§4.1 Fourier Series and Orthogonal Expansions

4.1.1 Find the Fourier Series on $-\pi < x < \pi$ for

(b) $f(x) = |\sin x|$, an even function.

$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos kx$ $\qquad$ $|\sin x| = \sin x$ on $0 \le x \le \pi$

$$\boxed{a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx} = \frac{1}{\pi}\int_0^{\pi} \sin x\,dx = -\frac{1}{\pi}\cos x\Big|_0^{\pi} = -\frac{1}{\pi}(-1-1) = 2/\pi$$

$$\boxed{a_k = \frac{2}{\pi}\int_0^{\pi} f(x)\cos kx}$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x \cos kx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\{\sin(1+k)x + \sin(1-k)x\}\,dx$$

$$= -\frac{1}{\pi}\left\{\frac{\cos(1+k)x}{1+k} + \frac{\cos(1-k)x}{1-k}\right\}\Big|_0^{\pi}$$

$$= -\frac{1}{\pi}\left\{\frac{\cos(1+k)\pi}{1+k} - \frac{1}{1+k} + \frac{\cos(1-k)\pi}{1-k} - \frac{1}{1-k}\right\}$$

$$= -\frac{1}{\pi}\left\{\frac{(-1)^{k+1}-1}{1+k} + \frac{(-1)^{k+1}-1}{1-k}\right\}$$

$$= \begin{cases} \frac{2}{\pi}\left\{\frac{1}{1+k} + \frac{1}{1-k}\right\}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$$

$k=1$	$\cos 2\pi = 1$	$\cos 0 = 1$
$k=2$	$\cos 3\pi = -1$	$\cos -\pi = -1$
$k=3$	$\cos 4\pi = 1$	$\cos -2\pi = 1$
$\vdots$	$\vdots$	$\vdots$

$$f(x) = |\sin x| = \frac{2}{\pi} + \sum_{\substack{k=2 \\ k \text{ even}}}^{\infty} \frac{2}{\pi}\left\{\frac{1}{1+k} + \frac{1}{1-k}\right\}\cos kx$$

$$= \frac{2}{\pi} + \frac{4}{\pi}\sum_{\substack{k=2 \\ k \text{ even}}}^{\infty} \frac{1}{1-k^2}\cos kx$$

$$= \boxed{\frac{2}{\pi} + \frac{4}{\pi}\sum_{j=1}^{\infty} \frac{1}{1-4j^2}\cos 2jx}$$

4.1.2 A square wave has $f(x) = -1$ on the left side $-\pi < x < 0$ and $f(x) = 1$ on the right side $0 < x < \pi$.

(a) Why are all the cosine coefficients $a_k = 0$?

Since f is an odd function, $a_0 = \int_{-\pi}^{\pi} f(x)\,dx = 0$.
For $k \geq 0$ $f(x)\cos kx$ is the product of an odd function with an even function so $f(x)\cos kx$ is an odd function $\longrightarrow a_k = \int_{-\pi}^{\pi} f(x)\cos kx\,dx = 0$.

(b) Find the sine series $\sum b_k \sin kx$ from equation (6):

$$\boxed{b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx} \qquad (6)$$

$$\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{0} -\sin kx\,dx + \frac{1}{\pi}\int_{0}^{\pi} \sin kx\,dx \\
&= \frac{1}{\pi}\int_{\pi}^{0} \sin -ky\,dy + \frac{1}{\pi}\int_{0}^{\pi} \sin kx\,dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} \sin ky\,dy + \frac{1}{\pi}\int_{0}^{\pi} \sin kx\,dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} \sin kx\,dx = -\frac{2}{\pi k}\cos kx\Big|_{0}^{\pi} = -\frac{2}{\pi k}\left\{(-1)^k - 1\right\} = \begin{cases} 4/\pi k, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}
\end{aligned}$$

