4.1.1 Find the Fourier Series on $-\pi < x < \pi$ for

(b) $f(x) = |\sin x|$, an even function.

$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos kx$ $\qquad |\sin x| = \sin x$ on $0 \le x \le \pi$

$$\boxed{a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx} = \frac{1}{\pi}\int_0^{\pi} \sin x\,dx = -\frac{1}{\pi}\cos x\Big|_0^{\pi} = -\frac{1}{\pi}(-1-1) = 2/\pi$$

$$\boxed{a_k = \frac{2}{\pi}\int_0^{\pi} f(x)\cos kx}$$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x \cos kx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \{\sin(1+k)x + \sin(1-k)x\}\,dx$$

$$= -\frac{1}{\pi}\left\{\frac{\cos(1+k)x}{1+k} + \frac{\cos(1-k)x}{1-k}\right\}\Big|_0^{\pi}$$

$$= -\frac{1}{\pi}\left\{\frac{\cos(1+k)\pi}{1+k} - \frac{1}{1+k} + \frac{\cos(1-k)\pi}{1-k} - \frac{1}{1-k}\right\}$$

$$= -\frac{1}{\pi}\left\{\frac{(-1)^{k+1}-1}{1+k} + \frac{(-1)^{k+1}-1}{1-k}\right\}$$

$k=1$	$\cos 2\pi = 1$	$\cos 0 = 1$
$k=2$	$\cos 3\pi = -1$	$\cos -\pi = -1$
$k=3$	$\cos 4\pi = 1$	$\cos -2\pi = 1$
$\vdots$	$\vdots$	$\vdots$

$$= \begin{cases} \frac{2}{\pi}\left\{\frac{1}{1+k} + \frac{1}{1-k}\right\}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$$

$$f(x) = |\sin x| = \frac{2}{\pi} + \sum_{\substack{k=2 \\ k\,\text{even}}}^{\infty} \frac{2}{\pi}\left\{\frac{1}{1+k} + \frac{1}{1-k}\right\}\cos kx$$

$$= \frac{2}{\pi} + \frac{4}{\pi}\sum_{\substack{k=2 \\ k\,\text{even}}}^{\infty} \frac{1}{1-k^2}\cos kx$$

$$= \boxed{\frac{2}{\pi} + \frac{4}{\pi}\sum_{j=1}^{\infty} \frac{1}{1-4j^2}\cos 2jx}$$

4.1.2 A square wave has $f(x) = -1$ on the left side $-\pi < x < 0$ and $f(x) = 1$ on the right side $0 < x < \pi$.

(a) Why are all the cosine coefficients $a_k = 0$?

Since f is an odd function, $a_0 = \int_{-\pi}^{\pi} f(x)\,dx = 0$.
For $k \geq 0$ $f(x)\cos kx$ is the product of an odd function with an even function so $f(x)\cos kx$ is an odd function $\longrightarrow a_k = \int_{-\pi}^{\pi} f(x)\cos kx\,dx = 0$.

(b) Find the sine series $\sum b_k \sin kx$ from equation (6):

$$\boxed{b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx} \qquad (6)$$

$$\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{0} -\sin kx\,dx + \frac{1}{\pi}\int_{0}^{\pi} \sin kx\,dx \\
&= \frac{1}{\pi}\int_{\pi}^{0} \sin -ky\,dy + \frac{1}{\pi}\int_{0}^{\pi} \sin kx\,dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} \sin ky\,dy + \frac{1}{\pi}\int_{0}^{\pi} \sin kx\,dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} \sin kx\,dx = -\frac{2}{\pi k}\cos kx\Big|_0^{\pi} = -\frac{2}{\pi k}\left\{(-1)^k - 1\right\} = \begin{cases} 4/\pi k, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}
\end{aligned}$$

1 $\cos\pi$

2 $\cos 2\pi$

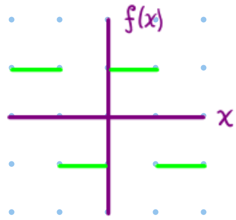
$$\boxed{f(x) = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi k}\sin kx = \sum_{j=1}^{\infty} \frac{4}{\pi(2j-1)}\sin(2j-1)x = \frac{4}{\pi}\left\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right\}}$$

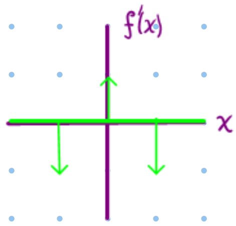
4.1.3 Find the sine series for the square wave in another way by showing

(a) $df/dx = 2\delta(x) - 2\delta(x+\pi)$ extended periodically.

