

AMATH 301 - Spring 2018

Homework #7

Due on Thursday, May 17, 2018

Note: All three problems in the homework have corresponding data held in a `.mat` file. You should download them to the same directory as your script file and access them by using the `load` command, but do **not** upload them to Scorelator. Scorelator has its own copy of the data.

1. In order to determine the half-life of Plutonium-239, scientists start with a sample of approximately 10 kg of Plutonium-239 and measure the remaining amount each year for 40 years. The data is contained in the file `Plutonium.mat` which is included with the homework. `Plutonium.mat` contains two vectors, `t` and `P`. The vector `t` contains the number of years since the beginning of the experiment, and the vector `P` contains the corresponding amounts of Plutonium-239 remaining measured in kg. Let the function $P(t)$ denote the amount of remaining Plutonium as a function of time.
 - (a) Use the second order difference formula that was derived in class to approximate the derivative $\frac{dP}{dt}$ at time $t = 0$. Save the result in `A1.dat`.
 - (b) Use a central difference to approximate the derivative $\frac{dP}{dt}$ at time $t = 10$. Save the result in `A2.dat`.
 - (c) Use a second order accurate difference scheme to approximate the derivative $\frac{dP}{dt}$ at time $t = 40$. Save the result in `A3.dat`.
 - (d) The decay rate of Plutonium-239 at a time t is given by $-\frac{1}{P} \frac{dP}{dt}$. Use a second order difference scheme to approximate $\frac{dP}{dt}$ at all 41 times in `t`. Then use these approximations to estimate the decay rate at all 41 times in `t`. Save the average of these decay rates in `A4.dat`.
 - (e) If λ is the average decay rate that you found in part (d), then the half-life of Plutonium-239, denoted by $t_{1/2}$, is given by the formula

$$t_{1/2} = \frac{\ln(2)}{\lambda}.$$

Calculate the half-life, and save it in `A5.dat`.

Things to think about: You can also estimate the half-life by finding the best fit exponential function of the form $P = P_0 e^{-\lambda t}$ where λ is the decay rate. Do you expect this to give a more accurate approximation of the half-life? Try estimating the half-life using this method. Then look up the actual half-life online to see which method gives a better approximation. Now look at the numbers in the vector P . How close are they to each other? Do you think that it would be important to have very precise measurements to capture the changes over these 40 years? How do you think measurement error would affect your derivative approximations?

2. Consider blood flow through an artery or vein. For laminar flow (i.e. flow in which the fluid moves in parallel layers), the velocity of the blood is given by the equation

$$v(r) = \frac{\Delta p}{4\mu L}(R^2 - r^2),$$

where r is the distance from the center of the blood vessel, R is the radius of the blood vessel (distance from the center to the wall), Δp is the change in pressure from the beginning of the blood vessel to the end, μ is the viscosity of the blood, and L is the length of the blood vessel. By examining this equation, we can see that the blood moves fastest in the center of the blood vessel ($r = 0$) and slowest near the walls (r is close to R). The volumetric flow rate Q (the volume of fluid that passes through a cross section per unit time) is given by

$$Q = \int_0^R 2\pi r v(r) dr.$$

If all of the parameters in the function $v(r)$ are known, this is a very easy integral to evaluate by hand. However, the parameters are often not known. Instead, measurements of the velocity can be taken at different values of r .

- (a) The file `BloodFlow.mat` contains two vectors, \mathbf{r} and \mathbf{v} . The vector \mathbf{r} contains the values of r at which measurements were taken, and the vector \mathbf{v} contains the corresponding velocities in m/s. Using this data, use the trapezoidal rule to approximate the volumetric flow rate Q and save the result in `A6.dat`.
- (b) The cross-sectional area A of the blood vessel can be calculated with the integral

$$A = \int_0^R 2\pi r dr.$$

Approximate this integral using the vector \mathbf{r} and the right-sided rectangle rule. Save the result in `A7.dat`.

- (c) The mean velocity is the volumetric flow rate Q divided by the cross-sectional area A . Calculate the mean velocity using your answers to part (a) and (b) and save the result in `A8.dat`.

Things to think about: Do you think it is realistic to have a function for blood velocity $v(r)$ that **only** depends on the variable r and no other variables? Why might this be too simple? What other variables might you include? What do you get when you take the average of the vector \mathbf{v} and how does it compare to your answer in part (c)? Why?

3. The cardiac output C_0 is the volume of blood pumped by the heart per unit time. One way to measure cardiac output is to inject dye into the right atrium and then measure the concentration of dye in the blood that is leaving the heart. If A is the amount of injected dye and $c(t)$ is the concentration of dye as a function of time, then the cardiac output is

$$C_0 = \frac{A}{\int_0^T c(t) dt}$$

where T is the final measurement time. The file `Dye.mat` contains two vectors. The vector \mathbf{t} contains the times (in seconds) at which the dye concentration was measured, and the vector c contains the concentration of dye in the blood at those times. Use Simpson's rule to evaluate the integral in the denominator of the formula above. Save the answer in `A9.dat`. Then use the value $A = 2$ ml to calculate the cardiac output in ml/s. Save the result in `A10.dat`.