## §4.1 Fourier Series and Orthogonal Expansions

**4.1.1** Find the Fourier Series on $-\pi < x < \pi$ for

(b) $f(x) = |\sin x|$, an even function.

$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos kx$ $\qquad |\sin x| = \sin x$ on $0 \le x \le \pi$

$\boxed{a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx} = \frac{1}{\pi}\int_0^{\pi} \sin x\,dx = -\frac{1}{\pi}\cos x\Big|_0^{\pi} = -\frac{1}{\pi}(-1-1) = 2/\pi$

$\boxed{a_k = \frac{2}{\pi}\int_0^{\pi} f(x)\cos kx}$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x \cos kx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\{\sin(1+k)x + \sin(1-k)x\}\,dx$$

$$= -\frac{1}{\pi}\left\{\frac{\cos(1+k)x}{1+k} + \frac{\cos(1-k)x}{1-k}\right\}\Big|_0^{\pi}$$

$$= -\frac{1}{\pi}\left\{\frac{\cos(1+k)\pi}{1+k} - \frac{1}{1+k} + \frac{\cos(1-k)\pi}{1-k} - \frac{1}{1-k}\right\}$$

$$= -\frac{1}{\pi}\left\{\frac{(-1)^{k+1}-1}{1+k} + \frac{(-1)^{k+1}-1}{1-k}\right\}$$

$$= \begin{cases} \frac{2}{\pi}\left\{\frac{1}{1+k} + \frac{1}{1-k}\right\}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$$

| | | |
|---|---|---|
| $k=1$ | $\cos 2\pi = 1$ | $\cos 0 = 1$ |
| $k=2$ | $\cos 3\pi = -1$ | $\cos -\pi = -1$ |
| $k=3$ | $\cos 4\pi = 1$ | $\cos -2\pi = 1$ |
| $\vdots$ | $\vdots$ | $\vdots$ |

$$f(x) = |\sin x| = \frac{2}{\pi} + \sum_{\substack{k=2 \\ k\text{ even}}}^{\infty} \frac{2}{\pi}\left\{\frac{1}{1+k} + \frac{1}{1-k}\right\}\cos kx$$

$$= \frac{2}{\pi} + \frac{4}{\pi}\sum_{\substack{k=2 \\ k\text{ even}}}^{\infty} \frac{1}{1-k^2}\cos kx$$

$$= \boxed{\frac{2}{\pi} + \frac{4}{\pi}\sum_{j=1}^{\infty} \frac{1}{1-4j^2}\cos 2jx}$$

4.1.2 A square wave has $f(x)=-1$ on the left side $-\pi<x<0$ and $f(x)=1$ on the right side $0<x<\pi$.

(a) Why are all the cosine coefficients $a_k=0$?

Since $f$ is an odd function, $a_0=\int_{-\pi}^{\pi} f(x)\,dx=0$.
For $k\geq 0$ $f(x)\cos kx$ is the product of an odd function with an even function so $f(x)\cos kx$ is an odd function $\longrightarrow a_k=\int_{-\pi}^{\pi} f(x)\cos kx\,dx=0$.

(b) Find the sine series $\sum b_k \sin kx$ from equation (6):

$$\boxed{b_k=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx} \qquad (6)$$

$$b_k=\frac{1}{\pi}\int_{-\pi}^{0} -\sin kx\,dx+\frac{1}{\pi}\int_0^{\pi}\sin kx\,dx$$

$$=\frac{1}{\pi}\int_{\pi}^{0}\sin -ky\,dy+\frac{1}{\pi}\int_0^{\pi}\sin kx\,dx$$

1 $\cos\pi$
2 $\cos 2\pi$

$$=\frac{1}{\pi}\int_0^{\pi}\sin ky\,dy+\frac{1}{\pi}\int_0^{\pi}\sin kx\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}\sin kx\,dx=-\frac{2}{\pi k}\cos kx\Big|_0^{\pi}=-\frac{2}{\pi k}\left\{(-1)^k-1\right\}=\begin{cases}4/\pi k, & k \text{ odd}\\ 0, & k \text{ even}\end{cases}$$

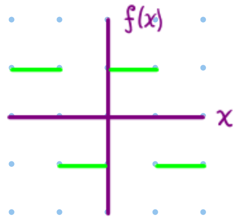
$$\boxed{f(x)=\sum_{\substack{k=1\\ k\text{ odd}}}^{\infty}\frac{4}{\pi k}\sin kx=\sum_{j=1}^{\infty}\frac{4}{\pi(2j-1)}\sin(2j-1)x=\frac{4}{\pi}\left\{\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\dots\right\}}$$

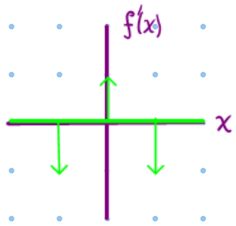
**4.1.3** Find the sine series for the square wave in another way by showing

(a) $df/dx = 2\delta(x) - 2\delta(x+\pi)$ extended periodically.

