

## § 4.4 Complex Variables and Conformal Mapping

### 4.4.2 Find the real and imaginary parts of

(d)  $i \log i \log \log i$

$$z = i \log i \log \log i$$

$$z = i \cdot i \pi/2 \log i \pi/2$$

$$\begin{aligned} z &= i \cdot i \pi/2 (\log \pi/2 + i \pi/2) \\ &= -\pi/2 (\log \pi/2 + i \pi/2) \end{aligned}$$

$$= \underbrace{-\pi/2 \log \pi/2}_{\text{real}} - \underbrace{i \pi^2/4}_{\text{imaginary}}$$

$$\log i = w \rightarrow e^w = i = e^{i\pi/2} \rightarrow \log i = i\pi/2$$

$$\log i \pi/2 = \log i + \log \pi/2 = i\pi/2 + \log \pi/2$$

### 4.4.3 What can you say about

(c) the product of two numbers on the unit circle  $z = e^{i\theta}$ ?

You can say the product is also on the unit circle  $|z| = 1$

$$v = e^{i\theta}, w = e^{i\phi} \rightarrow vw = e^{i(\theta+\phi)} \rightarrow |vw| = 1$$

(d) the sum of two numbers on the unit circle?

You can say the sum is on the disk  $|z| \leq 2$

$$v = e^{i\theta}, w = e^{i\phi} \rightarrow |v+w| \leq |v| + |w| = 1 + 1 = 2.$$

You cannot say the sum is on  $|z| = 1$  or  $|z| = 2$  generally. Consider the examples  $v = i, w = -i$  with  $|v+w| = 0$  or  $v = 1, w = i$  with  $|v+w| = \sqrt{2}$ .

### 4.4.4 Find the absolute value (or modulus) $|z|$ if

(a)  $z = e^i$

$$e^i = e^{i \cdot 1} = \cos 1 + i \sin 1 \rightarrow |e^i|^2 = \cos^2 1 + \sin^2 1 = \boxed{1}$$

(c)  $z = \frac{3+i}{3-i}$

$$\frac{3+i}{3-i} = \frac{(3+i)(3+i)}{(3-i)(3+i)} = \frac{9+6i+i^2}{9-i^2} = \frac{8+6i}{10} = \frac{4}{5} + \frac{3}{5}i \rightarrow |z| = \left(\frac{16}{25} + \frac{9}{25}\right)^{1/2} = \boxed{1}$$

(e)  $z = e^{3+4i}$

$$e^{3+4i} = e^3 e^{4i} \rightarrow |z| = |e^3 e^{4i}| = |e^3| |e^{4i}| = e^3 \cdot 1 = \boxed{e^3}$$

## Analytic Functions and Laplace's Equation

Laplace's Equation in 2-D:  $u_{xx} + u_{yy} = 0$  (1)

Any 'decent' function  $f(z) = f(x+iy)$  will be a solution.

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} \frac{\partial z}{\partial x} = \frac{\partial f}{\partial z} \quad \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} \frac{\partial z}{\partial y} = i \frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} \quad \frac{\partial f}{\partial y} = i \frac{\partial f}{\partial z}$$

$$i \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} \quad (2)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial z} \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial z} \right) = \frac{\partial^2 f}{\partial z^2} \frac{\partial z}{\partial x} = \frac{\partial^2 f}{\partial z^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial z^2} \quad (3)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial z} \frac{\partial z}{\partial y} \right) = i \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial z} \right) = i \frac{\partial^2 f}{\partial z^2} \frac{\partial z}{\partial y} = i^2 \frac{\partial^2 f}{\partial z^2}$$

$$\frac{\partial^2 f}{\partial y^2} = - \frac{\partial^2 f}{\partial z^2} \quad (4)$$

$$\therefore \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial z^2} - \frac{\partial^2 f}{\partial z^2} = 0$$

Let  $f(x+iy) = u(x,y) + i s(x,y)$  and substitute into (2):

$$i \left( \frac{\partial u}{\partial x} + i \frac{\partial s}{\partial x} \right) = \frac{\partial u}{\partial y} + i \frac{\partial s}{\partial y}$$

$$\text{Cauchy-Riemann Equations: } \frac{\partial u}{\partial x} = \frac{\partial s}{\partial y} \text{ and } \frac{\partial u}{\partial y} = - \frac{\partial s}{\partial x} \quad (5), (6)$$

Definition A function  $f(z)$  is analytic at  $z=a$  if in a neighborhood of  $a$ ,

- (1)  $f(z)$  depends on the combination  $z = x+iy$  and satisfies  $i \partial f / \partial x = \partial f / \partial y$ ,
- (2) the real and imaginary parts of  $f(z)$  are connected by the C-R equations  $u_x = s_y$  and  $u_y = -s_x$ ,
- (3)  $f(z)$  is the sum of a convergent power series  $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ .

