Problem Set 4

$$\begin{bmatrix} e^{-\alpha} & 1 & 0 \\ -1 & e^{-\alpha} & -1 \\ 1 & -2 & e^{-\alpha} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ e^{-\alpha} \\ e^{-\alpha} \end{Bmatrix}$$

1. Determine the solution in the limits $\alpha \to 0$, $\alpha \to \infty$.

$\alpha = 0$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -6 \end{array}\right]$$

$$x = \begin{bmatrix} 3 \\ -2 \\ 6 \end{bmatrix}$$

$\alpha \to \infty$:

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 1 & -2 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right], \quad x = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

2. Make a set of subroutines (or matlab functions) for the following subtasks associated with solving a general $n \times n$ system of equations:
   - Gaussian elimination without pivoting
   - Back-substitution
3. Make a program (C, Fortran or Matlab) using these subroutines to solve Eq. (1) for $\alpha = [0,\ 5,\ 10,\ 20,\ 40]$ and discuss the behavior of the solution for large α.
4. Modify your subroutines to use partial pivoting and redo the solution for the above series of values of α. Check the solution with the limits determined in Question 1.
5. Suggest a rearrangement of the unknowns which yields a stable solution with your original solver routines without pivoting. Demonstrate the stability using the series of α used above.

2. See PS4.m

3. See PS4.m

For me, solving this system was actually stable up to $\alpha = 709$ but became unstable for $\alpha \geq 710$ (I only tried integer values for α and no α s.t. $709 < \alpha < 710$).

4. By modifying my routine to include partial pivoting, the solution was stable for $\alpha \geq 710$. In fact there was no upper bound on the choice of α while maintaining stability.

5. Since $e^{-\alpha} \to 0$ as $\alpha \to \infty$, the first pivot $e^{-\alpha} \to 0$ for large α. Switching x_1 and x_2 will switch columns 1 and 2 in the system we want to solve. Then we have 1 in the first pivot instead of $e^{-\alpha}$ and this also keeps the second and third pivots away from 0 as well as elimination is performed.