1 $\cos \pi$
2 $\cos 2\pi$

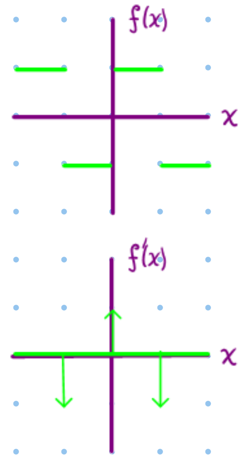
$$\boxed{f(x) = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi k}\sin kx = \sum_{j=1}^{\infty} \frac{4}{\pi(2j-1)}\sin(2j-1)x = \frac{4}{\pi}\left\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right\}}$$

4.1.3 Find the sine series for the square wave in another way by showing

(a) $df/dx = 2\delta(x) - 2\delta(x+\pi)$ extended periodically.

If $f(x)$ is the square wave extended periodically

$$f(x) = \begin{cases} \vdots & \\ 1 & -2\pi < x < -\pi \\ -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \\ -1 & \pi < x < 2\pi \\ \vdots & \end{cases} \qquad \frac{df}{dx} = \begin{cases} \vdots & \\ -\infty & x = -\pi \\ 0 & -\pi < x < 0 \\ \infty & x = 0 \\ 0 & 0 < x < \pi \\ -\infty & x = \pi \\ 0 & \pi < x < 2\pi \\ \vdots & \end{cases}$$



$\frac{df}{dx}$ is the 2π-periodic extension of $g(x) = \begin{cases} -\infty & x = -\pi \\ 0 & -\pi < x < 0 \\ \infty & x = 0 \\ 0 & 0 < x < \pi \end{cases}$

For $x \in [-\pi, \pi)$, $2\delta(x) - 2\delta(x+\pi) = \begin{cases} -\infty & x = -\pi \\ 0 & -\pi < x < 0 \\ \infty & x = 0 \\ 0 & 0 < x < \pi \end{cases} = g(x)$

That is, df/dx is the 2π-periodic extension of $2\delta(x) - 2\delta(x+\pi)$

(b) $2\delta(x) - 2\delta(x+\pi) = \frac{4}{\pi}\{\cos x + \cos 3x + \dots\}$

From page 269, for $\delta(x)$, $a_0 = 1/2\pi$, $a_k = 1/\pi$, $b_k = 0$. This implies:

$$2\delta(x) - 2\delta(x+\pi) = 2\left\{\frac{1}{2\pi} + \sum_{k=1}^{\infty} a_k \cos kx\right\} - 2\left\{\frac{1}{2\pi} + \sum_{k=1}^{\infty} a_k \cos k(x+\pi)\right\}$$

$$= 2\sum_{k=1}^{\infty} \frac{1}{\pi}\{\cos kx - \cos k(x+\pi)\} = \frac{4}{\pi}\sum_{k=1,\,k\text{ odd}}^{\infty} \cos kx = \frac{4}{\pi}\{\cos x + \cos 3x + \dots\}$$

(Since $\cos kx = -\cos k(x+\pi)$, k odd and $\cos kx = \cos k(x+\pi)$, k even)

From parts (a) and (b) conclude that the Fourier series is:

$$f(x) = \int_{-\pi}^{\pi} \{2\delta(x) - 2\delta(x+\pi)\}\,dx = \int \frac{4}{\pi}\sum_{k=1,\,k\text{ odd}}^{\infty} \cos kx\,dx = \frac{4}{\pi}\sum_{k=1,\,k\text{ odd}}^{\infty} \frac{1}{k}\sin kx$$

$$\boxed{f(x) = \frac{4}{\pi}\sum_{j=1}^{\infty} \frac{1}{2j-1}\sin(2j-1)x = \frac{4}{\pi}\left\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right\}}$$

Laplace's Equation in Cartesian coordinates is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy} = 0$$

