18.06SC Unit 3 Exam

1 (34 pts.) (a) If a square matrix A has all n of its *singular values* equal to 1 in the SVD, what basic classes of matrices does A belong to? (Singular, symmetric, orthogonal, positive definite or semidefinite, diagonal)

(b) Suppose the (orthonormal) columns of H are eigenvectors of B:

$$H = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \qquad H^{-1} = H^{\mathrm{T}}$$

The eigenvalues of B are $\lambda = 0, 1, 2, 3$. Write B as the product of 3 specific matrices. Write $C = (B + I)^{-1}$ as the product of 3 matrices.

(c) Using the list in question (a), which basic classes of matrices do B and C belong to? (Separate question for B and C)

(a) A is not singular: $|\det A| = \left|\prod_{i=1}^{n} \sigma_i\right| = 1$

A may not be symmetric: Consider $A = U\Sigma V^T$ with

$$A = \underbrace{\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T}_{V^T} = \underbrace{\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}}_{U}$$

$$A^T = U^T \neq U = A$$

A is orthogonal. Since A is square, $\Sigma = I$ so

$$A^TA = V\Sigma^TU^TU\Sigma V^T = VV^T = I, \quad AA^T = U\Sigma V^TV\Sigma^TU^T = I$$

A need not be positive (semi)definite. Consider

$$A = U\Sigma V^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \text{ which has}$$

$$\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0 \quad \text{and } \lambda = \pm i.$$

A need not be diagonal. Consider the A used to show A need not be symmetric.

(b) $BH = [Bh_0\ Bh_1\ Bh_2\ Bh_3] = [\lambda_0 h_0\ \lambda_1 h_1\ \lambda_2 h_2\ \lambda_3 h_3]$

$$= [h_0\ h_1\ h_2\ h_3]\begin{bmatrix}\lambda_0 & & & \\ & \lambda_1 & & \\ & & \lambda_2 & \\ & & & \lambda_3\end{bmatrix}$$

$$= H\Lambda$$

$B = H\Lambda H^T$ with $\Lambda = \begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&2&0\\0&0&0&3\end{bmatrix}$

For each eigenvalue λ of B with corresponding eigenvector h, $B+I$ has the eigenvalue $\lambda+1$ with the same corresponding eigenvector h: $(B+I)h = Bh+h = (\lambda+1)h$.

Then $B+I = H(\Lambda+I)H^T$ which gives

$$(B+I)^{-1} = H(\Lambda+I)^{-1}H^T$$

with $(\Lambda+I)^{-1} = \begin{bmatrix}1&&&\\&2&&\\&&3&\\&&&4\end{bmatrix}^{-1} = \begin{bmatrix}1&&&\\&1/2&&\\&&1/3&\\&&&1/4\end{bmatrix}$

(c) B is singular, symmetric, and positive semidefinite.

C is symmetric and positive definite.

2 (33 pts.) (a) Find three eigenvalues of A, and an eigenvector matrix S:

$$A = \begin{bmatrix} -1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Explain why $A^{1001} = A$. Is $A^{1000} = I$? Find the three diagonal entries of e^{At}.

(c) The matrix $A^{\mathrm{T}}A$ (for the same A) is

$$A^{\mathrm{T}}A = \begin{bmatrix} 1 & -2 & -4 \\ -2 & 4 & 8 \\ -4 & 8 & 42 \end{bmatrix}.$$

How many eigenvalues of $A^{\mathrm{T}}A$ are positive? zero? negative? (Don't compute them but explain your answer.) Does $A^{\mathrm{T}}A$ have the same eigenvectors as A?

(a) $\lambda = -1, 0, 1$

$A\begin{bmatrix}1\\0\\0\end{bmatrix} = -1\begin{bmatrix}1\\0\\0\end{bmatrix}$, $A\begin{bmatrix}2\\1\\0\end{bmatrix} = 0\begin{bmatrix}2\\1\\0\end{bmatrix}$, $A\begin{bmatrix}7\\5\\1\end{bmatrix} = 1\begin{bmatrix}7\\5\\1\end{bmatrix}$

$S = \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$, $AS = S\Lambda$, $A = S\Lambda S^{-1}$, $\Lambda = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $A^{1001} = S\Lambda^{1001}S^{-1} = S\Lambda S^{-1} = A$

$A^{1000} = S\Lambda^{1000}S^{-1} = S\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}S^{-1}$

