§ 6.5 Difference Methods for Initial-Value Problems

6.5.1 For $u' = -2u$ what is the largest Δt for which Euler's method is stable? What are the discrete solutions for $\Delta t = 1/2$ and $\Delta t = 1$?

When applied to $u' = au$, Euler's method approximates $u((n+1)\Delta t)$, $n \geq 0$, by $u_{n+1} = u_n + a\Delta t\, u_n$ with $u_0 = u(0)$ and step size $\Delta t > 0$.

Euler's method is stable for $|1 + a\Delta t| \leq 1$ (pg 652).

$$-1 \leq 1 - 2\Delta t \leq 1$$
$$-2 \leq -2\Delta t \leq 0$$
$$1 \geq \Delta t \geq 0$$

The largest Δt for which Euler's method is stable is $\boxed{\Delta t = 1}$.

$\Delta t = 1$:
$$u_1 = u_0 - 2u_0 = -u_0$$
$$u_2 = u_1 - 2u_1 = u_0$$
$$u_3 = u_2 - 2u_2 = -u_0$$
$$\vdots$$
$$u_n = (-1)^n u_0$$

If $u_0 = 1$, $u_n = (-1)^n$

$\Delta t = 1/2$:
$$u_1 = u_0 - u_0 = 0$$
$$u_2 = u_1 - u_1 = 0 - 0 = 0$$
$$\vdots$$
$$u_n = 0$$

For any u_0, $u_n = 0 \ \forall n > 0$

6.5.2 For $u' = -2u$ solve the backward Euler equation from $u_0 = 1$ with $\Delta t = 1/2$ and $\Delta t = 1$. At $t = 5$ which is closer to the solution $e^{-2t} = e^{-10}$?

Backward Euler: $u' = au$ is approximated by $\dfrac{u_{n+1} - u_n}{\Delta t} = a u_{n+1}$, $n \geq 0$

$$u_{n+1} = \frac{1}{1 - a\Delta t} u_n = G u_n, \quad u_0 = u(0)$$

$\Delta t = 1/2$:
$$u_1 = \frac{1}{1-(-2)1/2} u_0 = \frac{1}{2} u_0 = \frac{1}{2}$$
$$u_2 = \frac{1}{2} u_1 = \frac{1}{4}$$
$$\vdots$$
$$u_n = \left(\frac{1}{2}\right)^n$$

$\Delta t = 1$:
$$u_1 = \frac{1}{1-(-2)\cdot 1} u_0 = \frac{1}{3} u_0 = \frac{1}{3}$$
$$u_2 = \frac{1}{3} u_1 = \frac{1}{9}$$
$$\vdots$$
$$u_n = \left(\frac{1}{3}\right)^n$$

$$u(5) = u(10\Delta t) \approx u_{10} = \left(\frac{1}{2}\right)^{10} =$$

$$u(5) = u(5\Delta t) = \left(\frac{1}{3}\right)^5 =$$

At $t = 5$ the $\Delta t = 1/2$ approximation is closer since $|1/2^{10} - e^{-10}| < |1/3^5 - e^{-10}|$.

6.5.3 For $u' = -100u$ and $\Delta t = 1$, find the growth factors G for backward Euler and the trapezoidal rule. Which solution oscillates with slow decay?

Backward Euler: $G = (1 - a\Delta t)^{-1} = (1 - (-100)\cdot 1)^{-1} = 1/101$

Trapezoidal Rule: $G = (1 + \frac{1}{2}a\Delta t)(1 - \frac{1}{2}a\Delta t)^{-1} = (1-50)(1+50)^{-1} = -49/51$

Since G is negative for the trapezoidal rule the iterations oscillate since they alternate in sign. Since $|G| < 1$, the size of the iterates decays but this decay is slow since $|G|$ is still close to 1.

6.5.6 Find the growth factors G_1 and G_2 for the leapfrog method $u_{n+1} - u_{n-1} = 2a\Delta t\, u_n$ by solving $G^2 - 1 = 2a\Delta t\, G$. Show that one of the factors is below -1 if a is negative.

The growth factors are determined so that $u_k = G^k u_0$ satisfies the difference eqn.

$$G^{n+1}u_0 - G^{n-1}u_0 = 2a\Delta t\, G^n u_0$$

$$G^{n+1} - G^{n-1} = 2a\Delta t\, G^n$$

$$G^2 - 1 = 2a\Delta t\, G$$

$$G^2 - 2a\Delta t\, G - 1 = 0$$

$$G = \frac{2a\Delta t \pm \sqrt{4a^2\Delta t^2 + 4}}{2}$$

$$G = a\Delta t \pm \sqrt{1 + a^2\Delta t^2}$$

If $a < 0$, $G = a\Delta t - \sqrt{1 + a^2\Delta t^2} < -1$.

$$a < 0$$
$$2a\Delta t < 0$$
$$a^2\Delta t^2 + 2a\Delta t + 1 < 1 + a^2\Delta t^2$$
$$(a\Delta t + 1)^2 < 1 + a^2\Delta t^2$$
$$a\Delta t + 1 < \sqrt{1 + a^2\Delta t^2}$$
$$a\Delta t - \sqrt{1 + a^2\Delta t^2} < -1$$

6.5.7 Choose the constants in $u_{n+1} - u_{n-1} = 2\Delta t(c_0 u_{n+1} + c_1 u_n + c_2 u_{n-1})$ to achieve 3rd order accuracy in approximating the solution $u_n = e^{n\Delta t}$ of $u' = u$.

$$u_{n+1} = u + hu' + \tfrac{1}{2}h^2u'' + \tfrac{1}{6}h^3u''' + \mathcal{O}(h^4)$$
$$u_{n-1} = u - hu' + \tfrac{1}{2}h^2u'' - \tfrac{1}{6}h^3u''' + \mathcal{O}(h^4)$$
$$u_n = u$$

$$\begin{aligned} &u + hu' + \tfrac{1}{2}h^2u'' + \mathcal{O}(h^3) \\ &-u + hu' - \tfrac{1}{2}h^2u'' + \mathcal{O}(h^3) \\ &2hu' + \mathcal{O}(h^3) = 2hu + h^3u'' \end{aligned}$$

$$= 2hu(c_0 + c_1 + c_2) + 2h^2u'(c_0 - c_2) + h^3u''(c_0 + c_2) + \mathcal{O}(h^4)$$

$$\frac{2hu' + \frac{2}{6}h^3u''' + \mathcal{O}(h^5)}{2\Delta t} = \frac{u_{n+1} - u_{n-1}}{2\Delta t} = c_0u_{n+1} + c_1u_n + c_2u_{n-1}$$

$$\frac{hu' + \frac{1}{6}h^3u''' + \mathcal{O}(h^5)}{\Delta t} = u(c_0 + c_1 + c_2) + hu'(c_0 - c_2) + \tfrac{1}{2}h^2u''(c_0 + c_2) + \tfrac{1}{6}h^3u'''(c_0 - c_2) + \mathcal{O}(h^4)$$

Setting $c_0 = c_2 = 1/2$, $c_1 = 0$: $\quad u' + \frac{1}{6}h^2u''' + \mathcal{O}(h^4) = u + \frac{1}{2}h^2u'' + \mathcal{O}(h^4)$

assuming $h = \Delta t$? $\quad u = u' - \frac{1}{2}h^2u'' + \frac{1}{6}h^2u''' + \mathcal{O}(h^4)$