§6.1 Ordinary Differential Equations 1–5, 8, 11–14, 16, 20, 21

<u>6.1.1</u> Solve the differential equations. In each case $u_0 = 5$. Which solutions go to a steady state u_∞?

(a) $u' + u = e^{2t}$

$(ue^t)' = e^{3t}$

$\int_0^t (u(s)e^s)'\,ds = \int_0^t e^{3s}\,ds$

$u(t)e^t - u_0 e^0 = \frac{1}{3}e^{3t} - \frac{1}{3}$

$u(t)e^t - 5 = \frac{1}{3}e^{3t} - \frac{1}{3}$

$u(t) = \frac{1}{3}e^{2t} + \frac{14}{3}e^{-t}$ $u_\infty = +\infty$ unstable

(b) $u' + u = e^{i\omega t}$

$(ue^t)' = e^{(1+i\omega)t}$

$\int_0^t (u(s)e^s)'\,ds = \int_0^t e^{(1+i\omega)s}\,ds$

$u(t)e^t - u_0 = (1+i\omega)^{-1}\{e^{(1+i\omega)t} - 1\}$

$u(t) = \frac{1}{1+i\omega}e^{i\omega t} - \frac{1}{1+i\omega}e^{-t} + 5e^{-t} = \frac{1}{1+i\omega}e^{i\omega t} + \frac{4+5i\omega}{1+i\omega}e^{-t}$

$u_\infty = +\infty$ unstable

(c) $u' + u = e^{-t}$

$(ue^t)' = 1$

$ue^t - u_0 = t$

$u(t) = te^{-t} + 5e^{-t}$ $u_\infty = 0$ stable

6.1.2 If $u'+2u=\delta(t-1)+c\delta(t-4)$ find the solution u from eq.'s 4,5. What value of c will switch the solution off so $u=0$ for $t>4$?

For $u'-au=f(t)$:

$$\boxed{u(t)=\int_0^t e^{a(t-s)}f(s)\,ds + e^{at}u_0 \qquad (4)}$$

For an impulse δ acting at time T:

$$\boxed{\int_0^t e^{a(t-s)}\delta(s-T)\,ds = \begin{cases} 0 & t<T \\ e^{a(t-T)} & t\geq T \end{cases} \qquad (5)}$$

$\therefore$ $u'+2u=\delta(t-1)+c\delta(t-4)$ has the solution:

$$u(t) = \int_0^t e^{-2(t-s)}\{\delta(s-1)+c\,\delta(s-4)\}\,ds + e^{-2t}u_0$$

$$= \int_0^t e^{-2(t-s)}\delta(s-1)\,ds + c\int_0^t e^{-2(t-s)}\delta(s-4)\,ds + e^{-2t}u_0$$

$$= \begin{cases} e^{-2t}u_0, & t<1 \\ e^{2(1-t)}+e^{-2t}u_0, & 1\leq t<4 \\ ce^{2(4-t)}+e^{2(1-t)}+e^{-2t}u_0, & t\geq 4 \end{cases}$$

To find the value of c s.t. $u(t)=0$ for $t\geq 4$,

$$0 = ce^{2(4-t)}+e^{2(1-t)}+e^{-2t}u_0 = e^{-2t}(ce^8+e^2+u_0)$$

$$\boxed{c=-e^{-6}-u_0e^{-6}}$$

In the case $u_0=0$,

$$u(t)=\begin{cases} 0, & t<1 \\ e^{2(1-t)}, & 1\leq t<4 \\ ce^{2(4-t)}+e^{2(1-t)}, & t\geq 4 \end{cases} \qquad \boxed{c=-e^{-6}}$$

6.1.3 Solve $\frac{du}{dt} = u^{1-K}$ with $u_0 = 1$, $K \neq 0$ by separating $u^{K-1}du$ from dt and integrating. When does u blow up if $K<0$? Which of $u' = u^3$ and $u' = 1/u^3$ can be solved with $u_0 = 0$?

$$\frac{1}{K}u^K = \int u^{K-1}du = \int dt = t + C$$

$$\frac{1}{K} = \frac{1}{K}1^K = \frac{1}{K}u_0^K = 0 + C = C$$

$$\boxed{u(t) = (Kt+1)^{1/K}}$$

For $K<0$, $1/K<0$ so u blows up for $Kt+1=0 \to t = -1/K$.

