§ Integrating Factors

For each of the next three problems, find the general solution and the specific solution satisfying the IC.

Problem 1:

$$\frac{dy}{dx} + y = 2 \quad , \quad y(0) = 0$$

Answer:

$$\mu(x) = e^x$$

$$e^x \frac{dy}{dx} + e^x = 2e^x$$

$$ye^x = 2e^x + c$$

$$y(x) = 2 + ce^{-x}$$

$$0 = y(0) = 2 + c$$

$$-2 = c$$

$$y(x) = 2 - 2e^{-x}.$$

Problem 2:

$$xy' - y = x \quad , \quad x(1) = 7$$

Answer:

$$y' - \frac{1}{x} y = 1 \qquad \mu(x) = e^{\int -\frac{1}{x} dx} = 1/x$$

$$\frac{1}{x} y' - \frac{1}{x^2} y = \frac{1}{x}$$

$$\frac{1}{x} y = \ln|x| + c$$

$$y = x \ln|x| + cx$$

$$x(1) = 7 \longrightarrow \quad 1 = 7 \ln|7| + 7c$$

$$\frac{1 - 7\ln 7}{7} = c$$

$$y(x) = x \ln|x| + \frac{1 - 7\ln 7}{7} x$$

Problem 3:

$$y' = 1 + x + y + xy, \quad y(0) = 0$$

$$y' - (1+x)y = 1 + x \qquad u(x) = e^{\int -(1+x)\,dx} = e^{-(x+\frac{x^2}{2})}$$

$$e^{-(x+\frac{x^2}{2})} y' - (1+x) e^{-(x+\frac{x^2}{2})} y = (1+x) e^{-(x+\frac{x^2}{2})}$$

$$y e^{-(x+x^2/2)} = -e^{-(x+x^2/2)} + C$$

$$y(x) = Ce^{x+x^2/2} - 1$$

$$0 = y(0) = C - 1 \rightarrow C = 1$$

$$y(x) = e^{x+x^2/2} - 1$$

Problem 4:

Problem 4: Water flows into and out of a 100,000 liter (ℓ) reservoir at a constant rate of 10 ℓ/min. The reservoir initially contains pure water, but then the water coming in has a concentration of 10 grams/liter of a certain pollutant. The reservoir is well-stirred so that the concentration of pollutant in it is uniform at all times.

a) Set up the DE for the concentration $c = c(t)$ of salt in the reservoir at time t. Specify units.

b) Solve for $c(t)$ with the given initial condition, and graph the solution c vs. t.

c) How long will it take for the concentration of salt to be 5 $\frac{g}{\ell}$?

d) What happens in the long run?

Answer: I let c be the amt of salt/pollution in g (instead of concentration)

a) $\frac{dc}{dx} = 100 - \frac{c}{10{,}000}$, $c(0) = 0$

x : minutes

c : grams of salt/pollutant

b) $\frac{dc}{dx} + \frac{c}{10{,}000} = 100$

$\mu(x) = e^{x/10{,}000}$

$c e^{x/10{,}000} = \int 100 e^{x/10{,}000} \, dx$

$c e^{x/10{,}000} = 10^6 e^{x/10{,}000} + k$

$c(x) = 10^6 + k e^{-x/10{,}000}$

$0 = c(0) \rightarrow k = -10^6$

$c(x) = 10^6 - 10^6 e^{-x/10{,}000}$

(c) $5 = \frac{c(x)}{100{,}000} = 10 - 10e^{-x/10{,}000}$

$$\frac{1}{2} = e^{-x/10{,}000}$$

$$x = 10{,}000 \ln 2 \approx 6931.5 \text{ minutes.}$$

(d) $\lim_{x\to\infty} \frac{c(x)}{100{,}000} = \lim_{x\to\infty} 10 - 10e^{-x/10{,}000} = 10 \text{ g/liter}$