§ 6.5 Difference Methods for Initial-Value Problems

6.5.1 For $u' = -2u$ what is the largest Δt for which Euler's method is stable? What are the discrete solutions for $\Delta t = 1/2$ and $\Delta t = 1$?

When applied to $u' = au$, Euler's method approximates $u((n+1)\Delta t)$, $n \geq 0$, by $u_{n+1} = u_n + a\Delta t\, u_n$ with $u_0 = u(0)$ and step size $\Delta t > 0$.

Euler's method is stable for $|1 + a\Delta t| \leq 1$ (pg 652).

$$-1 \leq 1 - 2\Delta t \leq 1$$
$$-2 \leq -2\Delta t \leq 0$$
$$1 \geq \Delta t \geq 0$$

The largest Δt for which Euler's method is stable is $\boxed{\Delta t = 1}$.

$\Delta t = 1$:
$$u_1 = u_0 - 2u_0 = -u_0$$
$$u_2 = u_1 - 2u_1 = u_0$$
$$u_3 = u_2 - 2u_2 = -u_0$$
$$\vdots$$
$$u_n = (-1)^n u_0$$

If $u_0 = 1$, $u_n = (-1)^n$

$\Delta t = 1/2$:
$$u_1 = u_0 - u_0 = 0$$
$$u_2 = u_1 - u_1 = 0 - 0 = 0$$
$$\vdots$$
$$u_n = 0$$

For any u_0, $u_n = 0 \ \forall n > 0$

6.5.2 For $u' = -2u$ solve the backward Euler equation from $u_0 = 1$ with $\Delta t = 1/2$ and $\Delta t = 1$. At $t = 5$ which is closer to the solution $e^{-2t} = e^{-10}$?

Backward Euler: $u' = au$ is approximated by $\dfrac{u_{n+1} - u_n}{\Delta t} = a u_{n+1}$, $n \geq 0$

$$u_{n+1} = \frac{1}{1 - a\Delta t} u_n = G u_n, \quad u_0 = u(0)$$

$\Delta t = 1/2$:
$$u_1 = \frac{1}{1-(-2)\frac{1}{2}} u_0 = \tfrac{1}{2} u_0 = \tfrac{1}{2}$$
$$u_2 = \tfrac{1}{2} u_1 = \tfrac{1}{4}$$
$$\vdots$$
$$u_n = \left(\tfrac{1}{2}\right)^n$$

$\Delta t = 1$:
$$u_1 = \frac{1}{1-(-2)\cdot 1} u_0 = \tfrac{1}{3} u_0 = \tfrac{1}{3}$$
$$u_2 = \tfrac{1}{3} u_1 = \tfrac{1}{9}$$
$$\vdots$$
$$u_n = \left(\tfrac{1}{3}\right)^n$$

$u(5) = u(10\Delta t) \approx u_{10} = \left(\tfrac{1}{2}\right)^{10} =$

$u(5) = u(5\Delta t) = \left(\tfrac{1}{3}\right)^5 =$

At $t = 5$ the $\Delta t = 1/2$ approximation is closer since $|1/2^{10} - e^{-10}| < |1/3^5 - e^{-10}|$.

6.5.3 For $u' = -100u$ and $\Delta t = 1$, find the growth factors G for backward Euler and the trapezoidal rule. Which solution oscillates with slow decay?

Backward Euler: $G = (1 - a\Delta t)^{-1} = (1-(-100)\cdot 1)^{-1} = 1/101$

Trapezoidal Rule: $G = (1 + \frac{1}{2}a\Delta t)(1 - \frac{1}{2}a\Delta t)^{-1} = (1-50)(1+50)^{-1} = -49/51$

Since G is negative for the trapezoidal rule the iterations oscillate since they alternate in sign. Since $|G|<1$, the size of the iterates decays but this decay is slow since $|G|$ is still close to 1.

6.5.6 Find the growth factors G_1 and G_2 for the leapfrog method $u_{n+1} - u_{n-1} = 2a\Delta t\, u_n$ by solving $G^2 - 1 = 2a\Delta t G$. Show that one of the factors is below -1 if a is negative.

The growth factors are determined so that $u_k = G^k u_0$ satisfies the difference eqn.

$G^{n+1}u_0 - G^{n-1}u_0 = 2a\Delta t G^n u_0$

$G^{n+1} - G^{n-1} = 2a\Delta t G^n$

$G^2 - 1 = 2a\Delta t G$

$G^2 - 2a\Delta t G - 1 = 0$

$G = \dfrac{2a\Delta t \pm \sqrt{4a^2\Delta t^2 + 4}}{2}$

$G = a\Delta t \pm \sqrt{1 + a^2\Delta t^2}$

If $a<0$, $G = a\Delta t - \sqrt{1+a^2\Delta t^2} < -1$.

$$\begin{aligned}
a &< 0 \\
2a\Delta t &< 0 \\
a^2\Delta t^2 + 2a\Delta t + 1 &< 1 + a^2\Delta t^2 \\
(a\Delta t + 1)^2 &< 1 + a^2\Delta t^2 \\
a\Delta t + 1 &< \sqrt{1+a^2\Delta t^2} \\
a\Delta t - \sqrt{1+a^2\Delta t^2} &< -1
\end{aligned}$$

6.5.7 Choose the constants in $u_{n+1} - u_{n-1} = 2\Delta t(c_0 u_{n+1} + c_1 u_n + c_2 u_{n-1})$ to achieve 3rd order accuracy in approximating the solution $u_n = e^{n\Delta t}$ of $u' = u$.