Problem Set 11

6.1.1 $f(x,y) = x+2y$

(a) $\vec{x}(t) = (2-3t, 4t-1)$, $0 \le t \le 2$

$$\int_{\vec{x}} f\,ds = \int_0^2 f(\vec{x}(t))\,\|\vec{x}\,'(t)\|\,dt = \int_0^2 5t\sqrt{25}\,dt = 25 \cdot \frac{t^2}{2}\Big|_0^2 = 50$$

(b) $\vec{x}(t) = (\cos t, \sin t)$, $0 \le t \le \pi$

$$\int_{\vec{x}} f\,ds = \int_0^\pi (\cos t + 2\sin t)\,\|(-\sin t, \cos t)\|\,dt = \sin t - 2\cos t\Big|_0^\pi = 4$$

6.1.3 $f(x,y,z) = \frac{x+z}{y+z}$, $\vec{x}(t) = (t,t,t^{3/2})$, $1 \le t \le 3$

$$\int_{\vec{x}} f\,ds = \int_1^3 \frac{t+t^{3/2}}{t+t^{3/2}}\sqrt{2+9t/4}\,dt = \int_1^3 \sqrt{2+9t/4}\,dt = \int_{17/4}^{35/4} \frac{4}{9}\sqrt{u}\,du = \left(35\sqrt{35} - 17\sqrt{17}\right)/27$$

6.1.9 $\vec{F} = (y+2)\hat{\imath} + x\hat{\jmath}$, $\vec{x}(t) = (\sin t, -\cos t)$, $0 \le t \le \pi/2$

$$\int_{\vec{x}} \vec{F} \cdot d\vec{s} = \int_0^{\frac{\pi}{2}} \vec{F}(\vec{x}(t)) \cdot \vec{x}\,'(t)\,dt = \int_0^{\frac{\pi}{2}} (2\cos t - \cos^2 t + \sin^2 t)\,dt = \int_0^{\frac{\pi}{2}} 2\cos t\,dt = 2$$

since $\int_0^{\frac{\pi}{2}} (\sin^2 t - \cos^2 t)\,dt = \int_0^{\frac{\pi}{2}} -\cos 2t\,dt = -\frac{1}{2}\sin\pi + \frac{1}{2}\sin 0 = 0.$

6.1.17 $\vec{x}(t) = (\cos 3t, \sin 3t)$, $0 \le t \le \pi$

$$\int_{\vec{x}} x\,dy - y\,dx = \int_{\vec{x}} (-y(t), x(t)) \cdot d\vec{s} = \int_0^\pi (-\sin 3t, \cos 3t) \cdot (-3\sin 3t, 3\cos 3t)\,dt$$

$$= \int_0^\pi (3\sin^2 3t + 3\cos^2 3t)\,dt = 3\pi$$

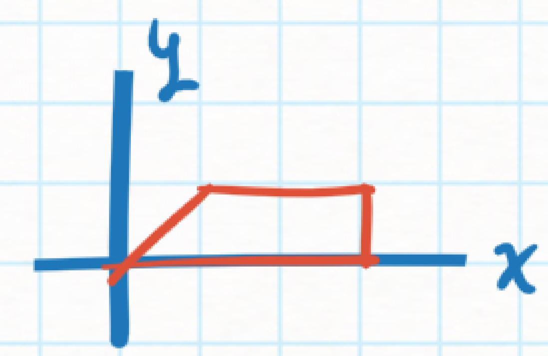
6.1.19 $\vec{x}(t) = (e^{2t}\cos 3t,\ e^{2t}\sin 3t)$, $0 \le t \le 2\pi$

$$\int_{\vec{x}} \frac{x\,dx + y\,dy}{(x^2+y^2)^{3/2}} = \int_{\vec{x}} \vec{F}\cdot d\vec{s}, \quad \vec{F} = \left(x/(x^2+y^2)^{3/2},\ y/(x^2+y^2)^{3/2} \right)$$

$$= \int_0^{2\pi} \left(\frac{\cos 3t}{e^{4t}}, \frac{\sin 3t}{e^{4t}} \right)\cdot\left(2e^{2t}\cos 3t - 3e^{2t}\sin 3t,\ 2e^{2t}\sin 3t + 3e^{2t}\cos 3t\right) dt$$

