Problem Set 6

1. $f:\mathbb{R}\to\mathbb{R}$, $f\in C^1$

Curve C parameterized by $\vec{r}(\theta)=\langle f(\theta)\cos\theta, f(\theta)\sin\theta\rangle$

The arc length s of C is given by

$$s(\theta)=\int_\alpha^\theta \|\vec{r}\,'(\tau)\|\,d\tau$$

$$=\int_\alpha^\theta \left[\left(f'(\tau)\cos\tau - f(\tau)\sin\tau\right)^2 + \left(f'(\tau)\sin\tau + f(\tau)\cos\tau\right)^2\right]^{1/2} d\tau$$

$$=\int_\alpha^\theta \Big[(f'(\tau))^2(\cos^2\tau+\sin^2\tau) - 2f'(\tau)f(\tau)\cos\tau\sin\tau$$
$$+ 2f'(\tau)\cos\tau\sin\tau + (f(\tau))^2(\sin^2\tau+\cos^2\tau)\Big]^{1/2} d\tau$$

$$=\int_\alpha^\theta \sqrt{[f(\tau)]^2+[f'(\tau)]^2}\,d\tau$$

3.1.18 $\vec{x}(t) = (\cos(e^t), 3-t^2, t)$, $t = 1$

$\vec{x}'(t) = (-e^t\sin(e^t), -2t, 1)$

$\vec{x}'(1) = (-e\sin(e), -2, 1)$

Tangent line $\ell(t) = \vec{x}(1) + (t-1)\vec{x}'(1)$

$\ell(t) = (\cos(e) + e\sin(e) - et\sin(e), -2t, t)$

3.1.26 $\vec{x}(t) = (t^2-2, \frac{t^2}{2}-1)$, $\vec{y}(t) = (t, 5-t^2)$

(a) For the balls to collide, there must exist a time t at which the components both match.

$t^2-2=t$ $\qquad$ $t^2/2 - 1 = 5-t^2$ $\qquad$ The balls collide at $t=2$

$t^2-t-2=0$ $\qquad$ $t^2-4=0$ $\qquad$ $\vec{x}(2) = y(2) = (2,1)$

$(t-2)(t+1)=0$ $\qquad$ $t = \pm 2$

$t=-1,2$

(b) The angle formed by the paths at the collision point is

$\theta = \arccos(\vec{x}'(1)\cdot\vec{y}'(1) / \|\vec{x}'(1)\|\|\vec{y}'(1)\|)$

$= \arccos((4,2)\cdot(1,-4)/(\sqrt{20}\sqrt{17})) = \arccos(-2/\sqrt{85})$

3.1.30 (a) To show that $\vec{x}(t) = (\cos t, \cos t \sin t, \sin^2 t)$ lies on a unit sphere, show $\|\vec{x}(t)\| = 1$ for all t.

$$\|\vec{x}(t)\|^2 = \cos^2 t + \cos^2 t \sin^2 t + \sin^4 t$$
$$= \cos^2 t + \sin^2 t(\cos^2 t + \sin^2 t)$$
$$= \cos^2 t + \sin^2 t = 1 \Rightarrow \|\vec{x}(t)\| = 1.$$

(b) $\vec{v}(t) = \vec{x}'(t) = (-\sin t, -\sin^2 t + \cos^2 t, 2\sin t \cos t)$

$$\vec{x}(t) \cdot \vec{v}(t) = -\cos t \sin t - \cos t \sin^3 t + \cos^3 t \sin t + 2\sin^3 t \cos t$$
$$= \sin^3 t \cos t + \cos^3 t \sin t - \cos t \sin t$$
$$= \cos t \sin t(\sin^2 t + \cos^2 t) - \cos t \sin t = 0$$

Since $\vec{x}(t) \cdot \vec{v}(t) = 0 \ \forall t$, $\vec{x}(t)$ is always perpendicular to $\vec{v}(t)$.

(c) Proposition 1.7 says that if $\vec{x}(t)$ is of constant length, then $\vec{x}(t)$ is perpendicular to $d\vec{x}/dt$ ($\forall t$).

Since $\|\vec{x}(t)\| = 1$ is constant, $\vec{x}(t)$ is perpendicular to $d\vec{x}/dt = \vec{v}(t)$.

