Final Exam

Operator Formulas

- Exponential Response Formula: $x_p = Ae^{rt}/p(r)$ solves $p(D)x = Ae^{rt}$ provided $p(r) \neq 0$.
- Resonant Response Formula: If $p(r) = 0$ then $x_p = Ate^{rt}/p'(r)$ solves $p(D)x = Ae^{rt}$ provided $p'(r) \neq 0$.

Properties of the Laplace transform

0. Definition: $\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$ for $\mathrm{Re}\, s >> 0$.

1. Linearity: $\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s)$.

2. Inverse transform: $F(s)$ essentially determines $f(t)$.

3. s-shift rule: $\mathcal{L}[e^{at}f(t)] = F(s-a)$.

4. t-shift rule: $\mathcal{L}[f_a(t)] = e^{-as}F(s)$, $\quad f_a(t) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$.

5. s-derivative rule: $\mathcal{L}[tf(t)] = -F'(s)$.

6. t-derivative rule: $\mathcal{L}[f'(t)] = sF(s)$ [generalized derivative]
$\mathcal{L}[f_r'(t)] = sF(s) - f(0+)$ [$f(t)$ continuous for $t > 0$]

7. Convolution rule: $\mathcal{L}[f(t) * g(t)] = F(s)G(s)$, $f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$.

8. Weight function: $\mathcal{L}[w(t)] = W(s) = \dfrac{1}{p(s)}$, $\quad w(t)$ the unit impulse response.

Formulas for the Laplace transform

$$\mathcal{L}[1] = \frac{1}{s} \qquad \mathcal{L}[e^{at}] = \frac{1}{s-a} \qquad \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2} \qquad \mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}[u_a(t)] = \frac{e^{-as}}{s} \qquad \mathcal{L}[\delta_a(t)] = e^{-as}$$

where $u(t)$ is the unit step function $u(t) = 1$ for $t > 0$, $u(t) = 0$ for $t < 0$.

Fourier series

$$f(t) = \frac{a_0}{2} + a_1\cos(t) + a_2\cos(2t) + \cdots + b_1\sin(t) + b_2\sin(2t) + \cdots$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(mt)\,dt, \qquad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(mt)\,dt$$

$$\int_{-\pi}^{\pi}\cos(mt)\cos(nt)\,dt = \int_{-\pi}^{\pi}\sin(mt)\sin(nt)\,dt = 0 \qquad \text{for} \quad m \neq n$$

$$\int_{-\pi}^{\pi}\cos^2(mt)\,dt = \int_{-\pi}^{\pi}\sin^2(mt)\,dt = \pi \qquad \text{for} \quad m > 0$$

If $\mathrm{sq}(t)$ is the odd function of period 2π which has value 1 between 0 and π, then

$$\mathrm{sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right)$$

1. This problem concerns the differential equation

$$\frac{dy}{dx} = x^2 - y^2 \qquad (*)$$

Let $y = f(x)$ be the solution with $f(-2) = 0$.

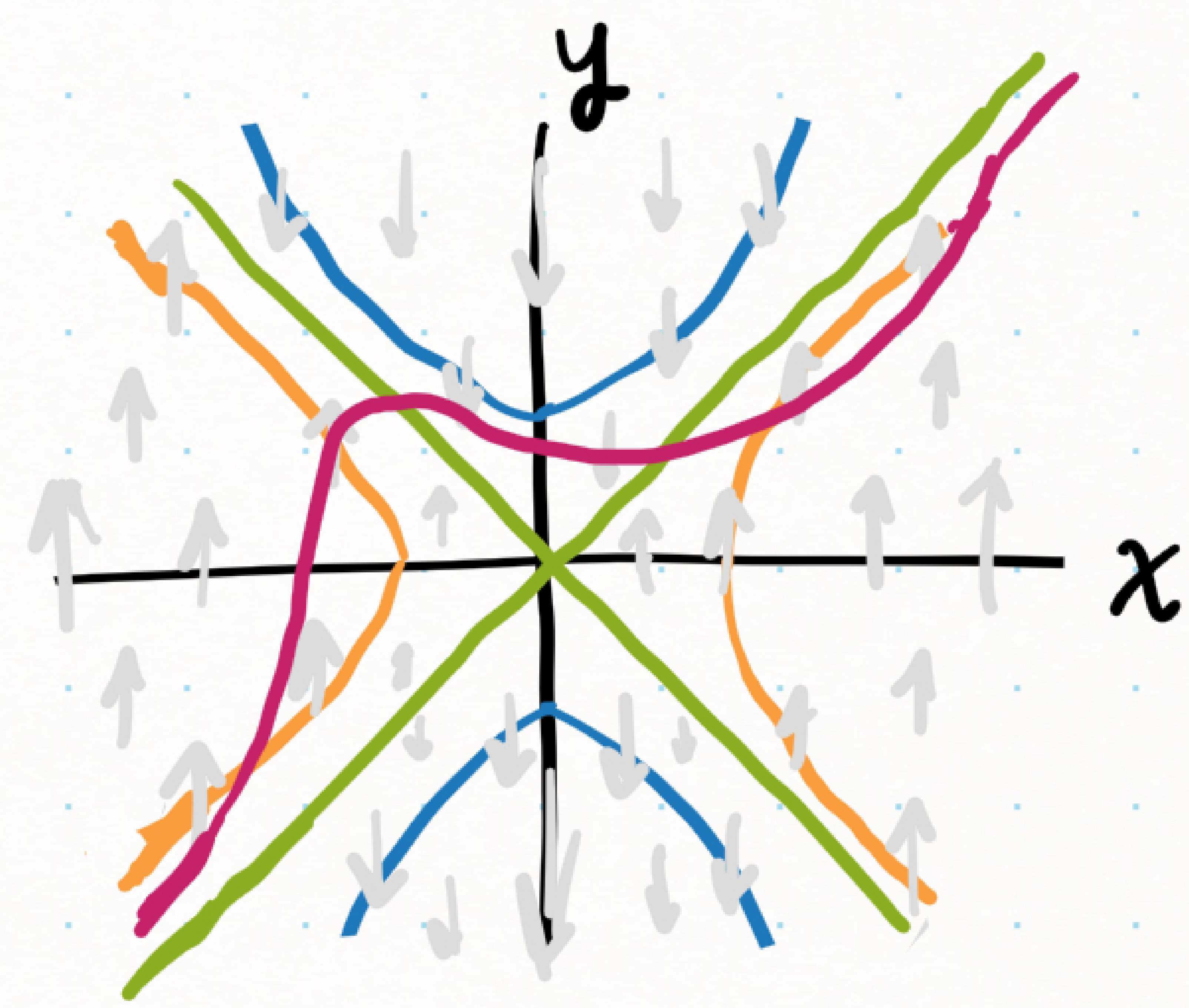
(a) Sketch the isoclines for slopes -2, 0, and 2, and sketch the direction field along them.