Let $u = u(r,\theta)$ with $x = r\cos\theta$, $y = r\sin\theta$

$$r = (x^2+y^2)^{1/2}, \quad \theta = \arctan y/x + c$$

depends on the quadrant

$u_x = u_r r_x + u_\theta \theta_x$

$\quad = u_r\cos\theta - \frac{1}{r}u_\theta \sin\theta$

$r_x = \frac{1}{2}(x^2+y^2)^{-1/2}\, 2x = x/(x^2+y^2)^{1/2} = x/r = \cos\theta$

$\theta_x = \dfrac{-y/x^2}{1+(y/x)^2} = -\dfrac{y}{x^2+y^2} = -y/r^2 = -\sin\theta/r$

$$u_{xx} = u_r(\cos\theta)_x + u_{rx}\cos\theta - u_\theta\left(\frac{\sin\theta}{r}\right)_x - u_{\theta x}\frac{\sin\theta}{r}$$

$$= u_r(-\sin\theta)(-\sin\theta/r) + u_{rx}\cos\theta - u_\theta\frac{(\cos\theta)(-\sin\theta/r)r - \sin\theta\cos\theta}{r^2} - u_{\theta x}\frac{\sin\theta}{r}$$

$$= \frac{\sin^2\theta}{r}u_r + u_{rx}\cos\theta + u_\theta\frac{2\cos\theta\sin\theta}{r^2} - u_{\theta x}\frac{\sin\theta}{r}$$

$$= \frac{\sin^2\theta}{r}u_r + u_{rr}\cos^2\theta - u_{r\theta}\frac{\sin\theta\cos\theta}{r} + u_\theta\frac{2\cos\theta\sin\theta}{r^2} - \left(u_{\theta\theta}\left(\frac{-\sin\theta}{r}\right) + u_{\theta r}\cos\theta\right)\left(\frac{\sin\theta}{r}\right)$$

$$\left\{\begin{aligned} u_{rx} &= u_{rr}r_x + u_{r\theta}\theta_x \\ &= u_{rr}\cos\theta - u_{r\theta}\frac{\sin\theta}{r} \\ u_{\theta x} &= u_{\theta\theta}\theta_x + u_{\theta r}r_x \\ &= u_{\theta\theta}\left(\frac{-\sin\theta}{r}\right) + u_{\theta r}\cos\theta \end{aligned}\right\}$$

$$\boxed{u_{xx} = \cos^2\theta\, u_{rr} + \frac{\sin^2\theta}{r}u_r - \frac{2\sin\theta\cos\theta}{r}u_{r\theta} + \frac{2\sin\theta\cos\theta}{r^2}u_\theta + \frac{\sin^2\theta}{r^2}u_{\theta\theta}}$$

$u_y = u_r r_y + u_\theta \theta_y$ $\qquad r_y = y/r = \sin\theta$

$= u_r \sin\theta + \frac{1}{r} u_\theta \cos\theta$ $\qquad \theta_y = \dfrac{1/x}{1 + y^2/x^2} = \dfrac{x}{x^2+y^2} = x/r^2 = \cos\theta/r$

$u_{yy} = u_{ry}\sin\theta + u_r(\sin\theta)_y + u_{\theta y}\dfrac{\cos\theta}{r} + u_\theta\left(\dfrac{\cos\theta}{r}\right)_y$

$= \left(u_{rr}\sin\theta + u_{r\theta}\dfrac{\cos\theta}{r}\right)\sin\theta$

$+ u_r \cos^2\theta/r + u_{\theta\theta}\cos^2\theta/r^2$

$+ u_{\theta r}\dfrac{\sin\theta\cos\theta}{r}$

$+ u_\theta\left(-\dfrac{2\sin\theta\cos\theta}{r^2}\right)$

$$\left\{\begin{aligned} u_{ry} &= u_{rr} r_y + u_{r\theta}\theta_y \\ &= u_{rr}\sin\theta + u_{r\theta}\frac{\cos\theta}{r} \\ u_{\theta y} &= u_{\theta\theta}\theta_y + u_{\theta r} r_y \\ &= u_{\theta\theta}\cos\theta/r + u_{\theta r}\sin\theta \end{aligned}\right\}$$

