1. Compute the following limits.

   (a)

$$\begin{aligned}
\lim_{h\to 0}\frac{4(x+h-3)^2-4(x-3)^2}{h} &= \lim_{h\to 0}\frac{4x^2+8xh-24x+4h^2-24h+36-4x^2+24x-36}{h}\\
&= \lim_{h\to 0}\frac{8xh+4h^2-24h}{h}\\
&= \lim_{h\to 0} 8x-24+4h\\
&= 8x-24
\end{aligned}$$

   (b)

$$\lim_{x\to\infty}\frac{1}{\sqrt{4x^2-2x-10}+2x}=0$$

   because the denominator grows to infinity.

2. Compute the derivatives of the following functions.

   (a)

$$\begin{aligned}
f(x) &= \frac{1}{1-x}\\
f'(x) &= -1(1-x)^{-2}(-1) \quad \text{Chain rule and power rule}\\
&= \frac{1}{(1-x)^2}
\end{aligned}$$

   (b)

$$\begin{aligned}
f(x) &= \sum_{k=1}^{7} ke^{-a_k x^3}\\
f'(x) &= \frac{d}{dx}\sum_{k=1}^{7} ke^{-a_k x^3}\\
&= \sum_{k=1}^{7}\frac{d}{dx}ke^{-a_k x^3} \quad \text{Linearity}\\
&= \sum_{k=1}^{7} -3x^2ka_ke^{-a_k x^3} \quad \text{Exponential, chain rule, power rule}
\end{aligned}$$

(c)

$$
\begin{aligned}
f(x) &= \frac{\log\left(\frac{x}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \\
f'(x) &= \frac{d}{dx}\frac{\log\left(\frac{x}{K}\right)}{\sigma\sqrt{T-t}} + \frac{d}{dx}\frac{\left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \quad \text{Linearity} \\
&= \frac{1/K}{\sigma\left(\frac{x}{K}\right)\sqrt{T-t}} + 0 \quad \text{Logarithm rule, chain rule} \\
&= \frac{1}{\sigma x\sqrt{T-t}}
\end{aligned}
$$

(d)

$$
\begin{aligned}
f(x) &= \frac{\log\left(\frac{S}{K}\right) + \left(r - q + \frac{x^2}{2}\right)(T-t)}{x\sqrt{T-t}} \\
f'(x) &= \frac{d}{dx}\left(\frac{\log\left(\frac{S}{K}\right) + (r-q)(T-t)}{x\sqrt{T-t}}\right) + \frac{d}{dx}\left(\frac{\frac{x^2}{2}(T-t)}{x\sqrt{T-t}}\right) \quad \text{Linearity} \\
&= \frac{\log\left(\frac{S}{K}\right) + (r-q)(T-t)}{\sqrt{T-t}}\frac{d}{dx}(x^{-1}) + \frac{T-t}{2\sqrt{T-t}}\frac{d}{dx}(x) \\
&= -\frac{\log\left(\frac{S}{K}\right) + (r-q)(T-t)}{x^2\sqrt{T-t}} + \frac{1}{2}\sqrt{T-t}
\end{aligned}
$$

(e)

$$
\begin{aligned}
f(x) &= \frac{\log\left(\frac{S}{K}\right) + \left(x - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \\
f'(x) &= \frac{d}{dx}\left(\frac{\log\left(\frac{S}{K}\right) + (\frac{\sigma^2}{2} - q)(t-T)}{\sigma\sqrt{T-t}}\right) + \frac{d}{dx}\left(\frac{x(T-t)}{\sigma\sqrt{T-t}}\right) \quad \text{Linearity} \\
&= 0 + \frac{T-t}{\sigma\sqrt{T-t}}\frac{d}{dx}(x) \\
&= \frac{\sqrt{T-t}}{\sigma}
\end{aligned}
$$

3. Recall that

$$
d_+(\cdot) = \frac{\log\left(\frac{S}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}
$$

(a) Parts (c), (d), and (e) of Problem 2 correspond to partial derivatives of d_+. What partial derivative does each correspond to?

Part (c) is $\frac{\partial d_+}{\partial S}$. Part (d) is $\frac{\partial d_+}{\partial \sigma}$. Part (d) is $\frac{\partial d_+}{\partial r}$.

