§Laplace Transform Basics

Rules for the Laplace transform

Definition:	$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt \quad \text{for} \quad \mathrm{Re}\,(s) \gg 0.$
Linearity:	$\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s).$
$\mathcal{L}^{-1}$:	$F(s)$ essentially determines $f(t)$ for $t > 0$.
s-shift rule:	$\mathcal{L}[e^{rt}f(t)] = F(s-r).$
s-derivative rule:	$\mathcal{L}[tf(t)] = -F'(s).$
t-derivative rule:	$\mathcal{L}[f'(t)] = sF(s) - f(0^-).$

Formulas for the Laplace transform

$$\mathcal{L}[1] = \frac{1}{s}, \quad \mathcal{L}[\delta(t-a)] = e^{-as}$$

$$\mathcal{L}[e^{rt}] = \frac{1}{s-r}, \quad \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2}, \quad \mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

Problem 1: Find $\mathcal{L}[e^{-t}(t^2+1)]$

Answer:

$$\mathcal{L}[e^{-t}] = \frac{1}{s+1}, \quad \mathcal{L}[t^2] = \frac{2}{s^3}, \quad \mathcal{L}[t^2 e^{-t}] =$$

$$\mathcal{L}[e^{-t}(t^2+1)] = \mathcal{L}[t^2 e^{-t}] + \mathcal{L}[e^{-t}]$$

$$= F(s+1) + \frac{1}{s+1}, \quad F(s) = \mathcal{L}[t^2]$$

$$= \frac{2}{(s+1)^3} + \frac{1}{s+1}$$

$$= \frac{2 + (s+1)^2}{(s+1)^3}$$

$$= \frac{s^2+2s+3}{(s+1)^3}$$

Problem 2: Let $z \in \mathbb{C}$. Find $\mathcal{L}[e^{-zt}]$ and its region of convergence.

Answer:

$$\mathcal{L}[e^{-zt}] = \int_0^\infty e^{-zt} e^{-st}\, dt$$

$$= \int_0^\infty e^{-(z+s)t}\, dt$$

$$= -\frac{1}{z+s} e^{-(z+s)t} \Big|_0^\infty$$

$$= \frac{1}{z+s}, \quad \mathrm{Re}(z+s) > 0$$

Problem 3: Find a) $\mathcal{L}[e^{-t}\sin 3t]$

b) $\mathcal{L}[e^{2t}(t^2-3t+2)]$

Answer:

a) $\mathcal{L}[\sin 3t] = \dfrac{3}{s^2+9}$

$$\mathcal{L}[e^{-t}\sin 3t] = \frac{3}{(s+1)^2+9}$$

b) $\mathcal{L}[t^2-3t+2] = \mathcal{L}[t^2] - 3\mathcal{L}[t] + 2\mathcal{L}[1]$

$$= \frac{2}{s^3} - \frac{3}{s^2} + \frac{2}{s}$$

$$\mathcal{L}[e^{2t}(t^2-3t+2)] = \frac{2}{(s-2)^3} - \frac{3}{(s-2)^2} + \frac{2}{s-2}$$

Problem 4: Find $\mathcal{L}[e^{-t}\sin 3t]$ by writing $e^{-t}\sin 3t$ as the sum of complex exponentials.

Answer:

$$e^{-t}\sin 3t = \frac{1}{2i}\left[e^{-t}(\cos 3t + i\sin 3t) - e^{-t}(\cos 3t - i\sin 3t)\right]$$

$$= \frac{1}{2i}\left[e^{(-1+3i)t} - e^{(-1-3i)t}\right]$$

$$\mathcal{L}[e^{-t}\sin 3t] = \frac{1}{2i}\left[\mathcal{L}[e^{(-1+3i)t}] - \mathcal{L}[e^{(-1-3i)t}]\right]$$

$$= \frac{1}{2i}\left[\frac{1}{s+1-3i} - \frac{1}{s+1+3i}\right]$$

$$= \frac{1}{2i}\left[\frac{s+1+3i-s-1+3i}{(s+1-3i)(s+1+3i)}\right]$$

$$= \frac{3}{s^2+2s+10} = \frac{3}{(s+1)^2+9} \checkmark$$

Part II Problems

Problem 1: [Laplace transform] **(a)** Suppose that $F(s)$ is the Laplace transform of $f(t)$, and let $a > 0$. Find a formula for the Laplace transform of $g(t) = f(at)$ in terms of $F(s)$, by using the integral definition and making a change of variable. Verify your formula by using formulas and rules to compute both $\mathcal{L}(f(t))$ and $\mathcal{L}(f(at))$ with $f(t) = t^n$.

(b) Use your calculus skills: Show that if $h(t) = f(t) * g(t)$ then $H(s) = F(s)G(s)$. Do this by writing $F(s) = \int_0^\infty f(x)e^{-sx}\,dx$ and $G(s) = \int_0^\infty g(y)e^{-sy}\,dy$; expressing the product as a double integral; and changing coordinates using $x = t - \tau$, $y = \tau$.

(c) Use the integral definition to find the Laplace transform of the function $f(t)$ with $f(t) = 1$ for $0 < t < 1$ and $f(t) = 0$ for $\cancel{t <} 0$. What is the region of convergence of the integral?

$t > 1$

Answer:

a)
$$G(s) = \mathcal{L}[g(t)] = \mathcal{L}[f(at)] = \int_0^\infty e^{-st} f(at)\,dt = \frac{1}{a}\int_0^\infty e^{-su/a} f(u)\,du = \frac{1}{a}F(s/a)$$

Let $f(t) = t^n$, $g(t) = (at)^n$

$$\mathcal{L}[f(t)] = F(s) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[g(t)] = G(s) = \mathcal{L}[a^n t^n] = a^n \frac{n!}{s^{n+1}} = \frac{1}{a}\frac{n!}{(s/a)^{n+1}} = \frac{1}{a}F(s/a).$$

b) $h(t) = (f * g)(t)$

$$F(s)G(s) = \int_0^\infty \int_0^\infty f(x)e^{-sx} g(y) e^{-sy} \, dx\, dy$$

Let $x = t - \tau$, $y = \tau$, $0 < t < \infty$, $0 < \tau < t$

$$J = \begin{vmatrix} \partial x/\partial t & \partial x/\partial \tau \\ \partial y/\partial t & \partial y/\partial \tau \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

$$F(s)G(s) = \int_0^\infty \int_0^t f(t-\tau)\, g(\tau)\, e^{-st} \, d\tau\, dt$$

$$= \int_0^\infty e^{-st} \left(\int_0^t f(t-\tau) g(\tau)\, d\tau \right) dt$$

$$= \int_0^\infty e^{-st} h(t)\, dt$$

$$= H(s).$$

c) $f(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st}\,dt$$

$$= \begin{cases} -\frac{1}{s} e^{-st} \Big|_0^1, & s \neq 0 \\ 1, & s = 0 \end{cases}$$

$$= \begin{cases} \frac{1 - e^{-s}}{s}, & s \neq 0 \\ 1, & s = 0 \end{cases}$$

Converges for any $s \in \mathbb{C}$.