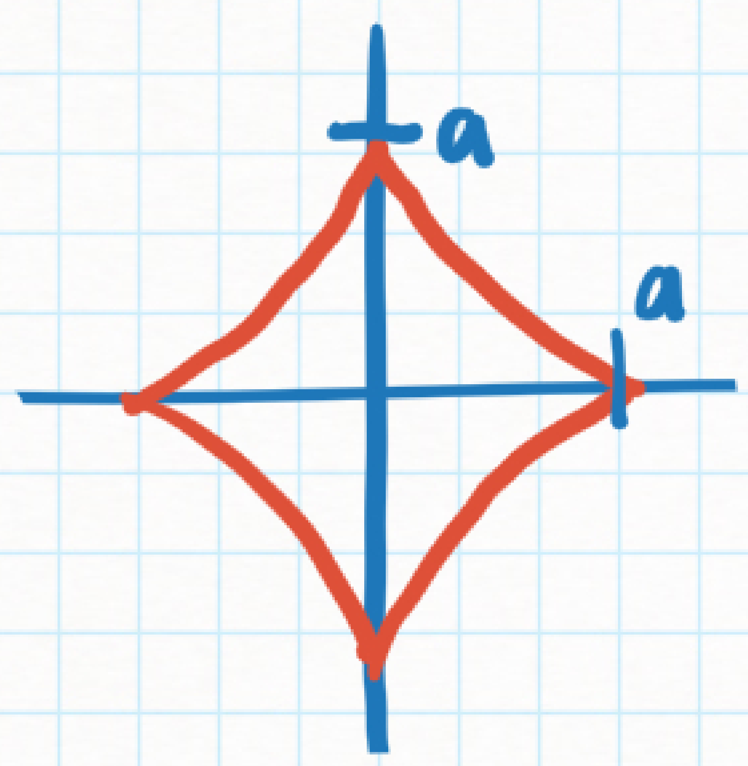
3.1.35 Let $\vec{x}(t)$ be a path of class $\mathcal{C}^1$ s.t. $\vec{x}(t)$ does not pass through $\vec{0} \in \mathbb{R}^3$. Let $\vec{x}(t_0)$ be the point on the image of $\vec{x}$ closest to the origin and suppose $\vec{x}'(t_0) \neq 0$.

If $\vec{x}(t)$ is closest to $\vec{0}$ at $t=t_0$, then $\|\vec{x}(t)-\vec{0}\|$ has a minimum at $t=t_0$ and therefore $\|\vec{x}(t)\|^2$ also has a minimum at $t=t_0$. This implies

$$0 = \frac{d}{dt}\|\vec{x}(t)\|^2\Big|_{t=t_0} = 2\,\vec{x}(t)\cdot\vec{x}'(t)\Big|_{t=t_0} = 2\vec{x}(t_0)\cdot\vec{x}'(t_0)$$

Since $\vec{x}(t_0) \neq 0$ and $\vec{x}'(t_0) \neq 0$, $\vec{x}(t_0)$ and $\vec{x}'(t_0)$ are orthogonal.

3.2.9 $\vec{x}(t) = (a\cos^3 t,\ a\sin^3 t)$, $a>0$ (asteroid or hypocycloid of four cusps).



$$L = 4\int_0^{\pi/2}\left(9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t\right)^{1/2} dt$$

$$= 4\int_0^{\pi/2} 3a\cos t\sin t\, dt$$

$$= \frac{12a}{2}\int_0^{\pi/2}\sin 2t\, dt$$

$$= -3a\cos 2t\Big|_0^{\pi/2} = 6a$$

3.2.16 $\vec{x}(t) = e^{at}\cos bt\,\hat{\imath} + e^{at}\sin bt\,\hat{\jmath} + e^{at}\hat{k}$

(a) $\vec{x}'(t) = (ae^{at}\cos bt - be^{at}\sin bt)\hat{\imath} + (ae^{at}\sin bt + be^{at}\cos bt)\hat{\jmath} + ae^{at}\hat{k}$

$$s(t) = \int_0^t \|\vec{x}'(\tau)\|\, d\tau$$

$$= \int_0^t \sqrt{2a^2+b^2}\, e^{a\tau}\, d\tau$$

$$= \frac{\sqrt{2a^2+b^2}}{a}\left(e^{at} - 1\right) = \left(e^{at}-1\right)\sqrt{2+(b/a)^2}$$

