5.2 Orthogonalization and Eigenvalue Problems

Notes

Orthonormal vectors

5A The components of b with respect to orthonormal vectors q_i in the equation $c_1 q_1 + \dots + c_n q_n = b$ are:

$$c_i = q_i^T b$$

The vector $c_i q_i = q_i q_i^T b$ is the projection of b in the direction of q_i.

5B A square orthogonal matrix Q satisfies:

- $Q^T Q = Q Q^T = I \iff Q^T = Q^{-1} \iff Q$ has orthonormal columns, rows
- $\|Qx\| = \|x\|$ for every vector x of appropriate dimension

Note: $Q^T Q = I \Rightarrow Q = QI = QQ^T Q \Rightarrow 0 = (QQ^T - I)Q \Rightarrow QQ^T = I$

$$\text{If } Qc = b, \quad c = Q^T b = \begin{bmatrix} - q_1^T - \\ \vdots \\ - q_n^T - \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} q_1^T b \\ \vdots \\ q_n^T b \end{bmatrix}$$

with $\|c\| = \|Qc\| = \|b\|$

Least Squares

$$Q = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}$$

$q_i \in \mathbb{R}^m$

$m > n$

orthonormal columns but not a basis

rows not orthonormal

$QQ^T \neq I_m$ but $Q^TQ = I_n$

$Qc = b$ usually has no solution

For a least squares solution, use the normal eqn's:

$$Q^TQc = Q^Tb \Rightarrow c = Q^Tb = \begin{bmatrix} - q_1^Tb - \\ \vdots \\ - q_n^Tb - \end{bmatrix}$$

Projection of b onto the column space of Q:

$$p = Qc = c_1q_1 + \ldots + c_nq_n = q_1^Tbq_1 + \ldots + q_n^Tbq_n$$

5C If the columns of $Q \in \mathbb{R}^{m\times n}$ $(n<m)$ are orthogonal so that $Q^TQ = I \in \mathbb{R}^{n\times n}$, the numbers $c_i = q_i^Tb$ give

(1) the best least squares solution to $Qc = b$

(2) the projection $p = c_1b_1 + \ldots + c_nb_n$ of b onto $col(A)$

(3) the coefficients that minimize the distance $\|c_1q_1 + \ldots + c_nq_n - b\|$ between b and any combination of the columns.

These all describe a projection onto the subspace of $\mathbb{R}^m$ spanned by $q_1, \ldots, q_n$.

Nonorthogonal Columns

Overdetermined system
more equations than unknowns

$$Ax = b$$
$A \in \mathbb{R}^{m \times n}$ $n < m$
$x \in \mathbb{R}^n$ $b \in \mathbb{R}^m$

If the columns of A are independent, the normal eqns give a least squares solution $A^TAx = A^Tb$ by elimination and back substitution.

For an ill-conditioned A^TA (when the columns of A are almost dependent, better to orthogonalize.

$$A = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix} = \begin{bmatrix} q_1 & \dots & q_n \end{bmatrix} \begin{bmatrix} r_{11} & \dots & r_{1n} \\ & \ddots & \\ & & r_{nn} \end{bmatrix} = QR$$

<u>5D</u> If $A = QR$ with $Q^TQ = I$, the least squares solution to $Ax = b$ is $\hat{x} = R^{-1}Q^Tb$

Proof: The least squares solution $\hat{x}$ satisfies $A^TA\hat{x} = A^Tb$.

$$A^TA\hat{x} = A^Tb$$
$$R^TQ^TQR\hat{x} = R^TQ^Tb$$
$$R^TR\hat{x} = R^TQ^Tb$$
$$R\hat{x} = Q^Tb \Rightarrow \hat{x} = R^{-1}Q^Tb$$

<u>Example</u> Forming a set q_1, q_2, q_3 of orthonormal vectors with the same span as a_1, a_2, a_3.

$$a_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad a_2 = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} \quad a_3 = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$$

