Exercises on symmetric matrices and positive definiteness

Problem 25.1: (6.4 #10. *Introduction to Linear Algebra:* Strang) Here is a quick "proof" that the eigenvalues of all real matrices are real:

False Proof: $A\mathbf{x} = \lambda\mathbf{x}$ gives $\mathbf{x}^T A\mathbf{x} = \lambda\mathbf{x}^T\mathbf{x}$ so $\lambda = \frac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T\mathbf{x}}$ is real.

There is a hidden assumption in this proof which is not justified. Find the flaw by testing each step on the 90° rotation matrix:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

with $\lambda = i$ and $\mathbf{x} = (i, 1)$.

Problem 25.2: (6.5 #32.) A *group* of nonsingular matrices includes AB and A^{-1} if it includes A and B. "Products and inverses stay in the group." Which of these are groups?

a) Positive definite symmetric matrices A.

b) Orthogonal matrices Q.

c) All exponentials e^{tA} of a fixed matrix A.

d) Matrices D with determinant 1.

25.1

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ i \end{bmatrix} = i\begin{bmatrix} i \\ 1 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} i & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} = i\begin{bmatrix} i & 1 \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$0 = i(0)$$

$$\frac{0}{0} = i \quad \times \quad \text{(Division by 0)}$$

25.2

(a) Not a group.

$A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ is SPD

$B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ is SPD

$AB = \begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}$ is not SPD

(b) Is a group.

$(Q_1 Q_2)^T (Q_1 Q_2) = Q_2^T Q_1^T Q_1 Q_2 = I$

If Q is orthogonal, Q^{-1} is too since $(Q^{-1})^T = (Q^T)^T = Q$.

(c) Is a group.

$$e^{t_1 A} e^{t_2 A} = e^{(t_1+t_2)A}$$

$$e^{tA} e^{-tA} = e^{-tA} e^{tA} = e^{0} = I.$$

(d) Is a group.

If $\det A = \det B = 1$,

$$\det(AB) = \det A \det B = 1 \cdot 1 = 1$$

$$1 = \det(I) = \det(AA^{-1}) = \det A \det A^{-1} = \det A^{-1}$$