§ Matrix Exponentials

The equation $\dot{\mathbf{u}} = A\mathbf{u}$ (or the matrix A) is

"stable" if all solutions tend to $\mathbf{0}$ as $t \to \infty$.

"unstable" if most solutions grow without bound as $t \to \infty$.

"neutrally stable" otherwise.

A **fundamental matrix** for a square matrix A is a matrix of functions, $\Phi(t)$, whose columns are linearly independent solutions to $\dot{\mathbf{u}} = A\mathbf{u}$.

The fundamental matrix whose value at $t = 0$ is the identity matrix is the **matrix exponential** e^{At}. It can be computed from any fundamental matrix $\Phi(t)$:

$$e^{At} = \Phi(t)\Phi(0)^{-1}.$$

The solution to $\dot{\mathbf{u}} = A\mathbf{u}$ with initial condition $\mathbf{u}(0)$ is given by $e^{At}\mathbf{u}(0)$.

If $\mathbf{q}$ is constant, and A is invertible, then $\mathbf{u_p}(t) = -A^{-1}\mathbf{q}$ is a solution to the inhomogeneous equation $\dot{\mathbf{u}} = A\mathbf{u} + \mathbf{q}$. The general solution is $\mathbf{u_p} + \mathbf{u_h}$, where $\mathbf{u_h}$ is the general solution of the associated homogeneous equation $\dot{\mathbf{u}} = A\mathbf{u}$.

1. In this problem, $A = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}$ and we are interested in the equation $\dot{\mathbf{u}} = A\mathbf{u}$.

(a) Find a fundamental matrix $\Phi(t)$ for A.

(b) Find the exponential matrix e^{At}.

(c) Find the solution to $\dot{\mathbf{u}} = A\mathbf{u}$ with $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(d) Find a solution to $\dot{\mathbf{u}} = A\mathbf{u} + \begin{bmatrix} 5 \\ 10 \end{bmatrix}$. What is the general solution? What is the solution with $\mathbf{u}(0) = \mathbf{0}$?

Answer:

a) $0 = \lambda^2 - 2\lambda + 5$

$\lambda = 1 \pm 2i$

$v = \begin{bmatrix} 1 \\ 2i \end{bmatrix}$

$$u(t) = c_1 e^t \begin{bmatrix} \cos 2t \\ -2\sin 2t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin 2t \\ 2\cos 2t \end{bmatrix}$$

$$\Phi(t) = \begin{bmatrix} e^t \cos 2t & e^t \sin 2t \\ -2e^t \sin 2t & 2e^t \cos 2t \end{bmatrix}$$

b) $\Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $\Phi(0)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$

$$e^{At} = \begin{bmatrix} e^t \cos 2t & \frac{1}{2} e^t \sin 2t \\ -2e^t \sin 2t & e^t \cos 2t \end{bmatrix}$$

c) $u(t) = e^{At} u(0) = e^t \begin{bmatrix} \cos 2t + \sin 2t \\ -2\sin 2t + 2\cos 2t \end{bmatrix}$

d) $u_p = -\frac{1}{5} \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$

$u_g(t) = e^{At} \begin{bmatrix} a \\ b \end{bmatrix} + u_p$, $u(0) = \vec{0} \rightarrow \begin{matrix} a = -1 \\ b = 6 \end{matrix}$

2. Suppose $\mathbf{u}_1(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (the constant trajectory) and $\mathbf{u}_2(t) = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$ are solutions to the equation $\dot{\mathbf{u}} = B\mathbf{u}$ for some matrix B.

(a) What is the general solution? What is the solution $\mathbf{u}(t)$ with $\mathbf{u}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$? What is the solution with $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$?

(b) Find a fundamental matrix, and compute the exponential e^{Bt}. What is e^B?

(c) What are the eigenvalues and eigenvectors of B?

(d) What is B?

Answer:

a) $u(t) = a u_1 + b u_2, \quad a, b \in \mathbb{R}$

$u(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} a+b \\ a-b \end{bmatrix}$

$\rightarrow u(t) = 2u_1 + 0u_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a+b \\ a-b \end{bmatrix}$

$\rightarrow u(t) = \frac{1}{2}u_1 + \frac{1}{2}u_2 = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}e^t \\ \frac{1}{2} - \frac{1}{2}e^t \end{bmatrix}$

b) $\Phi(t) = \begin{bmatrix} 1 & e^t \\ 1 & -e^t \end{bmatrix}$

$\Phi(0) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \Phi^{-1}(0) = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$

