§ Periodic Input and Resonance

Problem 1: For each spring-mass system, find whether pure resonance occurs, without actually calculating the solution.

a) $2x'' + 10x = F(t)$; $F(t) = 1$ on (0,1), $F(t)$ is odd, and of period 2;

b) $x'' + 4\pi^2 x = F(t)$; $F(t) = 2t$ on (0,1), $F(t)$ is odd, and of period 2;

c) $x'' + 9x = F(t)$; $F(t) = 1$ on $(0, \pi)$, $F(t)$ is odd, and of period 2π

Answer:

a) $\omega_0 = \sqrt{k/m} = \sqrt{10/2} = \sqrt{5}$. $F(t) = \sum_n b_n \sin(n\pi t)$.

Since $\sqrt{5} \neq n\pi \;\forall n$, no resonance.

b) $\omega_0 = \sqrt{4\pi^2} = 2\pi$. $F(t) = \sum_n b_n \sin(n\pi t)$

Since $2\pi = n\pi$ for $n=2$, resonance occurs.

c) $\omega_0 = \sqrt{\frac{9}{1}} = 3$. $F(t) = \sum_n b_n \sin(nt)$. Since all odd n are included in the sum, resonance occurs.

Problem 2: Find a periodic solution as a Fourier series to $x'' + 3x = F(t)$, where $F(t) = 2t$ on $(0, \pi)$, $F(t)$ is odd, and has period 2π.

Answer:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} 2t\sin(nt)\,dt$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}t\cos(nt)\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{1}{n}\cos(nt)\,dt\right]$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}\left(\pi\cos(n\pi) + \pi\cos(-n\pi)\right) + 0\right]$$

$$= -\frac{2\pi}{n\pi}2\cos(n\pi) = \frac{4}{n}(-1)^{n-1}$$

$$x'' + 3x = 4\left(\sin t - \frac{\sin 2t}{2} + \frac{\sin 3t}{3} - \dots\right)$$

$$x_n'' + 3x_n = b_n\sin(nt) \longrightarrow x_n = \frac{b_n\sin(nt)}{3-n^2}$$

$$x = \sum_n x_n = \sum_n \frac{b_n\sin(nt)}{3-n^2} = 4\sum_n \frac{(-1)^{n-1}\sin(nt)}{n(3-n^2)}$$