Problem Set 7

1 $\vec{F}:\mathbb{R}^2\to\mathbb{R}^2$, $\vec{F}(x,y)=x\hat{\imath}+y\hat{\jmath}$

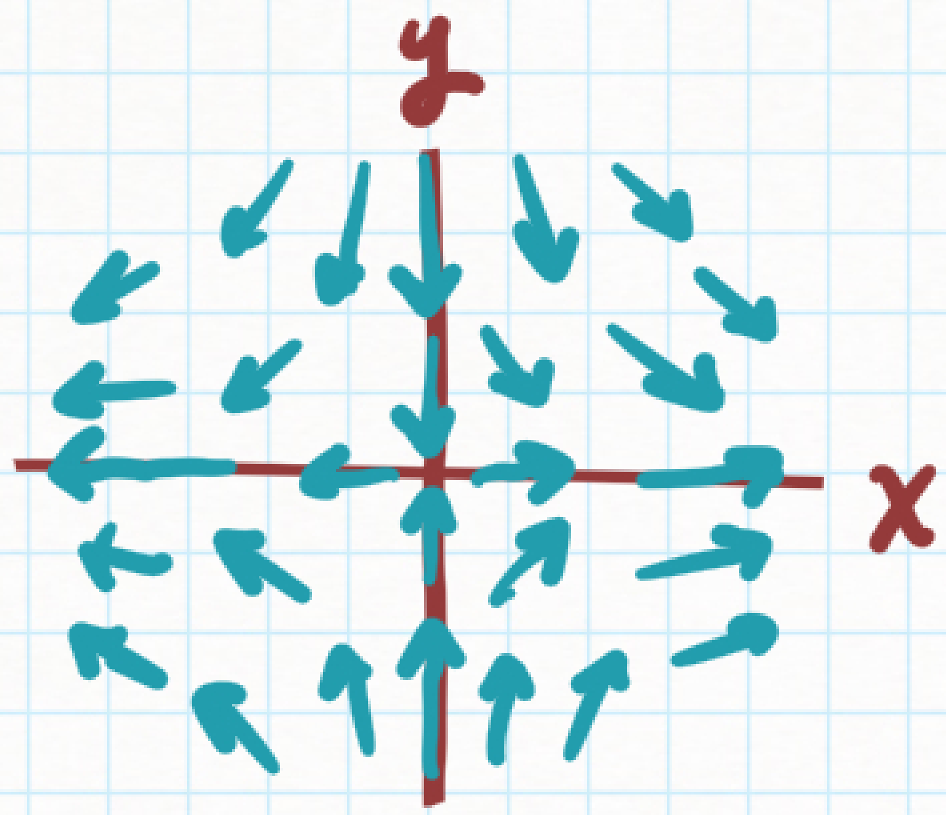
(a) $f(x,y)=x^2/2\,\hat{\imath}+y^2/2\,\hat{\jmath}+C$

(b) $\vec{r}(t)=(x(t),y(t))$, $x'(t)=x(t)$, $y'(t)=y(t)$,

$(x(0),y(0))=(a,b)\neq(0,0)$

$\to$ $x(t)=ae^t$, $y(t)=be^t$, $\vec{r}(t)=(ae^t,be^t)$

3.3.2 $\vec{F}=x\hat{\imath}-y\hat{\jmath}$



3.3.21 $\vec{F}(x,y)=(x^2,y)$, $\vec{x}(1)=(1,e)$

Using Defn 3.2, $\vec{x}'(t)=\vec{F}(\vec{x}(t))$.

