CFRM Homework 2

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1. Let K, T, σ, and r $\in \mathbb{R}$ be positive and let

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_0^{b(x)} e^{-\frac{y^2}{2}} dy \ ,$$

where $b(x) = \frac{1}{\sigma\sqrt{T}}[log(\frac{x}{K}) + (r + \frac{\sigma^2}{2})T]$. Compute $g'(x)$.

Let $f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$ and let $F(y)$ be an antiderivative of $f(y)$. Then by the Fundamental Theorem of Calculus

$$g(x) = F(b(x)) - F(0).$$

This implies by the chain rule for derivatives that

$$g'(x) = F'(b(x))b'(x) - F'(0)(0) = f(b(x))b'(x).$$

First compute

$$b'(x) = \frac{1}{\sigma\sqrt{T}} \frac{1}{\frac{x}{K}} \frac{1}{K} = \frac{1}{x\sigma\sqrt{T}} \ .$$

Also,

$$f(b(x)) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(b(x))^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{(log(\frac{x}{K} + (r + \frac{\sigma^2}{2})T)^2}{2\sigma^2 T}} \ .$$

Therefore,

$$g'(x) = \frac{1}{\sigma x \sqrt{2\pi T}} e^{-\frac{(log(\frac{x}{K} + (r + \frac{\sigma^2}{2})T)^2}{2\sigma^2 T}} \ .$$

2. Let $\phi(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}$ so that $\Phi(x) = \int_{-\infty}^{x} \phi(u) du$ (i.e., the $\Phi(x)$ Black-Scholes).

(a) For $x > 0$ show that $\phi(x) = \phi(-x)$.

$$\begin{aligned}
\phi(-x) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{(-x)^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{((-1)(x))^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{(-1)^2(x)^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \\
&= \phi(x).
\end{aligned}$$

(b) Given that $1 = \lim_{x\to\infty} \Phi(x) = \lim_{x\to\infty} \int_{-\infty}^{x} \phi(u)du = \int_{\infty}^{\infty} \phi(u)du$, we have

$$\begin{aligned}
\Phi(-x) &= \int_{-\infty}^{-x} \phi(u)du \\
&= \int_{-\infty}^{\infty} \phi(u)du - \int_{-x}^{\infty} \phi(u)du \\
&= 1 - \int_{-x}^{\infty} \phi(u)du \\
&= 1 - \int_{x}^{-\infty} \phi(-w)(-dw) \quad (\textit{by substition} : w = -u, dw = -du) \\
&= 1 - \int_{-\infty}^{x} \phi(-w)dw \\
&= 1 - \int_{-\infty}^{x} \phi(w)dw \\
&= 1 - \Phi(x)\ .
\end{aligned}$$

3. (a) Referencing pages 239 and 240 of the textbook (Stefanica), for the bounded and convex set $D \subset \mathbb{R}^2$ and function $f : D \longrightarrow \mathbb{R}$, by Thm 8.1 (Fubini),

$$\int\int_D f(x,y)\ dA = \int\int_D f(x,y)\ dx\, dy = \int\int_D f(x,y)\ dy\, dx$$

under the condition that $f(x,y)$ is continuous.

(b) Evaluate $\int\int_D e^{y^2} dA$ for $D = \{(x,y) : 0 \le y \le 1,\ 0 \le x < y\}$. By letting $f(x,y) = e^{y^2}$ we see that f and meets the condition needed to use

Fubini's Theorem on the region D and so

$$\begin{aligned}
\int\int_D e^{y^2}\,dA &= \int_0^1 \int_0^y e^{y^2}\,dx\,dy \\
&= \int_0^1 xe^{y^2}\Big|_{x=0}^{x=y}\,dy \\
&= \int_0^1 ye^{y^2}\,dy \\
&= \int_0^1 \frac{1}{2}e^u\,du\ (by\ substituting\ u = y^2,\ du = 2y\,dy) \\
&= \frac{1}{2}e^u\Big|_0^1 = \frac{1}{2}(e-1)\,.
\end{aligned}$$

4. (a) Transform $\int\int_D e^{\frac{x+y}{x-y}}\,dA$ into an integral in u and v using the change of variables

$$u = x + y, \quad v = x - y$$

and call the domain in the uv plane S.

