§ Damped Oscillation

1. Start with $\ddot{x} + \omega^2 x = 0$. What is the characteristic polynomial? What are its roots? What are the exponential solutions – the solutions of the form e^{at}? These may be complex exponentials. What are their real and imaginary parts? Check that these are also solutions to the original equation. What is the general real solution?

Answer: $r^2 + \omega^2 = 0$

$r = \pm i\omega$

$x_1(t) = e^{i\omega t}$

$\ddot{x}_1 + \omega^2 x_1 = (i\omega)^2 e^{i\omega t} + \omega^2 e^{i\omega t}$

$= 0$

$\mathrm{Re}(x_1) = \cos\omega t$

$\mathrm{Im}(x_1) = \sin\omega t$

$x_2(t) = e^{-i\omega t}$

$\ddot{x}_2 + \omega^2 x_2 = (-i\omega)^2 e^{-i\omega t} + \omega^2 e^{-i\omega t}$

$= 0$

$\mathrm{Re}(x_2) = \cos\omega t$

$\mathrm{Im}(x_2) = -\sin\omega t$

$x(t) = c_1 x_1(t) + c_2 x_2(t)$

2. Suppose that $e^{-t/2}\cos(3t)$ is a solution of the equation $m\ddot{x} + b\dot{x} + kx = 0$ (where m, b, k are real).

(a) What can you say about m, b, k?

(b) What are the exponential solutions (solutions of the form $e^{\alpha t}$) of this differential equation?

(c) Sketch the curve in the complex plane traced by one of the exponential solutions. Then sketch the graph of the real part, and explain the relationship.

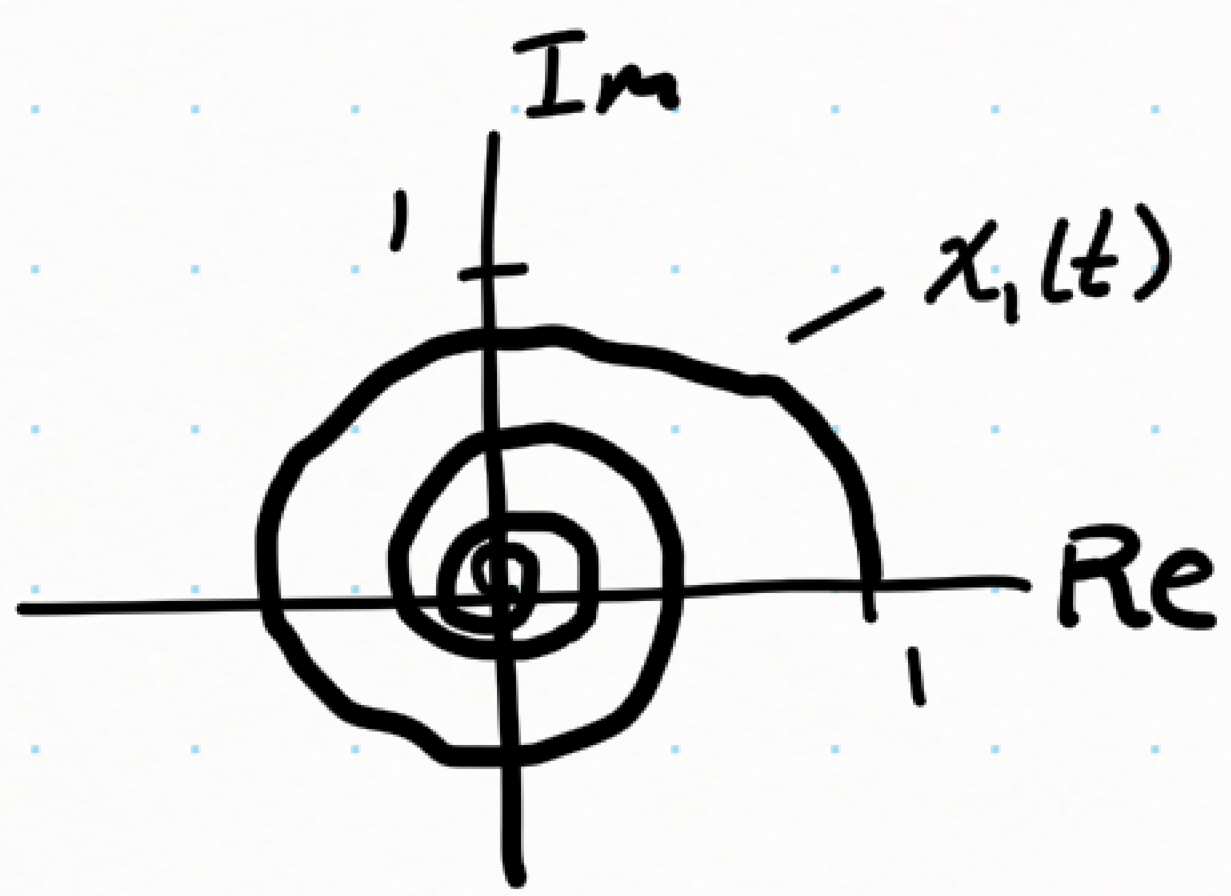
(d) What is the general solution?

