§ Qualitative Behavior: Phase Portraits

Problem 1: Give the general solution to the DE system $\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ -1 & -4 \end{bmatrix} \mathbf{x}$ and also give its phase-plane picture (i.e its direction field graph together with a few typical solution curves).

Answer:

$0 = \lambda^2 + 6\lambda + 9$

$\lambda = -3$ repeated

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right] \to v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

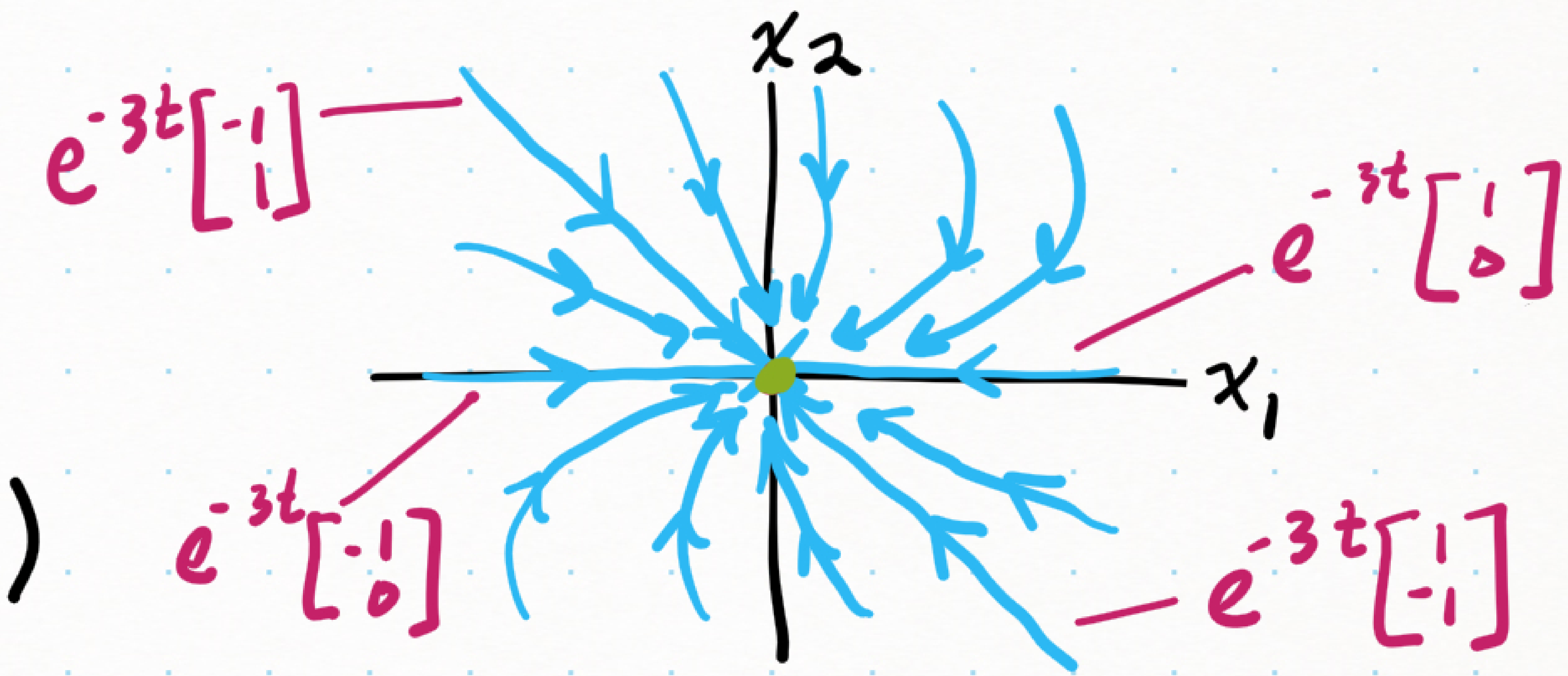
$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & -1 & -1 \end{array}\right] \to v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$x(t) = c_1 e^{-3t} v_1 + c_2 e^{-3t}(t v_1 + v_2)$$

$$x_1(t) = (c_1 + c_2 + t c_2) e^{-3t}$$

$$x_2(t) = (-c_1 - t c_1) e^{-3t}$$

$$x' = 0 \iff x = 0$$



Problem 2: For each of the following linear systems, carry out the graphing program laid out in this session, that is:

