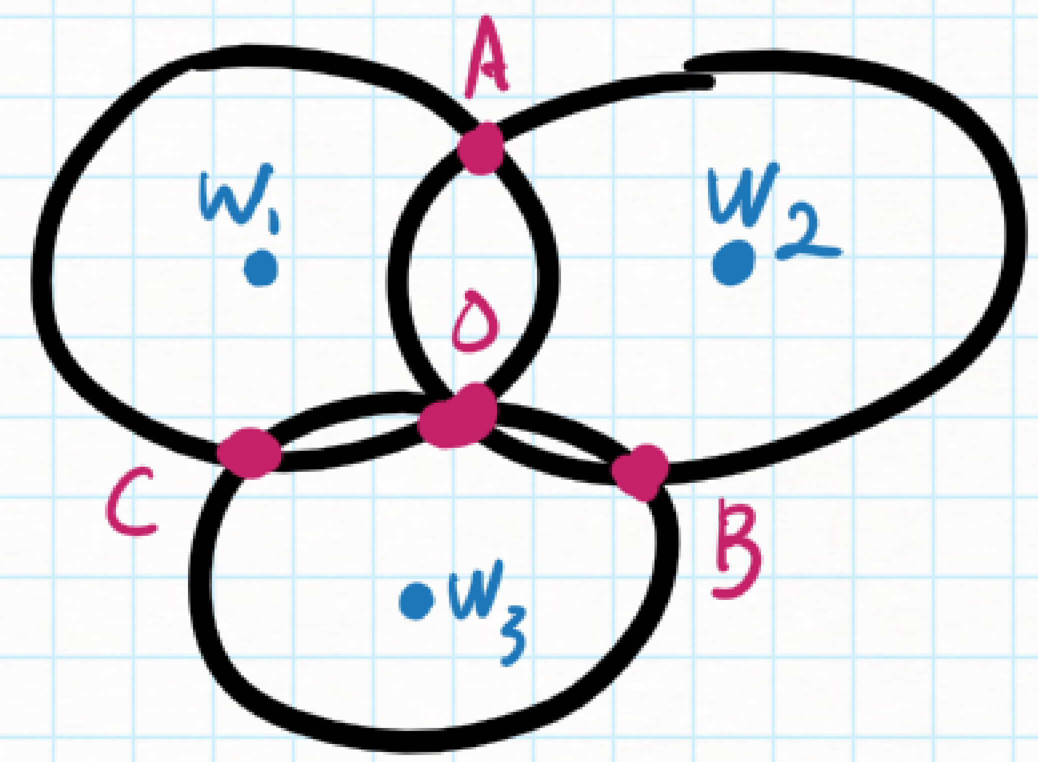
Problem Set 2

1.3.35



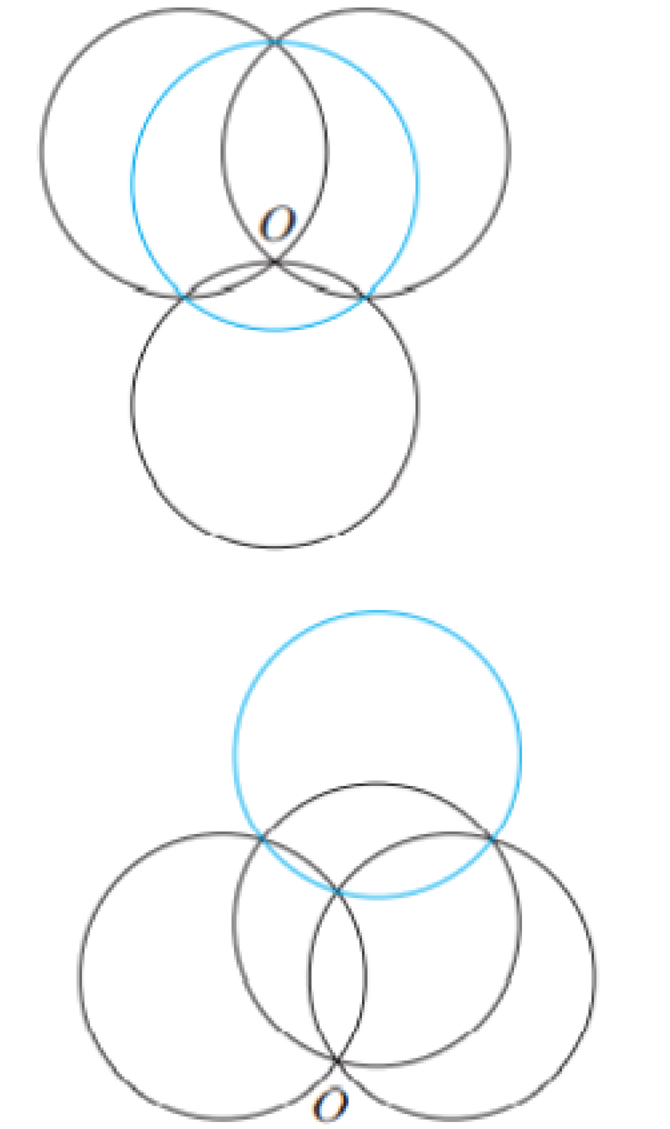


Figure 1.50 Two examples of three circles of equal radius intersecting in a single point O. (See Exercise 35.)

(a) $a = \vec{w}_1 + \vec{w}_2$

$b = \vec{w}_2 + \vec{w}_3$

$c = \vec{w}_1 + \vec{w}_3$

(b) Let $\mathcal{C}$ be the circle passing through A, B, C. The center of $\mathcal{C}$ is the point $P = \vec{w}_1 + \vec{w}_2 + \vec{w}_3$. To see that $\mathcal{C}$ has radius r,

$$|\overrightarrow{PA}| = |\vec{w}_1 + \vec{w}_2 + \vec{w}_3 - \vec{a}|$$

$$= |\vec{w}_1 + \vec{w}_2 + \vec{w}_3 - \vec{w}_1 - \vec{w}_2| = |\vec{w}_3| = r$$

Similarly, $|\overrightarrow{PB}| = |\overrightarrow{PC}| = r$.

2

$$\begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 0 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} = -3 - 2 + 8 = 3 > 0.$$

$\vec{u}, \vec{v}, \vec{w}$ are a right hand set.

3

(a) $\vec{w} = f(\vec{u}) = \begin{pmatrix} u_1\cos\theta + u_2\sin\theta \\ -u_1\sin\theta + u_2\cos\theta \end{pmatrix}$

$$\begin{aligned}\|\vec{w}\|^2 &= u_1^2\cos^2\theta + u_2^2\sin^2\theta + u_1^2\sin^2\theta + u_2^2\cos^2\theta \\ &= u_1^2 + u_2^2 \\ &= \|\vec{u}\|^2\end{aligned}$$