- q_1

$$q_1 = a_1 / \|a_1\| = a_1$$

- q_2

$$v = a_2 - (q_1^T a_2) q_1 = a_2 - q_1 = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$$

$$\|v\| = 4$$

$$q_2 = v / \|v\| = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

- q_3

$$v = a_3 - (q_1^T a_3) q_1 - (q_2^T a_3) q_2 = a_3 - q_1 - q_2 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$$

$$\|v\| = 5$$

$$q_3 = v / \|v\| = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 5 \\ 0 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 5 \end{bmatrix} = QR$$

Notice $a_1 = q_1$, $a_2 = q_1 + 4q_2$, $a_3 = q_1 + q_2 + 5q_3$

Modified Gram-Schmidt

A has independent columns $a_1, \dots, a_n$. We want orthonormal columns $q_1, \dots, q_n$ with the same span to factor $A = QR$.

$$v_j^{(1)} = a_j$$

$$v_j^{(2)} = P_{\perp q_1} v_j^{(1)} = v_j^{(1)} - (q_1^* v_j^{(1)}) q_1$$

$$v_j^{(3)} = P_{\perp q_2} v_j^{(2)} = v_j^{(2)} - (q_2^* v_j^{(2)}) q_2$$

$$\vdots$$

$$v_j = v_j^{(j)} = P_{\perp q_{j-1}} v_j^{(j-1)}$$
$$= v_j^{(j-1)} - (q_{j-1}^* v_j^{(j-1)}) q_{j-1}$$

$\rightarrow$

For $i = 1$ to n
  $v_i = a_i$
For $i = 1$ to n
  $r_{ii} = \|v_i\|$
  $q_i = v_i / r_{ii}$
  For $j = i+1$ to n
    $r_{ij} = q_i^* v_j$
    $v_j = v_j - r_{ij} q_i$

when implemented $P_{\perp q_i}$ can be applied as soon as q_i is known to each $v_j^{(i)}$ for $j > i$

Another pseudocode:

For $k = 1, \dots, n$
  $r_{kk} := \left(\sum_{i=1}^m a_{ik}^2 \right)^{1/2}$
  For $i = 1, \dots, m$
    $a_{ik} := a_{ik} / r_{kk}$
  For $j = k+1, \dots, n$
    $r_{kj} := \sum_{i=1}^m a_{ik} a_{ij}$
    For $i = 1, \dots, m$
      $a_{ij} := a_{ij} - a_{ik} r_{kj}$

The QR Algorithm

Given a square matrix A, factor into $A = QR$

Define $A_1 = RQ$. A_1 is similar to A:

$$A_1 = RQ = Q^{-1}QRQ = Q^{-1}AQ \quad (= Q^T A Q)$$

A_1 and A have the same eigenvalues:

$$A_1 x = \lambda x \Rightarrow QRQx = \lambda Qx \Rightarrow A(Qx) = \lambda(Qx)$$

$$Ay = \lambda y \Rightarrow QRy = \lambda y \Rightarrow QR(Qx) = \lambda Qx$$

$$\Rightarrow Q^T QRQx = \lambda Q^T Qx \Rightarrow RQx = \lambda x \Rightarrow A_1 x = \lambda x$$

For each $k = 0, 1, \ldots$ $(A_0 := A)$ factor $A_k = Q_k R_k$ and put $A_{k+1} = R_k Q_k$.

The sequence A_k approaches a diagonal or upper triangular matrix with eigenvalues on the diagonal.

Householder Matrices

$$\boxed{H = I - 2uu^T \text{ for some unit vector } u}$$

These matrices are symmetric and orthogonal:

