

§ 4.1 Fourier Series and Orthogonal Expansions

4.1.1 Find the Fourier Series on $-\pi < x < \pi$ for

(b) $f(x) = |\sin x|$, an even function.

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos kx \quad |\sin x| = \sin x \text{ on } 0 \leq x \leq \pi$$

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x) dx = \frac{1}{\pi} \int_0^{\pi} \sin x dx = -\frac{1}{\pi} \cos x \Big|_0^{\pi} = -\frac{1}{\pi} (-1 - 1) = \frac{2}{\pi}$$

$$a_k = \frac{2}{\pi} \int_0^{\pi} f(x) \cos kx$$

$$\begin{aligned} \sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\ \sin(\alpha - \beta) &= \sin \alpha \cos \beta - \cos \alpha \sin \beta \end{aligned}$$

$$= \frac{2}{\pi} \int_0^{\pi} \sin x \cos kx dx$$

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin \alpha \cos \beta$$

$$= \frac{1}{\pi} \int_0^{\pi} \{ \sin(1+k)x + \sin(1-k)x \} dx$$

$$= -\frac{1}{\pi} \left\{ \frac{\cos(1+k)x}{1+k} + \frac{\cos(1-k)x}{1-k} \right\} \Big|_0^{\pi}$$

$$= -\frac{1}{\pi} \left\{ \frac{\cos(1+k)\pi}{1+k} - \frac{1}{1+k} + \frac{\cos(1-k)\pi}{1-k} - \frac{1}{1-k} \right\}$$

$$= -\frac{1}{\pi} \left\{ \frac{(-1)^{k+1} - 1}{1+k} + \frac{(-1)^{k+1} - 1}{1-k} \right\}$$

$$= \begin{cases} \frac{2}{\pi} \left\{ \frac{1}{1+k} + \frac{1}{1-k} \right\}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$$

$k=1$	$\cos 2\pi = 1$	$\cos 0 = 1$
$k=2$	$\cos 3\pi = -1$	$\cos -\pi = -1$
$k=3$	$\cos 4\pi = 1$	$\cos -2\pi = 1$
\vdots	\vdots	\vdots

$$f(x) = |\sin x| = \frac{2}{\pi} + \sum_{\substack{k=2 \\ k \text{ even}}}^{\infty} \frac{2}{\pi} \left\{ \frac{1}{1+k} + \frac{1}{1-k} \right\} \cos kx$$

$$= \frac{2}{\pi} + \frac{4}{\pi} \sum_{\substack{k=2 \\ k \text{ even}}}^{\infty} \frac{1}{1-k^2} \cos kx$$

$$= \boxed{\frac{2}{\pi} + \frac{4}{\pi} \sum_{j=1}^{\infty} \frac{1}{1-4j^2} \cos 2jx}$$

4.1.2 A square wave has $f(x) = -1$ on the left side $-\pi < x < 0$ and $f(x) = 1$ on the right side $0 < x < \pi$.

(a) Why are all the cosine coefficients $a_k = 0$?

Since f is an odd function, $a_0 = \int_{-\pi}^{\pi} f(x) dx = 0$.

For $k \geq 0$ $f(x)\cos kx$ is the product of an odd function with an even function so $f(x)\cos kx$ is an odd function $\rightarrow a_k = \int_{-\pi}^{\pi} f(x)\cos kx dx = 0$.

(b) Find the sine series $\sum b_k \sin kx$ from equation (6):

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx \quad (6)$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^0 -\sin kx dx + \frac{1}{\pi} \int_0^{\pi} \sin kx dx$$

$$= \frac{1}{\pi} \int_{\pi}^0 \sin -ky dy + \frac{1}{\pi} \int_0^{\pi} \sin kx dx$$

$$\begin{array}{l} 1 \cos \pi \\ 2 \cos 2\pi \end{array}$$

$$= \frac{1}{\pi} \int_0^{\pi} \sin ky dy + \frac{1}{\pi} \int_0^{\pi} \sin kx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \sin kx dx = -\frac{2}{\pi k} \cos kx \Big|_0^{\pi} = -\frac{2}{\pi k} \{(-1)^k - 1\} = \begin{cases} 4/\pi k, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}$$

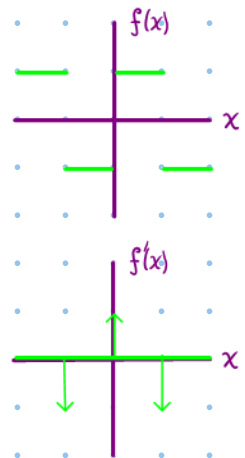
$$f(x) = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi k} \sin kx = \sum_{j=1}^{\infty} \frac{4}{\pi(2j-1)} \sin(2j-1)x = \frac{4}{\pi} \left\{ \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right\}$$

4.1.3 Find the sine series for the square wave in another way by showing.

(a) $df/dx = 2\delta(x) - 2\delta(x+\pi)$ extended periodically.

