§ Transfer and Weight Functions, Green's Formula

Part I Problems

Problems 1 and 2 are about the system

$$p(D)x = f(t) \qquad (1)$$

with rest IC's and with input $f(t)$.

Problem 1: In each of the following cases, find $p(D)$ such that $w(t)$ is the system unit impulse response.

(a) $w(t) = e^{-at}$. **(b)** $w(t) = \frac{1}{3}e^{-t/2}\sin t$. **(c)** $w(t) = 1$.

Answer: Use $W(s) = \frac{1}{p(s)}\mathcal{L}[\delta(t)] = \frac{1}{p(s)}$

a) $\frac{1}{s+a} = \frac{1}{p(s)}$

$p(D) = D + a$

b) $\frac{1}{3}\frac{1}{(s+\frac{1}{2})^2+1} = \frac{1}{p(s)}$

$3(s^2+s+\frac{1}{4})+3 = p(s)$

$p(D) = 3D^2 + 3D + \frac{15}{4}$

c) $\frac{1}{s} = \frac{1}{p(s)}$

$p(D) = D$

Problem 2: For $p(D) = D^2 + 4$:

(a) Find the system function $W(s)$;

(b) Find the weight function $w(t)$;

(c) Write down the convolution integral formula for the solution to the IVP (1).

Answer:

a) $W(s) = \dfrac{1}{p(s)} = \dfrac{1}{s^2+4}$

b) $w(t) = \mathcal{L}^{-1}[W(s)] = \mathcal{L}^{-1}\left[\frac{1}{2}\dfrac{2}{s^2+2^2}\right] = \frac{1}{2}\sin(2t)$

c) $x(t) = (w * f)(t) = (f * w)(t) = \int_0^{\infty} f(t-\tau)\dfrac{\sin(2\tau)}{2}\,d\tau.$

Part II Problems

Problem 1: [Second order ODEs via Laplace transform] Find the unit impulse response of the following operators by means of the Laplace transform.

(a) $3D^2 + 6D + 6I$.

(b) $D^4 - I$.

Answer:

a) $$W(s) = \frac{1}{3s^2+6s+6} = \frac{1}{3}\frac{1}{(s+1)^2+1}$$

$$w(t) = \mathcal{L}^{-1}[W(s)] = \frac{1}{3}e^{-t}\sin t\, u(t)$$

We use $u(t)$ to maintain $w(t)=0$ for $t<0$.

b) $$W(s) = \frac{1}{s^4-1} = \frac{1}{(s^2-1)(s^2+1)} = \frac{1}{2}\frac{1}{s^2-1} - \frac{1}{2}\frac{1}{s^2+1}$$

$$= \frac{1}{2}\left(\frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}\right) - \frac{1}{2}\frac{1}{s^2+1}$$

$$= \frac{1}{4}\frac{1}{s-1} - \frac{1}{4}\frac{1}{s+1} - \frac{1}{2}\frac{1}{s^2+1}$$

$$w(t) = \left(\frac{1}{4}e^{t} - \frac{1}{4}e^{-t} - \frac{1}{2}\sin t\right)u(t)$$