$\frac{dx}{dt}=x^2$, $x(1)=1\longrightarrow x(t)=\frac{1}{2-t}$

$\frac{dy}{dt}=y$, $y(1)=e\longrightarrow y(t)=e^t$

$\therefore\ \vec{x}(t)=\left(\frac{1}{2-t},e^t\right)$

3.3.24 $F = 2x\hat{\imath} + 2y\hat{\jmath} - 3\hat{k}$

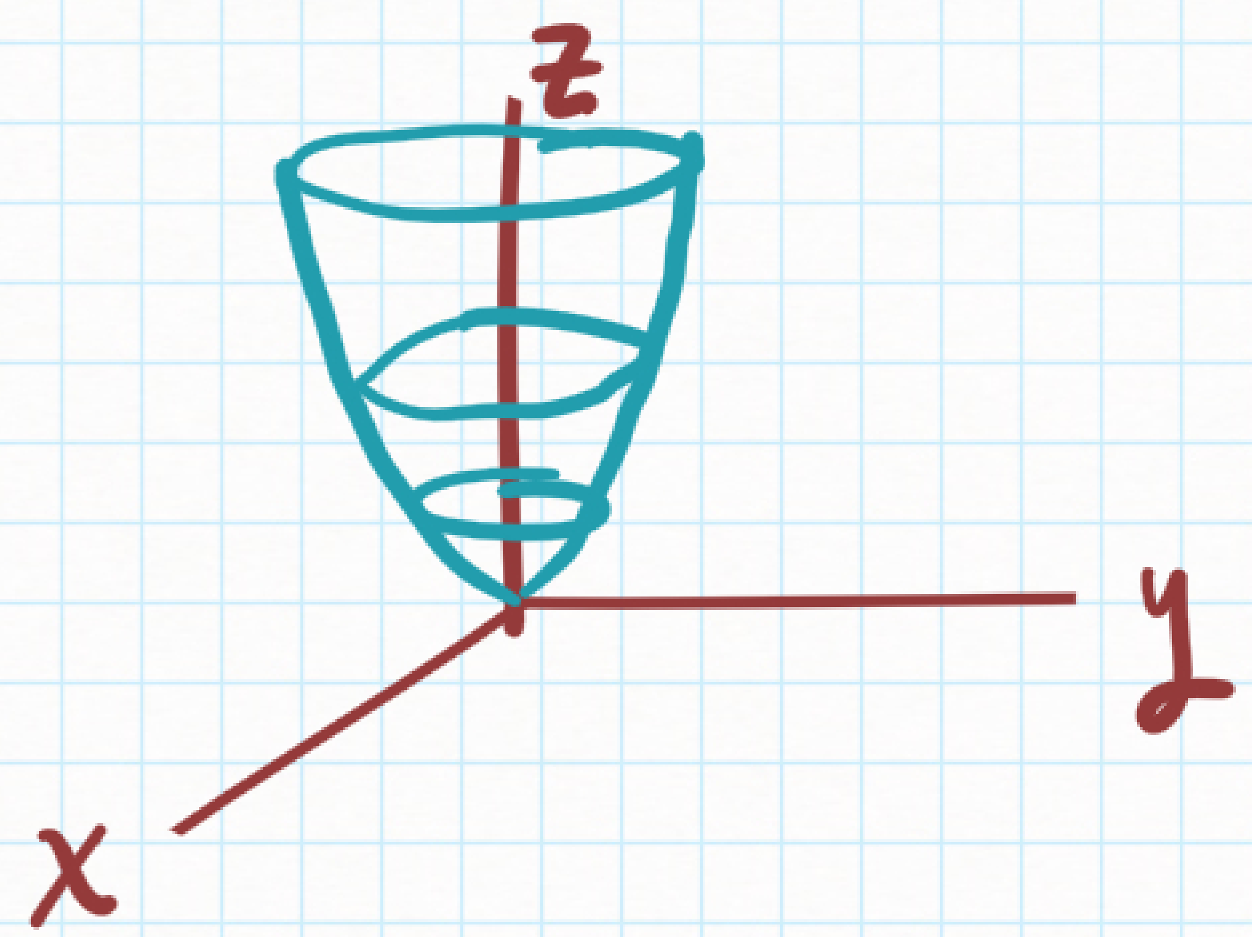
(a) Let $f(x,y,z) = x^2 + y^2 - 3z + d$, $d \in \mathbb{R}$.

$$\nabla f = 2x\hat{\imath} + 2y\hat{\jmath} - 3\hat{k} = F$$

(b) The equipotential sets of F are sets of the form

$$\{\vec{x} \mid c = f(\vec{x})\} = \{(x,y,z) \mid c = x^2 + y^2 - 3z\}$$

Rearranging to $z = x^2/3 + y^2/3 - c$ helps us to see that these surfaces are paraboloids opening upward along the positive z-axis, shifted down by a constant c. For $c = 0$, the surface looks like this:



