§ 4.4 Complex variables and Conformal Mapping

4.4.2 Find the real and imaginary parts of

(d) $i \log i \log \log i$

$z = i \log i \log \log i$ $\qquad$ $\log i = w \to e^w = i = e^{i\pi/2} \to \log i = i\pi/2$

$z = i \cdot i\pi/2 \log i\pi/2$ $\qquad$ $\log i\pi/2 = \log i + \log \pi/2 = i\pi/2 + \log \pi/2$

$$z = i \cdot i\pi/2 (\log \pi/2 + i\pi/2)$$
$$= -\pi/2 (\log \pi/2 + i\pi/2)$$
$$= \boxed{\underbrace{-\pi/2 \log \pi/2}_{\text{real}} \underbrace{- i\pi^2/4}_{\text{imaginary}}}$$

4.4.3 what can you say about

(c) the product of two numbers on the unit circle $z = e^{i\theta}$?

You can say the product is also on the unit circle $\boxed{|z| = 1}$

$v = e^{i\theta}, \; w = e^{i\phi} \to vw = e^{i(\theta+\phi)} \longrightarrow |vw| = 1$

(d) the sum of two numbers on the unit circle?

You can say the sum is on the disk $\boxed{|z| \le 2}$

$v = e^{i\theta}, \; w = e^{i\phi} \longrightarrow |v+w| \le |v| + |w| = 1 + 1 = 2.$

You cannot say the sum is on $|z| = 1$ or $|z| = 2$ generally. Consider the examples $v = i, w = -i$ with $|v+w| = 0$ or $v = 1, w = i$ with $|v+w| = \sqrt{2}$.

4.4.4 Find the absolute value (or modulus) $|z|$ if

(a) $z = e^i$

$e^i = e^{i \cdot 1} = \cos 1 + i \sin 1 \to |e^i|^2 = \cos^2 1 + \sin^2 1 = \boxed{1}$

(c) $z = \dfrac{3+i}{3-i}$

$$\frac{3+i}{3-i} = \frac{(3+i)(3+i)}{(3-i)(3+i)} = \frac{9+6i+i^2}{9-i^2} = \frac{8+6i}{10} = 4/5 + 3/5\, i \to |z| = (16/25 + 9/25)^{1/2} = \boxed{1}$$

(e) $z = e^{3+4i}$

$$e^{3+4i} = e^3 e^{4i} \longrightarrow |z| = |e^3 e^{4i}| = |e^3||e^{4i}| = e^3 \cdot 1 = \boxed{e^3}$$

Analytic Functions and Laplace's Equation

Laplace's Equation in 2-D: $\boxed{u_{xx}+u_{yy}=0 \quad (1)}$

Any 'decent' function $f(z)=f(x+iy)$ will be a solution.

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial z}\frac{\partial z}{\partial x}=\frac{\partial f}{\partial z} \qquad \frac{\partial f}{\partial y}=\frac{\partial f}{\partial z}\frac{\partial z}{\partial y}=i\frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial z} \qquad \frac{\partial f}{\partial y}=i\frac{\partial f}{\partial z}$$

$$\boxed{i\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y} \quad (2)}$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right)=\frac{\partial^2 f}{\partial z^2}\frac{\partial z}{\partial x}=\frac{\partial^2 f}{\partial z^2} \qquad \boxed{\frac{\partial^2 f}{\partial x^2}=\frac{\partial^2 f}{\partial z^2} \quad (3)}$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\frac{\partial z}{\partial y}\right)=i\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right)=i\frac{\partial^2 f}{\partial z^2}\frac{\partial z}{\partial y}=i^2\frac{\partial^2 f}{\partial z^2} \qquad \boxed{\frac{\partial^2 f}{\partial y^2}=-\frac{\partial^2 f}{\partial z^2} \quad (4)}$$

$$\therefore \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=\frac{\partial^2 f}{\partial z^2}-\frac{\partial^2 f}{\partial z^2}=0$$