If $f(x)$ is the square wave extended periodically

$$f(x) = \begin{cases} \vdots & \\ 1 & -2\pi < x < -\pi \\ -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \\ -1 & \pi < x < 2\pi \\ \vdots & \end{cases} \quad \frac{df}{dx} = \begin{cases} \vdots & \\ -\infty & x = -\pi \\ 0 & -\pi < x < 0 \\ \infty & x = 0 \\ 0 & 0 < x < \pi \\ -\infty & x = \pi \\ 0 & \pi < x < 2\pi \\ \vdots & \end{cases}$$



$\frac{df}{dx}$ is the 2π -periodic extension of $g(x) = \begin{cases} -\infty & x = -\pi \\ 0 & -\pi < x < 0 \\ \infty & x = 0 \\ 0 & 0 < x < \pi \end{cases}$

$$\text{For } x \in [-\pi, \pi), \quad 2\delta(x) - 2\delta(x+\pi) = \begin{cases} -\infty & x = -\pi \\ 0 & -\pi < x < 0 \\ \infty & x = 0 \\ 0 & 0 < x < \pi \end{cases} = g(x)$$

That is, df/dx is the 2π -periodic extension of $2\delta(x) - 2\delta(x+\pi)$

$$(b) \quad 2\delta(x) - 2\delta(x+\pi) = \frac{4}{\pi} \{ \cos x + \cos 3x + \dots \}$$

From page 269, for $\delta(x)$, $a_0 = 1/2\pi$, $a_k = 1/\pi$, $b_k = 0$. This implies:

$$\begin{aligned} 2\delta(x) - 2\delta(x+\pi) &= 2 \left\{ \frac{1}{2\pi} + \sum_{k=1}^{\infty} a_k \cos kx \right\} - 2 \left\{ \frac{1}{2\pi} + \sum_{k=1}^{\infty} a_k \cos k(x+\pi) \right\} \\ &= 2 \sum_{k=1}^{\infty} \frac{1}{\pi} \{ \cos kx - \cos k(x+\pi) \} = \frac{4}{\pi} \sum_{k=1, k \text{ odd}}^{\infty} \cos kx = \frac{4}{\pi} \{ \cos x + \cos 3x + \dots \} \end{aligned}$$

(Since $\cos kx = -\cos k(x+\pi)$, k odd and $\cos kx = \cos k(x+\pi)$, k even)

From parts (a) and (b) conclude that the Fourier series is:

$$f(x) = \int_{-\pi}^{\pi} \{ 2\delta(x) - 2\delta(x+\pi) \} dx = \int \frac{4}{\pi} \sum_{k=1, k \text{ odd}}^{\infty} \cos kx dx = \frac{4}{\pi} \sum_{k=1, k \text{ odd}}^{\infty} \frac{1}{k} \sin kx$$

$$f(x) = \frac{4}{\pi} \sum_{j=1}^{\infty} \frac{1}{2j-1} \sin(2j-1)x = \frac{4}{\pi} \left\{ \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right\}$$

Laplace's Equation in Cartesian coordinates is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy} = 0$$

Let $u = u(r, \theta)$ with $x = r \cos \theta$, $y = r \sin \theta$

$$r = (x^2 + y^2)^{1/2}, \quad \theta = \arctan y/x + c$$

depends on the quadrant

$$u_x = u_r r_x + u_\theta \theta_x \quad r_x = \frac{1}{2}(x^2 + y^2)^{-1/2} 2x = x / (x^2 + y^2)^{1/2} = x/r = \cos \theta$$

$$= u_r \cos \theta - \frac{1}{r} u_\theta \sin \theta \quad \theta_x = \frac{-y/x^2}{1 + (y/x)^2} = -\frac{y}{x^2 + y^2} = -y/r^2 = -\sin \theta / r$$

