$P(X \epsilon A, Y \epsilon B) = P(X \epsilon A)P(Y \epsilon B)$

$F_{X,Y}(x, y) = F_X(x)F_Y(y)$

$p_{X,Y}(x, y) = p_X(x)p_Y(y)$

$f_{X,Y}(x, y) = f_X(x)f_Y(y)$

Sums of Continuous Random Variables

$f_{X+Y}(z) = \int_{-\infty}^{\infty} f(x, z - x) dx = \int_{-\infty}^{\infty} f(z - y, y) dy$

$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x) dx = \int_{-\infty}^{\infty} f_X(z - y) f_Y(y) dy$ (if independent)

F_{X+Y} is called the convolution of F_x and F_Y

Expected Value $\quad E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y) dx dy$

$E(X + Y) = E(X) + E(Y)$

$E(X - Y) = E(X) - E(Y)$

$E(XY) = E(X) \cdot E(Y)$

Covariance

Need sums of independent random variables and all distribution from chart