$e^{Bt} = \begin{bmatrix} 1/2 + 1/2\, e^t & 1/2 - 1/2\, e^t \\ 1/2 - 1/2\, e^t & 1/2 + 1/2\, e^t \end{bmatrix}$

$e^{B} = \begin{bmatrix} 1/2 + 1/2\, e & 1/2 - 1/2\, e \\ 1/2 - 1/2\, e & 1/2 + 1/2\, e \end{bmatrix}$

c) $\lambda = 0, \quad v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 1, \quad v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

d) $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow B = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$

$a+b=0, \quad c+d=0$

$a-b=1, \quad c-d=-1$

Problem 3: Solve $x' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x$, $\quad x(0) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

Answer:

$$0 = \lambda^2 - 4\lambda + 3$$
$$\lambda = 1, 3$$

$$\left.\begin{array}{l} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \rightarrow v = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow x_1 = e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \rightarrow v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow x_2 = e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{array}\right\} \rightarrow \Phi = \begin{bmatrix} e^{t} & e^{3t} \\ -e^{t} & e^{3t} \end{bmatrix}$$

$$(5e^{t} + e^{3t})/2$$
$$(-5e^{t} + e^{3t})/2$$

$$\Phi(0) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad \Phi(0)^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$x(t) = e^{At}x(0) = \Phi(t)\Phi(0)^{-1}x(0) = \begin{bmatrix} \frac{5}{2}e^{t} + \frac{1}{2}e^{3t} \\ -\frac{5}{2}e^{t} + \frac{1}{2}e^{3t} \end{bmatrix}$$

Problem 4: Solve $x' = \begin{bmatrix} 2 & -5 \\ 4 & -2 \end{bmatrix} x$, $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Answer:

$$0 = \lambda^2 + 16 \rightarrow \lambda = \pm 4i$$

$$\left[\begin{array}{cc|c} 2-4i & -5 & 0 \\ 4 & -2-4i & 0 \end{array}\right] \longrightarrow v = \begin{bmatrix} 5 \\ 2-4i \end{bmatrix}$$

$$\left(\cos(4t) + i\sin(4t)\right)\begin{bmatrix} 5 \\ 2-4i \end{bmatrix} = \begin{bmatrix} 5\cos(4t) \\ 2\cos(4t) + 4\sin(4t) \end{bmatrix} + i\begin{bmatrix} 5\sin(4t) \\ -4\cos(4t) + 2\sin(4t) \end{bmatrix}$$

$$\Phi = \begin{bmatrix} 5\cos(4t) & 5\sin(4t) \\ 2\cos(4t) + 4\sin(4t) & -4\cos(4t) + 2\sin(4t) \end{bmatrix}$$

$$\Phi(0) = \begin{bmatrix} 5 & 5 \\ 2 & -4 \end{bmatrix}, \quad \Phi(0)^{-1} = \frac{1}{30}\begin{bmatrix} 4 & 5 \\ 2 & -5 \end{bmatrix}$$

$$x(t) = \Phi\,\Phi(0)^{-1} x(0) = \begin{bmatrix} 5/6 \cos 4t - 5/6 \sin 4t \\ \cos 4t + \frac{1}{3} \sin 4t \end{bmatrix}$$

Part II Problems

Problem 1: [Exponential matrix]

(a) We have seen that a complex number $z = a + bi$ determines a matrix $A(z)$ in the following way: $A(a+bi) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. This matrix represents the operation of multiplication by z, in the sense that if $z(x+yi) = v + wi$ then $A(z)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} v \\ w \end{bmatrix}$. What is $e^{A(z)t}$? What is $A(e^{zt})$?

(b) Say that a pair of solutions $x_1(t), x_2(t)$ of the equation $m\ddot{x} + b\dot{x} + kx = 0$ is *normalized at* $t = 0$ if:

$$x_1(0) = 1, \quad \dot{x}_1(0) = 0$$
$$x_2(0) = 0, \quad \dot{x}_2(0) = 1$$

For example, find the normalized pair of solutions to $\ddot{x} + 2\dot{x} + 2x = 0$. Then find e^{At} where A is the companion matrix for the operator $D^2 + 2D + 2I$.

(c) Suppose that $e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $e^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ satisfy the equation $\dot{\mathbf{u}} = A\mathbf{u}$.

(i) Find solutions $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ such that $\mathbf{u}_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{u}_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(ii) Find e^{At}.

(iii) Find A.

