18.03SC Unit 3 Exam

1. A certain periodic function has Fourier series

$$f(t) = 1 + \frac{\cos(\pi t)}{2} + \frac{\cos(2\pi t)}{4} + \frac{\cos(3\pi t)}{8} + \frac{\cos(4\pi t)}{16} + \cdots$$

(a) What is the minimal period of $f(t)$? [4]

Assuming this means the period of the first periodic nonconstant term, it is the period of $\frac{\cos(\pi t)}{2}$, which is 2.

(b) Is $f(t)$ even, odd, neither, or both? [4]

$f(-t) = f(t)$ $\quad f(t)$ is even.

$f(-t) \neq -f(t)$ $\quad f(t)$ is not odd.

(c) Please give the Fourier series of a periodic solution (if one exists) of [8]

$$\ddot{x} + \omega_n^2 x = f(t)$$

$\ddot{x}_p + \omega_n^2 x_p = (\cos k\pi t)/(2^k)$

$z_p = Ae^{k\pi i t}$

$A\left[(k\pi i)^2 + \omega_n^2\right] = \frac{1}{2^k}$

$A = 1 / \left[2^k(\omega_n^2 - k^2\pi^2)\right]$

$x_p = \operatorname{Re}(z_p)$

$= \dfrac{\cos k\pi t}{2^k(\omega_n^2 - k^2\pi^2)}$

Generalize for all k:

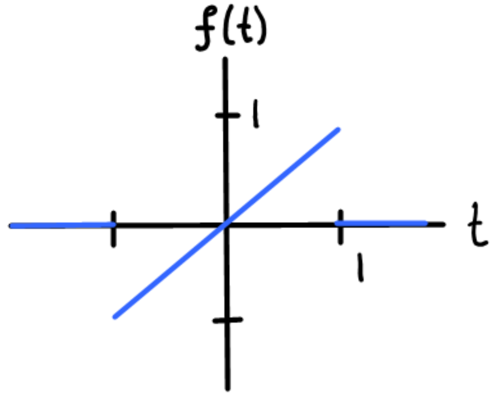
$$x_p = \sum_{k=0}^{\infty} \frac{\cos(k\pi t)}{2^k(\omega_n^2 - k^2\pi^2)}$$

(d) For what values of ω_n is there no periodic solution? [4]

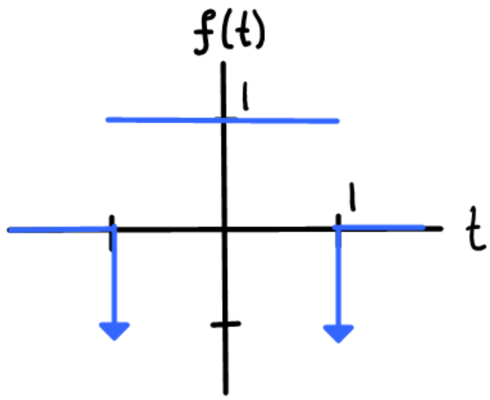
$\omega_n = k\pi, \quad k = 0, 1, 2, \ldots$

2. Let $f(t) = (u(t+1) - u(t-1))t$.

(a) Sketch a graph of $f(t)$. [6]



(b) Sketch a graph of the generalized derivative $f'(t)$. [6]



(c) Write a formula for the generalized derivative $f'(t)$. [8]

$$f'(t) = u(t+1) - u(t-1) + \delta(t+1) + \delta(t-1)$$

3. Let $p(D)$ be the operator whose unit impulse response is given by $w(t) = e^{-t} - e^{-3t}$.

(a) Using convolution, find the unit step response of this operator: the solution to $p(D)v = u(t)$ with rest initial conditions. [10]

$$v(t) = (w * u)(t) = \int_0^t w(t-\tau)\,u(\tau)\,d\tau$$

$$= \int_0^t \left(e^{-(t-\tau)} - e^{-3(t-\tau)}\right) d\tau$$

$$= e^{-t}e^{\tau} - \frac{1}{3}e^{-3t}e^{3\tau} \Big|_0^t$$

$$= e^0 - \frac{1}{3}e^0 - e^{-t}e^0 + \frac{1}{3}e^{-3t}e^0$$

$$= \frac{2}{3} - e^{-t} + \frac{1}{3}e^{-3t}$$

(b) What is the transfer function $W(s)$ of the operator $p(D)$? [5]

$$W(s) = \mathcal{L}[w(t)]$$

$$= \frac{1}{s+1} - \frac{1}{s+3}$$

(c) What is the characteristic polynomial $p(s)$? [5]