If $f(x)$ is the square wave extended periodically

$$f(x) = \begin{cases} \vdots & \\ 1 & -2\pi < x < -\pi \\ -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \\ -1 & \pi < x < 2\pi \\ \vdots & \end{cases} \qquad \frac{df}{dx} = \begin{cases} \vdots & \\ -\infty & x = -\pi \\ 0 & -\pi < x < 0 \\ \infty & x = 0 \\ 0 & 0 < x < \pi \\ -\infty & x = \pi \\ 0 & \pi < x < 2\pi \\ \vdots & \end{cases}$$




$\frac{df}{dx}$ is the $2\pi$-periodic extension of $g(x) = \begin{cases} -\infty & x = -\pi \\ 0 & -\pi < x < 0 \\ \infty & x = 0 \\ 0 & 0 < x < \pi \end{cases}$

For $x \in [-\pi, \pi)$, $2\delta(x) - 2\delta(x+\pi) = \begin{cases} -\infty & x = -\pi \\ 0 & -\pi < x < 0 \\ \infty & x = 0 \\ 0 & 0 < x < \pi \end{cases} = g(x)$

That is, $df/dx$ is the $2\pi$-periodic extension of $2\delta(x) - 2\delta(x+\pi)$

(b) $2\delta(x) - 2\delta(x+\pi) = \frac{4}{\pi}\{\cos x + \cos 3x + \dots\}$

From page 269, for $\delta(x)$, $a_0 = 1/2\pi$, $a_k = 1/\pi$, $b_k = 0$. This implies:

$$2\delta(x) - 2\delta(x+\pi) = 2\left\{\frac{1}{2\pi} + \sum_{k=1}^{\infty} a_k \cos kx\right\} - 2\left\{\frac{1}{2\pi} + \sum_{k=1}^{\infty} a_k \cos k(x+\pi)\right\}$$

$$= 2\sum_{k=1}^{\infty} \frac{1}{\pi}\{\cos kx - \cos k(x+\pi)\} = \frac{4}{\pi}\sum_{k=1,\, k \text{ odd}}^{\infty} \cos kx = \frac{4}{\pi}\{\cos x + \cos 3x + \dots\}$$

(Since $\cos kx = -\cos k(x+\pi)$, $k$ odd and $\cos kx = \cos k(x+\pi)$, $k$ even)

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From parts (a) and (b) conclude that the Fourier series is:

$$f(x) = \int_{-\pi}^{\pi} \{2\delta(x) - 2\delta(x+\pi)\}\,dx = \int \frac{4}{\pi}\sum_{k=1,\, k \text{ odd}}^{\infty} \cos kx\, dx = \frac{4}{\pi}\sum_{k=1,\, k \text{ odd}}^{\infty} \frac{1}{k}\sin kx$$

$$\boxed{f(x) = \frac{4}{\pi}\sum_{j=1}^{\infty} \frac{1}{2j-1}\sin(2j-1)x = \frac{4}{\pi}\left\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right\}}$$

Laplace's Equation in Cartesian coordinates is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=u_{xx}+u_{yy}=0$$

Let $u=u(r,\theta)$ with $x=r\cos\theta$, $y=r\sin\theta$

$$r=(x^2+y^2)^{1/2}, \quad \theta=\arctan y/x+c$$

depends on the quadrant

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$$u_x=u_r r_x+u_\theta \theta_x \qquad r_x=\tfrac{1}{2}(x^2+y^2)^{-1/2}\,2x=x/(x^2+y^2)^{1/2}=x/r=\cos\theta$$

$$=u_r\cos\theta-\tfrac{1}{r}u_\theta\sin\theta \qquad \theta_x=\frac{-y/x^2}{1+(y/x)^2}=-\frac{y}{x^2+y^2}=-y/r^2=-\sin\theta/r$$

