Problem Set 1

1. (a) $y = x^a$
$y' = a x^{a-1}$
$y'' = a(a-1) x^{a-2}$

$$x^2 y'' = 2y$$

$$x^2 a(a-1) x^{a-2} = 2x^a$$
$$[a(a-1) - 2] x^a = 0$$
$$a^2 - a - 2 = 0$$
$$(a-2)(a+1) = 0$$
$$a = -1, 2$$

$y = x^{-1}$, $x \in \mathbb{R}$, $x \neq 0$
$y = x^2$, $x \in \mathbb{R}$

(b) $x^2 y'''' = 2y''$

$$x^2 a(a-1)(a-2)(a-3) x^{a-4} = 2a(a-1) x^{a-2}$$
$$a(a-1)(a-2)(a-3) - 2a(a-1) = 0$$
$$a(a-1)[(a-2)(a-3) - 2] = 0$$
$$a(a-1)(a-1)(a-4) = 0$$
$$a = 1, 4 \quad (\text{since we seek } a \neq 0)$$

$\underline{a=1:}$ $y = x$ $\quad x^2 y'''' = 0 = 2y''$

$\underline{a=4:}$ $y = x^4$ $\quad x^2 y'''' = x^2 \cdot 24$
$= 2 \cdot 12 x^2$
$= 2y''$.

2. $dy = 4y\sin 2x\,dx$, $y(\pi) = e$

(a) $\frac{1}{y}dy = 4\sin 2x\,dx$

$\ln y = -2\cos 2x + c$, $y > 0$

$y(\pi) = e \rightarrow 1 = -2 + c$
$3 = c$

$\ln y = -2\cos 2x + 3$

$y(\pi/6) = ?$

$\ln y = -2\cos \pi/3 + 3$
$\ln y = 2$
$y = e^2$

(b) $\int_e^a \frac{dy}{y} = \int_\pi^{\pi/6} 4\sin 2x\,dx$

$\ln a - \ln e = -2\cos(2\frac{\pi}{6}) + 2\cos(2\pi)$

$\ln a - 1 = -2(\frac{1}{2}) + 2(1)$
$\ln a = 2$
$a = e^2$
$\therefore y(\pi/6) = e^2$

(c) $x\,dy + 3y\,dx = 0$, $y(-\pi) = e$.

$x\,dy = -3y\,dx$
$1/y\,dy = -3/x\,dx$

After separating variables we see that we must integrate for $x > 0$ or $x < 0$ since $1/x$ is not integrable over an interval containg $x = 0$. Since we have the initial condition at $x = -\pi$, we can only find a solution for y for $x < 0$. So we cannot solve for $y(\pi)$.

4. $y' = \frac{3}{2} y^{1/3}$, $x \geq 0$

(a) $y_1(x) = 0$, $y_1'(x) = 0$

$0 = \frac{3}{2} 0^{1/3}$ ✓

$y_2(x) = x^{3/2}$, $y_2'(x) = \frac{3}{2} x^{1/2}$

$\frac{3}{2} x^{1/2} = \frac{3}{2} \left(x^{3/2}\right)^{1/3}$ ✓

(b) $\frac{2}{3} y^{-1/3} dy = dx$

$y^{2/3} = x - c$

$y = (x-c)^{3/2}$

Let $c = \sup\{x : y(x) = 0\}$.
If $c = +\infty$, $y(x) = y_1(x) = 0$
If $c < +\infty$, $y(x) = 0$ for $x \leq c$ and $y(x) = (x-c)^{3/2}$ for $x > 0$.

6. Bernoulli Equation:

$$y' + p(x) y = q(x) y^n, \quad n \neq 1$$

(a) $y^{-n} y' + p y^{1-n} = q$ (for $y \neq 0$)

Let $u = y^{1-n}$, $u' = (1-n) y^{-n} y'$.

$\frac{1}{1-n} u' + p u = q$

(b) $y' + y = x y^3$, $u = y^{1-3} = y^{-2}$

$\frac{1}{1-3} u' + u = x$

$\frac{d}{dx}\left[u e^{-2x}\right] = x e^{-2x}$

$u(x) = e^{2x} \int x e^{-2x} dx$

$= e^{2x}\left(-\frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x} + c\right)$

$= -\frac{1}{2} x - \frac{1}{4} + c e^{2x}$

$y(x) = u^{-1/2} = \left(c e^{2x} - \frac{1}{2} x - \frac{1}{4}\right)^{-1/2}$, $y \neq 0$.