$$u_{xx} = u_r (\cos \theta)_x + u_{rx} \cos \theta - u_\theta \left(\frac{\sin \theta}{r} \right)_x - u_{\theta x} \frac{\sin \theta}{r}$$

$$= u_r (-\sin \theta) (-\sin \theta / r) + u_{rx} \cos \theta - u_\theta \frac{(\cos \theta)(-\sin \theta / r) r - \sin \theta \cos \theta}{r^2}$$

$$- u_{\theta x} \frac{\sin \theta}{r}$$

$$= \frac{\sin^2 \theta}{r} u_r + u_{rx} \cos \theta + u_\theta \frac{2 \cos \theta \sin \theta}{r^2} - u_{\theta x} \frac{\sin \theta}{r}$$

$$= \frac{\sin^2 \theta}{r} u_r + u_{rr} \cos^2 \theta - u_{r\theta} \frac{\sin \theta \cos \theta}{r}$$

$$+ u_\theta \frac{2 \cos \theta \sin \theta}{r^2} - (u_{\theta\theta} (-\frac{\sin \theta}{r}) + u_{\theta r} \cos \theta) \left(\frac{\sin \theta}{r} \right)$$

$$u_{xx} = \cos^2 \theta u_{rr} + \frac{\sin^2 \theta}{r} u_r - \frac{2 \sin \theta \cos \theta}{r} u_{r\theta} + \frac{2 \sin \theta \cos \theta}{r^2} u_\theta + \frac{\sin^2 \theta}{r^2} u_{\theta\theta}$$

$$\left\{ \begin{array}{l} u_{rx} = u_{rr} r_x + u_{r\theta} \theta_x \\ = u_{rr} \cos \theta - u_{r\theta} \frac{\sin \theta}{r} \\ u_{\theta x} = u_{\theta\theta} \theta_x + u_{\theta r} r_x \\ = u_{\theta\theta} (-\frac{\sin \theta}{r}) + u_{\theta r} \cos \theta \end{array} \right.$$

$$u_y = u_r r_y + u_\theta \theta_y \quad r_y = y/r = \sin \theta$$

$$= u_r \sin \theta + \frac{1}{r} u_\theta \cos \theta \quad \theta_y = \frac{1/x}{1 + y^2/x^2} = \frac{x}{x^2 + y^2} = x/r^2 = \cos \theta / r$$

$$u_{yy} = u_{ry} \sin \theta + u_r (\sin \theta)_y + u_{\theta y} \frac{\cos \theta}{r} + u_\theta \left(\frac{\cos \theta}{r} \right)_y$$

$$= (u_{rr} \sin \theta + u_{r\theta} \frac{\cos \theta}{r}) \sin \theta + u_r \cos^2 \theta / r + u_{\theta\theta} \cos^2 \theta / r^2 + u_{\theta r} \frac{\sin \theta \cos \theta}{r} + u_\theta \left(-\frac{2 \sin \theta \cos \theta}{r^2} \right)$$

$$\left\{ \begin{array}{l} u_{ry} = u_{rr} r_y + u_{r\theta} \theta_y = u_{rr} \sin \theta + u_{r\theta} \frac{\cos \theta}{r} \\ u_{\theta y} = u_{\theta\theta} \theta_y + u_{\theta r} r_y = u_{\theta\theta} \cos \theta / r + u_{\theta r} \sin \theta \end{array} \right.$$

$$u_{yy} = \sin^2 \theta u_{rr} + \frac{\cos^2 \theta}{r} u_r + \frac{2 \sin \theta \cos \theta}{r} u_{r\theta} - \frac{2 \sin \theta \cos \theta}{r^2} u_\theta + \frac{\cos^2 \theta}{r^2} u_{\theta\theta}$$

$$\Delta = u_{xx} + u_{yy}$$

$$= (\cos^2 \theta + \sin^2 \theta) u_{rr}$$

$$+ \frac{1}{r} (\sin^2 \theta + \cos^2 \theta) u_r + 0 \cdot u_{r\theta}$$

$$+ 0 \cdot u_\theta + \frac{1}{r^2} (\sin^2 \theta + \cos^2 \theta) u_{\theta\theta}$$

$$= u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = \frac{1}{r} (r u_r)_r + \frac{1}{r^2} u_{\theta\theta}$$

$$\left\{ \begin{array}{l} \left(\frac{\cos \theta}{r} \right)_y = \frac{-\sin \theta \theta_y r}{r^2} - \frac{\cos \theta r_y}{r^2} \\ = -\frac{\sin \theta \cos \theta}{r^2} - \frac{\cos \theta \sin \theta}{r^2} \\ = -\frac{2 \sin \theta \cos \theta}{r^2} \end{array} \right.$$

