§ 6.1 Ordinary Differential Equations

6.1.1 Solve the differential equations. In each case $u_0 = 5$. Which solutions go to a steady state u_∞?

(a) $u' + u = e^{2t}$

$(ue^t)' = e^{3t}$

$\int_0^t (u(s)e^s)'\,ds = \int_0^t e^{3s}\,ds$

$u(t)e^t - u_0 e^0 = \frac{1}{3}e^{3t} - \frac{1}{3}$

$u(t)e^t - 5 = \frac{1}{3}e^{3t} - \frac{1}{3}$

$u(t) = \frac{1}{3}e^{2t} + \frac{14}{3}e^{-t}$ $\qquad u_\infty = +\infty$ $\qquad$ unstable

(b) $u' + u = e^{i\omega t}$

$(ue^t)' = e^{(1+i\omega)t}$

$\int_0^t (u(s)e^s)'\,ds = \int_0^t e^{(1+i\omega)s}\,ds$

$u(t)e^t - u_0 = (1+i\omega)^{-1}\{e^{(1+i\omega)t} - 1\}$

$u(t) = \frac{1}{1+i\omega}e^{i\omega t} - \frac{1}{1+i\omega}e^{-t} + 5e^{-t} = \frac{1}{1+i\omega}e^{i\omega t} + \frac{4+5i\omega}{1+i\omega}e^{-t}$

$u_\infty = +\infty$ $\qquad$ unstable

(c) $u' + u = e^{-t}$

$(ue^t)' = 1$

$ue^t - u_0 = t$

$u(t) = te^{-t} + 5e^{-t}$ $\qquad u_\infty = 0$ $\qquad$ stable

6.1.2 If $u' + 2u = \delta(t-1) + c\,\delta(t-4)$ find the solution u from eq.'s 4,5. What value of c will switch the solution off so $u = 0$ for $t > 4$?

For $u' - au = f(t)$:

$$\boxed{u(t) = \int_0^t e^{a(t-s)} f(s)\,ds + e^{at} u_0 \qquad (4)}$$

For an impulse δ acting at time T:

$$\boxed{\int_0^t e^{a(t-s)} \delta(s-T)\,ds = \begin{cases} 0 & t < T \\ e^{a(t-T)} & t \geq T \end{cases} \qquad (5)}$$

$\therefore$ $u' + 2u = \delta(t-1) + c\,\delta(t-4)$ has the solution:

$$u(t) = \int_0^t e^{-2(t-s)} \{\delta(s-1) + c\,\delta(s-4)\}\,ds + e^{-2t} u_0$$

$$= \int_0^t e^{-2(t-s)} \delta(s-1)\,ds + c\int_0^t e^{-2(t-s)} \delta(s-4)\,ds + e^{-2t} u_0$$

$$= \begin{cases} e^{-2t} u_0 , & t < 1 \\ e^{2(1-t)} + e^{-2t} u_0 , & 1 \leq t < 4 \\ c e^{2(4-t)} + e^{2(1-t)} + e^{-2t} u_0 , & t \geq 4 \end{cases}$$

To find the value of c s.t. $u(t) = 0$ for $t \geq 4$,

$$0 = c e^{2(4-t)} + e^{2(1-t)} + e^{-2t} u_0 = e^{-2t}(c e^8 + e^2 + u_0)$$

$$\boxed{c = -e^{-6} - u_0 e^{-6}}$$

In the case $u_0 = 0$,

$$u(t) = \begin{cases} 0 , & t < 1 \\ e^{2(1-t)} , & 1 \leq t < 4 \\ c e^{2(4-t)} + e^{2(1-t)} , & t \geq 4 \end{cases} \qquad \boxed{c = -e^{-6}}$$

6.1.3 Solve $\frac{du}{dt} = u^{1-K}$ with $u_0 = 1$, $K \neq 0$ by separating $u^{K-1}du$ from dt and integrating. When does u blow up if $K<0$? Which of $u' = u^3$ and $u' = 1/u^3$ can be solved with $u_0 = 0$?

$$\frac{1}{K}u^K = \int u^{K-1}\,du = \int dt = t + C$$

$$\frac{1}{K} = \frac{1}{K}1^K = \frac{1}{K}u_0^K = 0 + C = C$$

$$\boxed{u(t) = (Kt+1)^{1/K}}$$

For $K<0$, $1/K<0$ so u blows up for $Kt+1=0 \rightarrow t = -1/K$.

