Unit III Practice Exam

1. Let ω be a positive constant. We drive a harmonic oscillator with a square wave of circular frequency ω: $\ddot{x} + 4x = \mathrm{sq}(\omega t)$.

(a) Write down a periodic solution to the equation, if ω is such that there is one.

(b) For what values of ω does there fail to be a periodic solution?

Answer:

a) $$\ddot{x} + 4x = \frac{4}{\pi}\left(\sin\omega t + \frac{\sin 3\omega t}{3} + \frac{\sin 5\omega t}{5} + \dots\right)$$

$$\ddot{x} + 4x = \frac{4}{\pi} A \sin\omega t \quad \text{(for a single term)}$$

$$\ddot{z} + 4z = \frac{4A}{\pi} e^{i\omega t}$$

$$B(i\omega)^2 e^{i\omega t} + 4Be^{i\omega t} = \ddot{z}_p + 4z_p = \frac{4A}{\pi} e^{i\omega t} \to B = \frac{4A}{\pi}\frac{1}{4-\omega^2}$$

$$x_p = \mathrm{Im}(z_p) = \frac{4A}{\pi}\frac{\sin\omega t}{4-\omega^2}.$$ Generalize to $\omega \mapsto k\omega$

$$x_p(t) = \frac{4}{\pi}\left(\frac{\sin\omega t}{4-\omega^2} + \frac{1}{3}\frac{\sin 3\omega t}{4-9\omega^2} + \frac{1}{5}\frac{\sin 5\omega t}{4-25\omega^2} + \dots\right)$$

b) There fails to be a periodic solution if

$$4 - \omega^2 = 0 \longrightarrow \omega = 2$$

$$4 - 9\omega^2 = 0 \longrightarrow \omega = 2/3$$

$$4 - 25\omega^2 = 0 \longrightarrow \omega = 2/5$$

$$4 - 49\omega^2 = 0 \longrightarrow \omega = 2/7$$

$$\vdots$$

$$\omega = 2/k, \quad k = 1, 3, 5, 7, 9, \ldots$$

Note that $\omega > 0$ by assumption.

2. Let $f(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 - t & \text{for} \quad 0 < t < 1 \\ 1 & \text{for} \quad t > 1 \end{cases}$.

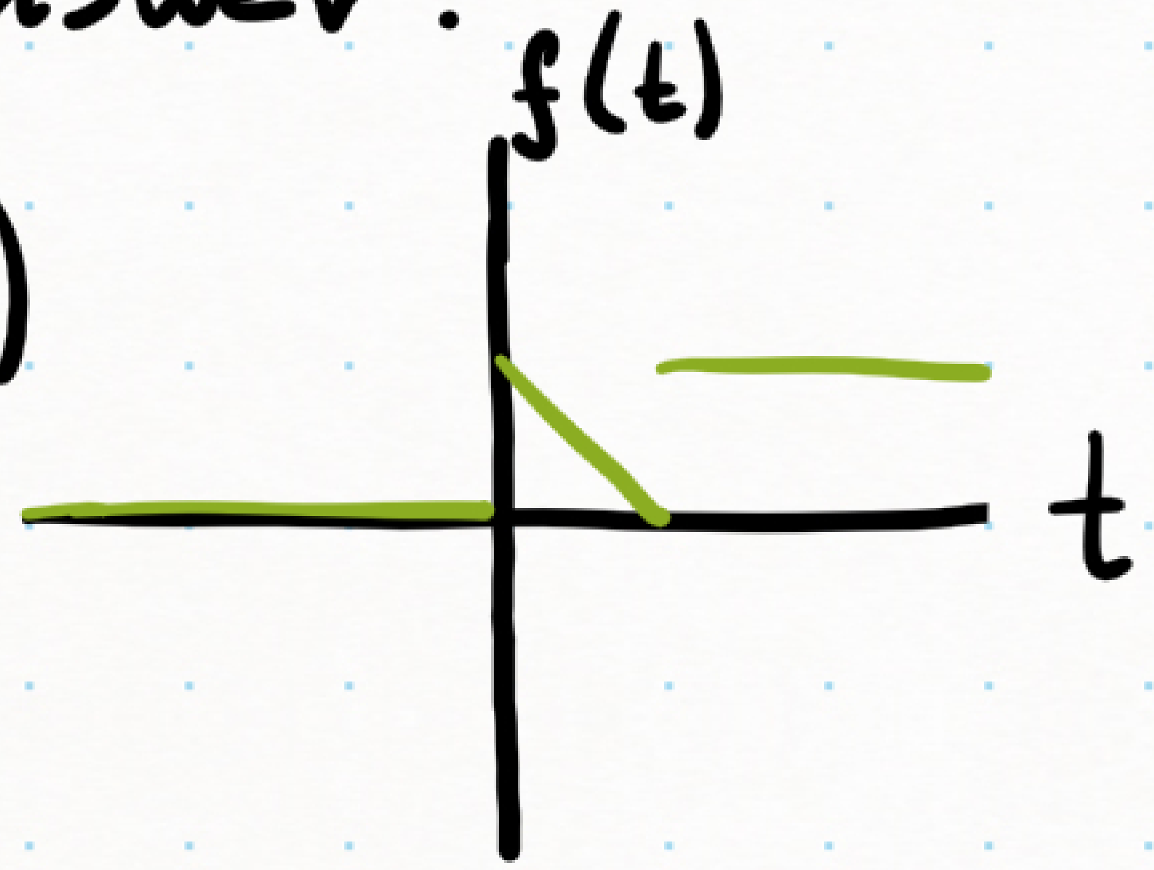
(a) Sketch the graph of $f(t)$.

