§ Unit Step and Unit Impulse Response

Problem 1: Find the unit step and unit impulse responses for the operator $2D+I$ and graph them.

Answer: The unit step response is the continuous function $v(t)$ s.t. $v(t)=0$ for $t<0$ and is a solution to $2\dot{v}+v=1$ for $t>0$.

$$v_p(t)=1$$

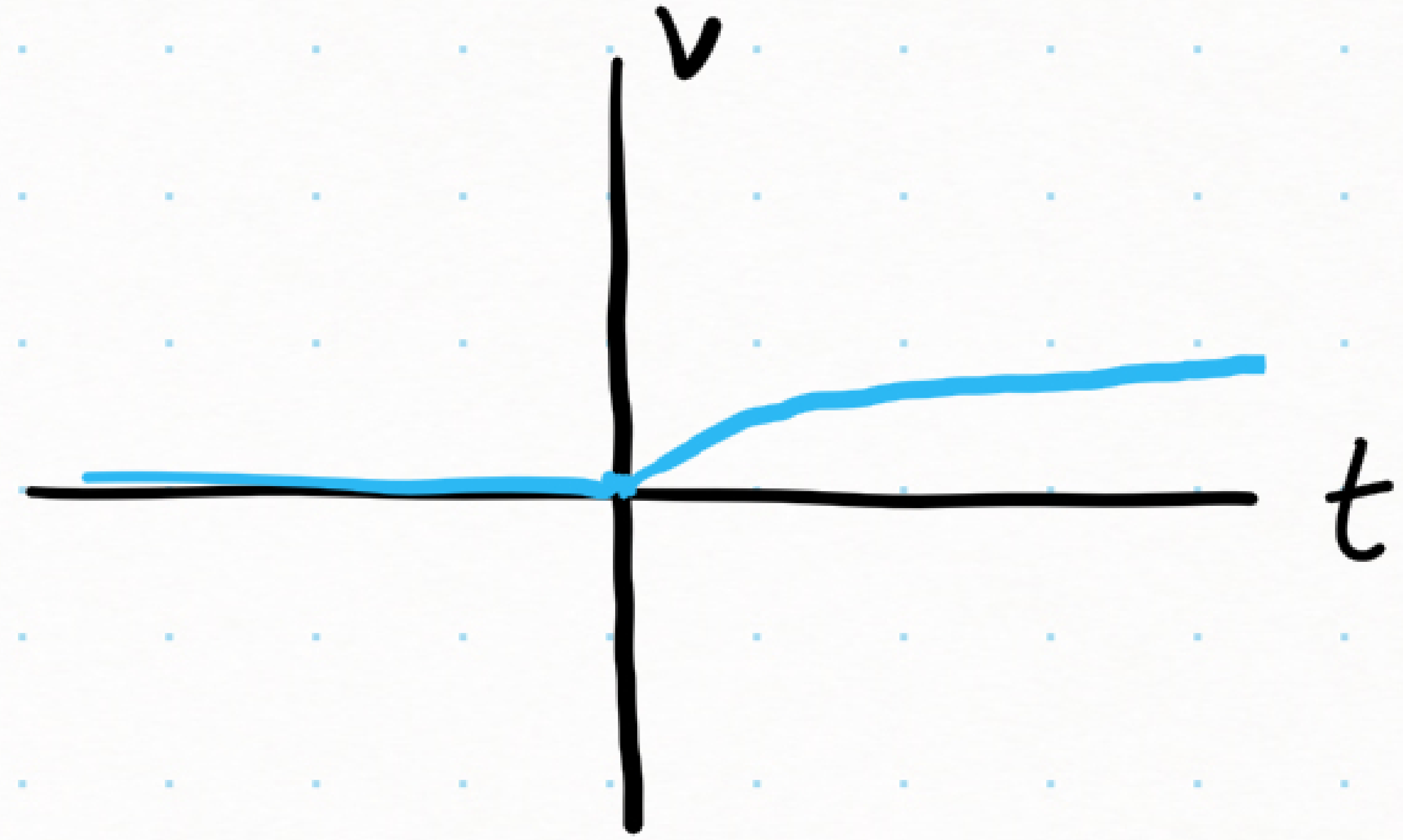
$$v(t)=ce^{-t/2}$$

$$v(t)=v_p+v_h, \quad t>0$$

$$0=v(0^+)=1+c$$

$$v(t)=1-e^{-t/2}, \quad t>0$$

$$v(t)=u(t)\left(1-e^{-t/2}\right)$$



