Exercises on Markov matrices; Fourier series

Problem 24.1: (6.4 #7. *Introduction to Linear Algebra:* Strang)

a) Find a symmetric matrix $\begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix}$ that has a negative eigenvalue.

b) How do you know it must have a negative pivot?

c) How do you know it can't have two negative eigenvalues?

Problem 24.2: (6.4 #23.) Which of these classes of matrices do A and B belong to: invertible, orthogonal, projection, permutation, diagonalizable, Markov?

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad B = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Which of these factorizations are possible for A and B: LU, QR, $S\Lambda S^{-1}$, or $Q\Lambda Q^T$?

24.1

a) Set $b = 2$.
$0 = (1-\lambda)^2 - 2^2$
$\lambda = -1, 3$

b) The number of positive pivots = number ~~of~~ of positive eigenvalues (pg. 333).

c) Same answer as part b.

24.2

	A	B
invertible	✓	✗
orthogonal	✓	✗
projection	✗	✓
permutation	✓	✗
diagonalizable	✓	✓
LU	✗	✓
QR	✓	✗
$S\Lambda S^{-1}$	✓	✓
$Q\Lambda Q^T$	✓	✓

Both A and B are Markov (forgot to put in the table).

Problem 24.3: (8.3 #11.) Complete A to a Markov matrix and find the steady state eigenvector. When A is a symmetric Markov matrix, why is $\mathbf{x}_1 = (1, \ldots, 1)$ its steady state?

$$A = \begin{bmatrix} .7 & .1 & .2 \\ .1 & .6 & .3 \\ \_ & \_ & \_ \end{bmatrix}.$$

24.3

$$A = \begin{bmatrix} .7 & .1 & .2 \\ .1 & .6 & .3 \\ .2 & .3 & .5 \end{bmatrix}$$

The eigenvector x_1 with $\lambda = 1$ is the steady state.

$$\left[\begin{array}{ccc|c} -.3 & .1 & .2 & 0 \\ .1 & -.4 & .3 & 0 \\ .2 & .3 & -.5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$ is the steady state vector for this A.

If A is a symmetric markov matrix, then since the columns add to 1 (markov), the rows must also add to 1 (symmetry) and the rows of $A - I$ add to 0. Then for (λ, x) with $\lambda = 1$,

$$(A - I)x = Ax - x = x - x = \vec{0}$$

has the solution

$$x = (1, 1, \ldots, 1)^T.$$