

## § Linear vs. Nonlinear

**Problem 1: [Linear vs Nonlinear]** Still working with the equation  $\dot{y} = (1-y)y - a$  with  $a = \frac{3}{16}$ , let  $y_0$  be the stable critical point. Write  $u = y - y_0$  for the population excess over equilibrium (so  $u < 0$  if the population is less than the equilibrium value).

(a) Rewrite the differential equation as a differential equation for  $u$ . Check that the new equation is again autonomous and that  $u = 0$  is a critical point for it.

(b) For small  $u$  we can neglect higher powers of  $u$  (such as  $u^2$ ). This process is "linearization near equilibrium." What is the linearized equation near  $u = 0$ ? What is the general solution of this linear autonomous equation?

(c) At least in this case, when solutions of the original autonomous equation get near equilibrium, they are well modeled by solutions of the linearization. Give an approximation of  $y$  near equilibrium. Use it to answer this question: if  $y(10) - y_0 = b$ , estimate  $y(11)$  and  $y(12)$ .

(d) Suppose that the linear equation  $\dot{x} + p(t)x = q(t)$  is autonomous. What can you say about  $p(t)$  and  $q(t)$ ?

Answer:

$$\begin{aligned} \text{a)} \quad 0 &= (1-y)y - \frac{3}{16} \rightarrow y = \frac{1}{4}, \frac{3}{4} \\ y_0 &= \frac{3}{4} \text{ is stable} \end{aligned}$$

$$u = y - y_0, \quad \dot{u} = \dot{y}$$

$$\begin{aligned} 0 &= \dot{u} = (1 - (u + \frac{3}{4}))(u + \frac{3}{4}) - \frac{3}{16} \\ &= -\frac{1}{2}u - u^2 \rightarrow u_0 = 0 \\ &\quad u_1 = -\frac{1}{2} \end{aligned}$$

$$\text{b)} \quad \text{If } u^2 \approx 0, \quad \dot{u} = f(u) = -\frac{u}{2}$$

$$\begin{aligned} \frac{1}{u} du &= -\frac{1}{2} dt \\ u(t) &= Ce^{-t/2} \end{aligned}$$

$$\text{c)} \quad \text{Near equilibrium,} \\ y(t) \approx \frac{3}{4} + Ce^{-t/2}$$

$$b + \frac{3}{4} = y(10) = \frac{3}{4} + Ce^{-5} \\ be^5 = C$$

$$y(11) \approx \frac{3}{4} + be^5 e^{-11/2}$$

$$y(12) \approx \frac{3}{4} + be^5 e^{-6}$$

$$\begin{aligned} \text{d)} \quad p(t) \text{ and } q(t) &\text{ are} \\ &\text{constant functions since} \\ \dot{x} &= f(x) = q(t) - p(t)x. \end{aligned}$$