The unit impulse response is satisfies $w(t)=0$ for $t>0$, $2\dot{w}+w=0$ for $t>0$ and $w(0^+)=1/2$.

$$w(t) = ce^{-t/2}, \quad t>0$$

$$1/2 = w(0^+) = c$$

$$w(t) = 0, \quad t<0$$

$$w(t) = \frac{1}{2}e^{-t/2}u(t)$$

Alternatively, use

$$w(t) = \dot{v}(t) = \frac{d}{dt}\left[u(t)\left(1-e^{-t/2}\right)\right]$$

$$= \frac{1}{2}e^{-t/2}u(t).$$

Problem 2: Find the unit impulse response for the operator $D^2 + 2D$, and graph it.

Answer:

$$w(t) = 0, \quad t < 0$$

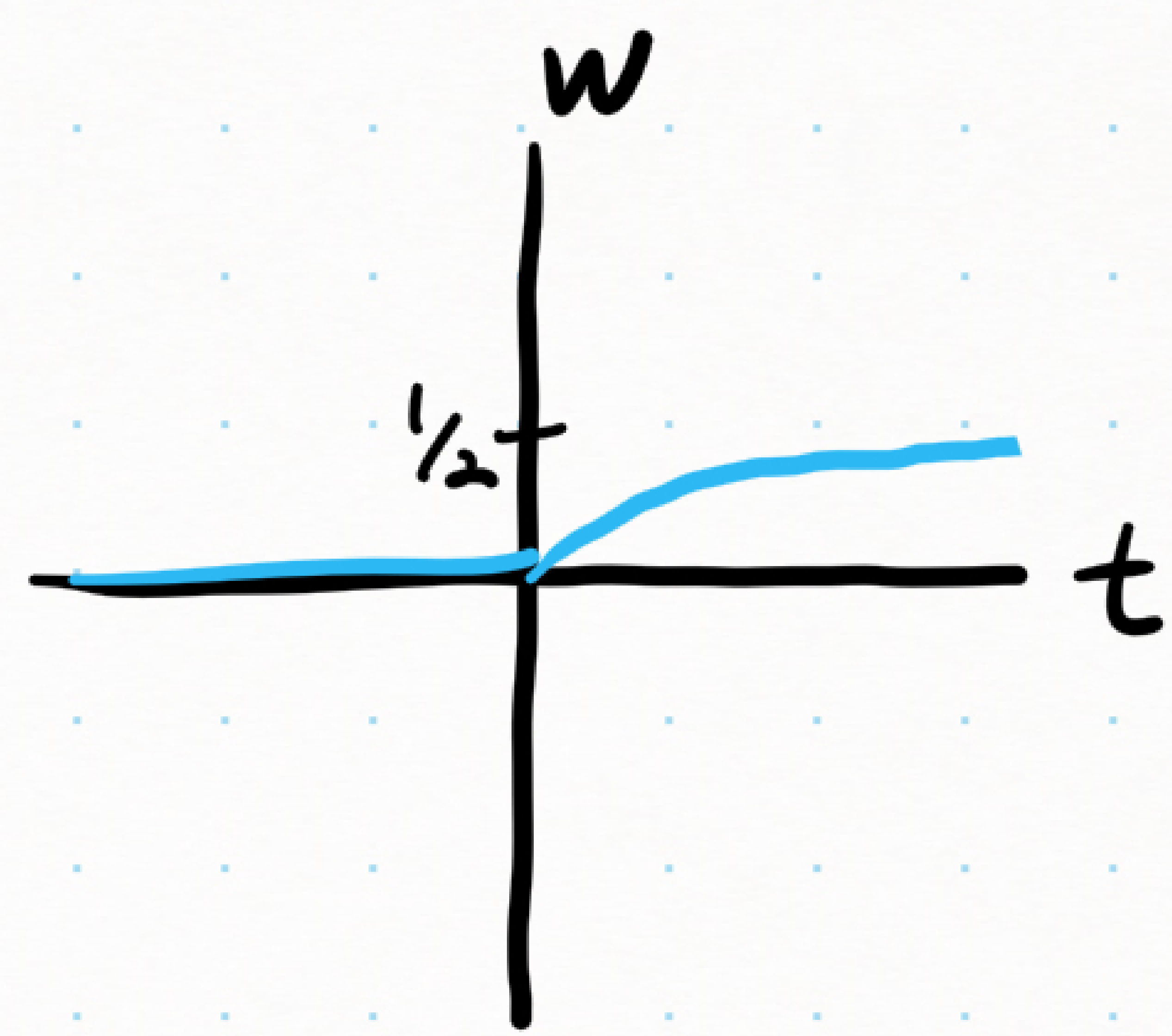
$$w(0^+) = 0, \quad \dot{w}(0^+) = 1/1 = 1$$