(c) On the same diagram, sketch the graph of the solution $f(x)$. What is its slope at $x = -2$?

(d) Estimate $f(100)$.

(e) Suppose that the function $f(x)$ reaches a maximum at $x = a$. What is $f(a)$?

(f) Use two steps of Euler's method to estimate $f(-1)$.



a) $-2 = x^2 - y^2 \rightarrow 1 = \frac{y^2}{2} - \frac{x^2}{2}$

$0 = x^2 - y^2 \rightarrow |y| = |x|$

$2 = x^2 - y^2 \longrightarrow 1 = \frac{x^2}{2} - \frac{y^2}{2}$

b) $f(x) = y \quad f'(-2) = 4$

c) $f(100) \approx 100$ since $\lim_{x\to\infty} f(x) = x$ for this solution.

d) $f(a) = -a$ since f reaches a local max on the line $y = -x$.

e) $f(-1.5) \approx 0 + 4 \cdot \frac{1}{2} = 2$, $f(-1) \approx 2 - \frac{7}{4}\frac{1}{2} = 2 - \frac{7}{8} = 9/8 = 1.125$

2. In **(a)–(c)** we consider the autonomous equation $\dot{x} = 2x - 3x^2 + x^3$.

(a) Sketch the phase line of this equation.

(b) Sketch the graphs of some solutions. Be sure to include at least one solution with values in each interval above, below, and between the critical points.

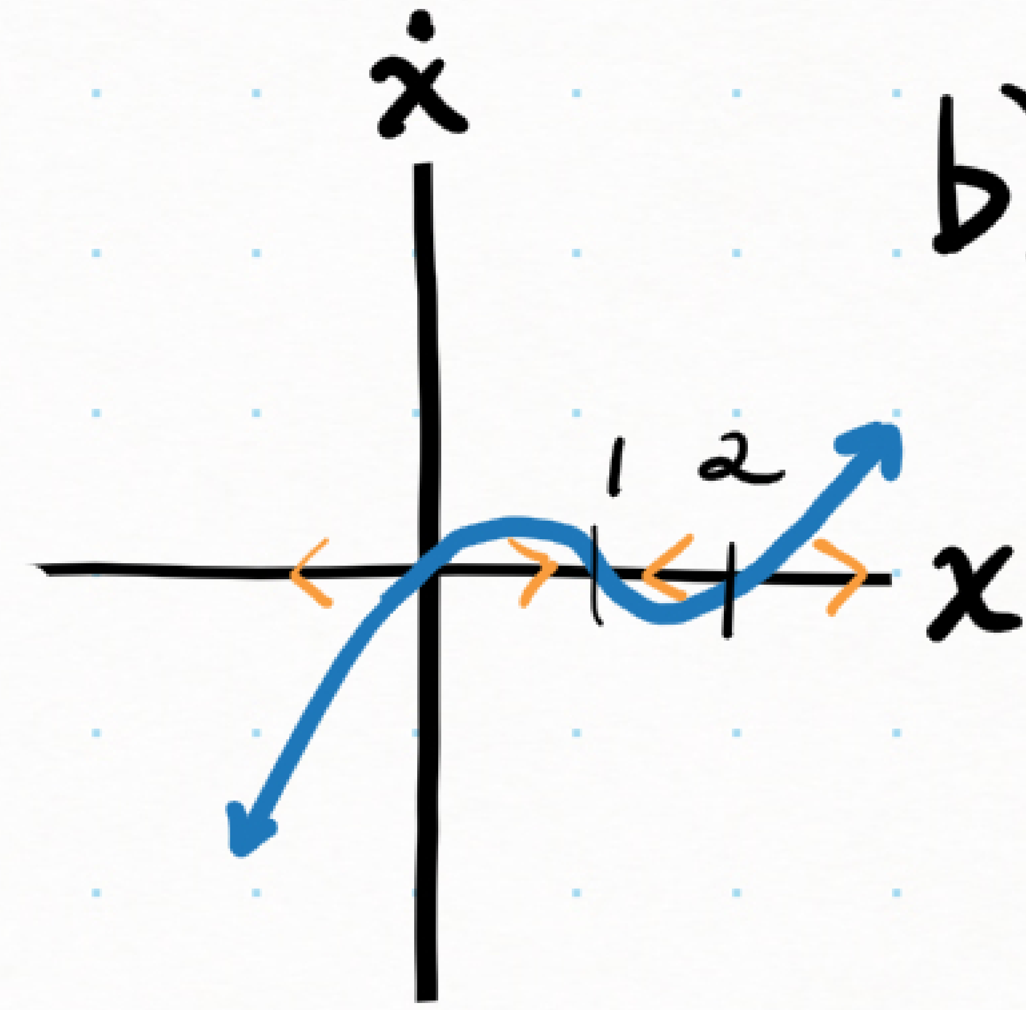
(c) Some solutions have points of inflection. What are the possible values of $x(a)$ if a non-constant solution $x(t)$ has a point of inflection at $t = a$?

(d) A radioactive isotope of the element Cantabrigium, Ct, decays with half life of two years. The MIT reactor runs on Cantabrigium.
At $t = 0$ there is no Ct in it, but starting at $t = 0$, Ct is added in such a way that the cumulative total amount inserted by time t years is t kg.

