§ Complex Arithmetic

Problem 1: Change to polar form.

a) $-1+i$
b) $\sqrt{3}-i$

Answer:

a) $|-1+i| = \sqrt{2}$

$\theta = 3\pi/4$

$-1+i = \sqrt{2}\, e^{3\pi i/4 + 2k\pi i}$

b) $|\sqrt{3}-i| = \sqrt{3+1} = 2$

$\tan\theta = -\sqrt{3}$

$\theta = -\pi/6$

$\sqrt{3}-i = 2e^{-\pi i/6 + 2k\pi i}$

Problem 2: Express $\frac{1-i}{1+i}$ in the form $a+bi$ via two methods: one using the Cartesian form throughout, and one changing numerator and denominator to polar form. Show the two answers agree.

Answer:

$$\frac{1-i}{1+i} = \frac{1-2i+i^2}{1-i^2}$$

$$= \frac{-2i}{2}$$

$$= 0-1i$$

$$|0-1i| = 1$$
$$\omega = -\pi/2$$

$$0-i = e^{-\pi i/2}$$

$$|1-i| = \sqrt{2}$$
$$\theta = -\pi/4$$

$$1-i = \sqrt{2}e^{-\pi i/4+2k\pi i}$$

$$|1+i| = \sqrt{2}$$
$$\varphi = \pi/4$$

$$1+i = \sqrt{2}\,e^{\pi i/4+2j\pi i}$$

$$\frac{1-i}{1+i} = \frac{\sqrt{2}\,e^{-\pi i/4}}{\sqrt{2}\,e^{\pi i/4}}$$

$$= e^{-\pi i/2}$$

Problem 3: Calculate each of the following two ways: first by using the binomial theorem and second by changing to polar form and using DeMoivre's formula:

a) $(1-i)^4$

b) $(1+i\sqrt{3})^3$

Answer:

a) $(1-i)^4 = 1 - 4i + 6i^2 - 4i^3 + i^4 = -4$

$1 - i = \sqrt{2}\, e^{-\pi/4\, i}$

$(1-i)^4 = \sqrt{2}^4 e^{-4\pi/4\, i}$

$= 4e^{-\pi i} = -4$

b) $(1+i\sqrt{3})^3 = 1 + 3i\sqrt{3} + 3i^2\sqrt{3}^2 + i^3\sqrt{3}^3 = 1 + 3\sqrt{3}i - 9 - 3\sqrt{3}i = -8$.

$1 + \sqrt{3}i = 2e^{\pi i/3}$ $\quad (1+\sqrt{3}i)^3 = 2^3 e^{3\pi i/3} = 8e^{\pi i} = -8$.

Problem 4: Express the 6^{th} roots of 1 in a+bi form.

Answer:

$$1 = e^{0\pi i + 2k\pi i} = e^{2k\pi i},\ k \in \mathbb{Z} \longrightarrow \sqrt[6]{1} = e^{k\pi/3\, i}$$

$$z_0 = e^{0\cdot\pi i/3} = 1 \qquad z_3 = e^{\pi i} = -1$$

$$z_1 = e^{\pi i/3} = \frac{1}{2} + \frac{\sqrt{3}}{2} i \qquad z_4 = e^{4\pi i/3} = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$$

$$z_2 = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i \qquad z_5 = e^{5\pi i/3} = \frac{1}{2} - \frac{\sqrt{3}}{2} i$$

Problem 5: Solve $x^4 + 16 = 0$

Answer: $r^4 e^{4i\theta} = 16(-1)$ $\quad (-1)^{1/4} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i,\ \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i,\ -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i,\ \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$

$r = 2,\ (-1)^{1/4} = e^{i\theta}$

$$\boxed{x = \pm\sqrt{2} \pm \sqrt{2}\, i}$$