§4.1 Fourier Series and Orthogonal Expansions

4.1.1 Find the Fourier Series on $-\pi < x < \pi$ for

(b) $f(x) = |\sin x|$, an even function.

$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos kx$ $\qquad |\sin x| = \sin x$ on $0 \le x \le \pi$

$$\boxed{a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx} = \frac{1}{\pi}\int_0^{\pi} \sin x\,dx = -\frac{1}{\pi}\cos x\Big|_0^{\pi} = -\frac{1}{\pi}(-1-1) = 2/\pi$$

$$\boxed{a_k = \frac{2}{\pi}\int_0^{\pi} f(x)\cos kx}$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x \cos kx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\left\{\sin(1+k)x + \sin(1-k)x\right\}dx$$

$$= -\frac{1}{\pi}\left\{\frac{\cos(1+k)x}{1+k} + \frac{\cos(1-k)x}{1-k}\right\}\Big|_0^{\pi}$$

$$= -\frac{1}{\pi}\left\{\frac{\cos(1+k)\pi}{1+k} - \frac{1}{1+k} + \frac{\cos(1-k)\pi}{1-k} - \frac{1}{1-k}\right\}$$

$$= -\frac{1}{\pi}\left\{\frac{(-1)^{k+1}-1}{1+k} + \frac{(-1)^{k+1}-1}{1-k}\right\}$$

$k=1$	$\cos 2\pi = 1$	$\cos 0 = 1$
$k=2$	$\cos 3\pi = -1$	$\cos -\pi = -1$
$k=3$	$\cos 4\pi = 1$	$\cos -2\pi = 1$
$\vdots$	$\vdots$	$\vdots$

$$= \begin{cases} \frac{2}{\pi}\left\{\frac{1}{1+k} + \frac{1}{1-k}\right\}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$$

$$f(x) = |\sin x| = \frac{2}{\pi} + \sum_{\substack{k=2 \\ k\,\text{even}}}^{\infty} \frac{2}{\pi}\left\{\frac{1}{1+k} + \frac{1}{1-k}\right\}\cos kx$$

$$= \frac{2}{\pi} + \frac{4}{\pi}\sum_{\substack{k=2 \\ k\,\text{even}}}^{\infty} \frac{1}{1-k^2}\cos kx$$

$$= \boxed{\frac{2}{\pi} + \frac{4}{\pi}\sum_{j=1}^{\infty} \frac{1}{1-4j^2}\cos 2jx}$$

4.1.2 A square wave has $f(x)=-1$ on the left side $-\pi<x<0$ and $f(x)=1$ on the right side $0<x<\pi$.

(a) Why are all the cosine coefficients $a_k=0$?

Since f is an odd function, $a_0=\int_{-\pi}^{\pi} f(x)\,dx=0$.
For $k\geq 0$ $f(x)\cos kx$ is the product of an odd function with an even function so $f(x)\cos kx$ is an odd function $\longrightarrow a_k=\int_{-\pi}^{\pi} f(x)\cos kx\,dx=0$.

(b) Find the sine series $\sum b_k \sin kx$ from equation (6):

$$\boxed{b_k=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx} \qquad (6)$$

$$\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{0} -\sin kx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\sin kx\,dx \\
&= \frac{1}{\pi}\int_{\pi}^{0}\sin -ky\,dy + \frac{1}{\pi}\int_{0}^{\pi}\sin kx\,dx \\
&= \frac{1}{\pi}\int_{0}^{\pi}\sin ky\,dy + \frac{1}{\pi}\int_{0}^{\pi}\sin kx\,dx \\
&= \frac{2}{\pi}\int_{0}^{\pi}\sin kx\,dx = -\frac{2}{\pi k}\cos kx\Big|_{0}^{\pi} = -\frac{2}{\pi k}\left\{(-1)^k-1\right\} = \begin{cases} 4/\pi k, & k \text{ odd} \\ 0, & k \text{ even}\end{cases}
\end{aligned}$$

1 $\cos\pi$
2 $\cos 2\pi$

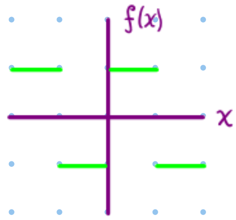
$$\boxed{f(x)=\sum_{\substack{k=1 \\ k\text{ odd}}}^{\infty}\frac{4}{\pi k}\sin kx=\sum_{j=1}^{\infty}\frac{4}{\pi(2j-1)}\sin(2j-1)x=\frac{4}{\pi}\left\{\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\dots\right\}}$$

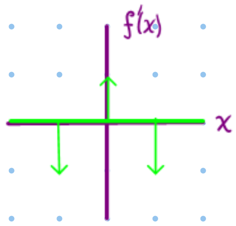
4.1.3 Find the sine series for the square wave in another way by showing

(a) $df/dx = 2\delta(x) - 2\delta(x+\pi)$ extended periodically.

