## Exercises on orthogonal matrices and Gram-Schmidt

**Problem 17.1:** (4.4 #10.b *Introduction to Linear Algebra:* Strang)
Orthonormal vectors are automatically linearly independent.
Matrix Proof: Show that $Q\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$. Since $Q$ may be rectangular, you can use $Q^T$ but not $Q^{-1}$.

**Problem 17.2:** (4.4 #18) Given the vectors **a**, **b** and **c** listed below, use the Gram-Schmidt process to find orthogonal vectors **A**, **B**, and **C** that span the same space.

$$\mathbf{a} = (1, -1, 0, 0), \mathbf{b} = (0, 1, -1, 0), \mathbf{c} = (0, 0, 1, -1).$$

Show that $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ and $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ are bases for the space of vectors perpendicular to $\mathbf{d} = (1, 1, 1, 1)$.

**17.1** Suppose $Qx = \vec{0}$.

Then $Q^T Q x = Q^T \vec{0}$

$$Ix = \vec{0}$$

$$x = \vec{0}.$$

**17.2**

$A = a$

$$B = b - \frac{A^T b}{A^T A} A = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \\ 0 \end{bmatrix}$$

$$C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} - 0A + \frac{2}{3} \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ -1 \end{bmatrix}$$

The row space of

$d = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is 1 dimensional so the nullspace of $d$ is $4-1=3$ dimensional. Since

$\{A, B, C\}$ and $\{a, b, c\}$ are each linearly independent sets of 3 vectors that are orthogonal to $d$, conclude that each set forms a basis for the nullspace of $d$.