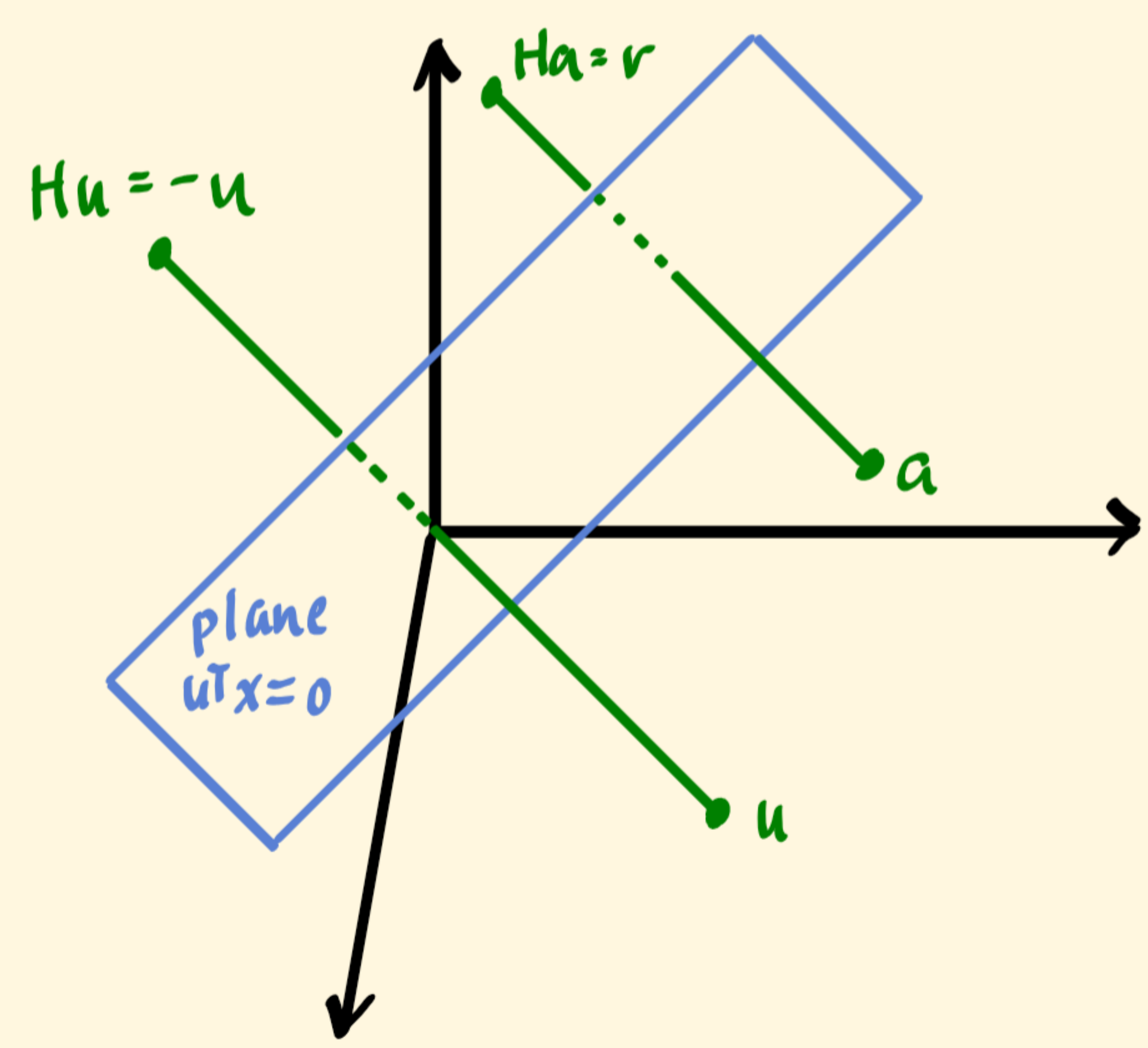
$$H^TH = (I^T - 2(u^T)^Tu)(I - 2uu^T) = (I - 2uu^T)(I - 2uu^T)$$
$$= I - 4uu^T + 4uu^Tuu^T = I - 4uu^T + 4uu^T = I$$

Examples

$$u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow H = I - 2uu^T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow H\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

$$u = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \rightarrow H = I - \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow H\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

$$u = \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix} \rightarrow H = \begin{bmatrix} 1 - 2\cos^2\theta & \sin 2\theta \\ \sin 2\theta & 1 - 2\sin^2\theta \end{bmatrix}$$
$$= \begin{bmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$$



For vectors in the plane $u^Tx = 0$, $Hx = x$.

