Practice Final Exam

1. For the DE $\frac{dy}{dx} = -\frac{y}{x} + 3x$:

a) Sketch the direction field for this DE, using (light or dotted) isoclines for the slopes -1 and 0.

b) For the solution curve passing through the point (1,2): if Euler's method with step-size $h = 0.1$ was used to approximate $y(1.1)$, would the approximation come out too high or too low? Explain.

c) Compute the Euler approximation to $y(1.1)$ using step-size $h = 0.1$.

d) The functions $y_1 = x^2$ and $y_2 = x^2 + \frac{1}{x}$ are solutions to this DE. If $y = y(x)$ is the solution satisfying the IC $y(1) = 1.5$, show that $100 \leq y(10) \leq 100.1$. Do we need to include the equal signs in this inequality? Why or why not?

Answer:

b) Since $\frac{d^2y}{dx^2} = 4$ at $(1,2)$, $y = y(x)$ is concave up. Euler's method will underestimate $y(1.1)$.

c) $y(1.1) \approx y(1) + h f(1, y(1)) = 2 + 0.1(-2+3) = 2.1$

d) $y_1(1) = 1$, $y_2(1) = 2$. Since $y_1(1) < y(1) < y_2(1)$ and solutions to the DE do not intersect by the Existence & Uniqueness Thm, $100 = y_1(10) < y(10) < y_2(10) = 100.1$.

2. Suppose that a population of variable size (in some suitable units) $p(t)$ follows the growth law $\frac{dp}{dt} = p^3 - 4p^2 + 4p$. Without solving the DE explicitly:

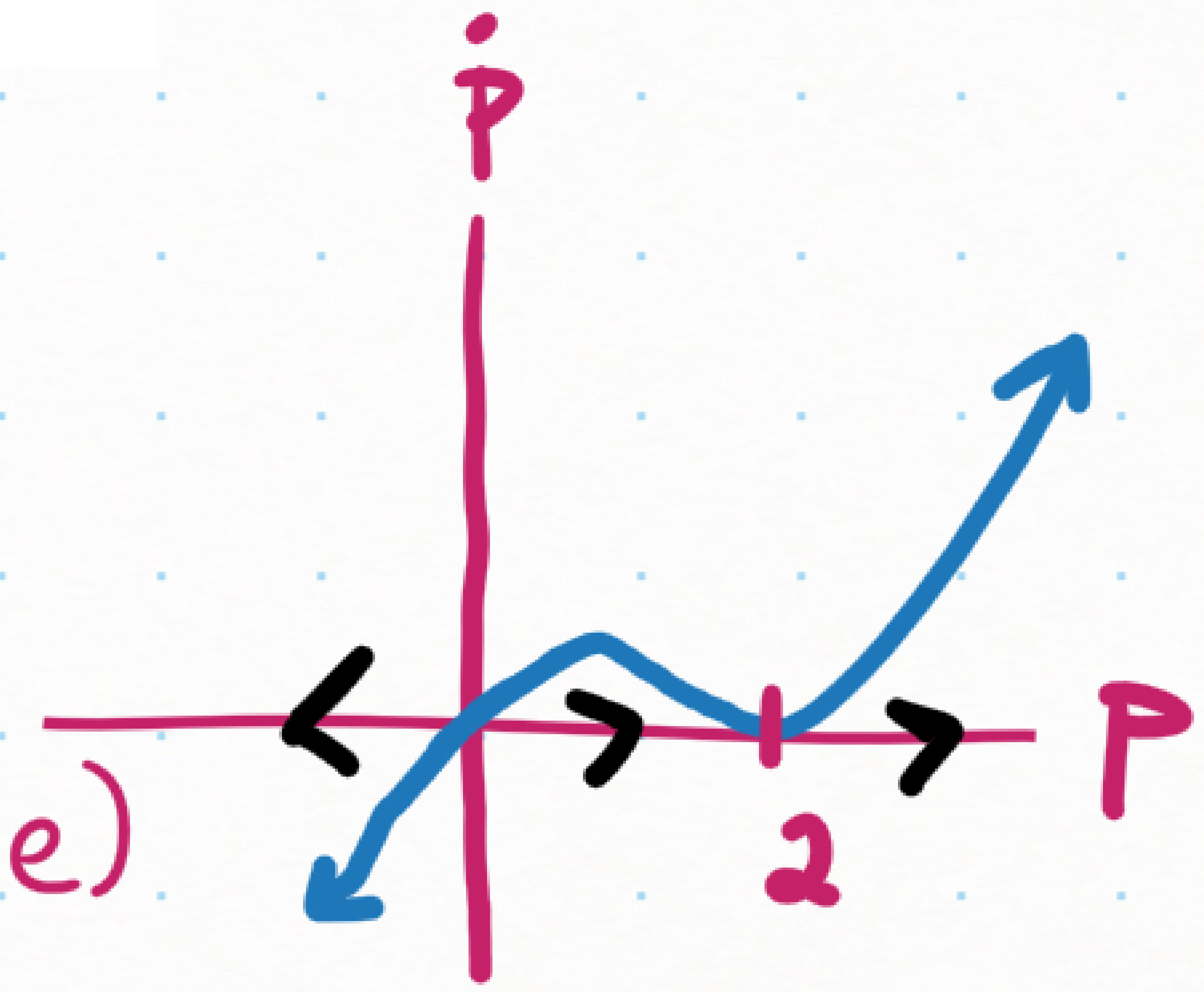
a) Find all critical points and classify each according to its stability type using a phase-line diagram.

