§ 4.3 Fourier Integrals

4.3.1 Find the transform $\hat{g}$ of the ascending pulse $g(x) = \begin{cases} e^{ax}, & x<0 \\ 0, & x>0 \end{cases}$

$$\hat{g}(k) = \int_{-\infty}^{\infty} g(x) e^{-ikx}\,dx = \int_{-\infty}^{0} e^{ax} e^{-ikx}\,dx = \left[\frac{e^{(a-ik)x}}{a-ik}\right]_{-\infty}^{0} = \frac{1}{a-ik}$$

4.3.2 Find the Fourier transforms (with $f=0$ outside the given regions) of

(b) $f(x) = 1$ for $x<0$

$\hat{g}(k) = 2\pi\delta(k)$

$g(x) = \mathcal{F}^{-1}\{\hat{g}(k)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(k) e^{ikx}\,dk = e^0 = 1 \quad \forall x$

$\therefore \mathcal{F}\{1\} = 2\pi\delta(k)$

$h(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x<0 \end{cases}$

$\hat{h}(k) = \lim_{a\to 0} \frac{-2ik}{a^2+k^2} = \frac{-2i}{k} = \frac{2}{ik} \quad$ (pg 311)

$\frac{1}{2}g(x) - \frac{1}{2}h(x) = \begin{cases} 0, & x \geq 0 \\ 1, & x<0 \end{cases} = f(x)$

$\therefore \hat{f}(k) = \mathcal{F}\{\frac{1}{2}g - \frac{1}{2}h\} = \frac{1}{2}\hat{g} - \frac{1}{2}\hat{h} = \pi\delta(k) - \frac{1}{ik}$

(c) $f(x) = \int_0^1 e^{ikx}\,dk$

Claim $\hat{f}(k) = \begin{cases} 2\pi, & k\in[0,1] \\ 0, & \text{else} \end{cases}$

Check $\mathcal{F}^{-1}\{\hat{f}(k)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k) e^{ikx}\,dk = \frac{1}{2\pi}\int_0^1 2\pi e^{ikx}\,dk = f(x).$ ✓

4.3.3

(b) Find the inverse transform of $\hat{f}(k) = e^{-|k|}$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{0} e^{k} e^{ikx}\,dk + \frac{1}{2\pi}\int_0^{\infty} e^{-k}e^{ikx}\,dk$$

$$= \frac{1}{2\pi}\left\{ \frac{1}{1+ix} e^{k(1+ix)}\Big|_{-\infty}^{0} + \frac{1}{ix-1} e^{k(ix-1)}\Big|_0^{\infty} \right\}$$

$$= \frac{1}{2\pi}\left(\frac{1}{1+ix} + \frac{1}{1-ix}\right) = \frac{1}{2\pi}\left(\frac{2}{1+x^2}\right) = \frac{1}{\pi}\frac{1}{1+x^2}$$

4.3.4 Apply Plancherel's formula $2\pi\int|f|^2dx = \int|\hat{f}|^2dk$ to

(1) the square pulse $f=1$ for $-1<x<1$ to find $\int_{-\infty}^{\infty}\frac{\sin^2 t}{t^2}dt$

By Example 2 (pg. 310), $\hat{f}(k) = \frac{2\sin k}{k}$

$$\int_{-\infty}^{\infty}\frac{\sin^2 t}{t^2}dt = \frac{1}{4}\int_{-\infty}^{\infty}\left(\frac{2\sin k}{k}\right)^2 dk = \frac{1}{4}\int_{-\infty}^{\infty}|\hat{f}|^2 dk = \frac{2\pi}{4}\int_{-\infty}^{\infty}|f|^2 dx = \frac{\pi}{2}\int_{-1}^{1}1^2 dx = \pi$$

(2) the even decaying pulse to find $\int_{-\infty}^{\infty}\frac{dt}{(a^2+t^2)^2}$

