CFRM Homework 3

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1. The Black-Scholes price for a European put option is

$$P(S,t,K,T,r,q,\sigma) = Ke^{-r(T-t)}\Phi(-d_-) - Se^{-q(T-t)}\Phi(-d_+)$$

where

$$d_+ = \frac{log(\frac{S}{K}) + (r - q + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_- = d_+ - \sigma\sqrt{T-t} = \frac{log(\frac{S}{K}) + (r - q - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

Please note that I will use the following equality throughout

$$\Phi'(d_\pm) = \phi(d_\pm) = \frac{1}{\sqrt{2\pi}}e^{\frac{-d_\pm^2}{2}}$$

(a)

$$\begin{aligned}
\Delta(P) = \frac{\partial P}{\partial S} &= \frac{\partial}{\partial S}[Ke^{-r(T-t)}\Phi(-d_-) - Se^{-q(T-t)}\Phi(-d_+)] \\
&= Ke^{-r(T-t)}\frac{\partial}{\partial S}\Phi(-d_-) - e^{-q(T-t)}\Phi(-d_+) - Se^{-q(T-t)}\frac{\partial}{\partial S}\Phi(-d_+) \\
&= Ke^{-r(T-t)}\phi(-d_-)\frac{\partial}{\partial S}(-d_-) - e^{-q(T-t)}\Phi(-d_+) - Se^{-q(T-t)}\phi(-d_+)\frac{\partial}{\partial S}(-d_+) \\
&= Ke^{-r(T-t)}\phi(d_-)\frac{\partial}{\partial S}(-d_-) - e^{-q(T-t)}\Phi(-d_+) - Se^{-q(T-t)}\phi(d_+)\frac{\partial}{\partial S}(-d_+) \\
&= Se^{-q(T-t)}\phi(d_+)\frac{\partial}{\partial S}(-d_-) - e^{-q(T-t)}\Phi(-d_+) - Se^{-q(T-t)}\phi(d_+)\frac{\partial}{\partial S}(-d_+) \quad (**) \\
&= -e^{-q(T-t)}\Phi(-d_+) - Se^{-q(T-t)}\phi(d_+)\frac{\partial}{\partial S}(d_+ - d_-) \\
&= -e^{-q(T-t)}\Phi(-d_+) - Se^{-q(T-t)}\phi(d_+)\frac{\partial}{\partial S}(d_+ - d_+ + \sigma\sqrt{T-t}) \\
&= -e^{-q(T-t)}\Phi(-d_+)
\end{aligned}$$

(**) Lemma 3.15 was used here: $Ke^{-r(T-t)}\phi(d_-) = Se^{-q(T-t)}\phi(d_+)$

The result matches the given expression for ΔP in the text.

Using Put-Call parity we have

$$\begin{aligned}\frac{\partial P}{\partial S} &= \frac{\partial}{\partial S}\left[C - Se^{-q(T-t)} + Ke^{-r(T-t)}\right] \\ &= \frac{\partial C}{\partial S} - e^{-q(T-t)}\end{aligned}$$

Substituting our previous result leads to

$$\begin{aligned}\frac{\partial C}{\partial S} &= -e^{-q(T-t)}\Phi(-d_+) + e^{-q(T-t)} \\ &= -e^{-q(T-t)}(1 - \Phi(d_+)) + e^{-q(T-t)} \\ &= e^{-q(T-t)}\Phi(d_+)\end{aligned}$$

This result agrees with the given value for the delta of a call option in the text.

(b)

$$\begin{aligned}\Gamma(P) = \frac{\partial^2 P}{\partial S^2} &= \frac{\partial^2}{\partial S^2}[Ke^{-r(T-t)}\Phi(-d_-) - Se^{-q(T-t)}\Phi(-d_+)] \\ &= \frac{\partial}{\partial S}[-e^{-q(T-t)}\Phi(-d_+)] \quad (by\, part\, (a)) \\ &= -e^{-q(T-t)}\Phi'(-d_+)\frac{\partial}{\partial S}[-d_+] \\ &= -e^{-q(T-t)}\phi(-d_+)(-1)\frac{\partial}{\partial S}[d_+] \quad (*) \\ &= \frac{e^{-q(T-t)}\phi(d_+)}{S\sigma\sqrt{T-t}}\end{aligned}$$