(i) find the eigenvalues of the associated matrix and from this determine the geometric type of the critical point at the origin, and its stability;

(ii) if the eigenvalues are real, find the associated eigenvectors and sketch the corresponding trajectories, showing the direction of motion for increasing t; then draw some nearby trajectories;

(iii) if the eigenvalues are complex, obtain the direction of motion and the approximate shape of the spiral by sketching in a few vectors from the vector field defined by the system.

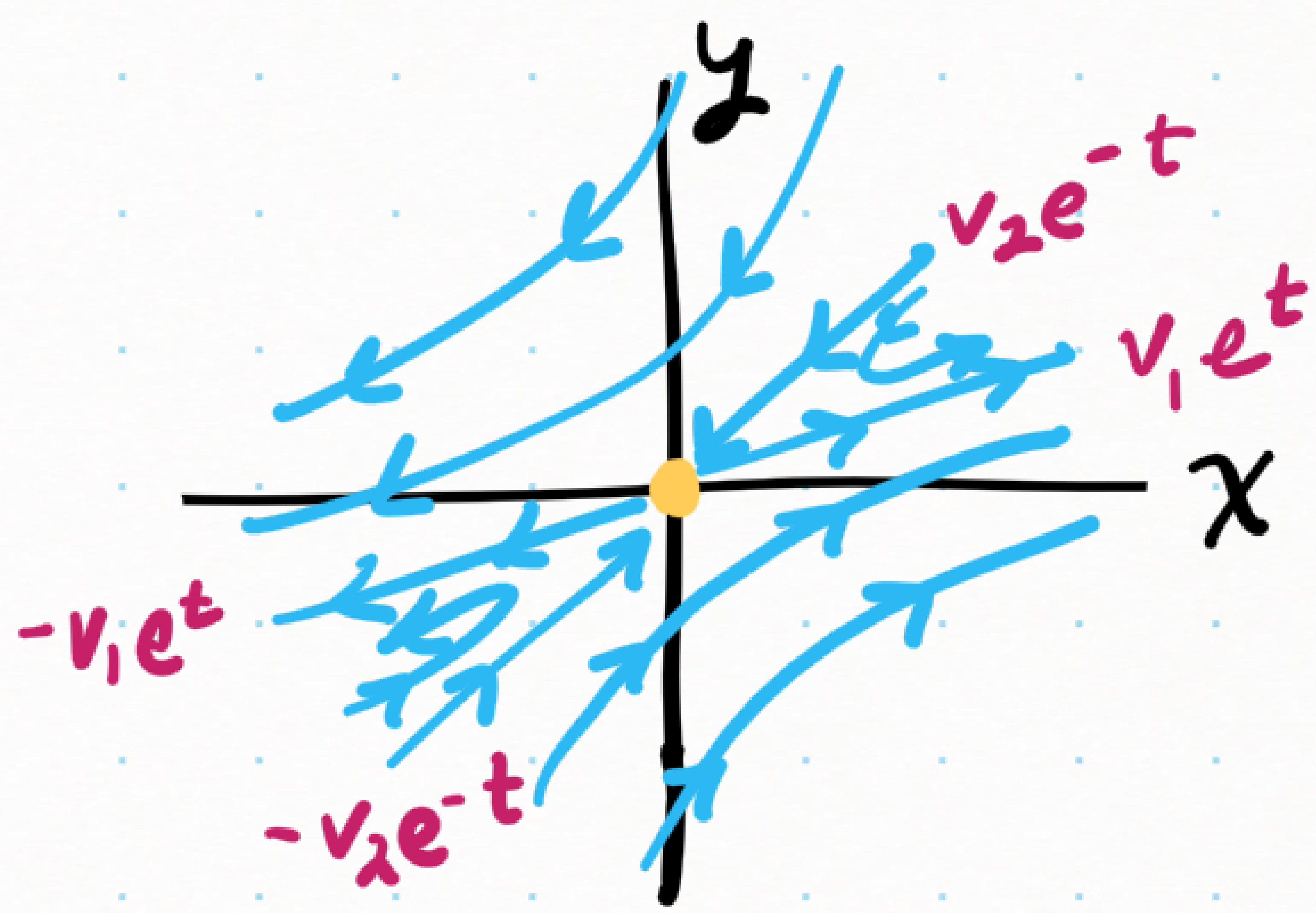
Answer:

a) $x' = 2x - 3y, \quad y' = x - 2y$

i. $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$

ii. $v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$0 = \lambda^2 - 1$
$\lambda = \pm 1$