$\therefore \|\vec{w}\| = \|\vec{u}\|$

Let φ denote the angle between $\vec{u}, \vec{w}$.

$$\cos\varphi = \frac{\vec{u}\cdot\vec{w}}{\|\vec{u}\|\,\|\vec{w}\|} = \frac{u_1^2\cos\theta + u_1u_2\sin\theta - u_1u_2\sin\theta + u_2^2\cos\theta}{\|\vec{u}\|\,\|\vec{u}\|}$$

$$= \|\vec{u}\|^2\cos\theta \,/\, \|\vec{u}\|^2 = \cos\theta$$

Since $\cos\varphi = \cos\theta$, $0 \le \varphi, \theta \le \pi$, and cosine is 1-1 on $[0,\pi]$, $\varphi = \theta$. Conclude that $\vec{w}$ is the vector produced by rotating $\vec{u}$ around the origin by an angle of θ.

(b)

$$BC = A(\theta)A(\phi) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & \cos\theta\sin\phi + \sin\theta\cos\phi \\ -(\sin\theta\cos\phi + \cos\theta\sin\phi) & \cos\theta\cos\phi - \sin\theta\sin\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta+\phi) & \sin(\theta+\phi) \\ -\sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix} = A(\theta+\phi).$$

(c) Applying the transformation by C and then B rotates by ϕ and then θ. This is the same as rotating by $\theta + \phi$ all at once.

