Problem Set 3

1 $D=\{(r,\theta,z) \mid -\pi/2 \le \theta \le \pi/2,\ 0 \le r \le 2a\cos\theta,\ -1 \le z \le 3\}$

2 $x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$, $z=\rho\cos\phi$

$D=\{(\rho,\theta,\phi) \mid 0 \le \rho \le 2a,\ -a \le \rho\sin\phi\cos\theta \le a\}$

3 $f: A \to B$, $g: B \to C$, $g \circ f : A \to C$

(i) True. Suppose f, g are surjective. If $c \in C$, $\exists$ $b \in B$ s.t. $g(b)=c$ and $\exists$ $a \in A$ s.t. $f(a)=b$. Thus $(g\circ f)(a) = g(f(a)) = g(b) = c$.

(ii) False. Let $A=B=\mathbb{R}$, $C=\{1\}$, $f(x)=1$, $g(x)=x$. Then since $(g\circ f)(x) = g(1) = 1$, $g\circ f$ is surjective yet $f: A \to B$ is not.

(iii) True. Suppose $g\circ f$ is surjective and $c \in C$. Then $\exists$ $a \in A$ s.t. $(g\circ f)(a) = g(f(a)) = c$. But since $f(a) \in B$, this means there is an element of B that g maps to c.

4 $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x) = \begin{cases} c, & x \in S \\ c-1, & x \notin S \end{cases}$. Then $f^{-1}(c)=S$, $f^{-1}(c-1)=S^c$, $\underset{(k \neq c, c-1)}{f^{-1}(k)} = \emptyset$

2.1.38 $x^2+xy-xz=2$

(a) $F(x,y,z)=\begin{cases}1, & x^2+xy-xz=2\\ 0, & x^2+xy-xz\neq 2\end{cases}$

Then $x^2+xy-xz=2$ is the level set of $F(x,y,z)=1$.

(b) $z=f(x,y)=\dfrac{x^2+xy-2}{x}=x+y-2, \quad x\neq 0$

2.2.9

$$\lim_{(x,y)\to(0,0)}\frac{(x+y)^2}{x^2+y^2} \text{ DNE.}$$

Along $y=x$, $\displaystyle\lim_{(x,y)\to(0,0)}\frac{(x+y)^2}{x^2+y^2}=\lim_{x\to 0}\frac{4x^2}{2x^2}=2.$

Along $y=-x$, $\displaystyle\lim_{(x,y)\to(0,0)}\frac{(x+y)^2}{x^2+y^2}=\lim_{x\to 0}\frac{0^2}{2x^2}=0.$

2.2.11

$\displaystyle\lim_{(x,y)\to(0,0)}\frac{2x^2+y^2}{x^2+y^2}$ DNE. Along $x=0$, $\displaystyle\lim_{(x,y)\to(0,0)}\frac{2x^2+y^2}{x^2+y^2}=\lim_{y\to 0}\frac{y^2}{y^2}=1.$

Along $y=0$ $\displaystyle\lim_{(x,y)\to(0,0)}\frac{2x^2+y^2}{x^2+y^2}=\lim_{x\to 0}\frac{2x^2}{x^2}=2.$

2.2.13

$$\lim_{(x,y)\to(0,0)} \frac{x^2+2xy+y^2}{x+y} = \lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x+y} = \lim_{(x,y)\to(0,0)} x+y = 0.$$

2.2.15

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)(x^2+y^2)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} x^2-y^2 = 0.$$

2.2.35

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} = \lim_{\rho\to 0} \frac{\rho^3 \sin^2\phi \cos\phi \sin\theta \cos\theta}{\rho^2} = 0$$

2.2.46

$$g(x,y) = \begin{cases} \dfrac{x^3+xy^2+2x^2+2y^2}{x^2+y^2} & , (x,y) \neq (0,0) \\ c & , (x,y) = (0,0) \end{cases}$$

g is continuous for $(x,y)\neq(0,0)$ since rational functions are continuous where defined. To make g continuous at $(0,0)$ set

$$c = \lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} \frac{(x+2)(x^2+y^2)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} x+2 = 2.$$