b)) Draw a rough sketch (on p-vs.-t axes) of the family of solutions. What happens to the population in the long-run if it starts out at size 1 unit; at size 3 units ?

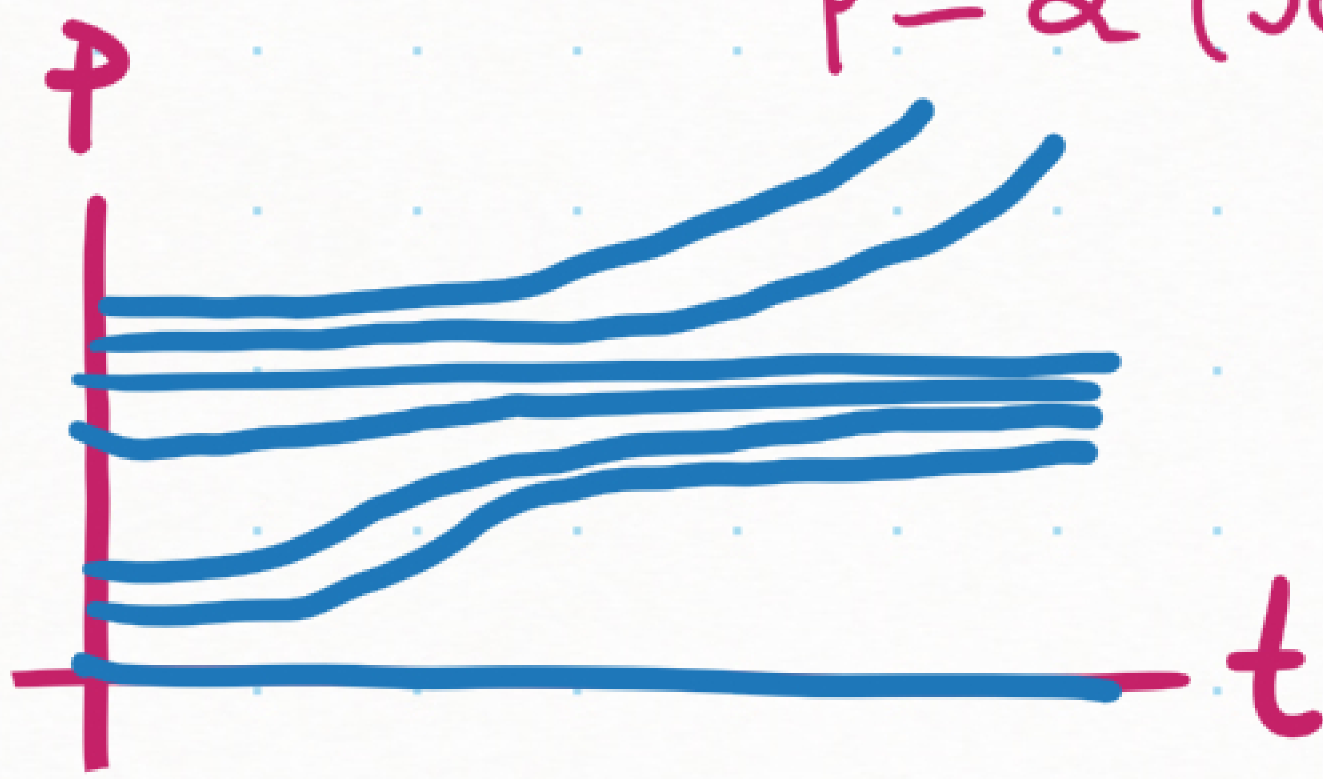
c) Explain why the rate information given by the DE was all we needed to get the answer to part (b).

