Exercises on orthogonal vectors and subspaces

Problem 16.1: (4.1 #7. *Introduction to Linear Algebra:* Strang) For every system of m equations with no solution, there are numbers $y_1, ..., y_m$ that multiply the equations so they add up to $0 = 1$. This is called *Fredholm's Alternative:*

> Exactly one of these problems has a solution:
> $A\mathbf{x} = \mathbf{b}$ OR $A^T\mathbf{y} = \mathbf{0}$ with $\mathbf{y}^T\mathbf{b} = 1$.

If $\mathbf{b}$ is not in the column space of A it is not orthogonal to the nullspace of A^T. Multiply the equations $x_1 - x_2 = 1$, $x_2 - x_3 = 1$ and $x_1 - x_3 = 1$ by numbers y_1, y_2 and y_3 chosen so that the equations add up to $0 = 1$.

$$\begin{aligned} & 1\cdot(x_1 - x_2 = 1) \\ + \; & 1\cdot(x_2 - x_3 = 1) \\ + \; & -1\cdot(x_1 - x_3 = 1) \\ \hline & 0 = 1 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{b} \qquad ✗$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{0}} \qquad ✓$$

$$\underbrace{\begin{bmatrix} 1 & 1 & -1 \end{bmatrix}}_{y^T} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{b} = 1 \qquad ✓$$

Problem 16.2: (4.1#32.) Suppose I give you four nonzero vectors $\mathbf{r}$, $\mathbf{n}$, $\mathbf{c}$ and $\mathbf{l}$ in $\mathbb{R}^2$.

a) What are the conditions for those to be bases for the four fundamental subspaces $C(A^T)$, $N(A)$, $C(A)$, and $N(A^T)$ of a 2 by 2 matrix?

b) What is one possible matrix A?

a) $r^T n = n^T r = 0$
$c^T \ell = \ell^T c = 0$
Since $r, n, c, \ell \neq \vec{0}$:
The rows are linearly dependent.
The columns are linearly dependent.

Note that this satisfies the FTLA:
$\dim C(A) + \dim N(A) = 1 + 1 = 2 = n$
$\dim C(A^T) + \dim N(A^T) = 1 + 1 = 2 = m$

b) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has $r = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $n = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$,
$c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\ell = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$