

1.5 Eigenvalues and Dynamical Systems

Notes

In this chapter the focus is on square matrices. Suppose A is an $n \times n$ matrix. Normally multiplication by A changes the direction of x . For certain exceptional vectors, however, Ax is a multiple of x :

$$Ax = \lambda x \quad \lambda \text{ (a scalar) is called an } \underline{\text{eigenvalue}} \text{ of } A \\ (x \neq 0) \quad x \text{ is called an } \underline{\text{eigenvector}} \text{ corresponding to } \lambda$$

For diagonal matrices the eigenvalues are the diagonal entries and the eigenvectors are (nonzero multiples of) the coordinate directions.

Example: $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \rightarrow A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix}, A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

For other matrices look for the eigenvalues first:

$$Ax = \lambda x \rightarrow (A - \lambda I)x = 0 \rightarrow A - \lambda I \text{ is singular since } x \neq 0$$

This means that an eigenvalue of A is a number that makes $\det(A - \lambda I) = 0$ and an eigenvector for this λ is a nonzero vector in the nullspace of $A - \lambda I$.

$P_A(\lambda) = \det(A - \lambda I)$ is called the characteristic Polynomial of A , a degree n polynomial in λ that begins with $(-\lambda)^n$. The roots of P_A are the eigenvalues.

The Diagonal Form

1J Suppose the $n \times n$ matrix A has n linearly independent eigenvectors. If these vectors are the columns of S ,

$$S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Proof: Let x_1, \dots, x_n be the eigenvectors so that $Ax_1 = \lambda_1 x_1, \dots, Ax_n = \lambda_n x_n$

$$AS = A \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \dots & \lambda_n x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} = \Lambda S$$

Since the columns of S (x_1, \dots, x_n) are independent, S is invertible.

$$\therefore AS = S\Lambda \quad \text{or} \quad S^{-1}AS = \Lambda \quad \text{or} \quad A = S\Lambda S^{-1}$$

TH The sum of the diagonal entries of A is the trace of A , denoted $\text{tr}(A)$. If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the diagonalizable matrix $A = S\Lambda S^{-1}$.

$$\text{tr}(A) = \sum_{i=1}^n \lambda_i$$

$$\det(A) = \prod_{i=1}^n \lambda_i$$

Proof: For any $n \times n$ matrices B and C , $\text{tr}(BC) = \text{tr}(CB)$.

$$\begin{aligned} \text{tr}(BC) &= \sum_{i=1}^n \sum_{j=1}^n b_{ij}c_{ji} = b_{11}c_{11} + b_{12}c_{21} + \dots + b_{1n}c_{n1} \\ &\quad + b_{21}c_{12} + b_{22}c_{22} + \dots + b_{2n}c_{n2} \\ &\quad \vdots \\ &\quad + b_{n1}c_{1n} + b_{n2}c_{2n} + \dots + b_{nn}c_{nn} \end{aligned} \quad \left. \begin{array}{l} \text{the order of} \\ \text{addition doesn't} \\ \text{matter - now sum} \\ \text{this 'vertically'.} \end{array} \right\}$$

$$\begin{aligned} &= \sum_{i=1}^n \sum_{j=1}^n c_{ij}b_{ji} \\ &= \text{tr}(CB) \end{aligned}$$

Choose $B = S$ and $C = \Lambda S^{-1}$:

$$\text{tr}(A) = \text{tr}(S\Lambda S^{-1}) = \text{tr}(\Lambda S^{-1}S) = \text{tr}(\Lambda) = \sum_{i=1}^n \lambda_i$$

Note that A need not be symmetric, just diagonalizable. We will see that not all matrices are diagonalizable.

To prove the second property use the fact that $\det(BC) = \det(B)\det(C)$. The proof of this fact is a bit too involved to go through here.

