1.3 Positive Definite Matrices and $A = LDL^T$

Exercises

<u>1.3.1</u> Write A in the forms $A = LDL^T$ and $A = l_1 d_1 l_1^T + l_2 d_2 l_2^T$. Are the pivots positive, so that A is symmetric positive definite?

Write $3x_1^2 - 6x_1x_2 + 5x_2^2$ as a sum of squares.

$$A = \begin{bmatrix} 3 & -3 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 3/3 \\ -3/3 \end{bmatrix} \cdot 3 \cdot \begin{bmatrix} 3/3 & -3/3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot 3 \cdot \begin{bmatrix} 1 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot 3 \cdot \begin{bmatrix} 1 & -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot 2 \cdot \begin{bmatrix} 0 & 1 \end{bmatrix} = l_1 d_1 l_1^T + l_2 d_2 l_2^T$$

$$A = LDL^T = \begin{bmatrix} l_1 & l_2 \end{bmatrix} \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \begin{bmatrix} l_1^T \\ l_2^T \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

The pivots $d_1 = 3$ and $d_2 = 2$ are positive $\longrightarrow$ A is positive definite.

$$x^T A x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 - 3x_2 & -3x_1 + 5x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 3x_1^2 - 6x_1x_2 + 5x_2^2 = 3(x_1^2 - 2x_1x_2 + x_2^2) + 2x_2^2$$

$$= 3(x_1 - x_2)^2 + 2x_2^2 \geq 0 \quad \text{with } x^TAx = 0 \text{ iff } x = 0.$$

Notice that the form $A = l_1 d_1 l_1^T + l_2 d_2 l_2^T$ already reveals the form of the completed square:

$$\left.\begin{aligned} l_1 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ d_1 = 3 \rightarrow 3(1x_1 - 1x_2)^2 = 3(x_1 - x_2)^2 \\ l_2 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ d_2 = 2 \rightarrow 2(0x_1 + 1x_2)^2 = 2x_2^2 \end{aligned}\right\} \rightarrow x^TAx = 3(x_1 - x_2)^2 + 2x_2^2$$

<u>1.3.2</u> Factor A into $A = LDL^T$. Is A positive definite? Write x^TAx as a combination of two squares.

$$A = \begin{bmatrix} 3 & 6 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot 3 \begin{bmatrix} 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot 3 \cdot \begin{bmatrix} 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot (-4) \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\left.\begin{aligned} l_1 &= \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ d_1 = 3 \\ l_2 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ d_2 = -4 \end{aligned}\right\} \longrightarrow \begin{aligned} &A = LDL^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \\ &x^TAx = 3(x_1 + 2x_2)^2 - 4x_2^2 \end{aligned}$$

A is <u>not</u> positive definite ($d_2 < 0$ so $x \neq 0$ does not ensure $x^TAx > 0$)

1.3.6 In the 2×2 case suppose $a, c > 0$ dominate b in the sense $a + c > 2b$. Is this enough to guarantee $ac > b^2$ and the matrix is positive definite? Give a proof or counterexample.

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} 1 \\ b/a \end{bmatrix} \cdot a \cdot \begin{bmatrix} 1 & b/a \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (c - b^2/a) \begin{bmatrix} 0 & 1 \end{bmatrix} = l_1 d_1 l_1^T + l_2 d_2 l_2^T$$

$$A = LDL^T = \begin{bmatrix} l_1 & l_2 \end{bmatrix} \begin{bmatrix} d_1 & \\ & d_2 \end{bmatrix} \begin{bmatrix} l_1^T \\ l_2^T \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b/a & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & c - b^2/a \end{bmatrix} \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix}$$

$a + c > 2b$ is not enough to guarantee $ac > b^2$.

Counterexample: $A = \begin{bmatrix} 1 & 3 \\ 3 & 6 \end{bmatrix}$ $\left.\begin{matrix} a = 1 \\ b = 3 \\ c = 6 \end{matrix}\right\} \longrightarrow \begin{matrix} a + c = 7 > 6 = 2b \\ ac = 6 \not> 9 = b^2 \end{matrix}$

1.3.8 If each diagonal entry a_{ii} is larger than the sum of the absolute values $|a_{ij}|$ along the rest of its row (diagonally dominant) then the symmetric matrix is positive definite.

For $A = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix}$, how large would c need to be for this statement to apply?

How large does c actually need to be to assure A is positive definite? Note:

This statement applies if $c > |1| + |1| = 2$. That is, the statement tells us $\underline{c > 2}$ guarantees A is positive definite. However, for this particular matrix,

$$x^T A x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} cx_1 + x_2 + x_3 & x_1 + cx_2 + x_3 & x_1 + x_2 + cx_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= cx_1^2 + x_1x_2 + x_1x_3 + x_1x_2 + cx_2^2 + x_2x_3 + x_1x_3 + x_2x_3 + cx_3^2$$

$$= cx_1^2 + cx_2^2 + cx_3^2 + 2x_1x_2 + 2x_1x_2 + 2x_2x_3$$

$$= cx_1^2 + cx_2^2 + cx_3^2 - x_1^2 - x_2^2 - x_3^2 + x_1^2 + x_2^2 + x_3^2 + 2(x_1x_2 + x_1x_3 + x_2x_3)$$

$$= (c-1)(x_1^2 + x_2^2 + x_3^2) + (x_1 + x_2 + x_3)^2$$

This shows that we need only $\underline{c > 1}$ to guarantee A is positive definite.

1.3.10 Inverting $A = LDL^T$ gives $A^{-1} = MD^{-1}M^T$, where $M = (L^T)^{-1}$. How could you refactor A itself, so that the first factor is upper and not lower triangular?

