§ Step and Delta Functions : Integrals and Generalized Derivatives

Problem 1: [Step and delta] For each of the following functions $f(t)$, (i) draw a graph, (ii) draw a graph of the generalized derivative, (iii) write a formula for $f(t)$ and for $f'(t)$ (with possibly a few values not defined) using $u(t-a)$, $\delta(t-a)$, and other functions.

(a) $f(t) = 0$ for $t < 0$, $f(t) = -t$ for $t > 0$.

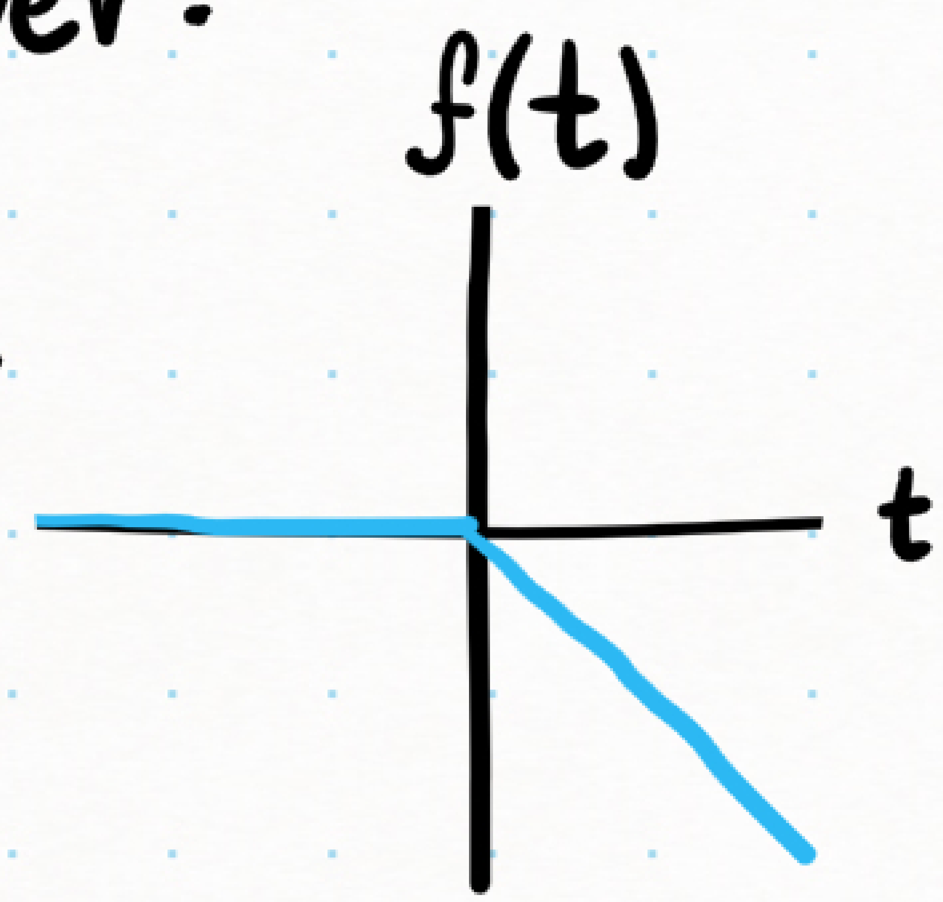
(b) $f(t) = 0$ for $t < 0$, $f(t) = 1 - t$ for $t > 0$.

(c) $f(t) = 0$ for $t < 0$, $f(t) = 2t - 1$ for $0 < t < 1$, $f(t) = 0$ for $t > 1$.