Also the determinant of a diagonal matrix is the product of the diagonal entries.

$$\begin{aligned} \det(A) &= \det(S\Lambda S^{-1}) = \det(S)\det(\Lambda S^{-1}) = \det(S)\det(\Lambda)\det(S^{-1}) \\ &= \det(ss^{-1})\det(\Lambda) = \det(I)\det(\Lambda) = \prod_{i=1}^n \lambda_i \end{aligned}$$

Differential Equations

1L If A has n linearly independent eigenvectors x_1, \dots, x_n the solution

$$\begin{aligned} \frac{du}{dt} = Au &\quad \rightarrow \quad u(t) = c_1 e^{\lambda_1 t} + \dots + c_n e^{\lambda_n t} \\ u(0) = u_0 &\quad \quad \quad c = S^{-1} u_0 \end{aligned}$$

Proof: Let $v = S^{-1} u \leftrightarrow Sv = u$

$$\frac{dv}{dt} = S^{-1} \frac{du}{dt} = S^{-1} Au = S^{-1} S \Lambda S^{-1} u = \Lambda v, \quad v(0) = S^{-1} u(0) = S^{-1} u_0.$$

We can solve the diagonal problem $\frac{dv}{dt} = \Lambda v, \quad v_0 = S^{-1} u_0$ as n uncoupled first order ODE's:

$$\left. \begin{aligned} \frac{dv_1}{dt} = \lambda_1 v_1 &\rightarrow v_1(t) = c_1 e^{\lambda_1 t} \\ \vdots &\vdots \\ \frac{dv_n}{dt} = \lambda_n v_n &\rightarrow v_n(t) = c_n e^{\lambda_n t} \end{aligned} \right\} \rightarrow v = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = e^{\Lambda t} c$$

$$u = Sv = Se^{\Lambda t} c = [x_1 \dots x_n] \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = c_1 e^{\lambda_1 t} x_1 + \dots + c_n e^{\lambda_n t} x_n$$

$$u_0 = Se^{\Lambda \cdot 0} c = SIc = Sc \rightarrow c = S^{-1} u_0$$

$\therefore \frac{du}{dt} = Au, \quad u(0) = u_0$ has the solution $u(t) = Se^{\Lambda t} c = c_1 e^{\lambda_1 t} x_1 + \dots + c_n e^{\lambda_n t} x_n$ with coefficients determined by $S^{-1} u_0$, so we could also write:

$$u(t) = Se^{\Lambda t} S^{-1} u_0$$

Note that since eigenvectors are not unique, S and S^{-1} are not unique and a different set of eigenvectors changes c. But $u = Se^{\Lambda t} c$ is still the same since the new eigenvectors are each just constant multiples of the original choices:

Replace x_1, \dots, x_n with $k_1 x_1, \dots, k_n x_n$.

Let $S_k = [k_1 x_1 \dots k_n x_n] = S \begin{bmatrix} k_1 & & \\ & \ddots & \\ & & k_n \end{bmatrix} = SK$. We find $u(t) = S_k e^{\Lambda t} S_k^{-1} u_0$

Matrix multiplication is commutative for diagonal matrices. The solution

$$u(t) = SK e^{\Lambda t} K^{-1} S^{-1} u_0 = SKK^{-1} e^{\Lambda t} S^{-1} u_0 = Se^{\Lambda t} S^{-1} u_0$$

is the same as before - the choice of eigenvectors. Use permutation matrices to show that the order of the eigenvalues/vectors doesn't matter either.

Second Order Equations

Consider the 1-d problem $\frac{d^2 u}{dt^2} + \lambda u = 0$, $u(0) = u_0$, $u'(0) = u'_0$, $\lambda > 0$.

The solution is:

$$u(t) = a \cos \omega t + b \sin \omega t \quad \text{with } \omega = \sqrt{\lambda}$$

$$u_0 = u(0) = a, \quad u'_0 = u'(0) = b\omega$$

$$\underline{u(t) = u_0 \cos \omega t + u'_0 / \omega \sin \omega t}$$

Consider now the n-dimensional problem $\frac{d^2 u}{dt^2} + A u = 0$, $u(0) = u_0$, $u'(0) = u'_0$.

Suppose λ, x are an eigenpair of A with $\lambda > 0$ so $Ax = \lambda x$ and $\sqrt{\lambda} \in \mathbb{R}$. Based on the 1-d solution, try $u(t) = (a \cos \omega t + b \sin \omega t)x$, $\omega = \sqrt{\lambda}$

$$\begin{aligned} u_{tt} + Au &= (-a\omega^2 \cos \omega t - b\omega^2 \sin \omega t)x + (a \cos \omega t + b \sin \omega t)\lambda x \\ &= (-a\lambda \cos \omega t - b\lambda \sin \omega t)x + (a \cos \omega t + b \sin \omega t)x = 0 \end{aligned}$$

This guess for $u(t)$ satisfies $u_{tt} + Au = 0$. The initial conditions u_0 and u'_0 present $2n$ equations that must be satisfied to find the complete solution to the differential equation. The 2 constants a, b cannot achieve this alone.