Answer:

a) $0 = dp/dt \longrightarrow p = 0$ (unstable)
$p = 2$ (semistable)



b)



If $p(0) = 1$, $\lim_{t \to \infty} p(t) = 2$

If $p(0) = 3$, $\lim_{t \to \infty} p(t) = \infty$.

c) Because the DE is autonomous and we classified $p = 2$ in part a. If $0 < p(0) \leq 2$, $p \longrightarrow 2$ while if $p(0) > 2$, $p \to \infty$.

3. Let $p(D) = D^2 + b\,D + 5$ where $D = \frac{d}{dt}$. a) For what range of the values of b will the solutions to $p(D)y = 0$ exhibit oscillatory behavior?

b) For $b = 4$, solve the DE's (i) $p(D)y = 4e^{2t}\sin t$
(ii) $p(D)y = 4e^{2t}\cos t$
using the Exponential Response formula. Write your answers in both amplitude-phase and rectangular form.

c) Given $b = 2$, for what ω does $p(D)y = \cos\omega t$ have the biggest response?

Answer:

a) $0 > b^2 - 4\cdot 5 \rightarrow |b| < 2\sqrt{5}$

b) i. $z_p = Ae^{(2+i)t} \longrightarrow A = \dfrac{4}{p(2+i)} = \dfrac{4}{(2+i)^2 + 4(2+i) + 5} = \dfrac{1}{5} - \dfrac{i}{10}$

$y_p = \mathrm{Im}(z_p) = e^{2t}\left[\frac{1}{5}\sin t - \frac{1}{10}\cos t\right] = \frac{1}{2\sqrt{5}}e^{2t}\sin(t - \tan^{-1}(1/2))$

$y_h = e^{-2t}[c_1\cos t + c_2\sin t]$

$y = y_h + y_p$

ii. $z_p = \dfrac{4}{p(2+i)}e^{(2+i)t} = \left(\frac{1}{5} - \frac{i}{10}\right)e^{(2+i)t}$

$y_p = \mathrm{Re}(z_p) = e^{2t}\left(\frac{1}{5}\cos t + \frac{1}{10}\sin t\right) = \frac{1}{2\sqrt{5}}e^{2t}\cos(t - \tan^{-1}(1/2))$

c) $A(\omega) = |p(\omega i)|^{-1} = |-\omega^2 + 5 + 2i\omega|^{-1} = 1/\sqrt{(5-\omega)^2 + 4\omega^2}$
is maximized with $\omega = 3$.

4. Find the general solution to $(D^3-1)y=e^x$.

Answer:

$$r^3-1=0$$
$$r=1,\ e^{2\pi i/3}=-\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ e^{4\pi i/3}=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

$$y_h(x)=c_1e^x+c_2e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)+c_3e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

Since $p(1)=0$, we have resonance.

