1. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 4 & 1 \\ -2 & 1 & 1 \end{bmatrix}$

(a) Use elimination to turn A into an upper triangular matrix. How many pivots does A have?

First subtract one times row one from row two:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 4 & 1 \\ -2 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix}$$

Next add two times row one to row three:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 3 & 1 \end{bmatrix}$$

Finally, subtract one times row two from row three:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We can read now that there are two pivots.

(b) Let $b = (1, 6, 3)$. Does $Ax = b$ have a solution?

Appending the vector b and repeating the same steps of reduction gives:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 4 & 1 & 6 \\ -2 & 1 & 1 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 3 & 1 & 5 \\ -2 & 1 & 1 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 3 & 1 & 5 \\ 0 & 3 & 1 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This system does have a solution, since there are no contradictions produced here. In other words, since b is in the row space of A, $Ax = b$ has a solution.

(c) Let $b = (1, 6, 5)$. Does $Ax = b$ have a solution?

Appending the vector b and repeating the same steps of reduction gives

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 4 & 1 & 6 \\ -2 & 1 & 1 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 3 & 1 & 5 \\ -2 & 1 & 1 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 3 & 1 & 5 \\ 0 & 3 & 1 & 7 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

This gives us a contradiction since it claims $0 = 2$. Thus, there is no solution to $Ax = b$.

(d) Can you find multiple solutions in either part (b) or part (c)? If so, find 2.

In part (b), there are multiple solutions. For one, choose $x_3 = 2$. Then back substitution says $3x_2 + 2 = 5 \implies x_2 = 1$ and $x_1 + x_2 = 1 \implies x_1 = 0$. So one solution is

$$x = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

For another solution, choose $x_3 = -1$. Then, $3x_2 - 1 = 5 \implies x_2 = 2$. Then, $x_1 + 2 = 1 \implies x_1 = -1$. Thus, a second solution is

$$x = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}.$$

(e) Does A have an inverse? Justify your answer using results from this exercise.

A does not have an inverse because it it is a 3×3 matrix that has fewer than 3 pivots.

2. Suppose $AB = I$ and $CA = I$ where I is the $n \times n$ identity matrix.

(a) What are the dimensions of the matrices A, B and C?

First consider $AB = I$. Since the result has n rows and n columns, A must have n rows and B must have n columns. Furthermore, A must have the same number of columns as B has rows (call this m).

Now consider $CA = I$. Since the result has n rows and n columns, C must have n rows and A must have n columns. Furthermore, C must have the same number of columns as A has rows (call this p).

Combining these two arguments, we can see that A is $n \times n$, which implies that B and C are also $n \times n$.

(b) Show that $B = C$.

$$\begin{aligned} AB &= I \quad \text{Multiply on the left by } C \text{ on both sides.} \\ CAB &= CI \\ (CA)B &= C \\ IB &= C \\ B &= C \end{aligned}$$

(c) Is A invertible?

A is invertible because there is a matrix B such that $AB = I$ and $BA = I$ (since we showed in (b) that B and C are the same).

3. Let A be a square matrix with the property that $A^2 = A$. Simplify $(I - A)^2$ and $(I - A)^7$.

$$\begin{aligned}
(I - A)^2 &= (I - A)(I - A) \\
&= I - A - A + A^2 \\
&= I - 2A + A \\
&= I - A \\
(I - A)^7 &= (I - A)^2(I - A)^2(I - A)^2(I - A) \\
&= (I - A)(I - A)(I - A)(I - A) \\
&= (I - A)^2(I - A)^2 \\
&= (I - A)(I - A) \\
&= I - A
\end{aligned}$$

Where we used the result from the first half to simplify the second half.

4. (a) Write the vector $(9, 2, -5)$ as a linear combination of the vectors $(1, 2, 3)$ and $(6, 4, 2)$ or explain why it can't be done.