However, suppose A has n eigenvalues $\lambda_1, \dots, \lambda_n$ with $\lambda_i > 0 \ \forall i$ and n independent eigenvectors x_1, \dots, x_n . Repeat the last calculation with each eigenpair to get a general solution:

$$\boxed{u(t) = (a_1 \cos \omega_1 t + b_1 \sin \omega_1 t)x_1 + \dots + (a_n \cos \omega_n t + b_n \sin \omega_n t)x_n, \omega_i = \sqrt{\lambda_i}}$$

Apply the initial conditions:

$$u_0 = u(0) = a_1 x_1 + \dots + a_n x_n = [x_1 \dots x_n] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = S a$$

$$u'_0 = u'(t) = b_1 \omega_1 x_1 + \dots + b_n \omega_n x_n = [x_1 \dots x_n] \begin{bmatrix} \sqrt{\lambda_1} \\ \vdots \\ \sqrt{\lambda_n} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = S \Lambda^{1/2} b$$

$$\boxed{a = S^{-1} u_0, \quad b = \Lambda^{1/2} S^{-1} u'_0}$$

Note: All of these calculations have relied upon real, positive eigenvalues. We will show later on that if A is symmetric positive definite then all of A 's eigenvalues are real and positive

Single Equations of Higher Order

Second Order ODE in $u: \mathbb{R} \rightarrow \mathbb{R}$

$$\frac{d^2 u}{dt^2} + p \frac{du}{dt} + q u = 0, \quad u(0) = u_0, \quad u'(0) = u'_0$$

One way to solve is to assume a solution of the form $u = e^{\lambda t}$:

$$\lambda^2 e^{\lambda t} + p\lambda^2 e^{\lambda t} + q e^{\lambda t} = 0 \rightarrow \lambda^2 + p\lambda + q = 0$$

This equation has two (possibly complex) roots λ_1, λ_2 . The general solution will be:

$$u(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \quad (\lambda_1 \neq \lambda_2)$$

$$u(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t} \quad (\lambda_1 = \lambda_2 = \lambda)$$

Alternatively, convert to a matrix equation.

$$\begin{aligned} U_1 &= u(t) \\ U_2 &= \frac{du}{dt} \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \rightarrow \begin{aligned} \frac{dU_1}{dt} &= U_2 \\ \frac{dU_2}{dt} &= -pU_2 - qU_1 \end{aligned}$$

$$U := \begin{bmatrix} u \\ \frac{du}{dt} \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, \quad U_0 := \begin{bmatrix} u_0 \\ u'_0 \end{bmatrix} = U(0)$$

$$\frac{dU}{dt} = \frac{d}{dt} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = AU$$

Similar to the method mentioned above, assume $U = e^{\lambda t} x$:

$$\lambda e^{\lambda t} x = \frac{dU}{dt} = AU = e^{\lambda t} Ax \rightarrow Ax = \lambda x$$

This shows that solving for U amounts to solving for the eigenvalues and eigenvectors of A .

$$0 = \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -q & -p - \lambda \end{vmatrix} = \lambda^2 + p\lambda + q$$

This is the same equation for λ as before. If $\lambda_1 \neq \lambda_2$ are eigenvalues satisfying this equation, with eigenvectors x_1, x_2 and $A = S \Lambda S^{-1}$ we have

$$U = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 = [x_1 \ x_2] \begin{bmatrix} e^{\lambda_1 t} & \\ & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = S e^{\Lambda t} c \rightarrow c = e^{-\Lambda t} S^{-1} U_0$$

Recall that $u(t) = U_1$. After determining U , you find u in the first entry.

Unfortunately it may be the case that there is a double root λ for the equation $0 = \lambda^2 + P\lambda^2 + q$ — A has only 1 eigenvalue and only one eigenvector. The nullspace of $A - \lambda I$ is only 1-dimensional. If we tried to write $A = S\Lambda S^{-1}$ then $\Lambda = \lambda I$ and so $A = S(\lambda I)S^{-1} = \lambda I$. But this doesn't agree with the form of A on the previous page so $A \neq S\Lambda S^{-1}$.