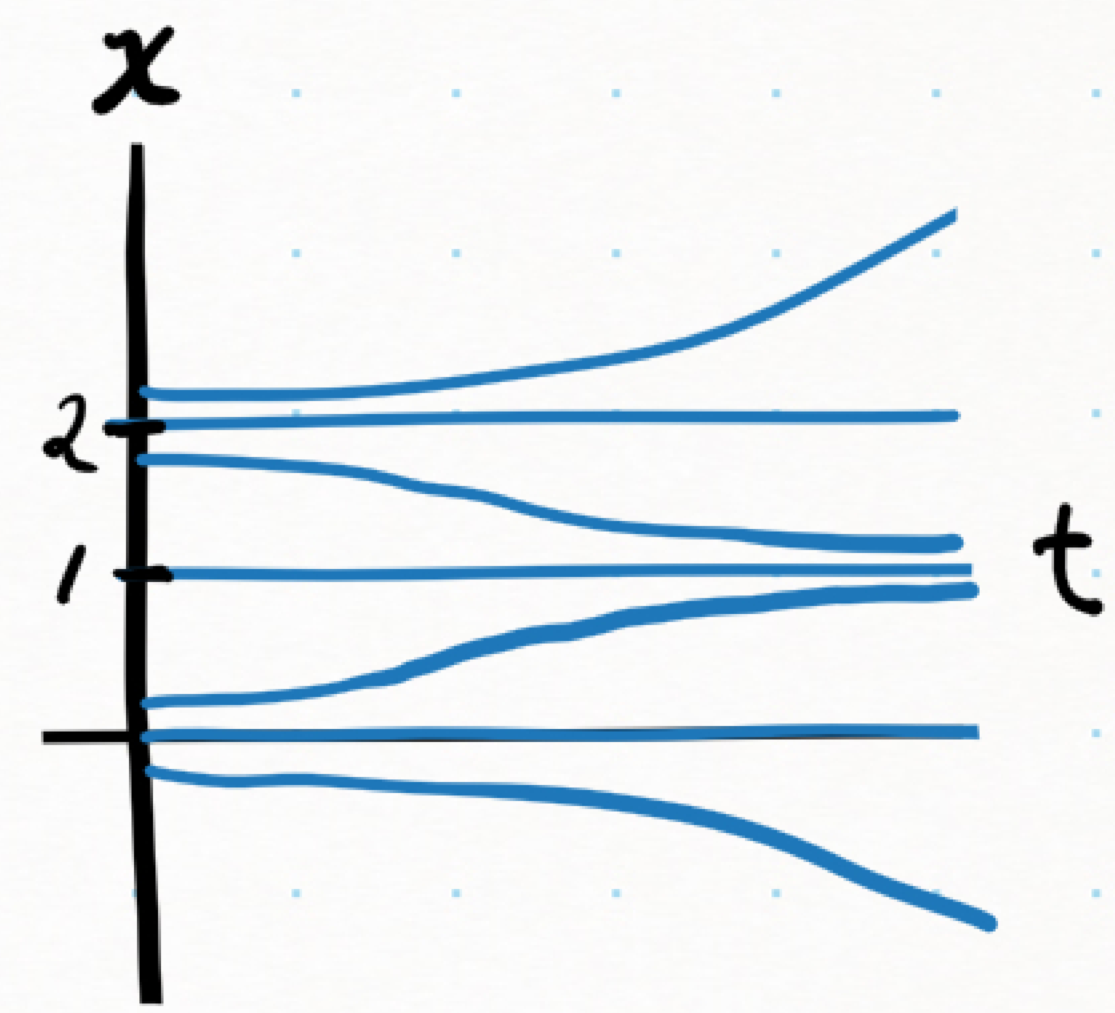
Write down a differential equation for the number of moles of Ct in the reactor as a function of time. What is the initial condition?

(e) Solve the initial value problem $x\dfrac{dy}{dx} + 3y = x^2, y(1) = 1$.

a)



b)



c) $0 = \ddot{x} = (2 - 6x + 3x^2)\dot{x} \longrightarrow x(a) = 1 \pm \frac{\sqrt{3}}{3}$.

d) $\dot{x} = -kx + 1$, $\frac{1}{2} = e^{-2k} \to k = \frac{\ln 2}{2}$

$\dot{x} = -\frac{\ln 2}{2}x + 1$, $x(0) = 0$

e) $\mu(x) = e^{\int 3/x \, dx} = x^3$

$y(x) = \frac{1}{x^3}\int x^4 \, dx = \frac{x^2}{5} + \frac{c}{x^3}$, $1 = y(1) = \frac{1}{5} + c$

$y(x) = \frac{1}{5}x^2 + \frac{4}{5}x^{-3}$.

3. (a) Find non-negative real numbers A, ω, and ϕ such that $\operatorname{Re}\left(\dfrac{ie^{2it}}{1+i}\right) = A\cos(\omega t - \phi)$.

(b) Sketch the trajectory of $e^{(1-\pi i)t}$.