Given any vector $v \neq 0$, $u = v/\|v\|$ gives a Householder matrix:

$$H = I - 2vv^T/\|v\|^2$$

5E The Householder matrix that takes a into r is $H = I - 2vv^T/\|v\|^2$ with $v = a - r$:

$$\text{if} \quad a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \quad \text{and} \quad r = \begin{bmatrix} \|a\| \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{then } Ha = r$$

Since H is orthogonal, $\|a\| = \|r\|$.

$$Ha = Ia - (a-r)\frac{2(a-r)^T a}{(a-r)^T(a-r)}$$

$$= a - (a-r)\frac{2a^Ta - 2r^Ta}{2a^Ta - 2r^Ta} \qquad \begin{pmatrix} a^Ta = r^Tr \\ a^Tr = r^Ta \end{pmatrix}$$

$$= a - a + r = r$$

<u>Example</u> $\quad A = \begin{bmatrix} 3 & 4 \\ 4 & 0 \end{bmatrix}$

$$a = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad r = \begin{bmatrix} \|a\| \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}, \quad v = a - r = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$H = I - 2vv^T/\|v\|^2 = I - \frac{2}{20}\begin{bmatrix} 4 & -8 \\ -8 & 16 \end{bmatrix} = \begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}$$

$$HA = \begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 2.4 \\ 0 & 3.2 \end{bmatrix} = R$$

In this one step case, $H = Q$. In general Q is the product of several Householders: $Q = H_1 H_2 \cdots H_n$

What would the second step look like if we wanted H_2?

$$H_1 A = \begin{bmatrix} r_{11} & * & * \\ 0 & * & * \\ 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & & * \\ 0 & \begin{bmatrix} a' \end{bmatrix} & * \\ 0 & & * \\ 0 & & * \end{bmatrix}$$

The numbers r_{12}, r_{13} just whatever is found in $H_1 A$.

Let $a = \begin{bmatrix} 0 \\ a' \end{bmatrix} \in \mathbb{R}^m$ (take the second column and replace top row by 0)

Let $r = (0, \|a\|, 0, \dots, 0) = (0, r_{22}, 0, \dots, 0)$

$v = a - r \longrightarrow H_2 = I - 2vv^T/\|v\|^2$

Notice that the first column of H_2 is $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ so $H_2(H_1 A)$ leaves the first column unchanged:

$$H_2(H_1 A) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & * \\ 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix}$$

In this 5×3 case we would apply one more Householder H_3 to get:

$H_3 H_2 H_1 A = R \rightarrow A = H_1 H_2 H_3 R = QR$

with Q square and R with 2 bottom rows of zeros.

<u>5F</u> The least squares solution to $Ax = b$ is computed by applying Householder transformations $H_1, \dots, H_n$ to both sides, leaving

$$\begin{bmatrix} R \\ \hline 0 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} c \\ \hline d \end{bmatrix}$$

$Q = H_1 \cdots H_n$ and the solution x is given by back substitution in $Rx = c$. The error $\|Ax - b\| = \|d\|$.

Exercises

5.2.1 If Q contains the first $n<m$ columns of the $m\times m$ identity matrix, what is the best least squares solution to $Qc=b$? Here $b\in\mathbb{R}^m$ and $c\in\mathbb{R}^n$.

By 5C, the numbers $c_i=q_i^Tb$ give the best least squares solution. But here the q_i are the first n standard unit basis vectors for $\mathbb{R}^m$: $q_1=e_1,\ldots,q_n=e_n$.

$$\therefore\ c_i=b_i \quad\text{and}\quad c=(b_1,\ldots,b_n)$$

5.2.2

(i) Find A^TA if the columns of A are orthogonal but not orthonormal with lengths $l_1>0,\ldots,l_n>0$.

$$A^TA=\begin{bmatrix}-\ a_1^T\ -\\ \vdots\\ -\ a_n^T\ -\end{bmatrix}\begin{bmatrix}| & & |\\ a_1 & \cdots & a_n\\ | & & |\end{bmatrix}=\begin{bmatrix}l_1 & & \\ & \ddots & \\ & & l_n\end{bmatrix}$$

(ii) If A has only one column a, $A^TAx=A^Tb$ gives $x=a^Tb/a^Ta$. Find x and the projection p of $b=(1,1,1)$ onto the line through $a=(3,0,4)$.

