§ Linearization Near Critical Points

For the following systems, the origin is clearly a critical point. Give its geometric type and stability, and sketch some nearby trajectories of the system.

Problem 1:

$$x' = x - y + xy$$
$$y' = 3x - 2y - xy$$

Problem 2:

$$x' = x + 2x^2 - y^2$$
$$y' = x - 2y + x^3$$

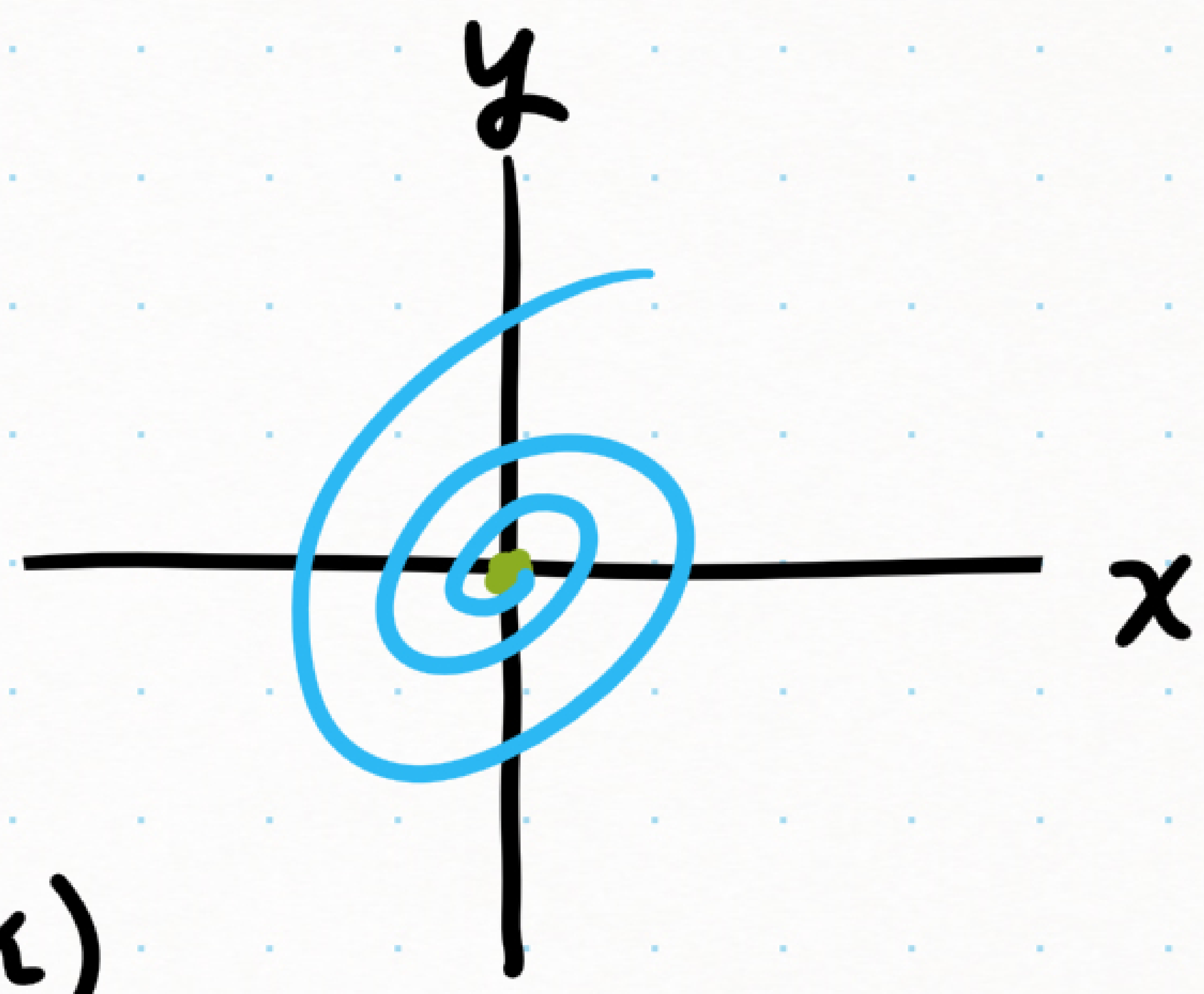
Answer:

1) Near $(0,0)$, $x' \approx x - y$
$y' \approx 3x - 2y$

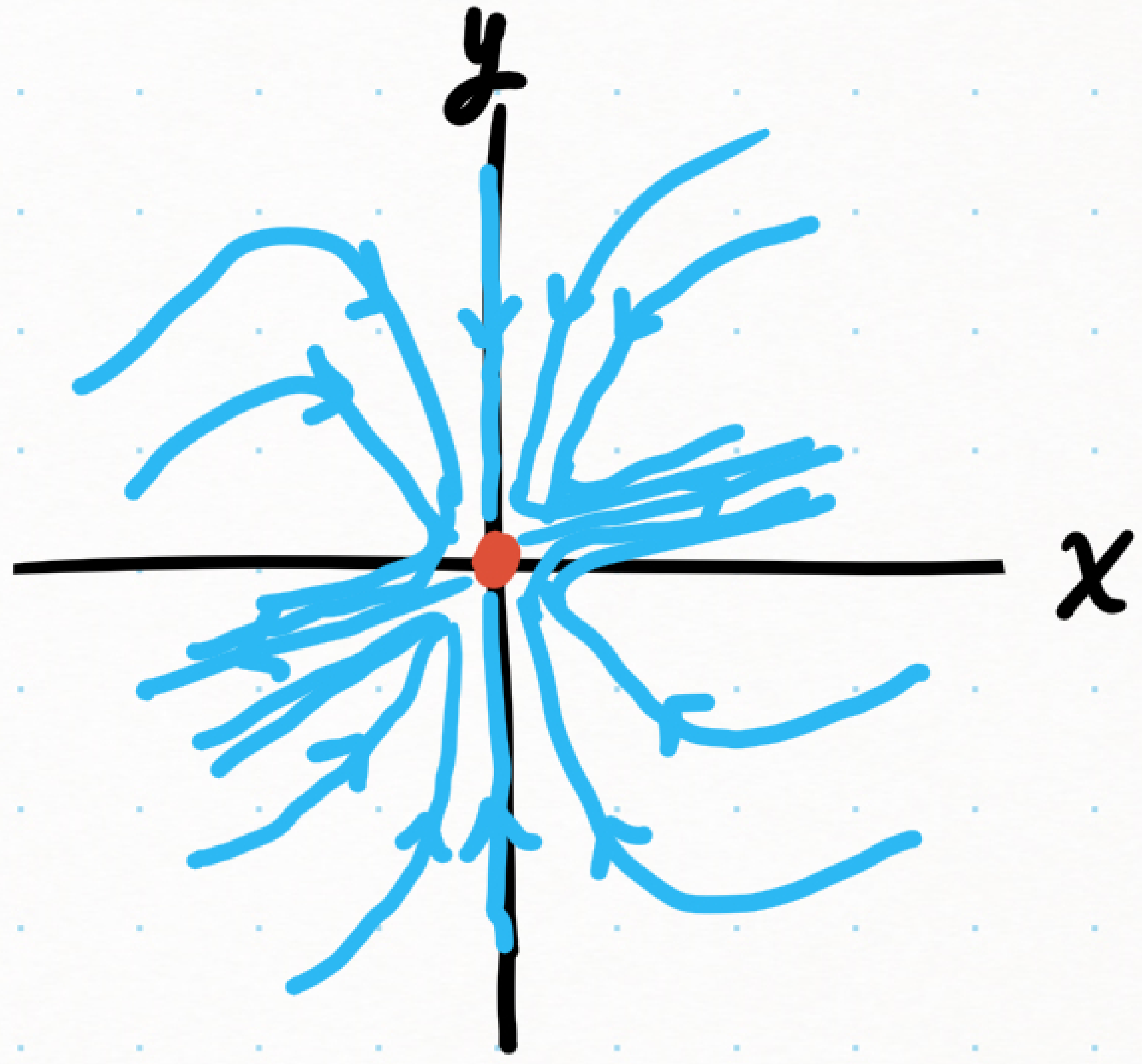
$$J(0,0) = \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix}$$

