CFRM 405: Mathematical Methods for Quantitative Finance Homework 2

Solve the exercises by hand.

1. Let K, T, σ, and r be positive constants and let

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_0^{b(x)} e^{-\frac{y^2}{2}}\, dy$$

where $b(x) = \frac{1}{\sigma\sqrt{T}} \left[\log\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T\right]$. Compute $g'(x)$.

2. Let $\phi(u) = \dfrac{1}{\sqrt{2\pi}} e^{-u^2/2}$ so that $\Phi(x) = \displaystyle\int_{-\infty}^{x} \phi(u)\, du$ (i.e., the $\Phi(x)$ in Black-Scholes).

   (a) For $x > 0$, show that $\phi(-x) = \phi(x)$.

   (b) Given that $\lim\limits_{x\to\infty} \Phi(x) = 1$, use the properties of the integral as well as a substitution to show that $\Phi(-x) = 1 - \Phi(x)$ (again, assuming $x > 0$).

3. (a) Under what condition does the following hold?

$$\iint_D f(x,y)\, dA = \iint_D f(x,y)\, dy\, dx = \iint_D f(x,y)\, dx\, dy$$

   (b) Evaluate the double integral

$$\iint_D e^{y^2}\, dA$$

   where $D = \{(x,y) : 0 \le y \le 1,\ \ 0 \le x \le y\}$.

4. (a) Transform the double integral

$$\iint_D e^{\frac{x+y}{x-y}}\, dA$$

into an integral of u and v using the change of variables

$$u = x + y \qquad\qquad v = x - y$$

and call the domain in the uv plane S.

(b) Let D be the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-2)$ and $(0,-1)$. Find the corresponding region S in the uv plane by evaluating the transformation at the vertices of D and connecting the dots. Sketch both regions.

(c) Compute the integral found in part (a) over the domain S from part (b).

5. (a) Let $D = \{(x,y) : 1 \leq x^2 + y^2 \leq 9,\ y \geq 0\}$. Compute the integral

$$\iint_D \sqrt{x^2 + y^2}\, dx\, dy$$

by changing to polar coordinates. Sketch the domains of integration in both the xy and $r\theta$ (that means r on one axis and θ on the other) planes.

(b) Compute the integral

$$\iint_D \sin(\sqrt{x^2 + y^2})\, dx\, dy$$

where $D = \{(x,y) : \pi^2 \leq x^2 + y^2 \leq 4\pi^2\}$.