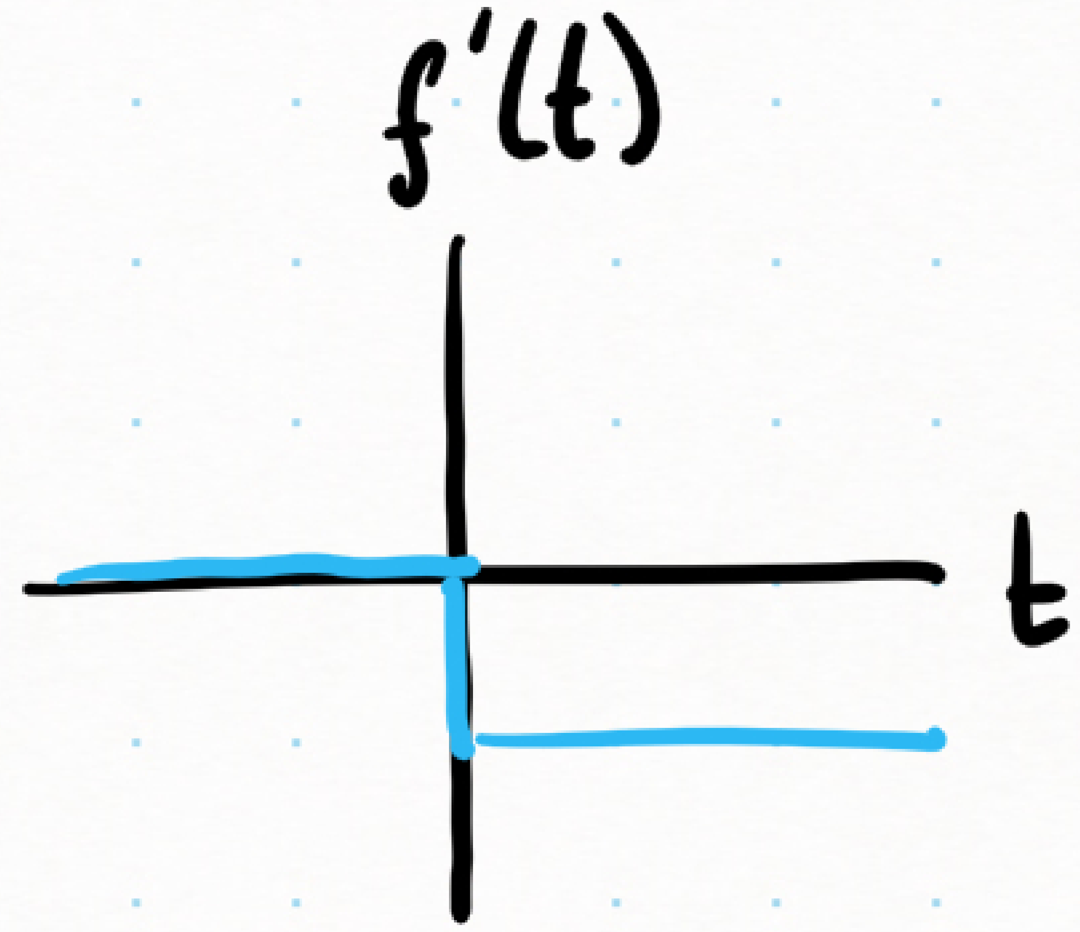
(d) $f(t) = 0$ for $t < 0$, $f(t) = t - \lfloor t \rfloor$ for $t > 0$, where $\lfloor t \rfloor$ denotes the greatest integer less than or equal to t.

Answer:

a) i.



ii.



