Exercises on similar matrices and Jordan form

Problem 28.1: (6.6 #12. *Introduction to Linear Algebra:* Strang) These Jordan matrices have eigenvalues 0, 0, 0, 0. They have two eigenvectors; one from each block. However, their block sizes don't match and they are *not similar*:

$$J = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ and } K = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right].$$

For a generic matrix M, show that if $JM = MK$ then M is not invertible and so J is not similar to K.

Problem 28.2: (6.6 #20.) Why are these statements all true?

a) If A is similar to B then A^2 is similar to B^2.

b) A^2 and B^2 can be similar when A and B are not similar (try $\lambda = 0, 0$.)

c) $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ is similar to $\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$.

d) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ is not similar to $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$.

e) Given a matrix A, let B be the matrix obtained by exchanging rows 1 and 2 of A and then exchanging columns 1 and 2 of A. Show that A is similar to B.

28.1 For $M = (m_{ij})$,

$$JM = \begin{bmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$MK = \begin{bmatrix} 0 & m_{11} & m_{12} & 0 \\ 0 & m_{21} & m_{22} & 0 \\ 0 & m_{31} & m_{32} & 0 \\ 0 & m_{41} & m_{42} & 0 \end{bmatrix}$$

If $JM = MK$,

$m_{11} = m_{22} = 0$, $m_{21} = 0$, $m_{31} = m_{42} = 0$, and $m_{41} = 0$. The first column of M is all 0's, so M is not invertible. If J were similar to K, $\exists\, M$ s.t. $K = M^{-1}JM \iff MK = JM$. Since no such M can exist, J is not similar to K.

28.2

a) $B = M^{-1}AM \Rightarrow B^2 = M^{-1}AMM^{-1}AM = M^{-1}A^2M$.

b) Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. If $B = M^{-1}AM$, then $AM = MB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, which implies $m_{21} = m_{22} = 0$. But this means M is not invertible. So A and B are not similar. However $A^2 = B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so for $M = I$, $B^2 = M^{-1}A^2M$.

c) $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ and $\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$ have the eigenvalues, $\lambda = 3, 4$ and these eigenvalues are not repeated.

Also, taking $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $M^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, we have

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}.$$

d) While these matrices have the same eigenvalues, $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ has only one independent eigenvector, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, while $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ has two: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

e) Let $P = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \ddots & \\ 0 & 0 & \cdots & & 0 & 1 \end{bmatrix}$ Then $P = P^{-1}$ and $B = PAP$ so $B = P^{-1}AP$ and we conclude that A and B are similar.