The suggested transformation rearranges to

$$x(u,v) = \frac{u+v}{2}, \quad y(u,v) = \frac{u-v}{2}\,.$$

Also,

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}\,.$$

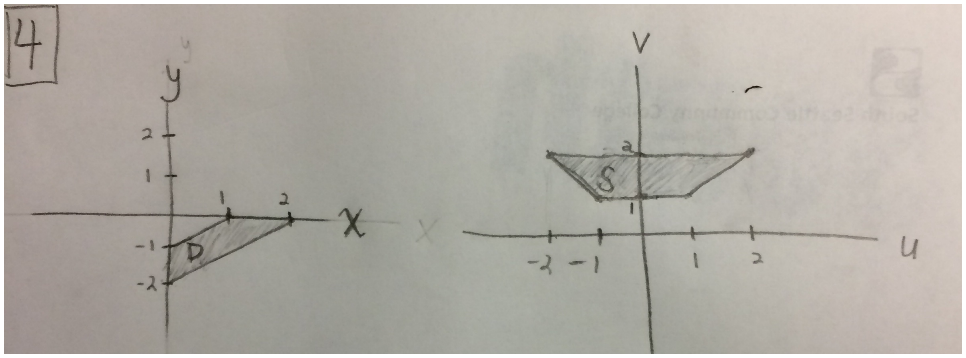
Therefore,

$$\int\int_D e^{\frac{x+y}{x-y}}\,dA = \int\int_S e^{\frac{u}{v}}|-\frac{1}{2}|\,du\,dv = \int\int_S \frac{1}{2}e^{\frac{u}{v}}\,du\,dv\,.$$

(b) Evaluating the transformation at the vertices of D yields the (u,v) points:

$$(1,1), (2,2), (-2,2), (-1,1).$$

Here are the sketches of the two regions of integration:



(c)

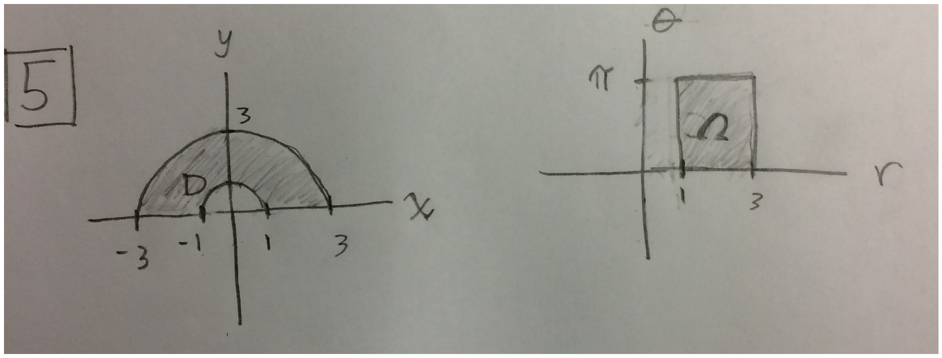
$$
\begin{aligned}
\int\int_D e^{\frac{x+y}{x-y}}\, dA &= \int\int_S \frac{1}{2} e^{\frac{u}{v}}\, du\, dv \\
&= \frac{1}{2}\int_1^2 \int_{-v}^{v} e^{\frac{u}{v}}\, du\, dv \\
&= \frac{1}{2}\int_1^2 \left(v e^{\frac{u}{v}} \right)\Big|_{-v}^{v}\, dv \\
&= \frac{1}{2}\left(e - \frac{1}{e}\right)\int_1^2 v\, dv \\
&= \frac{1}{4}\left(e - \frac{1}{e}\right) v^2\Big|_1^2 \\
&= \frac{3}{4}\left(e - \frac{1}{e}\right).
\end{aligned}
$$

5. (a) Compute $\int\int_D \sqrt{x^2+y^2}\,dx\,dy$ on $D = \{(x,y) : 1 \le x^2+y^2 \le 9, y \ge 0\}$ by changing to polar coordinates.

Let $x = rcos(\theta)\quad y = rsin(\theta)$. Then

$$
\begin{aligned}
\int\int_D \sqrt{x^2+y^2}\, dx\, dy &= \int_0^{\pi}\int_1^3 \sqrt{r^2}\, r\, dr\, d\theta \\
&= \left(\theta\Big|_0^{\pi}\right)\left(\frac{r^3}{3}\Big|_1^3\right) \\
&= \frac{26\pi}{3}.
\end{aligned}
$$

Here are the sketches of the two regions of integration:



(b) Compute

$$\int\int_D sin(\sqrt{x^2+y^2})\,dx\,dy \; on \; D = \{(x,y) : \pi^2 \leq x^2+y^2 \leq 4\pi^2\}$$

Using polar coordinates we have

$$\begin{aligned}
\int\int_D sin(\sqrt{x^2+y^2})\,dx\,dy &= \int_0^{2\pi}\int_\pi^{2\pi} r sin(r)\,dr\,d\theta \\
&= 2\pi \int_\pi^{2\pi} r sin(r)\,dr \\
&= 2\pi \left[-r cos(r)\Big|_\pi^{2\pi} + \int_{pi}^{2\pi} cos(r)\,dr \right] \\
&= 2\pi \left[-2\pi - \pi + sin(2\pi) - sin(\pi) \right] \\
&= -6\pi^2.
\end{aligned}$$

Sketches of the regions of integration were not requested for this last problem but note that the transformation of the annular region D in Cartesian coordinates becomes a square in (r,θ) coordinates (similar to the first part of problem 5 except the annulus is a full ring this time).