If $f(x)$ is the square wave extended periodically

$$f(x) = \begin{cases} \vdots & \\ 1 & -2\pi < x < -\pi \\ -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \\ -1 & \pi < x < 2\pi \\ \vdots & \end{cases} \qquad \frac{df}{dx} = \begin{cases} \vdots & \\ -\infty & x = -\pi \\ 0 & -\pi < x < 0 \\ \infty & x = 0 \\ 0 & 0 < x < \pi \\ -\infty & x = \pi \\ 0 & \pi < x < 2\pi \\ \vdots & \end{cases}$$





$\frac{df}{dx}$ is the 2π-periodic extension of $g(x) = \begin{cases} -\infty & x = -\pi \\ 0 & -\pi < x < 0 \\ \infty & x = 0 \\ 0 & 0 < x < \pi \end{cases}$

For $x \in [-\pi, \pi)$, $2\delta(x) - 2\delta(x+\pi) = \begin{cases} -\infty & x = -\pi \\ 0 & -\pi < x < 0 \\ \infty & x = 0 \\ 0 & 0 < x < \pi \end{cases} = g(x)$

That is, df/dx is the 2π-periodic extension of $2\delta(x) - 2\delta(x+\pi)$

(b) $2\delta(x) - 2\delta(x+\pi) = \frac{4}{\pi}\{\cos x + \cos 3x + \dots\}$

From page 269, for $\delta(x)$, $a_0 = 1/2\pi$, $a_k = 1/\pi$, $b_k = 0$. This implies:

$$2\delta(x) - 2\delta(x+\pi) = 2\left\{\frac{1}{2\pi} + \sum_{k=1}^{\infty} a_k \cos kx\right\} - 2\left\{\frac{1}{2\pi} + \sum_{k=1}^{\infty} a_k \cos k(x+\pi)\right\}$$

$$= 2\sum_{k=1}^{\infty} \frac{1}{\pi}\{\cos kx - \cos k(x+\pi)\} = \frac{4}{\pi}\sum_{k=1,\, k \text{ odd}}^{\infty} \cos kx = \frac{4}{\pi}\{\cos x + \cos 3x + \dots\}$$

(Since $\cos kx = -\cos k(x+\pi)$, k odd and $\cos kx = \cos k(x+\pi)$, k even)

From parts (a) and (b) conclude that the Fourier series is:

$$f(x) = \int_{-\pi}^{\pi} \{2\delta(x) - 2\delta(x+\pi)\}\, dx = \int \frac{4}{\pi}\sum_{k=1,\, k \text{ odd}}^{\infty} \cos kx \, dx = \frac{4}{\pi}\sum_{k=1,\, k \text{ odd}}^{\infty} \frac{1}{k}\sin kx$$

$$\boxed{f(x) = \frac{4}{\pi}\sum_{j=1}^{\infty} \frac{1}{2j-1}\sin(2j-1)x = \frac{4}{\pi}\left\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right\}}$$

Laplace's Equation in Cartesian coordinates is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy} = 0$$

Let $u = u(r,\theta)$ with $x = r\cos\theta$, $y = r\sin\theta$

$$r = (x^2+y^2)^{1/2}, \quad \theta = \arctan y/x + c$$

depends on the quadrant

$$u_x = u_r r_x + u_\theta \theta_x \qquad r_x = \tfrac{1}{2}(x^2+y^2)^{-1/2} 2x = x/(x^2+y^2)^{1/2} = x/r = \cos\theta$$

$$= u_r\cos\theta - \tfrac{1}{r} u_\theta \sin\theta \qquad \theta_x = \frac{-y/x^2}{1+(y/x)^2} = -\frac{y}{x^2+y^2} = -y/r^2 = -\sin\theta/r$$

$$u_{xx} = u_r(\cos\theta)_x + u_{rx}\cos\theta - u_\theta\left(\frac{\sin\theta}{r}\right)_x - u_{\theta x}\frac{\sin\theta}{r}$$

$$= u_r(-\sin\theta)(-\sin\theta/r) + u_{rx}\cos\theta - u_\theta\frac{(\cos\theta)(-\sin\theta/r)r - \sin\theta\cos\theta}{r^2}$$

