§ Partial Fractions and Inverse Laplace Transform

Rules for the Laplace transform

Definition:	$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt \quad \text{for} \quad \mathrm{Re}\,(s) \gg 0.$
Linearity:	$\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s).$
$\mathcal{L}^{-1}$:	$F(s)$ essentially determines $f(t)$ for $t > 0$.
s-shift rule:	$\mathcal{L}[e^{rt}f(t)] = F(s-r).$
s-derivative rule:	$\mathcal{L}[tf(t)] = -F'(s).$
t-derivative rule:	$\mathcal{L}[f'(t)] = sF(s) - f(0^-).$

Formulas for the Laplace transform

$$\mathcal{L}[1] = \frac{1}{s}, \quad \mathcal{L}[\delta(t-a)] = e^{-as}$$

$$\mathcal{L}[e^{rt}] = \frac{1}{s-r}, \quad \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2}, \quad \mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

Problem 1: Find $\mathcal{L}[t^4 e^{\pi t}]$

Answer:

$$\mathcal{L}[t^4 e^{\pi t}] = \frac{4!}{(s-\pi)^5}$$

Problem 2: Find $\mathcal{L}^{-1}\left[\frac{3}{2s-4}\right]$

Answer:

$$\mathcal{L}^{-1}\left[\frac{3}{2s-4}\right] = \frac{3}{2}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] = \frac{3}{2}e^{2t}$$

Problem 3: Find $\mathcal{L}^{-1}\left[\frac{1}{s^2+4s+4}\right]$

Answer:

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+4s+4}\right] = \mathcal{L}^{-1}\left[\frac{1}{(s+2)^2}\right] = te^{-2t}$$

Problem 5: Find $\mathcal{L}^{-1}\left[\frac{5s-6}{s^2-3s}\right]$

Answer:

$$\mathcal{L}^{-1}\left[\frac{5s-6}{s^2-3s}\right]=\mathcal{L}^{-1}\left[\frac{5s-6}{s(s-3)}\right]$$

$$=\mathcal{L}^{-1}\left[\frac{2}{s}+\frac{3}{s-3}\right]$$

$$=2\mathcal{L}^{-1}\left[\frac{1}{s}\right]+3\mathcal{L}^{-1}\left[\frac{1}{s-3}\right]$$

$$=2+3e^{3t}.$$

Problem 6: Find $\mathcal{L}^{-1}\left[\frac{5-2s}{s^2+7s+10}\right]$

Answer:

$$\mathcal{L}^{-1}\left[\frac{5-2s}{(s+2)(s+5)}\right]=3\mathcal{L}^{-1}\left[\frac{1}{s+2}\right]-5\mathcal{L}^{-1}\left[\frac{1}{s+5}\right]$$

$$=3e^{-2t}-5e^{-5t}$$