If $u_0 = 0$,

$$u' = u^3 \qquad\qquad u' = 1/u^3$$

$$u^{-3}u' = 1 \qquad\qquad u^3u' = 1$$

$$-2u^{-2} + 2u_0^{-2} = t \qquad\qquad \tfrac{1}{4}u^4 - \tfrac{1}{4}u_0^4 = t$$

u_0^{-2} undefined

$$\boxed{u(t) = (4t)^{1/4}}$$

6.1.4 Solve $u' - u\cos t = 1$ with $u_0 = 4$

$$(ue^{-\sin t})' = e^{-\sin t} \qquad \left\{ e^{-u(t)} = e^{\int -\cos t\,dt} = e^{-\sin t + C} \right\}$$

$$\int_0^t (u(s)e^{-\sin s})\,ds = \int_0^t e^{-\sin s}\,ds$$

$$u(t)e^{-\sin t} - 4e^{-\sin 0} = \int_0^t e^{-\sin s}\,ds$$

$$\boxed{u(t) = 4e^{\sin t} + \int_0^t e^{\sin t - \sin s}\,ds}$$

6.1.5 Find the general solution to the separable equation.

(b)
$$u' = -u/t$$
$$1/u\, u' = -1/t$$
$$\ln|u| = -\ln|t| + C$$
$\boxed{u(t) = C/t}$ on one of $t>0, t<0$.

(c)
$$uu' = \tfrac{1}{2}\cos t$$
$$u^2 = \sin t + C$$
$$\boxed{u(t) = (\sin t + C)^{1/2},\ C \geq 1}$$

6.1.7 Solve $u' + u/t = 3t$ with $u(1) = 0$.

$$(ut)' = 3t^2 \qquad \left\{ e^{-u(t)} = e^{\int 1/t\,dt} = t \right\}$$

$$\int_1^t (s\,u(s))'\,ds = \int_1^t 3s^2\,ds$$

$$tu(t) - 1u(1) = t^3 - 1$$

$$\boxed{u(t) = t^2 - 1/t,\ t>0}$$

The logistic equation $u' = au - bu^2$ is separable using partial fractions

$$\frac{1}{au - bu^2} = \frac{1}{au} + \frac{b/a}{a - bu}$$

Starting from $u_0 > 0$,

$$\int_{u_0}^{u(t)} \left\{ \frac{1}{au} + \frac{b/a}{a - bu} \right\} du = \int_0^t ds$$

$$\frac{1}{a} \ln u - \frac{1}{a} \ln u_0 - \frac{1}{a} \ln(a - bu) + \frac{1}{a} \ln(a - bu_0) = t$$

$$\ln \frac{u}{a - bu} = at + \ln \frac{u_0}{a - bu_0}$$

$$\frac{u}{a - bu} = e^{at} \frac{u_0}{a - bu_0}$$

$$\boxed{u(t) = \frac{a}{b + e^{-at}(a - bu_0)/u_0}}$$

6.1.8 Suppose a rumor starts with $u_0 = 1$ person and spreads according to $u' = u(N-u)$. Find $u(t)$ for this logistic equation. At what time T does the rumor reach half the population $(u(T) = \frac{1}{2}N)$?

$u' = Nu - u^2 \quad a = N, \; b = 1, \; u_0 = 1$

$$\boxed{u(t) = \frac{N}{1 + e^{-Nt}(N-1)}}$$

$$\frac{1}{2} N = \frac{N}{1 + e^{-NT}(N-1)}$$

$$1 + e^{-NT}(N-1) = 2$$

$$e^{-NT} = 1/(N-1)$$

$$\boxed{T = N^{-1} \ln(N-1)}$$

6.1.11 Find the solution with arbitrary constants to

(a) $u'' - 9u = 0$

Try $u = e^{\lambda t}$

$\lambda^2 e^{\lambda t} - 9e^{\lambda t} = 0$

$\lambda^2 - 9 = 0 \longrightarrow \; = \pm 3$

$$\boxed{u(t) = Ce^{3t} + De^{-3t}}$$

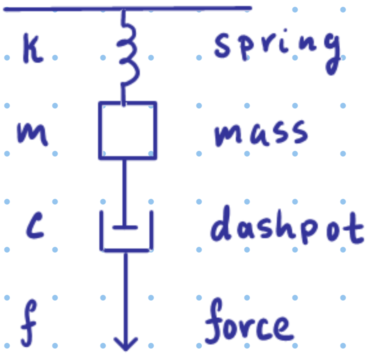
(c) $u'' + 2u' + 5u = 0$

$\lambda^2 + 2\lambda + 5 = 0$

$\lambda = -1 \pm \frac{1}{2}\sqrt{4 - 4 \cdot 5}$
$= -1 \pm \frac{1}{2}\sqrt{-16}$
$= -1 \pm 2i$

$$\boxed{u(t) = e^{-t}(C\sin 2t + D\cos 2t)}$$

Damped Spring $mu'' + cu' + ku = 0$ with free oscillations.



The displacement u is measured from the steady state position where the upward force kx balances downward gravitational, $mg = kx$.