Laplace's Equation in Polar coordinates is

$$0 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = \frac{1}{r} (r u_r)_r + \frac{1}{r^2} u_{\theta\theta}$$

$$u = u_0 \text{ (constant)} \quad \frac{1}{r} (r u_r)_r + \frac{1}{r^2} u_{\theta\theta} = \frac{1}{r} 0 + \frac{1}{r^2} 0 = 0.$$

$$u(r, \theta) = r^n \sin n\theta$$

$$\frac{1}{r} (r n r^{n-1} \sin n\theta)_r + \frac{1}{r^2} (-n^2 r^n \sin n\theta) = \frac{1}{r} n^2 r^{n-2} \sin n\theta - \frac{n^2}{r^2} r^n \sin n\theta = 0$$

$$u(r, \theta) = r^n \cos n\theta$$

$$\frac{1}{r} (r n r^{n-1} \cos n\theta)_r + \frac{1}{r^2} (-n^2 r^n \cos n\theta) = \frac{1}{r} n^2 r^{n-2} \cos n\theta - \frac{n^2}{r^2} r^n \cos n\theta = 0$$

$$u(r, \theta) = a_0 + a_1 r \cos \theta + b_1 r \sin \theta + a_2 r^2 \cos 2\theta + b_2 r^2 \sin 2\theta + \dots$$

Satisfies Laplace's equation in polar coordinates.

4.1.6 Around the unit circle suppose u is a square wave

$$u_0 = \begin{cases} +1 & \text{on the upper semicircle } 0 < \theta < \pi \\ -1 & \text{on the lower semicircle } -\pi < \theta < 0 \end{cases}$$

From the Fourier series for the square wave write down the Fourier series for u (the solution 21) to Laplace's equation). What is the value of u at the origin?

$$u(r, \theta) = a_0 + a_1 r \cos \theta + b_1 r \sin \theta + a_2 r^2 \cos 2\theta + b_2 r^2 \sin 2\theta + \dots$$

$$1 = u(1, \theta) = a_0 + a_1 \cos \theta + b_1 \sin \theta + \dots \quad 0 < \theta < \pi$$

$$-1 = u(1, \theta) = a_0 + a_1 \cos \theta + b_1 \sin \theta + \dots \quad -\pi < \theta < 0$$

$$1 = u(1, \pi/2) = a_0 + b_1 + b_2 + b_3 + \dots$$

$$-1 = u(1, -\pi/2) = a_0 - b_1 - b_2 - b_3 - \dots \rightarrow 0 = a_0$$

Since $\cos(\theta) = \cos(-\theta)$ and $-\sin(\theta) = \sin(-\theta)$, for $0 < \theta < \pi$:

$$1 = u(1, \theta) = a_1 \cos \theta + b_1 \sin \theta + a_2 \cos 2\theta + b_2 \sin 2\theta + \dots$$

$$-1 = u(1, -\theta) = a_1 \cos \theta - b_1 \sin \theta + a_2 \cos 2\theta - b_2 \sin 2\theta + \dots$$

$$0 = a_1 \cos \theta + a_2 \cos 2\theta + a_3 \cos 3\theta + \dots$$

Since $\{\cos k\theta\}_{k=1}^{\infty}$ are orthogonal wrt to the $L^2[-\pi, \pi]$ inner product, $a_i = 0$ for all i .

$$u(r, \theta) = b_1 r \sin \theta + b_2 r^2 \sin 2\theta + \dots$$

$$1 = u(1, \theta) = b_1 \sin \theta + b_2 \sin 2\theta + b_3 \sin 3\theta + b_4 \sin 4\theta + \dots \quad \text{for } 0 < \theta < \pi$$

From Exercises 4.1.2, 4.1.3,

$$1 = \frac{4}{\pi} \left\{ \sin \theta + \frac{1}{3} \sin 3\theta + \frac{1}{5} \sin 5\theta + \dots \right\} \quad \text{for } 0 < \theta < \pi$$

This implies $b_1 = 1$, $b_2 = 0$, $b_3 = 1/3$, $b_4 = 0$, $b_5 = 1/5$, ...

$$\therefore u(r, \theta) = \frac{4}{\pi} \left\{ r \sin \theta + \frac{1}{3} r^3 \sin 3\theta + \frac{1}{5} r^5 \sin 5\theta + \dots \right\}$$

$$\lim_{r \rightarrow 0} u(r, \theta) = u(0, \theta) = 0.$$

4.1.10 What constant function is closest in the least square sense to $f(x) = \cos^2 x$? What multiple of $\cos x$ is closest to $f(x) = \cos^3 x$?

