1. The Black-Scholes price for a European put option is

$$P(S,t,K,T,r,q,\sigma) = Ke^{-r(T-t)}\Phi(-d_-) - Se^{-q(T-t)}\Phi(-d_+) \tag{1}$$

where

$$d_+ = \frac{\log\left(\frac{S}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_- = d_+ - \sigma\sqrt{T-t} = \frac{\log\left(\frac{S}{K}\right) + \left(r - q - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

Compute each of

(a) $\Delta(P) = \dfrac{\partial P}{\partial S}$

(b) $\Gamma(P) = \dfrac{\partial^2 P}{\partial S^2}$

(c) $\theta(P) = \dfrac{\partial P}{\partial t}$

(d) $\rho(P) = \dfrac{\partial P}{\partial r}$

by taking derivatives of (1). Verify that your answers are correct using put-call parity.

You can find expressions for *The Greeks* on pages 92 and 93 of the Stefanica text. Verify that your answer matches the expression for the put option and that put-call parity gives the expression for the call option. And as always, verify your calculations with Mathematica.

Example

Compute the vega of a European put option.

$$\begin{aligned}
\text{vega}(P) = \frac{\partial P}{\partial \sigma} &= \frac{\partial}{\partial \sigma}\left[Ke^{-r(T-t)}\Phi(-d_-) - Se^{-q(T-t)}\Phi(-d_+)\right] \\
&= Ke^{-r(T-t)}\phi(-d_-)\frac{\partial}{\partial \sigma}(-d_-) - Se^{-q(T-t)}\phi(-d_+)\frac{\partial}{\partial \sigma}(-d_+) \\
&= Ke^{-r(T-t)}\phi(d_-)\frac{\partial}{\partial \sigma}(-d_-) - Se^{-q(T-t)}\phi(d_+)\frac{\partial}{\partial \sigma}(-d_+)
\end{aligned}$$

Lemma 3.15 states that $Ke^{-r(T-t)}\phi(d_-) = Se^{-q(T-t)}\phi(d_+)$, thus

$$\begin{aligned}
\text{vega}(P) = \frac{\partial P}{\partial \sigma} &= Se^{-q(T-t)}\phi(d_+)\frac{\partial}{\partial \sigma}(-d_-) - Se^{-q(T-t)}\phi(d_+)\frac{\partial}{\partial \sigma}(-d_+) \\
&= Se^{-q(T-t)}\phi(d_+)\left[\frac{\partial}{\partial \sigma}(-d_-) - \frac{\partial}{\partial \sigma}(-d_+)\right] \\
&= Se^{-q(T-t)}\phi(d_+)\left[\frac{\partial}{\partial \sigma}(d_+) - \frac{\partial}{\partial \sigma}(d_-)\right] \\
&= Se^{-q(T-t)}\phi(d_+)\ \frac{\partial}{\partial \sigma}\left[d_+ - d_-\right]
\end{aligned}$$

But $d_- = d_+ - \sigma\sqrt{T-t} \implies d_+ - d_- = \sigma\sqrt{T-t}$, thus

$$\begin{aligned}
\text{vega}(P) = \frac{\partial P}{\partial \sigma} &= Se^{-q(T-t)}\phi(d_+)\ \frac{\partial}{\partial \sigma}\left[\sigma\sqrt{T-t}\right] \\
\text{vega}(P) = \frac{\partial P}{\partial \sigma} &= Se^{-q(T-t)}\phi(d_+)\ \sqrt{T-t}
\end{aligned}$$

Check the result using put-call parity:

$$\begin{aligned}
P &= C - Se^{-q(T-t)} + Ke^{-r(T-t)} \\
\text{vega}(P) = \frac{\partial P}{\partial \sigma} &= \frac{\partial}{\partial \sigma}\left[C - Se^{-q(T-t)} + Ke^{-r(T-t)}\right] \\
&= \frac{\partial C}{\partial \sigma}
\end{aligned}$$

The vega of a European put option is the same as the vega for a European call option.

$$\Delta(P) = \frac{\partial P}{\partial S} = -Ke^{-r(T-t)}\phi(-d_-)\frac{\partial}{\partial S}(d_-) - e^{-q(T-t)}\Phi(-d_+) + Se^{-q(T-t)}\phi(-d_+)\frac{\partial}{\partial S}(d_+)$$