(*)Where $\frac{\partial}{\partial S}[d_+]$ is given by

$$\begin{aligned}\frac{\partial}{\partial S}[d_+] &= \frac{\partial}{\partial S}\left[\frac{log(\frac{S}{K}) + (r - q + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right] \\ &= \frac{1}{\sigma\sqrt{T-t}}\frac{1}{\frac{S}{K}}\frac{1}{K} \\ &= \frac{1}{S\sigma\sqrt{T-t}}\end{aligned}$$

Using Put-Call parity

$$\begin{aligned}\Gamma(P) = \frac{\partial^2 P}{\partial S^2} &= \frac{\partial^2}{\partial S^2}[C - Se^{-q(T-t)} + Ke^{-r(T-t)}] \\ &= \frac{\partial^2 C}{\partial S^2}\,.\end{aligned}$$

The gamma of a European put option is the same as the gamma of the European call option (for the same underlying asset).

(c)

$$\begin{aligned}\Theta(P) = \frac{\partial P}{\partial t} &= \frac{\partial}{\partial t}[Ke^{-r(T-t)}\Phi(-d_-) - Se^{-q(T-t)}\Phi(-d_+)] \\ &= rKe^{-r(T-t)}\Phi(-d_-) + Ke^{-r(T-t)}\Phi'(-d_-)\frac{\partial}{\partial t}[-d_-] - qSe^{-q(T-t)}\Phi(-d_+) \\ &\qquad - Se^{-q(T-t)}\Phi'(-d_+)\frac{\partial}{\partial t}[-d_+] \\ &= rKe^{-r(T-t)}\Phi(-d_-) - Ke^{-r(T-t)}\phi(d_-)\frac{\partial}{\partial t}[d_-] - qSe^{-q(T-t)}\Phi(-d_+) \\ &\qquad + Se^{-q(T-t)}\phi(d_+)\frac{\partial}{\partial t}[d_+] \\ &= rKe^{-r(T-t)}\Phi(-d_-) - Se^{-q(T-t)}\phi(d_+)\frac{\partial}{\partial t}[d_-] - qSe^{-q(T-t)}\Phi(-d_+) \quad (**) \\ &\qquad + Se^{-q(T-t)}\phi(d_+)\frac{\partial}{\partial t}[d_+] \\ &= rKe^{-r(T-t)}\Phi(-d_-) - qSe^{-q(T-t)}\Phi(-d_+) + Se^{-q(T-t)}\phi(d_+)\frac{\partial}{\partial t}[-d_- + d_+] \\ &= rKe^{-r(T-t)}\Phi(-d_-) - qSe^{-q(T-t)}\Phi(-d_+) - \frac{\sigma Se^{-q(T-t)}\phi(d_+)}{2\sqrt{T-t}}\,.\end{aligned}$$

For this last line we used

$$\begin{aligned}\frac{\partial}{\partial t}[-d_- + d_+] &= -\frac{\partial}{\partial t}[d_-] + \frac{\partial}{\partial t}[d_+] \\ &= -\frac{\partial}{\partial t}[d_+ - \sigma\sqrt{T-t}] + \frac{\partial}{\partial t}[d_+] \\ &= \frac{\partial}{\partial t}[\sigma\sqrt{T-t} - d_+ + d_+] \\ &= -\frac{\sigma}{2\sqrt{T-t}}\,.\end{aligned}$$

(**) Lemma 3.15 was used here: $Ke^{-r(T-t)}\phi(d_-) = Se^{-q(T-t)}\phi(d_+)$

Using Put-Call parity we have

$$\begin{aligned}\Theta(P) = \frac{\partial P}{\partial t} &= \frac{\partial}{\partial t}[C - Se^{-q(T-t)} + Ke^{-r(T-t)}] \\ &= \frac{\partial C}{\partial t} - qSe^{-q(T-t)} + rKe^{-r(T-t)}\end{aligned}$$