If $f(x)$ is the square wave extended periodically

$$f(x) = \begin{cases} \vdots & \\ 1 & -2\pi < x < -\pi \\ -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \\ -1 & \pi < x < 2\pi \\ \vdots & \end{cases} \qquad \frac{df}{dx} = \begin{cases} \vdots & \\ -\infty & x = -\pi \\ 0 & -\pi < x < 0 \\ \infty & x = 0 \\ 0 & 0 < x < \pi \\ -\infty & x = \pi \\ 0 & \pi < x < 2\pi \\ \vdots & \end{cases}$$





$\frac{df}{dx}$ is the 2π-periodic extension of $g(x) = \begin{cases} -\infty & x = -\pi \\ 0 & -\pi < x < 0 \\ \infty & x = 0 \\ 0 & 0 < x < \pi \end{cases}$

For $x \in [-\pi, \pi)$, $2\delta(x) - 2\delta(x+\pi) = \begin{cases} -\infty & x = -\pi \\ 0 & -\pi < x < 0 \\ \infty & x = 0 \\ 0 & 0 < x < \pi \end{cases} = g(x)$

That is, df/dx is the 2π-periodic extension of $2\delta(x) - 2\delta(x+\pi)$

(b) $2\delta(x) - 2\delta(x+\pi) = \frac{4}{\pi}\{\cos x + \cos 3x + \dots\}$

From page 269, for $\delta(x)$, $a_0 = 1/2\pi$, $a_k = 1/\pi$, $b_k = 0$. This implies:

$$2\delta(x) - 2\delta(x+\pi) = 2\left\{\frac{1}{2\pi} + \sum_{k=1}^{\infty} a_k \cos kx\right\} - 2\left\{\frac{1}{2\pi} + \sum_{k=1}^{\infty} a_k \cos k(x+\pi)\right\}$$

$$= 2\sum_{k=1}^{\infty} \frac{1}{\pi}\{\cos kx - \cos k(x+\pi)\} = \frac{4}{\pi}\sum_{k=1,\, k\text{ odd}}^{\infty} \cos kx = \frac{4}{\pi}\{\cos x + \cos 3x + \dots\}$$

(Since $\cos kx = -\cos k(x+\pi)$, k odd and $\cos kx = \cos k(x+\pi)$, k even)

From parts (a) and (b) conclude that the Fourier series is:

$$f(x) = \int_{-\pi}^{\pi} \{2\delta(x) - 2\delta(x+\pi)\}\, dx = \int \frac{4}{\pi} \sum_{k=1,\, k\text{ odd}}^{\infty} \cos kx \, dx = \frac{4}{\pi} \sum_{k=1,\, k\text{ odd}}^{\infty} \frac{1}{k} \sin kx$$

$$\boxed{f(x) = \frac{4}{\pi} \sum_{j=1}^{\infty} \frac{1}{2j-1} \sin(2j-1)x = \frac{4}{\pi}\left\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right\}}$$

Laplace's Equation in Cartesian coordinates is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy} = 0$$

Let $u = u(r,\theta)$ with $x = r\cos\theta$, $y = r\sin\theta$

depends on the quadrant

$$r = (x^2+y^2)^{1/2}, \quad \theta = \arctan y/x + c$$

$u_x = u_r r_x + u_\theta \theta_x$ $\qquad r_x = \frac{1}{2}(x^2+y^2)^{-1/2} 2x = x/(x^2+y^2)^{1/2} = x/r = \cos\theta$

$= u_r\cos\theta - \frac{1}{r}u_\theta\sin\theta$ $\qquad \theta_x = \frac{-y/x^2}{1+(y/x)^2} = -\frac{y}{x^2+y^2} = -y/r^2 = -\sin\theta/r$

$$u_{xx} = u_r(\cos\theta)_x + u_{rx}\cos\theta - u_\theta\left(\frac{\sin\theta}{r}\right)_x - u_{\theta x}\frac{\sin\theta}{r}$$

$$= u_r(-\sin\theta)(-\sin\theta/r) + u_{rx}\cos\theta - u_\theta\frac{(\cos\theta)(-\sin\theta/r)r - \sin\theta\cos\theta}{r^2} - u_{\theta x}\frac{\sin\theta}{r}$$

$$= \frac{\sin^2\theta}{r}u_r + u_{rx}\cos\theta + u_\theta\frac{2\cos\theta\sin\theta}{r^2} - u_{\theta x}\frac{\sin\theta}{r}$$

$$= \frac{\sin^2\theta}{r}u_r + u_{rr}\cos^2\theta - u_{r\theta}\frac{\sin\theta\cos\theta}{r} + u_\theta\frac{2\cos\theta\sin\theta}{r^2} - \left(u_{\theta\theta}\left(\frac{-\sin\theta}{r}\right) + u_{\theta r}\cos\theta\right)\left(\frac{\sin\theta}{r}\right)$$

$$\left\{\begin{aligned} u_{rx} &= u_{rr}r_x + u_{r\theta}\theta_x \\ &= u_{rr}\cos\theta - u_{r\theta}\frac{\sin\theta}{r} \\ u_{\theta x} &= u_{\theta\theta}\theta_x + u_{\theta r}r_x \\ &= u_{\theta\theta}\left(\frac{-\sin\theta}{r}\right) + u_{\theta r}\cos\theta \end{aligned}\right\}$$

$$\boxed{\begin{aligned} u_{xx} &= \cos^2\theta\, u_{rr} + \frac{\sin^2\theta}{r}u_r - \frac{2\sin\theta\cos\theta}{r}u_{r\theta} \\ &\quad + \frac{2\sin\theta\cos\theta}{r^2}u_\theta + \frac{\sin^2\theta}{r^2}u_{\theta\theta} \end{aligned}}$$

$$u_y = u_r r_y + u_\theta \theta_y \qquad r_y = y/r = \sin\theta$$

$$= u_r \sin\theta + \frac{1}{r} u_\theta \cos\theta \qquad \theta_y = \frac{1/x}{1 + y^2/x^2} = \frac{x}{x^2+y^2} = x/r^2 = \cos\theta/r$$