$I = ML^T$ implies $M = (L^T)^{-1}$ is upper triangular. A proof sketch is:

Write $M = \begin{bmatrix} - m_1^T - \\ \vdots \\ - m_n^T - \end{bmatrix}$, $L^T = \begin{bmatrix} | & & | \\ c_1 & \cdots & c_n \\ | & & | \end{bmatrix}$ Note L^T is upper triangular.

We have $0 = m_2^T c_1$ and c_1 has all zero entries except the first. This implies the first entry of m_2^T must be zero.

Similarly $0 = m_3^T c_1$ implies the first entry of m_3^T must be zero. Since $0 = m_3^T c_2$ and all but the first two entries of c_2 must be zero and we just concluded the first entry of m_3^T is zero, the second entry of m_3^T must also be zero.

Continue in this way to show all entries below the main diagonal of M must be zero. Conclude that M is upper triangular.

We could factor $A = UDU^T$ with U upper triangular using row operations similar to those used to factor $A = LDL^T$. Instead of starting by elimination of all entries below the upper left pivot, start with the bottom row and eliminate all entries above the lower right entry. Then eliminate all entries above the $(n-1),(n-1)$ entry of the result and so on. You arrive at $A = UL$. Divide out the pivots and get (for A symmetric) $A = UDU^T$.

1.3.11 A function $F(x,y)$ has a local minimum at any point where its first derivatives $\partial F/\partial x$ and $\partial F/\partial y$ are both zero and the matrix

$$A = \begin{bmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial y^2} \end{bmatrix}$$ evaluated at this point is positive definite.

Is this true for $F_1 = x^2 - x^2y^2 + y^2 + y^3$ and $F_2 = \cos x \cos y$ at $x = y = 0$? Does F_1 have a global minimum or can F_1 approach $-\infty$?

F_1:

$$\left.\begin{aligned} 0 &= \partial F_1/\partial x = 2x - 2xy^2 = 2x(1-y^2) \\ 0 &= \partial F_1/\partial y = -2x^2y + 2y + 3y^2 = y(3y + 2 - 2x^2) \end{aligned}\right\}$$ This confirms $(x,y) = (0,0)$ is a critical pt. (But not the only one).

$\partial^2 F/\partial x^2 = 2(1-y^2)$

$\partial^2 F/\partial x \partial y = -4xy$ $\qquad A\Big|_{(x,y)=0} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $\qquad$ This matrix is p.d. $\therefore$ <u>(0,0) is a local min</u>.

$\partial^2 F/\partial y^2 = 6y + 2 - 2x^2$

Note $\lim_{y \to -\infty} F_1(0,y) = -\infty$, so F_1 has <u>no global minimum</u>.

F_2:

$$\left.\begin{aligned} 0 &= \partial F_2/\partial x = -\sin x \cos y \\ 0 &= \partial F_2/\partial y = -\cos x \sin y \end{aligned}\right\}$$ This confirms $(x,y) = (0,0)$ is a critical pt. (but there are infinitely many others)

$\partial^2 F_2/\partial x^2 = -\cos x \cos y$

$\partial^2 F_2/\partial x \partial y = \sin x \sin y$ $\qquad A\Big|_{(x,y)=(0,0)} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $\qquad$ This matrix is <u>not</u> positive definite so (0,0) is not a local min.

$\partial^2 F_2/\partial y^2 = -\cos x \cos y$

In this case the matrix is negative definite $\rightarrow$ (0,0) is a local max!

Note that $F_2(x,y) = \cos x \cos y \le 1 \cdot 1 = 1 \;\; \forall (x,y)$ and $F_2(0,0) = 1$. So F_2 attains its global maximum at $(0,0)$, though this is true of any pt $(x,y) = (2n\pi, 2m\pi)$ with $n, m \in \mathbb{Z}$.

<u>1.3.15</u> Find the LDL^T and Cholesky's $\bar{L}\bar{L}^T$ factorization with $\bar{L} = LD^{1/2}$ for

$$A = \begin{bmatrix} 4 & 12 \\ 12 & 45 \end{bmatrix}$$

What is the connection to $x^TAx = (2x_1 + 6x_2)^2 + (3x_2)^2$?

$$A = \begin{bmatrix} 4 & 12 \\ 12 & 45 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot 4 \cdot \begin{bmatrix} 1 & 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot (45 - 144/4) \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\ell_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad d_1 = 4 \quad \ell_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad d_2 = 9$$

$$A = LDL^T = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$A = \bar{L}\bar{L}^T = \begin{bmatrix} 2 & 0 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 0 & 3 \end{bmatrix} \quad \text{(Cholesky)}$$

$$x^TAx = x^T\bar{L}\bar{L}^Tx = (\bar{L}^Tx)^T(\bar{L}^Tx) = \begin{bmatrix} 2x_1 + 6x_2 & 3x_2 \end{bmatrix} \begin{bmatrix} 2x_1 + 6x_2 \\ 3x_2 \end{bmatrix} = (2x_1 + 6x_2)^2 + (3x_2)^2$$

If we denote $\bar{\ell}_1 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, $\bar{\ell}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ from the Cholesky factorization, we have the coefficients of x_1, x_2 in each of the squares of x^TAx.

<u>1.3.20</u> If a new row v^T is added to A, what is the change to A^TA?

$$\text{If } A \mapsto \begin{bmatrix} A \\ v^T \end{bmatrix} \text{ then } A^TA \mapsto \begin{bmatrix} A^T & v \end{bmatrix} \begin{bmatrix} A \\ v^T \end{bmatrix} = A^TA + vv^T$$

So we could say A^TA is 'increased' by vv^T, which is a rank 1 matrix. Assuming A is $m \times n$ to start, $v^T = 1 \times n$ and vv^T is $n \times n$.