MATH 13.002J Exam

Problem 1.

$$A = \begin{bmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{bmatrix}$$

a) Compute the inverse A^{-1} and determine the solution to $Ax = b$ when $b = (4\ 3\ 3\ 1)^T$.

b) Assume that the right-hand side b is perturbed by a vector δb such that $\|\delta b\|_\infty \leq 0.01$ Give an upper bound for $\|\delta x\|_\infty$, where δx is the corresponding perturbation in the solution.

c) Compute the condition number $\kappa(A)$ and compare it with the bound for the quotient between $\frac{\|\delta x\|}{\|x\|}$ and $\frac{\|\delta b\|}{\|b\|}$ which can be derived from (b).

a)
$$A^{-1} = \begin{bmatrix} 25 & -41 & 10 & -6 \\ -41 & 68 & -17 & 10 \\ 10 & -17 & 5 & -3 \\ -6 & 10 & -3 & 2 \end{bmatrix}$$

$$x = A^{-1}b = (1, -1, 1, -1)^T$$

b)
$$\|\delta x\|_\infty := \|x - \hat{x}\|_\infty$$
$$\leq \|x\|_\infty \|A\|_\infty \|A^{-1}\|_\infty \frac{\|b - \hat{b}\|_\infty}{\|b\|_\infty}$$
$$= (1)(33)(136)\|\delta b\|_\infty / (4)$$
$$\leq 11.22$$

c)
$$\kappa_\infty = \|A\|_\infty \|A^{-1}\|_\infty = 4488$$

$$\frac{\|\delta x\|_\infty / \|x\|_\infty}{\|\delta b\|_\infty / \|b\|_\infty} = \frac{\|\delta x\|_\infty / 1}{\|\delta b\|_\infty / 4} \approx \frac{11.22}{0.01/4} = 4488.$$

Problem 2.

a. Make algorithms for finding the roots of the equation

$$x\tan(x) = 2\,, \tag{1}$$

in the interval $x \in [0, \pi/2]$, using:

1. Newton-Ralphson Iteration
2. The Secant Method

b. Make a graph of the relative errors vs iteration step for the two algorithms and compare their convergence behavior to that of a *quadratic* convergence.

a) Let $f(x) = x\tan x - 2$
Then $f'(x) = \tan x + x\sec^2 x$
Since $f'(x) \neq 0$, $x \in (0, \pi/2)$, $f(0) \neq 2$ and $\tan \pi/2$ is not defined, we can use Newton-Raphson iteration to solve $f(x^*) = 0$, $x^* \in (0, \pi/2)$.
(pg 71 of Numerical Methods using Matlab, Matthews and Fink).

See Exam.m

b) The convergence is approximately quadratic.