Problem Set 12

1 $\vec{F}:\mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $\vec{F}(x,y,z) = ay^2\hat{\imath} + 2y(x+z)\hat{\jmath} + (by^2+z^2)\hat{k}$

(i) $\vec{F}$ is conservative iff $\nabla \times \vec{F} = \vec{0}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ay^2 & 2y(x+z) & by^2+z^2 \end{vmatrix} = (2by - 2y,\ 0,\ 2y - 2ay)$$

$$\nabla \times \vec{F} = \vec{0} \rightarrow a = b = 1$$

(ii) $f:\mathbb{R}^3 \longrightarrow \mathbb{R}$, $\vec{F} = \nabla f$

$f_x = y^2$, $f_y = 2xy + 2yz$, $f_z = y^2 + z^2 \longrightarrow f(x,y,z) = xy^2 + y^2z + z^3/3$ is a potential

(iii) Since $\vec{F}$ is conservative with potential function f, $\int_C \vec{F}\cdot d\vec{s} = f(b) - f(a)$, where $a, b \in \mathbb{R}^3$ are the endpoints of a smooth oriented curve lying on the surface S. Let S be the surface in $\mathbb{R}^3$ defined by $f(x,y,z) = c$ for some constant $c \in \mathbb{R}$. Then $\int_C \vec{F}\cdot d\vec{s} = f(b) - f(a) = c - c = 0$. That is, for $S = \{(x,y,z) \mid xy^2 + y^2z + z^3/3 = c\}$ we have $\int_C \vec{F}\cdot d\vec{s}$ for any smooth oriented curve lying on S.

2 S is the rectangle with vertices $(0,0,0), (1,0,0), (1,2,2), (0,2,2)$.
$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$, $\vec{F}(x,y,z) = y^2\hat{\imath} - z^2\hat{\jmath} + x^2\hat{k}$
$\hat{n}$ is the unit normal vector for which $\hat{n}\cdot\hat{k} > 0$.

To parametrize the rectangle, let $\vec{X}(s,t) = (s,t,t)$, $0 \le s \le 1$, $0 \le t \le 2$.

$\vec{N} = \vec{X}_s \times \vec{X}_t = (0,-1,1)$

Flux $= \iint_S \vec{F}\cdot d\vec{S} = \iint_{\vec{X}} \vec{F}(\vec{X}(s,t))\cdot\vec{n}\, dS = \int_0^1\int_0^2 (t^2, -t^2, s^2)\cdot(0,-1,1)\,dt\,ds = 10/3$

6.3.24 $\vec{F} = (3x^2 + 3y^2 z \sin xz)\hat{\imath} + (ay\cos xz + bz)\hat{\jmath} + (3xy^2 \sin xz + 5y)\hat{k}$

$\vec{F}$ is conservative iff $\nabla \times \vec{F} = \vec{0}$.

$$\vec{0} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2+3y^2 z\sin xz & ay\cos xz + bz & 3xy^2\sin xz + 5y \end{vmatrix}$$

$$= (6xy\sin xz + 5 + axy\sin xz - b,\ 0,\ ayz\sin xz - 6yz\sin xz)$$

$\longrightarrow a = -6,\ b = 5$

7.1.4 $\vec{X}(s,t) = (s^2\cos t, s^2\sin t, s)$, $-3 \le s \le 3$, $0 \le t \le 2\pi$

(a)
$$\vec{X}_s \times \vec{X}_t = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2s\cos t & 2s\sin t & 1 \\ -s^2\sin t & s^2\cos t & 0 \end{vmatrix} = (-s^2\cos t, -s^2\sin t, 2s^3)$$

$\vec{X}_s \times \vec{X}_t(-1,0) = (-1, 0, -2)$ is normal to $\vec{X}$ at $(-1,0)$.

(b) $0 = (x-1, y, z+1)\cdot(-1,0,-2) = -(x-1) - 2(z+1)$ is a plane tangent to $\vec{X}$ when $(s,t) = (-1,0)$ and $\vec{X}(-1,0) = (1,0,-1)$. Simplified this is $x + 2z = -1$.

(c) If we let $x = s^2\cos t$, $y = s^2\sin t$, $z = s$, then the points on $\vec{X}$ satisfy $F(x,y,z) = x^2 + y^2 - z^4 = 0$.

