4.1.1 Find the Fourier Series on $-\pi < x < \pi$ for

(b) $f(x) = |\sin x|$, an even function.

$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos kx \qquad |\sin x| = \sin x$ on $0 \le x \le \pi$

$\boxed{a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx} = \frac{1}{\pi}\int_0^{\pi} \sin x\,dx = -\frac{1}{\pi}\cos x\Big|_0^{\pi} = -\frac{1}{\pi}(-1-1) = 2/\pi$

$\boxed{a_k = \frac{2}{\pi}\int_0^{\pi} f(x)\cos kx}$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\overline{\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta}$$

$= \frac{2}{\pi}\int_0^{\pi} \sin x \cos kx\,dx$

$= \frac{1}{\pi}\int_0^{\pi}\{\sin(1+k)x + \sin(1-k)x\}dx$

$= -\frac{1}{\pi}\left\{\frac{\cos(1+k)x}{1+k} + \frac{\cos(1-k)x}{1-k}\right\}\Big|_0^{\pi}$

$= -\frac{1}{\pi}\left\{\frac{\cos(1+k)\pi}{1+k} - \frac{1}{1+k} + \frac{\cos(1-k)\pi}{1-k} - \frac{1}{1-k}\right\}$

$= -\frac{1}{\pi}\left\{\frac{(-1)^{k+1}-1}{1+k} + \frac{(-1)^{k+1}-1}{1-k}\right\}$

$$= \begin{cases} \frac{2}{\pi}\left\{\frac{1}{1+k} + \frac{1}{1-k}\right\}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$$

$k=1$	$\cos 2\pi = 1$	$\cos 0 = 1$
$k=2$	$\cos 3\pi = -1$	$\cos -\pi = -1$
$k=3$	$\cos 4\pi = 1$	$\cos -2\pi = 1$
$\vdots$	$\vdots$	$\vdots$

$$f(x) = |\sin x| = \frac{2}{\pi} + \sum_{\substack{k=2 \\ k\text{ even}}}^{\infty} \frac{2}{\pi}\left\{\frac{1}{1+k} + \frac{1}{1-k}\right\}\cos kx$$

$$= \frac{2}{\pi} + \frac{4}{\pi}\sum_{\substack{k=2 \\ k\text{ even}}}^{\infty} \frac{1}{1-k^2}\cos kx$$

$$= \boxed{\frac{2}{\pi} + \frac{4}{\pi}\sum_{j=1}^{\infty} \frac{1}{1-4j^2}\cos 2jx}$$

4.1.2 A square wave has $f(x)=-1$ on the left side $-\pi<x<0$ and $f(x)=1$ on the right side $0<x<\pi$.

(a) Why are all the cosine coefficients $a_k=0$?

Since f is an odd function, $a_0=\int_{-\pi}^{\pi} f(x)\,dx=0$.
For $k\geq 0$ $f(x)\cos kx$ is the product of an odd function with an even function so $f(x)\cos kx$ is an odd function $\longrightarrow a_k=\int_{-\pi}^{\pi} f(x)\cos kx\,dx=0$.

(b) Find the sine series $\sum b_k \sin kx$ from equation (6):

$$\boxed{b_k=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx} \qquad (6)$$

$$\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{0} -\sin kx\,dx+\frac{1}{\pi}\int_{0}^{\pi}\sin kx\,dx\\
&= \frac{1}{\pi}\int_{\pi}^{0}\sin -ky\,dy+\frac{1}{\pi}\int_{0}^{\pi}\sin kx\,dx\\
&= \frac{1}{\pi}\int_{0}^{\pi}\sin ky\,dy+\frac{1}{\pi}\int_{0}^{\pi}\sin kx\,dx\\
&= \frac{2}{\pi}\int_{0}^{\pi}\sin kx\,dx=-\frac{2}{\pi k}\cos kx\Big|_0^{\pi}=-\frac{2}{\pi k}\{(-1)^k-1\}=\begin{cases}4/\pi k, & k \text{ odd}\\ 0, & k \text{ even}\end{cases}
\end{aligned}$$