$$= \int_0^{2\pi} \frac{2}{e^{2t}}\,dt = 1 - e^{-4\pi}$$

6.1.25 C is the trapezoid with vertices $(0,0)$, $(3,0)$, $(3,1)$, $(1,1)$. Since counterclockwise parameterizations produces complicated integrals, parameterize with clockwise orientation:



$$\vec{x}(t) = \begin{cases} (t,t), & 0 \le t \le 1 \\ (t,1), & 1 \le t \le 3 \\ (3,4-t), & 3 \le t \le 4 \\ (7-t,0), & 4 \le t \le 7 \end{cases} \qquad \vec{x}'(t) = \begin{cases} (1,1), & 0 \le t \le 1 \\ (1,0), & 1 \le t \le 3 \\ (0,-1), & 3 \le t \le 4 \\ (-1,0), & 4 \le t \le 7 \end{cases}$$

$$-\int_C x^2y\,dx - (x+y)\,dy = \int_{\vec{x}} \left(x^2(t)y(t),\ -x(t)-y(t)\right)\cdot \vec{x}'(t)\,dt$$

$$= \int_0^1 (t^3 - 2t)\,dt + \int_1^3 t^2\,dt + \int_3^4 (7-t)\,dt + \int_4^7 0\,dt$$

$$= 137/12$$

$$\therefore \int_C x^2y\,dx - (x+y)\,dy = -137/12$$

6.1.31 C is the line segment from (1,1,2) to (5,3,1).

$\vec{x}(t)=(1+4t,1+2t,2-t)$, $\vec{x}'(t)=(4,2,-1)$, $0\le t\le 1$

$$\int_C yz\,dx - xz\,dy + xy\,dz = \int_0^1 \left[4(1+2t)(2-t)-2(1+4t)(2-t)-(1+4t)(1+2t)\right]dt$$

$$= -\frac{11}{3}$$

6.2.5 $\vec{F}=3y\hat{\imath}-4x\hat{\jmath}$, $D=\{(x,y)\mid x^2+2y^2\le 4\}$

$$\oint_{\partial D} M\,dx+N\,dy=\int_0^{2\pi}(3y(t),-4x(t))\cdot\vec{x}'(t)\,dt, \quad \vec{x}(t)=(x(t),y(t))=(2\cos t,\sqrt{2}\sin t)$$

$$=\int_0^{2\pi}(3\sqrt{2}\sin t,-8\cos t)\cdot(-2\sin t,\sqrt{2}\cos t)\,dt$$

$$=\int_0^{2\pi}(-6\sqrt{2}\sin^2 t-8\sqrt{2}\cos^2 t)\,dt$$

$$=-14\sqrt{2}\pi$$

$\iint_D (N_x-M_y)\,dA=\iint_D(-4-3)\,dA=-7(\pi\cdot 2\cdot\sqrt{2})=-14\sqrt{2}\pi$ using the formula for the area of an ellipse with axes of length a and b Area $=\pi ab$. This verifies Green's Theorem for the given $\vec{F}$, D.

6.2.10 $\vec{F} = (4y-3x)\hat{\imath} + (x-4y)\hat{\jmath}$, $\partial D: x^2+4y^2=4$, $D: x^2+4y^2 \le 4$

The work done by $\vec{F}$ on a particle that travels counterclockwise once around the ellipse is, by Green's Theorem:

$$\oint_{\partial D} (M\,dx + N\,dy) = \iint_D (N_x - M_y)\,dA = \iint_D (1-4)\,dA = -3\cdot\pi\cdot 2\cdot 1 = -6\pi$$

6.2.12 $x(t) = a(t-\sin t)$, $y(t) = a(1-\cos t)$, $a > 0$

One arch for $0 \le t \le 2\pi$. Let D denote the area under this arch.

$\iint_D dA = \iint_D (N_x - M_y)\,dA$, $N_x - M_y = 1$, let $\vec{F} = (0, x)$ so $N_x = 1$, $M_y = 0$

$$= \oint_{\partial D} (M\,dx + N\,dy) = -\int_0^{2\pi} \vec{F}(\vec{x}(t)) \cdot \vec{x}\,'(t)\,dt$$

$$= -\int_0^{2\pi} (0, a(t-\sin t)) \cdot (a - a\cos t, a\sin t)\,dt$$

$$= -\int_0^{2\pi} (a^2 t\sin t - a^2 \sin^2 t)\,dt$$

$$= 3\pi a^2$$

Note the use of the negative sign since moving along ∂D is in the clockwise direction.

