Problem 1.

The definite integral

$$y_n = \int_0^1 \frac{x^n}{x+5} dx, n = 0, 1, \dots \infty$$

can be evaluated by recursion.

1. Show that the recurrence relation for the series is
$$y_n = \frac{1}{n} - 5y_{n-1}$$
2. Determine the starting value y_0
3. Show that $y_n > 0$ and that $y_n \to 0$ for $n \to \infty$.
4. Assume you have a (simple) computer using 3 digits after the decimal point. Calculate the results of the recursion up to $n = 4$. Discuss the results.

1. $$y_n + 5y_{n-1} = \int_0^1 \frac{x^n + 5x^{n-1}}{x+5} dx$$
$$= \int_0^1 \frac{x^{n-1}(x+5)}{x+5} dx$$
$$= \int_0^1 x^{n-1} dx$$
$$= \frac{1}{n} x^n \Big|_0^1 = \frac{1}{n}$$
$$\therefore y_n = \frac{1}{n} - 5y_{n-1}$$

2. $$y_0 = \int_0^1 \frac{1}{x+5} dx$$
$$= \ln|x+5| \Big|_0^1 = \ln(6/5) \approx 0.182$$

3. For $0 < x \leq 1$, $x^n/(x+5) > 0$ and $x^n/(x+5) = 0$ for $x = 0$. So y_n is the definite integral over a nonzero region of a function that is 0 at one pt and otherwise positive.
$\therefore y_n > 0$ for all $n = 0, 1, \dots$

Since $\lim y_n + 5y_{n-1} = \lim \frac{1}{n} = 0$ and $y_n, y_{n-1} > 0$, we must have $\lim y_n = \lim y_{n-1} = 0$ because if $\lim y_n = \lim y_{n-1} = l > 0$, we would have $0 < 6l = 0$.

4. $y_0 = \ln(6/5) \approx .182$

$y_1 = 1 - 5y_0 \approx .090$

$y_2 = 1/2 - 5y_1 \approx .050$

$y_3 = 1/3 - 5y_2 \approx .083$

$y_4 = 1/4 - 5y_3 \approx -.165$

This rounding scheme is too rough since we showed that the true value of y_n is positive for all n. Also we shouldn't see $y_3 > y_2$.

5. Make a computer program that uses the recursion to evaluate the integrals for $n = 0, \ldots 50$ and discuss your results.

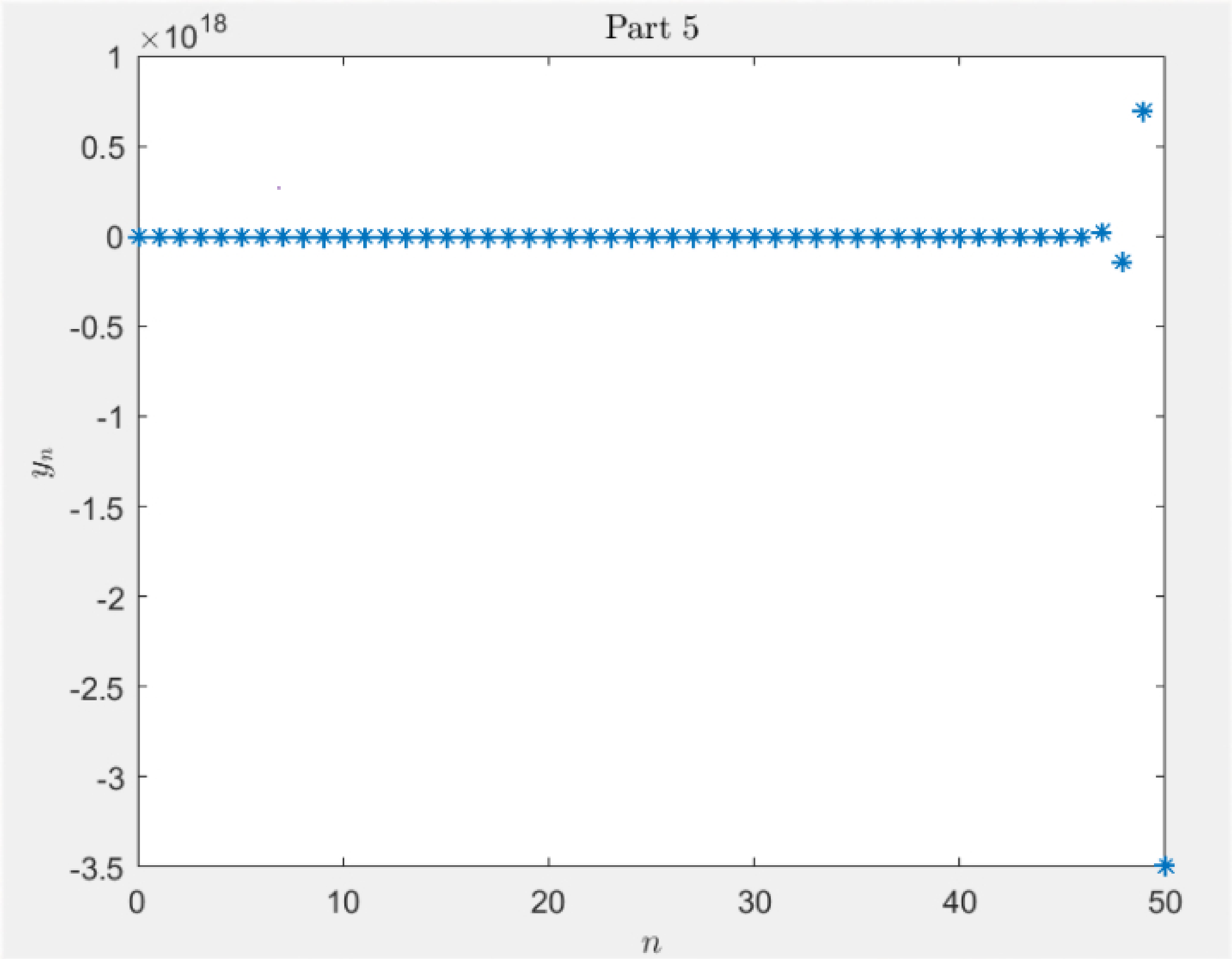
6. Rewrite your program to use 'backward recurrence',

$$\begin{aligned} y_{50} &= 0\,, \\ y_{n-1} &= \frac{1}{5n} - \frac{y_n}{5}\,. \end{aligned}$$

Discuss the results and speculate on what makes the difference.

5. See PS2.m

The iteration approaches 0 quickly but error accumulates a $n \to 50$.



6. See PS2.m

The error is more controlled as $n \to 0$. Instead of scaling any error by a factor of 5 as in the previous scheme, we divide possible error by 5. This allows us to accurately recover y_0 when starting with $y_{50} = 0 = \lim y_n$.