4.1.8 $f(x,y) = 1/(x^2+y^2+1)$, $\vec{a} = (0,0)$

$\vec{p}_1(\vec{x}) = f(\vec{a}) + Df(\vec{a})(\vec{x} - \vec{a})$

$= f(\vec{a}) + [f_x(\vec{a}) \ \ f_y(\vec{a})](\vec{x} - \vec{a})$

$= 1/(0^2+0^2+1) + [-2\cdot 0/(0^2+0^2+1)^2 \ \ -2\cdot 0/(0^2+0^2+1)^2]\begin{bmatrix} x-0 \\ y-0 \end{bmatrix}$

$\boxed{= 1}$

$\vec{p}_2(\vec{x}) = f(\vec{a}) + Df(\vec{a})\vec{h} + \frac{1}{2}\vec{h}^T Hf(\vec{a})\vec{h}$, $\vec{h} = \vec{x} - \vec{a}$

$f_{xx}(\vec{x}) = [-2(x^2+y^2+1)^2 + 8x^2(x^2+y^2+1)]/(x^2+y^2+1)^4$, $f_{xx}(\vec{a}) = -2$

$f_{yy}(\vec{x}) = [-2(x^2+y^2+1)^2 + 8y^2(x^2+y^2+1)]/(x^2+y^2+1)^4$, $f_{yy}(\vec{a}) = -2$

$f_{xy}(\vec{x}) = f_{yx}(\vec{x}) = 8xy/(x^2+y^2+1)$, $f_{xy}(\vec{a}) = f_{yx}(\vec{a}) = 0$

$p_2(\vec{x}) = 1 + 0 + \frac{1}{2}[x \ \ y]\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \boxed{1 - x^2 - y^2}$

4.1.19 $f(x,y,z) = x^3 + x^2y - yz^2 + 2z^3$, $\vec{a} = (1,0,1)$

$f_x = 3x^2 + 2xy$, $f_{xx} = 6x + 2y$, $f_{xy} = 2x$, $f_{xz} = 0$

$f_y = x^2 - z^2$, $f_{yx} = 2x$, $f_{yy} = 0$, $f_{yz} = -2z$

$f_z = -2yz + 6z^2$, $f_{zx} = 0$, $f_{zy} = -2z$, $f_{zz} = -2y + 12z$

$$Hf(\vec{a}) = \begin{bmatrix} f_{xx}(\vec{a}) & f_{xy}(\vec{a}) & f_{xz}(\vec{a}) \\ f_{yx}(\vec{a}) & f_{yy}(\vec{a}) & f_{yz}(\vec{a}) \\ f_{zx}(\vec{a}) & f_{zy}(\vec{a}) & f_{zz}(\vec{a}) \end{bmatrix} = \begin{bmatrix} 6 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 12 \end{bmatrix}$$

4.1.24 $P_2(\vec{x}) = f(\vec{a}) + Df(\vec{a})\vec{h} + \frac{1}{2}\vec{h}^T Hf(\vec{a})\vec{h}$

$$= 3 + \begin{bmatrix} 3 & 0 & 6 \end{bmatrix} \begin{bmatrix} x-1 \\ y \\ z-1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x-1 & y & z-1 \end{bmatrix} \begin{bmatrix} 6 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 12 \end{bmatrix} \begin{bmatrix} x-1 \\ y \\ z-1 \end{bmatrix}$$

$$= 3 + 3(x-1) + 6(z-1) + \begin{bmatrix} x-1 & y & z-1 \end{bmatrix} \begin{bmatrix} 3(x-1) + y \\ x-1 - (z-1) \\ -y + 6(z-1) \end{bmatrix}$$

$$= 3 + 3(x-1) + 6(z-1) + 3(x-1)^2 + 6(z-1)^2 + 2y(x-1) - 2y(z-1)$$

4.1.26 $f(x,y,z)=e^{x+2y+3z}$, $\vec{a}=\vec{0}=(0,0,0)$

$$P_3(\vec{x}) = f(\vec{0}) + f_x(\vec{0})x + f_y(\vec{0})y + f_z(\vec{0})z$$
$$+ \frac{1}{2}\Big[f_{xx}(\vec{0})x^2 + f_{xy}(\vec{0})xy + f_{xz}(\vec{0})xz + f_{yx}(\vec{0})yx + f_{yy}(\vec{0})y^2$$
$$+ f_{yz}(\vec{0})yz + f_{zx}(\vec{0})zx + f_{zy}(\vec{0})zy + f_{zz}(\vec{0})z^2\Big]$$
$$+ \frac{1}{3!}\sum_{i,j,k=1}^{3} f_{x_i x_j x_k}(\vec{0})\,x_i x_j x_k \quad (x_1 = x,\ x_2 = y,\ x_3 = z)$$

$$= 1 + x + y + z + \frac{1}{2}x^2 + 2y^2 + \frac{9}{2}z^2 + 2xy + 3xz + 6yz$$
$$+ \frac{1}{6}x^3 + \frac{4}{3}y^3 + \frac{9}{2}z^3 + x^2y + \frac{3}{2}x^2z + 2xy^2 + 6y^2z$$
$$+ \frac{9}{2}xz^2 + 9yz^2 + 6xyz$$

