1. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 4 & 1 \\ -2 & 1 & 1 \end{bmatrix}$

   (a) Use elimination to turn A into an upper triangular matrix. How many pivots does A have?

   (b) Let $b = (1, 6, 3)$. Does $Ax = b$ have a solution?

   (c) Let $b = (1, 6, 5)$. Does $Ax = b$ have a solution?

   (d) Can you find multiple solutions in either part (b) or part (c)? If so, find 2.

   (e) Does A have an inverse? Justify your answer using results from this exercise.

2. Suppose $AB = I$ and $CA = I$ where I is the $n \times n$ identity matrix.

   (a) What are the dimensions of the matrices A, B and C?

   (b) Show that $B = C$.

   [Hint: you can write $IB = B$]

   (c) Is A invertible?

3. Let A be a square matrix with the property that $A^2 = A$. Simplify $(I - A)^2$ and $(I - A)^7$.

4. (a) Write the vector $(9, 2, -5)$ as a linear combination of the vectors $(1, 2, 3)$ and $(6, 4, 2)$ or explain why it can't be done.

   (b) How many pivots does a system of equations with coefficient matrix

$$A = \begin{bmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ 3 & 2 & -5 \end{bmatrix}$$

   have?

5. Suppose A is a 6×20 matrix and B is a 20×7 matrix.

   (a) What are the dimensions of $C = AB$?

   (b) Suppose A, B, and C have been partitioned into block matrices like so:

$$A = \left[\begin{array}{c|c|c} A_{11} & A_{12} & A_{13} \\ \hline A_{21} & A_{22} & A_{23} \end{array}\right], \quad B = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \\ \hline B_{31} & B_{32} \end{array}\right], \quad C = \left[\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right],$$

   Suppose that A_{11} is 2×10, B_{22} is 4×3, and C_{11} is $? \times 4$. What are the dimensions of *each* block of A, B, and C such that all the resulting block matrix multiplications are valid?

   [Hint: Make note of every fact you know, sketch all three matrices, and fill in the unknowns step by step]

   (c) Write each block of C in terms of blocks of A and B.

6. Let A be an $m \times n$ matrix.

   (a) The full $A = QR$ factorization contains more information than necessary to reconstruct A. What are the smallest matrices $\tilde{Q}$ and $\tilde{R}$ such that $\tilde{Q}\tilde{R} = A$?

   (b) Let $\tilde{A}$ be an $m \times n$ matrix $(m > n)$ whose columns each sum to zero, and let $\tilde{A} = \tilde{Q}\tilde{R}$ be the reduced QR factorization of $\tilde{A}$. The squared *Mahalanobis* distance to the point $\tilde{x}_i^T$ (the i^{th} row of $\tilde{A}$) is

$$d_i^2 = \tilde{x}_i^T \hat{S}^{-1} \tilde{x}_i$$

   where $\hat{S} = \frac{1}{m-1}\tilde{A}^T\tilde{A}$ is a covariance matrix. Compute d_i^2 without inverting a matrix.