For a solution of the form $u = e^{\lambda t}$,

$m\lambda^2 + c\lambda + k = 0 \longrightarrow \lambda = -c/2m \pm 1/2m\sqrt{c^2 - 4mk}$

(I) Overdamping: $c^2 > 4mk$
(II) Critical Damping: $c^2 = 4mk$
(III) Underdamping: $c^2 < 4mk$

6.1.13

(a) What damping constants c in $\frac{1}{2}u'' + cu' + \frac{1}{2}u = 0$ produce overdamping, critical damping, underdamping, no damping, and negative damping?

$m = 1/2$, $k = 1/2$, $4mk = 1$

Overdamping: $c > 1$, critical damping: $c = 1$, underdamping: $0 < c < 1$, no damping: $c = 0$, negative damping $c < 0$

(b) Find the exponents λ_1, λ_2 and solve with $u_0 = 2$ and $u_0' = -2c$. For which c does $u(t) \to 0$?

$$\lambda = -c/2m \pm 1/2m\sqrt{c^2-4mk} = -c \pm \sqrt{c^2-1} \rightarrow \begin{array}{l}\lambda_1 = -c+\sqrt{c^2-1} \\ \lambda_2 = -c-\sqrt{c^2-1}\end{array}$$

$$u(t) = c_1 \exp[(-c+\sqrt{c^2-1})t] + c_2 \exp[(-c-\sqrt{c^2-1})t] = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$$u'(t) = c_1\lambda_1 e^{\lambda_1 t} + c_2 \lambda_2 e^{\lambda_2 t}$$

$$2 = u_0 = c_1 + c_2 \qquad -2c = u_0' = c_1\lambda_1 + c_2\lambda_2 = c_1\lambda_1 + 2\lambda_2 - c_1\lambda_2$$

$$c_2 = 2 - c_1 \qquad -2c = c_1(\lambda_1 - \lambda_2) + 2\lambda_2$$

$$c_2 = 2-2=0 \qquad -2c = 2c_1\sqrt{c^2-1} - 2c - 2\sqrt{c^2-1} \longrightarrow c_1 = 1$$

$$u(t) = e^{\lambda_1 t} = \exp[(-c+\sqrt{c^2-1})]$$

For $c \le -1$, $\lambda_1 > 0$ so $u(t) \not\to 0$

For $-1 < c < 0$, $\lambda_1 = -c + \sqrt{1-c^2}\, i$ and $-c > 0$. Oscillations increasing in amplitude. $u(t) \not\to 0$.

For $c=0$, λ_1 is pure imaginary. Oscillation a constant amplitude. $u(t) \not\to 0$.

For $0 < c < 1$, $\lambda_1 = -c + \sqrt{1-c^2}\, i$ and $-c < 0$. oscillations decreasing in amplitude. $u(t) \to 0$.

For $c = 1$, $\lambda_1 = -1$ so $u(t) = e^{-t} \longrightarrow 0$.

For $c > 1$, $0 < c^2 - 1 < c^2 \Rightarrow \lambda_1 = -c + \sqrt{c^2-1} < 0$. $u(t) \to 0$.

$\therefore$ $u(t) \to 0$ for $c > 0$. This confirms intuition. Since there is no forcing term the displacement will approach 0 whenever motion is (positively) damped.

<u>6.1.14</u> Find the undamped forced oscillation for

(a) $u'' + u = \cos 2t$, $u_0 = u_0' = 0$

$$u = a(\cos\omega t - \cos\omega_0 t) = a(\cos 2t - \cos t) \qquad u(0) = 0 \checkmark \quad u'(0) = 0 \checkmark$$

$$u'' = a(-4\cos 2t + \cos t)$$

$$-4a\cos 2t + a\cos t + a\cos 2t - a\cos t = \cos 2t \qquad \boxed{u(t) = -1/3(\cos 2t - \cos t)}$$

$$a = -1/3$$

6.1.15 Solve with $u_0 = 2$, $u_0' = 0$ and find the steady oscillation.

(a) $u'' + 2u = \cos\omega t$

Let $u_p = a\cos\omega t \longrightarrow \cos\omega t = u_p'' + 2u_p = a\cos\omega t\,(2-\omega^2)$

$$a(2-\omega^2) = 1$$
$$a = (2-\omega^2)^{-1}$$

By 6B (pg 486) $u(t) = a\cos\omega t + d_1\cos\sqrt{2}t + d_2\sin\sqrt{2}t$

$u'(t) = -\omega a\sin\omega t - \sqrt{2}d_1\sin\sqrt{2}t + \sqrt{2}d_2\cos\sqrt{2}t$

$$2 = u_0 = a + d_1 \longrightarrow d_1 = 2 - a = 2-(2-\omega^2)^{-1} = 3-2\omega$$
$$0 = u_0' = \sqrt{2}d_2 \rightarrow d_2 = 0$$

$$\boxed{u(t) = \frac{1}{2-\omega^2}\cos\omega t + \frac{3-2\omega^2}{2-\omega^2}\cos\sqrt{2}t}$$