1.4.11

$A = (1,2,3)$, $B = (4,-2,1)$, $C = (-3,1,0)$, $D = (0,-3,-2)$

Note $\overrightarrow{AB} = \overrightarrow{CD} = (3,-4,-2)$. The area of ABDC is

$$\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \left\| \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -4 & -2 \\ -4 & -1 & -3 \end{vmatrix} \right\| = \| 10\hat{\imath} + 17\hat{\jmath} - 19\hat{k} \| = 5\sqrt{30}.$$

1.4.18

$\vec{a} = 3\hat{\imath} - \hat{\jmath}$, $\vec{b} = -2\hat{\imath} + \hat{k}$, $\vec{c} = \hat{\imath} - 2\hat{\jmath} + 4\hat{k}$. These determine a parallelpiped with volume $|(\vec{a}\times\vec{b})\cdot\vec{c}|$.

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -1 & 0 \\ -2 & 0 & 1 \end{vmatrix} = \langle -1, -3, -2 \rangle$$

$$|(\vec{a}\times\vec{b})\cdot\vec{c}| = |-1+6-8| = 3.$$

1.5.7

$$(x+2) - y + 7(z-1) = 0$$

$x - y + 7z = 5$ is parallel to $x - y + 7z = 10$ and passes through $(-2, 0, 1)$.

1.5.9

Any plane parallel to $5x-3y+2z=10$ must must be perpendicular to $\vec{n} = \langle 5, -3, 2 \rangle$. Using $t=0$ and $t=1$, the parametric eqns produce the vector $\langle 1, -3, 2 \rangle$ and since $\vec{n} \cdot \langle 1, -3, 2 \rangle = 5 + 9 + 4 = 18 \neq 0$, the lines cannot lie in any plane parallel to $5x-3y+2z=10$.

1.5.13

$$x + 2y - 3z = 5 \quad , \quad 5x + 5y - z = 1$$

$$z = 0 \longrightarrow \begin{cases} x + 2y = 5 \\ 5x + 5y = 1 \end{cases} \longrightarrow (x, y, z) = (-23/5, 24/5, 0)$$

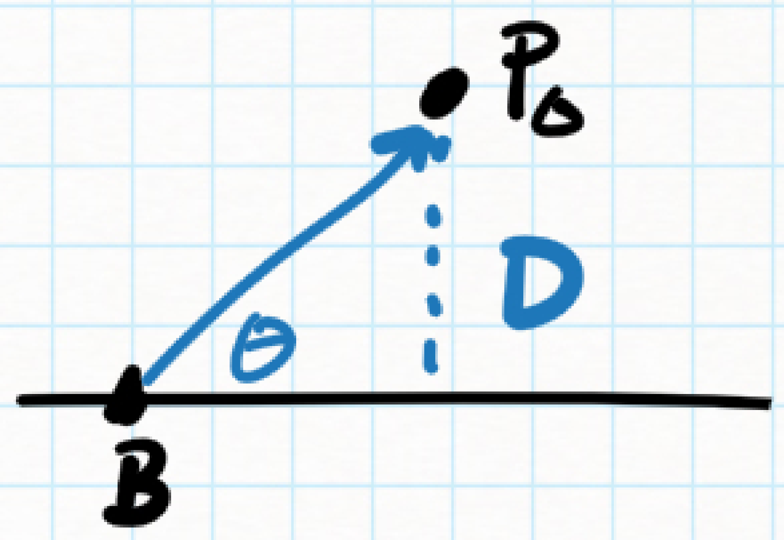
Since the line of intersection lies in both planes it must be perpendicular to both $\vec{n}_1 = \langle 1, 2, -3 \rangle$, $\vec{n}_2 = \langle 5, 5, -1 \rangle$.