We use row reduction to help with this

$$\left[\begin{array}{cc|c} 1 & 6 & 9 \\ 2 & 4 & 2 \\ 3 & 2 & -5 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 3 & 2 & -5 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & -16 & -32 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

Reading this off we see we need $x_2 = 2$ and $x_1 + 12 = 9 \implies x_1 = -3$. Thus,

$$-3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ -5 \end{bmatrix}$$

(b) How many pivots does a system of equations with coefficient matrix

$$A = \begin{bmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ 3 & 2 & -5 \end{bmatrix}$$

have?

Our work in (a) showed that this matrix has two pivots.

5. Suppose A is a 6×20 matrix and B is a 20×7 matrix.

(a) What are the dimensions of $C = AB$?

Because A is 6×20 and B is 20×7, their product, C, will be 6×7.

(b) Suppose A, B, and C have been partitioned into block matrices like so:

$$A = \left[\begin{array}{c|c|c} A_{11} & A_{12} & A_{13} \\ \hline A_{21} & A_{22} & A_{23} \end{array}\right], \quad B = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \\ \hline B_{31} & B_{32} \end{array}\right], \quad C = \left[\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right],$$

Suppose that A_{11} is 2×10, B_{22} is 4×3, and C_{11} is $? \times 4$. What are the dimensions of *each* block of A, B, and C?

[Hint: Make note of every fact you know, sketch all three matrices, and fill in the unknowns step by step]

First, we know that A_{11}, A_{12}, and A_{13} can only sit next to each other if they each have 2 rows. Thus, A_{12} is $2\times?$ and A_{13} is $2\times?$. In order for A to have 6 rows in total, A_{21}, A_{22}, and A_{23} should have 4 rows. In order for A_{21} to sit below A_{11}, it must have 10 columns. Thus, A_{21} is 4×10, and A_{22} and A_{23} are both $4\times?$.

Similarly for B, B_{12} and B_{32} can only line up with B_{22} if they have 3 columns. Thus, B_{12} and B_{32} are both $? \times 3$. In order for B to have 7 columns in total, B_{11}, B_{21}, and B_{31} must each have 4 columns. B_{21} must have 4 rows in order to line up with B_{22}. Thus, B_{11} and B_{31} are $? \times 4$ and B_{21} is 4×4.

In order for the products $A_{12}B_{21}$ and $A_{22}B_{21}$ to make sense, A_{12} and A_{22} must have 4 columns. Thus, A_{12} is 2×4 and A_{22} is 4×4. In order for the product $A_{11}B_{11}$ to make sense, B_{11} must have 10 rows. In order to fit next to B_{11}, B_{12} must also have 10 rows. Thus B_{11} is 10×4 and B_{12} is 10×3.

A will have 20 columns in total only if A_{13} and A_{23} have 6 columns each. Thus A_{13} is 2×6 and A_{23} is 4×6. Similarly, B will only have 20 rows in total if B_{31} and B_{32} each have 6 rows. Thus, B_{31} is 6×4 and B_{32} is 6×3.

$C_{11} = A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31}$. The dimensions of each of these products is 2×4.
$C_{12} = A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32}$. The dimensions of each of these products is 2×3.
$C_{21} = A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31}$. The dimensions of each of these products is 4×4.
$C_{22} = A_{21}B_{12} + A_{22}B_{22} + A_{23}B_{32}$. The dimensions of each of these products is 4×3.

In summary:

$$\left[\begin{array}{c|c|c} A_{11} & A_{12} & A_{13} \\ \hline A_{21} & A_{22} & A_{23} \end{array}\right] = \left[\begin{array}{c|c|c} 2\times 10 & 2\times 4 & 2\times 6 \\ \hline 4\times 10 & 4\times 4 & 4\times 6 \end{array}\right]$$

$$\left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \\ \hline B_{31} & B_{32} \end{array}\right] = \left[\begin{array}{c|c} 10\times 4 & 10\times 3 \\ \hline 4\times 4 & 4\times 3 \\ \hline 6\times 4 & 6\times 3 \end{array}\right]$$

$$\left[\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right] = \left[\begin{array}{c|c} 2\times 4 & 2\times 3 \\ \hline 4\times 4 & 4\times 3 \end{array}\right]$$

(c) Write each block of C in terms of blocks of A and B.

