§ Undetermined Coefficients

Problem 1: Find the solution to

$$y'' - y = x^2, \quad y(0) = 0, \quad y'(0) = -1$$

Answer: $r^2 - 1 = 0$

$r = \pm 1$

$y_h(x) = c_1 e^{x} + c_2 e^{-x}$

$y_p(x) = Ax^2 + Bx + C$

$$x^2 = y_p'' - y = 2A - Ax^2 - Bx - C \longrightarrow A = -1,\ B = 0,\ C = -2$$

$y_p(x) = -x^2 - 2, \quad y(x) = y_h + y_p$

$$\begin{aligned} 0 &= y(0) = c_1 + c_2 - 2 \\ -1 &= y'(0) = c_1 - c_2 \end{aligned} \rightarrow \begin{aligned} c_1 &= 1/2 \\ c_2 &= 3/2 \end{aligned} \rightarrow y(x) = \frac{1}{2}e^{x} + \frac{3}{2}e^{-x} - x^2 - 2.$$

Problem 3: Find a particular solution to

$$y^{(3)} + 4y' = 3x - 1$$

Answer: $y_p(x) = Ax^3 + Bx^2 + Cx + D$

$$3x - 1 = 6A + 4(3Ax^2 + 2Bx + C)$$

$$\begin{aligned} 0 &= 12A \\ 3 &= 8B \\ -1 &= 6A + 4C \end{aligned} \longrightarrow \begin{aligned} A &= 0 \\ B &= 3/8 \\ C &= -1/4 \\ D &= \text{Free} \end{aligned}$$

$$y_p(x) = \frac{3}{8}x^2 - \frac{1}{4}x$$

$$y_p''' + 4y_p' = 0 + 4\left(\frac{3}{4}x - \frac{1}{4}\right) = 3x - 1 \quad \checkmark$$

Problem 4: Find the general real solution of

$$\frac{d^3x}{dt^3} - \frac{dx}{dt} = t^2 + 1$$

Answer: $x_p(t) = At^3 + Bt^2 + Ct$

$$t^2+1 = 6A - (3At^2 + Bt + C)$$

$$\begin{matrix} 1 = -3A \\ 0 = -B \\ 1 = 6A - C \end{matrix} \rightarrow \begin{matrix} A = -1/3 \\ B = 0 \\ C = -3 \end{matrix} \rightarrow x_p(t) = -\frac{1}{3}t^3 - 3t$$

$$r^3 - r = 0 \rightarrow r = 0, \pm 1 \rightarrow x_h(t) = C_1 + c_2 e^x + c_3 e^{-x}$$

$$x(t) = x_h + x_p = c_1 + c_2 e^x + c_3 e^{-x} - \frac{1}{3}t^3 - 3t$$

Alternative way to find x_p: Let $u = \dot{x}$, so $u_p = \dot{x}_p$.
For $\ddot{u} - u = t^2 + 1$, try $u_p = At^2 + Bt + C$.

$$t^2+1 = \ddot{u}_p - u = 2A - (At^2+Bt+C)$$

$$\begin{aligned} 1 &= -A \\ 0 &= -B \\ 1 &= 2A-C \end{aligned} \longrightarrow \begin{aligned} A &= -1 \\ B &= 0 \\ C &= -3 \end{aligned}$$

$$u_p = -t^2 - 3$$

$$x_p = \int u_p \, dt = -\frac{t^3}{3} - 3t + \text{constant}$$

Any choice of constant satisfies the DE, so set

$$x_p(t) = -\tfrac{1}{3}t^3 - 3t\,.$$