§ Exponential Response

Problem 1: Find a particular solution to

$$y^{(3)} + y'' - y' + 2y = 2\cos t$$

Answer: $z(t) = Ce^{it}$

$$\begin{aligned} 2e^{it} &= z^{(3)} + z'' - z' + 2z \\ &= (i^3 + i^2 - i + 2)\,ce^{it} \\ &= (1-2i)\,ce^{it} \end{aligned}$$

$$C = \frac{2}{1-2i} = \frac{2+4i}{1+4} = \frac{2}{5} + \frac{4}{5}i$$

$$z(t) = \left(\frac{2}{5} + \frac{4}{5}i\right)e^{it} \qquad \left(\frac{2}{5} + \frac{4}{5}i\right)(\cos t + i\sin t)$$

$$y(t) = \operatorname{Re}(Z(t)) = \frac{2}{5}\cos t - \frac{4}{5}\sin t.$$

Problem 2: Find a particular solution to

$$y'' - 2y' + 4y = e^{x}\cos x$$

Answer: $z(x) = c\, e^{(1+i)x}$

$$e^{(1+i)x} = z'' - 2z' + 4z$$

$$= \left[(1+i)^2 - 2(1+i) + 4\right] c e^{(1+i)x}$$

$$1 = \left[(1+i)^2 - 2(1+i) + 4\right] c \longrightarrow c = 1/2$$

$$z_p(x) = \tfrac{1}{2} e^{(1+i)x} = \tfrac{1}{2} e^{x}(\cos x + i \sin x)$$

$$y_p(x) = \mathrm{Re}\left(z_p(x)\right) = \tfrac{1}{2} e^{x} \cos x.$$

Problem 4: Find the real general solution to

$$\frac{d^3x}{dt^3} - x = e^{2t}$$

Answer: $r^3 - 1 = 0$, $r = 1, e^{2\pi i/3}, e^{4\pi i/3}$

$$x_h(t) = c_1 e^t + c_2 e^{-t/2} \cos\frac{\sqrt{3}}{2}t + c_3 e^{-t/2} \sin\frac{\sqrt{3}}{2}t$$

$$x_p(t) = Ce^{2t}$$

$$e^{2t} = \dddot{x}_p - x_p = 7Ce^{2t} \longrightarrow C = 1/7$$

$$x_p(t) = \frac{1}{7}e^{2t}$$

$$x(t) = x_h(t) + x_p(t).$$

Problem 5: Find a particular solution to

$$y'' - 4y = \frac{1}{2}\left(e^{2x} + e^{-2x}\right)$$

Answer: $y'' - 4y = 0 \rightarrow r^2 - 4 = 0 \quad r = \pm 2$

$$y_h(x) = c_1 e^{2x} + c_2 e^{-2x}$$

$$y_{p,1}(x) = Cxe^{2x}$$

$$\frac{1}{2}e^{2x} = y_{p,1}'' - 4y_{p,1} = 4Ce^{2x} \longrightarrow C = 1/8.$$

$$y_{p,2}(x) = Dxe^{-2x}$$

$$\frac{1}{2}e^{-2x} = y_{p,2}'' - 4y_{p,2} = -4De^{-2x} \longrightarrow D = -1/8$$

$$y_p(x) = y_{p,1} + y_{p,2} = \frac{x}{8}e^{2x} - \frac{x}{8}e^{-2x}.$$