Exercises on projection matrices and least squares

Problem 16.1: (4.3 #17. *Introduction to Linear Algebra:* Strang) Write down three equations for the line $b = C + Dt$ to go through $b = 7$ at $t = -1$, $b = 7$ at $t = 1$, and $b = 21$ at $t = 2$. Find the least squares solution $\hat{\mathbf{x}} = (C, D)$ and draw the closest line.

Problem 16.2: (4.3 #18.) Find the projection $\mathbf{p} = A\hat{\mathbf{x}}$ in the previous problem. This gives the three heights of the closest line. Show that the error vector is $\mathbf{e} = (2, -6, 4)$. Why is $P\mathbf{e} = \mathbf{0}$?

Problem 16.3: (4.3 #19.) Suppose the measurements at $t = -1, 1, 2$ are the errors 2, -6, 4 in the previous problem. Compute $\hat{\mathbf{x}}$ and the closest line to these new measurements. Explain the answer: $\mathbf{b} = (2, -6, 4)$ is perpendicular to __________ so the projection is $\mathbf{p} = \mathbf{0}$.

Problem 16.4: (4.3 #20.) Suppose the measurements at $t = -1, 1, 2$ are $\mathbf{b} = (5, 13, 17)$. Compute $\hat{\mathbf{x}}$ and the closest line and $\mathbf{e}$. The error is $\mathbf{e} = \mathbf{0}$ because this $\mathbf{b}$ is __________.

Problem 16.5: (4.3 #21.) Which of the four subspaces contains the error vector $\mathbf{e}$? Which contains $\mathbf{p}$? Which contains $\hat{\mathbf{x}}$? What is the nullspace of A?

Problem 16.6: (4.3 #22.) Find the best line $C + Dt$ to fit $b = 4, 2, -1, 0, 0$ at times $t = -2, -1, 0, 1, 2$.

16.1

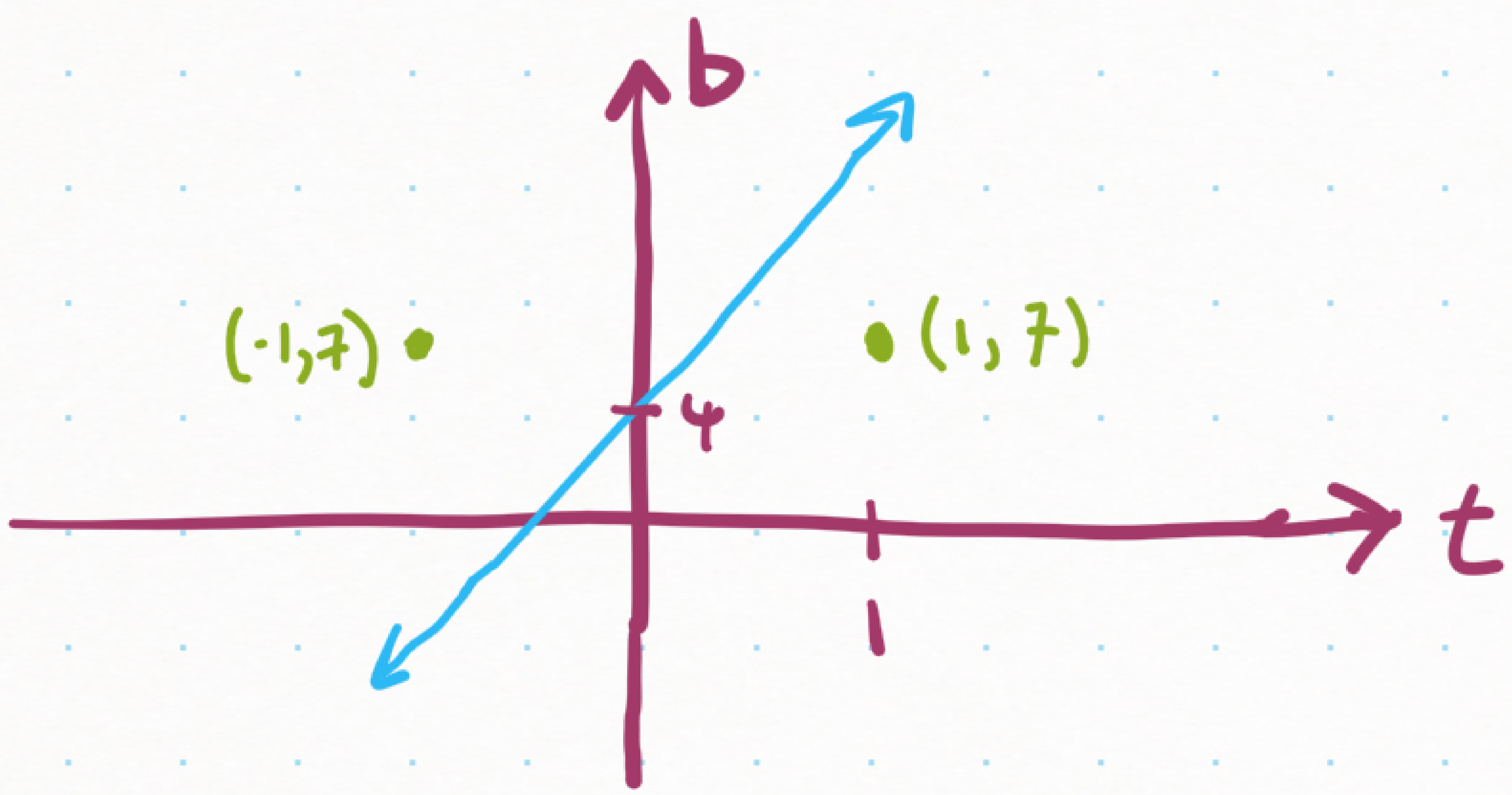
$$\begin{bmatrix}1 & 1\\ 1 & -1\\ 1 & 2\end{bmatrix}\begin{bmatrix}C\\ D\end{bmatrix} = \begin{bmatrix}7\\ 7\\ 21\end{bmatrix}$$

