18.03SC Unit 1 Exam

1. **(a)** In a perfect environment, the population of Norway rat that breeds on the MIT campus increases by a factor of $e \simeq 2.718281828459045\ldots$ each year. Model this natural growth by a differential equation. [8]

What is the growth rate k?

$x(t)$: # of rats after t yrs.

$\dot{x} = kx$

$\downarrow$

$x(t) = x(0)e^{kt}$

$e\,x(0) = x(1) = x(0)e^{k}$

$e\,x(0) = e^{k}x(0)$

$k = 1$

(b) MIT is a limited environment, with a maximal sustainable Norway rat population of $R = 1000$ rats. Write down the logistic equation modeling this. (You may use "k" for the natural growth rate here if you failed to find it in **(a)**.) [4]

$$\dot{x}(t) = kx\left(1 - \frac{x}{1000}\right) = x\left(1 - \frac{x}{1000}\right)$$

(c) The MIT pest control service intends to control these rats by killing them at a constant rate of a rats per year. If it wants to limit the rat population to 75% of the maximal sustainable population, what rate a it should aim for (in rats per year)? [8]

Set $0 = \dot{x} - a$ to obtain an equilibrium. To have equilibrium at 75% of the maximal sustainable population, set $R = 750$.

$$750\left(1 - \frac{750}{1000}\right) - a = 0$$

$$\frac{750}{4} = a$$

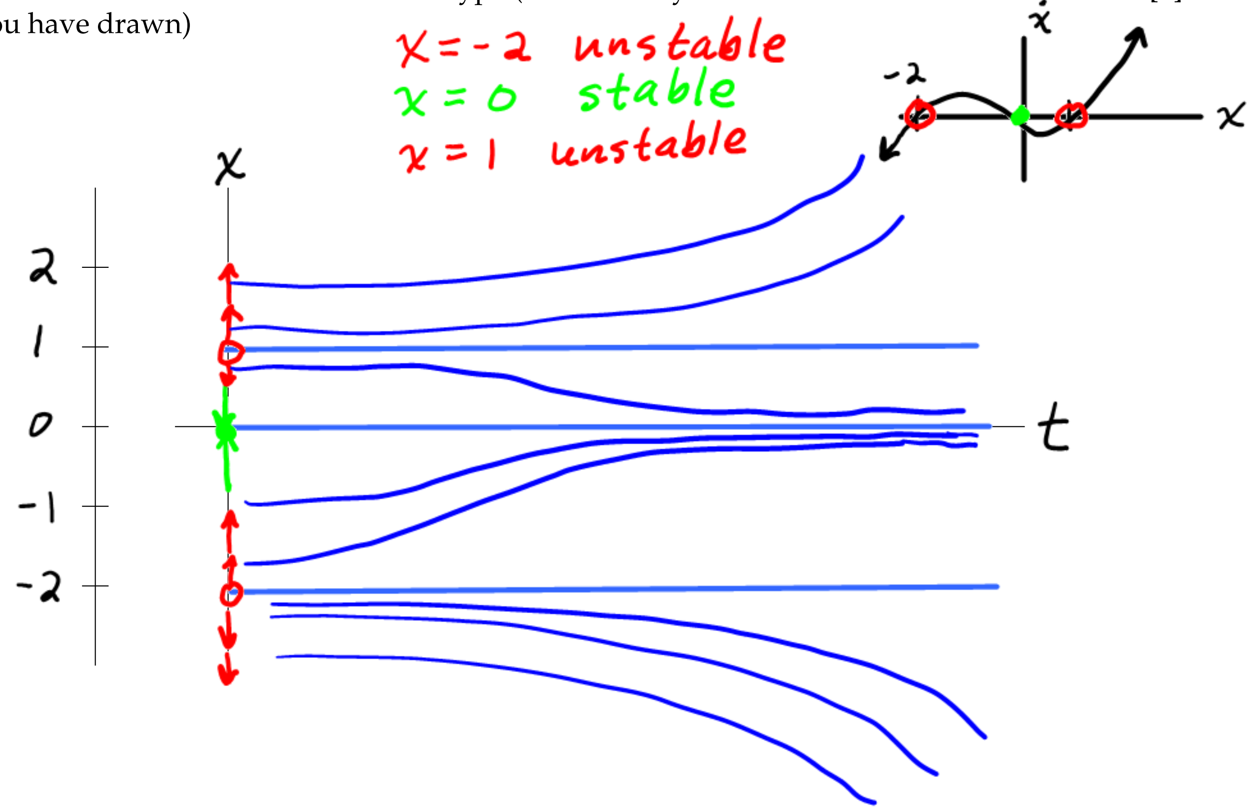
$$387.5 \text{ rats/yr} = a.$$

2. For the autonomous equation $\dot{x} = x(x-1)(x+2)$, please sketch:

(a) the phase line, identifying the critical points and whether they are stable, unstable, or neither. [4]

(b) at least one solution of each basic type (so that every solution is a time-translate of one you have drawn) [4]

