§ Laplace Transform: Solving Initial Value Problems

Rules for the Laplace transform

Definition:	$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt \quad \text{for} \quad \operatorname{Re}(s) \gg 0.$
Linearity:	$\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s).$
$\mathcal{L}^{-1}$:	$F(s)$ essentially determines $f(t)$ for $t > 0$.
s-shift rule:	$\mathcal{L}[e^{rt}f(t)] = F(s-r).$
s-derivative rule:	$\mathcal{L}[tf(t)] = -F'(s).$
t-derivative rule:	$\mathcal{L}[f'(t)] = sF(s) - f(0^-).$

Formulas for the Laplace transform

$$\mathcal{L}[1] = \frac{1}{s}, \quad \mathcal{L}[\delta(t-a)] = e^{-as}$$

$$\mathcal{L}[e^{rt}] = \frac{1}{s-r}, \quad \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2}, \quad \mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

For problems 1-7, solve the IVP.

Problem 1: $y' - y = e^{3t}$, $y(0^-) = 1$

Answer:

$$\mathcal{L}[y' - y] = \mathcal{L}[e^{3t}]$$

$$sY - y(0^-) - Y = \frac{1}{s-3}$$

$$Y(s-1) - 1 = \frac{1}{s-3}$$

$$Y = \frac{1}{s-1} + \frac{1}{(s-1)(s-3)}$$

$$Y = \frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s-3}$$

$$y(t) = \mathcal{L}^{-1}[Y] = \frac{1}{2}e^{t} + \frac{1}{2}e^{3t}.$$

Problem 2: $y'' - 3y' + 2y = 0$, $y(0^-) = 1$, $y'(0^-) = 1$

Answer:

$$\mathcal{L}[y'' - 3y' + 2y] = \mathcal{L}[0]$$

$$s\mathcal{L}[y'] - y'(0^-) - 3sY + 3y(0^-) + 2Y = 0$$

$$s(sY - y(0^-)) - y'(0^-) - 3sY + 3y(0^-) + 2Y = 0$$

$$s^2Y - s - 1 - 3sY + 3 + 2Y = 0$$

$$Y(s^2 - 3s + 2) = s - 2$$

$$Y = \frac{s-2}{(s-2)(s-1)} = \frac{1}{s-1}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] = e^t.$$

Problem 3: $y'' + 4y = \sin t$, $y(0^-) = 1$, $y'(0^-) = 0$

Answer:

$$s^2 Y - s y(0^-) - y'(0^-) + 4Y = \frac{1}{s^2+1}$$

$$(s^2+4)Y - s = \frac{1}{s^2+1}$$

$$Y = \frac{s}{s^2+4} + \frac{1}{(s^2+1)(s^2+4)}$$

$$Y = \frac{s}{s^2+4} - \frac{1}{3}\frac{1}{s^2+4} + \frac{1}{3}\frac{1}{s^2+1}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)]$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] - \frac{1}{6}\mathcal{L}^{-1}\left[\frac{2}{s^2+2^2}\right] + \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]$$

$$y(t) = \cos(2t) - \frac{1}{6}\sin 2t + \frac{1}{3}\sin t$$

Problem 4: $y'' - 2y' + 2y = 2e^t$, $y(0^-) = 0$, $y'(0^-) = 1$

Answer:

$$s^2 Y - s y(0^-) - y'(0^-) - 2sY + 2y(0^-) + 2Y = \frac{2}{s-1}$$

$$(s^2 - 2s + 2) Y = 1 + \frac{2}{s-1} = \frac{s+1}{s-1}$$

$$Y = \frac{s+1}{s-1} \frac{1}{(s-1)^2+1} = \frac{(s-1)+2}{(s-1)((s-1)^2+1)}$$

$$Y = \frac{2}{(s-1)((s-1)^2+1)} + \frac{1}{((s-1)^2+1)}$$

$$Y = \frac{2}{s-1} + \frac{3-2s}{(s-1)^2+1} = \frac{2}{s-1} + \frac{1}{(s-1)^2+1} - 2\frac{s-1}{(s-1)^2+1}$$

$$y(t) = 2e^t + e^t \sin t - 2e^t \cos t.$$

Problem 7: $x^{(4)} + 2x'' + x = e^{2t}$, $x(0^-) = x'(0^-) = x''(0^-) = x'''(0^-) = 0$

Answer: since all initial conditions are 0,

$$\mathcal{L}[x^{(4)} + 2x'' + x] = \mathcal{L}[e^{2t}]$$

$$(s^4 + 2s^2 + 1)X(s) = \frac{1}{s-2}$$

$$X(s) = \frac{1}{(s-2)}\frac{1}{(s^2+1)^2}$$

$$X(s) = \frac{1}{25}\left(\frac{1}{s-2} - \frac{s+2}{s^2+1} - 5\frac{s+2}{(s^2+1)^2}\right)$$

$$x(t) = \frac{1}{25}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] - \mathcal{L}\left[\frac{s}{s^2+1}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]$$

$$- 5\mathcal{L}^{-1}\left[\frac{s}{(s^2+1)^2}\right] - 10\mathcal{L}^{-1}\left[\frac{1}{(s^2+1)^2}\right]$$

$$x(t) = \frac{1}{25}\left[e^{2t} - \cos t - 2\sin t - 5t\sin t - 10(\sin t - t\cos t)\right)$$

Problem 8: Find $\mathcal{L}[u(t) - u(t-2\pi)\cdot \sin t]$ by use of the t-shift rule.

Answer:

t-translation Rule

We give the rule in two forms.

$$\begin{aligned}\mathcal{L}(u(t-a)f(t-a)) &= e^{-as}F(s)\\ \mathcal{L}(u(t-a)f(t)) &= e^{-as}\mathcal{L}(f(t+a)).\end{aligned}$$

$$\mathcal{L}[u(t)] - \mathcal{L}[u(t-2\pi)\sin t]$$

$$= \mathcal{L}[u(t)] - \mathcal{L}[u(t-2\pi)\sin(t-2\pi)]$$

$$= \frac{e^{0s}}{s} - e^{-2\pi s}\mathcal{L}[\sin t]$$

$$= \frac{1}{s} - e^{-2\pi s}\frac{1}{s^2+1}$$