Problem Set 5

1 Let $f:\mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = [x^2+y^2-1,\ y^2-x^2(x+1)]^T$

Then the points at which the curves $x^2+y^2=1$ and $x^2(x+1)=y^2$ intersect are the points at which $f(x,y)=(0,0)$.

$$Df(x,y) = \begin{bmatrix} 2x & 2y \\ -3x^2-2x & 2y \end{bmatrix} \quad \det(Df(x,y)) = 2xy(3x+4)$$

$$Df(x,y)^{-1} = \frac{1}{2xy(3x+4)} \begin{bmatrix} 2y & -2y \\ 3x^2+2x & 2x \end{bmatrix} \qquad \begin{matrix} x, y \neq 0 \\ x \neq -4/3 \end{matrix}$$

$$Df(x,y)^{-1} f(x,y) = \frac{1}{2xy(3x+4)} \begin{bmatrix} 4x^2 - 2y + 2x^3 y \\ x^4 + 4xy^2 + 3x^2y^2 - 2x - 3x^2 \end{bmatrix}$$

Newton's Method: If $\vec{x}_k = \vec{x}_{k-1} - Df(\vec{x}_{k-1})^{-1} f(\vec{x}_{k-1})$ converges, it converges to $\vec{x}$ s.t. $f(\vec{x}) = \vec{0}$. $(\vec{x} = \vec{x} - Df(\vec{x})^{-1} f(\vec{x}) \Rightarrow f(\vec{x}) = 0)$.

$$\therefore (x_{k+1}, y_{k+1}) = (x_k, y_k) - \frac{1}{2x_k y_k (3x_k+4)} \begin{bmatrix} 4x_k^2 - 2y_k + 2x_k^3 y_k \\ x_k^4 + 4x_k y_k^2 + 3x_k^2 y_k^2 - 2x_k - 3x_k^2 \end{bmatrix}$$

2 $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \cos(xy) + x^3 + y^2$

$$Df(x,y) = [-y\sin(xy) + 3x^2 \;,\; -x\sin(xy) + 2y] = \nabla f$$

Let $g:\mathbb{R}^2 \to \mathbb{R}^2$, $g(x,y) = Df(x,y)^T$ so that the critical points of f are the zeros of g. Use Newton's Method.

$$Dg(x,y) = \begin{bmatrix} -y^2\cos(xy) + 6x & -xy\cos(xy) - \sin(xy) \\ -xy\cos(xy) - \sin(xy) & -x^2\cos(xy) + 2 \end{bmatrix}$$

$$d = \det(Dg(x,y)) = -\cos(xy)(6x^3 + 2y^2) + 12x - xy\sin(2xy) - \sin^2(xy)$$

$$Dg(x,y)^{-1} g(x,y) = \frac{1}{d(x,y)} \begin{bmatrix} 2 - x^2\cos(xy) & xy\cos(xy) + \sin(xy) \\ xy\cos(xy) + \sin(xy) & 6x - y^2\cos(xy) \end{bmatrix} \begin{bmatrix} 3x^2 - y\sin xy \\ 2y - x\sin xy \end{bmatrix}$$

$$= \frac{1}{d(x,y)} \begin{bmatrix} 6x^2 - 3x^4\cos xy + 2xy^2\cos xy - x\sin^2 xy \\ 3x^3 y\cos xy - 3x^2\sin xy - 2y^3\cos xy + 12xy - y\sin^2 xy \end{bmatrix}$$

$$=: [X(x,y), Y(x,y)]^T.$$ Recursion: $\vec{x}_{k+1} = \vec{x}_k - \begin{bmatrix} X(x_k, y_k) \\ Y(x_k, y_k) \end{bmatrix}$

3 $f(-2,1)=(1,3)$ $Df(-2,1)=\begin{bmatrix}-2 & 3\\ -1 & 1\end{bmatrix}$

$g:\mathbb{R}^2\to\mathbb{R}$, $g(y_1,y_2)=y_1^2-y_2^2$

(a) $g\circ f:\mathbb{R}^2\to\mathbb{R}$ is differentiable at $(-2,1)$ by Theorem 5.3 (Chain Rule) since f is differentiable at $(-2,1)$ and g is differentiable at $f(-2,1)$. By the Chain Rule:

(b) $$D(g\circ f)(-2,1)=Dg(1,3)Df(-2,1)=\begin{bmatrix}2 & -6\end{bmatrix}\begin{bmatrix}-2 & 3\\ -1 & 1\end{bmatrix}=\begin{bmatrix}2 & 0\end{bmatrix}$$

4 $F:\mathbb{R}^3\to\mathbb{R}^2$, $F\in C^1$

$F(4,-1,2)=(0,0)$ $DF(4,-1,2)=\begin{pmatrix}1 & -1 & 4\\ 0 & 1 & -1\end{pmatrix}$

(a)