If $u_0 = 0$,

$$u' = u^3$$
$$u^{-3}u' = 1$$
$$-2u^{-2} + 2u_0^{-2} = t$$

u_0^{-2} undefined

$$u' = 1/u^3$$
$$u^3 u' = 1$$
$$\tfrac{1}{4}u^4 - \tfrac{1}{4}u_0^4 = t$$
$$\boxed{u(t) = (4t)^{1/4}}$$

6.1.4 Solve $u' - u\cos t = 1$ with $u_0 = 4$

$$(ue^{-\sin t})' = e^{-\sin t} \qquad \left\{ e^{-a(t)} = e^{\int -\cos t\,dt} = e^{-\sin t + c} \right\}$$

$$\int_0^t (u(s)e^{-\sin s})\,ds = \int_0^t e^{-\sin s}\,ds$$

$$u(t)e^{-\sin t} - 4e^{-\sin 0} = \int_0^t e^{-\sin s}\,ds$$

$$\boxed{u(t) = 4e^{\sin t} + \int_0^t e^{\sin t - \sin s}\,ds}$$

6.1.5 Find the general solution to the separable equation:

(b) $u' = -u/t$

$$\tfrac{1}{u}u' = -\tfrac{1}{t}$$

$$\ln|u| = -\ln|t| + C$$

$$u(t) = c/t$$ on one of either $t>0$ or $t<0$.

(c) $uu' = \frac{1}{2}\cos t$

$$u^2 = \sin t + C$$

$$u(t) = (\sin t + C)^{1/2}, \quad c \geq 1$$

6.1.7 Solve $u' + u/t = 3t$ with $u(1) = 0$.

$(ut)' = 3t^2$ $\qquad\qquad$ $\{e^{-h(t)} = e^{\int 1/t\, dt} = t\}$

$\int_1^t (s\,u(s))'\, ds = \int_1^t 3s^2\, ds$

$t u(t) - 1\, u(1) = t^3 - 1$

$u(t) = t^2 - 1/t, \quad t > 0$

Since we divide by t, $t \neq 0$. Since u is given at $t=1$, restrict $t > 0$.

The logistic equation $u' = au - bu^2$ is separable using partial fractions

$$\frac{1}{au - bu^2} = \frac{1}{au} + \frac{b/a}{a - bu}$$

Starting from $u_0 > 0$,

$$\int_{u_0}^{u(t)} \left\{ \frac{1}{au} + \frac{b/a}{a - bu} \right\} du = \int_0^t ds$$

$$\tfrac{1}{a} \ln u - \tfrac{1}{a} \ln u_0 - \tfrac{1}{a} \ln(a - bu) + \tfrac{1}{a} \ln(a - bu_0) = t$$

$$\ln \frac{u}{a - bu} = at + \ln \frac{u_0}{a - bu_0}$$

$$\frac{u}{a - bu} = e^{at} \frac{u_0}{a - bu_0}$$

$$\boxed{u(t) = \frac{a}{b + e^{-at}(a - bu_0)/u_0}}$$

6.1.8 Suppose a rumor starts with $u_0 = 1$ person and spreads according to $u' = u(N - u)$. Find $u(t)$ for this logistic equation. At what time T does the rumor reach half the population $(u(T) = \frac{1}{2} N)$?

$u' = Nu - u^2 \qquad a = N,\ b = 1,\ u_0 = 1$

$$\boxed{u(t) = \frac{N}{1 + e^{-Nt}(N-1)}}$$

$$\tfrac{1}{2} N = \frac{N}{1 + e^{-NT}(N-1)}$$

$$1 + e^{-NT}(N-1) = 2$$

$$e^{-NT} = 1/(N-1)$$

$$\boxed{T = N^{-1} \ln(N-1)}$$

6.1.11 Find the solution with arbitrary constants to

(a) $u'' - 9u = 0$

Try $u = e^{\lambda t}$

$\lambda^2 e^{\lambda t} - 9e^{\lambda t} = 0$

$\lambda^2 - 9 = 0 \longrightarrow \; = \pm 3$

$\boxed{u(t) = Ce^{3t} + De^{-3t}}$

(c) $u'' + 2u' + 5u = 0$

$\lambda^2 + 2\lambda + 5 = 0$

$$\begin{aligned}\lambda &= -1 \pm \tfrac{1}{2}\sqrt{4 - 4\cdot 5} \\ &= -1 \pm \tfrac{1}{2}\sqrt{-16} \\ &= -1 \pm 2i\end{aligned}$$

$\boxed{u(t) = e^{-t}(C\sin 2t + D\cos 2t)}$

Damped Spring $mu'' + cu' + ku = 0$ with free oscillations.

k: spring stiffness

m:

c:

f:

The displacement u is measured from the steady state position where the upward force kx

6.1.13

(a) What damping constants c in $\frac{1}{3}u'' + cu' + \frac{1}{2}u = 0$ produce overdamping, critical damping, underdamping, no damping, and negative damping?