Let $f(x+iy)=u(x,y)+is(x,y)$ and substitute into (2):

$$i\left(\frac{\partial u}{\partial x}+i\frac{\partial s}{\partial x}\right)=\frac{\partial u}{\partial y}+i\frac{\partial s}{\partial y}$$

$$\boxed{\text{Cauchy-Riemann Equations: } \frac{\partial u}{\partial x}=\frac{\partial s}{\partial y} \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial s}{\partial x} \quad (5),(6)}$$

<u>Definition</u> A function $f(z)$ is analytic at $z=a$ if in a neighborhood of a,

(1) $f(z)$ depends on the combination $z=x+iy$ and satisfies $i\,\partial f/\partial x=\partial f/\partial y$,
(2) the real and imaginary parts of $f(z)$ are connected by the C-R equations $u_x=s_y$ and $u_y=-s_x$,
(3) $f(z)$ is the sum of a convergent power series $f(z)=\sum_{n=0}^{\infty} c_n(z-a)^n$.

If these conditions are satisfied then the real functions u and s satisfy Laplace's equation and $u+is$ is a combination of the powers $(x+iy)^n$.

Ex) $f=(x+iy)^n$, $f=e^{x+iy}$, $f=\frac{1}{1-z}$ $(|z|\neq 1)$ are analytic at all admissible z.
$f=f(x-iy)$ is not analytic.

4.4.7 Are the following functions analytic?

(a) $f = |z|^2 = x^2 + y^2$

(b) $f = \mathrm{Re}\, z = x$

(c) $f = \sin z = \sin x \cosh y + i \cos x \sinh y$

Can a function satisfy Laplace's equation without being analytic?

(a) Not analytic since condition 1 is not satisfied: $i\frac{\partial f}{\partial x} = 2ix \neq 2y = \frac{\partial f}{\partial y}$

(b) Not analytic since condition 1 is not satisfied: $i\frac{\partial f}{\partial x} = i \neq 0 = \frac{\partial f}{\partial y}$

(c) Analytic at any point a:

(1) $f(z) = f(x+iy) = \sin(x+iy)$ depends on $x+iy$ and $i\partial f/\partial x = i\cos z = \partial f/\partial y$.

(2) $\partial u/\partial x = \partial/\partial x(\sin x \cosh y) = \cos x \cosh y = \partial/\partial y(\cos x \sinh y) = \partial s/\partial y$ ✓
$\partial u/\partial y = \partial/\partial y(\sin x \cosh y) = \sin x \sinh y = -\partial/\partial x(\cos x \sinh y) = -\partial s/\partial x$ ✓

(3) $f(z) = \sin z = z - z^3/3! + z^5/5! - \ldots = \sum_{n=0}^{\infty} z^{2n+1}/(2n+1)! \quad \forall z \in \mathbb{C}$

Yes a function can satisfy Laplace's equation w/o being analytic:

$f = \mathrm{Re}\, z = x$ is not analytic yet: $\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 = 0 + 0 = 0$.

4.4.8

(a) If $u(x,y) = x + 4y$, find its conjugate function $s(x,y)$.

(b) If $s(x,y) = (1+x)y$, find its conjugate function $u(x,y)$.

(c) If $u = x^2$, why does no s satisfy the C-R equations?

Answers:

(a) $\partial s/\partial y = \partial u/\partial x = 1 \rightarrow s = y + h_1(x)$
$\partial s/\partial x = -\partial u/\partial y = -4 \rightarrow s = -4x + h_2(y)$
$\Rightarrow$ $-4 = \partial s/\partial x = h_1'(x)$
$-4x = h_1(x) + c$
$\Rightarrow$ $\boxed{s(x,y) = y - 4x}$ $(+c)$

(b) $\partial u/\partial x = \partial s/\partial y = 1 + x \rightarrow u = x + \frac{1}{2}x^2 + g_1(y)$
$\partial u/\partial y = -\partial s/\partial x = y \rightarrow u = \frac{1}{2}y^2 + g_2(x)$
$\Rightarrow$ $y = \partial u/\partial y = g_1'(y)$
$\frac{1}{2}y^2 = g_1(y) + c$
$\Rightarrow$ $\boxed{u(x,y) = x + \frac{1}{2}(x^2 + y^2)}$

(c) $\left.\begin{array}{l}\partial s/\partial y = \partial u/\partial x = 2x \rightarrow s = 2xy + j(y) \\ \partial s/\partial x = -\partial u/\partial y = 0 \rightarrow s = \text{constant}\end{array}\right\}$ These conditions cannot be satisfied simultaneously. Also $u_{xx} + u_{yy} = 2 \neq 0$.