$x = -2$ unstable
$x = 0$ stable
$x = 1$ unstable



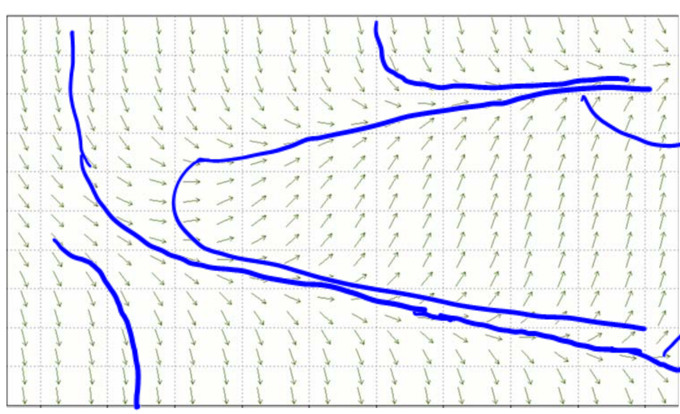
Below is a diagram of a direction field of the differential equation $y' = (1/4)(x - y^2)$. On it please plot and label:

(c) the nullcline [3]

(d) at least two quite different solutions [3]

(e) the separatrix (if there is one) [3]

(f) True or false: If $y(x)$ is a solution with a minimum, then for all large enough x, $y(x) < \sqrt{x}$. (No explanation needed: just circle one.) [3]



nullcline $y = \pm\sqrt{x}$

seperatrix

3. (a) Use Euler's method with stepsize $h = 1/2$ to estimate the value at $x = 3/2$ of the solution to $y' = x + y$ such $y(0) = 1$. [10]

$y(0) = 1$

$y(\frac{1}{2}) \approx 1 + \frac{1}{2}(0+1) = \frac{3}{2}$

$y(1) \approx \frac{3}{2} + \frac{1}{2}(\frac{1}{2} + \frac{3}{2}) = \frac{3}{2} + \frac{1}{2}\frac{4}{2} = \frac{5}{2}$

$y(\frac{3}{2}) \approx \frac{5}{2} + \frac{1}{2}(1 + \frac{5}{2}) = \frac{5}{2} + \frac{1}{2}\frac{7}{2} = \frac{10}{4} + \frac{7}{4} = \frac{17}{4}$

(b) Find the solution of $t\dot{x} + x = \cos t$ such that $x(\pi) = 1$. [10]

$t\dot{x} + x = \cos t$

$\frac{d}{dt}[tx] = \cos t$

$x(t) = \frac{1}{t}(\sin t + c)$

$1 = x(\pi) = \frac{1}{\pi}(\sin\pi + c) = \frac{c}{\pi}$

$\pi = c$

$x(t) = \frac{1}{t}(\sin t + \pi)$

4. (a) Find real a, b such that $\dfrac{1}{3+2i} = a + bi$. [3]

$$\frac{1}{3+2i} = \frac{3-2i}{3^2+2^2} = \frac{3}{13} - \frac{2}{13}i$$

$$a = \frac{3}{13}, \quad b = \frac{-2}{13}$$

(b) Find real r, θ such that $1 - i = re^{i\theta}$. [3]

