Exercises on differential equations and e^{At}

Problem 23.1: (6.3 #14.a *Introduction to Linear Algebra:* Strang) The matrix in this question is skew-symmetric ($A^T = -A$) :

$$\frac{d\mathbf{u}}{dt} = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \mathbf{u} \quad \text{or} \quad \begin{aligned} u_1' &= cu_2 - bu_3 \\ u_2' &= au_3 - cu_1 \\ u_3' &= bu_1 - au_2. \end{aligned}$$

Find the derivative of $||\mathbf{u}(t)||^2$ using the definition:

$$||\mathbf{u}(t)||^2 = u_1^2 + u_2^2 + u_3^2.$$

What does this tell you about the rate of change of the length of **u**? What does this tell you about the range of values of $\mathbf{u}(t)$?

Problem 23.2: (6.3 #24.) Write $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ as $S\Lambda S^{-1}$. Multiply $Se^{\Lambda t}S^{-1}$ to find the matrix exponential e^{At}. Check your work by evaluating e^{At} and the derivative of e^{At} when $t = 0$.

23.1

$$\frac{d}{dt}||\vec{u}(t)||^2 = \frac{d}{dt}\left[u_1^2 + u_2^2 + u_3^2\right]$$

$$= 2u_1(cu_2 - bu_3) + 2u_2(au_3 - cu_1) + 2u_3(bu_1 - au_2)$$

$$= 2cu_1u_2 - 2bu_1u_3 + 2au_2u_3 - 2cu_1u_2 + 2bu_1u_3 - 2au_2u_3$$

$$= 0.$$

The rate of change of the length of $\vec{u}$ is 0. The values of $\vec{u}$ must satisfy $u_1^2 + u_2^2 + u_3^2 = c$, where $c = ||\vec{u}(0)||$. The values of $\vec{u}(t)$ stay on a sphere of radius $||\vec{u}(0)||$.

23.2 $\lambda = 1, 3$

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 2 & 0 \end{array}\right], \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$2x = y$

$$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right], \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$A = S \Lambda S^{-1}$$

$$e^{At} = S e^{\Lambda t} S^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} e^t & e^{3t} \\ 0 & 2e^{3t} \end{bmatrix} \begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} e^t & \frac{1}{2}e^{3t} - \frac{1}{2}e^t \\ 0 & e^{3t} \end{bmatrix}.$$

Check: Set $t = 0$.

$$e^{At}\Big|_{t=0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left[\frac{d}{dt} e^{At}\right]\Big|_{t=0} = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = A$$