$$\boxed{\begin{aligned} u_{yy} = {} & \sin^2\theta\, u_{rr} + \frac{\cos^2\theta}{r} u_r + \frac{2\sin\theta\cos\theta}{r} u_{r\theta} \\ & - \frac{2\sin\theta\cos\theta}{r^2} u_\theta + \frac{\cos^2\theta}{r^2} u_{\theta\theta} \end{aligned}}$$

$$\left\{\begin{aligned} \left(\frac{\cos\theta}{r}\right)_y &= \frac{-\sin\theta\,\theta_y r}{r^2} - \frac{\cos\theta\, r_y}{r^2} \\ &= -\frac{\sin\theta\cos\theta}{r^2} - \frac{\cos\theta\sin\theta}{r^2} \\ &= -\frac{2\sin\theta\cos\theta}{r^2} \end{aligned}\right\}$$

$0 = u_{xx} + u_{yy}$

$= (\cos^2\theta + \sin^2\theta) u_{rr}$

$+ \frac{1}{r}(\sin^2\theta + \cos^2\theta) u_r + 0 \cdot u_{r\theta}$

$+ 0 \cdot u_\theta + \frac{1}{r^2}(\sin^2\theta + \cos^2\theta) u_{\theta\theta}$

$= u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = \frac{1}{r}(r u_r)_r + \frac{1}{r^2} u_{\theta\theta}$

Laplace's Equation in Polar coordinates is

$$0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{r}(r u_r)_r + \frac{1}{r^2} u_{\theta\theta}$$

$u = a_0$ (constant) $\quad \frac{1}{r}(r u_r)_r + \frac{1}{r^2} u_{\theta\theta} = \frac{1}{r} 0 + \frac{1}{r^2} 0 = 0.$

$u(r,\theta) = r^n \sin n\theta$

$\frac{1}{r}(r n r^{n-1}\sin n\theta)_r + \frac{1}{r^2}(-n^2 r^n \sin n\theta) = \frac{1}{r} n^2 r^{n-2}\sin n\theta - n^2/r^2\, r^n \sin n\theta = 0$

$u(r,\theta) = r^n \cos n\theta$

$\frac{1}{r}(r n r^{n-1}\cos n\theta)_r + \frac{1}{r^2}(-n^2 r^n \cos n\theta) = \frac{1}{r} n^2 r^{n-2}\cos n\theta - n^2/r^2\, r^n \cos n\theta = 0$

$u(r,\theta) = a_0 + a_1 r\cos\theta + b_1 r\sin\theta + a_2 r^2 \cos n\theta + b_2 r^2 \sin n\theta + \ldots$

satisfies Laplace's equation in polar coordinates.

4.1.6 Around the unit circle suppose u is a square wave

$$u_0 = \begin{cases} +1 \text{ on the upper semicircle} & 0<\theta<\pi \\ -1 \text{ on the lower semicircle} & -\pi<\theta<0 \end{cases}$$

From the Fourier series for the square wave write down the Fourier series for u (the solution 21) to Laplace's equation). What is the value of u at the origin?

$$u(r,\theta) = a_0 + a_1 r\cos\theta + b_1 r\sin\theta + a_2 r^2\cos 2\theta + b_2 r^2 \sin 2\theta + \dots$$

$$1 = u(1,\theta) = a_0 + a_1\cos\theta + b_1\sin\theta + \dots \quad 0<\theta<\pi$$
$$-1 = u(1,\theta) = a_0 + a_1\cos\theta + b_1\sin\theta + \dots \quad -\pi<\theta<0$$

$$1 = u(1,\pi/2) = a_0 + b_1 + b_2 + b_3 + \dots$$
$$-1 = u(1,-\pi/2) = a_0 - b_1 - b_2 - b_3 - \dots \quad \rightarrow \quad 0 = a_0$$