$$u_{yy} = u_{ry}\sin\theta + u_r(\sin\theta)_y + u_{\theta y}\frac{\cos\theta}{r} + u_\theta\left(\frac{\cos\theta}{r}\right)_y$$

$$= \left(u_{rr}\sin\theta + u_{r\theta}\frac{\cos\theta}{r}\right)\sin\theta$$
$$+ u_r \cos^2\theta/r + u_{\theta\theta}\cos^2\theta/r^2$$
$$+ u_{\theta r}\frac{\sin\theta\cos\theta}{r}$$
$$+ u_\theta\left(-\frac{2\sin\theta\cos\theta}{r^2}\right)$$

$$\left\{\begin{aligned} u_{ry} &= u_{rr} r_y + u_{r\theta}\theta_y \\ &= u_{rr}\sin\theta + u_{r\theta}\frac{\cos\theta}{r} \\ u_{\theta y} &= u_{\theta\theta}\theta_y + u_{\theta r} r_y \\ &= u_{\theta\theta}\cos\theta/r + u_{\theta r}\sin\theta \end{aligned}\right\}$$

$$\boxed{u_{yy} = \sin^2\theta\, u_{rr} + \frac{\cos^2\theta}{r}u_r + \frac{2\sin\theta\cos\theta}{r}u_{r\theta} - \frac{2\sin\theta\cos\theta}{r^2}u_\theta + \frac{\cos^2\theta}{r^2}u_{\theta\theta}}$$

$$0 = u_{xx} + u_{yy}$$
$$= (\cos^2\theta + \sin^2\theta)u_{rr}$$
$$+ \frac{1}{r}(\sin^2\theta + \cos^2\theta)u_r + 0\cdot u_{r\theta}$$
$$+ 0\cdot u_\theta + \frac{1}{r^2}(\sin^2\theta + \cos^2\theta)u_{\theta\theta}$$
$$= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}$$

$$\left\{\begin{aligned} \left(\frac{\cos\theta}{r}\right)_y &= \frac{-\sin\theta\,\theta_y r}{r^2} - \frac{\cos\theta\, r_y}{r^2} \\ &= -\frac{\sin\theta\cos\theta}{r^2} - \frac{\cos\theta\sin\theta}{r^2} \\ &= -\frac{2\sin\theta\cos\theta}{r^2} \end{aligned}\right\}$$

Laplace's Equation in Polar coordinates is

$$0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}$$

$u = a_0$ (constant) $\quad \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} = \frac{1}{r}0 + \frac{1}{r^2}0 = 0.$

$u(r,\theta) = r^n \sin n\theta$

$$\frac{1}{r}(r n r^{n-1}\sin n\theta)_r + \frac{1}{r^2}(-n^2 r^n \sin n\theta) = \frac{1}{r}n^2 r^{n-2}\sin n\theta - n^2/r^2\, r^n \sin n\theta = 0$$

$u(r,\theta) = r^n \cos n\theta$

$$\frac{1}{r}(r n r^{n-1}\cos n\theta)_r + \frac{1}{r^2}(-n^2 r^n \cos n\theta) = \frac{1}{r}n^2 r^{n-2}\cos n\theta - n^2/r^2\, r^n\cos n\theta = 0$$

$u(r,\theta) = a_0 + a_1 r\cos\theta + b_1 r\sin\theta + a_2 r^2\cos n\theta + b_2 r^2 \sin n\theta + \ldots$
Satisfies Laplace's equation in polar coordinates.

4.1.6 Around the unit circle suppose u is a square wave

$$u_0 = \begin{cases} +1 \text{ on the upper semicircle} & 0 < \theta < \pi \\ -1 \text{ on the lower semicircle} & -\pi < \theta < 0 \end{cases}$$

From the Fourier series for the square wave write down the Fourier series for u (the solution 21) to Laplace's equation). What is the value of u at the origin?

$$u(r,\theta) = a_0 + a_1 r\cos\theta + b_1 r\sin\theta + a_2 r^2\cos 2\theta + b_2 r^2 \sin 2\theta + \dots$$

$$1 = u(1,\theta) = a_0 + a_1\cos\theta + b_1\sin\theta + \dots \quad 0<\theta<\pi$$
$$-1 = u(1,\theta) = a_0 + a_1\cos\theta + b_1\sin\theta + \dots \quad -\pi<\theta<0$$

$$1 = u(1,\pi/2) = a_0 + b_1 + b_2 + b_3 + \dots$$
$$-1 = u(1,-\pi/2) = a_0 - b_1 - b_2 - b_3 - \dots \rightarrow 0 = a_0$$

Since $\cos(\theta) = \cos(-\theta)$ and $-\sin(\theta) = \sin(-\theta)$, for $0<\theta<\pi$:

$$1 = u(1,\theta) = a_1\cos\theta + b_1\sin\theta + a_2\cos 2\theta + b_2\sin 2\theta + \dots$$
$$-1 = u(1,-\theta) = a_1\cos\theta - b_1\sin\theta + a_2\cos 2\theta - b_2\sin 2\theta + \dots$$

$$0 = a_1\cos\theta + a_2\cos 2\theta + a_3\cos 3\theta + \dots$$

Since $\{\cos k\theta\}_{k=1}^{\infty}$ are orthogonal wrt to the $L^2[-\pi,\pi]$ inner product, $a_i = 0$ for all i.

