1.5 Eigenvalues and Dynamical Systems

Notes

In this chapter the focus is on square matrices. Suppose A is an $n \times n$ matrix. Normally multiplication by A changes the direction of x. For certain exceptional vectors, however, Ax is a multiple of x:

$Ax = \lambda x$ — λ (a scalar) is called an <u>eigenvalue</u> of A
$(x \neq 0)$ — x is called an <u>eigenvector</u> corresponding to λ

For diagonal matrices the eigenvalues are the diagonal entries and the eigenvectors are (nonzero multiples of) the coordinate directions.

Example: $$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \longrightarrow A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = b\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For other matrices look for the eigenvalues first:

$$Ax = \lambda x \rightarrow (A - \lambda I)x = 0 \longrightarrow A - \lambda I \text{ is singular since } x \neq 0$$

This means that an eigenvalue of A is a number that makes $\det(A - \lambda I) = 0$ and an eigenvector for this λ is a nonzero vector in the nullspace of $A - \lambda I$.

$P_A(\lambda) = \det(A - \lambda I)$ is called the <u>characteristic polynomial</u> of A, a degree n polynomial in λ that begins with $(-\lambda)^n$. The roots of P_A are the eigenvalues.

The Diagonal Form

<u>1J</u> Suppose the $n \times n$ matrix A has n linearly independent eigenvectors. If these vectors are the columns of S,

$$S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Proof: Let $x_1, \dots, x_n$ be the eigenvectors so that $Ax_1 = \lambda_1 x_1, \dots, Ax_n = \lambda x_n$

$$AS = A\begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \dots & \lambda_n x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} = \Lambda S$$

Since the columns of S $(x_1, \dots, x_n)$ are independent, S is invertible.

$$\therefore \boxed{AS = S\Lambda \quad \text{or} \quad S^{-1}AS = \Lambda \quad \text{or} \quad A = S\Lambda S^{-1}}$$

__1K__ The sum of the diagonal entries of A is the __trace__ of A, denoted $tr(A)$. If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the diagonalizable matrix $A = S\Lambda S^{-1}$:

$$tr(A) = \sum_{i=1}^{n} \lambda_i$$

$$det(A) = \prod_{i=n}^{n} \lambda_i$$

Proof: For any $n \times n$ matrices B and C, $tr(BC) = tr(CB)$. Let d_r be the r^{th} diagonal of BC.

$$tr(BC) = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} c_{ji} = \left.\begin{array}{l} b_{11}c_{11} + b_{12}c_{21} + \dots + b_{1n}c_{n1} \\ + b_{21}c_{12} + b_{22}c_{22} + \dots + b_{2n}c_{n2} \\ \vdots \\ + b_{n1}c_{1n} + b_{n2}c_{2n} + \dots + b_{nn}c_{nn} \end{array}\right\}$$

the order of addition doesn't matter – now sum this 'vertically'.

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} b_{ji}$$

$$= tr(CB)$$

Choose $B = S$ and $C = \Lambda S^{-1}$:

$$tr(A) = tr(S\Lambda S^{-1}) = tr(\Lambda S^{-1} S) = tr(\Lambda) = \sum_{i=1}^{n} \lambda_i$$

Note that A need not be symmetric, just diagonalizable. We will see that not all matrices are diagonalizable.

To prove the second property use the fact that $det(BC) = det(B)det(C)$. The proof of this fact is a bit too involved to go through here.

Also the determinant of a diagonal matrix is the product of the diagonal entries.

$$det(A) = det(S\Lambda S^{-1}) = det(S)det(\Lambda S^{-1}) = det(S)det(\Lambda)det(S^{-1})$$

$$= det(SS^{-1})det(\Lambda) = det(I)det(\Lambda) = \prod_{i=1}^{n} \lambda_i$$

Differential Equations

1L If A has n linearly independent eigenvectors $x_1, \dots, x_n$ the solution

$$\begin{matrix} \frac{du}{dt} = Au \\ u(0) = u_0 \end{matrix} \longrightarrow \begin{matrix} u(t) = c_1 e^{\lambda_1 t} + \dots + c_n e^{\lambda_n t} \\ c = S^{-1} u_0 \end{matrix}$$