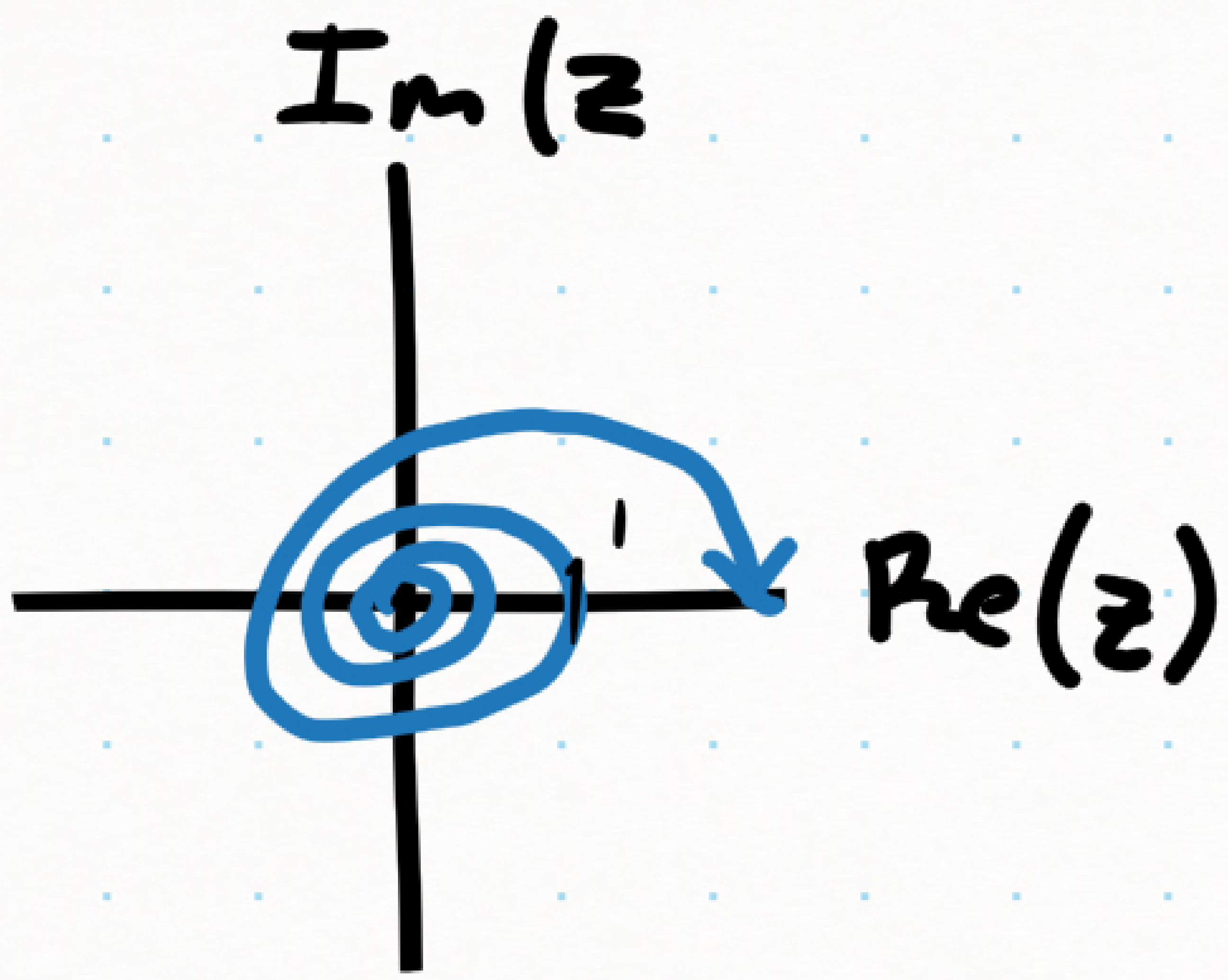
(c) Express the cube roots of $8i$ in the form $a + bi$ (with a and b real).

a) $$\operatorname{Re}\left(\frac{ie^{2it}}{1+i}\right) = \operatorname{Re}\left(\frac{i+1}{2}(\cos 2t + i\sin 2t)\right)$$

$$= \frac{1}{2}\cos 2t - \frac{1}{2}\sin 2t$$

$$= \frac{\sqrt{2}}{2}\cos\left(2t + \frac{\pi}{4}\right)$$

b)



c) $$8i = 8e^{(\pi/2 + 2k\pi)i}$$

$$\sqrt[3]{8i} = 2e^{(\pi/6 + 2k\pi/3)i}$$

$$= \sqrt{3} + i, \quad (k=0)$$
$$-\sqrt{3} + i, \quad (k=1)$$
$$0 - 2i \quad (k=2)$$

4. (a)–(c) Find one solution to $\ddot{x} + 2\dot{x} + 2x = q(t)$ for

(a) $q(t) = t^2 + 1$.

(b) $q(t) = e^{-2t} + 1$.

(c) $q(t) = \sin t$. What is the amplitude of the sinusoidal solution?

In **(d)** and **(e)**, suppose that that t^3 is a solution to $\ddot{x} + 2\dot{x} + 2x = q(t)$.

(d) What is $q(t)$?

(e) What is the general solution to $\ddot{x} + 2\dot{x} + 2x = q(t)$?

a) $x_p(t) = At^2 + Bt + C$

$2A + 4At + 2B + 2At^2 + 2Bt + 2C = t^2 + 1$

$\rightarrow A = 1/2,\ B = -1,\ C = 1$

$x_p(t) = \frac{1}{2}t^2 - t + 1$

b) Using the ERF, $x_1 = \frac{1}{2}e^{-2t}$ satisfies $\ddot{x} + 2\dot{x} + 2x = e^{-2t}$.

Since $x_2 = \frac{1}{2}$ satisfies $\ddot{x} + 2\dot{x} + 2x = 1$, $x_p = x_1 + x_2$.

c) Using the ERF, $z_p = \frac{1-2i}{5}e^{it}$ satisfies $\ddot{z} + 2\dot{z} + 2z = e^{it}$.

Then $x_p = \operatorname{Im}(z_p) = -\frac{2}{5}\cos t + \frac{1}{5}\sin t$

d) $q(t) = 6t + 6t^2 + 2t^3$

e) $r^2 + 2r + 2 = 0 \longrightarrow x_h(t) = e^{-t}(c_1\cos t + c_2 \sin t)$

$r = -1 \pm i$

$x(t) = x_h + x_p = e^{-t}(c_1\cos t + c_2\sin t) + t^3$.