The origin is an unstable saddlepoint.

b)

i. $x' = 2x$, $y' = 3x + y$

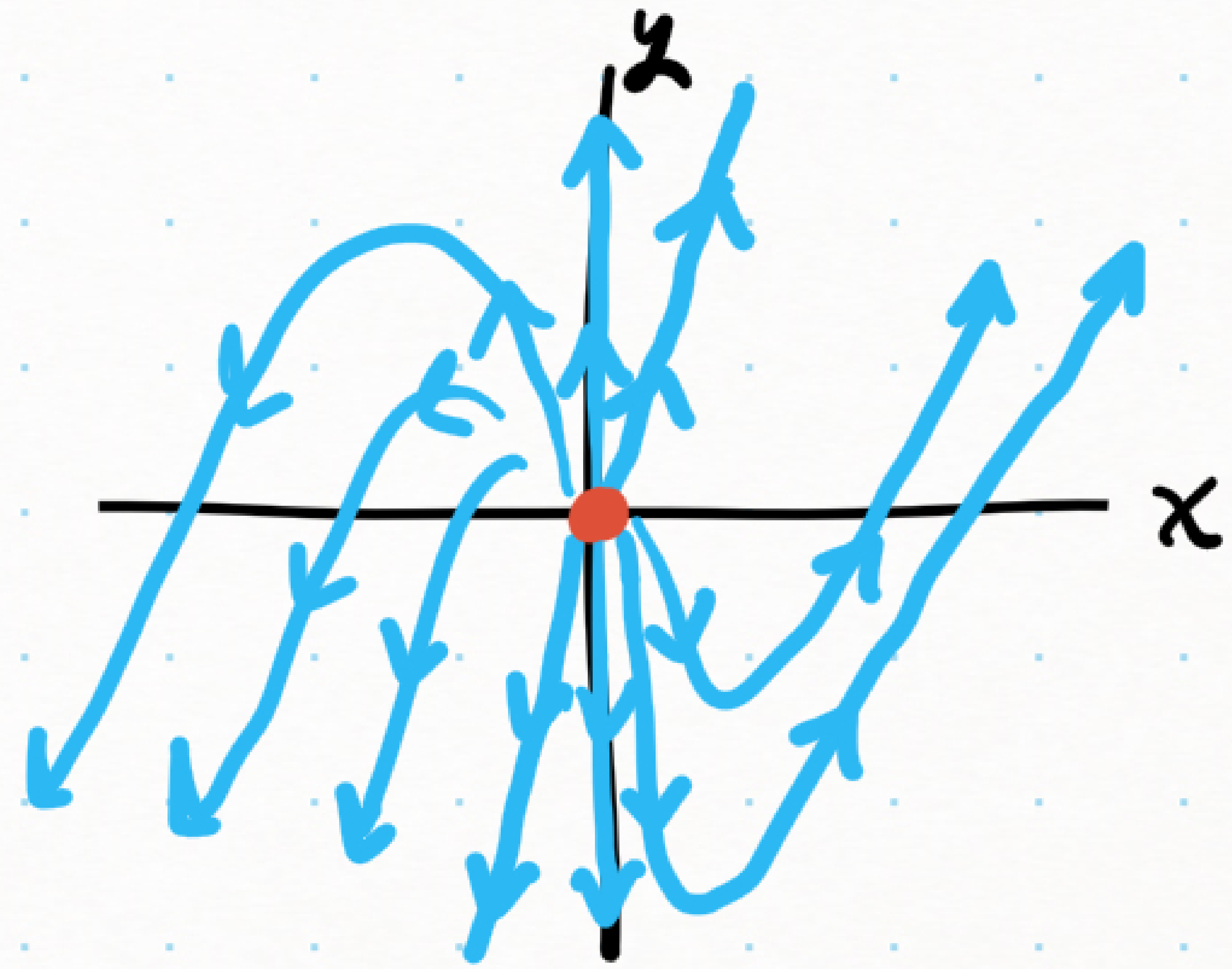
$$A = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$$

