Problem Set 10

1 Verify $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$. Let $I = \int_0^{\infty} e^{-x^2}\,dx$.

(i) Let $D_a = \{(x,y) \mid x^2+y^2 \le a^2, x \geq 0, y \geq 0\} \subseteq [0,a] \times [0,a]$,

$$\lim_{a\to\infty} \iint_{D_a} e^{-x^2-y^2}\,dA \le \lim_{a\to\infty} \int_0^a \int_0^a e^{-x^2-y^2}\,dy\,dx = I^2$$

$$\iint_{D_a} e^{-x^2-y^2}\,dA = \int_0^{\pi/2}\int_0^a e^{-r^2}\,r\,dr\,d\theta = (\pi/2)(1-e^{-a^2})/2 = \frac{\pi}{4}(1-e^{-a^2})$$

$$\therefore I^2 \geq \lim_{a\to\infty} \iint_{D_a} e^{-x^2-y^2}\,dA = \lim_{a\to\infty} \frac{\pi}{4}(1-e^{-a^2}) = \frac{\pi}{4}.$$

(ii) Let $D_a = \{(x,y) \mid x^2+y^2 \le 2a^2, x \geq 0, y \geq 0\} \supseteq [0,a] \times [0,a]$

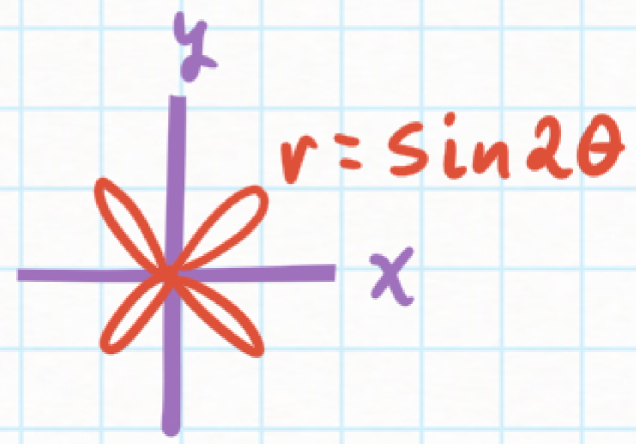
$$I^2 = \lim_{a\to\infty} \int_0^a \int_0^a e^{-x^2-y^2}\,dy\,dx \le \lim_{a\to\infty} \iint_{D_a} e^{-x^2-y^2}\,dA = \lim_{a\to\infty} \frac{\pi}{4}(1-e^{-2a^2}) = \frac{\pi}{4}.$$

By (i) and (ii), $I^2 = \pi/4$. Conclude $\int_{-\infty}^{\infty} e^{-x^2}\,dx = 2I = 2\sqrt{\frac{\pi}{4}} = \sqrt{\pi}$.

5.5.15 $\displaystyle\iint_{x^2+y^2\le 9} (x^2+y^2)^{3/2}\,dA = \int_0^{2\pi}\int_0^3 (r^2)^{3/2}\, r\,dr\,d\theta = 2\pi\int_0^3 r^4\,dr = \frac{486\pi}{5}$

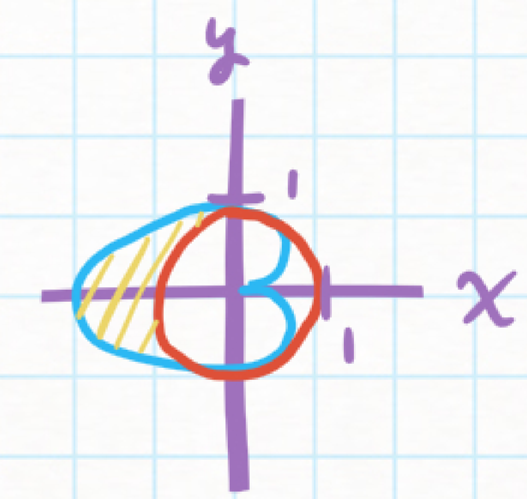
5.5.16 $\displaystyle\int_{-a}^{a}\int_0^{\sqrt{a^2-y^2}} e^{x^2+y^2}\,dx\,dy = 2\int_0^{\frac{\pi}{2}}\int_0^a e^{r^2}\, r\,dr\,d\theta = \frac{\pi}{2}\left(e^{a^2}-1\right)$

5.5.20 $\displaystyle\iint_{\text{rose}} dA = 4\int_0^{\pi/2}\int_0^{\sin 2\theta} r\,dr\,d\theta = 4\int_0^{\pi/2}\frac{\sin^2 2\theta}{2}\,d\theta = \int_0^{\pi/2}(1-\cos 4\theta)\,d\theta = \pi/2$

