**Exercises on positive definite matrices and minima**

**Problem 27.1:** (6.5 #33. *Introduction to Linear Algebra:* Strang) When $A$ and $B$ are symmetric positive definite, $AB$ might not even be symmetric, but its eigenvalues are still positive. Start from $AB\mathbf{x} = \lambda\mathbf{x}$ and take dot products with $B\mathbf{x}$. Then prove $\lambda > 0$.

**Problem 27.2:** Find the quadratic form associated with the matrix $\begin{bmatrix} 1 & 5 \\ 7 & 9 \end{bmatrix}$. Is this function $f(x,y)$ always positive, always negative, or sometimes positive and sometimes negative?

## 27.1

$$ABx = \lambda x$$

$$x^T B^T A B x = \lambda x^T B^T x$$

$$x^T B^T A B x = \lambda x^T B x$$

$$y^T A y = \lambda x^T B x, \quad y = Bx$$

$$\frac{y^T A y}{x^T B x} = \lambda \longrightarrow$$

$\lambda > 0$ since $y^T A y > 0$ and $x^T B x > 0$.

## 27.2

$$f(x,y) = \vec{x}^T A \vec{x}$$

$$= [x, y] \begin{bmatrix} 1 & 5 \\ 7 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= [x + 7y,\ 5x + 9y] \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= x^2 + 12xy + 9y^2.$$

$f(x,y)$ can be positive, negative, or zero.

$$f(-3, 1) = 9 - 36 + 9 = -18 < 0$$

$$f(1, 1) = 1 + 12 + 9 = 22 > 0$$

$$f(0, 0) = 0 + 0 + 0 = 0$$