§ 7.1 Parametrized Surfaces

7.1.1 $\vec{X}: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $\vec{X}(s,t) = (s^2 - t^2, s+t, s^2+3t)$

(a) $\vec{X}(2,-1) = (3, 1, 1)$.

$$\vec{N}(s,t) = \vec{X}_s \times \vec{X}_t = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2s & 1 & 2s \\ -2t & 1 & 3 \end{vmatrix} = (3-2s, -6s-4st, 2s+2t)$$

$$\vec{N}(2,-1) = (-1, -4, 2)$$

(b) $-(x-3) - 4(y-1) + 2(z-1) = 0 \longrightarrow x + 4y - 2z = 5$

7.1.13 The portion of the cylinder $x^2+z^2=4$ lying between $y=-1$ and $y=3$ can be parametrized by $\vec{X}(s,t) = (2\cos t, s, 2\sin t)$, $0 \leq t \leq 2\pi$, $-1 \leq s \leq 3$.

7.1.15 $z^2 - x^2 - y^2 = 1 \longrightarrow \begin{array}{l} \vec{X}_1(s,t) = (s, t, \sqrt{1+s^2+t^2}) \\ \vec{X}_2(s,t) = (s, t, -\sqrt{1+s^2+t^2}) \end{array}$, $(s,t) \in \mathbb{R}^2$

7.1.23 $\vec{X}: D \to \mathbb{R}^3$, $\vec{X}(s,t) = (s+t, s-t, s)$, D is the unit disk.

Surface Area $= \iint_D \|\vec{X}_s \times \vec{X}_t\| \, dA = \iint_D \|(1,1,-2)\| \, dA = \sqrt{6}\pi$

7.1.27 $z = 2x^2 + 2y^2$, $z = 2$, $z = 8$

surface area $= \iint_D (16x^2 + 16y^2 + 1)^{1/2} \, dA = \int_0^{2\pi} \int_1^2 4\sqrt{r^2 + 1/16} \, r \, dr \, d\theta$

$$= 8\pi \int_{17/16}^{65/16} \tfrac{1}{2} u^{1/2} \, du = \frac{8\pi}{3} \frac{65^{3/2} - 17^{3/2}}{16^{3/2}} = \frac{\pi(65^{3/2} - 17^{3/2})}{24}$$

§ 7.2 Surface Integrals

7.2.1 $\vec{X}(s,t) = (s, s+t, t)$, $0 \le s \le 1$, $0 \le t \le 2$

$\vec{X}_s = (1,1,0)$, $\vec{X}_t = (0,1,1)$, $\vec{N}(s,t) = \vec{X}_s \times \vec{X}_t = (1,-1,1)$

$$\iint_{\vec{X}} (x^2+y^2+z^2)\,dS = \int_0^2\int_0^1 (s^2+(s+t)^2+t^2)\,\|\vec{N}(s,t)\|\,ds\,dt$$

$$= \int_0^2\int_0^1 \sqrt{3}(2s^2+2st+2t^2)\,ds\,dt$$

$$= \int_0^2 2\sqrt{3}\left(\tfrac{1}{3}+\tfrac{1}{2}t+t^2\right)dt = 26\sqrt{3}/3$$

7.2.3 $\vec{F} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, $S = \{(x,y,z) \mid x,z \geq 0, y \leq 0, 2x-2y+z=2\}$, $\vec{N}(s,t) = 2\hat{\imath} - 2\hat{\jmath} + \hat{k}$

$\vec{X} = (s, t, 2-2s+2t)$

$$\text{Flux} = \iint_S \vec{F}\cdot d\vec{S} = \iint_{\vec{X}} \vec{F}\cdot d\vec{S} = \iint_{\vec{X}} (\vec{F}\cdot\vec{n})\,dS = \iint_{\vec{X}} \left(\vec{F}(\vec{X}(s,t))\cdot\vec{N}\right)dS$$

$$= \int_0^1\int_{s-1}^0 (s,t,2-2s+2t)\cdot(2,-2,1)\,dt\,ds = \int_0^1\int_{s-1}^0 2\,dt\,ds = 1$$