The best we do in this scenario is to find the eigenvector x for $(A - \lambda I)x = 0$ and then a generalized eigenvector y satisfying:

$$(A - \lambda I)y = x \rightarrow (A - \lambda I)^2 y = (A - \lambda I)x = 0$$

Instead of $A = S\Lambda S^{-1}$ we have the Jordan form $A = SJS^{-1}$ with

$$S = \begin{bmatrix} x & y \end{bmatrix} \quad J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad (\text{more on this later})$$

Symmetric Matrices

Definition An orthogonal matrix Q is a matrix with the property $Q^T = Q^{-1}$. The columns of Q are orthonormal, meaning $q_i^T q_j = \delta_{ij}$.

TM If A is real and symmetric, then we can write $A = Q\Lambda Q^T$ where all eigenvalues λ_i of A in the matrix Λ are real.

If A is positive definite, the eigenvalues are also positive.

Conversely, $A = Q\Lambda Q^T$ with Λ real implies $A = A^T$ and if all eigenvalues are positive, A is positive definite.

Proof: Suppose A is real and symmetric and let $\lambda = a + bi$ be an eigenvalue with eigenvector x .

$$\left\{ \begin{array}{l} Ax = \lambda x \\ A = A^T = \bar{A}^T \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \bar{x}^T(Ax) = \bar{x}^T(\lambda x) \rightarrow \bar{x}^T A x = \lambda \bar{x}^T x \\ \bar{A} \bar{x} = \bar{\lambda} \bar{x} \rightarrow \bar{x}^T A = \bar{\lambda} \bar{x}^T \rightarrow \bar{x}^T A x = \bar{\lambda} \bar{x}^T x \end{array} \right\}$$

Since $\lambda \bar{x}^T x = \bar{\lambda} \bar{x}^T x$ and $\bar{x}^T x = \|x\|^2 > 0$, $\lambda = \bar{\lambda}$ (λ is real). Then $A - \lambda I$ is real valued so we can solve $(A - \lambda I)x = 0$ for a real-valued x .

For eigenvectors $x_1 \neq x_2$ corresponding to $\lambda_1 \neq \lambda_2$, x_1 and x_2 are orthogonal:

$$\begin{aligned} Ax_1 = \lambda_1 x_1 &\rightarrow x_2^T A x_1 = \lambda_1 x_2^T x_1 \rightarrow \lambda_1 x_2^T x_1 = \lambda_2 x_2^T x_1 \rightarrow x_2^T x_1 = 0 \\ Ax_2 = \lambda_2 x_2 &\rightarrow x_2^T A^T x_1 = \lambda_2 x_2^T x_1 \quad (A^T = A) \quad (\lambda_1 \neq \lambda_2) \end{aligned}$$

Next consider the case that A has repeated eigenvalues. A slight perturbation makes the eigenvalues unique. The eigenvectors of the perturbed system are orthogonal and remain so in the limit as we shrink the perturbation and approach A . The perturbations should be defined so that the system is always orthogonal.

With this exceptional circumstance handled, we conclude that A has a full set of n orthogonal eigenvectors. To go from S to Q we choose to scale the eigenvectors so they are an orthonormal set. Therefore

$$A = Q \Lambda Q^{-1} = Q \Lambda Q^T$$

If also A is positive definite, $x^T A x > 0$ for any x . In particular, if λ is any eigenvalue of A and x a corresponding eigenvector,

$$\lambda \|x\|^2 = \lambda x^T x = x^T (\lambda x) = x^T A x > 0 \rightarrow \lambda > 0$$

\therefore All eigenvalues are positive.

For the converse, suppose $A = Q \Lambda Q^T$ with all real eigenvalues in Λ and with orthonormal eigenvectors in Q . Since all λ are real we can always solve $(A - \lambda I)x = 0$ for a real valued eigenvector x (by definition λ makes $A - \lambda I$ singular so there exist nonzero x as well). Then Q , Λ , and Q^T are real so the product $A = Q \Lambda Q^T$ is real. Also, $A^T = Q^T \Lambda^T Q^T = Q \Lambda Q^T = A$.

