Exercises on transposes, permutations, spaces

Problem 5.1: (2.7 #13. *Introduction to Linear Algebra:* Strang)

a) Find a 3 by 3 permutation matrix with $P^3 = I$ (but not $P = I$).

b) Find a 4 by 4 permutation $\widehat{P}$ with $\widehat{P}^4 \neq I$.

a) $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $\quad P^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b) $\hat{P} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ $\quad \hat{P}^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ $\quad \hat{P}^4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

Problem 5.2: Suppose A is a four by four matrix. How many entries of A can be chosen independently if:

a) A is symmetric?

b) A is *skew-symmetric*? ($A^T = -A$)

a) 4 choices for row 1
3 choices for row 2
2 choices for row 3
1 choice for row 4

If A is a 4×4 symmetric matrix, $4+3+2+1 = 10$ entries can be chosen independently.

b) $A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$ $\quad A^T = \begin{bmatrix} a & e & i & m \\ b & f & j & n \\ c & g & k & o \\ d & h & l & p \end{bmatrix} = (-1) \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$

All diagonals must be 0 ($a = -a \rightarrow a = 0$ for example).
must set $e = -b$, $i = -c$, $m = -d$
$j = -g$, $n = -h$
$o = -l$

$A = \begin{bmatrix} 0 & b & c & d \\ -b & 0 & g & h \\ -c & -g & 0 & l \\ -d & -h & -l & 0 \end{bmatrix}$

For a 4×4 skew symmetric A ($A^T = -A$), we can choose $3+2+1 = 6$ entries independently.

Problem 5.3: (3.1 #18.) True or false (check addition or give a counterexample):

a) The symmetric matrices in M (with $A^T = A$) form a subspace.

b) The skew-symmetric matrices in M (with $A^T = -A$) form a subspace.

c) The unsymmetric matrices in M (with $A^T \neq A$) form a subspace.

a) True. Let $A, B \in M$ $(n \times n)$ and $c \in \mathbb{R}$

Then $(A+B)^T = A^T + B^T = A + B$,

$(cA)^T = cA^T = cA$

b) True. Let A, B be skew symmetric matrices in M and $c \in \mathbb{R}$.

Then $(A+B)^T = A^T + B^T = -A - B = -(A+B)$,

$(cA)^T = cA^T = -cA$.

c) False. Consider the unsymmetric A, B:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 2 & 2 & 3 \\ 2 & 8 & 5 \\ 3 & 5 & 12 \end{bmatrix}$$ is symmetric.

Even more simply, for any unsymmetric matrix A, taking $c=0$ gives cA = zero matrix, which is of course symmetric.