$$u(r,\theta) = b_1 r\sin\theta + b_2 r^2 \sin 2\theta + \dots$$

$$1 = u(1,\theta) = b_1\sin\theta + b_2\sin 2\theta + b_3\sin 3\theta + b_4 \sin 4\theta + \dots \text{ for } 0<\theta<\pi$$

From Exercises 4.1.2, 4.1.3,

$$1 = \frac{4}{\pi}\left\{\sin\theta + \tfrac{1}{3}\sin 3\theta + \tfrac{1}{5}\sin 5\theta + \dots\right\} \text{ for } 0<\theta<\pi$$

This implies $b_1 = 1$, $b_2 = 0$, $b_3 = 1/3$, $b_4 = 0$, $b_5 = 1/5$, ...

$$\therefore u(r,\theta) = \frac{4}{\pi}\left\{r\sin\theta + \tfrac{1}{3}r^3\sin 3\theta + \tfrac{1}{5}r^5\sin 5\theta + \dots\right\}$$

$$\lim_{r\to 0} u(r,\theta) = u(0,\theta) = 0.$$

<u>4.1.10</u> What constant function is closest in the least square sense to $f(x) = \cos^2 x$? what multiple of $\cos x$ is closest to $f(x) = \cos^3 x$?

$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$ If we write out the Fourier series for $f(x)$, we would get $a_0 = 1/2$, $a_2 = 1/2$, $a_n = 0$ $\forall n \notin \{0, 2\}$ and all $b_n = 0$. So the constant function $\boxed{1/2}$ is closest.

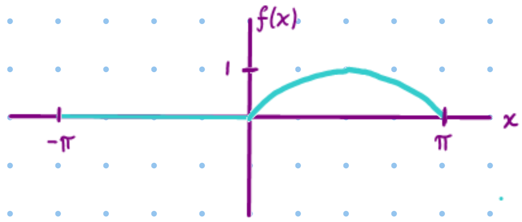
Let $K \in \mathbb{R}$. The least squares distance between $\cos^3 x$ and $K\cos x$ is:

$$\langle \cos^3 x, \cos x \rangle_{L^2[-\pi,\pi]} = \int_{-\pi}^{\pi} |\cos^3 x - K\cos x|^2 \, dx = \frac{\pi}{8}(8K^2 - 12K + 5).$$

This is minimized by $K = 3/4$. The multiple of $\cos x$ closest to $\cos^3 x$ in the least squares sense is $\boxed{\frac{3}{4}\cos x}$

<u>4.1.12</u> Sketch the 2π-periodic half wave $f(x)$ and find its Fourier series.

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \sin x & 0 < x < \pi \end{cases}$$



$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{\pi} \sin x\,dx = \frac{1}{\pi}$$

$$a_K = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos Kx = \frac{1}{\pi}\int_0^{\pi} \sin x \cos Kx\,dx = \frac{1}{2\pi}\int_0^{\pi}\left(\sin(1+K)x + \sin(1-K)x\right)dx$$

$$= -\frac{1}{2\pi}\left(\frac{1}{1+K}\cos(1+K)x + \frac{1}{1-K}\cos(1-K)x\right)\Big|_0^{\pi}$$

$$= -\frac{1}{2\pi}\left\{\frac{1}{1+K}(\cos(1+K)\pi - 1) + \frac{1}{1-K}(\cos(1-K)\pi - 1)\right\}$$

$$= -\frac{1}{2\pi}\left\{\frac{1}{1+K}((-1)^{K+1} - 1) + \frac{1}{1-K}((-1)^{K+1} - 1)\right\} = \begin{cases} -\frac{1}{2\pi}\left(\frac{-2}{1+K} + \frac{-2}{1-K}\right) & K \text{ even} \\ 0 & K \text{ odd} \end{cases}$$

$$= \begin{cases} 2(\pi(1+K)(1-K))^{-1} & K \text{ even} \\ 0 & K \text{ odd} \end{cases}$$

$$b_K = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin Kx\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x \sin Kx\,dx = \frac{1}{2\pi}\int_0^{\pi}(\cos(1-K)x - \cos(1+K)x)dx$$

$$= \frac{1}{2\pi}\left(\frac{1}{1-K}\sin(1-K)x - \frac{1}{1+K}\sin(1+K)x\right)\Big|_0^{\pi} = 0, \quad \forall K \in \{2, 3, 4, \dots\}$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\,dx = \frac{1}{\pi}\int_0^{\pi}\sin^2 x\,dx = \frac{1}{\pi}\frac{\pi}{2} = \frac{1}{2}$$

The Fourier series for $f(x)$ is:

$$a_0 + b_1 \sin x + \sum_{K \text{ even}} a_K \cos Kx = \boxed{\frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{j=0}^{\infty}\frac{\cos 2jx}{4j^2 - 1}}$$

Properties of the Fourier Series

1. Each Fourier coefficient a_k, b_k, or c_k is the best possible choice in the mean square sense. In other words, the error E is at a minimum when $A_k = a_k$, $B_k = b_k$ in

$$E = \int_{-\pi}^{\pi} \left[f(x) - \sum_0^N A_k \cos kx + B_k \sin kx \right]^2 dx$$

