# Exercises on solving $Ax = \mathbf{b}$ and row reduced form $R$

**Problem 8.1:** (3.4 #13.(a,b,d) *Introduction to Linear Algebra:* Strang) Explain why these are all false:

a) The complete solution is any linear combination of $\mathbf{x}_p$ and $\mathbf{x}_n$.

b) The system $Ax = \mathbf{b}$ has at most one particular solution.

c) If $A$ is invertible there is no solution $\mathbf{x}_n$ in the nullspace.

a) Suppose $Ax_p = b$ and $Ax_n = 0$. While it is true that $A(x_p + x_n) = Ax_p + Ax_n = b$, if $c_1, c_2$ are scalars we have $A(c_1 x_p + c_2 x_n) = c_1 Ax_p + c_2 Ax_n = c_1 b \neq b$ generally.

b) This is true iff $A$ has full rank. Otherwise, the nullspace is nontrivial, meaning $\exists\ x_n \neq 0$ s.t. $Ax_n = 0$. But then if $Ax_p = b$, $A(x_p + x_n) = b$ and $x_p \neq x_p + x_n$.

c) $x_n = \vec{0}$ will always be in the nullspace. It is the only vector in the nullspace if $A$ is invertible.

**Problem 8.2:** (3.4 #28.) Let

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \text{ and } \mathbf{c} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

Use Gauss-Jordan elimination to reduce the matrices $[U\ \ 0]$ and $[U\ \ \mathbf{c}]$ to $[R\ \ 0]$ and $[R\ \ \mathbf{d}]$. Solve $Rx = \mathbf{0}$ and $Rx = \mathbf{d}$.

Check your work by plugging your values into the equations $Ux = \mathbf{0}$ and $Ux = \mathbf{c}$.

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{matrix} x_1 = -2x_2 \\ x_3 = 0 \end{matrix}, \quad x = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad Ux = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \begin{matrix} x_1 = -1 - 2x_2 \\ x_3 = 2 \end{matrix}, \quad x = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \quad Ux = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \checkmark$$

**Problem 8.3:** (3.4 #36.) Suppose $Ax = \mathbf{b}$ and $Cx = \mathbf{b}$ have the same (complete) solutions for every $\mathbf{b}$. Is it true that $A = C$?

Yes. If $b$ is set to be the $j^{th}$ column of $A$, the vector $x = e_j$ solves $Ax = b$. But then since $Cx = Ce_j = b$, we have $c_{1j} = a_{1j}$, $c_{2j} = a_{2j}$, $\ldots$, $c_{mj} = a_{mj}$. That is, the $j^{th}$ column of $C$ matches the $j^{th}$ column of $A$. This holds for each $j$, so $A = C$.