1 $\cos\pi$
2 $\cos 2\pi$

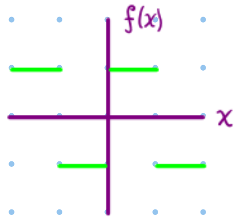
$$\boxed{f(x)=\sum_{\substack{k=1\\ k\text{ odd}}}^{\infty}\frac{4}{\pi k}\sin kx=\sum_{j=1}^{\infty}\frac{4}{\pi(2j-1)}\sin(2j-1)x=\frac{4}{\pi}\left\{\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\dots\right\}}$$

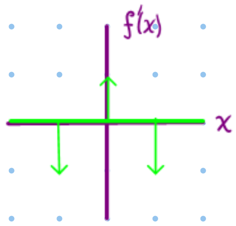
4.1.3 Find the sine series for the square wave in another way by showing

(a) $df/dx = 2\delta(x) - 2\delta(x+\pi)$ extended periodically.

If $f(x)$ is the square wave extended periodically

$$f(x) = \begin{cases} \vdots & \\ 1 & -2\pi < x < -\pi \\ -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \\ -1 & \pi < x < 2\pi \\ \vdots & \end{cases} \qquad \frac{df}{dx} = \begin{cases} \vdots & \\ -\infty & x = -\pi \\ 0 & -\pi < x < 0 \\ \infty & x = 0 \\ 0 & 0 < x < \pi \\ -\infty & x = \pi \\ 0 & \pi < x < 2\pi \\ \vdots & \end{cases}$$





$\frac{df}{dx}$ is the 2π-periodic extension of $g(x) = \begin{cases} -\infty & x = -\pi \\ 0 & -\pi < x < 0 \\ \infty & x = 0 \\ 0 & 0 < x < \pi \end{cases}$

For $x \in [-\pi, \pi)$, $2\delta(x) - 2\delta(x+\pi) = \begin{cases} -\infty & x = -\pi \\ 0 & -\pi < x < 0 \\ \infty & x = 0 \\ 0 & 0 < x < \pi \end{cases} = g(x)$

That is, df/dx is the 2π-periodic extension of $2\delta(x) - 2\delta(x+\pi)$

(b) $2\delta(x) - 2\delta(x+\pi) = \frac{4}{\pi}\{\cos x + \cos 3x + \dots\}$

From page 269, for $\delta(x)$, $a_0 = 1/2\pi$, $a_k = 1/\pi$, $b_k = 0$. This implies:

$$2\delta(x) - 2\delta(x+\pi) = 2\left\{\frac{1}{2\pi} + \sum_{k=1}^{\infty} a_k \cos kx\right\} - 2\left\{\frac{1}{2\pi} + \sum_{k=1}^{\infty} a_k \cos k(x+\pi)\right\}$$

$$= 2\sum_{k=1}^{\infty} \frac{1}{\pi}\{\cos kx - \cos k(x+\pi)\} = \frac{4}{\pi}\sum_{k=1,\, k \text{ odd}}^{\infty} \cos kx = \frac{4}{\pi}\{\cos x + \cos 3x + \dots\}$$

(Since $\cos kx = -\cos k(x+\pi)$, k odd and $\cos kx = \cos k(x+\pi)$, k even)

From parts (a) and (b) conclude that the Fourier series is:

$$f(x) = \int_{-\pi}^{\pi} \{2\delta(x) - 2\delta(x+\pi)\}\,dx = \int \frac{4}{\pi}\sum_{k=1,\, k\text{ odd}}^{\infty} \cos kx\,dx = \frac{4}{\pi}\sum_{k=1,\, k\text{ odd}}^{\infty} \frac{1}{k}\sin kx$$

$$\boxed{f(x) = \frac{4}{\pi}\sum_{j=1}^{\infty} \frac{1}{2j-1}\sin(2j-1)x = \frac{4}{\pi}\left\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right\}}$$