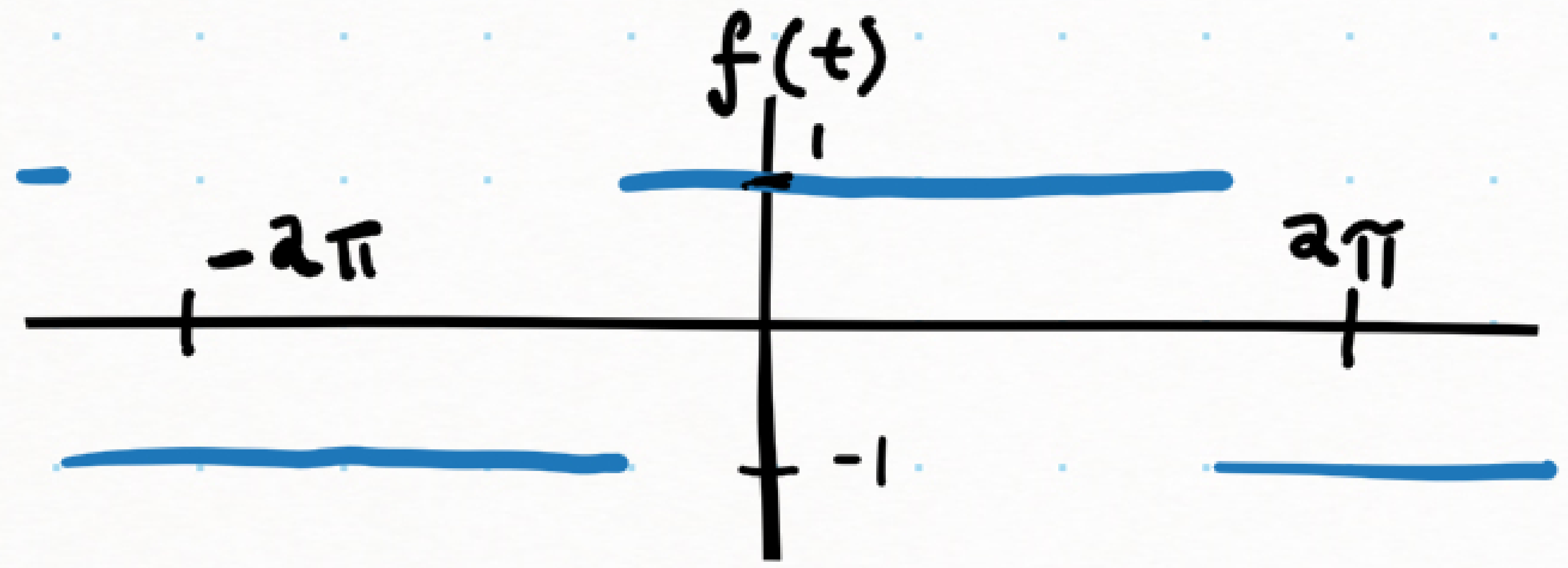
5. (a)–(b) concern the function $f(t) = \mathrm{sq}(t + \frac{\pi}{2})$.

(a) Graph $f(t)$.

(b) What is its Fourier series? (Simplify the trig functions.)

(c) Find a solution to $\ddot{x} + x = \mathrm{sq}(t)$.

a)



b) $$f(t) = \frac{4}{\pi}\left[\sin(t+\pi/2) + \frac{\sin(3(t+\pi/2))}{3} + \frac{\sin(5(t+\pi/2))}{5} + \dots\right]$$

$$= \frac{4}{\pi}\left[\cos t - \frac{\cos 3t}{3} + \frac{\cos 5t}{5} - \frac{\cos 7t}{7} + \dots\right]$$

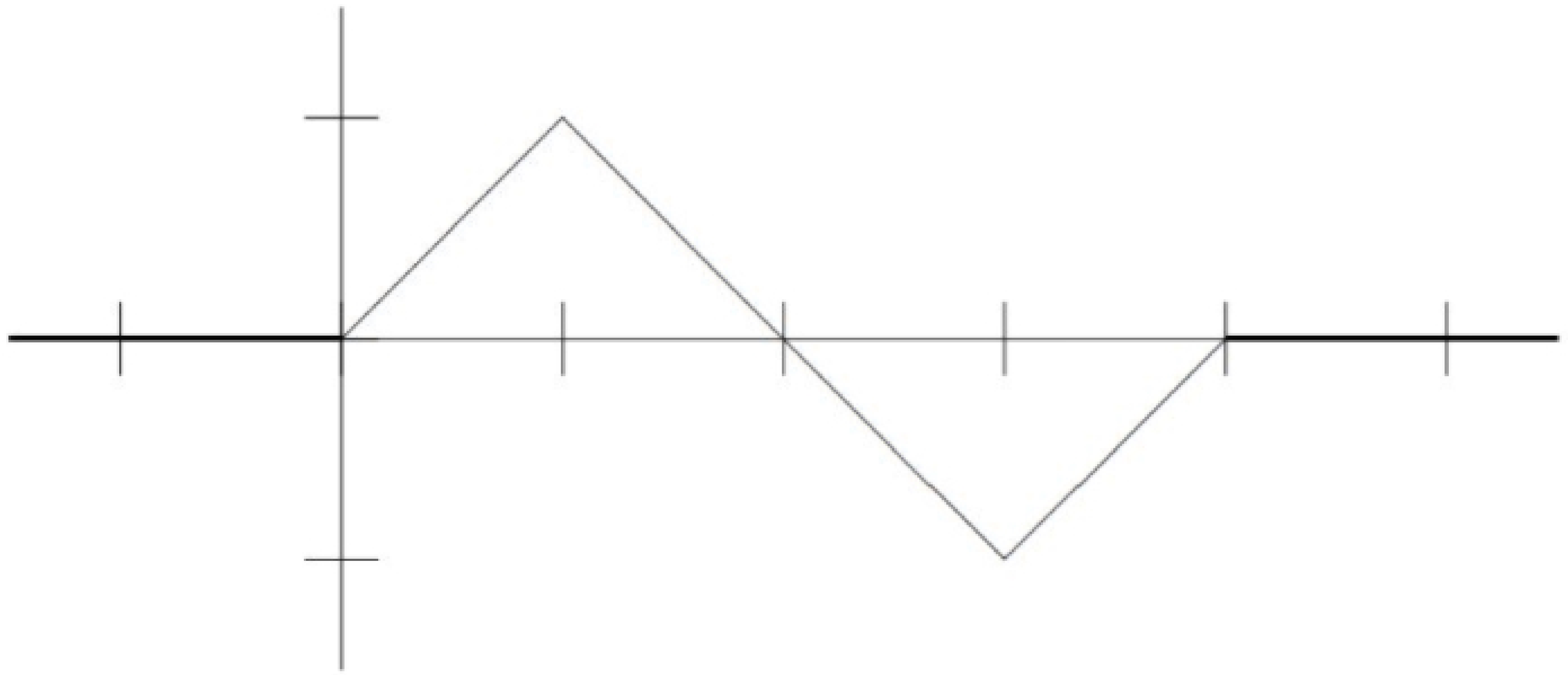
c) $\ddot{z} + z = e^{int}$

$$z_{p,n} = \begin{cases} \dfrac{e^{int}}{P(in)} = \dfrac{e^{int}}{1-n^2}, & n>1, \text{ odd} \\[2ex] \dfrac{te^{it}}{P'(i)} = \dfrac{te^{it}}{2i}, & n=1 \end{cases}$$

$$x_p(t) = \frac{4}{\pi}\sum_{n \text{ odd}} \mathrm{Im}(z_{p,n})/n$$

$$= \frac{4}{\pi}\left[-\frac{t}{2}\cos t + \frac{1}{3}\frac{1}{1-3^2}\sin 3t + \frac{1}{5}\frac{1}{1-5^2}\sin 5t + \dots\right]$$

6. (a)–(d) In a recent game of Capture the Flag, a certain student was observed to move according to the following graph, in which the hashmarks are at unit spacing.