6.1.16 What driving frequency ω will produce the largest amplitude A in equation (24)? For small R this is the "resonant frequency under damping".

$$A = \frac{V\omega}{\sqrt{L^2(\omega_0^2-\omega^2)^2 + \omega^2R^2}} \qquad (24)$$

$$0 = \frac{\partial A}{\partial\omega} = \frac{V}{\sqrt{L^2(\omega_0^2-\omega^2)^2+\omega^2R^2}} - \frac{V(\omega^2R^2 - 2L^2\omega^2(\omega_0^2-\omega^2)^2}{(L^2(\omega_0^2-\omega^2)^2+\omega^2R^2)^{3/2}}$$

$$0 = L^2(\omega_0^2-\omega^2)^2 + \omega^2R^2 - \omega^2R^2 + 2L^2\omega^2(\omega_0^2-\omega^2)^2 \qquad V, L \neq 0$$

$$0 = (\omega_0^2-\omega^2)^2(1+2\omega^2) \Rightarrow \omega = \omega_0 \qquad (\omega_0, \omega > 0)$$

6.1.18

(a) Solve $u'' + u' + u = t^2$ by assuming $u(t) = A + Bt + Ct^2$.

$u(t) = A + Bt + Ct^2$

$u'(t) = B + 2Ct$

$u''(t) = 2C$

$$t^2 = u'' + u' + u = Ct^2 + (B+2C)t + A+B+2C$$
$$1 = C$$
$$0 = B+2C = B+2 \rightarrow B = -2$$
$$0 = A+B+2C = A-2+2 = A$$

$$\boxed{u(t) = t^2 - 2t}$$

6.1.20 For $u''+u=\cos t$ show that $u(t)=A\cos t+B\sin t$ fails to give a solution. This is resonance. Solve with $u_0=0$ and $u_0'=1$ and $u(t)=A\cos t+B\sin t+Ct\cos t+Dt\sin t$.

Suppose $u(t)$ is of the form $u(t)=A\cos t+B\sin t$ so $u''(t)=-A\cos t-B\sin t$.

$$0=-A\cos t-B\sin t+A\cos t+B\sin t=u''+u=\cos t$$

This shows $\cos t\equiv 0$, which is a contradiction. So $u(t)$ cannot be of the form $u(t)=A\cos t+B\sin t$.

Try $u(t)=A\cos t+B\sin t+Ct\cos t+Dt\sin t$

$$u'(t)=-A\sin t+B\cos t+C\cos t-Ct\sin t+D\sin t+Dt\cos t$$

$$u''(t)=(2D-A)\cos t+(-B-2C)\sin t-Ct\cos t-Dt\sin t$$

$$\cos t=u''+u=2D\cos t-2C\sin t\rightarrow D=1/2,\ C=0$$

Apply initial conditions: $0=u_0=u(0)=A,\quad 1=u_0'=u'(0)=B$

$$\therefore\ \boxed{u(t)=\sin t+\tfrac{1}{2}t\sin t}$$

From 6C (pg 489), for the nonlinear oscillations $u''+V'(u)=0$, the energy E and the amplitude u_{max} are given by

$$\boxed{E=\tfrac{1}{2}(u')^2+V'(u)=V(u_{max})}$$

$$u'(u''+V'(u))=0\cdot u'$$

$$(u')'+(V(u))'=0$$

$$\underbrace{\tfrac{1}{2}(u')^2}_{\text{(kinetic energy)}}+\underbrace{V(u)}_{\text{(potential energy)}}=E$$ (constant total energy).

Kinetic energy is zero exactly when $u'=0$ and the oscillation is at full amplitude: $E=V(u_{max})$.

6.1.21 Find the energy function $E(u)$ for the equation. If $u_0=0$ and $u_0'=1$, what equation gives the amplitude u_{max}?

(a) $u''+\frac{1}{2}e^{u}-\frac{1}{2}e^{-u}=0$

$$E=\tfrac{1}{2}(u')^2+V(u)=\boxed{\tfrac{1}{2}(u')^2+\cosh u}$$

$$0=u''+\tfrac{1}{2}e^{u}-\tfrac{1}{2}e^{-u}=u''+\sinh u$$
$$=u''+(\cosh u)'=u''+V'(u)$$

Since energy is constant $E(u(0))=E(u(t))\ \forall t$

$E=E(0)=\frac{1}{2}(u_0')^2+\cosh u_0=3/2$. When $u'=0$,

$V(u_{max})=\boxed{\cosh u_{max}=3}$