(b) Compute the partial derivative of d_+ with respect to t.

$$\begin{aligned}
d_+(\cdot) &= \frac{\log\left(\frac{S}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \\
&= \frac{\log\left(\frac{S}{K}\right)}{\sigma}(T-t)^{-1/2} + \frac{r-q+\frac{\sigma^2}{2}}{\sigma}(T-t)^{1/2} \\
\frac{\partial d_+}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{\log\left(\frac{S}{K}\right)}{\sigma}(T-t)^{-1/2}\right) + \frac{\partial}{\partial t}\left(\frac{r-q+\frac{\sigma^2}{2}}{\sigma}(T-t)^{1/2}\right) \quad \text{Linearity} \\
&= \frac{\log\left(\frac{S}{K}\right)}{2\sigma}(T-t)^{-3/2} - \frac{r-q+\frac{\sigma^2}{2}}{2\sigma}(T-t)^{-1/2} \quad \text{Chain rule} \\
&= \frac{\log\left(\frac{S}{K}\right) - \left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{2\sigma(T-t)^{3/2}}
\end{aligned}$$

4. Compute the following antiderivatives.

(a) $\displaystyle\int x^2 \log(x)\, dx$

Integrate by parts with $F(x) = \log(x)$ and $g(x) = x^2$. Note: as usual, $\log(x)$ is assumed to be the natural logarithm.

$$F(x) = \log(x) \implies f(x) = F'(x) = \frac{1}{x}$$
$$g(x) = x^2 \implies G(x) = \int x^2 dx = \frac{x^3}{3}$$

Using these results, the antiderivative is:

$$\begin{aligned}
\int x^2 \log(x)\, dx &= F(x)G(x) - \int f(x)G(x)dx \\
&= \frac{x^3\log(x)}{3} - \int \frac{x^3}{3x}dx \\
&= \frac{x^3\log(x)}{3} - \frac{1}{3}\int x^2 dx \\
&= \frac{x^3\log(x)}{3} - \frac{x^3}{9} + C
\end{aligned}$$

(b) $\displaystyle\int x^2 e^x\, dx$

Integrate by parts with $F(x) = x^2$ and $g(x) = e^x$.

$$F(x) = x^2 \implies f(x) = F'(x) = 2x$$
$$g(x) = e^x \implies G(x) = \int e^x dx = e^x$$

This gives us

$$\int x^2 e^x \, dx \quad = \quad x^2 e^x - \int 2xe^x dx$$

Integrate the second integral by parts with $F(x) = 2x$ and $g(x) = e^x$.

$$F(x) = 2x \implies f(x) = F'(x) = 2$$
$$g(x) = e^x \implies G(x) = \int e^x dx = e^x$$

This gives as the final result:

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - \int 2xe^x dx \\ &= x^2 e^x - \left[2xe^x - \int 2e^x \right] \\ &= x^2 e^x - 2xe^x + 2e^x + C \end{aligned}$$

(c) $\int \left[\log(x) \right]^2 dx$

Integrate by parts with $F(x) = \log(x)^2$ and $g(x) = 1$.

$$F(x) = \log(x)^2 \implies f(x) = F'(x) = \frac{2\log(x)}{x}$$
$$g(x) = 1 \implies G(x) = \int 1 \, dx = x$$

This gives us:

$$\int \left[\log(x) \right]^2 dx \quad = \quad x\log(x)^2 - \int 2\log(x) dx$$

Integrate the second integral by parts with $F(x) = \log(x)$ and $g(x) = 2$.

$$F(x) = \log(x) \implies f(x) = F'(x) = \frac{1}{x}$$
$$g(x) = 2 \implies G(x) = \int 2 \, dx = 2x$$

This gives us:

$$\begin{aligned} \int \left[\log(x) \right]^2 dx &= x\log(x)^2 - \int 2\log(x) dx \\ &= x\log(x)^2 - \left[2x\log(x) - \int \frac{2x}{x} dx \right] \\ &= x\log(x)^2 - 2x\log(x) + 2x + C \end{aligned}$$

5. Evaluate the following definite integrals.

(a) $\int_4^7 x^2 \log(x)\, dx$

We use the antiderivative from 4(a):

$$\begin{aligned}\int_4^7 x^2 \log(x)\, dx &= \left(\frac{(7)^3}{3}\log(7) - \frac{(7)^3}{9}\right) - \left(\frac{(4)^3}{3}\log(4) - \frac{(4)^3}{9}\right)\\ &= \frac{343}{3}\log(7) - \frac{343}{9} - \frac{64}{3}\log(4) + \frac{64}{9}\\ &= \frac{343}{3}\log(7) - \frac{64}{3}\log(4) - 31.\end{aligned}$$

(b) $\int_0^2 \frac{1}{(1+x)^2}\, dx$

We use u-substitution with $u = 1 + x$. This means that $du = dx$ and $u(0) = 1$ while $u(2) = 3$. Thus:

$$\int_0^2 \frac{1}{(1+x)^2} dx = \int_1^3 \frac{1}{u^2} du = -\frac{1}{u}\bigg|_1^3 = -\frac{1}{3} + \frac{1}{1} = \frac{2}{3}.$$