$f(x) = \text{even decaying pulse} = e^{-a|x|}$, $a>0$

$\hat{f}(k) = \frac{2a}{a^2+k^2}$ by Example 4 (pg. 311)

$$\int_{-\infty}^{\infty}\frac{dt}{(a^2+t^2)^2} = \frac{1}{4a^2}\int_{-\infty}^{\infty}\left|\frac{2a}{(a^2+k^2)}\right|^2 dk = \frac{1}{4a^2}\int_{-\infty}^{\infty}|\hat{f}|^2 dk = \frac{\pi}{2a^2}\int_{-\infty}^{\infty}\left|e^{-a|x|}\right|^2 dx$$

$$= \frac{\pi}{2a^2}\int_{-\infty}^{\infty}e^{-2a|x|}dx = \frac{\pi}{a^2}\int_0^{\infty}e^{-2ax}dx = \frac{\pi}{a^2}\left(-\frac{1}{2a}\right)\left(e^{-2ax}\right)\Big|_0^{\infty}$$

$$= \frac{\pi}{2a^3}$$

4.3.9 The decaying pulse $f(x)$ has derivative $-ae^{-ax}$ for $x>0$ and derivative 0 for $x<0$ so that differentiation appears to multiply its transform by $-a$ instead of ik. How can this be?

$$f(x) = \begin{cases} e^{-ax}, & x>0 \\ 0, & x<0 \end{cases} \qquad f'(x) = \begin{cases} -ae^{-ax}, & x>0 \\ 0, & x<0 \end{cases} = \underset{\substack{\uparrow \\ \text{jump at } x=0 \text{ from} \\ 0 \text{ to } e^0=1}}{\delta(x)} - af(x)$$

$$\hat{f}(k) = \mathcal{F}\{f\} = \frac{1}{a+ik}$$

$$\mathcal{F}\{f'\} = \mathcal{F}\{\delta(x) - af(x)\} = \mathcal{F}\{\delta\} - a\mathcal{F}\{f\}$$

$$= 1 - \frac{a}{a+ik} = \frac{a+ik-a}{a+ik} = ik\frac{1}{a+ik} = ik\mathcal{F}\{f\}$$ No contradiction!

4.3.11 Take Fourier transforms of $\int_a^x u(t)\,dt - \frac{du}{dx} = \delta(x)$ to find $\hat{u}$ using 4L. Do you recognize u?

$$\mathcal{F}\{\textstyle\int_a^x u(t)\,dt\} = \frac{1}{ik}\hat{u}(k) + c\delta(k)$$

$$\mathcal{F}\{\tfrac{du}{dx}\} = ik\hat{u}(k)$$

$$\mathcal{F}\{\delta(x)\} = \int_{-\infty}^{\infty} \delta(x) e^{-ikx}\,dx = e^0 = 1$$

$$\frac{1}{ik}\hat{u}(k) + c\delta(k) - ik\hat{u}(k) = 1$$

since $\delta(k) = 0$, $k \neq 0$ and $\delta(k) = \infty$, $k = 0$ this equality implies $c = 0$.

$$\left(\frac{1}{ik} - ik\right)\hat{u}(k) = 1$$

$$\hat{u}(k) = \frac{ik}{1+k^2} = -\frac{1}{2}\,\frac{-2ik}{1+k^2}$$

odd decaying pulse $\mathcal{F}^{-1}\{-\frac{2ik}{a^2+k^2}\} = \begin{cases} e^{-ax}, & x>0 \\ -e^{ax}, & x<0 \end{cases}$

$$u(x) = -\frac{1}{2}\mathcal{F}^{-1}\left\{-\frac{2ik}{1+k^2}\right\} = \begin{cases} -\frac{1}{2}e^{-x}, & x>0 \\ \frac{1}{2}e^{x}, & x<0 \end{cases}$$

$u(x)$ is an odd decaying pulse with $a=1$, scaled by $-1/2$.

4.3.15 The derivative of the delta function is the doublet δ', a "distribution" concentrated at $x=0$ with

$$\int f(x)\delta'(x)\,dx = f(x)\delta(x) - \int f'(x)\delta(x)\,dx = -\int f'(x)\delta(x)\,dx = -f'(0)$$

(a) Why should the Fourier transform of δ' be ik?

$$\mathcal{F}\{\delta'(x)\} = \int_{-\infty}^{\infty} e^{-ikx}\delta'(x)\,dx = -\frac{d}{dx}\{e^{-ikx}\}\Big|_{x=0} = ike^{-ik\cdot 0} = ik.$$

Also $\mathcal{F}\{\delta(x)\} = 1$ and $\mathcal{F}\{g'(x)\} = ik\mathcal{F}\{g(x)\}$ so $\mathcal{F}\{\delta'(x)\}$ should be ik.