For proof, check a <u>typical</u> derivative, which should be zero at a minimum. Setting $0 = \partial E / \partial B_j$, $j \in \{0, 1, \dots, n\}$ (assume $b_0 := 0$)

$$0 = \frac{\partial E}{\partial B_j} = \int_{-\pi}^{\pi} 2 \left[f(x) - \sum_0^N A_k \cos kx + B_k \sin kx \right] (-\sin jx)\, dx$$

$$-\frac{1}{2} \frac{\partial E}{\partial B_j} = \int_{-\pi}^{\pi} \sin jx \left[f(x) - \sum_0^N A_k \cos kx + B_k \sin kx \right] dx$$

$$= \int_{-\pi}^{\pi} \sin jx \left[f(x) - B_j \sin jx \right] dx$$

$$\Rightarrow \quad B_j = \left(\int_{-\pi}^{\pi} f(x) \sin jx\, dx \right) \Big/ \left(\int_{-\pi}^{\pi} \sin^2 jx \right) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin jx\, dx = b_j$$

It's similar for any A_j. Take N as large as necessary to check any particular coefficient. $\therefore$ The claim holds for all $k = 0, 1, 2, \dots$

The sines and cosines are a perpendicular set of axes in function space. Fourier analysis projects f onto each of these axes.

Orthogonality allows you to find each coefficient separately and completeness that allows the sines and cosines (or e^{ikx}) to reproduce f.

2. Since projections are never larger, no piece of the Fourier series can be larger than f. In particular for a partial sum:

$$\boxed{\int_{-\pi}^{\pi} (a_0 + a_1 \cos x + \dots + b_N \sin Nx)^2 dx \le \int_{-\pi}^{\pi} (f(x))^2 dx \quad \text{(Bessel's Inequality)}}$$

Bessel's inequality is proved in Exercise 4.1.13. In the limit $N \to \infty$ it is

$$\boxed{2\pi \sum |c_k|^2 = 2\pi a_0^2 + \pi (a_1^2 + b_1^2 + a_2^2 + b_2^2 + \dots) = \int_{-\pi}^{\pi} (f(x))^2 dx \quad \text{(Parseval's Formula)}}$$

<u>Hilbert space</u> In one form contains functions and in another vectors with infinitely many components. The functions must have finite length – the integral of $(f(x))^2$ must be finite. For vectors the sum of squares is finite.

Informally, L^2: f s.t. $\int (f(x))^2 dx < \infty$, ℓ^2: x s.t. $\sum x_n^2 < \infty$. By Parseval's Formula, f is in L^2 exactly when the vector containing its Fourier coefficients is in ℓ^2.

3. We have found for various functions f its corresponding Fourier series and assuming it converges back to f. Convergence is in the mean-square sense, i.e. in Hilbert space. This is by squaring the difference and integrating. But this does not guarantee convergence at each point.

Suppose f is a function with Fourier coefficients a_0, a_k, b_k. Evaluating the series for f, $S(x) = a_0 + \sum(a_k \cos kx + b_k \sin kx)$ at $x=0$ gives:

$$a_0 + a_1 + a_2 + \ldots = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos 2x\,dx + \ldots$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left[1 + 2\cos x + 2\cos 2x + \ldots\right] dx$$

Since $1 + 2\cos x + 2\cos 2x + \ldots$ is the Fourier series for the delta function, we should expect:

$$a_0 + a_1 + a_2 + \ldots = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left[1 + 2\cos x + 2\cos 2x + \ldots\right] dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\delta(x) = f(0).$$

This proof is successful if f is smooth enough to have a derivative.

<u>4.1.13</u> Prove Bessel's Inequality by integrating the left side.

$$\int_{-\pi}^{\pi}(a_0 + a_1\cos x + \ldots + b_N \sin Nx)^2\,dx$$

$$= \int_{-\pi}^{\pi} a_0^2\,dx + \sum_{k=1}^{N}\int_{-\pi}^{\pi} a_k^2\cos^2 kx\,dx + \sum_{k=1}^{N}\int_{-\pi}^{\pi} b_k^2 \sin^2 kx\,dx \quad \text{(by orthogonality)}$$

$$= 2\pi a_0^2 + \pi\left(a_1^2 + b_1^2 + a_2^2 + b_2^2 + \ldots + a_N^2 + b_N^2\right)$$

Let $S_N = a_0 + a_1\cos x + b_1 \sin x + \ldots + b_N \sin Nx$. Consider $\int_{-\pi}^{\pi}(f(x) - S_N)^2\,dx$.

$$0 \le \left(f(x) - S_N(x)\right)^2 = (f(x))^2 - 2f(x)S_N(x) + S_N^2(x)$$

$$\int_{-\pi}^{\pi} f(x)S_N(x)\,dx = a_0\int_{-\pi}^{\pi} f(x)\,dx + a_1\int_{-\pi}^{\pi} f(x)\cos x\,dx + \ldots + b_N\int_{-\pi}^{\pi} f(x)\sin Nx\,dx$$
$$= 2\pi a_0^2 + \pi a_1^2 + \pi b_1^2 + \ldots + \pi a_N^2 + \pi b_N^2 = 2\pi a_0^2 + \pi(a_1^2 + \ldots + b_N^2)$$

$$\int_{-\pi}^{\pi} S_N^2(x)\,dx = 2\pi a_0^2 + \pi(a_1^2 + \ldots + b_N^2) \quad \text{(calculated previously).}$$

$$\therefore \int_{-\pi}^{\pi} f(x)S_N(x)\,dx = \int_{-\pi}^{\pi} S_N^2(x)\,dx$$

$$0 \le \int_{-\pi}^{\pi}(f(x))^2\,dx - 2\int_{-\pi}^{\pi} f(x)S_N(x)\,dx + \int_{-\pi}^{\pi} S_N^2(x)\,dx = \int_{-\pi}^{\pi}(f(x))^2\,dx - \int_{-\pi}^{\pi} S_N^2(x)\,dx$$

$$\int_{-\pi}^{\pi}(a_0 + a_1\cos x + \ldots + b_N\sin Nx)^2\,dx = \int_{-\pi}^{\pi} S_N^2(x)\,dx \le \int_{-\pi}^{\pi}(f(x))^2\,dx$$