(a) Graph the generalized derivative $v(t)$.

(b) Write a formula for $v(t)$ in terms of the unit step and (if necessary) the delta function.

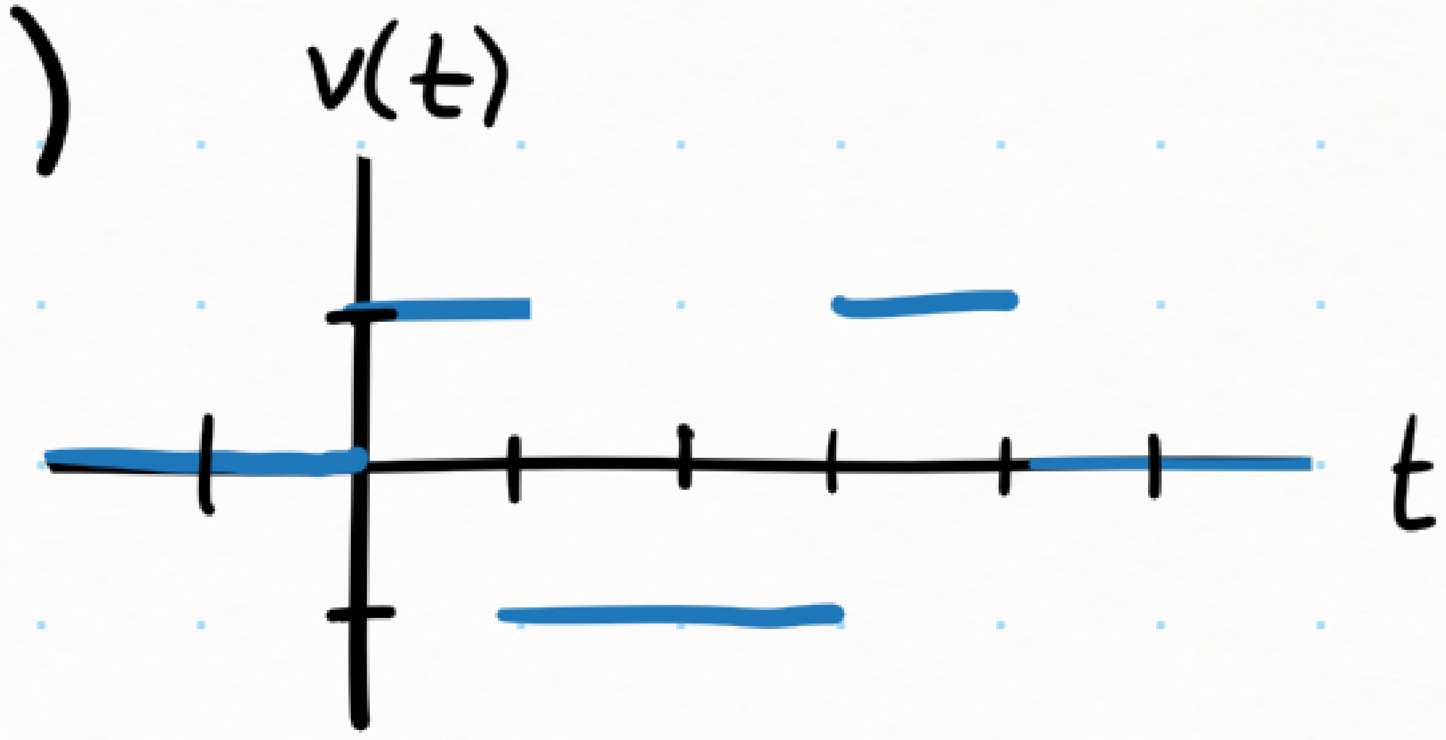
(c) Still with the same function as in **(a)**: Graph the generalized derivative $\dot{v}(t)$.

(d) Write a formula for the acceleration $\dot{v}(t)$ in terms of the unit step and (if necessary) the delta function.

(e) Suppose that the unit impulse response of a certain operator $p(D)$ is $w(t)$. Let $q(t) = 0$ for $t < 0$ and $t > 1$, and $q(t) = 1$ for $0 < t < 1$. Please find functions $a(t)$, $b(t)$ so that the solution $x(t)$ to $p(D)x = q(t)$, with rest initial conditions, is given by

$$x(t) = \int_{a(t)}^{b(t)} w(\tau)\,d\tau$$

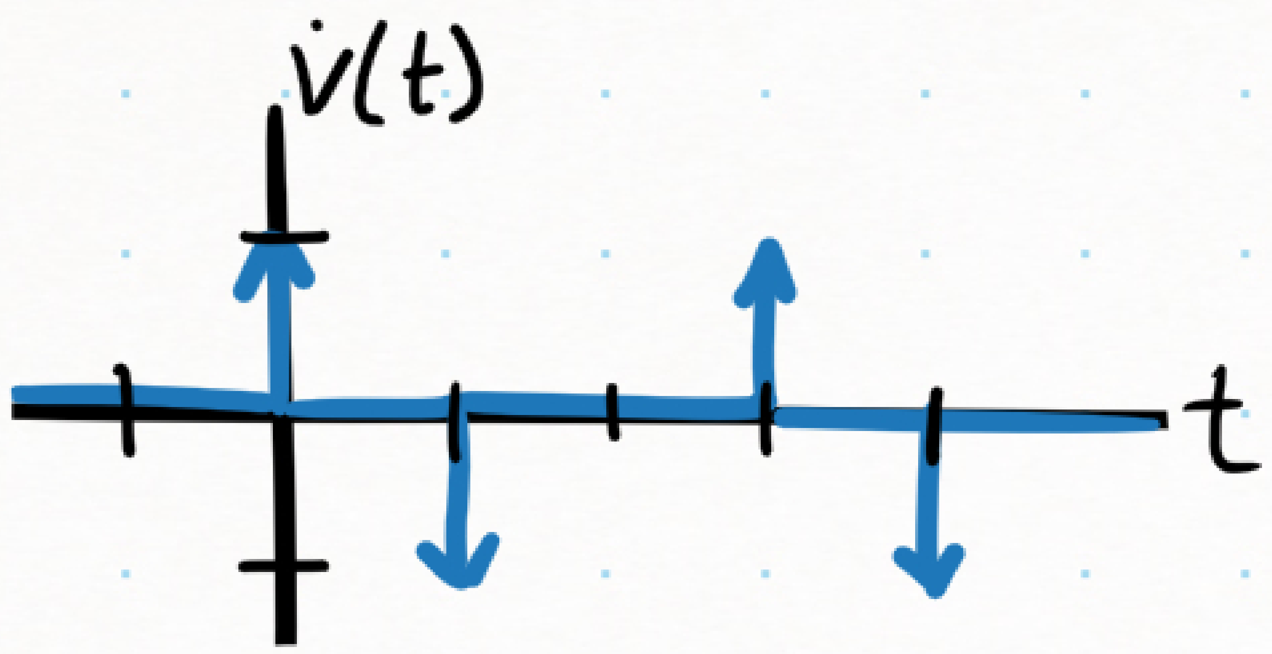
a)



