1.2 Gaussian Elimination

Exercises

1.2.1

Solve using elimination to reach $Ux = c$ and then back substitution to compute x.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

Elimination:

$$\begin{array}{c} \qquad A \qquad\qquad b \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 3 & 3 & 0 \\ 1 & 3 & 5 & 2 \end{array}\right] \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 2 & 2 & -2 \\ 0 & 2 & 4 & 0 \end{array}\right] \sim \begin{array}{c} \qquad U \qquad\qquad c \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 2 & 2 & -2 \\ 0 & 0 & 2 & 2 \end{array}\right] \end{array}$$

Back-substitution:

$$2x_3 = 2x_3 \rightarrow x_3 = 1$$

$$2x_2 + 2x_3 = -2 \rightarrow x_2 = -2$$

$$x_1 + x_2 + x_3 = 2 \rightarrow x_1 = 3$$

Solution:

$$x = (3, -2, 1)$$

1.2.4

(a) Find the value of q for which elimination fails in the system

$$\begin{bmatrix} 3 & 6 \\ 6 & q \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & 6 & 1 \\ 6 & q & 4 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & 6 & 1 \\ 0 & q-12 & 2 \end{array}\right] \longrightarrow \begin{array}{l} 3x_1 + 6x_2 = 1 \\ (q-12)x_2 = 2 \end{array}$$

Elimination fails if $q = 12$
We'd have $0x_2 = 2$

(d) What value should replace $b_2 = 4$ to make the system solvable for this q ($q = 12$)?

Replace $b_2 = 4$ with $b_2 = 2$. The system reduces to $3x_1 + 6x_2 = 1$ and $0x_2 = 0$. Then any choice of x_2 is ok and so $\exists$ infinitely many solutions for x.

1.2.9 Find examples of 2×2 matrices such that

(b) $A^2 = -I$ with real entries in A.

(c) $B^2 = 0$ with no 0's in B.

(d) $CD = -DC$ with $CD \neq 0$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

1.2.10

Factor A into LU and solve $Ax = b$ for the 3 right sides $b = e_1, e_2, e_3$.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$L_1^{-1} A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$L_2^{-1} L_1^{-1} A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$\rightarrow A = (L_2^{-1} L_1^{-1})^{-1} U \rightarrow L = (L_2^{-1} L_1^{-1})^{-1} = L_1 L_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Since $A = LU$, $Ax = b \rightarrow LUx = b$. Let $Ux = c$ so $Lc = b$. To find x, solve $Lc = b$ for c and then $Ux = c$ for x.

- $b = e_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \rightarrow c_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \rightarrow x_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

- $b = e_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{array}\right] \rightarrow c_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \rightarrow x_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

- $b = e_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{array}\right] \rightarrow c_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \rightarrow x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Note that we have found $A[x_1\ x_2\ x_3] = [e_1\ e_2\ e_3]$, $\therefore A^{-1} = [x_1\ x_2\ x_3]$.

1.2.12 What combination of the vectors v_1, v_2, v_3 gives b?

$$v_1 = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix} \quad v_2 = \begin{bmatrix} 3 \\ 4 \\ 9 \end{bmatrix} \quad v_3 = \begin{bmatrix} 2 \\ 0 \\ 7 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ -8 \\ 7 \end{bmatrix}$$

Let $V = [v_1 \ v_2 \ v_3]$. Solve $Vx = b$ for x.

$$\left[\begin{array}{ccc|c} 2 & 3 & 2 & 2 \\ 0 & 4 & 0 & -8 \\ 6 & 9 & 7 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 3 & 2 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$x = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \rightarrow b = 3v_1 - 2v_2 + v_3$$

1.2.13 What is the intersection point of the three planes?

$$\Pi_1 : \ 2x_1 + 3x_2 + 2x_3 = 2$$

$$\Pi_2 : \ 4x_2 = -8$$

$$\Pi_3 : \ 6x_1 + 9x_2 + 7x_3 = 7$$

If $x = (x_1, x_2, x_3)$ is the intersection point of the three planes, all three of the equations defining the planes must hold simultaneously. This means we solve as a system of equations. Written in matrix form:

$$\begin{bmatrix} 2 & 3 & 2 \\ 0 & 4 & 0 \\ 6 & 9 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -8 \\ 7 \end{bmatrix}$$

Notice that we already have the solution from Exercise 1.2.12. The intersection point is:

$$x = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$