Solve the exercises by hand and verify your answers using Mathematica.

1. Let K, T, σ, and r be positive constants and let

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_0^{b(x)} e^{-\frac{y^2}{2}}\,dy$$

where $b(x) = \frac{1}{\sigma\sqrt{T}}\left[\log\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right]$. Compute $g'(x)$.

We will need to have the derivative of $b(x)$ for our computation:

$$b'(x) = \frac{1}{\sigma\sqrt{T}}\left(\frac{d}{dx}\left[\log\left(\frac{x}{K}\right)\right] + \frac{d}{dx}\left[\left(r + \frac{\sigma^2}{2}\right)T\right]\right) = \frac{1}{\sigma\sqrt{T}}\left(\frac{1}{x/K}\frac{1}{K} + 0\right) = \frac{1}{\sigma x\sqrt{T}}.$$

Using this, we can now show that

$$g'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{b(x)^2}{2}}b'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{b(x)^2}{2}}\frac{1}{\sigma x\sqrt{T}} = \frac{1}{\sigma x\sqrt{2\pi T}}e^{-\frac{b(x)^2}{2}}.$$

2. Let $\phi(u) = \dfrac{1}{\sqrt{2\pi}}e^{-u^2/2}$ so that $\Phi(x) = \displaystyle\int_{-\infty}^{x} \phi(u)\,du$ (i.e., the $\Phi(x)$ in Black-Scholes).

(a) For $x > 0$, show that $\phi(-x) = \phi(x)$.

This can be shown directly:

$$\phi(-x) = \frac{1}{\sqrt{2\pi}}e^{-(-x)^2/2} = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} = \phi(x)$$

(b) Given that $\lim\limits_{x\to\infty} \Phi(x) = 1$, use the properties of the integral as well as a substitution to show that $\Phi(-x) = 1 - \Phi(x)$ (again, assuming $x > 0$).

Since $\lim_{x\to\infty}\Phi(x) = 1$, we can say $\int_{-\infty}^{\infty}\phi(u)\,du = 1$. Consider $\Phi(-x)$:

$$\Phi(-x) = \int_{-\infty}^{-x}\phi(u)\,du$$

Make the substitution $s = -u$ such that $\frac{ds}{du} = -1$ and $du = -ds$. As $u \to -\infty$, $s \to \infty$ so the lower limit becomes ∞. When $u = -x$, $s = x$, so the upper limit becomes x:

$$\Phi(-x) = \int_{-\infty}^{-x} \phi(u)\, du = -\int_{\infty}^{x} \phi(-s)\, ds$$

By the rules of integration, we can switch the limits and this introduces a minus sign. This minus sign cancels the one out front:

$$\Phi(-x) = -\int_{\infty}^{x} \phi(-s)\, ds = \int_{x}^{\infty} \phi(-s)\, ds$$

From part (a), we know $\phi(-s) = \phi(s)$. Also, we know from the rules of improper integrals that

$$\begin{aligned} \int_{-\infty}^{x} f(s)\, ds + \int_{x}^{\infty} f(s)\, ds &= \int_{-\infty}^{\infty} f(s)\, ds \\ \implies \int_{x}^{\infty} f(s)\, ds &= \int_{-\infty}^{\infty} f(s)\, ds - \int_{-\infty}^{x} f(s)\, ds \end{aligned}$$

Combining these two facts plus the fact about the integral from $-\infty$ to ∞:

$$\Phi(-x) = \int_{x}^{\infty} \phi(-s)\, ds = \int_{x}^{\infty} \phi(s)\, ds = \int_{-\infty}^{\infty} \phi(s)\, ds - \int_{-\infty}^{x} \phi(s)\, ds = 1 - \Phi(x)$$

3. (a) Under what condition does the following hold?

$$\iint_D f(x,y)\, dA = \iint_D f(x,y)\, dy\, dx = \iint_D f(x,y)\, dx\, dy$$

This holds when $f(x, y)$ is continuous on the domain D by Fubini's theorem.

(b) Evaluate the double integral

$$\iint_D e^{y^2}\, dA$$

where $D = \{(x, y) : 0 \le y \le 1,\ \ 0 \le x \le y\}$.

We can integrate this by substitution using

$$u = y^2, \quad \frac{du}{dy} = 2y \implies dy = \frac{du}{2y}, \quad u(0) = 0, \quad u(1) = 1$$

This gives us

$$\int\int_D e^{y^2}\, dA = \int_0^1 y e^u \frac{du}{2y} = \frac{1}{2}\int_0^1 e^u\, du = \frac{1}{2}(e-1)$$

4. (a) Transform the double integral

$$\iint_D e^{\frac{x+y}{x-y}}\, dA$$

into an integral of u and v using the change of variables

$$u = x + y \qquad\qquad v = x - y$$

and call the domain in the uv plane S.