(b) $$e^{at} = 1 + s/\sqrt{2+(b/a)^2}$$

$$t = \ln\left(1+s/\sqrt{2+(b/a)^2}\right)/a$$

$$\vec{x}(t) = \vec{x}\left(\ln\left(1+s/\sqrt{2+(b/a)^2}\right)/a\right)$$

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THEOREM 2.5 Let s be the arclength parameter and suppose C_1 and C_2 are two curves of class C^3 in $\mathbf{R}^3$. Assume that the corresponding curvature functions κ_1 and κ_2 are strictly positive. Then if $\kappa_1(s) \equiv \kappa_2(s)$ and $\tau_1(s) \equiv \tau_2(s)$, the two curves must be congruent (in the sense of high school geometry). In fact, given any two continuous functions κ and τ, where $\kappa(s) > 0$ for all s in the closed interval $[0, L]$, there is a unique curve parametrized by arclength on $[0, L]$ (up to position in space) whose curvature and torsion are κ and τ, respectively.

Suppose $\vec{r}_1, \vec{r}_2 : I \to \mathbb{R}^3$ are two smooth curves parameterized by arc length s and $\kappa_1(s) = \kappa_2(s)$, $\tau_1(s) = \tau_2(s)$ $\forall s \in I$.

Suppose $\exists a \in I$ s.t. $\vec{r}_1(a) = \vec{r}_2(a)$, $\vec{T}_1(a) = \vec{T}_2(a)$, $\vec{N}_1(a) = \vec{N}_2(a)$, $\vec{B}_1(a) = \vec{B}_2(a)$

(a) $\frac{d}{ds}\left[\|\vec{T}_1(s) - \vec{T}_2(s)\|^2 + \|\vec{N}_1(s) - \vec{N}_2(s)\|^2 + \|\vec{B}_1(s) - \vec{B}_2(s)\|^2\right]$

$= 2(\vec{T}_1(s) - \vec{T}_2(s)) \cdot (\vec{T}_1'(s) - \vec{T}_2'(s)) + 2(\vec{N}_1(s) - \vec{N}_2(s)) \cdot (\vec{N}_1'(s) - \vec{N}_2'(s))$
$\quad + 2(\vec{B}_1(s) - \vec{B}_2(s)) \cdot (\vec{B}_1'(s) - \vec{B}_2'(s))$

$= -2\Big(\kappa(\vec{T}_1 \cdot \vec{N}_2 + \vec{T}_2 \cdot \vec{N}_1) - \kappa(\vec{N}_1 \cdot \vec{T}_2 + \vec{N}_2 \cdot \vec{T}_1)$
$\quad + \tau(\vec{N}_1 \cdot \vec{B}_2 + \vec{N}_2 \cdot \vec{B}_1) - \tau(\vec{B}_1 \cdot \vec{N}_2 + \vec{B}_2 \cdot \vec{N}_1)\Big)$

$= 0$. The quantity is a constant function.

Frenet-Serret Formulas:
$\vec{T}'(s) = \kappa \vec{N}(s)$
$\vec{N}'(s) = -\kappa \vec{T}(s) + \tau \vec{B}(s)$
$\vec{B}'(s) = -\tau \vec{N}(s)$

(b) At $s=a$, the quantity we differentiated is 0 by hypothesis. But since the derivative is 0 as well, the quantity is identically zero. Thus the $\vec{T}$, $\vec{N}$, and $\vec{B}$ vectors are equal for all $s \in I$. We can calculate the position vectors by integrating velocity vectors:

$$\vec{r}_1(s) = \vec{r}_1(a) + \int_a^s \vec{T}_1(\xi)\,d\xi = \vec{r}_2(a) + \int_a^s \vec{T}_2(\xi)\,d\xi = \vec{r}_2(s)$$

