§ Convolution

Convolution product: The convolution product of two functions $f(t)$ and $g(t)$ is

$$(f * g)(t) = \int_{0^-}^{t^+} f(t-\tau)g(\tau)\, d\tau.$$

This is also a function. We define it only for $t > 0$.

Problem 1: [Convolution]

(a) Let $q(t) = \cos(\omega t)$. Compute $w(t) * q(t)$ (where $w(t)$ is the unit impulse response for $D + kI$ and verify that it is the solution to $\dot{x} + kx = q(t)$ with rest initial conditions.

(b) Let $q(t) = 1$. Compute $w(t) * q(t)$ (where $w(t)$ is the unit impulse response for $D^2 + \omega_0^2 I$ and verify that it is the solution to $\ddot{x} + \omega_0^2 x = q(t)$ with rest initial conditions.

(c) Compute $t^2 * t$ and $t * t^2$. Are they equal?

(d) Compute $(t * t) * t$ and $t * (t * t)$. Are they equal?

Answer:

a) $w(t) = e^{-kt}$

$$(w*q)(t) = \int_0^t w(t-\tau)q(\tau)d\tau$$

$$= \int_0^t e^{-k(t-\tau)}\cos(\omega\tau)d\tau$$

$$= e^{-kt}\int_0^t \mathrm{Re}\left[e^{(k+\omega i)\tau}\right]d\tau$$

$$= e^{-kt}\mathrm{Re}\left[\int_0^t e^{(k+\omega i)\tau}d\tau\right]$$

$$= e^{-kt}\mathrm{Re}\left[\frac{e^{(k+\omega i)t}-1}{k+\omega i}\right]$$

$$= e^{-kt}\left[e^{kt}(k\cos\omega t + \omega\sin\omega t) - k\right]/(k^2+\omega^2)$$

$$= \frac{k\cos\omega t + \omega\sin\omega t - ke^{-kt}}{k^2+\omega^2}$$

Let $x(t) = (w * q)(t)$

$$\dot{x} + kx = \left(-k\omega \sin\omega t + \omega^2 \cos\omega t + k^2 e^{-kt}\right)/(k^2+\omega^2)$$

$$+ k\left(k\cos\omega t + \omega \sin\omega t - ke^{-kt}\right)/(k^2+\omega^2)$$

$$= (\omega^2+k^2)\cos(\omega t)/(k^2+\omega^2)$$

$$= \cos(\omega t)$$

$$= q(t)$$

b) $q(t) = 1$, $p(D) = D^2 + \omega_0^2 I$,

$$w(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$

$$\begin{aligned} 0 &= w(0^+) = c_1 \\ 1 &= \dot{w}(0^+) = c_2 \omega_0 \end{aligned} \longrightarrow w(t) = \frac{1}{\omega_0} \sin(\omega_0 t)$$

$$(w*q)(t) = \int_0^t w(t-\tau)q(\tau)\,d\tau$$

$$= \int_0^t \frac{1}{\omega_0}\sin(\omega_0(t-\tau))\,d\tau$$

$$= \left.\frac{1}{\omega_0^2}\cos(\omega_0(t-\tau))\right|_{\tau=0}^{\tau=t}$$

$$= \frac{1}{\omega_0^2}\left[\,1-\cos(\omega_0 t)\,\right]$$

Let $x = w*q$

$$\ddot{x} + \omega_0^2 x = \cos\omega_0 t + \omega_0^2/\omega_0^2\left[\,1-\cos(\omega_0 t)\right]$$

$$= 1 = q(t) \checkmark$$

$x(0) = 0$, $\dot{x}(0) = 0$ Rest I.C.'s $\checkmark$

c) $f(t) = t^2$, $g(t) = t$

$$(f * g)(t) = \int_0^t (t-\tau)^2 \tau \, d\tau$$

$$= \int_0^t (t^2\tau - 2t\tau^2 + \tau^3)\, d\tau$$

$$= \frac{t^2\tau^2}{2} - \frac{2t\tau^3}{3} + \frac{\tau^4}{4}\Big|_0^t = \frac{t^4}{12}$$

$$(g*f)(t) = \int_0^t (t-\tau)\tau^2 \, d\tau$$

$$= \int_0^t (t\tau^2 - \tau^3)\, d\tau$$

$$= \frac{t\tau^3}{3} - \frac{\tau^4}{4}\Big|_0^t = \frac{1}{12}$$

✓

d) $f(t) = g(t) = h(t) = t$

$$(f*g)(t) = \int_0^t (t-\tau)\,\tau\, d\tau = \frac{t\tau^2}{2} - \frac{\tau^3}{3}\Big|_0^t = \frac{1}{6}t^3$$

$$[(f*g)*h](t) = \int_0^t \frac{1}{6}(t-\tau)^3\tau\, d\tau$$

$$= \int_0^t \frac{1}{6}\left(t^3\tau - 3t^2\tau^2 + 3t\tau^3 - \tau^4\right) d\tau$$

$$= \frac{1}{6}\left(\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5}\right)t^5 = \frac{1}{6}\,\frac{1}{20}t^5 = \frac{1}{120}t^5 .$$

$$[h*(f*g)](t) = \int_0^t (t-\tau)\,\frac{\tau^3}{6}\, d\tau$$

$$= \int_0^t \frac{1}{6}\left(t\tau^3 - \tau^4\right) d\tau$$

$$= \frac{1}{6}\left(\frac{1}{4} - \frac{1}{5}\right)t^5 = \frac{1}{120}t^5 .$$