13.002

Introduction to Numerical Methods for Engineers

Problem Set 3

2. a) Factor A into LU and solve $Ax = b$ for the 3 right sides:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

b) Verify that your solutions x_1, x_2, x_3 are the three columns of A^{-1}.
(A times this inverse matix should give the identity matrix.)

a)

$$E_2E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$A = E_1^{-1}E_2^{-1}U = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}U = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}\underset{U}{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$Ly_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \rightarrow y_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$Ux_1 = y_1 \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \rightarrow x_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$Ly_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{array}\right] \rightarrow y_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$Ux_2 = y_2 \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \rightarrow x_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$Ly_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{array}\right] \rightarrow y_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$Ux_3 = y_3 \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \rightarrow x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

b) $A\,[x_1\ x_2\ x_3]$

$$= \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = [x_1\ x_2\ x_3] = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

3. Use Cramer's Rule to solve:

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ -2x_1 + x_2 &= 0 \\ -4x_1 + x_3 &= 0. \end{aligned}$$

Let $B = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$

$|B| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} -2 & 0 \\ -4 & 1 \end{vmatrix} + \begin{vmatrix} -2 & 1 \\ -4 & 0 \end{vmatrix}$

$= 1 + 2 + 4 = 7$

$x_1 = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \Big/ |B| = \dfrac{1}{7}$

$x_2 = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 0 & 0 \\ -4 & 0 & 1 \end{vmatrix} \Big/ |B| = 2/7$

$x_3 = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 0 \end{vmatrix} \Big/ |B| = 4/7$

Check:

$1/7 + 2/7 + 4/7 = 1$ ✓

$-2/7 + 2/7 = 0$ ✓

$-4/7 + 4/7 = 0$ ✓