$$- u_{\theta x}\frac{\sin\theta}{r}$$

$$= \frac{\sin^2\theta}{r}u_r + u_{rx}\cos\theta + u_\theta\frac{2\cos\theta\sin\theta}{r^2} - u_{\theta x}\frac{\sin\theta}{r}$$

$$= \frac{\sin^2\theta}{r}u_r + u_{rr}\cos^2\theta - u_{r\theta}\frac{\sin\theta\cos\theta}{r}$$

$$+ u_\theta\frac{2\cos\theta\sin\theta}{r^2} - \left(u_{\theta\theta}\left(\frac{-\sin\theta}{r}\right) + u_{\theta r}\cos\theta\right)\left(\frac{\sin\theta}{r}\right)$$

$$\left\{\begin{aligned} u_{rx} &= u_{rr}r_x + u_{r\theta}\theta_x \\ &= u_{rr}\cos\theta - u_{r\theta}\frac{\sin\theta}{r} \\ u_{\theta x} &= u_{\theta\theta}\theta_x + u_{\theta r}r_x \\ &= u_{\theta\theta}\left(\frac{-\sin\theta}{r}\right) + u_{\theta r}\cos\theta \end{aligned}\right\}$$

$$\boxed{\begin{aligned} u_{xx} &= \cos^2\theta\, u_{rr} + \frac{\sin^2\theta}{r}u_r - \frac{2\sin\theta\cos\theta}{r}u_{r\theta} \\ &\quad + \frac{2\sin\theta\cos\theta}{r^2}u_\theta + \frac{\sin^2\theta}{r^2}u_{\theta\theta} \end{aligned}}$$

$$u_y = u_r r_y + u_\theta \theta_y \qquad r_y = y/r = \sin\theta$$

$$= u_r \sin\theta + \tfrac{1}{r} u_\theta \cos\theta \qquad \theta_y = \frac{1/x}{1 + y^2/x^2} = \frac{x}{x^2+y^2} = x/r^2 = \cos\theta/r$$

$$u_{yy} = u_{ry}\sin\theta + u_r(\sin\theta)_y + u_{\theta y}\frac{\cos\theta}{r} + u_\theta\left(\frac{\cos\theta}{r}\right)_y$$

$$= \left(u_{rr}\sin\theta + u_{r\theta}\frac{\cos\theta}{r}\right)\sin\theta$$

$$+ u_r \cos^2\theta/r + u_{\theta\theta}\cos^2\theta/r^2$$

$$+ u_{\theta r}\frac{\sin\theta\cos\theta}{r}$$

$$+ u_\theta\left(-\frac{2\sin\theta\cos\theta}{r^2}\right)$$

$$\left\{\begin{aligned} u_{ry} &= u_{rr} r_y + u_{r\theta}\theta_y \\ &= u_{rr}\sin\theta + u_{r\theta}\frac{\cos\theta}{r} \\ u_{\theta y} &= u_{\theta\theta}\theta_y + u_{\theta r} r_y \\ &= u_{\theta\theta}\cos\theta/r + u_{\theta r}\sin\theta \end{aligned}\right\}$$

$$\boxed{\begin{aligned} u_{yy} = {} & \sin^2\theta\, u_{rr} + \frac{\cos^2\theta}{r}u_r + \frac{2\sin\theta\cos\theta}{r}u_{r\theta} \\ & - \frac{2\sin\theta\cos\theta}{r^2}u_\theta + \frac{\cos^2\theta}{r^2}u_{\theta\theta} \end{aligned}}$$

$$0 = u_{xx} + u_{yy}$$

$$= (\cos^2\theta + \sin^2\theta)u_{rr}$$

$$+ \tfrac{1}{r}(\sin^2\theta + \cos^2\theta)u_r + 0\cdot u_{r\theta}$$

$$+ 0\cdot u_\theta + \tfrac{1}{r^2}(\sin^2\theta + \cos^2\theta)u_{\theta\theta}$$

$$= u_{rr} + \tfrac{1}{r}u_r + \tfrac{1}{r^2}u_{\theta\theta} = \tfrac{1}{r}(ru_r)_r + \tfrac{1}{r^2}u_{\theta\theta}$$

$$\left\{\begin{aligned} \left(\frac{\cos\theta}{r}\right)_y &= \frac{-\sin\theta\,\theta_y r}{r^2} - \frac{\cos\theta\, r_y}{r^2} \\ &= -\frac{\sin\theta\cos\theta}{r^2} - \frac{\cos\theta\sin\theta}{r^2} \\ &= -\frac{2\sin\theta\cos\theta}{r^2} \end{aligned}\right\}$$