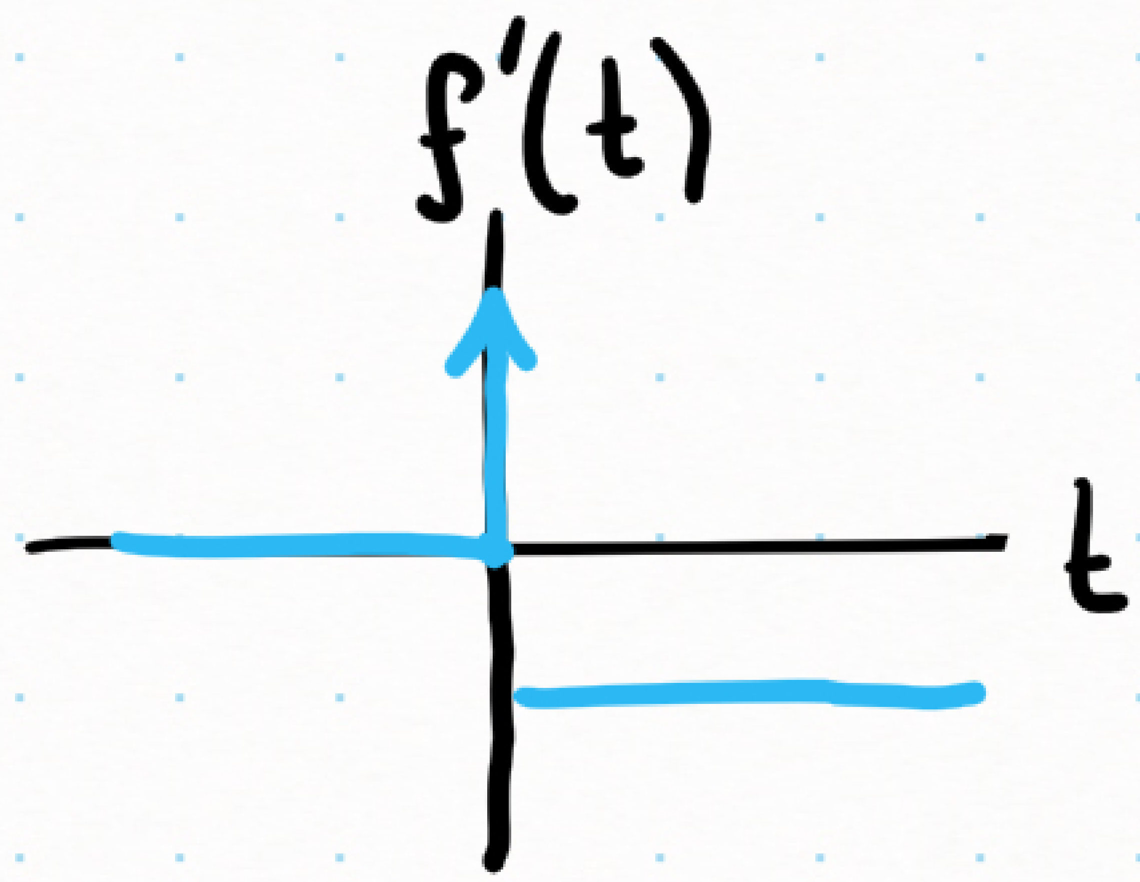
iii.

$f(t) = -t\,u(t)$

$f'(t) = -u(t)$

b) i.

ii.

