Exercises on left and right inverses; pseudoinverse

Problem 32.1: Find a right inverse for $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Problem 32.2: Does the matrix $A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$ have a left inverse? A right inverse? A pseudoinverse? If the answer to any of these questions is "yes", find the appropriate inverse.

32.1

One choice is $A^{-1}_{right} = A^T(AA^T)^{-1}$ so that $AA^{-1}_{right} = AA^T(AA^T)^{-1} = I$.

$$AA^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(AA^T)^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^T(AA^T)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \\ 1/2 & 0 \end{bmatrix}$$

Check: $AA^T(AA^T)^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \\ 1/2 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

32.2

A does not have a right inverse since the rows of A are dependent. A does not have a left inverse since the columns of A are dependent.

To find the pseudoinverse A^+ of A, use $A = U\Sigma V^T$

$$A = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}\begin{bmatrix} 5\sqrt{5} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 4/5 & 3/5 \\ -3/5 & 4/5 \end{bmatrix}$$

$$A^+ = V\Sigma^+U^T$$

$$= \begin{bmatrix} 4/5 & -3/5 \\ 3/5 & 4/5 \end{bmatrix}\begin{bmatrix} 1/5\sqrt{5} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

$$= \frac{1}{125}\begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

$$= \frac{1}{125}\begin{bmatrix} 4 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \frac{1}{125}\begin{bmatrix} 4 & 8 \\ 3 & 6 \end{bmatrix} = \frac{1}{125}A^T.$$