Laplace's Equation in Polar coordinates is

$$0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}$$

$u = a_0$ (constant) $\quad \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} = \frac{1}{r}0 + \frac{1}{r^2}0 = 0.$

$u(r,\theta) = r^n \sin n\theta$

$$\tfrac{1}{r}(r n r^{n-1}\sin n\theta)_r + \tfrac{1}{r^2}(-n^2 r^n \sin n\theta) = \tfrac{1}{r}n^2 r^{n-2}\sin n\theta - n^2/r^2\, r^n \sin n\theta = 0$$

$u(r,\theta) = r^n \cos n\theta$

$$\tfrac{1}{r}(r n r^{n-1}\cos n\theta)_r + \tfrac{1}{r^2}(-n^2 r^n \cos n\theta) = \tfrac{1}{r}n^2 r^{n-2}\cos n\theta - n^2/r^2\, r^n \cos n\theta = 0$$

$u(r,\theta) = a_0 + a_1 r\cos\theta + b_1 r\sin\theta + a_2 r^2\cos n\theta + b_2 r^2 \sin n\theta + \ldots$
Satisfies Laplace's equation in polar coordinates.

4.1.6 Around the unit circle suppose u is a square wave

$$u_0 = \begin{cases} +1 \text{ on the upper semicircle} & 0 < \theta < \pi \\ -1 \text{ on the lower semicircle} & -\pi < \theta < 0 \end{cases}$$

From the Fourier series for the square wave write down the Fourier series for u (the solution 21) to Laplace's equation). What is the value of u at the origin?

$$u(r,\theta) = a_0 + a_1 r\cos\theta + b_1 r\sin\theta + a_2 r^2\cos 2\theta + b_2 r^2 \sin 2\theta + \dots$$

$$1 = u(1,\theta) = a_0 + a_1\cos\theta + b_1\sin\theta + \dots \quad 0<\theta<\pi$$
$$-1 = u(1,\theta) = a_0 + a_1\cos\theta + b_1\sin\theta + \dots \quad -\pi<\theta<0$$

$$1 = u(1,\pi/2) = a_0 + b_1 + b_2 + b_3 + \dots$$
$$-1 = u(1,-\pi/2) = a_0 - b_1 - b_2 - b_3 - \dots \quad \to \quad 0 = a_0$$

Since $\cos(\theta) = \cos(-\theta)$ and $-\sin(\theta) = \sin(-\theta)$, for $0<\theta<\pi$:

$$1 = u(1,\theta) = a_1\cos\theta + b_1\sin\theta + a_2\cos 2\theta + b_2\sin 2\theta + \dots$$
$$-1 = u(1,-\theta) = a_1\cos\theta - b_1\sin\theta + a_2\cos 2\theta - b_2\sin 2\theta + \dots$$

$$0 = a_1\cos\theta + a_2\cos 2\theta + a_3\cos 3\theta + \dots$$

Since $\{\cos k\theta\}_{k=1}^{\infty}$ are orthogonal wrt to the $L^2[-\pi,\pi]$ inner product, $a_i = 0$ for all i.

$$u(r,\theta) = b_1 r\sin\theta + b_2 r^2 \sin 2\theta + \dots$$

$$1 = u(1,\theta) = b_1\sin\theta + b_2\sin 2\theta + b_3\sin 3\theta + b_4\sin 4\theta + \dots \text{ for } 0<\theta<\pi$$

From Exercises 4.1.2, 4.1.3,

$$1 = \frac{4}{\pi}\left\{\sin\theta + \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta + \dots\right\} \text{ for } 0<\theta<\pi$$

This implies $b_1 = 1$, $b_2 = 0$, $b_3 = 1/3$, $b_4 = 0$, $b_5 = 1/5$, ...

$$\therefore u(r,\theta) = \frac{4}{\pi}\left\{r\sin\theta + \frac{1}{3}r^3\sin 3\theta + \frac{1}{5}r^5\sin 5\theta + \dots\right\}$$

$$\lim_{r\to 0} u(r,\theta) = u(0,\theta) = 0.$$