Exercises on properties of determinants

Problem 18.1: (5.1 #10. *Introduction to Linear Algebra:* Strang) If the entries in every row of a square matrix A add to zero, solve $A\mathbf{x} = \mathbf{0}$ to prove that $\det A = 0$. If those entries add to one, show that $\det(A - I) = 0$. Does this mean that $\det A = 1$?

Problem 18.2: (5.1 #18.) Use row operations and the properties of the determinant to calculate the three by three "Vandermonde determinant":

$$\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (b-a)(c-a)(c-b).$$

18.1

$x = [1, 1, 1, 1]^T$ is a nonzero solution to $Ax = \vec{0}$. This means $\det A = 0$. If they add to one, then this same x satisfies

$$Ax = x$$
$$(A - I)x = \vec{0}$$

Since there is a nonzero solution to $(A - I)x = 0$, $\det(A - I) = 0$.

This doesn't mean $\det A = 1$. Consider the counterexample

$$A = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$$ with $\det A = 2$.

18.2

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & (c-b)(c-a) \end{vmatrix}$$

$$= (b-a)(c-b)(c-a)$$

Adding a multiple of one row to another doesn't change the determinant. The determinant of a diagonal matrix is the product of the diagonal elements.