Proof: Let $v = S^{-1} u \leftrightarrow Sv = u$

$$\frac{dv}{dt} = S^{-1}\frac{du}{dt} = S^{-1}Au = S^{-1}S\Lambda S^{-1}u = \Lambda v, \qquad v(0) = S^{-1}u(0) = S^{-1}u_0$$

We can solve the diagonal problem $\frac{dv}{dt} = \Lambda v$, $v_0 = S^{-1}u_0$ as n uncoupled first order ODE's:

$$\left.\begin{matrix} \frac{dv_1}{dt} = \lambda_1 v_1 \longrightarrow v_1(t) = c_1 e^{\lambda_1 t} \\ \vdots \qquad\qquad \vdots \\ \frac{dv_n}{dt} = \lambda_n v_n \longrightarrow v_n(t) = c_n e^{\lambda_n t} \end{matrix}\right\} \longrightarrow v = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = e^{\Lambda t} c$$

$$u = Sv = Se^{\Lambda t}c = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = c_1 e^{\lambda_1 t} x_1 + \dots + c_n e^{\lambda_n t} x_n$$

$$u_0 = Se^{\Lambda \cdot 0} c = SIc = Sc \longrightarrow c = S^{-1}u_0$$

$\therefore$ $\frac{du}{dt} = Au$, $u(0) = u_0$ has the solution $u(t) = Se^{\Lambda t}c = c_1 e^{\lambda_1 t} x_1 + \dots + c_n e^{\lambda_n t} x_n$ with coefficients determined by $S^{-1}u_0$, so we could also write:

$$u(t) = Se^{\Lambda t} S^{-1} u_0$$

Note that since eigenvectors are not unique, S and S^{-1} are not unique and a different set of eigenvectors changes c. But $u = Se^{\Lambda t}c$ is still the same since the new eigenvectors are each just constant multiples of the original choices:

Replace $x_1, \dots, x_n$ with $k_1 x_1, \dots, k_n x_n$.

Let $S_K = [k_1 x_1 \dots k_n x_n] = S\begin{bmatrix} k_1 & & \\ & \ddots & \\ & & k_n \end{bmatrix} = SK$. We find $u(t) = S_K e^{\Lambda t} S_K^{-1} u_0$

Matrix multiplication is commutative for diagonal matrices. The solution

$$u(t) = SKe^{\Lambda t}K^{-1}S^{-1}u_0 = SKK^{-1}e^{\Lambda t}S^{-1}u_0 = Se^{\Lambda t}S^{-1}u_0$$

is the same as before – the choice of eigenvectors. Use permutation matrices to show that the order of the eigenvalues/vectors doesn't matter either.

Second Order Equations

Consider the 1-d problem $\frac{d^2u}{dt^2} + \lambda u = 0$, $u(0) = u_0$, $u'(0) = u_0'$, $\lambda > 0$.

The solution is:

$$u(t) = a\cos\omega t + b\sin\omega t \quad \text{with } \omega = \sqrt{\lambda}$$

$$u_0 = u(0) = a \ , \ u_0' = u'(0) = b\lambda$$

$$\underline{u(t) = u_0\cos\omega t + u_0'/\lambda \sin\omega t}$$

Consider now the <u>n-dimensional problem</u> $\frac{d^2u}{dt^2} + Au = 0$, $u(0) = u_0$, $u'(0) = u_0'$.

Suppose λ, x are an eigenpair of A with $\lambda > 0$ so $Ax = \lambda x$ and $\sqrt{\lambda} \in \mathbb{R}$.
Based on the 1-d solution, try $u(t) = (a\cos\omega t + b\sin\omega t)x$, $\omega = \sqrt{\lambda}$

$$u_{tt} + Au = (-a\omega^2\cos\omega t - b\omega^2\sin\omega t)x + (a\cos\omega t + b\sin\omega t)\lambda x$$

$$= (-a\lambda\cos\omega t - b\lambda\sin\omega t)x + (a\lambda\cos\omega t + b\lambda\sin\omega t)x = 0$$

This guess for $u(t)$ satisfies $u_{tt} + Au = 0$. The initial conditions u_0 and u_0' present $2n$ equations that must be satisfied to find the complete solution to the differential equation. The 2 constants a, b cannot achieve this alone.