$0 = \lambda^2 + \lambda + 1$
$\lambda = (-1 \pm \sqrt{3}i)/2$

stable spiral (sink)



2) $J(0,0) = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}$



$0 = \lambda^2 + \lambda - 2$

$\lambda = \frac{-1 \pm 3}{2} = -2, 1$

unstable saddle node

$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -3 & 0 \end{array}\right] \rightarrow v = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \lambda = 1$

$\left[\begin{array}{cc|c} 3 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right] \rightarrow v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \lambda = -3$

For the following systems carry out the linearization for sketching trajectories. Find the critical points, analyze each, draw in nearby trajectories , then add some other trajectories compatible with the ones you have drawn; when necessary, sketch in a well-chosen vector from the vector field to help.

Problem 3:

$$x' = 1 - y$$
$$y' = x^2 - y^2$$

Problem 4:

$$x' = x - x^2 - xy$$
$$y' = 3y - xy - 2y^2$$

Answer:

3) $\begin{matrix} 0 = 1 - y \\ 0 = x^2 - y^2 \end{matrix} \rightarrow \begin{matrix} y = 1 \\ x = \pm 1 \end{matrix}$

$J(1,1) = \begin{bmatrix} 0 & -1 \\ 2 & -2 \end{bmatrix}$ $\lambda = -1 \pm i$ stable spiral sink near $(1,1)$

$J(-1,1) = \begin{bmatrix} 0 & -1 \\ -2 & -2 \end{bmatrix}$ $\lambda = -1 \pm \sqrt{3}$ unstable saddle near $(-1,1)$.

4) $\begin{aligned} 0 &= x - x^2 - xy \\ 0 &= 3y - xy - 2y^2 \end{aligned} \longrightarrow (x,y) = (0,0), (0,\tfrac{3}{2}), (1,0), (-1,2)$

$$J(x,y) = \begin{bmatrix} 1-2x-y & -x \\ -y & 3-x-4y \end{bmatrix}$$

$J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ $\lambda = 1, 3$ Unstable source node near $(0,0)$.

$J(0,\tfrac{3}{2}) = \begin{bmatrix} -1/2 & 0 \\ -3/2 & -3 \end{bmatrix}$ $\lambda = -\tfrac{1}{2}, -3$ asymptotically stable sink node near $(0,3)$.

$J(1,0) = \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}$ $\lambda = -1, 2$ unstable saddle node near $(1,0)$.

$J(-1,2) = \begin{bmatrix} 1 & 1 \\ -2 & -4 \end{bmatrix}$ $\lambda = \dfrac{-3 \pm \sqrt{17}}{2}$ unstable saddle node near $(-1,2)$.

Problem 5: Structural stability:
The following system has a critical point at the origin:

$$x' = 3x - y + x^2 + y^2, \qquad y' = -6x + 2y + 3xy.$$

For that critical point, find the geometric type and stability of the corresponding linearized system, and then tell what the possibilities would be for the corresponding critical point of the given non-linear system.

Answer: The linearized system is

$$x' = 3x - y$$
$$y' = -6x + 2y$$

$$J = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}, \quad \lambda = 0, 5$$

The linearized system has a line critical points passing through (0,0)
This can't be used to classify (0,0) for the nonlinear system.

Problem 6: Structural stability:
The following system has one critical point whose linearization is not structurally stable:

$$x' = y, \qquad y' = x(1-x).$$

Begin by finding the critical points and determining the type of the corresponding linearized system at each of the critical points. Then in each case, sketch several pictures showing the different ways the trajectories of the non-linear system *might* look.

* (0,0) is structurally stable.

† (1,0) is not structurally stable.

Answer: $x' = y' = 0 \rightarrow (x_0, y_0) = (0,0),\ (1,0)$

The linearized system $x' = y$, $y' = x$ has (0,0) as a critical point.

$J(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\lambda = \pm 1 \rightarrow$ unstable saddle near (0,0) (both systems). *

$J(1,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\lambda = \pm i \rightarrow$ stable center near (1,0) †

The linearized system has a stable center near (1,0) but the nonlinear system may not.