We did this during our work in part (b):

$C_{11} = A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31}$.

$C_{12} = A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32}$.

$C_{21} = A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31}$.

$C_{22} = A_{21}B_{12} + A_{22}B_{22} + A_{23}B_{32}$.

6. Let A be an $m \times n$ matrix.

(a) The full $A = QR$ factorization contains more information than necessary to reconstruct A. What are the smallest matrices $\tilde{Q}$ and $\tilde{R}$ such that $\tilde{Q}\tilde{R} = A$?

If A is $m \times n$, the dimensions of $\tilde{Q}$ are $m \times n$ and the dimensions of $\tilde{R}$ are $n \times n$. In order to span the columns of A we only need as many columns in $\tilde{Q}$ as were originally in A so we can excise the columns to the right of n. $\tilde{R}$ can be trimmed since it included only zeros below n anyway.

(b) Let $\tilde{A}$ be an $m \times n$ matrix ($m > n$) whose columns each sum to zero, and let $\tilde{A} = \tilde{Q}\tilde{R}$ be the reduced QR factorization of $\tilde{A}$. The squared *Mahalanobis* distance to the point $\tilde{x}_i^T$ (the i^{th} row of $\tilde{A}$) is

$$d_i^2 = \tilde{x}_i^T \hat{S}^{-1} \tilde{x}_i$$

where $\hat{S} = \frac{1}{m-1}\tilde{A}^T\tilde{A}$ is a covariance matrix. Compute d_i^2 without inverting a matrix.

We substitute in the QR factorization of $\tilde{A}$:

$$\begin{aligned} d_i^2 &= \tilde{x}_i^T \left(\frac{1}{m-1} (\tilde{Q}\tilde{R})^T \tilde{Q}\tilde{R} \right)^{-1} \tilde{x}_i \\ &= (m-1)\tilde{x}_i^T (\tilde{R}^T(\tilde{Q}^T\tilde{Q})\tilde{R})^{-1}\tilde{x}_i \\ &= (m-1)\tilde{x}_i^T \tilde{R}^{-1}(\tilde{R}^T)^{-1}\tilde{x}_i \\ &= (m-1)\left(\tilde{R}^{-T}\tilde{x}_i\right)^T \left(\tilde{R}^{-T}\tilde{x}_i\right) \end{aligned}$$

The term in parentheses can be computed with forward substitution without computing a matrix inverse.

Alternatively, we can recognize that, since $\tilde{x}_i$ is the i^{th} row of $\tilde{A}$, we can use the row definition of matrix multiplication to recognize that $\tilde{x}_i^T = \tilde{Q}_i\tilde{R}$ where $\tilde{Q}_i$ is the i^{th} row of $\tilde{Q}$. Then,

$$\begin{aligned} d_i^2 =&(m-1)\tilde{x}_i^T \tilde{R}^{-1} \tilde{R}^{-T} \tilde{x}_i \\ =&(m-1)(\tilde{Q}_i \tilde{R}) \tilde{R}^{-1} \tilde{R}^{-T} (\tilde{Q}_i \tilde{R})^T \\ =&(m-1)\tilde{Q}_i (\tilde{R}\tilde{R}^{-1})(\tilde{R}^{-T}\tilde{R}^T)\tilde{Q}_i^T \\ =&(m-1)\tilde{Q}_i \tilde{Q}_i^T. \end{aligned}$$

Thus, d_i^2 is equal to the dot product of the i^{th} row of $\tilde{Q}$ with itself. If this were the full QR factorization, this would be one, because Q would be orthogonal. But since this is the reduced decomposition, we do not have that guarantee! This formulation also does not require any inverses to be taken.