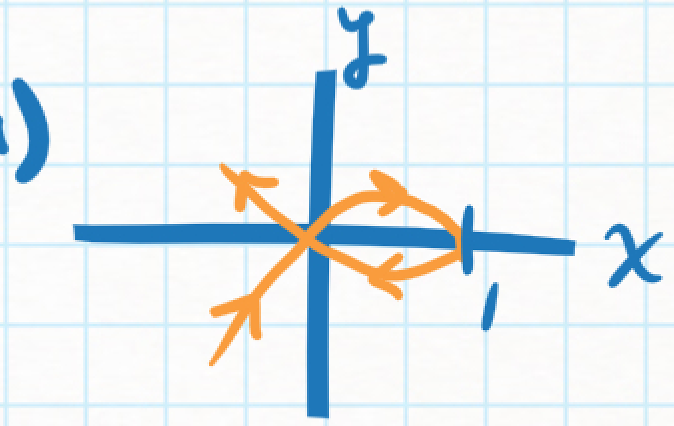
6.2.15 $\vec{x}(t) = (1-t^2,\ t^3-t)$

(b) Let D be the region enclosed by one loop. The area of D is

$$\iint_D dA = -\oint_{\partial D} M\,dx + N\,dy = -2\int_0^1 (0, 1-t^2)\cdot(-2t,\ 3t^2-1)\,dt = 2\int_0^1 (3t^4-4t^2+1)\,dt = \frac{8}{15}$$

The negative sign accounts for the clockwise orientation.
The factor of 2 accounts for the fact that integrating from 0 to 1 only collects the area below the x-axis of the loop (we used symmetry). Alternatively integrate from -1 to 1.

(a)



6.2.16 Let D be the region between $x^2+y^2=25$ and $x^2/9+y^2/4=1$.
The area of D is

$$\iint_D dA = \iint_D (N_x - M_y)\,dA, \quad \text{where } \vec{F} = (M,N) = (0,x)$$

$$= 25\pi - \iint_{\text{ellipse}} (N_x - M_y)\,dA$$

$$= 25\pi - \int_0^{2\pi} (0, 3\cos t)\cdot(-3\sin t,\ 2\cos t)\,dt$$

$$= 25\pi - \int_0^{2\pi} 6\cos^2 t\,dt$$

$$= 25\pi - \left(3t + \tfrac{3}{2}\sin 2t\right)\Big|_0^{2\pi}$$

$$= 19\pi$$

6.2.25 Let C be any simple closed curve in the plane. By Green's Theorem,

$$\oint_C 3x^2y\,dx + x^3dy = \iint_D (3x^2 - 3x^2)\,dA = \iint_D 0\,dA = 0,$$ where D is the region enclosed by C.

6.2.26 Let C be any closed curve to which Green's Theorem applies.

$$\oint_C -y^3dx + (x^3 + 2x + y)\,dy = \iint_D (3x^2 + 2 + 3y^2)\,dA > \iint_D 0\,dA = 0,$$

where D is the region enclosed by C.

6.2.31 D is a region in the plane to which Green's Theorem applies.
$\vec{n}$ is the outward unit normal vector to D.
$f(x,y) \in C^2$

$$\begin{aligned}
\iint_D \nabla^2 f\,dA &= \iint_D \nabla \cdot \nabla f\,dA \\
&= \iint_D \nabla \cdot \vec{F}\,dA, \quad \vec{F} := \nabla f \\
&= \oint_{\partial D} \vec{F} \cdot \vec{n}\,ds, \quad \text{by Thm 2.3 (Divergence in the plane)} \\
&= \oint_{\partial D} \nabla f \cdot \vec{n}\,ds \\
&= \oint_{\partial D} \frac{\partial f}{\partial n}\,ds, \quad \frac{\partial f}{\partial n} := \nabla f \cdot \vec{n}
\end{aligned}$$