If $A^{1000} = I$, $S\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = S$. But $S\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq S$, so we must not have $A^{1000} = I$

$$e^{At} = e^{S\Lambda S^{-1}t} = I + S\Lambda S^{-1}t + \frac{1}{2}(S\Lambda S^{-1}t)^2 + \frac{1}{3!}(S\Lambda S^{-1}t)^3 + \dots$$
$$= I + S\Lambda S^{-1}t + \frac{1}{2}S\Lambda^2 S^{-1}t^2 + \frac{1}{3!}S\Lambda^3 S^{-1}t^3 + \dots$$
$$= S(I + \Lambda t + \frac{1}{2}\Lambda^2 t^2 + \frac{1}{3!}\Lambda^3 t^3 + \frac{1}{4!}\Lambda^4 t^4 + \dots)S^{-1}$$

For an eigenvalue λ of A with eigenvector x,

$$e^{At}x = S(I + \Lambda t + \frac{1}{2}\Lambda^2 t^2 + \frac{1}{3!}\Lambda^3 t^3 + \dots)S^{-1}x$$
$$= x + Atx + \frac{A^2t^2x}{2} + \frac{A^3t^3x}{3!} + \dots$$
$$= x + t\lambda x + \frac{t^2\lambda^2 x}{2} + \frac{t^3\lambda^3 x}{3!} + \dots$$
$$= \left(1 + t\lambda + \frac{t^2\lambda^2}{2} + \frac{t^3\lambda^3}{3!} + \dots\right)x = e^{\lambda t}x$$

That is, $e^{\lambda t}$ is an eigenvalue of e^{At} with eigenvector x. so,

$$e^{At} = Se^{\Lambda t}S^{-1} \quad \text{where} \quad e^{\Lambda t} = \begin{bmatrix} e^{-t} & \\ & e^{t} \end{bmatrix}$$

(c) A^TA is symmetric so the signs of the eigenvalues are the same as the pivots.

$$\begin{bmatrix} 1 & -2 & -4 \\ -2 & 4 & 8 \\ -4 & 8 & 42 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 26 \end{bmatrix}$$

$\therefore$ 2 positive eigenvalues and 1 zero eigenvalue.

3 (33 pts.) Suppose the n by n matrix A has n orthonormal eigenvectors $q_1, \ldots, q_n$ and n positive eigenvalues $\lambda_1, \ldots, \lambda_n$. Thus $Aq_j = \lambda_j q_j$.

(a) What are the eigenvalues and eigenvectors of A^{-1}? *Prove that your answer is correct.*

(b) Any vector b is a combination of the eigenvectors:

$$b = c_1 q_1 + c_2 q_2 + \cdots + c_n q_n \,.$$

What is a quick formula for c_1 using orthogonality of the q's?

(c) The solution to $Ax = b$ is also a combination of the eigenvectors:

$$A^{-1} b = d_1 q_1 + d_2 q_2 + \cdots + d_n q_n \,.$$

What is a quick formula for d_1? You can use the c's even if you didn't answer part (b).

(a) A^{-1} has eigenvalues $1/\lambda_1, 1/\lambda_2, \ldots, 1/\lambda_n$ corresponding to the same eigenvectors $q_1, \ldots, q_n$.

Proof: Let λ_i be an eigenvalue of A with eigenvector q_i. Since $Aq_i = \lambda_i q_i$, we have $q_i = \lambda_i A^{-1} q_i$. It follows that $\frac{1}{\lambda_i} q_i = A^{-1} q_i$.

(b) $b = c_1 q_1 + c_2 q_2 + \cdots + c_n q_n$

$q_1^T b = c_1 q_1^T q_1 + c_2 q_1^T q_2 + \cdots + c_n q_1^T q_n$

$c_1 = \dfrac{q_1^T b}{\|q_1\|^2} = \dfrac{q_1^T b}{1} = q_1^T b$.

(c) $A^{-1} b = d_1 q_1 + d_2 q_2 + \cdots + d_n q_n$

$b = d_1 A q_1 + d_2 A q_2 + \cdots + d_n A q_n$

$c_1 q_1 + \cdots + c_n q_n = d_1 \lambda_1 q_1 + \cdots + d_n \lambda_n q_n$

$c_1 q_1^T q_1 = d_1 \lambda_1 q_1^T q_1 \longrightarrow d_1 = c_1 / \lambda_1$.