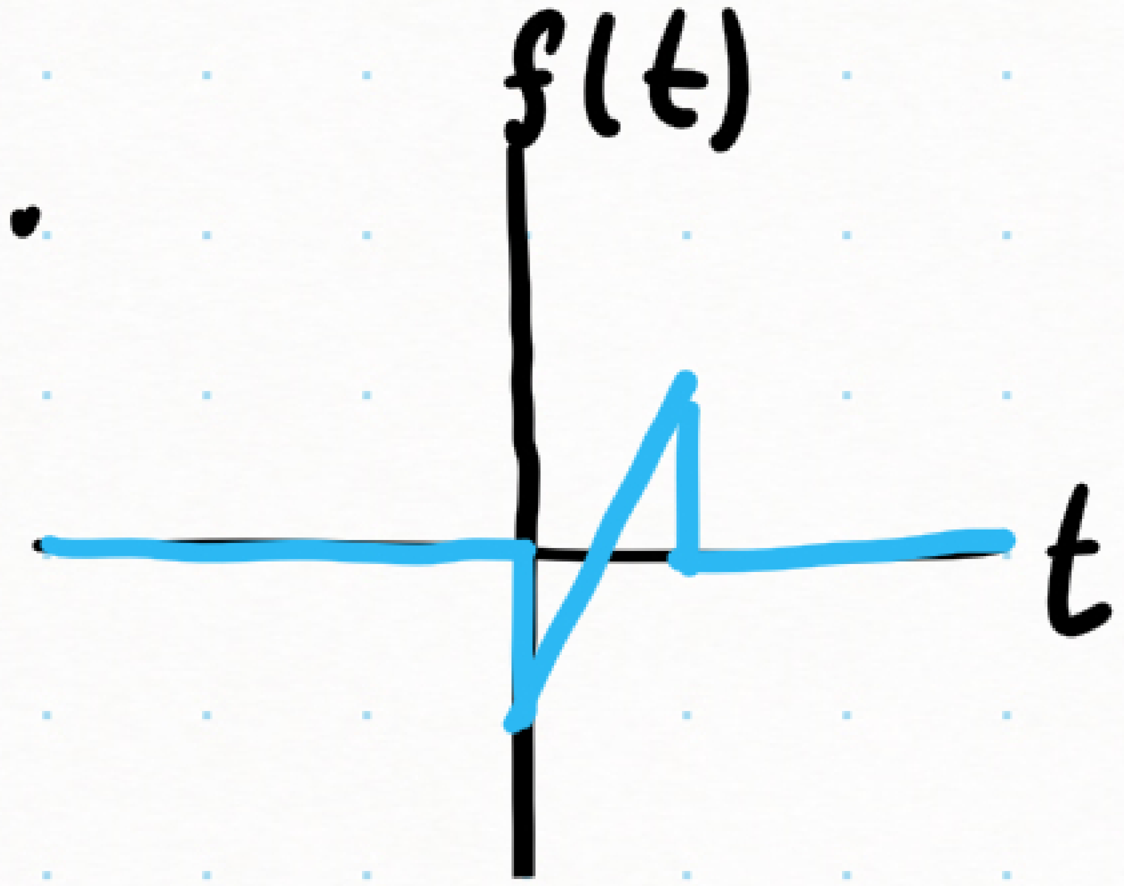


iii.