Finally suppose also that all eigenvalues of A are positive. Since the eigenvectors q_1, \dots, q_n in Q are independent, any vector $x \in \mathbb{R}^n$ can be written as a linear combination $x = d_1 q_1 + \dots + d_n q_n$ (the q_i form a basis for that space).

$$\begin{aligned} x^T A x &= (d_1 q_1^T + \dots + d_n q_n^T) Q \Lambda Q^T (d_1 q_1 + \dots + d_n q_n) \\ &= (d_1 q_1^T Q + \dots + d_n q_n^T Q) \Lambda (d_1 Q^T q_1 + \dots + d_n Q^T q_n) \\ &= (d_1 e_1^T + \dots + d_n e_n^T) \Lambda (d_1 e_1 + \dots + d_n e_n) && e_i : \text{standard unit basis vectors} \\ &= [d_1 \dots d_n] \Lambda \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = [d_1 \dots d_n] \begin{bmatrix} \lambda_1 d_1 \\ \vdots \\ \lambda_n d_n \end{bmatrix} \\ &= \sum_{i=1}^n \lambda_i |d_i|^2 > 0 \quad \text{since } \lambda_i > 0 \ \forall i \rightarrow A \text{ is symmetric positive definite.} \end{aligned}$$

Remark: We relied upon $\bar{A}^T = A$ in the proof. For a real matrix A , $A = \bar{A}$ so this is the same as symmetry $A^T = A$. For a complex valued matrix A , if $\bar{A}^T = A$, A is called Hermitian.

Exercises

1.5.3 Find all eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

$$P_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 3\lambda^2 - \lambda^3 \rightarrow \lambda=0, \lambda=3$$

$$\lambda=0: Ax=0 \rightarrow x = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad a+b+c=0 \rightarrow x = \begin{bmatrix} c \\ d \\ -c-d \end{bmatrix}$$

$$\lambda=3: (A-3I)x=0 \rightarrow \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \begin{array}{l} a=c \\ b=c \end{array} \rightarrow x = \begin{bmatrix} c \\ c \\ c \end{bmatrix}$$

1.5.5 Solve the first order system $\frac{du}{dt} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}u$ with $u_0 = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$.

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

$$0 = (3-\lambda)(-3-\lambda) - 16 = \lambda^2 - 25 \rightarrow \lambda = \pm 5$$

$$\lambda_1 = -5 \quad \left[\begin{array}{cc|c} 3+5 & 4 & 0 \\ 4 & -3+5 & 0 \end{array} \right] \sim \left[\begin{array}{cc|c} 8 & 4 & 0 \\ 4 & 2 & 0 \end{array} \right] \sim \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] \quad s_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\lambda_2 = 5 \quad \left[\begin{array}{cc|c} 3-5 & 4 & 0 \\ 4 & -3-5 & 0 \end{array} \right] \sim \left[\begin{array}{cc|c} -2 & 4 & 0 \\ 4 & -8 & 0 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array} \right] \quad s_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} -5 & 0 \\ 0 & 5 \end{bmatrix} \quad S^{-1} = \begin{bmatrix} 1/5 & -2/5 \\ 2/5 & 1/5 \end{bmatrix}$$

In the notes we found that the solution is $u(t) = Se^{\Lambda t}S^{-1}u_0$:

$$u(t) = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} e^{-5t} & 0 \\ 0 & e^{5t} \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 6 \end{bmatrix} = -\frac{12}{5} \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-5t} + \frac{12}{5} \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t}$$

1.5.8 For A given below, add the three equations $du/dt = Au$ to show $u_1 + u_2 + u_3$ is constant. What is the general solution for u ?

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix} \quad \frac{du}{dt} = Au \rightarrow \begin{aligned} du_1/dt &= 3u_1 - u_2 \\ du_2/dt &= -2u_1 + 2u_2 - u_3 \\ du_3/dt &= -u_1 - u_2 + u_3 \end{aligned}$$

$$\frac{d}{dt}(u_1 + u_2 + u_3) = 0$$

$$\rightarrow u_1 + u_2 + u_3 = \text{constant wrt } t$$

For $u' = Au$ try solutions of the form $u = e^{\lambda t} x$ with λ, x an eigenpair.