(b) Sketch the graph of the generalized derivative $f'(t)$.

(c) Write down a formula for $f'(t)$ in terms of step and delta functions.

Answer:

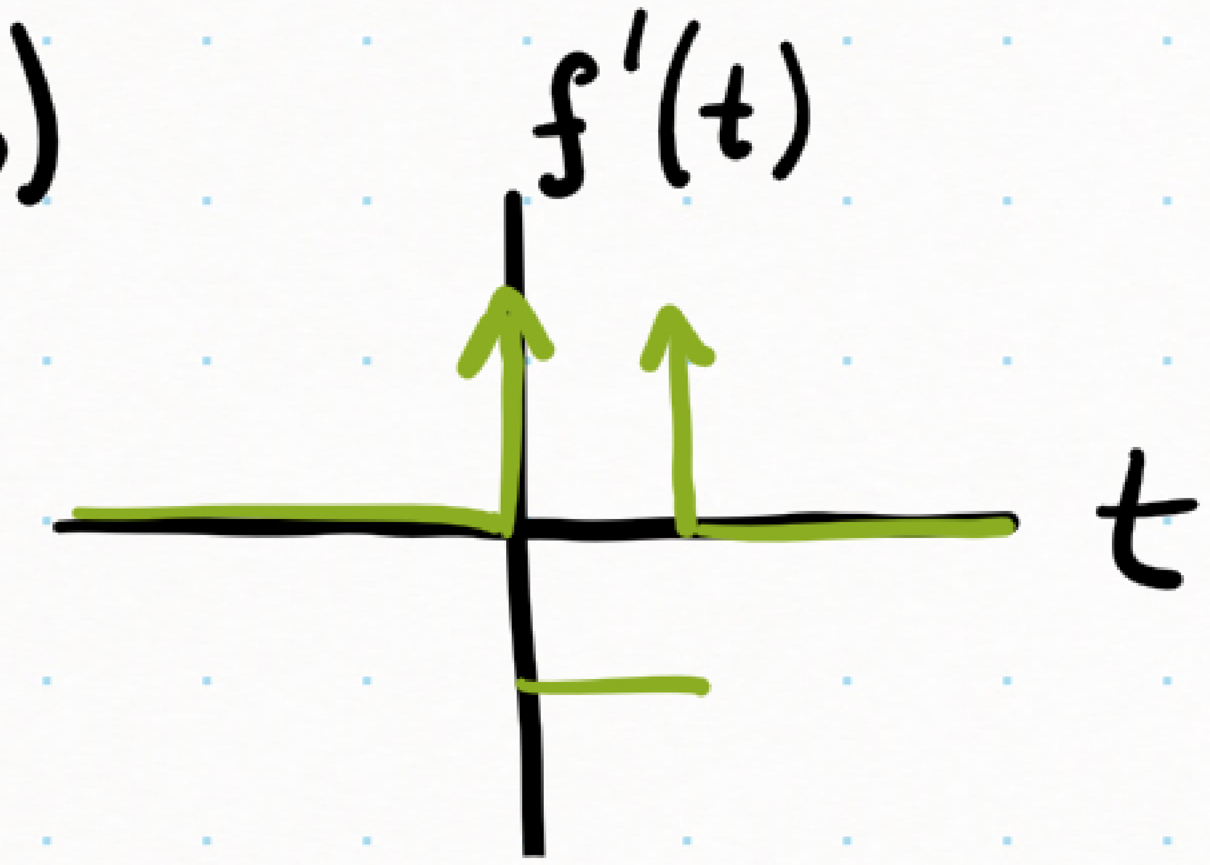
a)



c) $f'(t) = -1\,(u(t) - u(t-1)) + \delta(t) + \delta(t-1)$

$= u(t-1) - u(t) + \delta(t) + \delta(t-1)$

b)



3. (a) Compute the convolution product $t * t^6$.

