1.5 Eigenvalues and Dynamical Systems

Notes

In this chapter the focus is on square matrices. Suppose A is an $n \times n$ matrix. Normally multiplication by A changes the direction of x. For certain exceptional vectors, however, Ax is a multiple of x:

$Ax = \lambda x$ $(x \neq 0)$ — λ (a scalar) is called an <u>eigenvalue</u> of A; x is called an <u>eigenvector</u> corresponding to λ

For diagonal matrices the eigenvalues are the diagonal entries and the eigenvectors are (nonzero multiples of) the coordinate directions.

Example: $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \longrightarrow A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = b\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

For other matrices look for the eigenvalues first:

$$Ax = \lambda x \rightarrow (A - \lambda I)x = 0 \longrightarrow A - \lambda I \text{ is singular since } x \neq 0$$

This means that an eigenvalue of A is a number that makes $\det(A - \lambda I) = 0$ and an eigenvector for this λ is a nonzero vector in the nullspace of $A - \lambda I$.

$P_A(\lambda) = \det(A - \lambda I)$ is called the <u>characteristic polynomial</u> of A, a degree n polynomial in λ that begins with $(-\lambda)^n$. The roots of P_A are the eigenvalues.

The Diagonal Form

<u>1J</u> Suppose the $n \times n$ matrix A has n linearly independent eigenvectors. If these vectors are the columns of S,

$$S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Proof: Let $x_1, \ldots, x_n$ be the eigenvectors so that $Ax_1 = \lambda_1 x_1, \ldots, Ax_n = \lambda x_n$

$$AS = A\begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \ldots & \lambda_n x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix} = \Lambda S$$

Since the columns of S $(x_1, \ldots, x_n)$ are independent, S is invertible.

$$\therefore \boxed{AS = S\Lambda \quad \text{or} \quad S^{-1}AS = \Lambda \quad \text{or} \quad A = S\Lambda S^{-1}}$$

1K The sum of the diagonal entries of A is the trace of A, denoted $tr(A)$. If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the diagonalizable matrix $A = S\Lambda S^{-1}$:

$$tr(A) = \sum_{i=1}^{n} \lambda_i$$

$$det(A) = \prod_{i=n}^{n} \lambda_i$$

Proof: For any $n \times n$ matrices B and C, $tr(BC) = tr(CB)$. Let d_r be the r^{th} diagonal of BC.

$$\left.\begin{aligned} tr(BC) = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}c_{ji} = \; & b_{11}c_{11} + b_{12}c_{21} + \dots + b_{1n}c_{n1} \\ & + b_{21}c_{12} + b_{22}c_{22} + \dots + b_{2n}c_{n2} \\ & \vdots \\ & + b_{n1}c_{1n} + b_{n2}c_{2n} + \dots + b_{nn}c_{nn} \end{aligned}\right\}$$

the order of addition doesn't matter - now sum this 'vertically'.

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}b_{ji}$$

$$= tr(CB)$$

Choose $B = S$ and $C = \Lambda S^{-1}$:

$$tr(A) = tr(S\Lambda S^{-1}) = tr(\Lambda S^{-1}S) = tr(\Lambda) = \sum_{i=1}^{n} \lambda_i$$

Note that A need not be symmetric, just diagonalizable. We will see that not all matrices are diagonalizable.

To prove the second property use the fact that $det(BC) = det(B)det(C)$. The proof of this fact is a bit too involved to go through here.

Also the determinant of a diagonal matrix is the product of the diagonal entries.

$$det(A) = det(S\Lambda S^{-1}) = det(S)det(\Lambda S^{-1}) = det(S)det(\Lambda)det(S^{-1})$$

$$= det(SS^{-1})det(\Lambda) = det(I)det(\Lambda) = \prod_{i=1}^{n} \lambda_i$$

Differential Equations

1L If A has n linearly independent eigenvectors $x_1, \ldots, x_n$ the solution

$$\frac{du}{dt} = Au, \quad u(0) = u_0 \longrightarrow u(t) = c_1 e^{\lambda_1 t} + \ldots + c_n e^{\lambda_n t}, \quad c = S^{-1}u_0$$

Proof: Let $v = S^{-1}u \longleftrightarrow Sv = u$

$$\frac{dv}{dt} = S^{-1}\frac{du}{dt} = S^{-1}Au = S^{-1}S\Lambda S^{-1}u = \Lambda v, \qquad v(0) = S^{-1}u(0) = S^{-1}u_0$$

We can solve the diagonal problem $\frac{dv}{dt} = \Lambda v$, $v_0 = S^{-1}u_0$ as n uncoupled first order ODE's:

$$\left.\begin{aligned} \frac{dv_1}{dt} &= \lambda_1 v_1 \longrightarrow v_1(t) = c_1 e^{\lambda_1 t} \\ &\vdots \qquad\qquad\qquad \vdots \\ \frac{dv_n}{dt} &= \lambda_n v_n \longrightarrow v_n(t) = c_n e^{\lambda_n t} \end{aligned}\right\} \longrightarrow v = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = e^{\Lambda t} c$$

$$u = Sv = Se^{\Lambda t}c = \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = c_1 e^{\lambda_1 t} x_1 + \ldots + c_n e^{\lambda_n t} x_n$$

$$u_0 = Se^{\Lambda \cdot 0}c = SIc = Sc \longrightarrow c = S^{-1}u_0$$

$\therefore$ $\frac{du}{dt} = Au$, $u(0) = u_0$ has the solution $u(t) = Se^{\Lambda t}c = c_1 e^{\lambda_1 t}x_1 + \ldots + c_n e^{\lambda_n t} x_n$ with coefficients determined by $S^{-1}u_0$, so we could also write:

$$u(t) = Se^{\Lambda t}S^{-1}u_0$$

Note that since eigenvectors are not unique, S and S^{-1} are not unique and a different set of eigenvectors changes c. But $u = Se^{\Lambda t}c$ is still the same since the new eigenvectors are each just constant multiples of the original choices:

Replace $x_1, \ldots, x_n$ with $k_1 x_1, \ldots, k_n x_n$.

Let $S_K = [k_1 x_1 \ldots k_n x_n] = S\begin{bmatrix} k_1 & & \\ & \ddots & \\ & & k_n \end{bmatrix} = SK$. We find $u(t) = S_K e^{\Lambda t} S_K^{-1} u_0$

Matrix multiplication is commutative for diagonal matrices. The solution

$$u(t) = SKe^{\Lambda t}K^{-1}S^{-1}u_0 = SKK^{-1}e^{\Lambda t}S^{-1}u_0 = Se^{\Lambda t}S^{-1}u_0$$

is the same as before — the choice of eigenvectors. Use permutation matrices to show that the order of the eigenvalues/vectors doesn't matter either.

Second Order Equations

Exercises

1.5.3 Find all eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

1.5.5 Solve the first order system $\frac{du}{dt} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} u$ with $u_0 = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$.

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