4.4.14

(a) Find the lengths of the vectors $u = (1, 1/2, 1/4, 1/8, \ldots)$ and $v = (1, 1/3, 1/9, \ldots)$ in Hilbert Space and test the inequality $|u^T v|^2 \le (u^T u)(v^T v)$

$$\|u\|^2 = u^T u = \sum u_i^2 = 1 + 1/4 + 1/16 + \ldots = \frac{1}{1-1/4} = 4/3 \longrightarrow \|u\| = 2/\sqrt{3}$$

$$\|v\|^2 = v^T v = \sum v_i^2 = 1 + 1/9 + 1/81 + \ldots = \frac{1}{1-1/9} = 9/8 \longrightarrow \|v\| = 3/2\sqrt{2}$$

$$|u^T v|^2 = \left(1 + \tfrac{1}{2}\tfrac{1}{3} + \tfrac{1}{4}\tfrac{1}{9} + \ldots\right)^2 = \left(1 + 1/6 + 1/36 + 1/216 + \ldots\right)^2 = \left(\frac{1}{1-1/6}\right)^2 = 36/25$$

$$(u^T u)(v^T v) = \|u\|^2 \|v\|^2 = 4/3 \cdot 9/8 = 36/24 > 36/25 = |u^T v|^2.$$

(b) For the functions $f = 1 + \frac{1}{2}e^{ix} + \frac{1}{4}e^{2ix} + \ldots$, $g = 1 + \frac{1}{3}e^{ix} + \frac{1}{9}e^{2ix} + \ldots$ use part a to find the numerical value of each term in:

$$\left|\int_{-\pi}^{\pi} \bar{f}(x) g(x)\, dx\right|^2 \le \int_{-\pi}^{\pi} |f(x)|^2 dx \int_{-\pi}^{\pi} |g(x)|^2 dx$$

By Parseval's Formula,

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = 2\pi \sum_0^\infty |c_k|^2 = 2\pi\left(1^2 + (\tfrac{1}{2})^2 + (\tfrac{1}{4})^2 + \ldots\right) = 2\pi \|u\|^2 = 8\pi/3$$

$$\int_{-\pi}^{\pi} |g(x)|^2 dx = 2\pi \|v\|^2 = 9\pi/4$$

$$\left|\int_{-\pi}^{\pi} \bar{f}(x) g(x) dx\right|^2 = \left|\int_{-\pi}^{\pi} \left(1 + \tfrac{1}{2}e^{-ix} + \tfrac{1}{4}e^{-2ix} + \ldots\right)\left(1 + \tfrac{1}{3}e^{ix} + \tfrac{1}{9}e^{2ix} + \ldots\right) dx\right|^2$$

$$= \left|\int_{-\pi}^{\pi} \left(1 + \tfrac{1}{2}\tfrac{1}{3}e^0 + \tfrac{1}{4}\tfrac{1}{9}e^0 + \ldots\right) dx\right|^2 \quad \text{(by orthogonality)}$$

$$= \left|\int_{-\pi}^{\pi} \left(1 + 1/6 + 1/36 + \ldots\right) dx\right|^2 = 2\pi |u^T v| = 72\pi/25$$

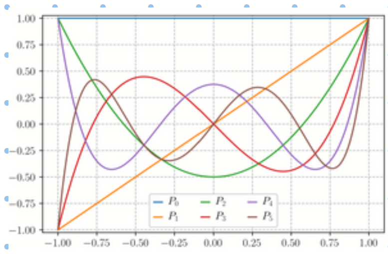
Orthogonal Functions

(1) Legendre Polynomials are a system of complete and orthogonal polynomials. They are orthogonal wrt weight function $w(x) = 1$ on the interval $[-1, 1]$:

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0, \quad m \neq n$$

The Legendre polynomials can be calculated explicitly using

$$\boxed{\text{Rodrigues' Formula: } P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n}$$



$P_0(x) = 1$

$P_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1) = x$

$P_2(x) = \frac{1}{4\cdot 2}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\frac{d}{dx}\left[2(x^2-1)2x\right] = \frac{1}{2}\frac{d}{dx}\left[x(x^2-1)\right] = \frac{1}{2}(3x^2-1)$

$P_3(x) = \frac{1}{8\cdot 6}\frac{d^3}{dx^3}(x^2-1)^3 = \frac{1}{48}\frac{d^3}{dx^3}\left[x^6 - 3x^4 + 3x^2 - 1\right] = \frac{1}{48}(120x^3 - 72x) = \frac{1}{2}(5x^3 - 3x)$

$\vdots$

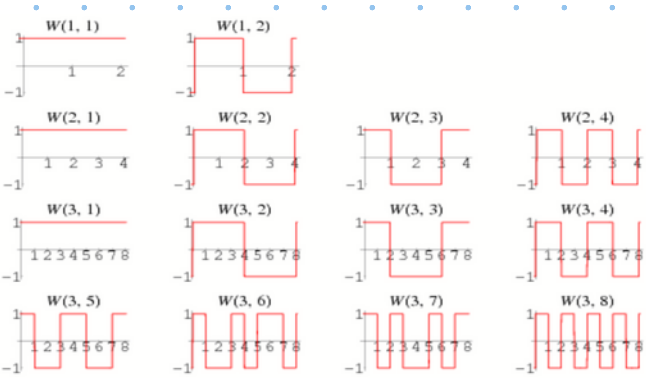
Ex) $\int_{-1}^{1} P_1(x) P_3(x)\, dx = \int_{-1}^{1}\left(\tfrac{5}{2}x^4 - \tfrac{3}{2}x^2\right) dx = 2\int_0^1 (5x^4 - 3x^2)\, dx$

