Exercises on matrix spaces; rank 1; small world graphs

Problem 11.1: [Optional] (3.5 #41. *Introduction to Linear Algebra:* Strang) Write the 3 by 3 identity matrix as a combination of the other five permutation matrices. Then show that those five matrices are linearly independent. (Assume a combination gives $c_1P_1 + \cdots + c_5P_5 = 0$ and check entries to prove c_i is zero.) The five permutation matrices are a basis for the subspace of three by three matrices with row and column sums all equal.

Example 5 There are six 3 by 3 permutation matrices. Here they are without the zeros:

$$I = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad P_{21} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} \quad P_{32}P_{21} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$$

$$P_{31} = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \quad P_{32} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} \quad P_{21}P_{32} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}.$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Found by inspection / strategic guess and check. This shows

$$I = P_{21} - P_{32}P_{21} + P_{31} + P_{32} - P_{21}P_{32}.$$
$$= P_1 - P_2 + P_3 + P_4 - P_5$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{matrix} c_4 = 0 \\ c_1 + c_2 = 0 \\ c_3 + c_5 = 0 \\ c_1 + c_5 = 0 \\ c_3 = 0 \\ c_2 + c_4 = 0 \\ c_4 + c_5 = 0 \\ c_1 = 0 \end{matrix} \longrightarrow \begin{matrix} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \\ c_4 = 0 \\ c_5 = 0 \end{matrix}$$

Problem 11.2: (3.6 #31.) **M** is the space of three by three matrices. Multiply each matrix X in **M** by:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Notice that $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

a) Which matrices X lead to $AX = 0$?

b) Which matrices have the form AX for some matrix X?

c) Part (a) finds the "nullspace" of the operation AX and part (b) finds the "column space." What are the dimensions of those two subspaces of **M**? Why do the dimensions add to $(n - r) + r = 9$?

a) $$\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} a-g & b-h & c-i \\ d-a & e-b & f-c \\ g-d & h-e & i-f \end{bmatrix}}_{0}$$

$a = d = g$
$b = e = h$
$c = f = i$

b) $$AX = \begin{bmatrix} a-g & b-h & c-i \\ d-a & e-b & f-c \\ g-d & h-e & i-f \end{bmatrix}$$

These are matrices whose columns each individually sum to 0.

c) The dimension of the null space is 3 and the column space is 6. They add up to 9 because the set of 3×3 matrices is a 9 dimensional vector space.