§ Operations on Fourier Series

Problem 1: Find the Fourier series of the function $f(t)$ of period 2π which is given over the interval $-\pi < t \leq \pi$ by

$$f(t) = \begin{cases} 0, & -\pi < t \leq 0 \\ 1, & 0 < t \leq \pi \end{cases}$$

as in the same problem in the previous session – but this time use the known Fourier series for $sq(t)$ = the standard square wave.

Answer:

$$sq(t) = \begin{cases} -1 & , -\pi \leq t < 0 \\ 1 & , 0 \leq t < \pi \end{cases} = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt)}{n}$$

To obtain $f(t)$ from $sq(t)$, shift $sq(t)$ up by 1 and then scale the result by ½. Applying this transformation to the Fourier series for $sq(t)$ gives

$$\frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt)}{n} \longmapsto \frac{1}{2}\left(1 + \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt)}{n}\right)$$

$$= \frac{1}{2} + \frac{2}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt)}{n} = f(t).$$

Problem 2: Find the Fourier series of the function $f(t)$ with period 2π given by $f(t) = |t|$ on $(-\pi, \pi)$ by integrating the Fourier series of the derivative $f'(t)$.

Answer:

$$f'(t) = \begin{cases} -1, & -\pi < t < 0 \\ 1, & 0 < t < \pi \end{cases} = sq(t) \quad \left(\begin{array}{l}\text{except at the} \\ \text{discontinuity at } t=0\end{array}\right)$$

$$f(t) = \int f'(t)\,dt = \int \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt)}{n}\,dt$$

$$= \frac{a_0}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos(nt)}{n^2}, \quad \text{where}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\,dt = \pi$$