Unit II Practice Exam

1. The mass and spring constant in a certain mass/spring/dashpot system are known—$m = 1$, $k = 25$—but the damping constant b is not known. It's observed that for a certain solution $x(t)$ of $\ddot{x} + b\dot{x} + 25x = 0$, $x(\frac{\pi}{6}) = 0$ and $x(\frac{\pi}{2}) = 0$, but $x(t) > 0$ for $\frac{\pi}{6} < t < \frac{\pi}{2}$.

(a) Is the system underdamped, critically damped, or overdamped?

(b) Determine the value of b.

Answer:

a) The characteristic equation gives

$$r = \frac{-b \pm \sqrt{b^2 - 100}}{2}$$

The solution $x(t) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ cannot satisfy the description unless $b < 10$. The system is underdamped.

b) The pseudoperiod is $2\left(\pi/2 - \pi/6\right) = 2\pi/3$

$\omega = 2\pi/(2\pi/3) = 3$, $9 = \omega^2 = k - (b/2)^2 = 25 - \frac{b^2}{4}$

$\longrightarrow$ $b = 8$.

2. Find a solution of $3\ddot{x} + 2\dot{x} + x = t^2$.

Answer: To find a particular solution, let

$$x_p = At^2 + Bt + C$$

$$t^2 = 3 \cdot 2 \cdot A + 2 \cdot (2At + B) + At^2 + Bt + C.$$

$$\left.\begin{aligned} 1 &= A \\ 0 &= 4A + B \\ 0 &= 6A + 2B + C \end{aligned}\right\} \longrightarrow \begin{aligned} A &= 1 \\ B &= -4 \\ C &= 2 \end{aligned}$$

$x_p(t) = t^2 - 4t + 2$ is a solution.

3. Find a solution to $\ddot{x}+3\dot{x}+2x=e^{-t}$.

Answer: Note that $r=-1$ is a solution to the characteristic equation so $x_p = Ae^{-t}$ won't work.

$$x_p = \frac{1\cdot t\cdot e^{-t}}{|2(-1)+3|} = \frac{te^{-t}}{|1|} = te^{-t}.$$

4. This problem concerns the sinusoidal solution $x(t)$ of $\ddot{x} + 4\dot{x} + 9x = \cos(\omega t)$.

(a) For what value of ω is the amplitude of $x(t)$ maximal?

(b) For what value of ω is the phase lag exactly $\frac{\pi}{4}$?

Answer:

a) Amplitude $= 1/|p(i\omega)| = 1/|(i\omega)^2 + 4i\omega + 9|$

$$= 1/|9-\omega^2 + 4i\omega|$$

$$= 1/[(9-\omega^2)^2 + 16\omega^2] = 1/(\omega^4 - 2\omega^2 + 81)$$

Minimize $\omega^4 - 2\omega^2 + 81$. Set $0 = 4\omega^3 - 4\omega = 4\omega(\omega^2 - 1)$.
Since $\omega > 0$, take $\omega = 1$. the max. amplitude is
$1/(1-2+81) = 1/80$.

b) $\phi = \mathrm{Arg}(p(i\omega)) = \mathrm{Arg}(9-\omega^2 + i(4\omega)) = \arctan\left(\frac{4\omega}{9-\omega^2}\right)$

$1 = \tan \pi/4 = \tan\phi = \frac{4\omega}{9-\omega^2} \longrightarrow \omega = 2 \pm \sqrt{13}$.

5. The equation $2\ddot{x} + \dot{x} + x = \dot{y}$ models a certain system in which the input signal is y and the system response is x. We drive it with a sinusoidal input signal of circular frequency ω. Determine the complex gain as a function of ω, and the gain and phase lag at $\omega = 1$.

Answer: $y = A\cos(\omega t)$, $y_{cx} = Ae^{i\omega t}$

$$2\ddot{z} + \dot{z} + z = i\omega A e^{i\omega t}$$

$$p(i\omega) = 2(i\omega)^2 + i\omega + 1 = 1 - 2\omega^2 + i\omega$$

$$z_p = \frac{Ai\omega}{1-2\omega^2+i\omega} e^{i\omega t} \quad \text{by ERF}$$

Complex Gain $H(\omega) = \dfrac{i\omega}{1-2\omega^2+i\omega}$

At $\omega=1$, gain $g(\omega) = g(1) = |H(1)| = \left|\dfrac{i}{1-2+i}\right| = \left|\frac{1}{2} - \frac{1}{2}i\right| = \frac{1}{\sqrt{2}}$.

Phase Lag $-\operatorname{Arg}(H(1)) = -\operatorname{Arg}\left(\frac{1}{2} - \frac{1}{2}i\right) = -(-\pi/4) = \pi/4$.

6. Find a solution to $\dddot{x} + x = e^{-t}\cos t$.

Answer: $z_p = Ae^{(-1+i)t}$

$$e^{(-1+i)t} = A(-1+i)^3 e^{(-1+i)t} + Ae^{(-1+i)t}$$

$$= (3+2i)Ae^{(-1+i)t}$$

$$A = \frac{1}{3+2i} = \frac{3-2i}{13} = \frac{3}{13} - \frac{2}{13}i$$

$$x_p = \mathrm{Re}(z_p) = \mathrm{Re}\left(\left(\frac{3}{13} - \frac{2}{13}i\right)e^{-t}(\cos t + i\sin t)\right)$$

$$= e^{-t}\left(\frac{3}{13}\cos t + \frac{2}{13}\sin t\right)$$

7. Assume that $\cos t$ and t are both solutions of the equation $p(D)x = q(t)$, for a certain polynomial $p(s)$ and a certain function $q(t)$.

(a) Write down a nonzero solution of the equation $p(D)x = 0$.

(b) Write down a solution $x(t)$ of $p(D)x = q(t)$ such that $x(0) = 2$.

(c) Write down a solution of the equation $p(D)x = q(t-1)$.

Answer:

a) $p(D)\cos t - p(D)t = q(t) - q(t) = 0$.

b) $x(t) = 2\cos t - t$

$$p(D)x = 2q(t) - q(t) = q(t)$$
$$x(0) = 2 \cdot 1 - 0 = 2.$$

c) $x(t) = t - 1$

$$p(D)u = q(u)$$

$$p(D)x = p(D)(t-1) = q(t-1)$$

Also, $x(t) = \cos(t-1) \rightarrow p(D)\cos(t-1) = q(t-1)$.

$x(t) = a\cos(t-1) + (1-a)(t-1) \longrightarrow p(D)x = aq(t-1) + (1-a)q(t-1) = q(t-1)$.