$$p(s) = \frac{1}{W(s)}$$

$$= \frac{1}{\frac{1}{s+1} - \frac{1}{s+3}}$$

$$= \frac{1}{2}s^2 + 2s + \frac{3}{2}$$

4 (a) Find a generalized function $f(t)$ with Laplace transform $F(s) = \dfrac{e^{-s}(s-1)}{s}$. [10]

$$\mathcal{L}[u(t-1)] = e^{-s}\mathcal{L}[1] = \frac{e^{-s}}{s}$$

$$\mathcal{L}[\delta(t-1)] = \int_0^\infty e^{-st}\delta(t-1) = e^{-s}$$

$$F(s) = e^{-s} - \frac{e^{-s}}{s}$$

$$f(t) = \mathcal{L}^{-1}[F(s)] = \delta(t-1) - u(t-1)$$

(b) Find a function $f(t)$ with Laplace transform $F(s) = \dfrac{s+10}{s^3+2s^2+10s}$. [10]

$$F(s) = \frac{s+10}{s((s+1)^2+9)}$$

$$= \frac{1}{s} - \frac{s+1}{(s+1)^2+3^2} \qquad \text{(By partial fractions)}$$

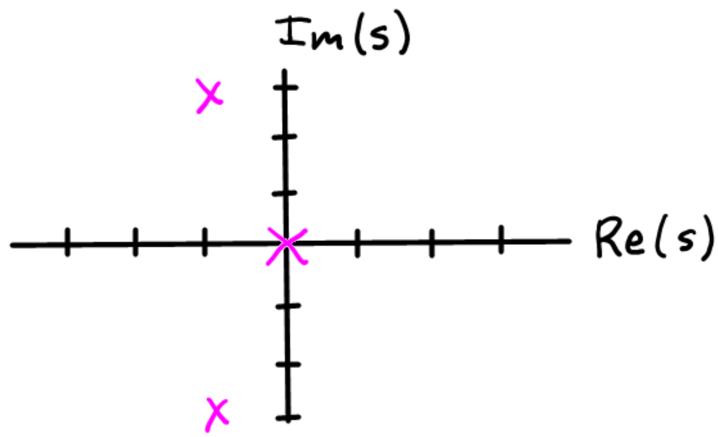
$$f(t) = \mathcal{L}^{-1}[F(s)]$$

$$= 1 - e^{-t}\cos(3t)$$

5. Let $W(s) = \dfrac{s+10}{s^3+2s^2+10s}$.

(a) Sketch the pole diagram of $W(s)$. [10]

$$0 = s\left((s+1)^2+9\right) \longrightarrow s = 0,\ -1 \pm 3i$$



(b) If $W(s)$ is the transfer function of an LTI system , what is the Laplace transform of the response from rest initial conditions to the input $\sin(2t)$? [10]

$$X(s) = W(s)\,\mathcal{L}[\sin 2t]$$

$$= \frac{s+10}{s^3+2s^2+10s}\,\frac{2}{s^2+4}$$

Properties of the Laplace transform

0. Definition: $\mathcal{L}[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt$ for $\mathrm{Re}\, s \gg 0$.

1. Linearity: $\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s)$.

2. Inverse transform: $F(s)$ essentially determines $f(t)$.

3. s-shift rule: $\mathcal{L}[e^{at}f(t)] = F(s-a)$.

4. t-shift rule: $\mathcal{L}[f_a(t)] = e^{-as}F(s)$, $\quad f_a(t) = u(t-a)f(t-a) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$.

5. s-derivative rule: $\mathcal{L}[tf(t)] = -F'(s)$.

6. t-derivative rule: $\mathcal{L}[f'(t)] = sF(s) - f(0^-)$, where $f'(t)$ denotes the generalized derivative.

7. Convolution rule: $\mathcal{L}[f(t) * g(t)] = F(s)G(s)$, $f(t) * g(t) = \int_{0^-}^{t^+} f(t-\tau)g(\tau)d\tau$.

8. Weight function: $\mathcal{L}[w(t)] = W(s) = 1/p(s)$, $w(t)$ the unit impulse response.

Formulas for the Laplace transform

$$\mathcal{L}[1] = \frac{1}{s} \qquad \mathcal{L}[e^{at}] = \frac{1}{s-a} \qquad \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2} \qquad \mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}[t\cos(\omega t)] = \frac{2\omega s}{(s^2+\omega^2)^2} \qquad \mathcal{L}[t\sin(\omega t)] = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

Fourier coefficients for periodic functions of period 2π:

$$f(t) = \frac{a_0}{2} + a_1\cos(t) + a_2\cos(2t) + \cdots + b_1\sin(t) + b_2\sin(2t) + \cdots$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(mt)\,dt, \qquad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(mt)\,dt$$

If $\mathrm{sq}(t)$ is the odd function of period 2π which has value 1 between 0 and π, then

$$\mathrm{sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right)$$