§ Matrix Methods : Eigenvalues and Normal Modes.

Problem 1: Solve $x' = Ax$, where $A = \begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix}$.

Answer: $0 = \lambda^2 - 0\lambda - 1$

$\lambda = \pm 1$

$$\left[\begin{array}{cc|c} -3-1 & 4 & 0 \\ -2 & 3-1 & 0 \end{array}\right] \qquad \left[\begin{array}{cc|c} -3+1 & 4 & 0 \\ -2 & 3+1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \sim \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$x(t) = c_1 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Problem 2: Solve $x' = Ax$, where $A = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix}$.

Answer: $0 = \lambda^2 + 2\lambda + 0$

$\lambda = 0, -2$

$$\left[\begin{array}{cc|c} 4 & -3 & 0 \\ 8 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{cc|c} 6 & -3 & 0 \\ 8 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$v = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$x(t) = c_1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Problem 3: Solve $x' = Ax$, where $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{bmatrix}$

Answer: $\lambda = \pm 1, 2$

$\underline{\lambda = -1}$

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ -2 & 1 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 7 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$v = \begin{bmatrix} 1 \\ 2 \\ -7 \end{bmatrix}$$

$\underline{\lambda = 1}$

$$\left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ -2 & 1 & -2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$v = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$\underline{\lambda = 2}$

$$\left[\begin{array}{ccc|c} -1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -2 & 1 & -3 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$v = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$x(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 2 \\ -7 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

Problem 4: Find the real solutions to

$$x' = Ax = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} x.$$

Answer: $0 = \lambda^2 - 6\lambda + 25$

$\lambda = 3 \pm 4i$

$\underline{\lambda = 3 + 4i}$

$$\left[\begin{array}{cc|c} -4i & -4 & 0 \\ 4 & -4i & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$e^{3t}\left(\cos(4t) + i\sin(4t)\right)\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$= e^{3t}\left(\begin{bmatrix} \cos(4t) \\ \sin(4t) \end{bmatrix} + i \begin{bmatrix} \sin(4t) \\ -\cos(4t) \end{bmatrix}\right)$$

$\longrightarrow$

$$x(t) = c_1 e^{3t} \begin{bmatrix} \cos 4t \\ \sin 4t \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} \sin 4t \\ -\cos 4t \end{bmatrix}$$