If these conditions are satisfied then the real functions  $u$  and  $s$  satisfy Laplace's equation and  $u+is$  is a combination of the powers  $(x+iy)^n$ .

Ex)  $f = (x+iy)^n$ ,  $f = e^{x+iy}$ ,  $f = \frac{1}{1-z}$  ( $|z| \neq 1$ ) are analytic at all admissible  $z$ .  
 $f = f(x-iy)$  is not analytic.

#### 4.4.7 Are the following functions analytic?

(a)  $f = |z|^2 = x^2 + y^2$

$\begin{matrix} a, c, e \\ 2, 3, 4, 7, 8, 10, 11, 13, 17, 18, 20, 21, 23 \\ d \quad c, d \end{matrix}$

(b)  $f = \operatorname{Re} z = x$

(c)  $f = \sin z = \sin x \cosh y + i \cos x \sinh y$

Can a function satisfy Laplace's equation without being analytic?

(a) Not analytic since condition 1 is not satisfied:  $i \frac{\partial f}{\partial x} = 2ix \neq 2y = \frac{\partial f}{\partial y}$

(b) Not analytic since condition 1 is not satisfied:  $i \frac{\partial f}{\partial x} = i \neq 0 = \frac{\partial f}{\partial y}$

(c) Analytic at any point a:

(1)  $f(z) = f(x+iy) = \sin(x+iy)$  depends on  $x+iy$  and  $i \frac{\partial f}{\partial x} = i \cos z = \frac{\partial f}{\partial y}$ .

(2)  $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(\sin x \cosh y) = \cos x \cosh y = \frac{\partial}{\partial y}(\cos x \sinh y) = \frac{\partial s}{\partial y} \checkmark$   
 $\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(\sin x \cosh y) = \sin x \sinh y = -\frac{\partial}{\partial x}(\cos x \sinh y) = -\frac{\partial s}{\partial x} \checkmark$

(3)  $f(z) = \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \quad \forall z \in \mathbb{C}$

Yes a function can satisfy Laplace's equation w/o being analytic:

$f = \operatorname{Re} z = x$  is not analytic yet:  $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 + 0 = 0$ .

#### 4.4.8

(a) If  $u(x,y) = x+4y$ , find its conjugate function  $s(x,y)$ .

(b) If  $s(x,y) = (1+x)y$ , find its conjugate function  $u(x,y)$ .

(c) If  $u = x^2$ , why does no  $s$  satisfy the C-R equations?

Answers:

(a)  $\frac{\partial s}{\partial y} = \frac{\partial u}{\partial x} = 1 \rightarrow s = y + h_1(x)$   
 $\frac{\partial s}{\partial x} = -\frac{\partial u}{\partial y} = -4 \rightarrow s = -4x + h_2(y) \Rightarrow \begin{cases} -4 = \frac{\partial s}{\partial x} = h_1'(x) \\ -4x = h_1(x) + C \end{cases} \Rightarrow \boxed{s(x,y) = y - 4x} (+C)$

(b)  $\frac{\partial u}{\partial x} = \frac{\partial s}{\partial y} = 1+x \rightarrow u = x + \frac{1}{2}x^2 + g_1(y)$   
 $\frac{\partial u}{\partial y} = -\frac{\partial s}{\partial x} = y \rightarrow u = \frac{1}{2}y^2 + g_2(x) \Rightarrow \begin{cases} y = \frac{\partial u}{\partial y} = g_1'(y) \\ \frac{1}{2}y^2 = g_1(y) + C \end{cases} \Rightarrow \boxed{u(x,y) = x + \frac{1}{2}(x^2 + y^2)}$

(c)  $\frac{\partial s}{\partial y} = \frac{\partial u}{\partial x} = 2x \rightarrow s = 2xy + j(y)$   
 $\frac{\partial s}{\partial x} = -\frac{\partial u}{\partial y} = 0 \rightarrow s = \text{constant}$  } These conditions cannot be satisfied simultaneously. Also  $u_{xx} + u_{yy} = 2 \neq 0$ .