$$r = |1-i| = \sqrt{2}$$

$$\tan\theta = -1 \longrightarrow$$

$$\theta = -\pi/4 + 2k\pi$$

$$1 - i = \sqrt{2}e^{i(-\pi/4+2k\pi)}$$

$$k \in \mathbb{Z}$$

(c) Find real a, b such that $(1-i)^8 = a + bi$. [3]

$$(1-i)^8 = \sqrt{2}^8 e^{i(-\pi/4)8}$$

$$= 16e^{-2\pi i}$$

$$= 16$$

$$(1-i)^8 = 16 + 0i$$

$$= a + bi$$

(d) Find real a, b such that $b > 0$ and $a + bi$ is a cube root of -1. [3]

$$-1 = e^{i(\pi+2k\pi)}$$

$$\sqrt[3]{-1} = e^{i(\pi+2k\pi)/3}$$

$$= \underline{e^{i\pi/3}}, e^{i\pi}, e^{5\pi i/3}$$

$$e^{i\pi/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$= a + bi$$

is a cube root of -1 with $b > 0$.

(e) Find real a, b such that $e^{\ln 2 + i\pi} = a + bi$. [3]

$$e^{\ln 2 + i\pi} = e^{\ln 2}e^{i\pi}$$

$$= (2)(-1)$$

$$= -2 + 0i = a + bi$$

(f) Write $f(t) = 2\cos(4t) - 2\sin(4t)$ in the form $A\cos(\omega t - \phi)$. [5]

$$A = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$$

$$\omega = 4$$

$$\tan\theta = -1$$

$$\theta = -\pi/4$$

$$f(t) = 2\sqrt{2}\cos(4t + \pi/4).$$

5. (a) Find a particular solution to the equation $\dot{x} + 3x = e^{2t}$. [5]

$x_p(t) = Ae^{2t}$

$\dot{x}_p(t) = 2Ae^{2t}$

$e^{2t} = 2Ae^{2t} + 3Ae^{2t}$

$A = \frac{1}{5}$

$x_p(t) = \frac{1}{5}e^{2t}$

(b) Find the solution to the same equation such that $x(0) = 1$. [5]

$\dot{x} + 3x = 0$

$\frac{1}{x}dx = -3dt$

$x_h(t) = Ce^{-3t}$

$x(t) = \frac{1}{5}e^{2t} + ce^{-3t}$

$1 = x(0) = \frac{1}{5} + c$

$\frac{4}{5} = c$

$x(t) = \frac{1}{5}e^{2t} + \frac{4}{5}e^{-3t}$

$\dot{x} + 3x = \frac{2}{5}e^{2t} - \frac{12}{5}e^{-3t}$

$+ \frac{3}{5}e^{2t} + \frac{12}{5}e^{-3t}$

$= e^{2t}$ ✓

$x(t)$ satisfies:

$\dot{x} + 3x = e^{2t}$, $x(0) = 1$

(c) Write down a linear equation with exponential right hand side of which $\dot{x} + 3x = \cos(2t)$ is the real part. [5]

$\dot{z} + 3z = e^{2it} = e^{(0+2i)t} = e^{0}(\cos 2t + i\sin 2t)$

$\text{Re}(\cos 2t + i\sin 2t) = \cos 2t$

(d) Find a particular solution to the equation $\dot{x} + 3x = \cos(2t)$. [5]

$z(t) = Ae^{2it}$

$\dot{z} + 3z = 2iAe^{2it} + 3Ae^{2it} = e^{2it}$

$2iA + 3A = 1$

$A = \frac{1}{3+2i}$

$A = \frac{3}{13} - \frac{2}{13}i$

$$z_p(t) = \left(\frac{3}{13} - \frac{2}{13}i\right)\left(\cos 2t + i\sin 2t\right)$$

$$= \frac{3}{13}\cos 2t + \frac{2}{13}\sin 2t$$

$$+ \frac{3}{13} i \sin 2t - \frac{2}{13} i \cos 2t$$

$$x_p = \mathrm{Re}(z_p) = \frac{3}{13}\cos 2t + \frac{2}{13}\sin 2t$$

$$\dot{x}_p + 3x = \frac{-6}{13}\sin 2t + \frac{4}{13}\cos 2t$$

$$+ \frac{9}{13}\cos 2t + \frac{6}{13}\sin 2t = \cos 2t \checkmark$$