$$y_p(x)=\frac{xe^x}{p'(1)}=\frac{xe^x}{3}.$$

$$y(x)=y_h(x)+y_p(x).$$

5. For $f(t) = t$ on $-1 < t < 1$, periodic with period $P = 2$:

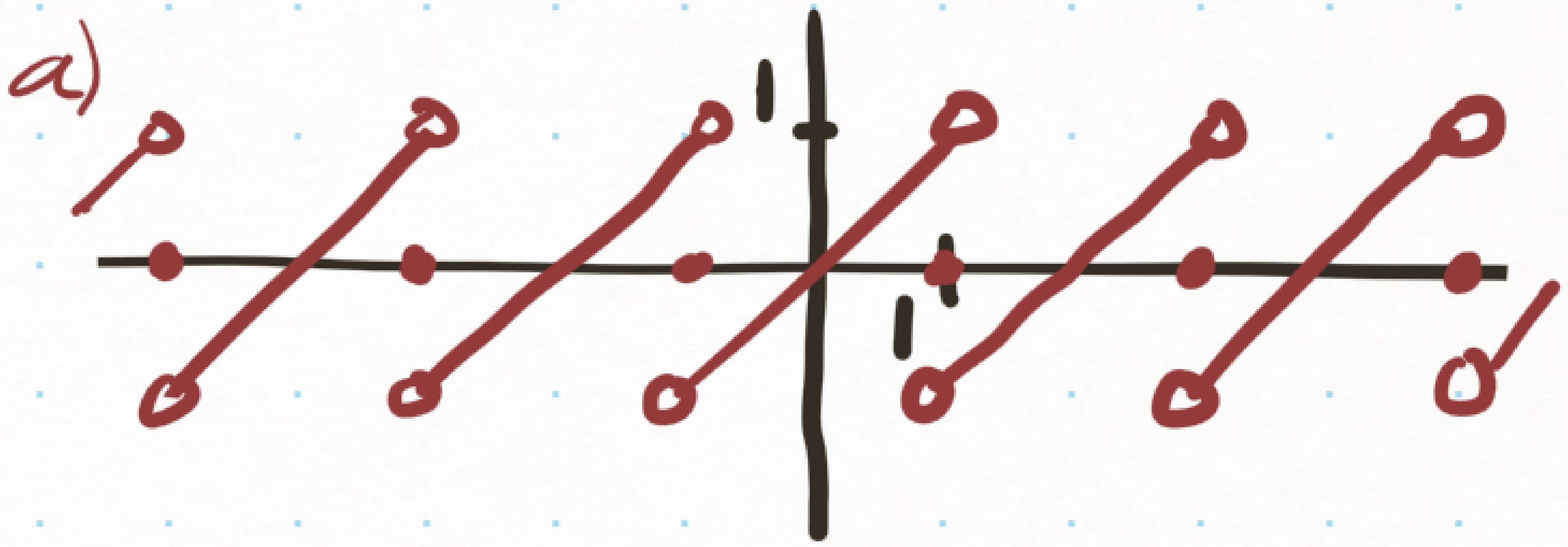
a) Sketch f over three or more full periods P

Choose endpoint values that show where the Fourier series expansion will converge (*without* computing the Fourier series).

b) Compute the Fourier series of f

c) Compute the steady-periodic solution to the DE $x''(t) + 10\,x(t) = f(t)$. Does near-resonance occur in this situation? If so, which frequency in the 'driving force' $\tilde{f}_{\text{odd}}$ produces it?

a)



Answer:

b) $a_n = 0 \;\; \forall n$ since $f(t)$ is odd.

$$b_n = \frac{1}{1}\int_{-1}^{1} t\sin(n\pi t)\,dt = 2\int_0^1 t\sin(n\pi t) = \frac{2(-1)^{n+1}}{n\pi}$$

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(n\pi t) = \frac{2}{\pi}\left(\sin\pi t - \frac{\sin 2\pi t}{2} + \frac{\sin 3\pi t}{3} - \dots\right)$$

c) $x_p'' + 10x_p = \sin(at) \to z'' + 10z = e^{ait} \to z = \frac{1}{10-a^2}e^{ait}$

$x_p = \operatorname{Im} z = \frac{1}{10-a^2}\sin(at)$, $a^2 \neq 10$. By superposition

$$x = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(\frac{1}{10-n^2\pi^2}\right)\sin(n\pi t).$$

near resonance.

$n=1$ produces the largest response since $n^2\pi^2 \approx 10$ when $n=1$.

6. a) Suppose that starting at $t = 0$ a radioactive material is continuously flowing into a container at a rate $f(t)$ in curies per unit time, and that one uses the standard exponential model for continuous radioactive decay, with rate constant k (in $\frac{1}{\text{time}}$). Let $R = R(t)$ denote the total amount of radioactive material in the container at time t. Give the DE for $R(t)$ and solve it (in terms of k, $f(t)$ and $R_0 = R(0)$).

b) Show that the solution satisfying the IC $R(0) = 0$ can be written as a convolution integral. What is the weight function w in this case? What DE does w satisfy?

