Problem Set 3

1. $y'' + \omega^2 y = \sin\omega_0 t$, $\omega \neq 0$

ω_0 close to ω but $\omega_0 \neq \omega$

(a) $$y_1(t) = \frac{\sin\omega_0 t}{\omega^2 - \omega_0^2}$$

$$y_1''(t) = \frac{-\omega_0^2 \sin\omega_0 t}{\omega^2 - \omega_0^2}$$

$$y_1'' + \omega^2 y_1 = \frac{\sin\omega_0 t\,(-\omega_0^2 + \omega^2)}{\omega^2 - \omega_0^2}$$

$$= \sin\omega_0 t \quad ✓$$

(b) $$\lim_{\omega_0 \to \omega} y_1(0) = \lim_{\omega_0 \to \omega} \frac{0}{\omega^2 - \omega_0^2}$$

$$= \lim_{\omega_0 \to \omega} \frac{0}{-2\omega_0} = 0, \quad \omega \neq 0$$

$$\lim_{\omega_0 \to \omega^+} y_1'(0) = \lim_{\omega_0 \to \omega^+} \frac{\omega_0}{\omega + \omega_0} \frac{1}{\omega - \omega_0}$$

$$= -\infty$$

$$\lim_{\omega_0 \to \omega^-} y_1'(0) = +\infty$$

(c) $$y_2(t) = \frac{\sin\omega_0 t - \sin\omega t}{\omega^2 - \omega_0^2}$$

$$y_2''(t) = \frac{-\omega_0^2 \sin\omega_0 t + \omega^2 \sin\omega t}{\omega^2 - \omega_0^2}$$

$$y_2'' + \omega^2 y_2 = \frac{(-\omega_0^2 + \omega^2)\sin\omega_0 t}{\omega_2^2 - \omega_0^2}$$

$$+ \frac{(\omega^2 - \omega^2)\sin\omega t}{\omega^2 - \omega_0^2}$$

$$= \sin\omega_0 t \quad ✓$$

$$\lim_{\omega_0\to\omega} y_2(0) = \lim_{\omega_0\to\omega} \frac{0}{\omega^2-\omega_0^2} = 0, \quad \omega \neq 0$$

$$\lim_{\omega_0\to\omega} y_2'(0) = \lim_{\omega_0\to\omega} \frac{\omega_0-\omega}{(\omega+\omega_0)(\omega-\omega_0)} = \frac{-1}{2\omega}$$

$\neq \pm\infty$ since $\omega \neq 0$.

(d) $$\lim_{\omega_0\to\omega} y_2(t) = \lim_{\omega_0\to\omega} \frac{t\cos\omega t}{-2\omega} =: \tilde{y}_2(t)$$

$$\tilde{y}_2''(t) = \frac{-2\omega\sin\omega t - \omega^2 t\cos\omega t}{-2\omega}$$

$$= \sin\omega t + \frac{\omega}{2} t\cos\omega t$$

$$\tilde{y}_2'' + \omega^2\tilde{y}_2 = \sin\omega t + t\cos\omega t\left(\frac{\omega}{2} - \frac{\omega^2}{2\omega}\right)$$

$$= \sin\omega t \left(= \lim_{\omega_0\to\omega} \sin\omega_0 t\right)$$

3. $y_1(x) = \frac{\sin x}{x}$

(a) $xy'' + 2y' + xy = 0, \quad x > 0$

$$y_1'(x) = \frac{x\cos x - \sin x}{x^2}$$

$$= \frac{\cos x}{x} - \frac{\sin x}{x^2}$$

$$y_1''(x) = \frac{-x\sin x - \cos x}{x^2} - \frac{x^2\cos x - 2x\sin x}{x^4}$$

$$= \frac{-x^2\sin x - 2x\cos x + 2\sin x}{x^3}$$

$$xy_1'' + 2y_1' + xy_1 =$$

$$\frac{-x^2\sin x - 2x\cos x + 2\sin x}{x^2}$$

$$+ \frac{2x\cos x - 2\sin x}{x^2}$$

$$+ \frac{x^2 \sin x}{x^2} = 0 \checkmark$$

$$y_2(x) = \frac{\cos x}{x}$$

$$y_2'(x) = \frac{-\sin x}{x} - \frac{\cos x}{x^2}$$

$$y_2''(x) = \frac{-x\cos x + \sin x}{x^2} - \frac{-x^2\sin x - 2x\cos x}{x^4}$$

$$= \frac{-x^2\cos x + x\sin x + x\sin x + 2\cos x}{x^3}$$

$$xy_2'' + 2y_2' + xy_2 =$$

$$(-x^2\cos x + 2x\sin x + 2\cos x)/x^2$$

$$+ (-2x\sin x - 2\cos x)/x^2$$

$$+ \quad x^2\cos x / x^2 \quad = 0 \checkmark$$

(b) $y_1(x) = x$, $y_1'(x) = 1$, $y_1''(x) = 0$

$$(2x-1)y_1'' - 4xy_1' + 4y \qquad (2x>1)$$

$$= 0 - 4x + 4x = 0 \checkmark$$

$y_2(x) = e^{2x}$, $y_2'(x) = 2e^{2x}$, $y_2''(x) = 4e^{2x}$

$$(2x-1)y_2'' - 4xy_2' + 4y =$$

$$(2x-1)(4e^{2x}) - 8xe^{2x} + 4e^{2x}$$

$$= 0 \checkmark$$

5. $(\cosh x)y'' + (\cos x)y' = (1+x^2)y$ for $a < x < b$ and $y(a) = y(b) = 1$.

Suppose $y(x) > 1$ for some $x \in (a,b)$. Then $\exists\ c \in (a,b)$ s.t. y is maximized over (a,b) with $y(c) > 1$, $y'(c) = 0$, and $y''(c) \leq 0$. Then,

$$(\cosh c)y''(c) + (\cos c)y'(c) = (1+c^2)y(c)$$

$$(\cosh c)y''(c) = (1+c^2)y(c)$$

This is a contradiction since $y''(c)\cosh c \leq 0$ while $(1+c^2)y(c) > 1+c^2 > 0$. Conclude that if y satisfies the differential equation on $a < x < b$, then $y(x) < 1$.

Similarly if $y(x) < 0$ for some $x \in (a,b)$, $\exists\ c \in (a,b)$ s.t. $y(c) < 0$, $y'(c) = 0$, $y''(c) \geq 0$. The differential equation reduces to:

$$0 \leq (\cosh c)y''(c) = (1+c^2)y(c) < 0.$$

From this contradiction conclude that $y(x) > 0\ \forall x \in (a,b)$.