However, suppose A has n eigenvalues $\lambda_1, \dots, \lambda_n$ with $\lambda_i > 0 \ \forall i$ and n independent eigenvectors $x_1, \dots, x_n$. Repeat the last calculation with each eigenpair to get a general solution:

$$\boxed{u(t) = (a_1\cos\omega_1 t + b_1\sin\omega_1 t)x_1 + \dots + (a_n\cos\omega_n t + b_n\sin\omega_n t)x_n \ , \ \omega_i = \sqrt{\lambda_i}}$$

Apply the initial conditions:

$$u_0 = u(0) = a_1x_1 + \dots + a_nx_n = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = Sa$$

$$u_0' = u'(t) = b_1\omega_1x_1 + \dots + b_n\omega_nx_n = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}\begin{bmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{bmatrix}\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = S\Lambda^{1/2}b$$

$$\boxed{a = S^{-1}u_0 \ , \ b = \Lambda^{-1/2}S^{-1}u_0'}$$

<u>Note</u>: All of these calculations have relied upon real, positive eigenvalues. We will show later on that if A is symmetric positive definite then all of A's eigenvalues are real and positive

Single Equations of Higher Order

Second Order ODE in $u: \mathbb{R} \to \mathbb{R}$

$$\frac{d^2u}{dt^2} + p\frac{du}{dt} + qu = 0 \quad , \quad u(0) = u_0 \; , \; u'(0) = u_0'$$

One way to solve is to assume a solution of the form $u = e^{\lambda t}$:

$$\lambda^2 e^{\lambda t} + p\lambda^2 e^{\lambda t} + q e^{\lambda t} = 0 \longrightarrow \lambda^2 + p\lambda + q = 0$$

This equation has two (possibly complex) roots λ_1, λ_2. The general solution will be:

$$u(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \quad (\lambda_1 \neq \lambda_2)$$

$$u(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t} \quad (\lambda_1 = \lambda_2 = \lambda)$$

Alternatively, convert to a matrix equation.

$$\left.\begin{aligned} U_1 &= u(t) \\ U_2 &= du/dt \end{aligned}\right\} \longrightarrow \begin{aligned} dU_1/dt &= U_2 \\ dU_2/dt &= -pU_2 - qU_1 \end{aligned}$$

$$U := \begin{bmatrix} u \\ du/dt \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \quad , \quad U_0 := \begin{bmatrix} u_0 \\ u_0' \end{bmatrix} = U(0)$$

$$\frac{dU}{dt} = \frac{d}{dt}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = AU$$

Similar to the method mentioned above, assume $U = e^{\lambda t}x$:

$$\lambda e^{\lambda t} x = \frac{dU}{dt} = AU = e^{\lambda t} A x \rightarrow Ax = \lambda x$$

This shows that solving for U amounts to solving for the eigenvalues and eigenvectors of A.

$$0 = \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -q & -p-\lambda \end{vmatrix} = \lambda^2 + p\lambda + q$$

This is the same equation for λ as before. If $\lambda_1 \neq \lambda_2$ are eigenvalues satisfying this equation, with eigenvectors x_1, x_2 and $A = S\Lambda S^{-1}$ we have

$$U = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 = [x_1 \; x_2]\begin{bmatrix} e^{\lambda_1 t} & \\ & e^{\lambda_2 t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = S e^{\Lambda} c \rightarrow c = e^{-\Lambda} S^{-1} U_0$$

Recall that $u(t) = U_1$. After determining U, you find u in the first entry.

