Unit I: First Order Differential Equations

§ Basic DE's

Problem 1: Find the general solution by separation of variables.

$$\frac{dy}{dx} = 2 - y, \quad y(0) = 0$$

Answer:

$$\frac{1}{2-y}\, dy = dx$$

$$-\ln|2-y| = x + c$$

$$2 - y = e^{-c} e^{-x}$$

$$y = 2 - ke^{-x}$$

$$0 = y(0) = 2 - k$$

$$k = 2$$

$$y(x) = 2 - 2e^{-x}$$

$$\frac{dy}{dx} = 2e^{-x}$$

$$= 2 - (2 - 2e^{-x})$$

$$= 2 - y \ \checkmark$$

Problem 2: Find the general solution by separation of variables.

$$\frac{dy}{dx} = \frac{(y-1)^2}{(x+1)^2}$$

Answer: $$\int \frac{1}{(y-1)^2}\,dy = \int \frac{1}{(x+1)^2}\,dx$$

$$(y-1)^{-1} = (x+1)^{-1} + c$$

$$y(x) = 1 + \frac{x+1}{1+c(x+1)}$$

Problem 3: The rate of change of a population is proportional to the square root of its size. Model this situation with a differential equation.

Answer: Let $P(t)$ be the population size at time t.

$$\frac{dP}{dt} = k\sqrt{P}\ , \ k > 0\ , \ P(t_0) = P_0$$

Problem 4: The rate of change of the velocity is proportional to the square of the velocity. Model this situation with a differential equation.

Answer: Let $v(t)$ be the velocity of the object at time t.

$$\frac{dv}{dt} = kv^2\ , \ k > 0\ , \ v(t_0) = v_0$$

Problem 5: In a population of fixed size S, the rate of change of the number N of persons who have heard a rumor is proportional to the number of those who have not yet heard it. Model this situation with a differential equation.

Answer:

$$\frac{dN}{dt} = k(S-N) \quad , \quad k>0 \; , \; N(t_0) = N_0$$

Problem 6: The amount of a certain medicine in the bloodstream decays exponentially with a half-life of 5 hours. In order to keep a patient safe during a one-hour procedure there needs to be at least 50-mg of the medicine per kg of bodyweight. How much should be administered to a 60kg patient at the start of the procedure?

Answer: The amount A of medicine in kg in the bloodstream t hours after administration is

$$A(t) = A_0 \left(\tfrac{1}{2}\right)^{t/5}, \quad A_0 = A(0)$$

We want to find A_0 s.t. $A(1) \geq 50 \cdot 60 = 3000\text{mg}$.

$$3000 \leq A_0 \left(\tfrac{1}{2}\right)^{1/5}$$

$$A_0 \geq 3000\sqrt[5]{2} \approx 3446g$$

To keep the patient safe, administer at least $3000\sqrt[5]{2}$ mg to start.

Problem 7: At 7am a snowplow begans to clear the road. At 8am it has gone 2 miles. It takes an additional 2 hours to go another 2 miles. Let $t=0$ when it begins to snow and let x denote the distance traveled by the plow at time t. Assume the snowplow clears snow at a constant rate in m^3/hr.

a) Find the DE modeling the value of x.
b) When did it start snowing?

Answer: The velocity $\frac{dx}{dt}$ of the snowplow is the ratio of the volume of snow on the ground to the rate at which the snowplow clears snow. Assuming constant snowfall rate, the volume of snow on the ground is proportional to t. Let t_s be how long after it began snowing that the snowplow started.

$$\frac{dx}{dt} = k/t \quad , \quad k \in \mathbb{R}, \qquad x(t_s+1)=2, \quad x(t_s+3)=4$$

$$x(t) = k\ln t + c$$

$$2 = x(t_s+1) - x(t_s) = k\ln\left(\frac{t_s+1}{t_s}\right)$$

$$4 = x(t_s+3) - x(t_s) = k\ln\left(\frac{t_s+3}{t_s}\right)$$

$$\longrightarrow \quad 2\ln\left(\frac{t_s+1}{t_s}\right) = \ln\left(\frac{t_s+3}{t_s}\right)$$

$$\left(\frac{t_s+1}{t_s}\right)^2 = \frac{t_s+3}{t_s}$$

$$t_s^2 + 2t_s + 1 = t_s^2 + 3t_s$$

$$1 = t_s$$

This shows that the snowplow started 1 hr after the snowfall began. It started snowing at 6AM.

Problem 8: A tank holds 100 L of water which contains 25g of salt initially. Pure water flows into the tank and saltwater flows out, both at 5L/min. The mixture is kept uniform at all times by stirring.

a) Write down the DE with the IC for this situation.
b) How long until 1 gram of salt remains in the tank?

Answer: Let $A(t)$ be the amount of salt in the tank at time t.

$\ln\left(\frac{1}{25}\right) = -\frac{t}{20}$

$$\frac{dA}{dt} = -\frac{5A}{100} = -\frac{A}{20}, \quad A(0) = 25$$

$$\ln A = -\frac{t}{20} + c \qquad 25 = A(0) = e^{c} \qquad 1 = A(t^*) = 25e^{-t^*/20}$$

$$A(t) = e^{c}e^{-t/20} \qquad A(t) = 25e^{-t/20} \qquad t^* = 20\ln 25$$

≈ 64.38 minutes.

Problem 9:

A mysterious virus infects the oryxes of the Tana River area in Kenya, which causes the growth rate to decrease as time goes on according to the formula $k(t) = k_0/(a+t)^2$ for $t \geq 0$, where a and k_0 are certain positive constants.

(a) What are the units of the constant a in "$a+t$," and of the constant k_0?

(b) Write down the differential equation modeling this situation.

(c) Write down the general solution to your differential equation. Don't restrict yourself to the values of t and of x that are relevant to the oryx problem; take care of all values of these variables. Points to be careful about: use absolute values in $\int \frac{dx}{x} = \ln|x| + c$ correctly, and don't forget about any "lost" solutions.

(d) Now suppose that at $t = 0$ there is a positive population x_0 of oryx. Does the progressive decline in growth rate cause the population stabilize for large time, or does it grow without bound? If it does stabilize, what is the limiting population as $t \to \infty$?

Answer:

(a) The units of a are years. The units of K_0 are years.

(b) $\dot{P} = k(t)P = \frac{K_0 P}{(a+t)^2}$ P: Population size in kilo-oryxes.

(c) $\frac{1}{P} dP = \frac{K_0}{(a+t)^2} dt$ $\quad |P| = e^c e^{-\frac{K_0}{a+t}}$ $\quad$ Allow $C = 0$ to account for $\dot{P} = 0$.

$\ln|P| = -\frac{K_0}{a+t} + c$ $\quad P(t) = Ce^{-\frac{K_0}{a+t}}, \; C := \pm e^c$

(d) $0 < x_0 = P(0) = Ce^{-k_0/a} \longrightarrow P(t) = x_0 e^{k_0/a} e^{-\frac{k_0}{a+t}}$

$$x_\infty = \lim_{t \to \infty} P(t) = x_0 e^{k_0/a} < +\infty .$$