$\lambda = 0, 2, 4$ (computed using Matlab)

$$\lambda_1 = 0 : \begin{bmatrix} 3 & -1 & 0 & 0 \\ -2 & 2 & -1 & 0 \\ -1 & -1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/4 & 0 \\ 0 & 1 & -3/4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow x_1 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$$\lambda_2 = 2 : \begin{bmatrix} 1 & -1 & 0 & 0 \\ -2 & 0 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow x_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$\lambda_3 = 4 : \begin{bmatrix} -1 & -1 & 0 & 0 \\ -2 & -2 & -1 & 0 \\ -1 & -1 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow x_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

The general solution to $du/dt = Au$ is

$$u(t) = c_1 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{4t}$$

1.5.11 Why is the sum of entries on the diagonal of BC equal to the sum along the diagonal of CB ?

$$\begin{aligned} \text{tr}(BC) &= \sum_{i=1}^n \sum_{j=1}^n b_{ij}c_{ji} = b_{11}c_{11} + b_{12}c_{21} + \dots + b_{1n}c_{n1} \\ &\quad + b_{21}c_{12} + b_{22}c_{22} + \dots + b_{2n}c_{n2} \\ &\quad \vdots \\ &\quad + b_{n1}c_{1n} + b_{n2}c_{2n} + \dots + b_{nn}c_{nn} \end{aligned} \quad \left\{ \begin{array}{l} \text{the order of} \\ \text{addition doesn't} \\ \text{matter - now sum} \\ \text{this 'vertically'.} \end{array} \right.$$

$$\begin{aligned} &= \sum_{i=1}^n \sum_{j=1}^n c_{ij}b_{ji} \\ &= \text{tr}(CB) \end{aligned}$$

1.5.12 Show that the determinant equals the product of the eigenvalues by imagining that the characteristic polynomial is factored into:

$$P_A(\lambda) = \det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$$

The fundamental theorem of algebra says that the n^{th} degree polynomial P_A has n roots in \mathbb{C} . The eigenvalues of A are by definition the roots of P_A . That explains this factorization. Evaluate P_A at $\lambda = 0$:

$$\det(A) = P_A(0) = \lambda_1 \lambda_2 \cdots \lambda_n = \prod_{i=1}^n \lambda_i$$

1.5.14

(a) Show how $u' = Au$ becomes $v' = Jv$ if $A = SJS^{-1}$ and $v = S^{-1}u$.

$$v = S^{-1}u \rightarrow v' = S^{-1}u' = S^{-1}Au = S^{-1}(SJS^{-1})(Sv) = Jv$$

(b) By back substitution solve for v when

$$\begin{aligned} \frac{dv_1}{dt} &= 3v_1 + v_2 \\ \frac{dv_2}{dt} &= 3v_2 \end{aligned} \quad \text{with } v(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\left. \begin{aligned} \frac{dv_2}{dt} &= 3v_2 \rightarrow v_2(t) = e^{3t} + c \\ 1 &= v_2(0) = 1 + c \rightarrow c = 0 \end{aligned} \right\} \underline{v_2(t) = e^{3t}}$$

Back substitute v_2 into $\frac{dv_1}{dt} = 3v_1 + v_2$

$$\left. \begin{aligned} \frac{dv_1}{dt} - 3v_1 &= e^{3t} & v_1(t) &= te^{3t} + de^{3t} \\ (v_1 e^{-3t})' &= 1 & 1 &= v_1(0) = d \end{aligned} \right\} \underline{v_1(t) = te^{3t} + e^{3t}}$$

1.5.20 Suppose there is a damping term, F , proportional to velocity in

$$M \frac{d^2u}{dt^2} + F \frac{du}{dt} + Ku = 0$$

When will $u(t) = e^{\lambda t}x$ be a solution?

$u(t)$ is a solution if it satisfies the differential equation (nothing profound about this statement). In this context this just means:

$$0 = M\lambda^2 e^{\lambda t}x + F\lambda e^{\lambda t}x + Ke^{\lambda t}x$$

$$\rightarrow 0 = (M\lambda^2 + F\lambda + K)x \text{ since } e^{\lambda t} > 0 \ \forall t$$

0 denotes here the zero vector of appropriate length.

