§ Autonomous Equations

Problem 1: For each of the following autonomous equations $dx/dt = f(x)$, obtain a qualitatitive picture of the solutions as follows:

(i) Draw horizontally the axis of the dependent variable x, indiciating the critical points of the equation; put arrows on the axis indicating the direction of motion between the critical points and label each critical point as stable, unstable, or semi-stable. Indicate where this information comes from by including in the same picture the graph of $f(x)$, drawn with dashed lines.

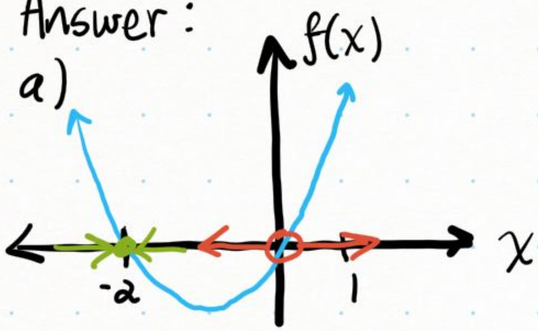
(ii) Use the information in the first picture to make a second picture showing the tx-plane, with a set of typical solutions to the ODE. The sketch should show the main qualitative features (e.g., the constant solutions, asymptotic behavior of the non-constant solutions).

a) $x' = x^2 + 2x$

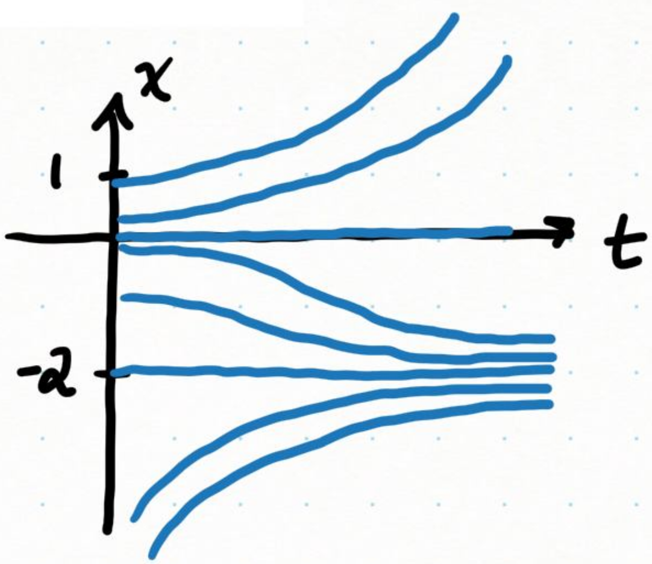
b) $x' = -(x-1)^2$

Answer:

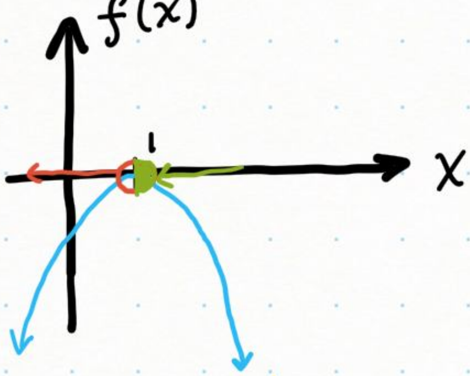
a)

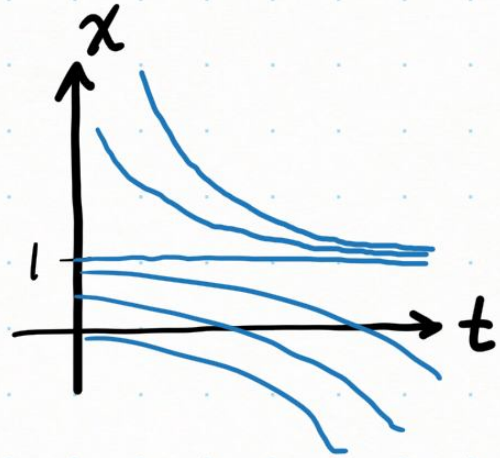


$x = -2$ stable $x = 0$ unstable



b) $x' = -(x-1)^2$





$x = 1$ semistable

Problem 2: Consider the differential equation $\dot{x} + 2x = 1$.

a) Find the general solution three ways: (i) by separation of variables, (ii) by use of an integrating factor, (iii) by regarding the right hand side as e^{0t} and using the method of optimism (i.e. look for a solution of the form Ae^{0t}) to find a particular solution, and then adding in a transient.

b) This equation is also autonomous. Sketch its phase line and some solutions (including the equilibrium solution). Is the equilibrium stable, unstable, or neither?

Answer:

a) i. $\dfrac{dx}{1-2x} = dt$

$-\frac{1}{2}\ln|1-2x| = t + c$

$x(t) = \frac{1}{2} + Ce^{-2x}$

ii. $u(x) = e^{2x}$

$xe^{2x} = \int e^{2x}\,dx$

$x(t) = e^{-2x}\left(\frac{1}{2}e^{2x} + c\right)$

$= \frac{1}{2} + ce^{-2x}$

iii. $x_p = Ae^{0t}$

$\dot{x}_p = 0$

$0 + 2Ae^{0t} = 1$

$A = 1/2$

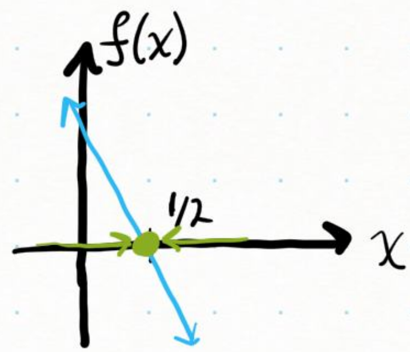
$x_p(t) = \frac{1}{2}$

$\dot{x} + 2x = 0$

$\rightarrow x_h(t) = ce^{-2x}$

$x(t) = \frac{1}{2} + ce^{-2x}$

b) $\dot{x} = f(x) = 1 - 2x$



$x = \frac{1}{2}$

Stable