CFRM Homework 1

Dane Johnson

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1. Compute the following limits.

(a)

$$\lim_{h\to 0} \frac{4(x+h-3)^2 - 4(x-3)^2}{h}$$

$$= 4\lim_{h\to 0} \frac{(x+h)^2 - 6(x+h) + x - x^2 + 6x - 9}{h}$$

$$= 4\lim_{h\to 0} \frac{x^2 + 2hx + h^2 - 6x - 6h + x - x^2 + 6x - 9}{h}$$

$$= 4\lim_{h\to 0} \frac{2hx + h^2 - 6h}{h}$$

$$= 4\lim_{h\to 0} 2x + h - 6$$

$$= 8x - 24.$$

(b)

$$\lim_{x\to\infty} \frac{1}{\sqrt{4x^2 - 2x - 10} + 2x}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{4x^4}}{\frac{1}{4x^4}(\sqrt{4x^2 - 2x - 10} + 2x)}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{4x^4}}{\sqrt{4 - \frac{2}{x} - \frac{10}{x^2}} + \frac{2}{x^3}}$$

$$= \frac{\lim_{x\to\infty} \frac{1}{4x^4}}{\lim_{x\to\infty} \sqrt{4 - \frac{2}{x} - \frac{10}{x^2}} + \frac{2}{x^3}}$$

$$= \frac{0}{\sqrt{4 - 0 - 0} + 0}$$

$$= \frac{0}{4} = 0.$$

2. Compute the derivatives of the following functions.

(a) $f(x) = \frac{1}{1-x}$

$$f'(x) = -(1-x)^{-2}(-1)$$
$$= \frac{1}{(1-x)^2}\,.$$

(b) $f(x) = \sum\limits_{n=1}^{7} k e^{-a_k x^3}, \quad (\{a_k\}\ constant)$

$$f'(x) = \sum_{n=1}^{7} k e^{-a_k x^3}(-3a_k x^2)$$

$$= \sum_{n=1}^{7} -3a_k k x^2 e^{-a_k x^3}.$$

(c) $f(x) = \frac{\log(\frac{x}{K})+(r-q+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, (K, r, q, \sigma > 0\ and\, T > t\ constant).$

$$[\,Rewrite : f(x) = \frac{\log(\frac{x}{K})}{\sigma\sqrt{T-t}} + \frac{(r-q+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\,]$$

$$f'(x) = \frac{1}{\sigma\sqrt{T-t}}\frac{1}{\frac{x}{K}}\frac{1}{K} + 0$$
$$= \frac{1}{x\sigma\sqrt{T-t}}\,.$$

(d) $f(x) = \frac{\log(\frac{S}{K})+(r-q+\frac{x^2}{2})(T-t)}{x\sqrt{T-t}} \quad (S > 0, K > 0, r, q and T > t constant).$

$$[\,Rewrite : f(x) = \frac{\log(\frac{S}{K})}{\sqrt{T-t}}\frac{1}{x} + \frac{T-t}{\sqrt{T-t}}\frac{r-q-\frac{x^2}{2}}{x}\,]$$

$$f'(x) = \frac{-log(\frac{S}{K})}{\sqrt{T-t}}\frac{1}{x^2} + \frac{T-t}{\sqrt{T-t}}\frac{x^2-(r-q+\frac{x^2}{2})}{x^2}$$
$$= \frac{-log(\frac{S}{K}) + (T-t)(\frac{x^2}{2} - r + q)}{x^2\sqrt{T-t}}\,.$$

(e) $f(x) = \frac{\log(\frac{S}{K})+(x-q+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$ $(S > 0, K > 0, \sigma, q > 0 \; and \; T > t \; constant)$.

$$[\, Rewrite : f(x) = \frac{log(\frac{S}{K})}{\sigma\sqrt{T-t}} + \frac{(x-q+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \,]$$

$$f'(x) = 0 + \frac{T-t}{\sigma\sqrt{T-t}} = \frac{T-t}{\sigma\sqrt{T-t}} \, .$$

3.

(a) By inspection we can see that 2(c) corresponds to $\frac{\partial}{\partial S}[d_+(\cdot)]$, 2(d) corresponds to $\frac{\partial}{\partial \sigma}[d_+(\cdot)]$, and 2(e) corresponds to $\frac{\partial}{\partial r}[d_+(\cdot)]$.

(b)

$$\frac{\partial}{\partial t}[d_+(\cdot)] = \frac{\partial}{\partial t}\left[\frac{log(\frac{S}{K})}{\sigma\sqrt{T-t}}\right] + \frac{\partial}{\partial t}\left[\frac{r-q+\frac{\sigma^2}{2}}{\sigma}\sqrt{T-t}\right]$$

$$= -\frac{1}{2}\frac{log(\frac{S}{K})}{\sigma(T-t)^{\frac{3}{2}}}(-1) + \frac{1}{2}\frac{r-q+\frac{\sigma^2}{2}}{\sigma}\frac{1}{\sqrt{T-t}}(-1)$$

$$= \frac{log(\frac{S}{K}) - (r-q+\frac{\sigma^2}{2})(T-t)}{2\sigma(T-t)^{\frac{3}{2}}} \, .$$

4. Compute the following indefinite integrals.

(a) $\int x^2 log(x)\, dx$

$$= \frac{x^3}{3}log(x) - \int \frac{x^3}{3}\frac{1}{x}\, dx$$

$$= \frac{x^3}{3}log(x) - \int \frac{x^2}{3}\, dx$$

$$= \frac{x^3 log(x)}{3} - \frac{1}{9}x^3 + c.$$

(b) $\int x^2 e^x\, dx$

$$= x^2e^x - \int 2xe^x\, dx$$

$$= x^2e^x - 2(xe^x - \int e^x\, dx)$$

$$= x^2e^x - 2xe^x + 2e^x + c$$

$$= e^x(x^2 - 2x + 2) + c.$$

(c) $\int [log(x)]^2\, dx = \int log(x)log(x)\, dx$

$$= (xlog(x) - x)log(x) - \int (xlog(x) - x)\frac{1}{x}\, dx$$

$$= xlog(x)(log(x) - 1) - \int (log(x) - 1)\, dx$$

$$xlog(x)(log(x) - 1) - x(log(x) - 1) + x + c$$

$$= xlog^2(x) - xlog(x) - xlog(x) + x + x + c$$

$$= x(log^2(x) - 2log(x) + 2) + c.$$

5. Evaluate the following definite integrals.

(a) $\int_4^7 x^2 log(x)\, dx$

$$= \left(\frac{x^3 log(x)}{3} - \frac{1}{9}x^3 \right) \Big|_4^7$$

$$= \frac{343log(7)}{3} - \frac{343}{9} - \frac{64log(4)}{3} + \frac{64}{9}$$

$$= \frac{343log(7) - 64log(4)}{3} - 31 \approx 161.91.$$

(b) $\int_0^2 \frac{1}{(1+x)^2}\, dx$

$$= -\frac{1}{1+x}\Big|_0^2$$

$$= -\frac{1}{3} + \frac{1}{1} = \frac{2}{3}\,.$$