$0 = \lambda^2 - 3\lambda + 2$

$\lambda = 1, 2$

The origin is an unstable node.

ii. $v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$



c)

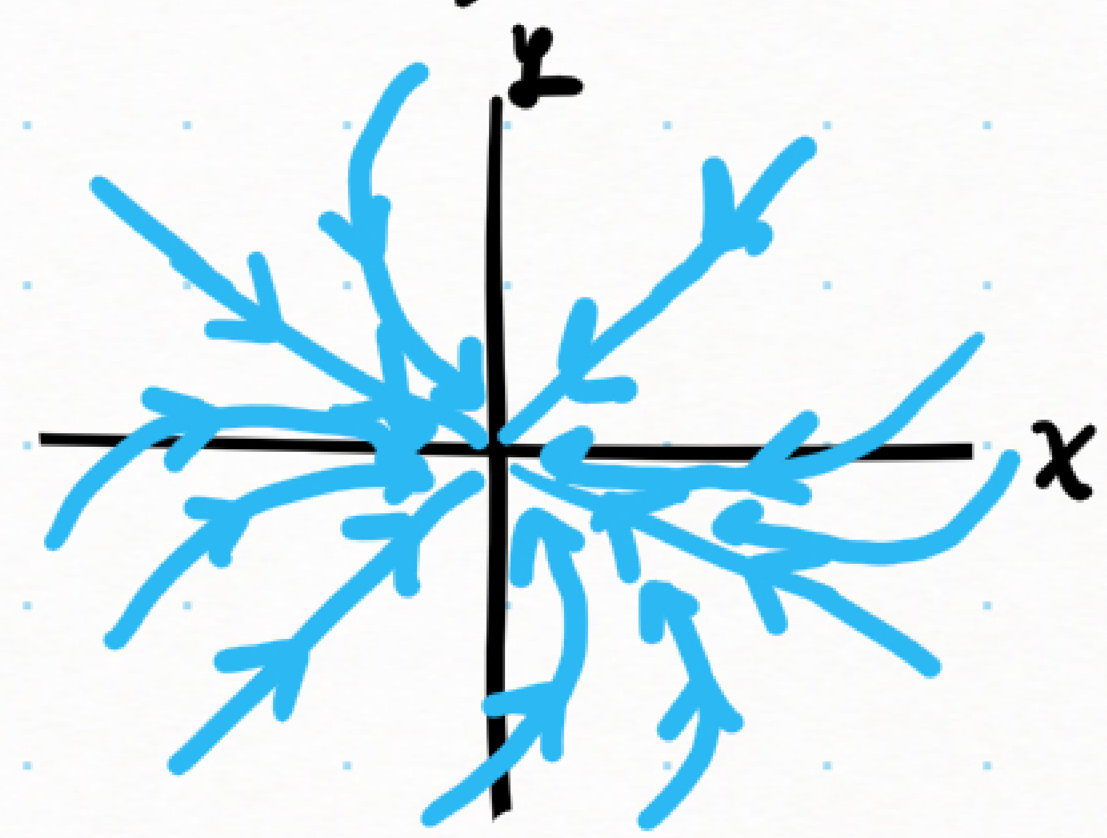
i. $x' = -2x - 2y$, $y' = -x - 3y$

$$A = \begin{bmatrix} -2 & -2 \\ -1 & -3 \end{bmatrix}$$

$0 = \lambda^2 + 5\lambda + 4 \rightarrow \lambda = -1, -4$

The origin is an asymptotically stable node.