We first solve for x and y so we can compute the Jacobian:

$$\begin{aligned} u = x + y \implies x = u - y, \quad v = x - y &\implies x = v + y \\ u - y = v + y &\implies y = \frac{u-v}{2} \\ x = u - y = u - \frac{u-v}{2} &= \frac{u+v}{2} \end{aligned}$$

Computing partial derivatives:

$$\frac{\partial x}{\partial u} = \frac{1}{2}, \quad \frac{\partial x}{\partial v} = \frac{1}{2}, \quad \frac{\partial y}{\partial u} = \frac{1}{2}, \quad \frac{\partial y}{\partial v} = -\frac{1}{2}$$

$$J = \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right| = \left|\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\right| = \frac{1}{2}$$

Thus, the transformed integral becomes:

$$\int\int_D e^{\frac{x+y}{x-y}} dA = \int\int_S e^{\frac{u}{v}} \left(\frac{1}{2}\right) dudv$$

(b) Let D be the trapezoidal region with vertices $(1, 0)$, $(2, 0)$, $(0, -2)$ and $(0, -1)$. Find the corresponding region S in the uv plane by evaluating the transformation at the vertices of D and connecting the dots. Sketch both regions.

The transformed region is also a trapezoid. It has vertices $(1, 1)$, $(2, 2)$, $(-2, 2)$, and $(-1, 1)$. The sketches are not included, but one must simply plot the four vertices and connect them.

(c) Compute the integral found in part (a) over the domain S from part (b).

In our new trapezoid, v varies from 1 to 2. The right side of the trapezoid is the line $u = v$ and the left side of the trapezoid is the line $u = -v$. Plugging these in and integrating u first, we find:

$$\begin{aligned}
\int\int_D e^{\frac{x+y}{x-y}} dA &= \frac{1}{2}\int_1^2 \int_{-v}^{v} e^{\frac{u}{v}} du dv \\
&= \frac{1}{2}\int_1^2 ve^{\frac{u}{v}}\Big|_{u=-v}^{u=v} dv \\
&= \frac{1}{2}\int_1^2 v(e-e^{-1})dv \\
&= \frac{e-e^{-1}}{2}\frac{1}{2}v^2\Big|_{v=1}^{v=2} \\
&= \frac{e-e^{-1}}{4}(4-1) = \frac{3}{4}\left(e-\frac{1}{e}\right)
\end{aligned}$$

5. (a) Let $D = \{(x,y) : 1 \leq x^2+y^2 \leq 9,\ y \geq 0\}$. Compute the integral

$$\iint_D \sqrt{x^2+y^2}\, dx\, dy$$

by changing to polar coordinates. Sketch the domains of integration in both the xy and $r\theta$ (that means r on one axis and θ on the other) planes.

In the xy plane, the region of integration is the upper half of a donut centered on the origin with an inner radius of 1 and an outer radius of 3. In the $r\theta$ plane, the region of integration is the rectangle $[1,3]\times[0,\pi]$.

$$\begin{aligned}
\int\int_D \sqrt{x^2+y^2} dxdy &= \int_0^\pi \int_1^3 \sqrt{r^2\cos^2(\theta)+r^2\sin^2(\theta)}r\, dr d\theta \\
&= \int_0^\pi \int_1^3 \sqrt{r^2(\sin^2(\theta)+\cos^2(\theta))}r\, dr d\theta \\
&= \int_0^\pi \int_1^3 r^2 dr d\theta = \int_0^\pi \frac{1}{3}r^3\Big|_{r=1}^{r=3} d\theta \\
&= \frac{26}{3}\int_0^\pi 1 d\theta = \frac{26}{3}\left(\theta\Big|_{\theta=0}^{\theta=\pi}\right) = \frac{26\pi}{3}
\end{aligned}$$

(b) Compute the integral

$$\iint_D \sin(\sqrt{x^2+y^2})\, dx\, dy$$

where $D = \{(x,y) : \pi^2 \le x^2 + y^2 \le 4\pi^2\}$.

$$\begin{aligned}
\int\int_D \sin(\sqrt{x^2+y^2})dxdy &= \int_0^{2\pi}\int_\pi^{2\pi} \sin\left(\sqrt{(r^2\cos^2(\theta) + r^2\sin^2(\theta)}\right) rdrd\theta \\
&= r\sin\left(\sqrt{r^2(\sin^2(\theta)+\cos^2(\theta))}\right) drd\theta \\
&= \int_0^{2\pi}\int_\pi^{2\pi} r\sin(r)drd\theta
\end{aligned}$$

We can use integration by parts for the inner integral:

$$G(r) = r \implies g(r) = 1, \;\; f(r) = \sin(r) \implies F(r) = -\cos(r)$$

Plugging this in gives us

$$\begin{aligned}
\int\int_D \sin(\sqrt{x^2+y^2})dxdy &= \int_0^{2\pi} \left[-r\cos(r)|_{r=\pi}^{r=2\pi} + \int_\pi^{2\pi} \cos(r)dr \right] \\
&= \int_0^{2\pi} \left[-2\pi(1) + \pi(-1) + \sin(r)|_{r=\pi}^{r=2\pi} \right] d\theta \\
&= \int_0^{2\pi} -3\pi d\theta \\
&= -3\pi\,\theta|_{\theta=0}^{\theta=2\pi} = -6\pi^2.
\end{aligned}$$