b) $v(t) = [u(t) - u(t-1)] - [u(t-1) - u(t-3)] + [u(t-3) - u(t-4)]$

$= u(t) - 2u(t-1) + 2u(t-3) - u(t-4)$

c)



d) $\dot{v}(t) = \delta(t) - 2\delta(t-1) + 2\delta(t-3) - \delta(t-4)$

e) $x(t) = (w * q)(t) = \int_0^t w(\tau)q(t-\tau)\,d\tau$

$= \int_{(t-1)u(t-1)}^{t} w(\tau)\,d\tau$ since

$q(t-\tau) = 1$ if $t-1 < \tau < t$ and convolution is defined for $\tau \geq 0$.

$\therefore\ a(t) = (t-1)u(t-1),\ b(t) = t.$

7. This problem concerns the operator $p(D) = 2D^2 + 8D + 16I$.

(a) What is the transfer function of the operator $p(D)$?

(b) What is the unit impulse response of this operator?

(c) What is the Laplace transform of the solution to $p(D)x = \sin(t)$ with rest initial conditions?

a) $\mathcal{L}[2\ddot{w} + 8\dot{w} + 16w] = \mathcal{L}[\delta(t)]$

$$2s^2W + 8sW + 16W = 1$$

$$W(s) = \frac{1}{2s^2 + 8s + 16}$$

b) $W(s) = \frac{1}{4} \frac{2}{(s+2)^2 + 2^2} \longrightarrow w(t) = \frac{1}{4} e^{-2t} \sin 2t$

c) $X(s) = W(s)\,F(s) = W(s)\,\mathcal{L}[\sin t]$

$$= \frac{1}{4} \frac{2}{(s+2)^2 + 2^2} \frac{1}{s^2 + 1}$$

Suppose we have an equation

$$p(D)x = f(t), \quad \text{with rest IC.}$$

Taking Laplace transform of both sides gives

$$p(s)X(s) = F(s) \quad \Rightarrow \quad X(s) = \frac{1}{p(s)}F(s) = W(s)F(s).$$

Note: possible error in official solutions, which assumes IC $x(0^+) = 1$, $\dot{x}(0^+) = 2$. The course notes disagree with the official solutions

8. In **(a)** and **(b)**, $A = \begin{bmatrix} 2 & 12 \\ 3 & 2 \end{bmatrix}$.

(a) What are the eigenvalues of A?

(b) For each eigenvalue, find a nonzero eigenvector.

(c) Suppose that the matrix B has eigenvalues 1 and 2, with eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ respectively. Calculate e^{Bt}.

d) Solve $\dot{u} = Bu$, $u(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

a) $0 = \lambda^2 - 4\lambda - 32 \longrightarrow \lambda = -4, 8$

b) $\lambda = -4$, $v = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\qquad$ $\lambda = 8$, $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\left[\begin{array}{cc|c} 6 & 12 & 0 \\ 3 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\qquad$ $\left[\begin{array}{cc|c} -6 & 12 & 0 \\ 3 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

c) $\Phi(t) = \begin{bmatrix} e^t & -e^{2t} \\ e^t & e^{2t} \end{bmatrix}$, $\Phi(0)^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $e^{Bt} = \frac{1}{2}\begin{bmatrix} e^t + e^{2t} & e^t - e^{2t} \\ e^t - e^{2t} & e^t + e^{2t} \end{bmatrix}$

