**Exercises on Cramer's rule, inverse matrix, and volume**

**Problem 20.1:** (5.3 #8. *Introduction to Linear Algebra:* Strang) Suppose

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}.$$

Find its cofactor matrix $C$ and multiply $AC^T$ to find $\det(A)$.

$$C = \begin{bmatrix} 6 & -3 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \text{ and } AC^T = \_\_\_\_\_.$$

If you change $a_{1,3} = 4$ to 100, why is $\det(A)$ unchanged?

**Problem 20.2:** (5.3 #28.) Spherical coordinates $\rho, \phi, \theta$ satisfy

$$x = \rho \sin\phi \cos\theta, \ y = \rho \sin\phi \sin\theta \text{ and } z = \rho \cos\phi.$$

Find the three by three matrix of partial derivatives:

$$\begin{bmatrix} \partial x/\partial\rho & \partial x/\partial\phi & \partial x/\partial\theta \\ \partial y/\partial\rho & \partial y/\partial\phi & \partial y/\partial\theta \\ \partial z/\partial\rho & \partial z/\partial\phi & \partial z/\partial\theta \end{bmatrix}.$$

Simplify its determinant to $J = \rho^2 \sin\phi$. In spherical coordinates,

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

is the volume of an infinitesimal "coordinate box."

20.1

$$C = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 1 & -1 \\ -6 & 2 & 1 \end{bmatrix}$$

$$AC^T = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 6 & 3 & -6 \\ -3 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$\det(A) = 3$.

Changing $a_{1,3}$ doesn't change $\det A$ since its cofactor is 0.

<u>20.2</u> $x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$, $z=\rho\cos\phi$

$$J=\begin{vmatrix} x_\rho & x_\phi & x_\theta \\ y_\rho & y_\phi & y_\theta \\ z_\rho & z_\phi & z_\theta \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$=\cos\phi\left(\rho^2\cos^2\theta\cos\phi\sin\phi+\rho^2\sin^2\theta\cos\phi\sin\phi\right)$$
$$+\rho\sin\phi\left(\rho\sin^2\phi\cos^2\theta+\rho\sin^2\phi\sin^2\theta\right)$$
$$=\rho^2\cos^2\phi\sin\phi+\rho^2\sin^3\phi$$
$$=\rho^2\sin\phi.$$