Since $\cos(\theta)=\cos(-\theta)$ and $-\sin(\theta)=\sin(-\theta)$, for $0<\theta<\pi$:

$$1 = u(1,\theta) = a_1\cos\theta + b_1\sin\theta + a_2\cos 2\theta + b_2\sin 2\theta + \dots$$
$$-1 = u(1,-\theta) = a_1\cos\theta - b_1\sin\theta + a_2\cos 2\theta - b_2\sin 2\theta + \dots$$

$$0 = a_1\cos\theta + a_2\cos 2\theta + a_3\cos 3\theta + \dots$$

Since $\{\cos k\theta\}_{k=1}^{\infty}$ are orthogonal wrt to the $L^2[-\pi,\pi]$ inner product, $a_i = 0$ for all i.

$$u(r,\theta) = b_1 r\sin\theta + b_2 r^2 \sin 2\theta + \dots$$

$$1 = u(1,\theta) = b_1\sin\theta + b_2\sin 2\theta + b_3\sin 3\theta + b_4 \sin 4\theta + \dots \text{ for } 0<\theta<\pi$$

From Exercises 4.1.2, 4.1.3,

$$1 = \frac{4}{\pi}\left\{\sin\theta + \tfrac{1}{3}\sin 3\theta + \tfrac{1}{5}\sin 5\theta + \dots\right\} \text{ for } 0<\theta<\pi$$

This implies $b_1 = 1$, $b_2 = 0$, $b_3 = 1/3$, $b_4 = 0$, $b_5 = 1/5, \dots$

$$\therefore u(r,\theta) = \frac{4}{\pi}\left\{r\sin\theta + \tfrac{1}{3}r^3\sin 3\theta + \tfrac{1}{5}r^5\sin 5\theta + \dots\right\}$$

$$\lim_{r\to 0} u(r,\theta) = u(0,\theta) = 0.$$

4.1.10 What constant function is closest in the least square sense to $f(x) = \cos^2 x$? what multiple of $\cos x$ is closest to $f(x) = \cos^3 x$?

$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$ If we write out the Fourier series for $f(x)$, we would get $a_0 = 1/2$, $a_2 = 1/2$, $a_n = 0$ $\forall n \notin \{0,2\}$ and all $b_n = 0$. So the constant function $\boxed{1/2}$ is closest.

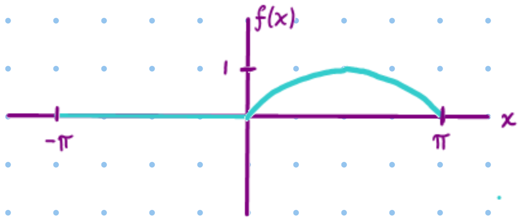
Let $K \in \mathbb{R}$. The least squares distance between $\cos^3 x$ and $K\cos x$ is:

$$\langle \cos^3 x, \cos x\rangle_{L^2[-\pi,\pi]} = \int_{-\pi}^{\pi} |\cos^3 x - K\cos x|^2\, dx = \frac{\pi}{8}(8K^2 - 12K + 5).$$

This is minimized by $K = 3/4$. The multiple of $\cos x$ closest to $\cos^3 x$ in the least squares sense is $\boxed{\frac{3}{4}\cos x}$

4.1.12 Sketch the 2π-periodic half wave $f(x)$ and find its Fourier series.

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \sin x & 0 < x < \pi \end{cases}$$