7.2.7 S is a sphere of radius a. $\vec{X}(s,t) = (a\cos s\sin t, a\sin s\sin t, a\cos t)$

(a) $\vec{N}(s,t) = \vec{X}_s \times \vec{X}_t = -a^2\sin t(\cos s\sin t\,\hat{\imath} + \sin s\sin t\,\hat{\jmath} + a\cos t\,\hat{k})$

$$\iint_S (x^2+y^2+z^2)\,dS = \iint_{\vec{X}} a^2\|\vec{N}(s,t)\|\,ds\,dt = \int_0^{2\pi}\int_0^{\pi} a^2\cdot a^2\sin t\,dt\,ds = 4\pi a^4$$

(b) $$\iint_S y^2\,dS = \tfrac{1}{3}\left(\iint_S x^2\,dS + \iint_S y^2\,dS + \iint_S z^2\,dS\right) = \tfrac{1}{3}\iint_S (x^2+y^2+z^2)\,dS = 4\pi a^4/3$$

§ 7.3 Stokes's and Gauss's Theorems

7.3.1 $S: x^2+y^2+5z=1,\ z \geq 0$, oriented upward normal, $\partial S: x^2+y^2=1,\ z=0$

$\vec{F} = xz\hat{\imath} + yz\hat{\jmath} + (x^2+y^2)\hat{k}$

$\oint_{\partial S} \vec{F}\cdot d\vec{s} = \int_0^{2\pi} \vec{F}(\vec{x}(t))\cdot \vec{x}'(t)\,dt = \int_0^{2\pi} (0,0,1)\cdot(-\sin t, \cos t, 0)\,dt = 0$

$\nabla\times\vec{F} = (y,-x,0)$, $\vec{X}(s,t) = (s,t,(1-s^2-t^2)/5) \longrightarrow \vec{N}(s,t) = \vec{X}_s \times \vec{X}_t = (2/5, 2/5, 1)$

$\iint_S \nabla\times\vec{F}\cdot d\vec{S} = \iint_S (t,-s,0)\cdot d\vec{S} = \iint_S (t,-s,0)\cdot(2/5,2/5,1)\,dS$

$= \int_0^1\int_0^1 (2/5\,t - 2/5\,s)\,ds\,dt = 0$

7.3.3 $S: x = \sqrt{16-y^2-z^2}$, $\partial S: y^2+z^2=16,\ x=0$

$\vec{F} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$\oint_{\partial S}\vec{F}\cdot d\vec{s} = \int_0^{2\pi} (0, 4\cos\theta, 4\sin\theta)\cdot(0,-4\sin\theta, 4\cos\theta)\,dt = \int_0^{2\pi} 0\,dt = 0$

$\nabla\times\vec{F} = (0,0,0) \rightarrow \iint_S \nabla\times\vec{F}\cdot d\vec{S} = \iint_S 0\,dS = 0$

7.3.8 $\vec{F} = x^2\hat{\imath} + y\hat{\jmath} + z\hat{k}$, $D = \{(x,y,z) \mid x^2+y^2+1 \leq z \leq 5\}$, $X(s,t) = (s\cos t, s\sin t, s^2+1)$

$0 \leq t \leq 2\pi,\ 0 \leq s \leq 2$

$\iiint_D \nabla\cdot\vec{F}\,dV = \int_0^{2\pi}\int_0^2\int_{1+r^2}^5 (2r\cos\theta + 2)\,r\,dz\,dr\,d\theta = 16\pi$

$\iint_{\partial D_1}\vec{F}\cdot d\vec{S} = \iint_{\partial D_1}(\vec{F}\cdot\vec{N})\,dS = \int_0^{2\pi}\int_0^2 \left[(r^2\cos^2\theta, r\sin\theta, 5)\cdot\hat{k}\right] r\,dr\,d\theta = 20\pi$

$\iint_{\partial D_2}\vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^2 (s^2\cos^2 t, s\sin t, s^2+1)\cdot(2s^2\cos t, 2s^2\sin t, -s)\,ds\,dt = -4\pi$

$\iiint_D \nabla\cdot\vec{F}\,dV = \oiint_{\partial D}\vec{F}\cdot d\vec{S} = \iint_{\partial D_1}\vec{F}\cdot d\vec{S} + \iint_{\partial D_2}\vec{F}\cdot d\vec{S}$ ✓