§ 4.3 Fourier Integrals

4.3.1 Find the transform $\hat{g}$ of the ascending pulse $g(x) = \begin{cases} e^{ax} \,, & x < 0 \\ 0 \,, & x > 0 \end{cases}$

$$\hat{g}(k) = \int_{-\infty}^{\infty} g(x) e^{-ikx} \, dx = \int_{-\infty}^{0} e^{ax} e^{-ikx} \, dx = \left[\frac{e^{(a-ik)x}}{a-ik} \right]_{-\infty}^{0} = \frac{1}{a-ik}$$

4.3.2 Find the Fourier transforms (with $f = 0$ outside the given regions) of

(b) $f(x) = 1$ for $x < 0$

$$\hat{g}(k) = 2\pi \delta(k)$$

$$g(x) = \mathcal{F}^{-1}\{\hat{g}(k)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi \delta(k) e^{ikx} \, dk = e^{0} = 1 \quad \forall x$$

$$\therefore \mathcal{F}\{1\} = 2\pi \delta(k)$$

$$h(x) = \begin{cases} 1 \,, & x \geq 0 \\ -1 \,, & x < 0 \end{cases}$$

$$\hat{h}(k) = \lim_{a \to 0} \frac{-2ik}{a^2 + k^2} = \frac{-2i}{k} = \frac{2}{ik} \quad (\text{pg } 311)$$

$$\tfrac{1}{2} g(x) - \tfrac{1}{2} h(x) = \begin{cases} 0 \,, & x \geq 0 \\ 1 \,, & x < 0 \end{cases} = f(x)$$

$$\therefore \hat{f}(k) = \mathcal{F}\{\tfrac{1}{2} g - \tfrac{1}{2} h\} = \tfrac{1}{2}\hat{g} - \tfrac{1}{2}\hat{h} = \pi \delta(k) - \frac{1}{ik}$$

(c) $f(x) = \int_{0}^{1} e^{ikx} \, dk$

Claim $\hat{f}(k) = \begin{cases} 2\pi \,, & k \in [0,1] \\ 0 \,, & \text{else} \end{cases}$

Check $\mathcal{F}^{-1}\{\hat{f}(k)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} \, dk = \frac{1}{2\pi} \int_{0}^{1} 2\pi e^{ikx} \, dk = f(x)$. ✓

4.3.3

(b) Find the inverse transform of $\hat{f}(k) = e^{-|k|}$