Exercises on eigenvalues and eigenvectors

Problem 21.1: (6.1 #19. *Introduction to Linear Algebra:* Strang) A three by three matrix B is known to have eigenvalues 0, 1 and 2. This information is enough to find three of these (give the answers where possible):

a) The rank of B

b) The determinant of $B^T B$

c) The eigenvalues of $B^T B$

d) The eigenvalues of $(B^2 + I)^{-1}$

Problem 21.2: (6.1 #29.) Find the eigenvalues of A, B, and C when

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

21.1

a) $\text{rank}(B) = 2$

b) $\det(B^T B) = \det B^T \det B = 0.$

d) $(0^2 + 1)^{-1} = 1,$

$(1^2 + 1)^{-1} = \tfrac{1}{2},$ and

$(2^2 + 1)^{-1} = \tfrac{1}{5}$ are eigenvalues

of $(B^2 + I)^{-1}$.

21.2

a) $\lambda = 1, 4, 6$

b) $0 = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 3 & 0 & -\lambda \end{vmatrix} = -\lambda(2-\lambda)(-\lambda) - 3(2-\lambda) = (2-\lambda)(\lambda^2 - 3)$

$\lambda = 2, \pm\sqrt{3}$

c)

$$0 = \begin{vmatrix} 2-\lambda & 2 & 2 \\ 2 & 2-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)\left[(2-\lambda)^2 - 4\right] - 2\left[2(2-\lambda) - 4\right] + 2\left[4 - 2(2-\lambda)\right]$$

$$= (2-\lambda)(4 - 4\lambda + \lambda^2 - 4) + 2\left[4 - 2(2-\lambda)\right] + 2\left[4 - 2(2-\lambda)\right]$$

$$= (2-\lambda)(\lambda^2 - 4\lambda) + 4\lambda + 4\lambda$$

$$= 2\lambda^2 - 8\lambda - \lambda^3 + 4\lambda^2 + 8\lambda$$

$$= -\lambda^3 + 6\lambda^2 = \lambda^2(6-\lambda)$$

$$\lambda = 0, 0, 6.$$