18.06SC Unit 2 Exam

1 (24 pts.) Suppose q_1, q_2, q_3 are orthonormal vectors in $\mathbb{R}^3$. Find **all possible values** for these 3 by 3 determinants and explain your thinking in 1 sentence each.

(a) $\det \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} = \pm 1$

(b) $\det \begin{bmatrix} q_1 + q_2 & q_2 + q_3 & q_3 + q_1 \end{bmatrix} = \pm 2$

(c) $\det \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}$ times $\det \begin{bmatrix} q_2 & q_3 & q_1 \end{bmatrix} = 1$

(a) $1 = \det(I) = \det(Q^T Q) = \det Q^T \det Q = (\det Q)^2.$

(b) Let $A = \begin{bmatrix} q_1+q_2 & q_2+q_3 & q_3+q_1 \end{bmatrix}$

$$(\det A)^2 = \det(A^T A) = \begin{bmatrix} q_1^T + q_2^T \\ q_2^T + q_3^T \\ q_3^T + q_1^T \end{bmatrix} \begin{bmatrix} q_1+q_2 & q_2+q_3 & q_3+q_1 \end{bmatrix}$$

$$= \det \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = 4$$

(c) The determinant changes signs when two columns are exchanged. There are two exchanges between $[q_1 \ q_2 \ q_3]$ and $[q_2 \ q_3 \ q_1]$ so these matrices have the same determinant. By part (a), determinant $= (-1)^2 = 1$ or determinant $= 1^2 = 1$.

2 (24 pts.) Suppose we take measurements at the 21 equally spaced times $t = -10, -9, \ldots, 9, 10$. All measurements are $b_i = 0$ except that $b_{11} = 1$ at the middle time $t = 0$.

(a) Using least squares, what are the best $\widehat{C}$ and $\widehat{D}$ to fit those 21 points by a straight line $C + Dt$?

(b) You are projecting the vector b onto what subspace? (*Give a basis.*) Find a nonzero vector perpendicular to that subspace.

(a) $A = \begin{bmatrix} 1 & -10 \\ 1 & -9 \\ \vdots & \vdots \\ 1 & 9 \\ 1 & 10 \end{bmatrix}$, $x = \begin{bmatrix} C \\ D \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, $\hat{x} = \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix}$.

$$A^T A \hat{x} = A^T b$$

$$\begin{bmatrix} 21 & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad S = 2 \sum_{i=1}^{10} i^2$$

$$\begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 1/21 \\ 0 \end{bmatrix}, \quad \text{Best fit line: } y = \frac{1}{21}.$$

(b) The column space of A. We have $A^T v = \vec{0}$ for $v = [10, 9, \ldots, 0, \ldots, -9, -10]^T$, meaning v is orthogonal to the columns of A and thus the column space of A.

$$A^T v = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ -10 & -9 & \cdots & 9 & 10 \end{bmatrix} \begin{bmatrix} 10 \\ 9 \\ \vdots \\ -9 \\ -10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

3 (9 + 12 + 9 pts.) The Gram-Schmidt method produces orthonormal vectors q_1, q_2, q_3 from independent vectors a_1, a_2, a_3 in $\mathbb{R}^5$. Put those vectors into the columns of 5 by 3 matrices Q and A.

(a) Give formulas using Q and A for the projection matrices P_Q and P_A onto the column spaces of Q and A.

(b) ***Is $P_Q = P_A$ and why ? What is P_Q times Q ? What is $\det P_Q$?***

(c) Suppose a_4 is a new vector and a_1, a_2, a_3, a_4 are independent. Which of these (if any) is the new Gram-Schmidt vector q_4? (P_A and P_Q from above)

1. $\dfrac{P_Q a_4}{\|P_Q a_4\|}$ **2.** $\dfrac{a_4 - \frac{a_4^T a_1}{a_1^T a_1} a_1 - \frac{a_4^T a_2}{a_2^T a_2} a_2 - \frac{a_4^T a_3}{a_3^T a_3} a_3}{\|\text{ norm of that vector }\|}$ **3.** $\dfrac{a_4 - P_A a_4}{\|a_4 - P_A a_4\|}$

(a) $P_Q = Q(Q^TQ)^{-1}Q^T = Q I_3 Q^T = QQ^T$.

$P_A = A(A^TA)^{-1}A^T$

(b) $P_Q = P_A$ since both project onto the same subspace.

$P_Q Q = QQ^TQ = Q$.

$\det P_Q = 0$. All vectors orthogonal to column space of Q are projected to 0.

(c) 3. Since $P_Q = P_A$, $q_4 = \dfrac{a_4 - P_Q a_4}{\|a_4 - P_Q a_4\|} = \dfrac{a_4 - P_A a_4}{\|a_4 - P_A a_4\|}$.

4 (22 pts.) Suppose a 4 by 4 matrix has the same entry $\times$ throughout its first row and column. The other 9 numbers could be anything like $1, 5, 7, 2, 3, 99, \pi, e, 4$.

$$A = \begin{bmatrix} \times & \times & \times & \times \\ \times & & \text{any numbers} & \\ \times & & \text{any numbers} & \\ \times & & \text{any numbers} & \end{bmatrix}$$

(a) The determinant of A is a polynomial in $\times$. What is the largest possible degree of that polynomial? **Explain your answer**.

(b) If those 9 numbers give the identity matrix I, what is $\det A$? Which values of $\times$ give $\det A = 0$?

$$A = \begin{bmatrix} \times & \times & \times & \times \\ \times & 1 & 0 & 0 \\ \times & 0 & 1 & 0 \\ \times & 0 & 0 & 1 \end{bmatrix}$$

(a) Each term in the formula for $\det A$ takes an entry from each row and each column. This means we can take at most 2 x's in any term – degree 2.

(b) $$\det A = x|I| - x\begin{vmatrix} x & 0 & 0 \\ x & 1 & 0 \\ x & 0 & 1 \end{vmatrix} + x\begin{vmatrix} x & 1 & 0 \\ x & 0 & 0 \\ x & 0 & 1 \end{vmatrix} - x\begin{vmatrix} x & 1 & 0 \\ x & 0 & 1 \\ x & 0 & 0 \end{vmatrix}$$

$$= x - x^2 + x(-1(x)) - x(-1(-x))$$

$$= x - x^2 - x^2 - x^2 = x - 3x^2$$

$$0 = \det A = x - 3x^2 = x(1 - 3x) \text{ if } x = 0,\ x = \tfrac{1}{3}.$$