Problem Set 2

1. $$f(x) = \begin{cases} x/|x| & , x \neq 0 \\ K & , x = 0 \end{cases}$$

K constant

$y' = f(x) \rightarrow dy = \frac{x}{|x|} dx, x \neq 0$

$$y(x) = \begin{cases} x + c_1 & , x > 0 \\ -x + c_2 & , x < 0 \end{cases}$$

To satisfy $\lim_{x \to 0^-} y(x) = \lim_{x \to 0^+} y(x)$, we must have $c_1 = c_2 = 0$.

But then $y(x) = |x|$, which is not differentiable at $x = 0$.
So no matter our choice for K, assuming that a solution exists over $x = 0$ produces this contradiction.

2. Suppose f is a continuous and bounded function on $\mathbb{R}$ and f' is continuous.

Let y be the nonzero solution of $y' = y f(y)$, $y(0) = y_0 \neq 0$.

Since $y \not\equiv 0$, we can solve $\frac{y'}{y} = f(y)$ for all $y \neq 0$. Note

$\frac{y'}{y} = (\log y)' = f(y)$

$\rightarrow |(\log y)'| \leq |f(y)| \leq M \in \mathbb{R}$.

$\rightarrow |y(x)| \leq |y_0| e^{Mx}$

For any $a > 0$, consider $\{(x, y) : |x| \leq a, |y| \leq |y_0| e^{Ma}\}$. If y does not exist on $x \in (-a, a)$, $y(a) = |y_0| e^{Ma}$, contradicting bddness.

4. Ricatti Equation:

$y'(x) = a(x) + b(x)y + c(x)y^2$

(a) Suppose y_1 is a solution and $y = y_1 + u$ is the general solution.

$$y_1' + u' = a(x) + b(x)(y_1 + u) + c(x)(y_1 + u)^2$$

$$u' = a(x) + b(x)y_1 + c(x)y_1^2 - y_1' + b(x)u + 2c(x)uy_1 + c(x)u^2$$

$$u' = b(x)u + 2c(x)uy_1 + c(x)u^2$$

$$u' + (-b(x) - 2c(x)y_1)u = c(x)u^2$$

That is, u satisfies the Bernoulli Equation with $p(x) = -b(x) - c(x)y_1$, $q(x) = c(x)$, and $n = 2$.

(b) $y' = 1 - x^2 + y^2$

$a(x) = 1 - x^2$, $b(x) = 0$, $c(x) = 1$

If $y_1(x) = x$, $y_1'(x) = 1$ and

$$y_1'(x) = 1 = 1 - x^2 + x^2 = 1 - x^2 + y_1^2.$$

So $y_1(x) = x$ is a solution.

The general solution is $y = y_1 + u$, where u satisfies the Bernoulli Equation:

$$u' - 2xu = u^2.$$

The substition is $w = u^{-1}$, so $y = y_1 + 1/w$. Putting this into the original equation

$$y_1' + \left(\frac{1}{w}\right)' = 1 - x^2 + y_1^2 + 2\frac{y_1}{w} + \frac{1}{w^2}$$

Since $y_1' = 1 - x^2 + y_1^2$, we have

$$\left(\frac{1}{w}\right)' = \frac{2y_1}{w} + \frac{1}{w^2}$$

$$-\frac{1}{w^2}w' = \frac{2x}{w} + \frac{1}{w^2}$$

$$w' = -2xw - 1$$

$$w' + 2xw = -1$$

$\therefore\ y = y_1 + \frac{1}{w}$, where

w satisfies this equation, which requires advanced methods.

5. Let $Ly = y'' + y$

$$Ly = 3\sin 2x + 3 + 4e^x$$

$$y(0) = 0 \quad y'(0) = 0$$

(Rest solution)

(a) $y'' + y = 0$ has the solution:

$$y(x) = c_1 \sin x + c_2 \cos x$$

(b) $y'' + y = 3\sin 2x \rightarrow y_1(x) = -\sin 2x$

$y'' + y = 3 \rightarrow y_2(x) = 3$

$y'' + y = 4e^x \rightarrow y_3(x) = 2e^x$

$\therefore\ Ly = y'' + y$ has the general soln:

$$y(x) = c_1 \sin x + c_2 \cos x - \sin 2x + 3 + 2e^x.$$

(c) $0 = y(0) = c_2 + 3 + 2 \rightarrow c_2 = -5$

$0 = y'(0) = c_1 - 2 + 2 \rightarrow c_1 = 0$

$$y(x) = -5\cos x - 2\sin 2x + 3 + 2e^x.$$

6. Euler's Equi-dimensional Equation: $xy'' + pxy' + qy = 0$, p, q constants

(a) $x = e^t \longleftrightarrow \ln x = t$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dt}\frac{dy}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}$$

$$= \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$0 = xy'' + pxy' + qy$$

$$= \frac{1}{x}\ddot{y} - \frac{1}{x}\dot{y} + p\dot{y} + qy$$

Multiplying through by $1/x$ yields:

$$0 = \ddot{y} + (p-1)\dot{y} + qy$$

(b) $x^2y'' + xy' + y = 0$, $\quad x = e^t$

$$0 = \ddot{y} + y \longrightarrow y(t) = c_1 \sin t + c_2 \cos t$$

$$y(x) = c_1 \sin(\ln x) + c_2 \cos(\ln x)$$