Substituting our previous result leads to

$$\begin{aligned}\frac{\partial C}{\partial t} &= rKe^{-r(T-t)}\Phi(-d_-) - qSe^{-q(T-t)}\Phi(-d_+) - \frac{\sigma Se^{-q(T-t)}\phi(d_+)}{2\sqrt{T-t}} \\ &\quad + qSe^{-q(T-t)} - rKe^{-r(T-t)} \\ &= rKe^{-r(T-t)}(1-\Phi(d_-)) - qSe^{-q(T-t)}(1-\Phi(d_+)) - \frac{\sigma Se^{-q(T-t)}\phi(d_+)}{2\sqrt{T-t}} \\ &\quad + qSe^{-q(T-t)} - rKe^{-r(T-t)} \\ &= -rKe^{-r(T-t)}\Phi(d_-) + qSe^{-q(T-t)}\Phi(d_+) - \frac{\sigma Se^{-q(T-t)}\phi(d_+)}{2\sqrt{T-t}}\ .\end{aligned}$$

This agrees with the given value for the theta of a call option in the text.

(d)

$$\begin{aligned}\rho(P) &= \frac{\partial P}{\partial r} \\ &= \frac{\partial}{\partial r}[Ke^{-r(T-t)}\Phi(-d_-) - Se^{-q(T-t)}\Phi(-d_+)] \\ &= \frac{\partial}{\partial r}[Ke^{-r(T-t)}\Phi(-d_-) - Se^{-q(T-t)}\Phi(-d_+)] \\ &= -(T-t)Ke^{-r(T-t)}\Phi(-d_-) + Ke^{-r(T-t)}\Phi'(-d_-)\frac{\partial}{\partial r}[-d_-] - Se^{-q(T-t)}\Phi'(-d_+)\frac{\partial}{\partial r}[-d_+] \\ &= -(T-t)Ke^{-r(T-t)}\Phi(-d_-) + Ke^{-r(T-t)}\phi(-d_-)\frac{\partial}{\partial r}[-d_-] - Se^{-q(T-t)}\phi(-d_+)\frac{\partial}{\partial r}[-d_+] \\ &= -(T-t)Ke^{-r(T-t)}\Phi(-d_-) + Ke^{-r(T-t)}\phi(d_-)\frac{\partial}{\partial r}[-d_-] - Se^{-q(T-t)}\phi(d_+)\frac{\partial}{\partial r}[-d_+] \\ &= -(T-t)Ke^{-r(T-t)}\Phi(-d_-) + Se^{-q(T-t)}\phi(d_+)\frac{\partial}{\partial r}[-d_-] - Se^{-q(T-t)}\phi(d_+)\frac{\partial}{\partial r}[-d_+] \qquad (**) \\ &= -(T-t)Ke^{-r(T-t)}\Phi(-d_-) + Se^{-q(T-t)}\phi(d_+)\frac{\partial}{\partial r}[d_+ - d_-] \\ &= -(T-t)Ke^{-r(T-t)}\Phi(-d_-) + Se^{-q(T-t)}\phi(d_+)\frac{\partial}{\partial r}[d_+ - (d_+ - \sigma\sqrt{T-t})] \\ &= -(T-t)Ke^{-r(T-t)}\Phi(-d_-) + Se^{-q(T-t)}\phi(d_+)\frac{\partial}{\partial r}[\sigma\sqrt{T-t}] \\ &= -(T-t)Ke^{-r(T-t)}\Phi(-d_-)\ .\end{aligned}$$

(**) Lemma 3.15 was used here: $Ke^{-r(T-t)}\phi(d_-) = Se^{-q(T-t)}\phi(d_+)$

Using Put-Call parity we have

$$\begin{aligned}\rho(P) = \frac{\partial P}{\partial r} &= \frac{\partial}{\partial r}[C - Se^{-q(T-t)} + Ke^{-r(T-t)}] \\ &= \frac{\partial C}{\partial r} - (T-t)Ke^{-r(T-t)}\ .\end{aligned}$$

Substituting the previous result for $\frac{\partial P}{\partial r}$ leads to

$$\frac{\partial C}{\partial r} = -(T-t)Ke^{-r(T-t)}\Phi(-d_-) - (T-t)Ke^{-r(T-t)}$$

$$\frac{\partial C}{\partial r} = (T-t)Ke^{-r(T-t)}\Phi(d_+)$$

This matches the $\rho(C)$ in the text.