$$u_{xx}=u_r(\cos\theta)_x+u_{rx}\cos\theta-u_\theta\left(\frac{\sin\theta}{r}\right)_x-u_{\theta x}\frac{\sin\theta}{r}$$

$$=u_r(-\sin\theta)(-\sin\theta/r)+u_{rx}\cos\theta-u_\theta\frac{(\cos\theta)(-\sin\theta/r)r-\sin\theta\cos\theta}{r^2}$$

$$-u_{\theta x}\frac{\sin\theta}{r}$$

$$=\frac{\sin^2\theta}{r}u_r+u_{rx}\cos\theta+u_\theta\frac{2\cos\theta\sin\theta}{r^2}-u_{\theta x}\frac{\sin\theta}{r}$$

$$=\frac{\sin^2\theta}{r}u_r+u_{rr}\cos^2\theta-u_{r\theta}\frac{\sin\theta\cos\theta}{r}$$

$$+u_\theta\frac{2\cos\theta\sin\theta}{r^2}-\left(u_{\theta\theta}\left(\frac{-\sin\theta}{r}\right)+u_{\theta r}\cos\theta\right)\left(\frac{\sin\theta}{r}\right)$$

$$\boxed{\begin{aligned}u_{xx}&=\cos^2\theta\, u_{rr}+\frac{\sin^2\theta}{r}u_r-\frac{2\sin\theta\cos\theta}{r}u_{r\theta}\\&\quad+\frac{2\sin\theta\cos\theta}{r^2}u_\theta+\frac{\sin^2\theta}{r^2}u_{\theta\theta}\end{aligned}}$$

$$\left\{\begin{aligned}u_{rx}&=u_{rr}r_x+u_{r\theta}\theta_x\\&=u_{rr}\cos\theta-u_{r\theta}\frac{\sin\theta}{r}\\u_{\theta x}&=u_{\theta\theta}\theta_x+u_{\theta r}r_x\\&=u_{\theta\theta}\left(\frac{-\sin\theta}{r}\right)+u_{\theta r}\cos\theta\end{aligned}\right\}$$

$u_y = u_r r_y + u_\theta \theta_y \qquad r_y = y/r = \sin\theta$

$= u_r \sin\theta + \frac{1}{r} u_\theta \cos\theta \qquad \theta_y = \frac{1/x}{1 + y^2/x^2} = \frac{x}{x^2+y^2} = x/r^2 = \cos\theta/r$

$u_{yy} = u_{ry}\sin\theta + u_r(\sin\theta)_y + u_{\theta y}\frac{\cos\theta}{r} + u_\theta\left(\frac{\cos\theta}{r}\right)_y$

$= \left(u_{rr}\sin\theta + u_{r\theta}\frac{\cos\theta}{r}\right)\sin\theta$

$+ u_r \cos^2\theta/r + u_{\theta\theta}\cos^2\theta/r^2$

$+ u_{\theta r}\frac{\sin\theta\cos\theta}{r}$

$+ u_\theta\left(-\frac{2\sin\theta\cos\theta}{r^2}\right)$

$$\left\{\begin{aligned} u_{ry} &= u_{rr} r_y + u_{r\theta}\theta_y \\ &= u_{rr}\sin\theta + u_{r\theta}\frac{\cos\theta}{r} \\ u_{\theta y} &= u_{\theta\theta}\theta_y + u_{\theta r} r_y \\ &= u_{\theta\theta}\cos\theta/r + u_{\theta r}\sin\theta \end{aligned}\right\}$$

$$\boxed{u_{yy} = \sin^2\theta\, u_{rr} + \frac{\cos^2\theta}{r}u_r + \frac{2\sin\theta\cos\theta}{r}u_{r\theta} - \frac{2\sin\theta\cos\theta}{r^2}u_\theta + \frac{\cos^2\theta}{r^2}u_{\theta\theta}}$$

$0 = u_{xx} + u_{yy}$

$= (\cos^2\theta + \sin^2\theta)u_{rr}$

$+ \frac{1}{r}(\sin^2\theta + \cos^2\theta)u_r + 0 \cdot u_{r\theta}$

$+ 0 \cdot u_\theta + \frac{1}{r^2}(\sin^2\theta + \cos^2\theta)u_{\theta\theta}$

$= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}$

$$\left\{\begin{aligned} \left(\frac{\cos\theta}{r}\right)_y &= \frac{-\sin\theta\,\theta_y r}{r^2} - \frac{\cos\theta\, r_y}{r^2} \\ &= -\frac{\sin\theta\cos\theta}{r^2} - \frac{\cos\theta\sin\theta}{r^2} \\ &= -\frac{2\sin\theta\cos\theta}{r^2} \end{aligned}\right\}$$