$$\ddot{w} + 2\dot{w} = 0, \quad t > 0.$$

$$w(t) = u(t)\left(c_1 + c_2 e^{-2t}\right)$$

$$0 = w(0^+) = c_1 + c_2$$
$$1 = \dot{w}(0^+) = -2c_2$$

$$w(t) = \frac{u(t)}{2}\left(1 - e^{-2t}\right)$$



Problem 3: Find the solution to $\ddot{x} + 2\dot{x} = 3\delta(t-1)$ with rest initial conditions.

Answer: Using $w_2 = w$ of problem 2,

$$w_3(t) = 3w_2(t-1)$$
$$= \frac{3}{2}u(t-1)\left(1 - e^{-2(t-1)}\right)$$

Problem 4: Find the unit step and unit impulse response for the operator $D + kI$.

Answer:

$v(t) = 0, \; t < 0$

$\dot{v} + kv = 1, \; t > 0$

$v(0^+) = 0$

$v_p = 1/k$

$v_h = ce^{-kt}$

$0 = v(0^+) = \frac{1}{k} + c$

$v(t) = \frac{u(t)}{k}\left[1 - e^{-kt}\right]$

$w(t) = 0, \; t < 0$

$\dot{w} + kw = 0, \; t > 0$

$w(0^+) = 1$

$w(t) = u(t)\, d e^{-kt}$

$1 = w(0^+) \Rightarrow d = 1$

$w(t) = u(t) e^{-kt}$.

Also, $w(t) = \dot{v}(t) = u(t) e^{-kt}$.

v: step response

w: impulse response

Problem 5: Find the unit step and unit impulse response for the operator $D^2 + \omega_0^2 I$.

Answer:

$v(t) = 0, \ t < 0$

$\ddot{v} + \omega_0^2 v = 1, \ t > 0$

$v(0^+) = 0, \ \dot{v}(0^+) = 0$

$V_p(t) = 1/\omega_0^2$

$V_h(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$

$0 = V_h(0^+) = c_1 + 1/\omega_0^2$

$0 = \dot{V}_h(0^+) = \omega_0 c_2$

$$v(t) = u(t)\left(\frac{1}{\omega_0^2} - \frac{1}{\omega_0^2}\cos(\omega_0 t)\right)$$

$w(t) = 0, \ t < 0$

$\ddot{w} + \omega_0^2 w = 0, \ t > 0$

$w(0^+) = 0, \ \dot{w}(0^+) = 1$

$w(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$, $t > 0$

$0 = w(0^+) = c_1, \ 1 = \dot{w}(0^+) = \omega_0 c_2$

$$w(t) = \frac{u(t)}{\omega_0} \sin(\omega_0 t)$$

Also,

$$w(t) = \dot{v}(t) = \frac{u(t)}{\omega_0}\sin(\omega_0 t)$$