THEOREM 6.5 (THE IMPLICIT FUNCTION THEOREM) Let $F\colon X\subseteq\mathbf{R}^n\to\mathbf{R}$ be of class C^1 and let $\mathbf{a}$ be a point of the level set $S=\{\mathbf{x}\in\mathbf{R}^n \mid F(\mathbf{x})=c\}$. If $F_{x_n}(\mathbf{a})\neq 0$, then there is a neighborhood U of $(a_1,a_2,\ldots,a_{n-1})$ in $\mathbf{R}^{n-1}$, a neighborhood V of a_n in $\mathbf{R}$, and a function $f\colon U\subseteq\mathbf{R}^{n-1}\to V$ of class C^1 such that if $(x_1,x_2,\ldots,x_{n-1})\in U$ and $x_n\in V$ satisfy $F(x_1,x_2,\ldots,x_n)=c$ (i.e., $(x_1,x_2,\ldots,x_n)\in S$), then $x_n=f(x_1,x_2,\ldots,x_{n-1})$.

Since the submatrix $\begin{pmatrix}-1 & 4\\ 1 & -1\end{pmatrix}$ is invertible the existence of such a function f follows from the Implicit Function Theorem.

(b) Let $g: \mathbb{R} \to \mathbb{R}^2$, $g(x) = F(x, f(x))$ so $g(x) = (0,0)$ $\forall x \in U$

and $dg_1/dx = 0$, $dg_2/dx = 0$. But by the Chain Rule,

$$\frac{dg_1}{dx} = \frac{\partial F_1}{\partial x}\frac{dx}{dx} + \frac{\partial F_1}{\partial y_1}\frac{df_1(x)}{dx} + \frac{\partial F_1}{\partial y_2}\frac{df_2(x)}{dx}$$

$$\frac{dg_1}{dx}(4,-1,2) = 1 - \frac{df_1(4)}{dx} + 4\frac{df_2(4)}{dx} = 0$$

Similarly, $\frac{dg_2}{dx}(4,-1,2) = 0 + \frac{df_1(4)}{dx} - \frac{df_2(4)}{dx} = 0$

$$1 = \frac{df_1(4)}{dx} - 4\frac{df_2(4)}{dx}$$

$$0 = \frac{df_1(4)}{dx} - \frac{df_2(4)}{dx}$$

$\longrightarrow$ $\frac{df_1(4)}{dx} = 1/3$, $\frac{df_2(4)}{dx} = -1/3$

$$Df(4) = \left(1/3, -1/3\right)$$

5 $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^3y^3 + y^3z^3 + x^3z^3 = -1\}$

(a) Let $F : \mathbb{R}^3 \to \mathbb{R}$, $F(x,y,z) = x^3y^3 + y^3z^3 + x^3z^3 - 1$

$$DF(4,-1,2) = \left(3x^2(y^3+z^3),\ 3y^2(x^3+z^3),\ 3z^2(x^3+y^3)\right)\Big|_{(4,-1,2)}$$
$$= (0, 27, 21)$$

Since (21) is an invertible matrix, the result follows from the Implicit Function Theorem.

(b) $$\frac{\partial f}{\partial x}(2,-1) = -\frac{\frac{\partial F}{\partial x}(2,-1,1)}{\frac{\partial F}{\partial y}(2,-1,1)} = 0.$$

$$\frac{\partial f}{\partial y}(2,-1) = -\frac{\frac{\partial F}{\partial y}(2,-1,1)}{\frac{\partial F}{\partial z}(2,-1,1)} = -\frac{27}{21} = -\frac{9}{7}.$$

2.5.11 $z=f(x,y)$ has continuous partial derivatives.

$x=e^r\cos\theta \quad , y=e^r\sin\theta$

$$\left(\frac{\partial z}{\partial r}\right)^2=\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}\right)^2=\left(e^r\cos\theta\frac{\partial z}{\partial x}+e^r\sin\theta\frac{\partial z}{\partial y}\right)^2$$

$$=e^{2r}\cos^2\theta\left(\frac{\partial z}{\partial x}\right)^2+e^{2r}\sin^2\theta\left(\frac{\partial z}{\partial y}\right)^2+2e^r\cos\theta\sin\theta\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$

$$\left(\frac{\partial z}{\partial \theta}\right)^2=\left(-e^r\sin\theta\frac{\partial z}{\partial x}+e^r\cos\theta\frac{\partial z}{\partial y}\right)^2$$

$$=e^{2r}\sin^2\theta\left(\frac{\partial z}{\partial x}\right)^2+e^{2r}\sin^2\theta\left(\frac{\partial z}{\partial y}\right)^2-2e^r\cos\theta\sin\theta\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$

$$e^{-2r}\left(\left(\frac{\partial z}{\partial r}\right)^2+\left(\frac{\partial z}{\partial \theta}\right)^2\right)=e^{-2r}\left[\left(e^{2r}\left(\frac{\partial z}{\partial x}\right)^2+e^{2r}\left(\frac{\partial z}{\partial y}\right)^2\right)\left(\cos^2\theta+\sin^2\theta\right)\right]$$

$$=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2$$

2.5.15 $w=f(u)$, $u=\dfrac{xy}{x^2+y^2}$

$$\frac{\partial w}{\partial x}=\frac{dw}{du}\frac{\partial u}{\partial x}=\frac{dw}{du}\left[\frac{y(y^2-x^2)}{(x^2+y^2)^2}\right]$$

$$\frac{\partial w}{\partial y}=\frac{dw}{du}\frac{\partial u}{\partial y}=\frac{dw}{du}\left[\frac{x(x^2-y^2)}{(x^2+y^2)^2}\right]$$

$$x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=\left[xy(y^2-x^2)+xy(x^2-y^2)\right]/(x^2+y^2)^2=0.$$

2.5.30 $z=f(x,y)$ has continuous partial derivatives.