$= 2(x^5 - x^3)\Big|_0^1 = 0$

$\int_{-1}^{1} P_2(x) P_2(x)\, dx = \frac{1}{4}\int_{-1}^{1}(3x^2-1)^2\, dx = \frac{1}{4}\int_{-1}^{1}(9x^4 - 6x^2 + 1)\, dx$

$= \frac{1}{2}\left(9/5 - 6/3 + 1\right) = \frac{1}{2}\left(9/5 - 5/5\right) = 2/5$

(2) The Walsh functions form a complete orthogonal set of functions that can be used to represent any discrete function. Similar to how the trigonometric functions can be used to represent any continuous function in Fourier Analysis. The Walsh functions look like digital sines and cosines or vectors of 1's and -1's:



(3) Double Fourier Series

For an odd function periodic in both x and y there's the double Fourier sine series:

$$f(x,y) = b_{11}\sin x \sin y + b_{21}\sin 2x \sin y + b_{12}\sin x \sin 2y + \dots$$

If we know $f(x,y)$ on the period square $-\pi \le x, y \le \pi$ we know $f(x,y)$ everywhere. The functions $\sin kx \sin ly$ are orthogonal over the unit square. Similarly, for an even function periodic in x, y there is a double Fourier Cosine Series:

$$f(x,y) = a_{11}\cos x \cos y + a_{21}\cos 2x \cos y + a_{12}\cos x \cos 2y + \dots$$

For functions neither even nor odd a mix of sines and cosines could be used, or a complex exponential double Fourier series:

$$f(x,y) = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} c_{kl}\, e^{ikx} e^{ily}$$

(4) We could rewrite each of the cosines as a polynomial in $\cos\theta$:

$$\cos 2\theta = 2\cos^2\theta - 1, \quad \cos 3\theta = 4\cos^3\theta - 3\cos\theta, \dots$$

Changing variables by $x = \cos\theta$ gives the <u>Chebyshev polynomials</u>:

$T_0 = 1$
$T_1 = x$
$T_2 = 2x^2 - 1$
$T_3 = 4x^3 - 3x$
$\vdots$

They are orthogonal because the cosines are orthogonal

$$0 = \int_{-\pi}^{\pi} \cos k\theta \cos l\theta \, d\theta$$

there exists some polynomial in $\cos\theta$, $T_k(\cos\theta)$, s.t. $T_k(\cos\theta) = \cos k\theta$. Same for l.

$$= \int_{-\pi}^{\pi} T_k(\cos\theta) T_l(\cos\theta)\, d\theta$$

$$= \int_{-1}^{1} T_k(x) T_l(x) \frac{1}{\sqrt{1-x^2}}\, dx$$

Substitute $x = \cos\theta$
$dx = \sin\theta \, d\theta = \sqrt{1-x^2}\, d\theta$

However, these polynomials are orthogonal weight function $w(x) = 1/\sqrt{1-x^2}$.

There are many other examples of orthogonal functions. In one way they all work similarly. Suppose a function $f(x)$ is to be expanded into a sum of orthogonal functions: $f(x) = c_0 T_0 + c_1 T_1 + c_2 T_2 + \dots$ To find a coefficient c_k, multiply by $T_k(x)$ and the weight function. Integrate over the set Θ where the functions are orthogonal. Only one term survives on the right.

$$\int_\Theta f T_k \, dx = \int_\Theta c_k T_k^2 \, dx \longrightarrow \boxed{c_k = \frac{\int_\Theta f T_k \, dx}{\int_\Theta T_k^2 \, dx}}$$

Bessel Functions

Physically the best example is a circular drum. When you strike it, its motion contains a mixture of oscillations. The problem is governed by Laplace's equation in polar coordinates:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\lambda u \ , \ u(1,\theta) = 0$$

The boundary condition $u=0$ at $r=1$ is imposed to say the drum is fastened down along the circumference of the drumhead. The drum has a radius of 1 unit. Look for eigenfunctions of the form $u = A(\theta)B(r)$.

$$AB'' + \frac{1}{r}AB' + \frac{1}{r^2}A''B = -\lambda AB$$

$$r^2AB'' + rAB' + A''B = -r^2\lambda AB$$

Since the left side is a function of r only and the right side a function of θ only, both are constant, say n^2. Then

$$r^2B''/B + rB'/B + A''/A = -r^2\lambda$$

$$(r^2B'' + rB' + \lambda r^2 B)/B = -A''/A$$

$$-A''/A = n^2$$
$$A'' = -n^2 A \longrightarrow A(\theta) = e^{in\theta} \text{ or } A(\theta) = e^{-in\theta}$$

Note that each of $A(\theta) = e^{in\theta}$ and $A(\theta) = e^{-in\theta}$ are periodic. If we had tried $-n^2$ instead the result would be $A(\theta) = e^{\pm n\theta}$, not periodic. Based on the physical problem, $A(\theta)$ must be periodic. Also $A(0) = A(2\pi)$ means $n \in \mathbb{Z}$. Considering $B(r)$:

$$\boxed{r^2B'' + rB' + \lambda r^2 B = n^2 B}$$

This is one form of <u>Bessel's equation</u>. We are interested in solutions B (for each n) that are zero on the boundary $r=1$. The eigenfunctions $u = AB$ of the Laplacian will be $B\cos n\theta$ and $B\sin n\theta$. The direct approach is to assume an infinite series solution $B = \sum c_m r^m$ and determine the c_m. In the easiest case, $n=0$:

$$\left.\begin{array}{l} A'' = 0 \\ A(\theta) = c_1\theta + c_2 \end{array}\right\}$$
Periodic B.C.'s mean $c_1 = c_2 = 0$. $A(\theta) \equiv 0$ means the eigenfunction $u = u(r)$ is radially symmetric.