Answer:

a) $\dot{R} = f(t) - kR, \quad R(0) = R_0$

$$Re^{kt} = \int_0^t f(u)e^{ku}\,du + R_0$$

$$R(t) = e^{-kt}\left[\int_0^t f(u)e^{ku}\,du + R_0\right]$$

b) $0 = R(0) = R_0 \rightarrow R(t) = \int_0^t f(\tau)e^{-k(t-\tau)}\,d\tau = (f * w)(t)$,
where $w(t) = e^{-kt}$ is the weight function satisfying

$$\dot{w} + kw = \delta, \quad \text{Rest IC.}$$

7. Derive $\mathcal{L}[\cos t]$ using complex exponentials.

Answer:

$$\mathcal{L}[\cos t] = \mathcal{L}\left[\tfrac{1}{2}e^{it} + \tfrac{1}{2}e^{-it}\right]$$

$$= \tfrac{1}{2}\mathcal{L}[e^{it}] + \tfrac{1}{2}\mathcal{L}[e^{-it}]$$

$$= \frac{1}{2}\frac{1}{s-i} + \frac{1}{2}\frac{1}{s+i}$$

$$= \frac{1}{2}\left(\frac{s+i+s-i}{s^2+1}\right)$$

$$= \frac{s}{s^2+1}$$

8. Compute $f(t) = \mathcal{L}^{-1}\left[1/s - e^{-s}/(s(s+1))\right]$.

Answer: $\dfrac{1}{s(s+1)} = \dfrac{1}{s} - \dfrac{1}{s+1}$

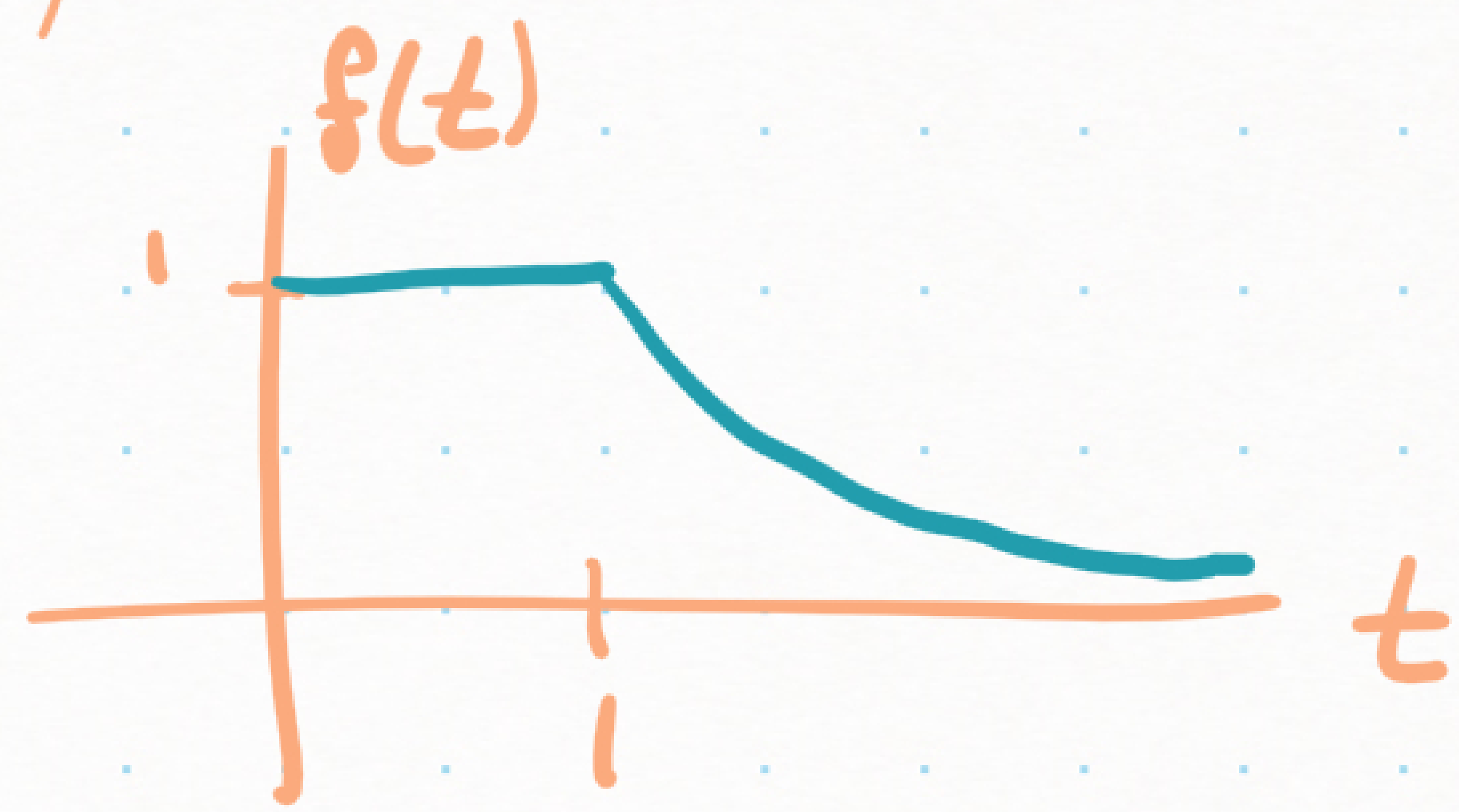
$$\mathcal{L}[u(t-a)] = \int_0^\infty e^{-st} u(t-a)\,dt = \frac{e^{-as}}{s}.$$