$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2\pi}\int_0^{\pi} \sin x\, dx = \frac{1}{\pi}$$

$$a_K = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos Kx = \frac{1}{\pi}\int_0^{\pi} \sin x \cos Kx\, dx = \frac{1}{2\pi}\int_0^{\pi}\left(\sin(1+K)x + \sin(1-K)x\right)dx$$

$$= -\frac{1}{2\pi}\left(\frac{1}{1+K}\cos(1+K)x + \frac{1}{1-K}\cos(1-K)x\right)\Big|_0^{\pi}$$

$$= -\frac{1}{2\pi}\left\{\frac{1}{1+K}\left(\cos(1+K)\pi - 1\right) + \frac{1}{1-K}\left(\cos(1-K)\pi - 1\right)\right\}$$

$$= -\frac{1}{2\pi}\left\{\frac{1}{1+K}\left((-1)^{K+1} - 1\right) + \frac{1}{1-K}\left((-1)^{K+1} - 1\right)\right\} = \begin{cases} -\frac{1}{2\pi}\left(\frac{-2}{1+K} + \frac{-2}{1-K}\right) & K \text{ even} \\ 0 & K \text{ odd} \end{cases}$$

$$= \begin{cases} 2\left(\pi(1+K)(1-K)\right)^{-1} & K \text{ even} \\ 0 & K \text{ odd} \end{cases}$$

$$b_K = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin Kx\, dx = \frac{1}{\pi}\int_0^{\pi} \sin x \sin Kx\, dx = \frac{1}{2\pi}\int_0^{\pi}\left(\cos(1-K)x - \cos(1+K)x\right)dx$$

$$= \frac{1}{2\pi}\left(\frac{1}{1-K}\sin(1-K)x - \frac{1}{1+K}\sin(1+K)x\right)\Big|_0^{\pi} = 0, \quad \forall K \in \{2,3,4,\dots\}$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\, dx = \frac{1}{\pi}\int_0^{\pi}\sin^2 x\, dx = \frac{1}{\pi}\frac{\pi}{2} = \frac{1}{2}$$

The Fourier series for $f(x)$ is:

$$a_0 + b_1 \sin x + \sum_{K\text{ even}} a_K \cos Kx = \boxed{\frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{j=0}^{\infty}\frac{\cos 2jx}{4j^2 - 1}}$$

Properties of the Fourier Series

1. Each Fourier coefficient a_k, b_k, or c_k is the best possible choice in the mean square sense. In other words, the error E is at a minimum when $A_k = a_k$, $B_k = b_k$ in

$$E = \int_{-\pi}^{\pi} \left[f(x) - \sum_0^N A_k \cos kx + B_k \sin kx \right]^2 dx$$

For proof, check a <u>typical</u> derivative, which should be zero at a minimum. Setting $0 = \partial E / \partial B_j$, $j \in \{0, 1, \dots, n\}$ (assume $b_0 := 0$)

$$0 = \frac{\partial E}{\partial B_j} = \int_{-\pi}^{\pi} 2 \left[f(x) - \sum_0^N A_k \cos kx + B_k \sin kx \right] (-\sin jx)\, dx$$

$$-\frac{1}{2} \frac{\partial E}{\partial B_j} = \int_{-\pi}^{\pi} \sin jx \left[f(x) - \sum_0^N A_k \cos kx + B_k \sin kx \right] dx$$

$$= \int_{-\pi}^{\pi} \sin jx \left[f(x) - B_j \sin jx \right] dx$$

$$\Rightarrow \quad B_j = \left(\int_{-\pi}^{\pi} f(x) \sin jx\, dx \right) / \left(\int_{-\pi}^{\pi} \sin^2 jx \right) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin jx\, dx = b_j$$

It's similar for any A_j. Take N as large as necessary to check any particular coefficient. $\therefore$ The claim holds for all $k = 0, 1, 2, \dots$

The sines and cosines are a perpendicular set of axes in function space. Fourier analysis projects f onto each of these axes.

Orthogonality allows you to find each coefficient separately and completeness that allows the sines and cosines (or e^{ikx}) to reproduce f.

