# 18.06SC Unit 1 Exam

**1 (24 pts.)** This question is about an $m$ by $n$ matrix $A$ for which

$$Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ has } \textbf{no solutions} \quad \text{and} \quad Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ has } \textbf{exactly one solution}.$$

(a) Give all possible information about $m$ and $n$ and the rank $r$ of $A$.

(b) Find all solutions to $Ax = 0$ and **explain your answer**.

(c) Write down an example of a matrix $A$ that fits the description in part (a).

(a) $1 \leq r = n < m = 3$.

So $1 = r = n, m = 3$ or $2 = r = n, m = 3$.

(b) $x = [0]$ if $n = 1$, $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ if $n = 2$.

(c) $A = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is a $3 \times 1$ example.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a $3 \times 2$ example.

**2 (24 pts.)** The 3 by 3 matrix $A$ reduces to the identity matrix $I$ by the following three row operations (in order):

$E_{21}$: Subtract 4 (row 1) from row 2.

$E_{31}$: Subtract 3 (row 1) from row 3.

$E_{23}$: Subtract row 3 from row 2.

(a) Write the inverse matrix $A^{-1}$ in terms of the $E$'s. **Then compute $A^{-1}$.**

(b) What is the original matrix $A$?

(c) What is the lower triangular factor $L$ in $A = LU$?

(a) $A^{-1}A = E_{23}E_{31}E_{21}A = I_{3\times 3}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = I_{3\times 3}$$

$$A^{-1} = E_{23}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ -3 & 0 & 1 \end{bmatrix}$$

(b) $A = (A^{-1})^{-1} = E_{21}^{-1}E_{31}^{-1}E_{23}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix}$$

(c) $E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U$

$$A = E_{21}^{-1}E_{31}^{-1}U = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = LU.$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}.$$

**3 (28 pts.)** This 3 by 4 matrix depends on $c$:

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 3 & c & 2 & 8 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

(a) *For each* $c$ find a basis for the column space of $A$.

(b) *For each* $c$ find a basis for the nullspace of $A$.

(c) *For each* $c$ find the complete solution $x$ to $Ax = \begin{bmatrix} 1 \\ c \\ 0 \end{bmatrix}$.

(a) Note that column 4 is a combination of columns 1 and 3, which do not depend on c. Whether we have 2 or 3 vectors in the basis depends on whether $c = 3$ or not.

If $c = 3$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$

If $c \neq 3$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ c \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$

(b) $A \sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & c-3 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

If $c = 3$,
$x_1 + x_2 + 2x_4 = 0$
$x_3 + x_4 = 0$
$\longrightarrow \mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

If $c \neq 3$,
$x_1 + x_2 + x_4 = 0$
$(c-3)x_2 = 0$
$x_3 + x_4 = 0$
$\longrightarrow \mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

(c) $x_p = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

If $c = 3$, $x = x_p + c \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$, $c, d \in \mathbb{R}$

If $c \neq 3$, $x = x_p + e \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$, $e \in \mathbb{R}$

**4 (24 pts.)** (a) If $A$ is a 3 by 5 matrix, what information do you have about the nullspace of $A$?

(b) Suppose row operations on $A$ lead to this matrix $R = \texttt{rref}(A)$:

$$R = \begin{bmatrix} 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Write all known information about the columns of $A$.

(c) In the vector space $M$ of all 3 by 3 matrices (you could call this a matrix space), what subspace $S$ is spanned by all possible row reduced echelon forms $R$?

(a) $\mathcal{N}(A) \subseteq \mathbb{R}^5$ (or $\mathbb{C}^5$ if we allow complex numbers).

$$2 \leq N(A) := \dim(\mathcal{N}(A)) \leq 5$$

(b) The columns of $A$ are linearly dependent. There are nevertheless 3 independent columns: Columns 1, 4, and 5. These 3 columns give us a pivot in each row, which means the columns of $A$ (specifically columns 1, 4, 5) span $\mathbb{R}^3$.

(c) $S$ is the set of upper triangular matrices, which is a subspace of $M$.