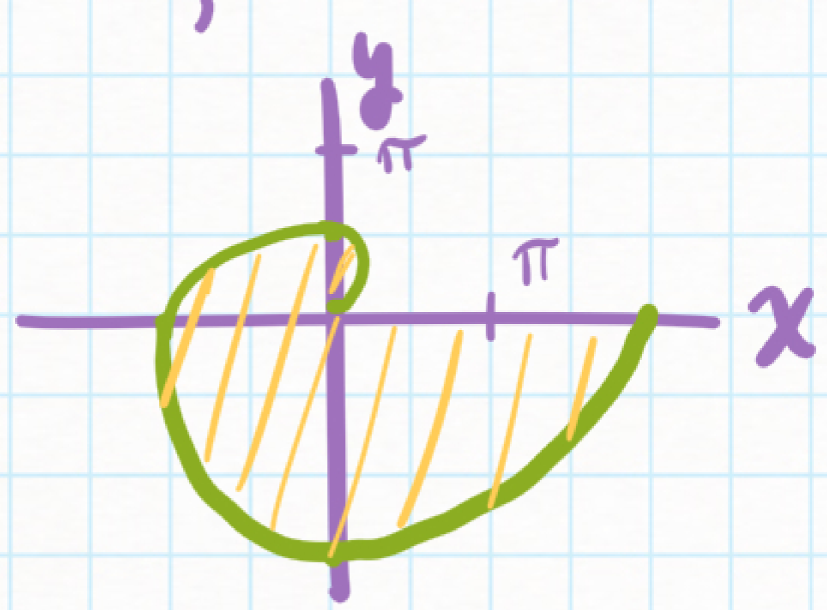


5.5.23 Area inside the cardioid $r = 1-\cos\theta$ and outside the circle $r=1$.

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_1^{1-\cos\theta} r\,dr\,d\theta = 2\int_{\pi/2}^{\pi}\frac{1}{2}\left[(1-\cos\theta)^2-1\right]d\theta$$
$$= \int_{\pi/2}^{\pi}\left(-2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right)d\theta$$
$$= -2\sin\theta + \theta/2 + \frac{1}{4}\sin 2\theta\Big|_{\pi/2}^{\pi}$$
$$= 2 + \frac{\pi}{4}$$



5.5.24 $r = 3\theta,\ 0\le\theta\le 2\pi$



$$\int_0^{2\pi}\int_0^{3\theta} r\,dr\,d\theta$$
$$= \int_0^{2\pi} 9\theta^2/2\,d\theta$$
$$= 12\pi^3$$

5.5.39 $x^2+y^2\le 1,\ x^2/5 + y^2/5 + z^2/10 \le 1$

$$\iiint_W dV = 2\int_0^{2\pi}\int_0^1\int_0^{\sqrt{10-2r^2}} r\,dz\,dr\,d\theta$$
$$= 4\pi\int_0^1 r\sqrt{10-2r^2}\,dr$$
$$= \pi\int_8^{10}\sqrt{u}\,du$$
$$= \frac{4\pi\sqrt{2}}{3}\left(5\sqrt{5}-8\right)$$

5.5.40 $z=9-x^2-y^2$, $z=0$, $x^2+y^2=4$

$$\int_0^{2\pi}\int_0^{2}\int_0^{9-r^2} r\,dz\,dr\,d\theta = 2\pi\int_0^2 (9r-r^3)\,dr = 28\pi$$

5.5.41 W is the region inside $x^2+y^2+z^2=25$ and above $z=3$.

$$\iiint_W (2+x^2+y^2)\,dV = \int_0^{2\pi}\int_3^5\int_0^{\sqrt{25-z^2}} (2+r^2)\,r\,dr\,dz\,d\theta$$

$$= 2\pi\int_3^5 \left(r^2+\frac{r^4}{4}\right)\Big|_0^{\sqrt{25-z^2}} dz$$

$$= 2\pi\int_3^5 \left(25-z^2+(625-50z^2+z^4)/4\right)dz$$

$$= 2\pi\left(25z-\frac{z^3}{3}+\frac{625}{4}z-\frac{25}{6}z^3+\frac{z^5}{20}\right)\Big|_3^5$$

$$= 656\pi/5$$

5.5.42 Volume $= 16\int_0^{\frac{\pi}{4}}\int_0^{a}\int_0^{\sqrt{a^2-r^2\cos^2\theta}} r\,dz\,dr\,d\theta$

$$= 16\int_0^{\frac{\pi}{4}}\int_0^{a} r\sqrt{a^2-r^2\cos^2\theta}\,dr\,d\theta$$

$$= \frac{16}{3}\int_0^{\frac{\pi}{4}} \frac{1-\sin^3\theta}{\cos^2\theta}\,d\theta$$

$$= (16-8\sqrt{2})a^3$$