Problem Set 4

1 (i) $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\begin{cases}0 & ; x\in\mathbb{Q}\\ 1 & ; x\notin\mathbb{Q}\end{cases}$

f is nowhere continuous. Let $x\in\mathbb{R}$. If $x\in\mathbb{Q}$, $\exists$ a sequence $(x_n)\subset\mathbb{R}\setminus\mathbb{Q}$ s.t. $x_n\to x$ so $\lim\limits_{x_n\to x} f(x_n)=1\neq f(x)$. If $x\notin\mathbb{Q}$, $\exists$ $(x_n)\subset\mathbb{Q}$ s.t. $x_n\to x$ so $\lim\limits_{x_n\to x} f(x_n)=0\neq f(x)$.

(ii) $g:\mathbb{R}\to\mathbb{R}$, $g(x)=\begin{cases}x & , x\in\mathbb{Q}\\ 2x & , x\notin\mathbb{Q}\end{cases}$

g is continuous at 0. Let $\varepsilon>0$ and set $\delta=\varepsilon/2$. Then for any x_0 with $|x_0-0|<\delta$ (rational or irrational),

$$|g(x_0)-g(0)| = |g(x_0)| \leq |2x_0| = 2|x_0| < 2\delta = \varepsilon.$$

g is not continuous for any $x\neq 0$. If $x\in\mathbb{Q}$, $\exists$ $x_n\to x$ s.t. $x_n\notin\mathbb{Q}$ $\forall n$ so $\lim f(x_n)=2x\neq x=f(x)$. Similarly if $x\notin\mathbb{Q}$ $\exists$ $x_n\to x$ s.t. $x_n\in\mathbb{Q}$ $\forall n$ so $\lim f(x_n)=x\neq 2x=f(x)$.

2 $f:\mathbb{R}^n\to\mathbb{R}^m$

$f(P)=(f_1(P),\dots f_m(P))$, $P\in\mathbb{R}^n$

$f_k:\mathbb{R}^n\to\mathbb{R}$, $k=1,\dots,m$

(i) Assume f is continuous and let $\varepsilon>0$. Then $\exists\,\delta>0$ s.t. if $\|P-P_0\|<\delta$, $\|f(P)-f(P_0)\|<\varepsilon$. But then $|f_k(P)-f_k(P_0)|\le\|f(P)-f(P_0)\|<\varepsilon$ for each k by the triangle inequality. Conclude that each f_k is continuous $\forall P_0$.

(ii) Assume $f_1,\dots,f_m$ are each continuous. Let $P_0\in\mathbb{R}^n$ and $\varepsilon>0$ be arbitrary. Then $\exists\,\delta_k>0$ s.t. $|f_k(P)-f_k(P_0)|<\varepsilon/\sqrt{m}$ if $\|P-P_0\|<\delta$ for each k. Let $\delta=\min\delta_k$. Then for $\|P-P_0\|<\delta$,

$$\|f(P)-f(P_0)\|^2=(f_1(P)-f_1(P_0))^2+\dots+(f_m(P)-f_m(P_0))^2<m\varepsilon^2/m=\varepsilon^2$$

Since $0\le\|f(P)-f(P_0)\|^2<\varepsilon^2$, $\|f(P)-f(P_0)\|<\varepsilon$.

3 $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=|xy|$

(a) $$f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{0}{h}=0$$

$$f_y(0,0)=\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}=0$$

$$h(x,y)=f(0,0)+f_x(0,0)(x-0)+f_y(0,0)(y-0)=0$$

$$\lim_{(x,y)\to(0,0)}\frac{f(x,y)-h(x,y)}{\|(x,y)-(0,0)\|}=\lim_{(x,y)\to(0,0)}\frac{|xy|}{\sqrt{x^2+y^2}}=\lim_{r\to 0}\frac{r^2|\cos\theta\sin\theta|}{r}=0.$$

DEFINITION 3.4 Let X be open in $\mathbf{R}^2$ and $f: X\subseteq \mathbf{R}^2\to\mathbf{R}$ be a scalar-valued function of two variables. We say that f is **differentiable at** $(a,b)\in X$ if the partial derivatives $f_x(a,b)$ and $f_y(a,b)$ exist and if the function

$$h(x,y)=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$$

is a good linear approximation to f near (a,b)—that is, if

$$\lim_{(x,y)\to(a,b)}\frac{f(x,y)-h(x,y)}{\|(x,y)-(a,b)\|}=0.$$

Moreover, if f is differentiable at (a,b), then the equation $z=h(x,y)$ defines the **tangent plane** to the graph of f at the point $(a,b,f(a,b))$. If f is differentiable at all points of its domain, then we simply say that f is **differentiable**.

