18.03SC Unit 2 Exam

1. (a) For what value of k is the system represented by $\ddot{x} + \dot{x} + kx = 0$ critically damped? [8]

$$0 = 1^2 - 4k$$
$$k = 1/4$$

(b) For k greater than that value, is the system overdamped or underdamped? [4]

$$k > 1/4$$
$$4k > 1$$
$$0 > 1^2 - 4k$$

Underdamped

(c) Suppose a solution of $\ddot{x} + \dot{x} + kx = 0$ vanishes at $t = 1$, and then again for $t = 2$ (but not in between). What is k? [8]

$$\omega = \frac{2\pi}{2(2-1)} = \pi$$
$$\pi^2 = \omega^2 = k - (b/2)^2 = k - 1/4$$
$$\pi^2 + 1/4 = k$$

2. (a) Find a solution of $\ddot{x} + x = 5te^{2t}$. [10]

Let $x_p = (At + B)e^{2t}$

$\dot{x}_p = Ae^{2t} + 2(At + B)e^{2t}$

$\ddot{x}_p = 2Ae^{2t} + 2[Ae^{2t} + 2(At+B)e^{2t}]$

$5te^{2t} = e^{2t}[5At + 4A + 5B]$

$$\left.\begin{array}{l} 5A = 5 \\ 4A + 5B = 0 \end{array}\right\} \longrightarrow \begin{array}{l} A = 1 \\ B = -4/5 \end{array}$$

$x_p(t) = (t - 4/5)e^{2t}$

(b) Suppose that $y(t)$ is a solution of the same equation, $\ddot{x} + x = 5te^{2t}$, such that $y(0) = 1$ and $\dot{y}(0) = 2$. (This is probably *not* the solution you found in **(a)**.) Use $y(t)$ and other functions to write down a solution $x(t)$ such that $x(0) = 3$ and $\dot{x}(0) = 5$. [10]

From (a), $x_p(0) = -4/5$, $\dot{x}_p(0) = -3/5$

$$\left[\begin{array}{cc|c} 1 & -4/5 & 3 \\ 2 & -3/5 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -4/5 & 3 \\ 0 & 1 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 11/5 \\ 0 & 1 & -1 \end{array}\right]$$

$3 - \frac{4}{5} = \frac{15}{15} - \frac{12}{15}$

$3 - \frac{4}{5} = \frac{15}{5} - \frac{4}{5} = \frac{3}{5}$

$x(t) = \frac{11}{5}y(t) - x_p(t)$

$x(0) = \frac{11}{5} \cdot 1 + \frac{4}{5} = 3$

$\dot{x}(0) = \frac{11}{5} \cdot 2 + \frac{3}{5} = \frac{25}{5} = 5.$

3. **(a)** Consider the equation $\ddot{x} + b\dot{x} + kx = \cos(\omega t)$. We will vary the spring constant but keep b fixed. For what value of k is the amplitude of the sinusoidal solution of $\ddot{x} + b\dot{x} + kx = \cos(\omega t)$ maximal? (Your answer will be a function of ω and may depend upon b as well.) [10]

$$\text{Amplitude} = \frac{1}{|p(i\omega)|} = \frac{1}{|(i\omega)^2 + bi\omega + k|}$$

$$= \frac{1}{|k - \omega^2 + ib\omega|}$$

$$= \frac{1}{\sqrt{(k-\omega^2)^2 + b^2\omega^2}}$$

minimize $(k-\omega^2)^2 + b^2\omega^2 = \omega^4 + (b^2 - 2k)\omega^2 + k^2$

$$0 = \frac{d}{dk}\left[\omega^4 + (b^2-2k)\omega^2 + k^2\right] = -2\omega^2 + 2k$$

$k = \omega^2$ maximizes amplitude.

(b) (Unrelated to the above.) Find the general solution of $\dfrac{d^3x}{dt^3} - \dfrac{dx}{dt} = 0$. [10]

$r^3 - r = 0$

$r(r^2-1) = 0$

$r = 0,\ r = \pm 1$

$x(t) = c_1 + c_2 e^{t} + c_3 e^{-t}$.

4. A certain system has input signal y and system response x related by the differential equation $\ddot{x} + \dot{x} + 6x = 6y$. It is subjected to a sinusoidal input signal.

(a) Calculate the complex gain $H(\omega)$. [10]

By linearity and time invariance, we can assume $y(t) = \cos(\omega t)$ and $y_{cx}(t) = e^{i\omega t}$

$$\ddot{z} + \dot{z} + 6z = 6e^{i\omega t}$$

$$z(t) = \frac{6}{(i\omega)^2 + i\omega + 6} e^{i\omega t} = \frac{6}{p(i\omega)} y_{cx}$$

$$H(\omega) = \frac{6}{p(i\omega)} = \frac{6}{i^2\omega^2 + i\omega + 6} = \frac{6}{6 - \omega^2 + i\omega}$$

(b) Compute the gain at $\omega = 2$. [5]

$$H(2) = \frac{6}{2+2i} = \frac{12 - 12i}{8} = \frac{3}{2} - \frac{3}{2}i$$

(c) Compute the phase lag at $\omega = 2$. [5]

$$\phi = \mathrm{Arg}(p(i\omega)) = \mathrm{Arg}(2 + 2i) = \pi/4 .$$

or $$\phi = -\mathrm{Arg}(H(\omega)) = -\mathrm{Arg}\left(\frac{3}{2} - \frac{3}{2}i\right) = -(-\pi/4) = \pi/4 .$$

5. Suppose that $\frac{1}{2}t\sin(2t)$ is a solution to a certain equation $m\ddot{x} + b\dot{x} + kx = 4\cos(2t)$.

(a) Write down a solution to $m\ddot{x} + b\dot{x} + kx = 4\cos(2t-1)$. [4]

$2t \to 2t-1$

$t \to t-\frac{1}{2}$

$\frac{1}{2}t\sin(2t) \to \frac{1}{2}(t-\frac{1}{2})\sin(2t-1)$

(b) Write down a solution to $m\ddot{x} + b\dot{x} + kx = 8\cos(2t)$. [4]

$x(t) = t\sin(2t)$

By linearity $m\ddot{x} + b\dot{x} + kx = 2\cdot 4\cos(2t)$.

(c) Determine m, b, and k. [12]

$0 = m(s-2i)(s+2i) = m(s^2+4)$

$b=0, \quad k=4m$

$m\ddot{z} + b\dot{z} + kz = 4e^{2it}$

$m\ddot{z} + kz = 4e^{2it}$

$$z = \frac{4te^{2it}}{4mi} = \frac{te^{2it}}{mi}$$

$x(t) = \text{Re}(z) = \frac{t}{m}\sin 2t$

Comparing to $x(t) = \frac{t}{2}\sin 2t$,

$m=2, \quad b=0, \quad k=8$.