1. Give an example of a $2 \times 2$ matrix that has no real eigenvectors. Justify your solution with intuition (without solving completely for the eigenvectors and eigenvalues).

2. Consider an $n \times p$ matrix $A$. Show that the number of linear independent rows is the same as the number of linearly independent columns.
   Hint: Write $A = CR$ where $C$ is a matrix of the linearly independent columns of $A$. Why can we write $A$ like this? Then consider the $CR$ product in the "row" interpretation of matrix multiplication.

3. Let $A$ be an $m \times n$ matrix (assume $m > n$). The full singular value factorization $A = U\Sigma V^{\mathrm{T}}$ contains more information than necessary to reconstruct $A$.
   (a) What are the smallest matrices $\tilde{U}$, $\tilde{\Sigma}$ and $\tilde{V}^{\mathrm{T}}$ such that $\tilde{U}\tilde{\Sigma}\tilde{V}^{\mathrm{T}} = A$?
   (b) Let $U = \begin{bmatrix} \tilde{U} & \hat{U} \end{bmatrix}$. That is, think about $U$ from the full singular value factorization as a block matrix consisting of the matrix $\tilde{U}$ found in part (a) and the remaining (unneeded) columns $\hat{U}$.

       Find expressions for $\tilde{U}^{\mathrm{T}}\tilde{U}$ and $\tilde{U}\tilde{U}^{\mathrm{T}}$.
   (c) Use the *reduced* singular value factorization obtained in part (a) to find an expression for the matrix $H = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. How many matrices must be inverted (diagonal and orthogonal matrices don't count)?

4. Let $x$ and $y$ be vectors of $m$ elements. The least squares solution for a best-fit line for a plot of $y$ versus $x$ is

$$\hat{\beta} = (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}y$$

   where

$$X = \begin{bmatrix} | & | \\ 1 & x \\ | & | \end{bmatrix}$$

   (a) Suppose you know the **full** singular value factorization $X = U\Sigma V^{\mathrm{T}}$. Find an expression for $\hat{\beta}$ in terms of $U$, $\Sigma$, and $V$. Hint: Only square matrices can be invertible.
   (b) Repeat part (a) using the reduced singular value factorization $X = \tilde{U}\tilde{\Sigma}\tilde{V}^{\mathrm{T}}$.

5. Let $\tilde{X}$ be an $m \times n$ matrix ($m > n$) whose columns have sample mean zero, and let $\tilde{X} = \tilde{U}\tilde{\Sigma}\tilde{V}^{\mathrm{T}}$ be a reduced singular value factorization of $\tilde{X}$. The squared *Mahalanobis* distance to the point $\tilde{x}_i^{\mathrm{T}}$ (the $i^{\text{th}}$ row of $\tilde{X}$) is

$$d_i^2 = \tilde{x}_i^{\mathrm{T}} \hat{S}^{-1} \tilde{x}_i$$

where $\hat{S} = \frac{1}{m-1}\tilde{X}^{\mathrm{T}}\tilde{X} = \mathrm{cov}(\tilde{X})$. Explain how to compute $d_i^2$ without inverting a matrix.

6. (a) Suppose $A = LU$ where $L$ is lower triangular and $U$ is upper triangular. Explain how you would solve the problem $Ax = b$ using $L$, $U$, and the concepts of forward and backward substitution.

   (b) Compute the LU factorization of

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 2 \\ -2 & 1 & 1 \end{bmatrix}$$

   by hand using elimination matrices.