4.1.41 $f(x,y) = \cos x \sin y$, $\vec{a} = (0, \pi/2)$

(a) $f_x = -\sin x \sin y$, $f_y = \cos x \cos y$

$f_{xx} = -\cos x \sin y$, $f_{yy} = -\cos x \sin y$, $f_{xy} = f_{yx} = -\sin x \cos y$

$$P_2(\vec{x}) = f(\vec{a}) + Df(\vec{a})\vec{h} + \frac{1}{2} h^T Hf(\vec{a}) h$$

$$= f(\vec{a}) + \begin{bmatrix} f_x(\vec{a}) & f_y(\vec{a}) \end{bmatrix} \begin{bmatrix} x \\ y - \pi/2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x & y - \pi/2 \end{bmatrix} \begin{bmatrix} f_{xx}(\vec{a}) & f_{xy}(\vec{a}) \\ f_{yx}(\vec{a}) & f_{yy}(\vec{a}) \end{bmatrix} \begin{bmatrix} x \\ y - \pi/2 \end{bmatrix}$$

$$= 1 + 0x + 0(y - \pi/2) + \frac{1}{2} \begin{bmatrix} x & y - \pi/2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y - \pi/2 \end{bmatrix}$$

$$= 1 - \frac{1}{2} x^2 - \frac{1}{2} (y - \pi/2)^2$$

(b) $\vec{h} = (h_1, h_2) = (x, y) - (0, \pi/2)$, $|h_1|, |h_2| \leq 0.3$

$$|R_2(x, y, 0, \pi/2)| = \frac{1}{3!} \left| \sum_{i,j,k=1}^{2} f_{x_i x_j x_k}(\vec{z}) h_i h_j h_k \right|$$

$$\leq \frac{1}{3!} \left| \sum 1 \cdot h_i h_j h_k \right| \quad (|\pm \cos|, |\pm \sin| \leq 1)$$

$$\leq \frac{1}{6} \left(|h_1|^3 + 3|h_1|^2 |h_2| + 3|h_1| |h_2|^2 + |h_2|^3 \right)$$

$$\leq \frac{1}{6} \left(.027 + 3 \cdot 0.027 + 3 \cdot 0.027 + 0.027 \right)$$

$$= 0.036$$

4.1.42 $f(x,y) = e^{x+2y}$, $\vec{a} = \vec{0} = (0,0)$

(a) $f_x = f$, $f_y = 2f$, $f_{xx} = f$, $f_{yy} = 4f$, $f_{xy} = f_{yx} = 2f$

$$p_2(\vec{x}) = 1 + \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= 1 + x + 2y + \frac{1}{2} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} x+2y \\ 2x+4y \end{bmatrix}$$

$$= 1 + x + 2y + \frac{1}{2}\left(x^2 + 2xy + 2xy + 4y^2\right)$$

$$= 1 + x + 2y + \frac{1}{2}x^2 + 2xy + 2y^2$$

(b) $$|R_3(x, y, 0, 0)| = \frac{1}{3!}\left| f_{xxx}(\vec{z})|h_1|^3 + 3f_{xxy}(\vec{z})|h_1|^2|h_2| + 3f_{yyx}(\vec{z})|h_1||h_2|^2 + f_{yyy}(\vec{z})|h_2|^3 \right|$$

$$\leq \frac{1}{6}\left(8 f_{yyy}(\vec{z})\right)(0.1)^3$$

$$\leq \frac{8}{6} e^{0.3} (0.001)$$

$$\approx 1.8 \times 10^{-3}.$$

(since $f_{yyy} = 8f$ is the largest 3^{rd} order derivative and $|x| = |h_1|$, $|y| = |h_2| \leq 0.1$ $\rightarrow e^{x+2y} \leq e^{0.3}$)

A tighter bound could be found if we considered each 3^{rd} order derivative separately.