Problem Set 8

4.2.1 $f(x,y) = 4x+6y-12-x^2-y^2$

(a) $(0,0) = \nabla f = (4-2x, 6-2y) \longrightarrow (x,y)=(2,3)$

(b) $\Delta f(2,3) = f(2+h, 3+k) - f(2,3)$

$$= 8+4h+18+6k-12-4-4h-h^2-9-6k-k^2$$
$$-8-18+12+4+9$$
$$= -h^2-k^2 \le 0 \quad \forall h,k$$

Since $\Delta f(2,3)$ is always nonpositive, all points away from $(2,3)$ lead to a decrease in f. $\therefore (2,3)$ is a maximum.

(c) $Hf(2,3) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$

$d_1 = |f_{xx}(2,3)| = -2 < 0$

$d_2 = |Hf(2,3)| = 4 > 0$

Since $d_k < 0$ for odd k and $d_k > 0$ for even k, the second derivative test for local extrema (pg 268) confirms that $(2,3)$ is a local maximum. Since $(2,3)$ is the only critical point and $\Delta f(2,3)$ is always nonpositive, $(2,3)$ must be a global maximum.

4.2.6 $f(x,y) = y^4 - 2xy^2 + x^3 - x$

$(0,0) = \nabla f(x,y) = (-2y^2 + 3x^2 - 1,\ 4y^3 - 4xy)$

$\longrightarrow (x,y) = (\pm 1/\sqrt{3},\ 0),\ (1, \pm 1)$

$$Hf(x,y) = \begin{bmatrix} 6x & -4y \\ -4y & 12y^2 - 4x \end{bmatrix}$$

$Hf(1/\sqrt{3}, 0) = \begin{bmatrix} 6/\sqrt{3} & 0 \\ 0 & -4/\sqrt{3} \end{bmatrix}$ $\begin{matrix} d_1 > 0 \\ d_2 < 0 \end{matrix}$

saddle point

$Hf(1,1) = \begin{bmatrix} 6 & -4 \\ -4 & 8 \end{bmatrix}$ $\begin{matrix} d_1 > 0 \\ d_2 > 0 \end{matrix}$

local minimum

$Hf(-1/\sqrt{3}, 0) = \begin{bmatrix} -6/\sqrt{3} & 0 \\ 0 & 4/\sqrt{3} \end{bmatrix}$ $\begin{matrix} d_1 < 0 \\ d_2 < 0 \end{matrix}$

saddle point

$Hf(1,-1) = \begin{bmatrix} 6 & 4 \\ 4 & 8 \end{bmatrix}$ $\begin{matrix} d_1 > 0 \\ d_2 > 0 \end{matrix}$

local minimum

<u>4.2.8</u> $f(x,y) = e^x \sin y$

$(0,0) = \nabla f = (e^x \sin y, e^x \cos y) \longrightarrow \sin y = 0$ and $\cos y = 0$

There is no $y \in \mathbb{R}$ s.t. both $\sin y = 0$ and $\cos y = 0$. If (x,y) is a critical point of f, we must have $\nabla f(x,y) = (0,0)$.

$\therefore f$ has no critical points

4.2.22

(a) $f(x,y) = kx^2 - 2xy + ky^2$

$$Hf(0,0) = \begin{bmatrix} 2k & -2 \\ -2 & 2k \end{bmatrix} \qquad \begin{aligned} d_1 &= 2k \\ d_2 &= 4(k^2-1) \end{aligned}$$

nondegenerate local minimum iff $d_1, d_2 > 0 \longrightarrow k > 1$

nondegenerate local maximum iff $d_1 < 0,\ d_2 > 0 \longrightarrow k < -1$

(b) $g(x,y,z) = kx^2 + kxz - 2yz - y^2 + \frac{k}{2} z^2$

$$Hg(0,0,0) = \begin{bmatrix} 2k & 0 & k \\ 0 & -2 & -2 \\ k & -2 & k \end{bmatrix} \qquad \begin{aligned} d_1 &= 2k \\ d_2 &= -4k \\ d_3 &= -2k^2 - 8k \end{aligned}$$

nondegenerate local minimum iff $d_1, d_2, d_3 > 0 \longrightarrow$ No k works

nondegenerate local maximum iff $d_1 < 0,\ d_2 > 0,\ d_3 < 0 \longrightarrow k < -4$

4.2.23

(a) $f(x,y) = ax^2 + by^2, \quad a,b \neq 0$

$(0,0) = \nabla f(x,y) = (2ax, 2by) \longrightarrow (x,y) = (0,0)$

$Hf(0,0) = \begin{bmatrix} 2a & 0 \\ 0 & 2b \end{bmatrix}$ $\quad d_1 = 2a, \quad d_2 = 4ab$

local minimum iff $a,b > 0$
local maximum iff $a,b < 0$
saddle point otherwise

(b) $f(x,y,z) = ax^2 + by^2 + cz^2, \quad a,b,c \neq 0$

$\vec{0} = \nabla f(x,y,z) = (2ax, 2by, 2cz) \longrightarrow (x,y,z) = \vec{0}$

$Hf(\vec{0}) = \begin{bmatrix} 2a & 0 & 0 \\ 0 & 2b & 0 \\ 0 & 0 & 2c \end{bmatrix}$ $\quad d_1 = 2a, \quad d_2 = 4ab, \quad d_3 = 8abc$

local minimum iff $a,b,c > 0$
local maximum iff $a,b,c < 0$
saddle point otherwise

(c) $f(x_1,\dots,x_n) = a_1x_1^2 + \dots + a_nx_n^2, \quad a_i \neq 0 \ \forall i \in \{1,\dots,n\}$

$\vec{0} = \nabla f(\vec{x}) = (2a_1x_1,\dots,2a_nx_n) \longrightarrow \vec{x} = \vec{0}$

$Hf(\vec{0}) = \begin{bmatrix} 2a_1 & & \\ & \ddots & \\ & & 2a_n \end{bmatrix}$ $\quad d_1 = 2a_1, \ \vdots \ , \ d_n = 2^n(a_1 \dots a_n)$

local minimum iff $a_i > 0 \ \forall i$
local maximum iff $a_i < 0 \ \forall i$
saddle point otherwise