Exercises on singular value decomposition

Problem 29.1: (Based on 6.7 #4. *Introduction to Linear Algebra:* Strang) Verify that if we compute the singular value decomposition $A = U\Sigma V^T$ of the Fibonacci matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$,

$$\Sigma = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{\sqrt{5}-1}{2} \end{bmatrix}.$$

Problem 29.2: (6.7 #11.) Suppose A has orthogonal columns $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ of lengths $\sigma_1, \sigma_2, \ldots, \sigma_n$. Calculate A^TA. What are U, Σ, and V in the SVD?

29.1

$$A^TA = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

has eigenvalues $\sigma^2 = \frac{3 \pm \sqrt{5}}{2}$.

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}$$

$$\left(\frac{\sqrt{5}-1}{2}\right)^2 = \frac{6-2\sqrt{5}}{2} = \frac{3-\sqrt{5}}{2}$$

confirms that $\sigma = \frac{1 \pm \sqrt{5}}{2}$ (since we take $\sigma \geqslant 0$).

29.2

$$A^TA = \begin{bmatrix} - & w_1^T & - \\ & \vdots & \\ - & w_n^T & - \end{bmatrix} \begin{bmatrix} | & & | \\ w_1 & \cdots & w_n \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_n^2 \end{bmatrix} = V\Sigma^T\Sigma V^T$$

$$A = U\Sigma V^T$$

$$= \begin{bmatrix} | & & | \\ w_1/\sigma_1 & \cdots & w_n/\sigma_n \\ | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix} I$$

is a possible SVD of A, where $V = V^T = I$ is the $n \times n$ identity.