Problem 6.1: (3.1 #30. *Introduction to Linear Algebra:* Strang) Suppose $\mathbf{S}$ and $\mathbf{T}$ are two subspaces of a vector space $\mathbf{V}$.

a) **Definition:** The sum $\mathbf{S}+\mathbf{T}$ contains all sums $\mathbf{s}+\mathbf{t}$ of a vector $\mathbf{s}$ in $\mathbf{S}$ and a vector $\mathbf{t}$ in $\mathbf{T}$. Show that $\mathbf{S}+\mathbf{T}$ satisfies the requirements (addition and scalar multiplication) for a vector space.

b) If $\mathbf{S}$ and $\mathbf{T}$ are lines in $\mathbf{R}^m$, what is the difference between $\mathbf{S}+\mathbf{T}$ and $\mathbf{S}\cup\mathbf{T}$? That union contains all vectors from $\mathbf{S}$ and $\mathbf{T}$ or both. Explain this statement: *The span of* $\mathbf{S}\cup\mathbf{T}$ *is* $\mathbf{S}+\mathbf{T}$.

a) Let $v_1 = s_1 + t_1$, $v_2 = s_2 + t_2 \in S + T$ and c be a scalar.

$v_1 + v_2 = s_1 + t_1 + s_2 + t_2 = (s_1 + s_2) + (t_1 + t_2) \in S + T$ since $s_1 + s_2 \in S$ and $t_1 + t_2 \in T$.

$cv_1 = c(s_1 + t_1) = cs_1 + ct_1 \in S + T$ since $cs_1 \in S$ and $ct_1 \in T$.

b) $S+T$ contains all vectors $s+t$ where $s \in S$ and $t \in T$. $S \cup T$ contains all vectors that are in S or in T. Geometrically, this means $S+T$ is a plane in $\mathbb{R}^m$ (unless $S=T$) while $S \cup T$ is two lines in $\mathbb{R}^m$ (again unless $S=T$). $S \cup T$ spans $S+T$ since for any $v \in S+T$, $\exists$ $s \in S$ and $t \in T$ s.t. $s+t=v$ (this is immediate from the definition of $S+T$).

Problem 6.2: (3.2 #18.) The plane $x - 3y - z = 12$ is parallel to the plane $x - 3y - z = 0$. One particular point on this plane is $(12, 0, 0)$. All points on the plane have the form (fill in the first components)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \ \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \ \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \ \\ 0 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$ since $x = 12 + 3y + z$.

Problem 6.3: (3.2 #36.) How is the nullspace $\mathbf{N}(C)$ related to the spaces $\mathbf{N}(A)$ and $\mathbf{N}(B)$, if $C = \begin{bmatrix} A \\ B \end{bmatrix}$?

$N(C) = N(A) \cap N(B)$ since in order for $Cv = 0$ we need:

$$0 = Cv \begin{bmatrix} A \\ B \end{bmatrix} v = \begin{bmatrix} Av \\ Bv \end{bmatrix}.$$

Then we must have $Av = 0$ and $Bv = 0$ (of appropriate dimensions, also we must assume A and B have the same number of columns).