$x=r\cos\theta$, $y=r\sin\theta$

$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta$$

$$\frac{\partial z}{\partial \theta}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}=\frac{\partial z}{\partial x}(-r\sin\theta)+\frac{\partial z}{\partial y}(r\cos\theta)$$

$$\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2=\left(\frac{\partial z}{\partial x}\right)^2\left(\cos^2\theta+\frac{r^2}{r^2}\sin^2\theta\right)+\left(\frac{\partial z}{\partial y}\right)^2\left(\sin^2\theta+\frac{r^2}{r^2}\cos^2\theta\right)$$

$$+2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\left(\cos\theta\sin\theta-\frac{r^2}{r^2}\cos\theta\sin\theta\right)=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2.$$

2.5.34 $F(x,y)=0$, F and y differentiable, $F_y(x,y)\neq 0$

(a) $\frac{d}{dx}F = \frac{d}{dx}0$

$F_x\frac{dx}{dx} + F_y\frac{dy}{dx} = 0$

$F_y\frac{dy}{dx} = -F_x\cdot 1$

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)}$$

(b) $0=F(x,y)=x^3-y^2$

$\frac{dy}{dx} = -\frac{3x^2}{-2y} = \frac{3x^2}{2y}$ (Implicit Function theorem)

$y^2 = x^3$

$2y\frac{dy}{dx} = 3x^2 \rightarrow \frac{dy}{dx} = \frac{3x^2}{2y}$ (isolating y and differentiation)

2.5.35 $0 = F(x,y) = \sin(xy) - x^2y^7 + e^y$

$$\frac{dy}{dx} = -\frac{y\cos(xy) - 2xy^7}{x\cos(xy) - 7x^2y^6 + e^y} = \frac{2xy^7 - y\cos(xy)}{x\cos(xy) - 7x^2y^6 + e^y}$$

2.6.21 $x\sin y + xz^2 = 2e^{yz}$, point $(2, \pi/2, 0)$

(a) $x = 2e^{yz}/(\sin y + z^2) = f(y,z)$

Variant of (4) from Theorem 3.3 for tangent plane:

$$x = f(\pi/2, 0) + f_y(\pi/2, 0)(y - \pi/2) + f_z(\pi/2, 0)\,z$$

$$f_y = \left(2ze^{yz}(\sin y + z^2) - 2e^{yz}\cos y\right) / (\sin y + z^2)^2$$

$$f_z = \left(2ye^{yz}(\sin y + z^2) - 4ze^{yz}\right) / (\sin y + z^2)^2$$

$$x = 2 + 0(y - \pi/2) + \pi z = 2 + \pi z$$

(b) $0 = g(x,y,z) = x\sin y + xz^2 - 2e^{yz}$. Equation (6) of §2.6:

$$0 = g_x(2, \pi/2, 0)(x-2) + g_y(2, \pi/2, 0)(y - \pi/2) + g_z(2, \pi/2, 0)\,z$$

$$0 = (1 + 0^2)(x-2) + (2\cdot 0 - 2\cdot 0\cdot 1)(y - \pi/2) + \left(2\cdot 2\cdot 0 - \frac{2\pi}{2}\cdot 1\right)z = x - 2 - \pi z$$

2.6.24 Surfaces: $z = 7x^2 - 12x - 5y^2$, $xyz^2 = 2$, point: $(2,1,-1)$

Let $f = z - 7x^2 + 12x + 5y^2$, $g = xyz^2 - 2$

$\nabla f(2,1,-1) = \langle -14x+12, 10y, 1 \rangle \big|_{(2,1,-1)} = \langle -16, 10, 1 \rangle$

$\nabla g(2,1,-1) = \langle yz^2, xz^2, 2xyz \rangle \big|_{(2,1,-1)} = \langle 1, 2, -4 \rangle$

∇f is orthogonal to $7x^2 - 12x - 5y^2 = z$

∇g is orthogonal to $xyz^2 = 2$

$0 = \nabla f(2,1,-1) \cdot \nabla g(2,1,-1)$ means that ∇f and ∇g are orthogonal at $(2,1,-1)$

Since $7x^2 - 12x - 5y^2 = z$ is orthogonal to ∇f, ∇f is orthogonal to ∇g, and ∇g is orthogonal to $xyz^2 = 2$, $7x^2 - 12x - 5y^2 = z$ is orthogonal to $xyz^2 = 2$ at $(2,1,-1)$. This relationship (orthogonality) is symmetric, so we can say the planes intersect orthogonally at $(2,1,-1)$.