4.4.10 The Cauchy-Riemann equations in polar coordinates, where $z = re^{i\theta}$, must still come from the chain rule:

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial r} = \frac{\partial f}{\partial z}e^{i\theta} \quad \text{and} \quad \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial \theta} = \frac{\partial f}{\partial z}ire^{i\theta}.$$

(a) Multiply the first by ir to find the relationship between $\partial f/\partial r$ and $\partial f/\partial \theta$.

$$ir\,\partial f/\partial r = \partial f/\partial \theta$$

(b) Substituting $f = u(r,\theta) + is(r,\theta)$ into that relation, find the C-R equations connecting u and s.

$$ir\left(\partial u/\partial r + i\,\partial s/\partial r\right) = \partial u/\partial \theta + i\,\partial s/\partial \theta$$

$$\boxed{r\frac{\partial u}{\partial r} = \frac{\partial s}{\partial \theta} \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial s}{\partial r}}$$

(c) Show that these equations are satisfied by the powers $f = z^n = r^n e^{in\theta}$, for which $u = r^n \cos n\theta$ and $s = r^n \sin n\theta$, and also by $f = \log z$ for which $u = \log r$ and $s = \theta$.

$$f = z^n = r^n e^{in\theta} = r^n \cos n\theta + ir^n \sin n\theta = u + is$$

$$r\,\partial u/\partial r = r \cdot nr^{n-1}\cos n\theta = nr^n \cos n\theta = \partial s/\partial \theta \quad ✓$$
$$\partial u/\partial \theta = -nr^n \sin n\theta = -r\left(nr^{n-1}\sin n\theta\right) = -r\,\partial s/\partial r \quad ✓$$

$$f = \log z = \log re^{i\theta} = \log r + \log e^{i\theta} = \log r + i\theta = u + is$$

$$r\,\partial u/\partial r = r \cdot 1/r = 1 = \partial s/\partial \theta \quad ✓$$
$$\partial u/\partial \theta = 0 = -r \cdot 0 = -r\,\partial s/\partial r \quad ✓$$

(d) Combine the C-R equations in part (b) into the polar coordinate form of Laplace's equation:

$$\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial r}{\partial r}\right) = \frac{\partial^2 s}{\partial r\,\partial \theta} \qquad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial s}{\partial r} \rightarrow \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} = -\frac{\partial^2 s}{\partial \theta\,\partial r}$$

Assuming the second partial derivatives are continuous, $\dfrac{\partial^2 s}{\partial \theta\,\partial r} = \dfrac{\partial^2 s}{\partial r\,\partial \theta}$.

$$\therefore \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial r}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 s}{\partial r\,\partial \theta} - \frac{\partial^2 s}{\partial \theta\,\partial r} = 0$$

4.4.11 The function $1/(1-z)$ has a singularity at $z=1$, but around any other point a it admits the power series

$$\frac{1}{1-z} = \frac{1}{(1-a)-(z-a)} = \frac{1}{1-a}\left(\frac{1}{1-(z-a)/(1-a)}\right) = \frac{1}{1-a}\left(1+\frac{z-a}{1-a}+\left(\frac{z-a}{1-a}\right)^2+\dots\right).$$

This geometric series converges when $r=(z-a)/(1-a)$ has a magnitude $|r|<1$. Sketch the regions in the complex plane given by $|r|<1$ for the three cases $a=0$, $a=2$, $a=i$.

a, c, e

2, 3, 4, 7, 8, 10, 11, 13, 17, 18, 20, 21, 23

d c, d