(b) What does the inverse formula (5) give for $\int ke^{ikx}\,dk$?

$$\int_{-\infty}^{\infty} ke^{ikx}\,dk = \frac{2\pi}{i}\frac{1}{2\pi}\int_{-\infty}^{\infty} ike^{ikx}\,dk = \frac{2\pi}{i}\mathcal{F}^{-1}\{ik\} = \frac{2\pi}{i}\delta'(x) = -2\pi i\delta'$$

(c) Exchanging k and x, what is the Fourier transform of $f(x) = x$?

$$\hat{f}(k) = \int_{-\infty}^{\infty} xe^{-ikx}\,dx = \int_{\infty}^{-\infty} -ye^{iky}(-dy) = -\int_{-\infty}^{\infty} ye^{iky}\,dy = 2\pi i\delta'(k), \text{ by (b)}.$$

4.3.23(b) Find $g(x)$ if $\hat{g}(\omega) = \left[\frac{8}{1+\omega^2} + 1\right]^{-1}$.

$$\hat{g}(\omega) = \frac{1+\omega^2}{8+1+\omega^2} = \frac{9+\omega^2}{9+\omega^2} - \frac{8}{9+\omega^2} = 1 - \frac{8}{9+\omega^2}$$

$$g(x) = \mathcal{F}^{-1}\{1\} - 8/6\,\mathcal{F}^{-1}\left\{\frac{2\cdot 3}{3^2+\omega^2}\right\} = \delta(x) - 4/3\, e^{-3|x|}$$

using even decaying pulse Example 4 pg. 311

4.3.27 Take Fourier transforms in the equation to find the transform $\hat{G}$ of the fundamental solution G. How would it be possible to find G?

$$\frac{d^4G}{dx^4} - 2a^2\frac{d^2G}{dx^2} + a^4 G = \delta$$

$$(ik)^4\hat{G} - 2a^2(ik)^2\hat{G} + a^4\hat{G} = 1$$

$$k^4\hat{G} + 2a^2k^2\hat{G} + a^4\hat{G} = 1$$

$$\hat{G}(k) = \frac{1}{(k^2+a^2)^2} = \frac{1}{4a^2}\frac{2a}{k^2+a^2}\,\frac{2a}{k^2+a^2}$$

$G(x) = \frac{1}{4a^2}\left(e^{-a|x|} * e^{-a|x|}\right)$ using the identity $\mathcal{F}^{-1}\{\hat{f}^2\} = f*f$:

$$\mathcal{F}\{f*g\} = \mathcal{F}\left\{\int_{-\infty}^{\infty} f(x-y)g(y)\,dy\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)g(y)\,dy\, e^{-ikx}\,dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)e^{-ik(x-y)}g(y)e^{-iky}\,dy\,dx$$

$$= \int_{-\infty}^{\infty} f(z)e^{-ikz}\,dz\int_{-\infty}^{\infty} g(y)e^{-iky}\,dy = \hat{f}(k)\hat{g}(k) = (\hat{f}\hat{g})(k)$$

$\therefore \mathcal{F}^{-1}\{\hat{f}\hat{g}\} = (f*g)(x)$ and in particular $\mathcal{F}^{-1}\{\hat{f}^2\} = f*f$.

4.3.28 What is $\delta * \delta$?

$$u(x) = (\delta*\delta)(x)$$

$$\hat{u}(k) = \mathcal{F}\{u\} = \mathcal{F}\{\delta*\delta\} = \left(\mathcal{F}\{\delta\}\right)^2 = 1^2 = 1$$

$$u(x) = \mathcal{F}^{-1}\{\hat{u}\} = \mathcal{F}^{-1}\{1\} = \delta(x)$$

$$\therefore \delta*\delta = \delta$$