4.2.35 $f: R \to \mathbb{R}$, $f(x,y) = \cos x \sin y$, $R = [0, 2\pi] \times [0, 2\pi]$

The minimum value f can take on is -1. On R this occurs if $\cos x = -1$ and $\sin y = 1$ or $\cos x = 1$ and $\sin y = -1$. This means $(x,y) = (\pi, \pi/2), (0, 3\pi/2), (2\pi, 3\pi/2)$

The maximum value f can take on is 1. On R this occurs if $\cos x = 1$ and $\sin y = 1$ or $\cos x = -1$ and $\sin y = -1$. This means $(x,y) = (0, \pi/2), (2\pi, \pi/2), (\pi, 3\pi/2)$

Note: In the book, $f(x,y) = \sin x \cos y$. I switched sine and cosine when copying down the problem. This switches the position of x and y in the answers but since the reasoning is the same, it's not worth reworking the problem to fix this.

4.2.52

(b) $f(x,y) = 3ye^{x} - e^{3x} - y^{3}$

$\vec{0} = \nabla f(x,y) = (3ye^{x} - 3e^{3x}, 3e^{x} - 3y^{2})$

$\longrightarrow\ y = e^{2x},\ y^{2} = e^{x} \longrightarrow 0 = (e^{x})^{2} - e^{x} \longrightarrow (x,y) = (0,1)$

$Hf(0,1) = \begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix}$ $d_1 = -6 < 0$, $d_2 = 27 > 0$ local maximum at $(0,1)$ with $f(0,1) = 1$.

Along the y-axis, $x = 0$ and $f = f(0,y) = 3y - 1 - y^{3}$. Then $f(0,y) > -y^{3} > M$ $\forall$ $M \in \mathbb{R}$ if $y < -\sqrt[3]{M}$. That is, $\lim_{y \to -\infty} f(0,y) = +\infty$ so f has no global maximum. This shows that unlike the 1-variable case, having a unique critical point with a strict local extremum does not mean that the function also has a global extremum at that point.

8 $V: \mathbb{R}^3 \to \mathbb{R}$, $V(x,y,z) = xyz$

We maximize V on the set A of points $x,y,z > 0$ and $2(xy+xz+yz) = a$.

(i) Let $g = 2(xy+xz+yz)$ and set the constraint $g = a$.

$(yz, xz, xy) = \nabla V = \lambda \nabla g = 2\lambda(y+z, x+z, x+y)$

$z(y-x) = yz - xz = 2\lambda(y+z-x-z) = 2\lambda(y-x) \to z = 2\lambda$ if $y \neq x$

Then $2y\lambda = yz = 2\lambda(y+2\lambda)$, which implies $\lambda = 0$ and so $z = 0$.

But this means $xy = 2\lambda(x+y) = 0$, contradicting $x,y>0$. $\therefore y = x$.

We have $x^2 = 2x\lambda \to x = 0$ or $x = 2\lambda$. Since $x > 0$, $x = y = 2\lambda$. Repeating this argument with $x(y-z) = 2\lambda(y-z)$ shows $z = y = x = 2\lambda$. By $g = a$, $a = 2(xy+xz+yz) = 6x^2 \to x = y = z = \sqrt{a/6}$.

$$P = \left(\sqrt{a/6}, \sqrt{a/6}, \sqrt{a/6}\right).$$

(ii) Let $K \subseteq A$ be the set of points where $x,y,z \geq \sqrt{a}/3\sqrt{6}$.

Let $Q \in A - K$. WLOG assume $x < \sqrt{a}/3\sqrt{6}$ for $Q = (x,y,z)$. Since $g = a$, $a/2 = xy+xz+yz > yz$ and $V(Q) = xyz < \sqrt{a}/3\sqrt{6} \cdot a/2 = (a/6)^{3/2} = V(P)$.

(iii) Since $K = \{(x,y,z) \mid x,y,z \geq \sqrt{a}/3\sqrt{6} \wedge 2(xy+xz+yz) = a\}$, K can be defined as the intersection of two closed sets. Thus K is closed. Since $a/2 = xy+xz+yz \geq x(y+z) \geq x\, 2\sqrt{a}/3\sqrt{6}$, x is bounded. Similarly, y and z are bounded. Since K is a closed and bounded subset of $\mathbb{R}^3$, K is compact.