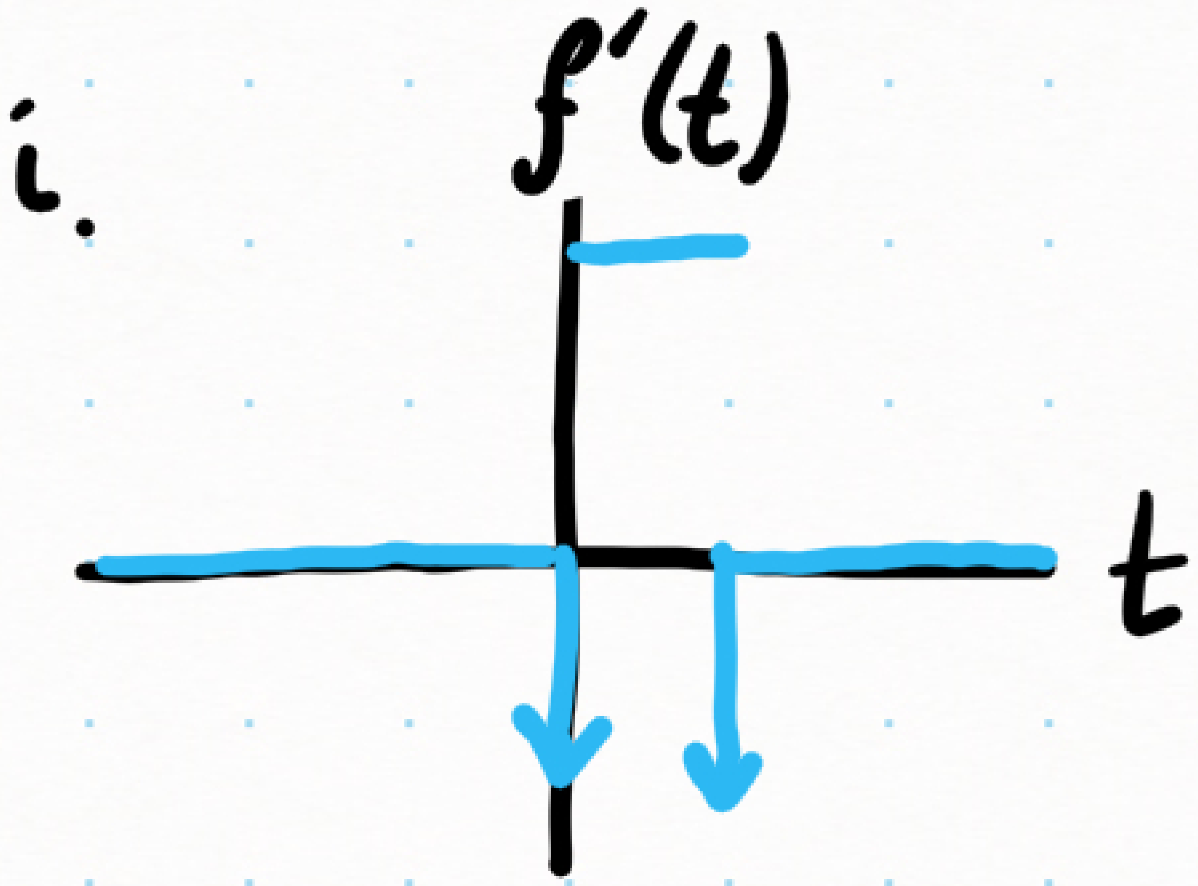
$f(t) = (1-t)\,u(t)$

$f'(t) = -u(t) + \delta(t)$

c) i.



ii.

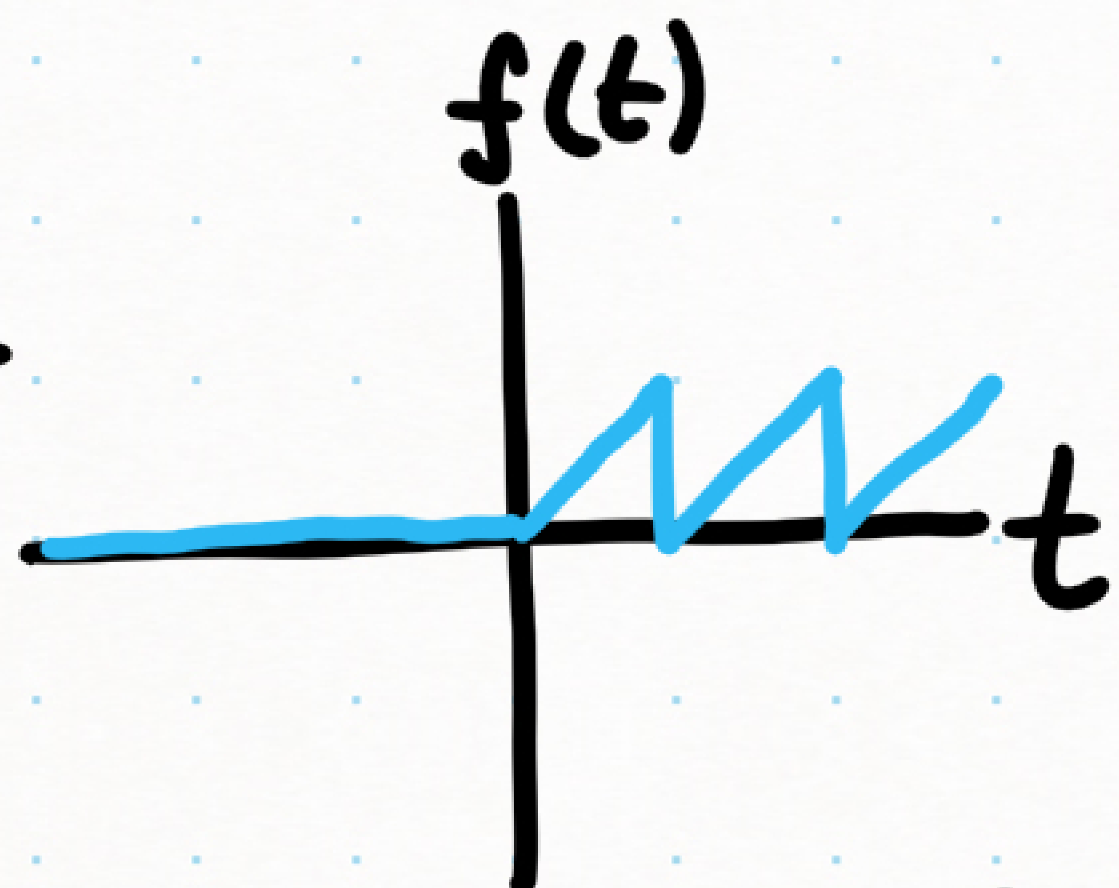


iii.