Answer:

a) $0 = \lambda^2 - 2a\lambda + a^2 + b^2 = (\lambda - a)^2 + b^2$

$\lambda = a \pm bi$

$$\left[\begin{array}{cc|c} -bi & -b & 0 \\ b & -bi & 0 \end{array}\right] \to v = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$e^{at}(\cos bt + i\sin bt)\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\to e^{at}\begin{bmatrix} \cos bt \\ \sin bt \end{bmatrix},\ e^{at}\begin{bmatrix} \sin bt \\ -\cos bt \end{bmatrix}$$

$$\Phi = \begin{bmatrix} e^{at}\cos bt & e^{at}\sin bt \\ e^{at}\sin bt & -e^{at}\cos bt \end{bmatrix}$$

$$\Phi(0)^{-1} = \Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$e^{A(z)t} = \Phi\,\Phi(0)^{-1}$$
$$= e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$$

$$A(e^{zt}) = \begin{bmatrix} e^{at}\cos bt & -e^{at}\sin bt \\ e^{at}\sin bt & e^{at}\cos bt \end{bmatrix}$$
$$= e^{A(z)t}$$

b) $\ddot{x}+2\dot{x}+2x=0$

$r^2+2r+2=0$

$r=-1\pm i$

$y_1=e^{-t}\cos t$

$y_2=e^{-t}\sin t$

$y_1(0)=1,\ \dot{y}_1(0)=-1$

$y_2(0)=0,\ \dot{y}_2(0)=1$

$$x_1(t)=y_1+y_2$$
$$=e^{-t}(\cos t+\sin t)$$

$$x_2(t)=y_2=e^{-t}\sin t$$

$$\begin{bmatrix}\dot{x}\\ \ddot{x}\end{bmatrix}=\begin{bmatrix}0 & 1\\ -2 & -2\end{bmatrix}\begin{bmatrix}x\\ \dot{x}\end{bmatrix}\longrightarrow A=\begin{bmatrix}0 & 1\\ -2 & -2\end{bmatrix}$$

$$\lambda^2+2\lambda+2=0\longrightarrow\lambda=-1\pm i$$

$$\left[\begin{array}{cc|c}1-i & 1 & 0\\ -2 & -1-i & 0\end{array}\right]\longrightarrow v=\begin{bmatrix}1\\ i-1\end{bmatrix}$$

$$e^{-t}(\cos t+i\sin t)\begin{bmatrix}1\\ i-1\end{bmatrix}$$

$$=e^{-t}\left(\begin{bmatrix}\cos t\\ -\cos t-\sin t\end{bmatrix}+i\begin{bmatrix}\sin t\\ \cos t-\sin t\end{bmatrix}\right)$$

$$\Phi(t)=e^{-t}\begin{bmatrix}\cos t & \sin t\\ -\cos t-\sin t & \cos t-\sin t\end{bmatrix}$$

$$\Phi(0)=\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix},\ \Phi(0)^{-1}=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$$

$$e^{At}=e^{-t}\begin{bmatrix}\cos t+\sin t & \sin t\\ -2\sin t & \cos t-\sin t\end{bmatrix}$$

c) i. $y_1(t) = e^{3t}\begin{bmatrix}1\\1\end{bmatrix}$, $y_2(t) = e^{2t}\begin{bmatrix}1\\2\end{bmatrix}$

$y_1(0) = \begin{bmatrix}1\\1\end{bmatrix}$, $y_2(0) = \begin{bmatrix}1\\2\end{bmatrix}$

$u_1 = 2y_1 - y_2$, $u_1(0) = \begin{bmatrix}1\\0\end{bmatrix}$

$u_2 = -y_1 + y_2$, $u_2(0) = \begin{bmatrix}0\\1\end{bmatrix}$

ii. $\Phi(t) = [u_1 \ u_2]$, $\Phi(0) = I_{2\times2}$, $\Phi(0)^{-1} = I_{2\times2}$.

$$e^{At} = \Phi\Phi(0)^{-1} = \Phi = \begin{bmatrix} 2e^{3t} - e^{2t} & e^{2t} - e^{3t} \\ 2e^{3t} - 2e^{2t} & 2e^{2t} - e^{3t} \end{bmatrix}$$

iii. From the given soln's y_1, y_2 we know A has $\lambda = 3$, $v = \begin{bmatrix}1\\1\end{bmatrix}$ and $\lambda = 2$, $v = \begin{bmatrix}1\\2\end{bmatrix}$.

$A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3\\3\end{bmatrix}$, $A\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix} \rightarrow A = \begin{bmatrix}4 & -1\\2 & 1\end{bmatrix}$