(b) Let X be any neighborhood of $(0,0)$ and $(0,b)$ be any point in X with $b\neq 0$.

$$f_x(a,b)=\lim_{h\to 0}\frac{|(a+h)b|-|ab|}{h}=|b|\lim_{h\to 0}\frac{|a+h|-|a|}{h}=\begin{cases}|b|, & a>0\\ -|b|, & a<0\end{cases}$$

This means $\lim_{a\to 0^+}f_x(a,b)\neq\lim_{a\to 0^-}f_x(a,b)$ so $\lim_{a\to 0}f_x(a,b)$ DNE.

Since $\lim_{a\to 0}f_x(a,b)$ DNE, we cannot have $\lim_{a\to 0}f_x(a,b)=f(0,b)$. Thus f_x is not continuous at $(0,b)$ (and so not continuous on X). By symmetry, neither is f_y continuous on any neighborhood of $(0,0)$.

4 $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^3y^2 + x\cos(xy) + y^3 + 12345$

$$\frac{\partial f}{\partial x} = 3x^2y^2 - xy\sin(xy) + \cos(xy)$$

$$\frac{\partial f}{\partial y} = 2x^3y - x^2\sin(xy) + 3y^2$$

2.3.33

$$f(s,t) = (s^2, st, t^2) \quad , \quad a = (-1,1)$$

$$Df(s,t) = \begin{pmatrix} 2s & 0 \\ t & s \\ 0 & 2t \end{pmatrix} \quad , \quad Df(a) = \begin{pmatrix} -2 & 0 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$$

2.3.38 $z = 4\cos(xy)$, $P = (\pi/3, 1, 2)$, $z_x = -4y\sin(xy)$, $z_y = -4x\cos($

$$h(x,y) = z(\pi/3,1) + z_x(\pi/3,1)(x-\pi/3) + z_y(\pi/3,1)(y-1)$$

$$= 2 - 4\sin(\pi/3)(x-\pi/3) - \frac{4\pi}{3}\sin(\pi/3)(y-1)$$

$$= 2 - 2\sqrt{3}(x-\pi/3) - \frac{2\pi\sqrt{3}}{3}(y-1)$$

$$= 2 - 2\sqrt{3}x - \frac{2\pi\sqrt{3}}{3}y + \frac{4\pi\sqrt{3}}{3}$$

2.3.41 $x_5 = 10 - (x_1^2 + 3x_2^2 + 2x_3^2 + x_4^2)$, $a = (2, -1, 1, 3, -8)$

The tangent hyperplane is $h(x) = f(a) + Df(a)(x-a)$, $x \in \mathbb{R}^4$
where $f(x) = 10 - (x_1^2 + 3x_2^2 + 2x_3^2 + x_4^2)$

$$Df(x) = [-2x_1, -6x_2, -4x_3, -2x_4]$$

$$h(x) = 10 - (4 + 3 + 2 + 9) + [-4, 6, -4, -6]\begin{bmatrix} x_1 - 2 \\ x_2 + 1 \\ x_3 - 1 \\ x_4 - 3 \end{bmatrix}$$

$$h(x) = -8 - 4(x_1 - 2) + 6(x_2 + 1) - 4(x_3 - 1) - 6(x_4 - 3)$$

$$h(x) = -4x_1 + 6x_2 - 4x_3 - 6x_4 + 28$$

2.3.59

$f: \mathbb{R}^n \to \mathbb{R}^m$, $f(x) = Ax$, $x \in \mathbb{R}^n$ $A \in \mathbb{R}^{m \times n}$

$$f(x) = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix}$$

$$Df(x) = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = A.$$

This generalizes $f'(x) = a$ for $f(x) = ax$, which could be considered the case where $m = n = 1$.