2. Since projections are never larger, no piece of the Fourier series can be larger than f. In particular for a partial sum:

$$\boxed{\int_{-\pi}^{\pi} (a_0 + a_1 \cos x + \dots + b_N \sin Nx)^2 dx \leq \int_{-\pi}^{\pi} (f(x))^2 dx \quad \text{(Bessel's Inequality)}}$$

Bessel's inequality is proved in Exercise 4.1.13. In the limit $N \to \infty$ it is

$$\boxed{2\pi \sum |c_k|^2 = 2\pi a_0^2 + \pi (a_1^2 + b_1^2 + a_2^2 + b_2^2 + \dots) = \int_{-\pi}^{\pi} (f(x))^2 dx \quad \text{(Parseval's Formula)}}$$

<u>Hilbert space</u> In one form contains functions and in another vectors with infinitely many components. The functions must have finite length – the integral of $(f(x))^2$ must be finite. For vectors the sum of squares is finite.

Informally, L^2: f s.t. $\int (f(x))^2 dx < \infty$, ℓ^2: x s.t. $\sum x_n^2 < \infty$. By Parseval's Formula, f is in L^2 exactly when the vector containing its Fourier coefficients is in ℓ^2.

3. We have found for various functions f its corresponding Fourier series and assuming it converges back to f. Convergence is in the mean-square sense, i.e. in Hilbert space. This is by squaring the difference and integrating. But this does not guarantee convergence at each point.

Suppose f is a function with Fourier coefficients a_0, a_k, b_k. Evaluating the series for f, $S(x) = a_0 + \sum(a_k \cos kx + b_k \sin kx)$ at $x = 0$ gives:

$$a_0 + a_1 + a_2 + \ldots = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos 2x\,dx + \ldots$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left[1 + 2\cos x + 2\cos 2x + \ldots\right] dx$$

Since $1 + 2\cos x + 2\cos 2x + \ldots$ is the Fourier series for the delta function, we should expect:

$$a_0 + a_1 + a_2 + \ldots = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left[1 + 2\cos x + 2\cos 2x + \ldots\right] dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\delta(x) = f(0).$$

This proof is successful if f is smooth enough to have a derivative.

<u>4.1.13</u> Prove Bessel's Inequality by integrating the left side.

$$\int_{-\pi}^{\pi}(a_0 + a_1\cos x + \ldots + b_N \sin Nx)^2\,dx$$

$$= \int_{-\pi}^{\pi} a_0^2\,dx + \sum_{k=1}^{N}\int_{-\pi}^{\pi} a_k^2\cos^2 kx\,dx + \sum_{k=1}^{N}\int_{-\pi}^{\pi} b_k^2 \sin^2 kx\,dx \quad \text{(by orthogonality)}$$

$$= 2\pi a_0^2 + \pi\left(a_1^2 + b_1^2 + a_2^2 + b_2^2 + \ldots + a_N^2 + b_N^2\right)$$

Let $S_N = a_0 + a_1\cos x + b_1 \sin x + \ldots + b_N \sin Nx$. Consider $\int_{-\pi}^{\pi}(f(x) - S_N)^2\,dx$.

$$0 \le \left(f(x) - S_N(x)\right)^2 = (f(x))^2 - 2f(x)S_N(x) + S_N^2(x)$$

$$\int_{-\pi}^{\pi} f(x)S_N(x)\,dx = a_0\int_{-\pi}^{\pi} f(x)\,dx + a_1\int_{-\pi}^{\pi} f(x)\cos x\,dx + \ldots + b_N\int_{-\pi}^{\pi} f(x)\sin Nx\,dx$$
$$= 2\pi a_0^2 + \pi a_1^2 + \pi b_1^2 + \ldots + \pi a_N^2 + \pi b_N^2 = 2\pi a_0^2 + \pi(a_1^2 + \ldots + b_N^2)$$

$$\int_{-\pi}^{\pi} S_N^2(x)\,dx = 2\pi a_0^2 + \pi(a_1^2 + \ldots + b_N^2) \quad \text{(calculated previously).}$$