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 5 & 5 & -1 \end{vmatrix} = \langle 13, -14, -5 \rangle. \quad \ell = \begin{cases} x = -23/5 + 13t \\ y = 24/5 - 14t \\ z = -5t \end{cases}$$

1.5.24

$P_0 = (1, -2, 3)$, ℓ : $x = 2t - 5$, $y = 3 - t$, $z = 4$

$B = (-5, 3, 4)$, $\vec{a} = \langle 2, -1, 0 \rangle$



$\sin\theta = D / \|\overrightarrow{BP_0}\|$

$$D = \frac{\|\vec{a}\| \|\overrightarrow{BP_0}\| \sin\theta}{\|\vec{a}\|} = \frac{\|\vec{a} \times \overrightarrow{BP_0}\|}{\|\vec{a}\|} = \frac{\sqrt{21}}{\sqrt{5}} = \sqrt{\frac{21}{5}}$$

Alternatively,

$$D = \|\overrightarrow{BP_0} - \text{proj}_{\vec{a}} \overrightarrow{BP_0}\| = \left\|\overrightarrow{BP_0} - \frac{\overrightarrow{BP_0} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a}\right\|$$

$$= \left\| \langle 6, -5, -1 \rangle - \frac{17}{5} \langle 2, -1, 0 \rangle \right\|$$

$$= \| \langle -4, -8, -5 \rangle / 5 \|$$

$$= \sqrt{105}/5 = \sqrt{5}\sqrt{21}/5 = \sqrt{21}/\sqrt{5} = \sqrt{\frac{21}{5}}.$$

1.5.28

$\ell_1(t) = \langle t-7,\ 5t+1,\ 3-2t \rangle \qquad B_1 = \langle -7, 1, 3 \rangle$

$\ell_2(t) = \langle 4t,\ 2-t,\ 8t+1 \rangle \qquad B_2 = \langle 0, 2, 1 \rangle$

$\vec{a}_1 = \langle 1, 5, -2 \rangle \quad , \vec{a}_2 = \langle 4, -1, 8 \rangle$

$\vec{n} = \vec{a}_1 \times \vec{a}_2 = \langle 38, -16, -21 \rangle \quad , \overrightarrow{B_1B_2} = \langle 7, 1, -2 \rangle$

$$D = \|\text{proj}_{\vec{n}} \overrightarrow{B_1B_2}\| = \left\| \frac{\vec{n} \cdot \overrightarrow{B_1B_2}}{\|\vec{n}\|^2} \vec{n} \right\| = \frac{\vec{n} \cdot \overrightarrow{B_1B_2}}{\|\vec{n}\|} = \frac{292}{\sqrt{2141}}$$

1.5.32

$\Pi_1 : 5x - 2y + 2z = 12$, $\Pi_2 : -10x + 4y - 4z = 8$ are two parallel planes.

Take $P_1(0,0,6)$ and $P_2(0,0,-2)$ as points on Π_1 and Π_2 respectively. Then $\overrightarrow{P_1P_2} = \langle 0, 0, -8 \rangle$.

$\vec{n} = \langle 5, -2, 2 \rangle$ is perpendicular to both planes.

$$D = \|\text{proj}_{\vec{n}} \overrightarrow{P_1P_2}\| = \left\| \frac{-16}{\vec{n} \cdot \vec{n}} \vec{n} \right\| = \frac{16}{\|\vec{n}\|} = \frac{16}{\sqrt{33}}$$

1.6.9

For $a, b, c \in \mathbb{R}^n$,

$\|a-b\| = \|(a-c)+(c-b)\| \leq \|a-c\| + \|c-b\|$ by the triangle inequality (pg. 51).

1.6.11

Suppose $\|a+b\| = \|a-b\|$ for $a, b \in \mathbb{R}$. Then,

$$\|a+b\|^2 = \|a-b\|^2$$

$$(a+b)\cdot(a+b) = (a-b)\cdot(a-b)$$

$$a\cdot a + 2a\cdot b + b\cdot b = a\cdot a - 2a\cdot b + b\cdot b$$

$$a\cdot b = -a\cdot b$$

$$2a\cdot b = 0$$

$$a\cdot b = 0$$

Since $a\cdot b = 0$, a and b are orthogonal.

1.6.14

Half of the friends inventory is

$\frac{1}{2}(30, 16, 20, 28)\cdot(10, 10, 12, 15) = \$560$ ← the friend should not trade.

Half of my inventory is not worth as much:

$\frac{1}{2}(20, 30, 24, 20)\cdot(8, 10, 12, 15) = \$524$