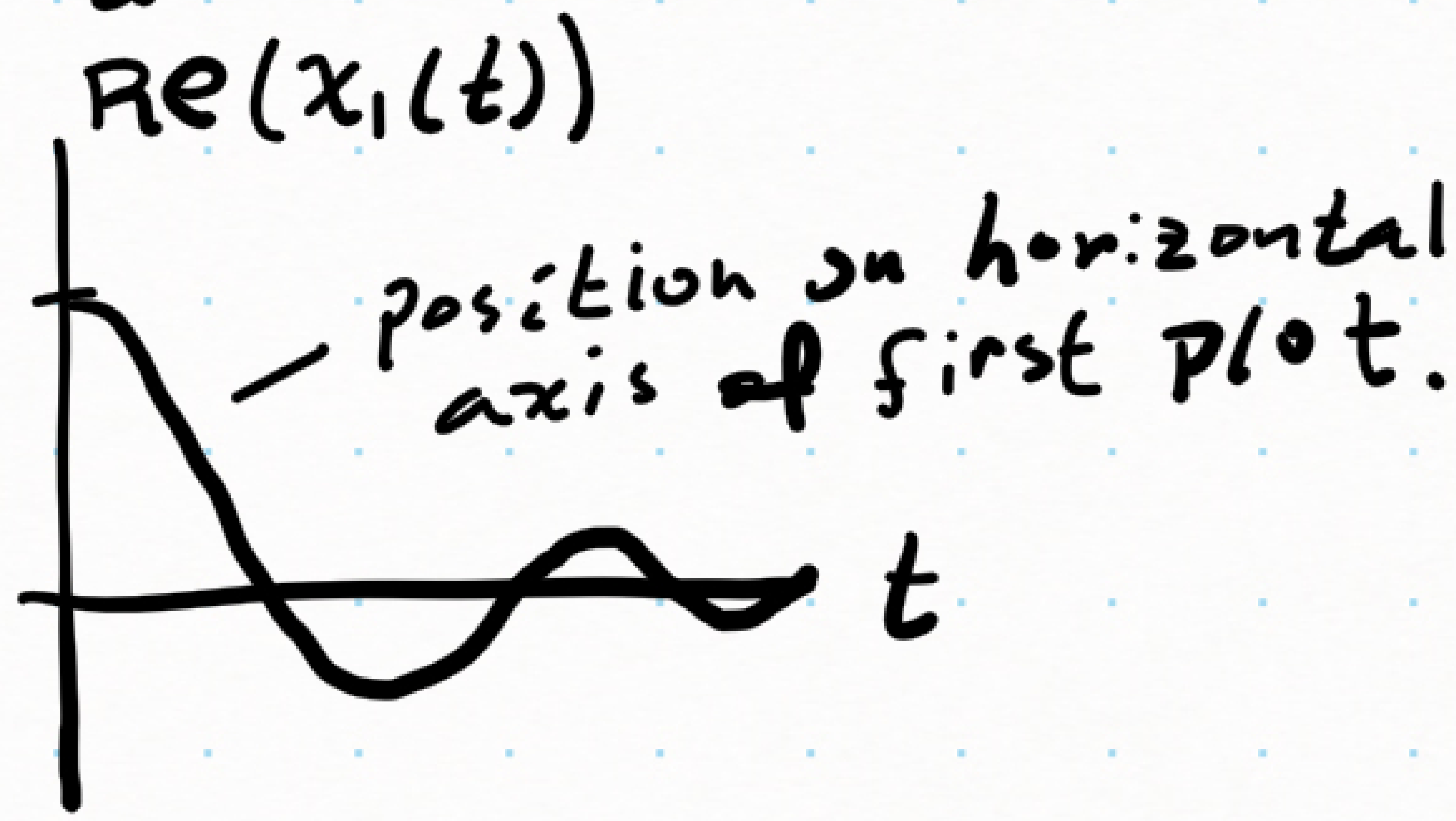
Answer:

a) $-\frac{t}{2} \pm 3i = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$

b) $x_1(t) = e^{(-1/2+3i)t}$, $x_2(t) = e^{(-1/2-3i)t}$

c)





d) $x(t) = c_1 x_1(t) + c_2 x_2(t)$

or $x(t) = e^{-t/2}\left(d_1 \cos(3t) + d_2 \sin(3t)\right)$

3. Let $\omega > 0$. A damped sinusoid $x(t) = Ae^{-at}\cos(\omega t)$ has "pseudo-period" $2\pi/\omega$. The pseudo-period, and hence ω, can be measured from the graph: it is twice the distance between successive zeros of $x(t)$, which is always the same. Now what is the spacing between successive maxima of $x(t)$? Is it always the same, or does it differ from one successive pair of maxima to the next?

Answer: The successive maxima occur at $\overbrace{t = 0,\ 2\pi/\omega,\ 4\pi/\omega,\ \ldots}^{A>0}$
The distance between successive maxima is $2\pi/\omega$.

$$0 = \dot{x}(t) = Ae^{-at}[-a\cos\omega t - \omega\sin\omega t]$$

$$\rightarrow \quad \tan\omega t = -a/\omega$$

The maxima/minima alternate, so the maxima occur every time this equation is satisfied. Since $\tan\omega t$ has period π/ω, the spacing of maxima is $2\pi/\omega$.

4. Suppose that successive maxima of $x(t) = Ae^{-at}\cos(\omega t)$ occur at $t = t_0$ and $t = t_1$. What is the ratio $x(t_1)/x(t_0)$? (Hint: Compare $\cos(\omega t_0)$ and $\cos(\omega t_1)$.) Does this offer a means of determining the value of a from the graph?

Answer:

$$
\begin{aligned}
x(t_1)/x(t_2) &= \left(Ae^{-at_1}\cos(\omega t_1)\right)/\left(Ae^{-at_2}\cos(\omega t_2)\right) \\
&= e^{-a(t_1-t_2)}\cos(\omega t_1)/\cos(\omega t_2) \\
&= e^{-a(t_1-t_2)} \quad (\text{since } \cos(\omega t_1) = \cos(\omega t_2) = 1) \\
&= e^{-2a\pi/\omega} \quad (t_1 - t_2 = 2\pi/\omega)
\end{aligned}
$$

Using a graph to find $x(t_1)$ and $x(t_2)$ then gives

$$a = \frac{-\omega \ln\left(x(t_1)/x(t_2)\right)}{2\pi}.$$

5. *For what value of b does* $\ddot{x} + b\dot{x} + x = 0$ *exhibit critical damping? For this value of b, what is the solution* x_1 *with* $x_1(0) = 1$, $\dot{x}_1(0) = 0$*? What is the solution* x_2 *with* $x_2(0) = 0$, $\dot{x}_2(0) = 1$*? (This is a "normalized pair" of solutions.) What is the solution such that* $x(0) = 2$ *and* $\dot{x}(0) = 3$*?*

Answer: For $m\ddot{x} + b\dot{x} + kx = 0$, critical damping occurs when $b^2 = 4mk$ (discriminant $= 0 \rightarrow$ double root).

The solution to $\ddot{x} + b\dot{x} + x = 0$ exhibits critical damping when $b^2 = 4$, $b = \pm 2$. Since we take $m > 0$, $b \geq 0$, $k > 0$, $b = 2$ and the root of the characteristic equation is $r = -1$.

Let $x_1(t) = e^{-t} + te^{-t}$ so $x_1(0) = 1$ and $\dot{x}_1(0) = 0$.
Let $x_2(t) = te^{-t}$ so $x_2(0) = 0$ and $\dot{x}_2(0) = 1$.
Let $x(t) = 2x_1(t) + 3x_2(t)$ so $x(0) = 2x_1(0) + 3x_2(0) = 2$ and $\dot{x}(0) = 2\dot{x}_1(0) + 3\dot{x}_2(0) = 3$. Given explicitly,

$$x(t) = 2e^{-t} + 5te^{-t}.$$