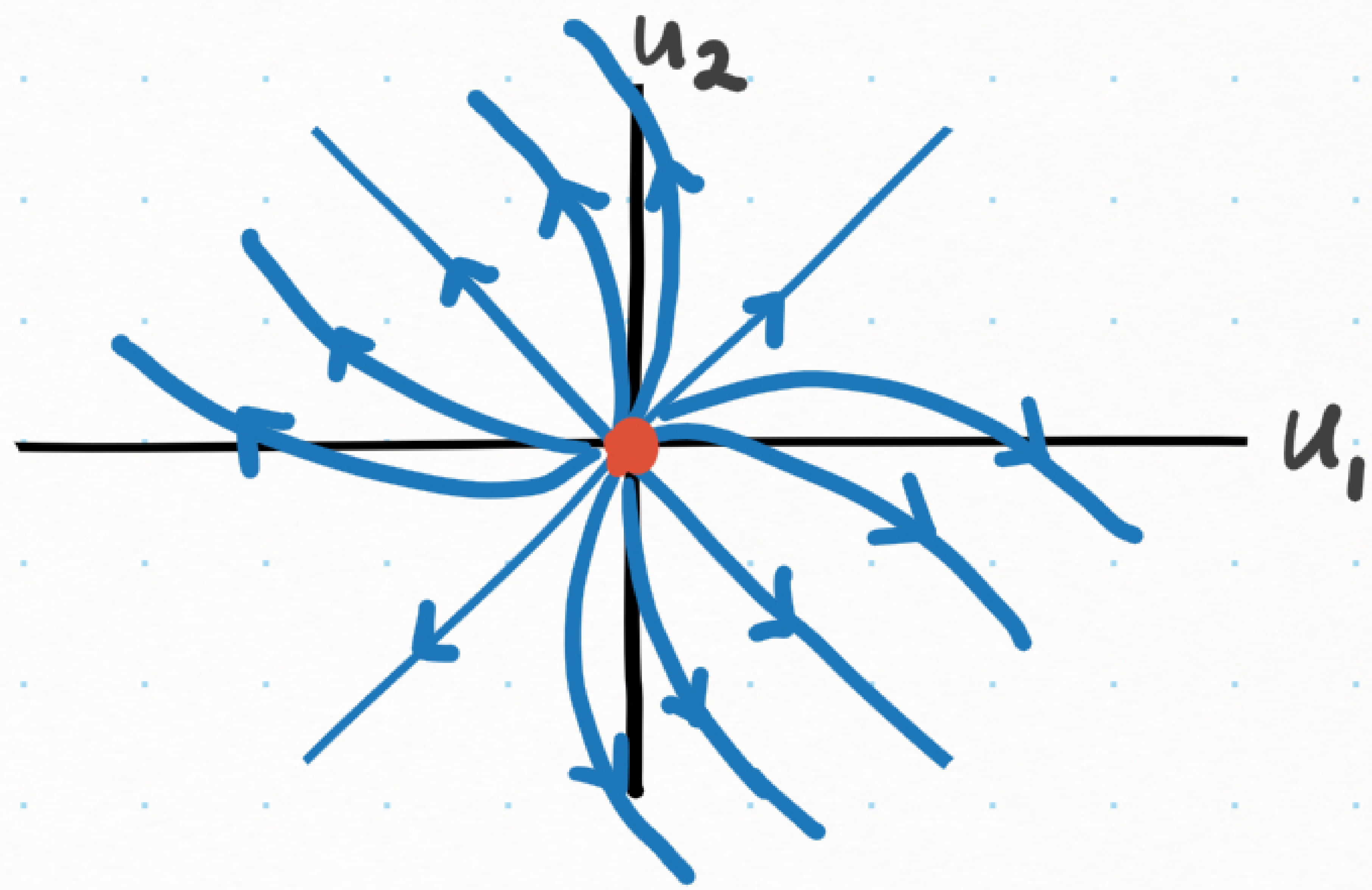
d) $u(t) = e^{Bt} u(0) = \frac{1}{2}\begin{bmatrix} 3e^t + e^{2t} \\ 3e^t - e^{2t} \end{bmatrix}$

9. (a) Suppose again that the matrix B has eigenvalues 1 and 2, with eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ respectively. Sketch the phase portrait on the graph below.

(b) Let $A = \begin{bmatrix} a & -2 \\ 2 & 1 \end{bmatrix}$, and consider the homogeneous linear system $\dot{\mathbf{u}} = A\mathbf{u}$. For each of the following conditions, determine all values of a (if any) which are such that the system satisfies the condition.

(i) Saddle
(ii) Star
(iii) Stable node
(iv) Stable spiral. What is the direction of rotation?
(v) Unstable spiral.
(vi) Unstable defective node

a)



b) $\lambda = \dfrac{a+1 \pm \sqrt{(a+1)^2 - 4(a+4)}}{2} = \dfrac{a+1 \pm \sqrt{(a+3)(a-5)}}{2}$

$\Delta = (a+3)(a-5)$, $\det A = a+4$

i. $\det A < 0 \longrightarrow a < -4$

ii. $\Delta = 0$ and 2 ind. eigenvectors $\longrightarrow$ no a

iii. $\Delta > 0$ and $\det > 0$ and $\operatorname{tr} A < 0 \longrightarrow -4 < a < -3$

iv. $\Delta < 0$ and $\operatorname{tr} A < 0 \longrightarrow -3 < a < -1$ counterclockwise

v. $\Delta < 0$ and $\operatorname{tr} A > 0 \longrightarrow -1 < a < 5$

vi. $\Delta = 0$ and $\operatorname{tr} A > 0 \longrightarrow a = 5$

10. Parts **(a)–(c)** deal with the nonlinear autonomous system $\begin{cases} \dot{x} = x^2 - y^2 \\ \dot{y} = x^2 + y^2 - 8 \end{cases}$.

(a) Find the equilibria of this system.

(b) There is one equilibrium in the south-west quadrant. Find the Jacobian at this equilibrium.

(c) The equilibrium you found in **(b)** is a stable spiral. For large t, the solutions which converge to this equilibrium have x-coordinate which are well-approximated by the function $Ae^{at}\cos(\omega t - \phi)$ for some constants A, ϕ, a, and ω. Some of these constants depend upon the particular solution, and some are common to all solutions of this type. Find the values of the ones which are common to all such solutions.

(d) Finally, return to the autonomous equation $\dot{x} = 2x - 3x^2 + x^3$ that you studied in problem **2**. Write down a formula approximating the solutions converging to the stable equilibrium when t is large.

a) $0 = \dot{x} = \dot{y}$

$\rightarrow x^2 = y^2, \quad 8 = 2x^2 \rightarrow \pm 2 = x$

$(x,y) = (\pm 2, \pm 2)$ (4 pts).

b) $J(x,y) = \begin{bmatrix} 2x & -2y \\ 2x & 2y \end{bmatrix} \quad J(-2,-2) = \begin{bmatrix} -4 & 4 \\ -4 & -4 \end{bmatrix}$

c) $J(-2,2)$ has $\lambda = -4 \pm 4i$. So the soln u to the linearized problem $\dot{u} = J(-2,-2)u$ has the general solution

$$u(t) = c_1 e^{-4t}\cos 4t + c_2 e^{-4t}\sin 4t$$
$$= \sqrt{c_1^2 + c_2^2}\, e^{-4t}\cos\left(4t - \arctan\frac{c_2}{c_1}\right).$$

So $a = -4$, $\omega = 4$, but A and ϕ may change.

$x(t) \approx -2 + u(t)$

d) Stable equilibrium at $x = 1$. Linearizing: $\dot{u} = f'(1)u$ for $f(x) = 2x - 3x^2 + x^3$. Then $\dot{u} = -u \longrightarrow u(t) = Ae^{-t} \rightarrow x(t) \approx 1 + Ae^{-t}$.