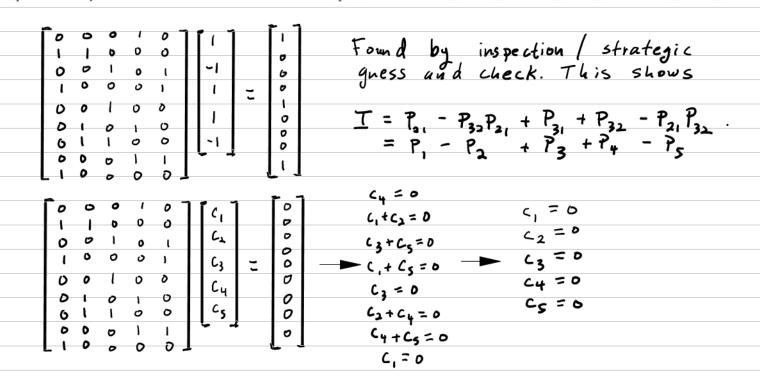
Exercises on matrix spaces; rank 1; small world graphs

Example 5 There are six 3 by 3 permutation matrices. Here they are without the zeros:

Problem 11.1: [Optional] (3.5 #41. *Introduction to Linear Algebra:* Strang) Write the 3 by 3 identity matrix as a combination of the other five permutation matrices. Then show that those five matrices are linearly independent. (Assume a combination gives $c_1P_1 + \cdots + c_5P_5 = 0$ and check entries to prove c_i is zero.) The five permutation matrices are a basis for the subspace of three by three matrices with row and column sums all equal.

$$I = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 \end{bmatrix} \qquad P_{21} = \begin{bmatrix} & 1 & \\ & 1 & \\ & & 1 \end{bmatrix} \qquad P_{32}P_{21} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$$
$$P_{31} = \begin{bmatrix} & & 1 \\ & 1 & \\ & 1 & \end{bmatrix} \qquad P_{32} = \begin{bmatrix} 1 & & \\ & & 1 \\ & & 1 \end{bmatrix} \qquad P_{21}P_{32} = \begin{bmatrix} & & 1 \\ & & 1 \\ & & 1 \end{bmatrix}.$$



Problem 11.2: (3.6 #31.) \mathbf{M} is the space of three by three matrices. Multiply each matrix X in \mathbf{M} by:

$$A = \left[\begin{array}{rrr} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{array} \right]$$

Notice that
$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- a) Which matrices X lead to AX = 0?
- b) Which matrices have the form AX for some matrix X?
- c) Part (a) finds the "nullspace" of the operation AX and part (b) finds the "column space." What are the dimensions of those two subspaces of **M**? Why do the dimensions add to (n-r)+r=9?

[1 0 -1] [a b c] [a-q b-h (-i]
$$a = d = q$$

[a) -1 1 0 d e f = d-a e-b f-c $b = e = h$

[a-q b-h (-i] $a = d = q$

[b-h (-i] $b = e = h$

[a-q b-h (-i] $c = f = i$

[b-h (-i] $c = f = i$

[a-q b-h (-i] $c = f = i$

c) The dimension of the null space is 3 and the column space is 6. They add up to 9 because the set of 3×3 matrices is a 9 dimensional vector space.