First let's verify the claim $x=a^Tb/a^Ta$. Since a^Ta is a scalar:

$$a^Tax=A^TAx=A^Tb=a^Tb\longrightarrow x=a^Tb/a^Ta$$

Given $b=(1,1,1)$ and $a=(3,0,4)$, $x=(3+4)/(9+16)=7/25$.

The one-dimensional projection onto the line through a is the multiple $p=xa$ that is closest to b: $p=7a/25=(21/25,0,28/25)$.

5.2.3 The distance (squared) from the point b to the line through a is:

$$\|b-p\|^2 = \left\| b - \frac{a^Tb}{a^Ta} a \right\|^2 = \frac{(b^Tb)(a^Ta)-(a^Tb)^2}{a^Ta}$$

Since $\|b-p\|^2 \geq 0$ this gives the Schwarz inequality $|a^Tb| \leq \|a\|\|b\|$.

Verify:

$$0 \leq \frac{(b^Tb)(a^Ta)-(a^Tb)^2}{a^Ta} \Leftrightarrow 0 \leq (b^Tb)(a^Ta)-(a^Tb)^2 \text{ since } a^Ta>0$$

$$\Leftrightarrow (a^Tb)^2 \leq (b^Tb)(a^Ta) = \|b\|^2\|a\|^2$$

$$\Leftrightarrow |a^Tb| \leq \|a\|\|b\|$$

(i) How does the Schwarz inequality come from the geometric formula $a^Tb = \|a\|\|b\|\cos\theta$?

Just take absolute values:

$$|a^Tb| = |\|a\|\|b\|\cos\theta| = \|a\|\|b\||\cos\theta| \leq \|a\|\|b\| \cdot 1 = \|a\|\|b\|$$

(ii) By making the right choice of b, show that for any $a = (a_1, \ldots, a_m)$

$$(a_1 + \ldots + a_m)^2 \leq m(a_1^2 + \ldots + a_m^2)$$

Try $b = (1, 1, \ldots, 1)$:

$$(a_1 + \ldots + a_m)^2 = |a^Tb|^2 \leq \|a\|^2\|b\|^2 = (a^Ta)(b^Tb) = (a_1^2 + \ldots + a_m^2) \cdot m$$

(iii) When does equality hold in part (ii)? In $|a^Tb| \leq \|a\|\|b\|$?

$$(a_1 + \ldots + a_m)^2 = m(a_1^2 + \ldots + a_m^2)$$

$$a_1^2 + \ldots + a_m^2 + \Sigma_{i \neq j} 2a_{ij} = ma_1^2 + \ldots + ma_m^2$$

$$\|a\| = a_1^2 + \ldots + a_m^2 = \frac{\Sigma_{i \neq j} 2a_{ij}}{1-m}$$

$$a^Tb = \|a\|\|b\|\cos\theta \Rightarrow |a^Tb| = \|a\|\|b\| \text{ when } \cos\theta = \pm 1$$

$$\Rightarrow a \parallel b \text{ (parallel)}$$

5.2.5 Find the best least squares solution to

$$\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \\ -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = b$$

What is the projection of b onto the space spanned by these two columns

The least squares solution $\hat{x}$ satisfies $A^TA\hat{x} = A^Tb$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\hat{x} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\hat{x} = A^TA\hat{x} = A^Tb = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

The projection is $p = A\hat{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

5.2.10 The projection matrix onto the column space of A is $P = A(A^TA)^{-1}A^T$ (since $p = A\hat{x}$, $\hat{x} = (A^TA)^{-1}A^Tb$). If $A = QR$, find a simpler formula for P.

