Exercises on change of basis; image compression

Problem 31.1: Verify that the vectors of the Haar wavelet basis, given in lecture, are orthogonal. Adjust their lengths so that the resulting basis vectors are orthonormal.

Problem 31.2: We can think of the set of all two by two matrices with real valued entries as a vector space. Describe two different bases for this space. Is one of your bases better than the other for describing diagonal matrices? What about triangular matrices? Symmetric matrices?

31.1

The Haar wavelet basis

$$\underset{h_1}{\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix}},\underset{h_2}{\begin{bmatrix}1\\1\\1\\1\\-1\\-1\\-1\\-1\end{bmatrix}},\underset{h_3}{\begin{bmatrix}1\\1\\-1\\-1\\0\\0\\0\\0\end{bmatrix}},\underset{h_4}{\begin{bmatrix}0\\0\\0\\0\\1\\1\\-1\\-1\end{bmatrix}},\underset{h_5}{\begin{bmatrix}1\\-1\\0\\0\\0\\0\\0\\0\end{bmatrix}},\ldots,\underset{h_8}{\begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\-1\end{bmatrix}}$$

h_1 is orthogonal to all others since for $h_2,\ldots,h_8$ there are an equal number of 1's as -1's and any other entries are 0.

$$h_6=\begin{bmatrix}0\\0\\1\\-1\\0\\0\\0\\0\end{bmatrix}\quad h_7=\begin{bmatrix}0\\0\\0\\0\\1\\-1\\0\\0\end{bmatrix}$$

Quick mental calculations show that all $h_i \cdot h_j = 0$ for $i \neq j$.

The orthonormal basis is:

$$\left\{\frac{h_1}{\sqrt{8}},\frac{h_2}{\sqrt{8}},\frac{h_3}{2},\frac{h_4}{2},\frac{h_5}{\sqrt{2}},\frac{h_6}{\sqrt{2}},\frac{h_7}{\sqrt{2}},\frac{h_8}{\sqrt{2}}\right\}$$

31.2

$$\left\{\underset{M_1}{\begin{bmatrix}1&0\\0&0\end{bmatrix}},\underset{M_2}{\begin{bmatrix}0&1\\0&0\end{bmatrix}},\underset{M_3}{\begin{bmatrix}0&0\\1&0\end{bmatrix}},\underset{M_4}{\begin{bmatrix}0&0\\0&1\end{bmatrix}}\right\}$$

Diagonal: $\{M_1, M_4\}$

Upper Triangular: $\{M_1, M_2, M_4\}$

Lower Triangular: $\{M_1, M_3, M_4\}$

Symmetric: $\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix}\right\}$