(b) A certain operator $p(D)$ has unit impulse response $w(t) = 2u(t)te^{-t}$. What is the solution to $p(D)x = e^{-t}$ with rest initial conditions?

Answer:

a) $t * t^6 = \int_0^t (t-\tau)\tau^6 \, d\tau = \frac{t\tau^7}{7} - \frac{\tau^8}{8}\Big|_0^t = \frac{t^8}{56}$

b) $x(t) = w(t) * q(t) = 2u(t)te^{-t} * e^{-t}$

$= \int_0^t 2\tau e^{-\tau} e^{-(t-\tau)} \, d\tau$

$= 2\int_0^t \tau e^{-t} \, d\tau$

$= \tau^2 e^{-t} \Big|_0^t$

$= t^2 e^{-t}$

4. (a) What is the Laplace transform of the solution to the equation $\ddot{x} + 2\dot{x} + 2x = 1$ having rest initial conditions?

(b) What function $f(t)$ has Laplace transform $F(s) = \dfrac{2s}{(s+1)(s^2+2s+5)}$?

Answer: Since $x(0) = \dot{x}(0) = 0$,

a) $\mathcal{L}[\ddot{x} + 2\dot{x} + 2x] = s^2X + 2sX + X = \frac{1}{s} = \mathcal{L}[1]$

b) $$F(s) = \frac{2(s+1)-2}{(s+1)((s+1)^2+2^2)} = \frac{-\frac{1}{2}}{s+1} + \frac{\frac{1}{2}(s+1)+2}{(s+1)^2+2^2}$$

$$= -\frac{1}{2}\frac{1}{s+1} + \frac{1}{2}\frac{s+1}{(s+1)^2+2^2} + \frac{2}{(s+1)^2+2^2}$$

$$f(t) = \mathcal{L}^{-1}[F(s)]$$

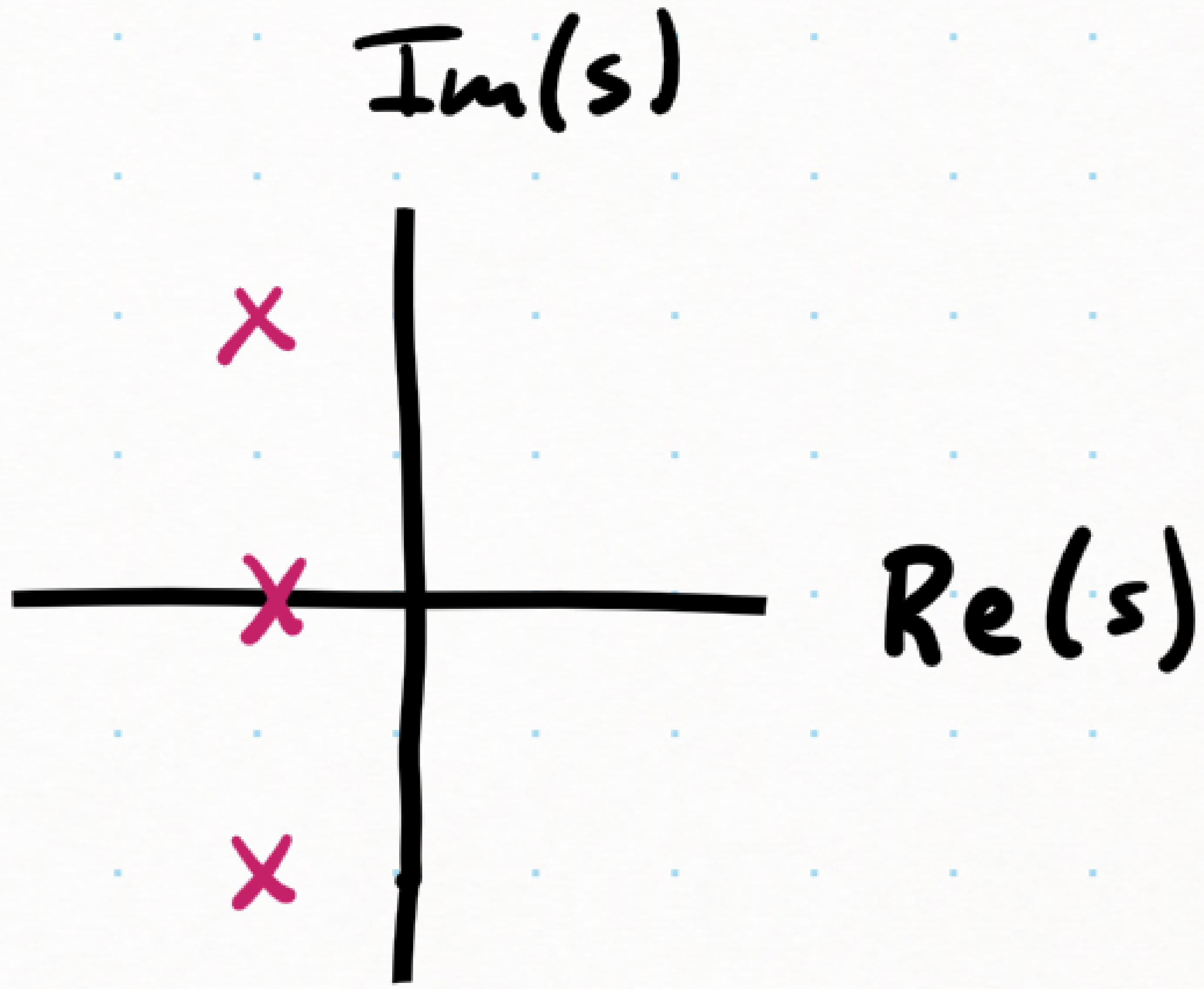
$$= -\frac{1}{2}e^{-t} + \frac{1}{2}e^{-t}\cos(2t) + e^{-t}\sin(2t).$$