9 Let $\vec{r} : \mathbb{R} \to \mathbb{R}^3$ be the helix given by

$$\vec{r}(s) = a\cos s/c\ \hat{\imath} + a\sin s/c\ \hat{\jmath} + bs/c\ \hat{k}$$

$$a > 0 \qquad c > 0 \qquad c^2 = a^2 + b^2$$

(a) A vector valued function $\vec{r}(s)$ is parameterized by arclength if $s(t) = \int_a^t \|\vec{r}\,'(\tau)\|\,d\tau$. In this case the length of $\vec{r}(s)$, $a \le s \le t$ is $s(t) - s(a) = t - a$. By the FTC, $1 = \frac{d}{dt}[t-a] = \frac{d}{dt}\int_a^t \|\vec{r}\,'(\tau)\|\,d\tau = \|\vec{r}\,'(t)\|$. Conversely if $\|\vec{r}\,'(t)\| = 1$ then the length of the curve traced out by $\vec{r}(t)$ between a and t is $t-a$. $\therefore$ Check if $\|\vec{r}\,'(s)\|$

$$\|\vec{r}\,'(s)\| = \left(\left(-a/c \sin s/c\right)^2 + \left(a/c \cos s/c\right)^2 + \left(\frac{b}{c}\right)^2 \right)^{1/2}$$

$$= (a^2+b^2)/c^2 = 1.$$

(b) $\vec{T}(s) = \vec{r}'(s) = \left(-a/c \sin s/c,\ a/c \cos s/c,\ b/c\right)$

$$\vec{N}(s) = \frac{d\vec{T}/ds}{\|d\vec{T}/ds\|} = \left(-a/c^2 \cos s/c,\ -a/c^2 \sin s/c,\ 0\right)/\sqrt{a^2/c^4}$$

$$= \left(-\cos s/c,\ -\sin s/c,\ 0\right)$$

$\vec{B}(s) = \vec{T}(s) \times \vec{N}(s)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a/c \sin s/c & a/c \cos s/c & b/c \\ -\cos s/c & -\sin s/c & 0 \end{vmatrix}$$

$$= \left(\frac{b}{c} \sin s/c,\ -\frac{b}{c} \cos s/c,\ a/c\right)$$

(c) $\kappa(s) = \|d\vec{T}/ds\| = a/c^2$

$d\vec{B}/ds = -\tau \vec{N}(s)$

$\left(b/c^2 \cos s/c,\ b/c^2 \sin s/c,\ 0\right) = -\tau\left(-\cos s/c,\ -\sin s/c,\ 0\right)$

$\Longrightarrow \tau = b/c^2$

10 $\vec{r}:\mathbb{R}\to\mathbb{R}^3$ is a smooth regular curve parametrized by arclength.

$\kappa(s)=\kappa$ (constant) , $\tau(s)=\tau$ (constant)

From problem 9, if

$$\kappa = a/c^2 \quad , \quad \tau = b/c^2 \quad , \quad a^2+b^2=c^2$$

$$\kappa^2+\tau^2 = (a^2+b^2)/c^4 = 1/c^2$$

$$\longrightarrow \quad a=\frac{\kappa}{\kappa^2+\tau^2} \quad , \quad b=\frac{\tau}{\kappa^2+\tau^2} \quad , \quad c=\frac{1}{\sqrt{\kappa^2+\tau^2}}$$

11 $\vec{r}: I \to \mathbb{R}$ smooth regular curve

$\vec{T}(a) = \frac{2}{3}\hat{\imath} + \frac{2}{3}\hat{\jmath} - \frac{1}{3}\hat{k}$, $\vec{B}(a) = -\frac{1}{3}\hat{\imath} + \frac{2}{3}\hat{\jmath} + \frac{2}{3}\hat{k}$,

$\frac{d\vec{N}}{ds}(a) = -4\hat{\imath} + 2\hat{\jmath} + 5\hat{k}$ for some $a \in I$.

(a) $\vec{N}(a) = \vec{B}(a) \times \vec{T}(a)$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{vmatrix} = \left(-2/3, 1/3, -2/3\right)$$

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix}$$

(b) $\vec{N}' = -\kappa\vec{T} + \tau\vec{B}$

$\vec{T} \cdot \vec{N}' = -\kappa\vec{T} \cdot \vec{T} + \tau\vec{T}\vec{B}$

$\vec{T} \cdot \vec{N}' = -\kappa$

$\kappa = -(2/3, 2/3, -1/3) \cdot (-4, 2, 5)$

$\kappa = 3$

(c) $\vec{N}' = -\kappa\vec{T} + \tau\vec{B}$

$\vec{N}' \cdot B = \tau$

$\tau = (-4, 2, 5) \cdot (-1/3, 2/3, 2/3)$

$\tau = 4/3 + 4/3 + 10/3 = 6$