Exercises on independence, basis, and dimension

Problem 9.1: (3.5 #2. *Introduction to Linear Algebra:* Strang) Find the largest possible number of independent vectors among:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix},$$

$$\mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \text{ and } \mathbf{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

One approach is to find the number of pivots in

$$V = [v_1\ v_4\ v_6\ v_5\ v_2\ v_3].$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This reveals that V has 3 pivots. There exists at most 3 independent vectors. In particular, we could take v_1, v_4, and v_6.

Problem 9.2: (3.5 #20.) Find a basis for the plane $x - 2y + 3z = 0$ in $\mathbb{R}^3$. Then find a basis for the intersection of that plane with the xy plane. Then find a basis for all vectors perpendicular to the plane.

- $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$, Basis: $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

- $\begin{matrix} x - 2y + 3z = 0 \\ z = 0 \end{matrix} \longrightarrow \begin{matrix} x - 2y = 0 \\ z = 0 \end{matrix}$, Basis: $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

- $v_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ is perpendicular to the plane and forms a basis on its own.