Unfortunately it may be the case that there is a double root λ for the equation $0 = \lambda^2 + p\lambda^2 + q$ — A has only 1 eigenvalue and only one eigenvector. The nullspace of $A - \lambda I$ is only 1-dimensional. If we tried to write $A = S\Lambda S^{-1}$ then $\Lambda = \lambda I$ and so $A = S(\lambda I)S^{-1} = \lambda I$. But this doesn't agree with the form of A on the previous page so $A \neq S\Lambda S^{-1}$.

The best we do in this scenario is to find the eigenvector x for $(A-\lambda I)x = 0$ and then a <u>generalized eigenvector</u> y satisfying:

$$(A - \lambda I)y = x \longrightarrow (A - \lambda I)^2 y = (A - \lambda I)x = 0$$

Instead of $A = S\Lambda S^{-1}$ we have the <u>Jordan form</u> $A = SJS^{-1}$ with

$$S = \begin{bmatrix} x & y \end{bmatrix} \qquad J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \qquad \text{(more on this later)}$$

Symmetric Matrices

<u>Definition</u> An <u>orthogonal matrix</u> Q is a matrix with the property $Q^T = Q^{-1}$. The columns of Q are <u>orthonormal</u>, meaning $q_i^T q_j = \delta_{ij}$.

<u>1M</u> If A is real and symmetric, then we can write $A = Q\Lambda Q^T$ where all eigenvalues λ_i of A in the matrix Λ are real.

If A is positive definite, the eigenvalues are also positive.

Conversely, $A = Q\Lambda Q^T$ with Λ real implies $A = A^T$ and if all eigenvalues are positive, A is positive definite.

Proof: Suppose A is real and symmetric and let $\lambda = a + bi$ be an eigenvalue with eigenvector x.

$$\left\{ \begin{array}{l} Ax = \lambda x \\ \\ A = A^T = \bar{A}^T \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \bar{x}^T(Ax) = \bar{x}^T(\lambda x) \longrightarrow \bar{x}^T A x = \lambda \bar{x}^T x \\ \\ \bar{A}\bar{x} = \bar{\lambda}\bar{x} \longrightarrow \bar{x}^T A = \bar{\lambda}\bar{x}^T \longrightarrow \bar{x}^T A x = \bar{\lambda}\bar{x}^T x \end{array} \right\}$$

Since $\lambda \bar{x}^T x = \bar{\lambda}\bar{x}^T x$ and $\bar{x}^T x = \|x\|^2 > 0$, $\lambda = \bar{\lambda}$ (λ is real). Then $A - \lambda I$ is real valued so we can solve $(A - \lambda I)x = 0$ for a real-valued x.

For eigenvectors x_1 & x_2 corresponding to $\lambda_1 \neq \lambda_2$, x_1 and x_2 are orthogonal:

$$\begin{array}{l} Ax_1 = \lambda_1 x_1 \\ Ax_2 = \lambda_2 x_2 \end{array} \longrightarrow \begin{array}{l} x_2^T A x_1 = \lambda_1 x_2^T x_1 \\ x_2^T A^T x_1 = \lambda_2 x_2^T x_1 \end{array} \longrightarrow \underset{(A^T = A)}{\lambda_1 x_2^T x_1 = \lambda_2 x_2^T x_1} \longrightarrow \underset{(\lambda_1 \neq \lambda_2)}{x_2^T x_1 = 0}$$

Next consider the case that A has repeated eigenvalues.

Exercises

1.5.3 Find all eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

$$p_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 3\lambda^2 - \lambda^3 \longrightarrow \lambda = 0, \ \lambda = 3$$

$$\lambda = 0: \ Ax = 0 \longrightarrow x = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \ a+b+c=0 \longrightarrow x = \begin{bmatrix} c \\ d \\ -c-d \end{bmatrix}$$

$$\lambda = 3: \ (A - 3I)x = 0 \longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \ \begin{matrix} a = c \\ b = c \end{matrix} \longrightarrow x = \begin{bmatrix} c \\ c \\ c \end{bmatrix}$$

1.5.5 Solve the first order system $\frac{du}{dt} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} u$ with $u_0 = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$.

1.5.8

1.5.11

1.5.12

1.5.14

1.5.20

1.5.23

1.5.24

1.5.25