ii. $v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



d)

i. $x' = x - 2y$, $y' = x + y$

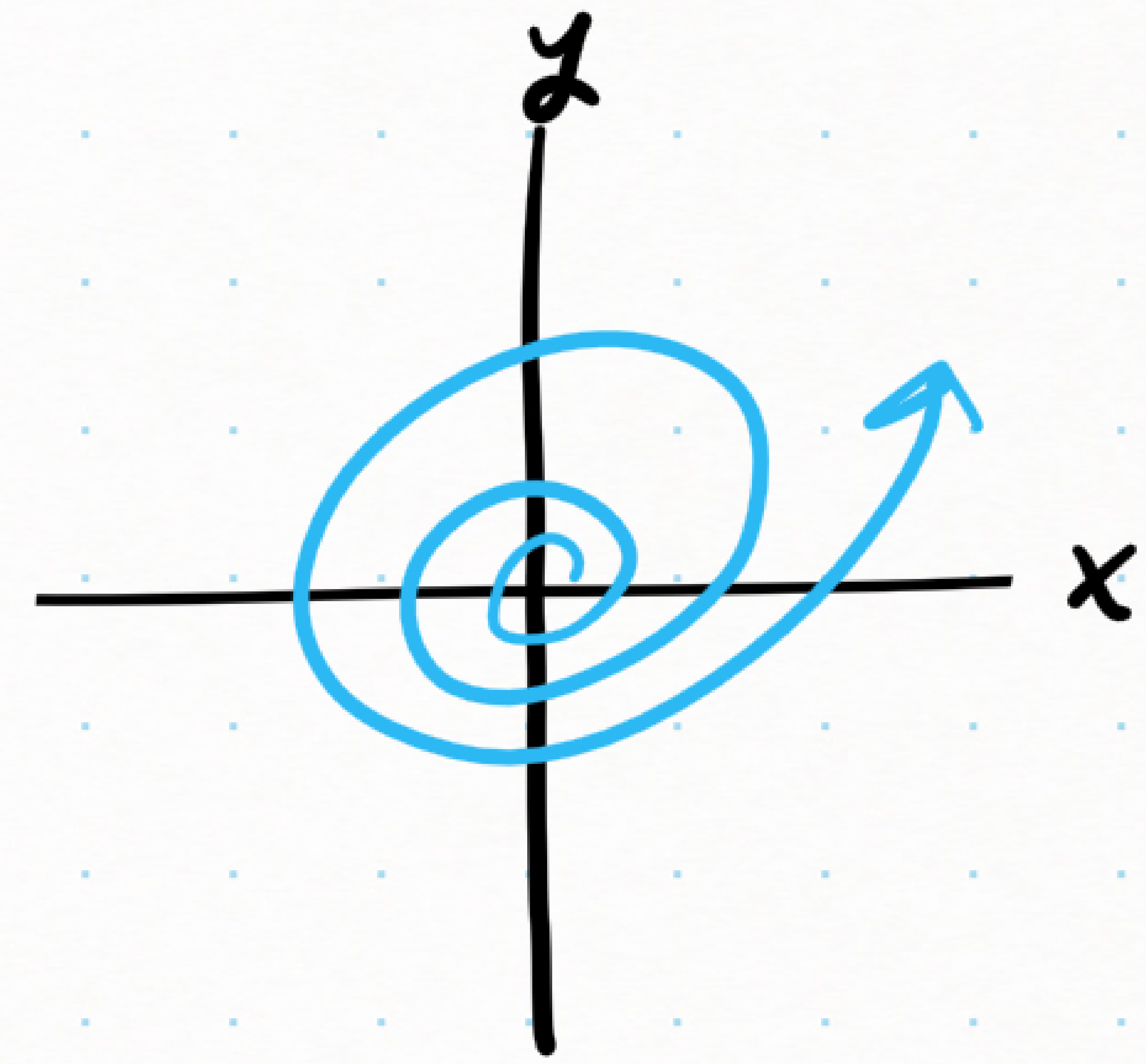
$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$

$0 = \lambda^2 - 2\lambda + 3$

$\lambda = 1 \pm \sqrt{2}\, i$

There is an unstable spiral about the origin.

iii.



e)

i. $x' = x + y$, $y' = -2x - y$

$A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$

$0 = \lambda^2 + 1$

$\lambda = \pm i$

The origin is a stable center.

iii.