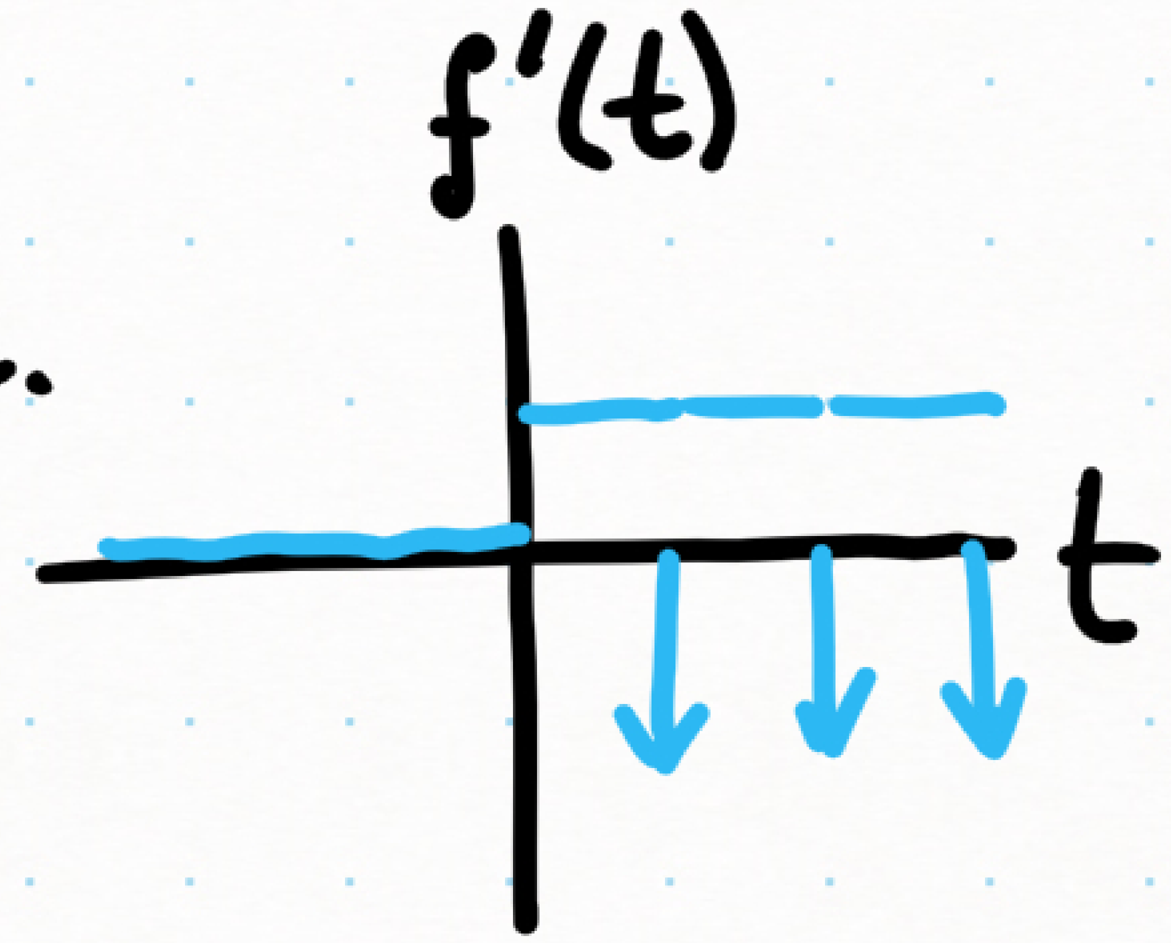
$$f(t) = (2t-1)\big(u(t) - u(t-1)\big)$$

$$f'(t) = 2\big(u(t) - u(t-1)\big) - \delta(t) - \delta(t-1)$$

d) i.



ii.



iii. $$f(t) = \sum_{n=0}^{\infty} (t-n)\big(u(t-n) - u(t-n-1)\big)$$

$$= t\,u(t) - \sum_{n=1}^{\infty} u(t-n)$$

$$f'(t) = u(t) - \sum_{n=1}^{\infty} \delta(t-n)$$