Exercises on the four fundamental subspaces

Problem 10.1: (3.6 #11. *Introduction to Linear Algebra:* Strang) A is an m by n matrix of rank r. Suppose there are right sides $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has *no solution*.

a) What are all the inequalities ($<$ or $\leq$) that must be true between m, n, and r?

b) How do you know that $A^T\mathbf{y} = \mathbf{0}$ has solutions other than $\mathbf{y} = \mathbf{0}$?

a) We must have $r < m$. It is possible for $r = n$, but this necessitates $m > n$. So $r \leq n$. Never $r > m, n$.

b) A reduces to a matrix with a row of 0's, which means A^T has a column of zeros, which corresponds to a free variable. Also, the dimension of the left nullspace, $N(A^T)$, is $m - r > 0$ since $m > r$. This means $\exists$ nonzero soln's to $A^T y = 0$.

Problem 10.2: (3.6 #24.) $A^T\mathbf{y} = \mathbf{d}$ is solvable when $\mathbf{d}$ is in which of the four subspaces? The solution $\mathbf{y}$ is unique when the __________ contains only the zero vector.

$A^T y = d$ is solvable (for y) when d is in the row space of A. The solution is unique when the left null space contains only the zero vector.