Laplace's Equation in Polar coordinates is

$$0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}$$

$u = a_0$ (constant) $\quad \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} = \frac{1}{r}0 + \frac{1}{r^2}0 = 0.$

$u(r,\theta) = r^n \sin n\theta$

$\frac{1}{r}(r n r^{n-1}\sin n\theta)_r + \frac{1}{r^2}(-n^2 r^n \sin n\theta) = \frac{1}{r}n^2 r^{n-2}\sin n\theta - n^2/r^2\, r^n \sin n\theta = 0$

$u(r,\theta) = r^n \cos n\theta$

$\frac{1}{r}(r n r^{n-1}\cos n\theta)_r + \frac{1}{r^2}(-n^2 r^n \cos n\theta) = \frac{1}{r}n^2 r^{n-2}\cos n\theta - n^2/r^2\, r^n \cos n\theta = 0$

$u(r,\theta) = a_0 + a_1 r\cos\theta + b_1 r\sin\theta + a_2 r^2\cos n\theta + b_2 r^2 \sin n\theta + \ldots$

Satisfies Laplace's equation in polar coordinates.

4.1.6 Around the unit circle suppose $u$ is a square wave

$$u_0 = \begin{cases} +1 \text{ on the upper semicircle} & 0<\theta<\pi \\ -1 \text{ on the lower semicircle} & -\pi<\theta<0 \end{cases}$$

From the Fourier series for the square wave write down the Fourier series for $u$ (the solution 21) to Laplace's equation). What is the value of $u$ at the origin?

$$u(r,\theta) = a_0 + a_1 r\cos\theta + b_1 r\sin\theta + a_2 r^2\cos 2\theta + b_2 r^2 \sin 2\theta + \ldots$$

$$1 = u(1,\theta) = a_0 + a_1\cos\theta + b_1\sin\theta + \ldots \quad 0<\theta<\pi$$
$$-1 = u(1,\theta) = a_0 + a_1\cos\theta + b_1\sin\theta + \ldots \quad -\pi<\theta<0$$

$$1 = u(1,\pi/2) = a_0 + b_1 + b_2 + b_3 + \ldots$$
$$-1 = u(1,-\pi/2) = a_0 - b_1 - b_2 - b_3 - \ldots \rightarrow 0 = a_0$$

Since $\cos(\theta) = \cos(-\theta)$ and $-\sin(\theta) = \sin(-\theta)$, for $0<\theta<\pi$:

$$1 = u(1,\theta) = a_1\cos\theta + b_1\sin\theta + a_2\cos 2\theta + b_2\sin 2\theta + \ldots$$
$$-1 = u(1,-\theta) = a_1\cos\theta - b_1\sin\theta + a_2\cos 2\theta - b_2\sin 2\theta + \ldots$$

$$0 = a_1\cos\theta + a_2\cos 2\theta + a_3\cos 3\theta + \ldots$$

since $\{\cos k\theta\}_{k=1}^{\infty}$ are orthogonal wrt to the $L^2[-\pi,\pi]$ inner product, $a_i = 0$ for all $i$.

$$u(r,\theta) = b_1 r\sin\theta + b_2 r^2 \sin 2\theta + \ldots$$

$$1 = u(1,\theta) = b_1\sin\theta + b_2\sin 2\theta + b_3\sin 3\theta + b_4\sin 4\theta + \ldots \text{ for } 0<\theta<\pi$$

From Exercises 4.1.2, 4.1.3,

$$1 = \frac{4}{\pi}\left\{\sin\theta + \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta + \ldots\right\} \text{ for } 0<\theta<\pi$$

This implies $b_1 = 1$, $b_2 = 0$, $b_3 = 1/3$, $b_4 = 0$, $b_5 = 1/5$, ...

$$\therefore u(r,\theta) = \frac{4}{\pi}\left\{r\sin\theta + \frac{1}{3}r^3\sin 3\theta + \frac{1}{5}r^5\sin 5\theta + \ldots\right\}$$

$$\lim_{r\to 0} u(r,\theta) = u(0,\theta) = 0.$$

4.1.10 What constant function is closest in the least square sense to $f(x) = \cos^2 x$? What multiple of $\cos x$ is closest to $f(x) = \cos^3 x$?