1.5.23 (Positive Definite Square Root)

Suppose $A = Q \Lambda Q^T$ is positive definite (implies $\lambda_i > 0$ for each λ_i in Λ). Let $A^{1/2} = Q \Lambda^{1/2} Q^T$ be the matrix with the same eigenvectors in Q and with eigenvalues $\lambda_i^{1/2}$. Explain why $A^{1/2}$ is symmetric positive definite and why its square is A . Find $A^{1/2}$ if

$$A = \begin{bmatrix} 10 & -6 \\ -6 & 10 \end{bmatrix}$$

$A^{1/2}$ is symmetric because $(A^{1/2})^T = (Q \Lambda^{1/2} Q^T)^T = Q^T (\Lambda^{1/2})^T Q^T = Q \Lambda^{1/2} Q^T = A^{1/2}$.

A is the square of $A^{1/2}$ since $(A^{1/2})^2 = Q \Lambda^{1/2} Q^T Q \Lambda^{1/2} Q^T = Q \Lambda^{1/2} \Lambda^{1/2} Q^T = Q \Lambda Q^T = A$.

$A^{1/2}$ is positive definite since $x^T A^{1/2} x > 0 \quad \forall x \neq 0$, which we prove next. Following the idea from the end of the proof of IM, let x be arbitrary. Since the columns q_1, \dots, q_n form a basis for \mathbb{R}^n , $x = a_1 q_1 + \dots + a_n q_n$.

$$\begin{aligned} x^T A^{1/2} x &= (a_1 q_1^T + \dots + a_n q_n^T) Q \Lambda^{1/2} Q^T (a_1 q_1 + \dots + a_n q_n) \\ &= (a_1 q_1^T Q + \dots + a_n q_n^T Q) \Lambda^{1/2} (a_1 Q^T q_1 + \dots + a_n Q^T q_n) \\ &= [a_1 \dots a_n] \Lambda^{1/2} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = [a_1 \dots a_n] \begin{bmatrix} \lambda_1^{1/2} a_1 \\ \vdots \\ \lambda_n^{1/2} a_n \end{bmatrix} \\ &= \sum_{i=1}^n \lambda_i^{1/2} a_i^2 > 0 \text{ since } \lambda_i^{1/2} > 0 \quad \forall i \end{aligned}$$

For the given A matrix,

$$0 = (10 - \lambda)^2 - 36 = \lambda^2 - 20\lambda + 64 = (\lambda - 16)(\lambda - 4) \rightarrow \lambda = 4, 16$$

$$\underline{\lambda_1 = 4}: \quad \left[\begin{array}{cc|c} 6 & -6 & 0 \\ -6 & 6 & 0 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right] \rightarrow q_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\underline{\lambda_2 = 16}: \quad \left[\begin{array}{cc|c} -6 & -6 & 0 \\ -6 & -6 & 0 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] \rightarrow q_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$A = Q \Lambda Q^T = \begin{bmatrix} K & K \\ K & -K \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} K & K \\ K & -K \end{bmatrix}, \quad K := 1/\sqrt{2}$$

$$A^{1/2} = Q \Lambda^{1/2} Q^T = \begin{bmatrix} K & K \\ K & -K \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} K & K \\ K & -K \end{bmatrix} = K^2 \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 4 & -4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 6 & -2 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

1.5.24 If K, M are positive definite and $Kx = \lambda Mx$, prove that $\lambda > 0$. This is the generalized eigenvalue problem, with two matrices. Find the two eigenvalues when

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad K = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$Kx = \lambda Mx$$

$$x^T K x = \lambda x^T M x$$

$$\lambda = \frac{x^T K x}{x^T M x} > 0 \quad \text{since } x^T K x \text{ and } x^T M x \text{ are both positive scalars.}$$

To find nonzero x satisfying $Kx = \lambda Mx$, we need λ s.t. $(K - \lambda M)x = 0$. That is, $K - \lambda M$ must be singular.

$$0 = \det(K - \lambda M) = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-2\lambda \end{vmatrix} = (2-\lambda)(2-2\lambda) - 1 = 2\lambda^2 - 6\lambda + 3$$

$$\lambda = (6 \pm \sqrt{36-24})/4 = 3/2 \pm \sqrt{3}/2$$