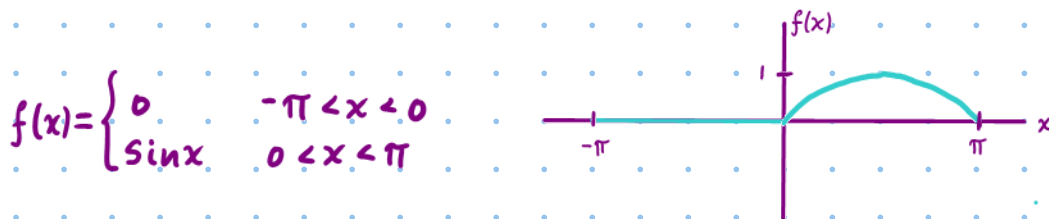
$\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x$ If we write out the Fourier series for $f(x)$, we would get $a_0 = \frac{1}{2}$, $a_2 = \frac{1}{2}$, $a_n = 0 \ \forall n \notin \{0, 2\}$ and all $b_n = 0$. So the constant function $\frac{1}{2}$ is closest.

Let $k \in \mathbb{R}$. The least squares distance between $\cos^3 x$ and $k \cos x$ is:

$$\langle \cos^3 x, \cos x \rangle_{L^2[-\pi, \pi]} = \int_{-\pi}^{\pi} |\cos^3 x - k \cos x|^2 dx = \frac{\pi}{8} (8k^2 - 12k + 5).$$

This is minimized by $k = \frac{3}{4}$. The multiple of $\cos x$ closest to $\cos^3 x$ in the least squares sense is $\frac{3}{4} \cos x$.

4.1.12 Sketch the 2π -periodic half wave $f(x)$ and find its Fourier series.



$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi} \int_0^{\pi} \sin x dx = \frac{1}{\pi}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx = \frac{1}{\pi} \int_0^{\pi} \sin x \cos kx dx = \frac{1}{2\pi} \int_0^{\pi} (\sin(1+k)x + \sin(1-k)x) dx$$

$$= -\frac{1}{2\pi} \left(\frac{1}{1+k} \cos(1+k)x + \frac{1}{1-k} \cos(1-k)x \right) \Big|_0^{\pi}$$

$$= -\frac{1}{2\pi} \left\{ \frac{1}{1+k} (\cos(1+k)\pi - 1) + \frac{1}{1-k} (\cos(1-k)\pi - 1) \right\}$$

$$= -\frac{1}{2\pi} \left\{ \frac{1}{1+k} ((-1)^{k+1} - 1) + \frac{1}{1-k} ((-1)^{k+1} - 1) \right\} = \begin{cases} -\frac{1}{2\pi} \left(\frac{-2}{1+k} + \frac{-2}{1-k} \right) & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

$$= \begin{cases} 2(\pi(1+k)(1-k))^{-1} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx = \frac{1}{\pi} \int_0^{\pi} \sin x \sin kx dx = \frac{1}{2\pi} \int_0^{\pi} (\cos(1-k)x - \cos(1+k)x) dx$$

$$= \frac{1}{2\pi} \left(\frac{1}{1-k} \sin(1-k)x - \frac{1}{1+k} \sin(1+k)x \right) \Big|_0^{\pi} = 0, \quad \forall k \in \{2, 3, 4, \dots\}$$

$$b_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin x dx = \frac{1}{\pi} \int_0^{\pi} \sin^2 x dx = \frac{1}{\pi} \frac{\pi}{2} = \frac{1}{2}$$

The Fourier series for $f(x)$ is:

$$a_0 + b_1 \sin x + \sum_{k \text{ even}} a_k \cos kx = \frac{1}{\pi} + \frac{1}{2} \sin x - \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{\cos 2jx}{4j^2 - 1}$$