$$f(t) = \mathcal{L}^{-1}[1/s] - \mathcal{L}^{-1}[e^{-s}/s] + e\mathcal{L}^{-1}[e^{-(s+1)}/(s+1)]$$

$$= 1 - u(t-1) + ee^{-t}u(t-1)$$

$$= 1 + u(t-1)\left(e^{-(t-1)} - 1\right)$$

$$= \begin{cases} 1, & 0 \le t < 1 \\ e^{-(t-1)}, & t \geq 1 \end{cases}$$



9. Use the Laplace transform method to solve the IVP's

a) $y'' - y' - 2y = 0, \quad y(0) = 0, \quad y'(0) = 2$

b) $y'' + 4y = \cos t, \quad y(0) = y'(0) = 0$

Answer:

a) $s^2Y - sy(0) - y'(0) - sY + y(0) - 2Y = 0$

$$Y(s) = \frac{2}{s^2 - s - 2} = \frac{2}{3}\frac{1}{s-2} - \frac{2}{3}\frac{1}{s+1}$$

$$y(t) = \frac{2}{3}e^{2t} - \frac{2}{3}e^{-t}.$$

b) $s^2Y + 4Y = \frac{s}{s^2+1}$

$$Y(s) = \frac{s}{(s^2+4)(s^2+1)} = -\frac{1}{3}\frac{s}{s^2+4} + \frac{1}{3}\frac{s}{s^2+1}$$

$$y(t) = -\frac{1}{3}\cos 2t + \frac{1}{3}\cos t$$

10. Consider an undamped spring-mass system $Lx = x'' + x = f(t)$, where $f(t)$ is an external applied force, and suppose that the system starts out at time $t = 0$ at its equilibrium position $x = 0$ with a velocity $x'(0) = 1$ (in some suitable units). Using the Laplace transform method, solve for the position function $x = x(t)$ for the following forcing function $f(t)$:

$f(t)$ is an *impulsive* force of magnitude F_0 at time $t = \pi$ and $f(t) = 0$ otherwise.
Graph the general solution. What happens in the special case $F_0 = 1$, and why?
Express your answer in u-form, and in 'cases' form as well.

Answer:

$$s^2 X - x'(0) + X = F_0 e^{-\pi s}$$

$$(s^2+1)X - 1 = F_0 e^{-\pi s}$$

$$X(s) = \frac{F_0 e^{-\pi s}}{s^2+1} + \frac{1}{s^2+1}$$

$* F(s) = \frac{1}{s^2+1}$, $a = \pi$

$f(t) = \sin t$

$$x(t) = F_0 \sin(t-\pi)u(t-\pi) + \sin t = \begin{cases} F_0 \sin(t-\pi) + \sin t, & t \geq \pi \\ \sin t, & 0 \leq t < \pi \end{cases}$$

$$* \mathcal{L}[f(t-a)u(t-a)] = \int_a^\infty f(t-a)e^{-st}\,dt = \int_0^\infty f(u)e^{-su}e^{-sa}\,du$$

$$= e^{-sa}F(s)$$

11. Find the general solution to

$$x' = x - 2y, \quad y' = 4x + 3y$$

Answer:

$$A = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix} \quad \text{has} \quad \lambda = 2 \pm \sqrt{7}\, i$$

$$\left[\begin{array}{cc|c} -1-\sqrt{7}i & -2 & 0 \\ 4 & 1-\sqrt{7}i & 0 \end{array}\right] \longrightarrow \quad v = \begin{bmatrix} 2 \\ -1-\sqrt{7}i \end{bmatrix}$$

$$e^{2t}\left(\cos\sqrt{7}t + i\sin\sqrt{7}t\right)\begin{bmatrix} 2 \\ -1-\sqrt{7}i \end{bmatrix}$$

$$= e^{2t}\left(\begin{bmatrix} 2\cos\sqrt{7}t \\ -\cos\sqrt{7}t + \sqrt{7}\sin\sqrt{7}t \end{bmatrix} + i\begin{bmatrix} 2\sin\sqrt{7}t \\ -\sqrt{7}\cos\sqrt{7}t - \sin\sqrt{7}t \end{bmatrix}\right)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{2t}\begin{bmatrix} 2\cos\sqrt{7}t \\ -\cos\sqrt{7}t + \sqrt{7}\sin\sqrt{7}t \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 2\sin\sqrt{7}t \\ -\sqrt{7}\cos\sqrt{7}t - \sin\sqrt{7}t \end{bmatrix}$$

12. For the DE system $\mathbf{x}' = A_a\mathbf{x}$ with $A_a = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}$:

a) Find the range of the values of a for which the critical point at (0,0) will be:
(i) a source node (ii) a sink node (iii) a saddle.

b) Choose a convenient value for a for each of the types above, solve, and sketch the trajectories in the vicinity of the critical point, showing the direction of increasing t.

$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \vec{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$

Answer:

a) $0 = \lambda^2 - 2a\lambda + (a^2 - 1) \longrightarrow \lambda = a \pm 1$

i. $\lambda_1, \lambda_2 > 0$ if $a > 1$
ii. $\lambda_1, \lambda_2 < 0$ if $a < -1$
iii. $\lambda_1 > 0 > \lambda_2$ if $-1 < a < 1$

b)

i. $a = 2,\ \lambda = 1, 3$
$\vec{x} = c_1 e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

ii. $a = -2,\ \lambda = -3, -1$
$\vec{x} = c_1 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

iii. $a = 0,\ \lambda = \pm 1$
$\vec{x} = c_1 e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

13. For the DE system $x' = x - 2y \qquad y' = 4x - x^3$:

a) Compute the critical points of this system.

b) Find the type of each of the critical points using the linearized system which approximates this non-linear system and classify them according to their stability type and also their structural stability type. (Use the Jacobian.)

c) Using the results of part(b), compute the eigenvectors as needed. Now put it all together into a reasonable sketch of the phase-plane portrait of this system. Is there more than one possiblity for the general shape and stability type of the trajectories around each of the critical points in this case? Why/why not?

14. Same instructions as **13** for the DE system $x' = y \qquad y' = 2x - x^2$.

c)

Answer:

13. a) $0 = x' = y' \rightarrow x = 2y,\ 0 = 2y(4-4y^2) \rightarrow (x,y) = (0,0), (2,1), (-2,-1)$

b) $J(x,y) = \begin{bmatrix} 1 & -2 \\ 4-3x^2 & 0 \end{bmatrix}$

$J(0,0) = \begin{bmatrix} 1 & -2 \\ 4 & 0 \end{bmatrix}$ $\quad J(2,1) = \begin{bmatrix} 1 & -2 \\ -8 & 0 \end{bmatrix}$ $\quad J(-2,-1) = \begin{bmatrix} 1 & -2 \\ -8 & 0 \end{bmatrix}$

$\lambda = \dfrac{1 \pm \sqrt{31}\, i}{2}$ $\quad \lambda = (1 \pm \sqrt{65})/2$ $\quad \lambda = (1 \pm \sqrt{65})/2$

Spiral source, structurally stable $\quad$ Saddle, structurally stable $\quad$ Saddle, structurally stable