5. (a) Sketch the pole diagram for the function $F(s) = \dfrac{2}{(s+1)(s^2+2s+5)}$.

(b) Give an example of a function $f(t)$ whose Laplace transform has poles at $s = 2$ and $s = -3 \pm 4i$ and nowhere else.

Answer:

a) Poles at $s = -1, -1 \pm 2i$



b) Consider $F(s) = \dfrac{1}{s-2} + \dfrac{4}{(s+3)^2 + 4^2}$, which has poles at $s = 2, -3 \pm 4i$ and nowhere else. Then,

$f(t) = \mathcal{L}^{-1}[F(s)] = e^{2t} + e^{-3t} \sin 4t$ is one example.

Properties of the Laplace transform

0. Definition: $\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$ for $\mathrm{Re}\, s \gg 0$.

1. Linearity: $\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s)$.

2. Inverse transform: $F(s)$ essentially determines $f(t)$.

3. s-shift rule: $\mathcal{L}[e^{at}f(t)] = F(s-a)$.

4. t-shift rule: $\mathcal{L}[f_a(t)] = e^{-as}F(s)$, $\quad f_a(t) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$.

5. s-derivative rule: $\mathcal{L}[tf(t)] = -F'(s)$.

6. t-derivative rule: $\mathcal{L}[f'(t)] = sF(s)$, where $f'(t)$ denotes the generalized derivative.

$\mathcal{L}[f_r'(t)] = sF(s) - f(0+)$ if $f(t)$ is continuous for $t > 0$.

7. Convolution rule: $\mathcal{L}[f(t) * g(t)] = F(s)G(s)$, $f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$.

8. Weight function: $\mathcal{L}[w(t)] = W(s) = 1/p(s)$, $w(t)$ the unit impulse response.

Formulas for the Laplace transform

$$\mathcal{L}[1] = \frac{1}{s} \qquad \mathcal{L}[e^{at}] = \frac{1}{s-a} \qquad \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2} \qquad \mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}[t\cos(\omega t)] = \frac{2\omega s}{(s^2+\omega^2)^2} \qquad \mathcal{L}[t\sin(\omega t)] = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

Fourier coefficients for periodic functions of period 2π:

$$f(t) = \frac{a_0}{2} + a_1\cos(t) + a_2\cos(2t) + \cdots + b_1\sin(t) + b_2\sin(2t) + \cdots$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(mt)\,dt, \qquad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(mt)\,dt$$

If $\mathrm{sq}(t)$ is the odd function of period 2π which has value 1 between 0 and π, then

$$\mathrm{sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right)$$