$$A \quad \hat{x} = b$$

$$\hat{x} = (A^TA)^{-1}A^Tb$$

$$= \begin{bmatrix}3 & 2\\ 2 & 6\end{bmatrix}^{-1}\begin{bmatrix}1 & 1 & 1\\ 1 & -1 & 2\end{bmatrix}\begin{bmatrix}7\\ 7\\ 21\end{bmatrix}$$

$$= \begin{bmatrix}6/14 & -2/14\\ -2/14 & 3/14\end{bmatrix}\begin{bmatrix}35\\ 42\end{bmatrix} = \begin{bmatrix}9\\ 4\end{bmatrix}$$



16.2

$$p = A\hat{x} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 9 \\ 4 \end{bmatrix} = \begin{bmatrix} 13 \\ 5 \\ 17 \end{bmatrix}$$

$$e = b - p = \begin{bmatrix} -6 \\ 2 \\ 4 \end{bmatrix}$$

$Pe = \vec{0}$ since the error e is perpendicular to the column space of A

$Pe = Pb - Pp = p - p = 0.$

16.3

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 4 \end{bmatrix}$$

$b = (2, -6, 4)$ is perpendicular to the column space of A, so the projection $p = \vec{0}$.

$$\hat{x} = \begin{bmatrix} 3/7 & -1/7 \\ -1/7 & 3/14 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -6 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3/7 & -1/7 \\ -1/7 & 3/14 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

16.4 $\underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} C \\ D \end{bmatrix}}_{\hat{x}} = \underbrace{\begin{bmatrix} 5 \\ 13 \\ 17 \end{bmatrix}}_{b} \longrightarrow \hat{x} = \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$

$$p = A\hat{x} = b \ , \quad e = b - p = \vec{0}$$

The error $e = \vec{0}$ since this b is in the column space of A.

16.5 e is in the left nullspace of A.
p is in the column space of A.
$\hat{x}$ is in the row space of A.

Since $r = 2 = n$, the nullspace of A is $\{\vec{0}\}$.

16.6

$$\begin{array}{ccc} A & \hat{x} & b \\ \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} & \begin{bmatrix} C \\ D \end{bmatrix} = & \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\hat{x} = (A^T A)^{-1} A^T b$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/5 & 0 \\ 0 & 1/10 \end{bmatrix} \begin{bmatrix} 5 \\ -10 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} C \\ D \end{bmatrix}$$

Note: using symmetric spacing of t's leads to a diagonal $A^T A$.