§Pure Resonance

Problem 1: Find a particular solution to

$$\dddot{x} + \dot{x} = 2\cos t$$

The problem is thus to find a particular solution the DE

$$x'' + \omega_0^2 x = F_0 \cos \omega t.$$

The steps, as in the example in the last note, are

Complex replacement: $z'' + \omega_0^2 z = F_0 e^{i\omega t}$, $x = \mathrm{Re}(z)$.

Characteristic polynonial: $p(r) = r^2 + \omega_0^2 \Rightarrow p(i\omega) = \omega_0^2 - \omega^2$.

Exponential Response formula $\Rightarrow z_p = \begin{cases} \dfrac{F_0 e^{i\omega t}}{p(i\omega)} = \dfrac{F_0 e^{i\omega t}}{\omega_0^2 - \omega^2} & \text{if } w \neq \omega_0 \\ \dfrac{F_0\, t e^{i\omega t}}{p'(i\omega)} = \dfrac{F_0\, t e^{i\omega t}}{2i\omega} & \text{if } \omega = \omega_0. \end{cases}$

$$\Rightarrow x_p = \begin{cases} \dfrac{F_0 \cos \omega t}{\omega_0^2 - \omega^2} & \text{if } \omega \neq \omega_0 \\ \dfrac{F_0\, t \sin \omega_0 t}{2\omega_0} & \text{if } \omega = \omega_0 \quad \text{(resonant case).} \end{cases}$$

Answer: Let $u = \dot{x}$. Then

$\ddot{u} + u = 2\cos t$. Note that $\omega = \omega_0 = 1$.

$$z_p = \frac{2te^{it}}{2i} = \frac{-4ite^{it}}{-4i^2} = -ite^{it} = -t(i\cos t + i^2 \sin t)$$

$$z_p = t\sin t - it\cos t$$

$$u_p = \mathrm{Re}(z_p) = t\sin t$$

$$x_p = \int u_p\, dt = -t\cos t + \sin t$$

<u>Check:</u> $\dot{x}_p = t\sin t$

$\ddot{x}_p = \sin t + t\cos t$

$\dddot{x}_p = 2\cos t - t\sin t$

$\dddot{x}_p + \dot{x}_p = 2\cos t$ ✓.