$$\therefore \int_{-\pi}^{\pi} f(x)S_N(x)\,dx = \int_{-\pi}^{\pi} S_N^2(x)\,dx$$

$$0 \le \int_{-\pi}^{\pi}(f(x))^2\,dx - 2\int_{-\pi}^{\pi} f(x)S_N(x)\,dx + \int_{-\pi}^{\pi} S_N^2(x)\,dx = \int_{-\pi}^{\pi}(f(x))^2\,dx - \int_{-\pi}^{\pi} S_N^2(x)\,dx$$

$$\int_{-\pi}^{\pi}(a_0 + a_1\cos x + \ldots + b_N \sin Nx)^2\,dx = \int_{-\pi}^{\pi} S_N^2(x)\,dx \le \int_{-\pi}^{\pi}(f(x))^2\,dx$$

4.4.14

(a) Find the lengths of the vectors $u=(1, 1/2, 1/4, 1/8, \dots)$ and $v=(1, 1/3, 1/9, \dots)$ in Hilbert Space and test the inequality $|u^Tv|^2 \le (u^Tu)(v^Tv)$

$$\|u\|^2 = u^Tu = \sum u_i^2 = 1 + 1/4 + 1/16 + \dots = \frac{1}{1-1/4} = 4/3 \longrightarrow \|u\| = 2/\sqrt{3}$$

$$\|v\|^2 = v^Tv = \sum v_i^2 = 1 + 1/9 + 1/81 + \dots = \frac{1}{1-1/9} = 9/8 \longrightarrow \|v\| = 3/2\sqrt{2}$$

$$|u^Tv|^2 = \left(1 + \tfrac{1}{2}\tfrac{1}{3} + \tfrac{1}{4}\tfrac{1}{9} + \dots\right)^2 = \left(1 + 1/6 + 1/36 + 1/216 + \dots\right)^2 = \left(\frac{1}{1-1/6}\right)^2 = 36/25$$

$$(u^Tu)(v^Tv) = \|u\|^2\|v\|^2 = 4/3 \cdot 9/8 = 36/24 > 36/25 = |u^Tv|^2.$$

(b) For the functions $f = 1 + \frac{1}{2}e^{ix} + \frac{1}{4}e^{2ix} + \dots$, $g = 1 + \frac{1}{3}e^{ix} + \frac{1}{9}e^{2ix} + \dots$ use part a to find the numerical value of each term in:

$$\left|\int_{-\pi}^{\pi} \bar{f}(x)g(x)dx\right|^2 \le \int_{-\pi}^{\pi} |f(x)|^2 dx \int_{-\pi}^{\pi} |g(x)|^2 dx$$

By Parseval's Formula,

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = 2\pi \sum_0^\infty |c_k|^2 = 2\pi\left(1^2 + (\tfrac{1}{2})^2 + (\tfrac{1}{4})^2 + \dots\right) = 2\pi\|u\|^2 = 8\pi/3$$

$$\int_{-\pi}^{\pi} |g(x)|^2 dx = 2\pi\|v\|^2 = 9\pi/4$$

$$\left|\int_{-\pi}^{\pi} \bar{f}(x)g(x)dx\right|^2 = \left|\int_{-\pi}^{\pi} \left(1 + \tfrac{1}{2}e^{-ix} + \tfrac{1}{4}e^{-2ix} + \dots\right)\left(1 + \tfrac{1}{3}e^{ix} + \tfrac{1}{9}e^{2ix} + \dots\right)dx\right|^2$$

$$= \left|\int_{-\pi}^{\pi} \left(1 + \tfrac{1}{2}\tfrac{1}{3}e^0 + \tfrac{1}{4}\tfrac{1}{9}e^0 + \dots\right)dx\right|^2 \quad \text{(by orthogonality)}$$

$$= \left|\int_{-\pi}^{\pi} \left(1 + 1/6 + 1/36 + \dots\right)dx\right|^2 = 2\pi|u^Tv| = 72\pi/25$$

4.1.16 If the boundary