7.1.24 $\vec{X}: D \to \mathbb{R}^3$, $\vec{X}(r,\theta) = (r\cos\theta, r\sin\theta, \theta)$, $D = \{(r,\theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi n\}$, $n \in \mathbb{N}$

$$\vec{X}_r \times \vec{X}_\theta = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 1 \end{vmatrix} = (\sin\theta, -\cos\theta, r)$$

The surface area of the helicoid parameterized by $\vec{X}$ is

$$\iint_D \|\vec{X}_r \times \vec{X}_\theta\| \, dA = \int_0^{2n\pi}\int_0^1 \sqrt{1+r^2}\, dr\, d\theta$$

$$= 2n\pi \int_0^{\operatorname{arcsinh}(1)} \cosh^2 t \, dt$$

$$= n\pi \int_0^{\operatorname{arcsinh}(1)} (1+\cosh 2t)\, dt$$

$$= n\pi \left(t + \tfrac{1}{2}\sinh 2t\right)\Big|_0^{\operatorname{arcsinh}(1)}$$

$$= n\pi \left(\operatorname{arcsinh}(1) + \tfrac{1}{2} 2 \cdot 1\sqrt{1^2+1}\right)$$

$$= n\pi \left(\operatorname{arcsinh}(1) + \sqrt{2}\right)$$

7.2.13 S is the closed cylinder formed by $S_1: x^2+y^2=9$, $S_2: z=0$, $S_3: z=4$. Orient S with outward normals. Parameterize as

$S_1: (3\cos\theta, 3\sin\theta, z)$, $0 \le \theta \le 2\pi$, $0 \le z \le 4$ $\quad$ $S_2: (r\cos\theta, r\sin\theta, 0)$, $0 \le r \le 3$, $0 \le \theta \le 2\pi$ $\quad$ $S_3: (r\cos\theta, r\sin\theta, 4)$, $0 \le r \le 3$, $0 \le \theta \le 2\pi$

$\frac{\partial S_1}{\partial \theta} \times \frac{\partial S_1}{\partial z} = (3\cos\theta, 3\sin\theta, 0)$ $\quad$ $\frac{\partial S_2}{\partial r} \times \frac{\partial S_2}{\partial \theta} = (0,0,r)$ $\quad$ $\frac{\partial S_3}{\partial r} \times \frac{\partial S_3}{\partial \theta} = (0,0,r)$

$$\iint_{S_1} x^2\,dS = \int_0^{2\pi}\int_0^4 9\cos^2\theta\,\|(3\cos\theta, 3\sin\theta, 0)\|\,dz\,d\theta$$

$$\iint_{S_2} x^2\,dS = \iint_{S_3} x^2\,dS = \int_0^{2\pi}\int_0^3 r^2\cos^2\theta\,\|(0,0,r)\|\,dr\,d\theta = \frac{81\pi}{4}$$

$$\iint_S x^2\,dS = \iint_{S_1} x^2\,dS + \iint_{S_2} x^2\,dS + \iint_{S_3} x^2\,dS = \frac{297\pi}{2}$$

7.2.17 Let S be the same as in exercise 7.2.13.

$$\iint_S (-y\hat{\imath} + x\hat{\jmath})\cdot d\vec{S} = \iint_{S_1} (-3\sin\theta, 3\cos\theta, z)\cdot(3\cos\theta, 3\sin\theta, 0)\,dS$$
$$+ \iint_{S_2} (-r\sin\theta, r\cos\theta, 0)\cdot(0,0,r)\,dS$$
$$+ \iint_{S_3} (-r\sin\theta, r\cos\theta, 4)\cdot(0,0,r)\,dS$$
$$= 0$$

using $\iint_{\vec{X}} \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{X}(s,t))\cdot\vec{N}(s,t)\,dS$

7.3.11 $S: y = 10 - x^2 - z^2$, $y \geq 1$, oriented with rightward pointing normal

$\vec{F} = (2xyz + 5z)\hat{\imath} + e^x\cos yz\,\hat{\jmath} + x^2y\hat{k}$

Let D be the disk $x^2+z^2=9$, $y=1$, which has $\vec{n} = \hat{\jmath}$.

$$\iint_S \nabla\times\vec{F}\,d\vec{S} = \oint_{\partial S} \vec{F}\cdot d\vec{s} = \iint_D \nabla\times\vec{F}\,d\vec{S} = \iint_D (\nabla\times\vec{F})\cdot\hat{\jmath}\,dS = \iint_D 5\,dS = 5\cdot 9\pi = 45\pi$$