§ Linear Operators

Problem 1: Find a particular solution to the ODE.

$$\ddot{x} + x = t^2 + \cos(2t-1)$$

Answer:

$x_1 = At^2 + Bt + C$

$t^2 = 2A + At^2 + Bt + C$

$A = 1,\ B = 0,\ C = -2$

$x_1 = t^2 - 2$

$x_2 = A\cos(2t-1) + B\sin(2t-1)$

$\cos(2t-1) = -4A\cos(2t-1)$
$-4B\sin(2t-1) + A\cos(2t-1)$
$+ B\sin(2t-1)$

$A = -1/3,\ B = 0$
$x_2 = -1/3\cos(2t-1)$

$$x_p(t) = t^2 - 2 - \frac{1}{3}\cos(2t-1)$$

Problem 2: Find the general solution to

$$y'' + y' + y = 2xe^{x}$$

Answer:

$$r^2 + r + 1 = 0$$

$$r = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$y_h(x) = e^{-x/2}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right).$$

$$y_p(x) = Axe^{x} + Be^{x}$$

$$2xe^{x} = Ae^{x}[(x+2) + (x+1) + x] + 3Be^{x}$$

$$= 3Axe^{x} + (3A + 3B)e^{x}$$

$$A = 2/3, \quad B = -2/3$$

$$y_p(x) = \frac{2}{3}e^{x}(x-1)$$

$$y(x) = e^{-x/2}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right) + \frac{2}{3}e^{x}(x-1).$$

Problem 3: Find a particular solution to

$$y^{(4)} - 2y'' + y = xe^x$$

Answer: First checking $y_h(x)$ for redundancies.

$$r^4 - 2r^2 + 1 = 0 \rightarrow (r^2 - 1)^2 = 0 \rightarrow r = \pm 1 \text{ (repeated)}$$

$$y_p(x) = x^2(Ax + B)e^x = (Ax^3 + Bx^2)e^x$$

$$\begin{aligned} xe^x &= e^x\left[(D+1)^4 - 2(D+1)^2 + 1\right](Ax^3 + Bx^2) \\ &= e^x\left[D^4 + 4D^3 + 4D^2\right](Ax^3 + Bx^2) \\ &= e^x(24A + 24Ax + 8B) \longrightarrow A = \frac{1}{24},\ B = -\frac{1}{8} \end{aligned}$$

$$y_p(x) = \frac{x^2 e^x}{24}(x - 3).$$