(iv) Since K is compact, V attains its maximum on K. By (i), we know P is the only critical point of V on $K \subseteq A$. It remains to check V on the boundary of K. Suppose $x = \sqrt{a}/3\sqrt{6}$ for some point $R = (x, y, z) \in K$. Then $V(R) = xyz = \sqrt{a}/3\sqrt{6}\, yz < \sqrt{a}/3\sqrt{6} \cdot a/2 = (a/6)^{3/2} = V(P)$. The argument is similar along the boundaries $y = \sqrt{a}/3\sqrt{6}$ and $z = \sqrt{a}/3\sqrt{6}$. So V is not maximized on the boundary of K. Conclude P is the maximizer of V on K.

(v) By (ii) $V(Q) < V(P)$ $\forall\, Q \in A - K$. Since P maximizes V on K and $V(P)$ is greater than $V(Q)$ for $Q \in A-K$, $V(P) > V(Q)$ $\forall\, Q \in A$.

4.3.2 $f(x,y) = y \,,\; g(x,y) = 2x^2 + y^2 = 4$

$$(0,1) = \nabla f = \lambda \nabla g = (4\lambda x, 2\lambda y)$$

$$\longrightarrow \quad x = 0 \,,\; y = 1/2\lambda$$

$$2 \cdot 0^2 + (1/2\lambda)^2 = 4$$

$$\lambda = \pm 1/4$$

$$(x,y) = (0, \pm 2)$$

4.3.12 $f(x,y,z) = x + y + z \,,\; g_1(x,y,z) = y^2 - x^2 = 1 \,,\; g_2(x,y,z) = x + 2z = 1$

$$(1,1,1) = \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 = (-2\lambda_1 x, 2\lambda_1 y, 0) + (\lambda_2, 0, 2\lambda_2)$$

$$x = (\lambda_2 - 1)/2\lambda_1 \qquad x = -1/4\lambda_1 \qquad \lambda_1 = \pm\sqrt{3}/4$$

$$y = 1/2\lambda_1 \quad \longrightarrow \quad y = 1/2\lambda_1 \quad \longrightarrow \quad x = -1/\sqrt{3},\; 1/\sqrt{3}$$

$$\lambda_2 = 1/2 \qquad 1 = y^2 - x^2 = 3/16\lambda_1^2 \qquad y = 2/\sqrt{3},\; -2/\sqrt{3}$$

$$z = (1-x)/2$$

Critical Points: $(x_0, y_0, z_0) = \left(-1/\sqrt{3},\; 2/\sqrt{3},\; (1 + 1/\sqrt{3})/2\right),$

$\left(1/\sqrt{3},\; -2/\sqrt{3},\; (1 - 1/\sqrt{3})/2\right)$

4.3.22 $f(x,y,z) = x + y - z$, $g(x,y,z) = x^2 + y^2 + z^2 = 81$

$\nabla f = \lambda \nabla g$

$(1,1,-1) = (2\lambda x, 2\lambda y, 2\lambda z)$

$x = 1/2\lambda$, $y = 1/2\lambda$, $z = -1/2\lambda$

$81 = 3/4\lambda^2 \longrightarrow \lambda = \pm 1/6\sqrt{3}$

$f(x_0, y_0, z_0) = f(3\sqrt{3}, 3\sqrt{3}, -3\sqrt{3}) = 9\sqrt{3}$ maximum

$f(x_1, y_1, z_1) = f(-3\sqrt{3}, -3\sqrt{3}, 3\sqrt{3}) = -9\sqrt{3}$ minimum

Since the sphere $x^2+y^2+z^2=81$ is closed (b/c the complement is open) and bounded, it is compact. This means that the maximum and minimum values of f on this compact set are attained for some point(s) on the sphere. Since there are two critical points of f subject to the constraint, these points must produce the extrema of f on the sphere.