1. The Black-Scholes price for a European put option is

$$P(S,t,K,T,r,q,\sigma) = Ke^{-r(T-t)}\Phi(-d_-) - Se^{-q(T-t)}\Phi(-d_+) \tag{1}$$

where

$$d_+ = \frac{\log\left(\frac{S}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_- = d_+ - \sigma\sqrt{T-t} = \frac{\log\left(\frac{S}{K}\right) + \left(r - q - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

Compute each of

(a) $\Delta(P) = \dfrac{\partial P}{\partial S}$

(b) $\Gamma(P) = \dfrac{\partial^2 P}{\partial S^2}$

(c) $\theta(P) = \dfrac{\partial P}{\partial t}$

(d) $\rho(P) = \dfrac{\partial P}{\partial r}$

by taking derivatives of (1). Verify that your answers are correct using put-call parity.

You can find expressions for *The Greeks* on pages 92 and 93 of the Stefanica text. Verify that your answer matches the expression for the put option and that put-call parity gives the expression for the call option. And as always, verify your calculations with Mathematica.

Example

Compute the vega of a European put option.

$$\begin{aligned}
\text{vega}(P) = \frac{\partial P}{\partial \sigma} &= \frac{\partial}{\partial \sigma}\left[Ke^{-r(T-t)}\Phi(-d_-) - Se^{-q(T-t)}\Phi(-d_+)\right] \\
&= Ke^{-r(T-t)}\phi(-d_-)\frac{\partial}{\partial \sigma}(-d_-) - Se^{-q(T-t)}\phi(-d_+)\frac{\partial}{\partial \sigma}(-d_+) \\
&= Ke^{-r(T-t)}\phi(d_-)\frac{\partial}{\partial \sigma}(-d_-) - Se^{-q(T-t)}\phi(d_+)\frac{\partial}{\partial \sigma}(-d_+)
\end{aligned}$$

Lemma 3.15 states that $Ke^{-r(T-t)}\phi(d_-) = Se^{-q(T-t)}\phi(d_+)$, thus

$$\begin{aligned}
\text{vega}(P) = \frac{\partial P}{\partial \sigma} &= Se^{-q(T-t)}\phi(d_+)\frac{\partial}{\partial \sigma}(-d_-) - Se^{-q(T-t)}\phi(d_+)\frac{\partial}{\partial \sigma}(-d_+) \\
&= Se^{-q(T-t)}\phi(d_+)\left[\frac{\partial}{\partial \sigma}(-d_-) - \frac{\partial}{\partial \sigma}(-d_+)\right] \\
&= Se^{-q(T-t)}\phi(d_+)\left[\frac{\partial}{\partial \sigma}(d_+) - \frac{\partial}{\partial \sigma}(d_-)\right] \\
&= Se^{-q(T-t)}\phi(d_+)\ \frac{\partial}{\partial \sigma}\left[d_+ - d_-\right]
\end{aligned}$$

But $d_- = d_+ - \sigma\sqrt{T-t} \implies d_+ - d_- = \sigma\sqrt{T-t}$, thus

$$\begin{aligned}
\text{vega}(P) = \frac{\partial P}{\partial \sigma} &= Se^{-q(T-t)}\phi(d_+)\ \frac{\partial}{\partial \sigma}\left[\sigma\sqrt{T-t}\right] \\
\text{vega}(P) = \frac{\partial P}{\partial \sigma} &= Se^{-q(T-t)}\phi(d_+)\ \sqrt{T-t}
\end{aligned}$$

Check the result using put-call parity:

$$\begin{aligned}
P &= C - Se^{-q(T-t)} + Ke^{-r(T-t)} \\
\text{vega}(P) = \frac{\partial P}{\partial \sigma} &= \frac{\partial}{\partial \sigma}\left[C - Se^{-q(T-t)} + Ke^{-r(T-t)}\right] \\
&= \frac{\partial C}{\partial \sigma}
\end{aligned}$$

The vega of a European put option is the same as the vega for a European call option.