$$P = QR(R^TQ^TQR)^{-1}R^TQ^T = QR(R^TR)^{-1}R^TQ^T$$
$$= QRR^{-1}(R^T)^{-1}R^TQ^T = QQ^T$$

5.2.12 What rotation angle θ produces a zero below the diagonal in the product QA?

$$QA = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$QA = \begin{bmatrix} a\cos\theta - c\sin\theta & b\cos\theta - d\sin\theta \\ a\sin\theta + c\cos\theta & b\sin\theta + d\cos\theta \end{bmatrix}$$

$$0 = a\sin\theta + c\cos\theta \Rightarrow \theta \text{ s.t. } \tan\theta = -c/a$$

? **5.2.17** If A_0 is symmetric and tridiagonal show that $A_1 = RQ = Q^{-1}A_0Q$ is also symmetric and tridiagonal. Hint: Indicate by 'x' the (generally) nonzero entries of a 4×4 $A_0 = QR$. Each column of Q comes by G-S from the current column and previous column of A_0.

$$A_0 = \begin{bmatrix} x & x & 0 & 0 \\ x & x & x & 0 \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix} \overset{?}{=} \begin{bmatrix} x & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} x & x & x & x \\ & x & x & x \\ & & x & x \\ & & & x \end{bmatrix}$$

5.2.18 If H is a Householder matrix and $Hx=y$, show that also $Hy=x$. What 4x4 matrix gives a reflection across the plane $v^T x = x_1+x_2+x_3+x_4=0$?

- $Hy = HHx = H^T Hx = x$ (since $H=H^T$ and $H^T H = I$).

- $0 = x_1+x_2+x_3+x_4 = v^T x \Rightarrow v=(1,1,1,1)$

$$H = I - 2vv^T/\|v\|^2 = I - \frac{2}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

H is orthogonal and $Hx=0$ for $0=x_1+x_2+x_3+x_4$

5.2.21 Find the Householder matrix H_1 that produces zeros below the first entry in A. Compute $H_1 A$.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 2 & 2 \end{bmatrix}$$

$$a = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \rightarrow r = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \rightarrow v = a - r = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \quad \|v\|^2 = 6$$

$$H_1 = I - 2vv^T/\|v\|^2 = I - \frac{2}{6}\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$H_1 A = \begin{bmatrix} 3 & 2 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} = R \longrightarrow A = \frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \\ 2 & -2 & -1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} = H_1 R = QR$$

No H_2 needed since $H_1 A$ is already upper triangular

__5.2.22__ If $H = I - 2uu^T$ show that the unit vector u is an eigenvector and find the eigenvalue. Show that any $x \perp u$ is also an eigenvector with $\lambda = 1$. Find the trace and determinant.

- $Hu = Iu - 2uu^Tu = u - 2u = -u \rightarrow \begin{matrix} Hu = -1u \\ \lambda_1 = -1 \end{matrix}$

- $u^Tx = x^Tu = 0 \rightarrow Hx = Ix - 2uu^Tx = x \rightarrow \begin{matrix} Hx = 1x \\ \lambda_2 = 1 \end{matrix}$

- $\det H = \lambda_1 \lambda_2 = -1$, $\text{trace } H = \lambda_1 + \lambda_2 = 0$

- What if $u \in \mathbb{R}^n$, $n > 2$ so $H \in \mathbb{R}^{m \times m}$?

  Do we find $x_1, \ldots, x_{n-1} \perp u$?

  Then $\det H = -1$ but $\text{trace } H = -1 + (n-1)1 = n-2$?

5.2.24 Find H and the tridiagonal form $A_0 = HAH$.

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 2 & 1 \\ 4 & 1 & 0 \end{bmatrix}$$

$$a = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \rightarrow \quad r = \begin{bmatrix} a_{11} \\ \|a\| \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} \rightarrow \quad v = a - r = \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix}$$

$$H = I - 2vv^T/\|v\|^2 = \frac{1}{5}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix}$$

$$A_0 = HAH = \begin{bmatrix} 1 & 5 & 0 \\ 5 & 42/25 & 31/25 \\ 0 & 31/25 & 8/25 \end{bmatrix} \quad \text{is tridiagonal}$$