14. a) $0 = x' = y$, $0 = y' = 2x - x^2$

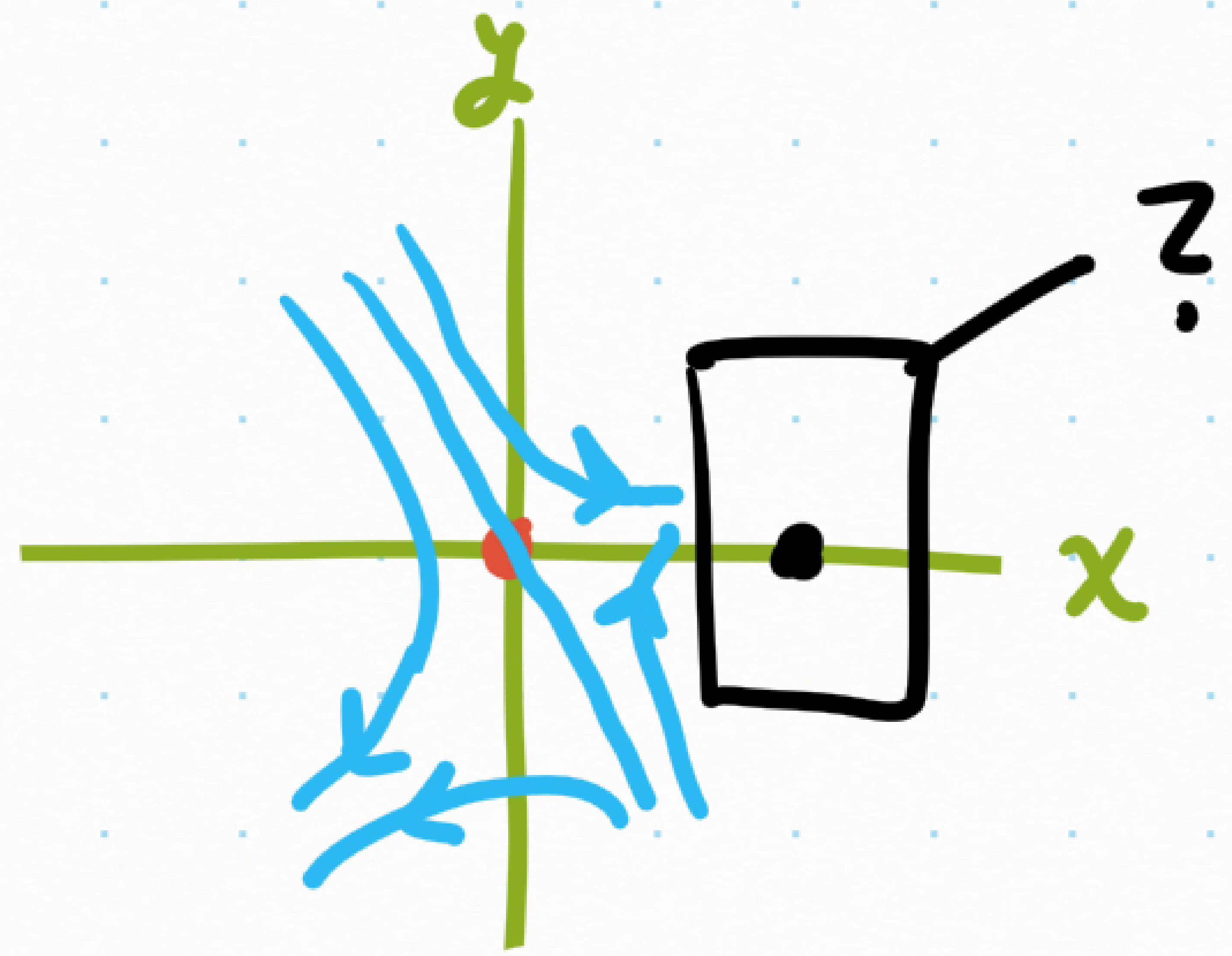
$\rightarrow (x,y) = (0,0), (2,0)$

b) $J(x,y) = \begin{bmatrix} 0 & 1 \\ 2-2x & 0 \end{bmatrix}$

$J(0,0) = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ $\quad$ $J(2,0) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$

$\lambda = \pm\sqrt{2}$
saddle (unstable)
Structurally stable

$\lambda = \pm\sqrt{2}\, i$
center (stable)
structurally unstable



Needs more analysis near $(2,0)$ since this critical pt. is structurally unstable.