Substitute $B = \sum c_m r^m$:

$$\sum_{m=0}^{\infty} c_m m(m-1) r^m + \sum_{m=0}^{\infty} c_m m r^m + \lambda \sum_{m=0}^{\infty} c_m r^{m+2} = 0 \nearrow \begin{array}{c} c_m m(m-1) + c_m m + \lambda c_{m-2} = 0 \\ \lambda c_{m-2} = -m^2 c_m \end{array}$$

$c_0 := 1 \longrightarrow B(r) = 1 - \lambda r^2/2^2 + \lambda^2 r^4/2^2 4^2 - \lambda^3 r^6/2^2 4^2 6^2 + \ldots$ is the Bessel function of order 0, $J_0(\sqrt{\lambda}\, r)$. This is an eigenfunction. The eigenvalues λ come from $B=0$ at $r=1$: $J_0(\sqrt{\lambda}) = 0 \longrightarrow \sqrt{\lambda} \approx 2.4, 5.5, 8.65, 11.8, 14.9, \ldots$

The Bessel functions $B_k = J_0(\sqrt{\lambda_k}\, r)$ are orthogonal over the unit circle,

$$\int_0^{2\pi}\int_0^1 B_k(r) B_l(r)\, r\, dr\, d\theta = 0 \ , \ k \neq l.$$

4.1.30 Show that two eigenfunctions u_1 and u_2 of a Sturm-Liouville problem $(pu')' + qu + \lambda w u = 0$ are orthogonal with weight w with boundary conditions $u(0) = u'(1) = 0$.

Suppose u_1, u_2 are eigenfunctions with eigenvalues λ_1, λ_2 respectively, with $\lambda_1 \neq \lambda_2$.

$$\begin{array}{l} (pu_1')' u_2 + q u_1 u_2 + \lambda_1 w u_1 u_2 = 0 \\ (pu_2')' u_1 + q u_1 u_2 + \lambda_2 w u_1 u_2 = 0 \\ \hline (pu_1')' u_2 - (pu_2')' u_1 + (\lambda_1 - \lambda_2) u_1 u_2 w = 0 \end{array}$$

$$\int_0^1 \{ (pu_1')' u_2 - (pu_2')' u_1 \} dx + (\lambda_1 - \lambda_2) \int_0^1 u_1 u_2 w \, dx = 0$$

$$(pu_1' u_2 - pu_2' u_1)\Big|_0^1 - \int_0^1 \{ pu_1' u_2' - pu_1' u_2' \} dx + (\lambda_1 - \lambda_2) \int_0^1 u_1 u_2 w \, dx = 0$$

$$0 - 0 + (\lambda_1 - \lambda_2) \int_0^1 u_1 u_2 w \, dx = 0, \quad \lambda_1 - \lambda_2 \neq 0$$

$\therefore \int_0^1 u_1 u_2 w \, dx = 0$ $\quad u_1$ and u_2 orthogonal wrt $w(x)$ on $[0,1]$

4.1.31 Fit the Bessel equation (40) into the framework of a Sturm-Liouville problem $(pu')' + qu + \lambda w u = 0$. What are $p, q,$ and w?

$$r^2 B'' + rB' + \lambda r^2 B = n^2 B \longrightarrow rB'' + B' + (\lambda r - n^2/r) B = 0 \quad (40)$$

$$0 = p'u' + pu'' + qu + \lambda w u = pu'' + p'u' + (\lambda w + q) u$$

$$\boxed{p = r, \; q = -n^2/r, \; w = r}$$

4.1.36 Explain why the third Bessel eigenfunction $B = J_0(\sqrt{\lambda_3}\, r)$ is zero at $r = (\lambda_1/\lambda_3)^{1/2}$, $r = (\lambda_2/\lambda_3)^{1/2}$, and $r = 1$.

The eigenvalues λ satisfy $J_0(\sqrt{\lambda}) = 0$. Order these as $0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots$
That is, $0 = J_0(\sqrt{\lambda_1}) = J_0(\sqrt{\lambda_2}) = J_0(\sqrt{\lambda_3}) = \dots$

$$r = \sqrt{\lambda_1/\lambda_3}: \quad J_0(\sqrt{\lambda_3} \cdot \sqrt{\lambda_1/\lambda_3}) = J_0(\sqrt{\lambda_1}) = 0$$

$$r = \sqrt{\lambda_2/\lambda_3}: \quad J_0(\sqrt{\lambda_3} \cdot \sqrt{\lambda_2/\lambda_3}) = J_0(\sqrt{\lambda_2}) = 0$$

$$r = 1: \quad J_0(\sqrt{\lambda_3} \cdot 1) = J_0(\sqrt{\lambda_3}) = 0$$