Now we use the proven fact that $\phi(-x) = \phi(x)$. We also prove the useful equality we found in class:

$$\begin{aligned}
Ke^{-r(T-t)}\phi(-d_-) &= Ke^{-r(T-t)}\phi(d_-) = Ke^{-r(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_-^2}{2}} \\
&= Ke^{-r(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{(d_+-\sigma\sqrt{T-t})^2}{2}} = Ke^{-r(T-t)}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{d_+^2}{2}}\right)e^{d_+\sigma\sqrt{T-t}}e^{-\frac{\sigma^2(T-t)}{2}} \\
&= Ke^{-r(T-t)}\phi(d_+)e^{\left[\log\left(\frac{S}{K}\right)+\left(r-q+\frac{\sigma^2}{2}\right)(T-t)\right]}e^{-\frac{\sigma^2(T-t)}{2}} \\
&= K\left(\frac{S}{K}\right)e^{-r(T-t)}e^{r(T-t)}\phi(-d_+)e^{-q(T-t)} = Se^{-q(T-t)}\phi(-d_+)
\end{aligned}$$

Making this substitution gives us:

$$\begin{aligned}
\Delta(P) &= -e^{-q(T-t)}\Phi(-d_+) + Se^{-q(T-t)}\phi(-d_+)\left[\frac{\partial d_+}{\partial S} - \frac{\partial d_-}{\partial S}\right] \\
&= -e^{-q(T-t)}\Phi(-d_+) + Se^{-q(T-t)}\phi(-d_+)\frac{\partial}{\partial S}\left[d_+ - d_-\right] \\
&= -e^{-q(T-t)}[1 - \Phi(d_+)] + Se^{-q(T-t)}\phi(-d_+)\frac{\partial}{\partial S}[\sigma\sqrt{T-t}] \\
&= e^{-q(T-t)}\Phi(d_+) - e^{-q(T-t)}
\end{aligned}$$

Checking using Put-Call parity:

$$\begin{aligned}
P &= C - Se^{-q(T-t)} + Ke^{-r(T-t)} \\
\Delta(P) &= \Delta(C) - e^{-q(T-t)} \\
\Delta(P) &= e^{-q(T-t)}\Phi(d_+) - e^{-q(T-t)}
\end{aligned}$$

We now take the derivative again to find $\Gamma(P)$:

$$\Gamma(P) = \frac{\partial}{\partial S}(\Delta(P)) = \frac{\partial}{\partial S}\left(e^{-q(T-t)}\Phi(d_+) - e^{-q(T-t)}\right) = e^{-q(T-t)}\phi(d_+)\frac{\partial}{\partial S}(d_+)$$

We can find the partial derivative we need easily:

$$\frac{\partial d_+}{\partial S} = \frac{\partial}{\partial S}\left(\frac{\log\left(\frac{S}{K}\right)}{\sigma\sqrt{T-t}} + \frac{\left(r-q-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) = \frac{1}{\sigma\left(\frac{S}{K}\right)\sqrt{T-t}}\left(\frac{1}{K}\right) = \frac{1}{\sigma S\sqrt{T-t}}$$

Plugging this in, we find,

$$\Gamma(P) = \frac{e^{-q(T-t)}\phi(d_+)}{\sigma S\sqrt{T-t}}$$

Checking with Put-Call parity:

$$\Gamma(P) = \frac{\partial}{\partial S}\Delta(P) \quad = \quad \frac{\partial}{\partial S}\Delta(C) - \frac{\partial}{\partial S}e^{-q(T-t)} = \Gamma(C) = \frac{e^{-q(T-t)}}{S\sigma\sqrt{T-t}}\phi(d_+)$$

We continue now onto $\theta(P)$:

$$\begin{aligned}
\theta(P) &= \frac{\partial P}{\partial t} = \frac{\partial}{\partial t}\left(Ke^{-r(T-t)}\Phi(-d_-)\right) - \frac{\partial}{\partial t}\left(Se^{-q(T-t)}\Phi(-d_+)\right) \\
&= rKe^{-r(T-t)}\Phi(-d_-) - Ke^{-r(T-t)}\phi(-d_-)\frac{\partial d_-}{\partial t} - qSe^{-q(T-t)}\Phi(-d_+) + Se^{-q(T-t)}\phi(-d_+)\frac{\partial d_+}{\partial t} \\
&= rKe^{-r(T-t)}\Phi(-d_-) - qSe^{-q(T-t)}\Phi(-d_+) + Se^{-q(T-t)}\frac{\partial}{\partial t}\phi(-d_+)[d_+ - d_-] \\
&= rKe^{-r(T-t)}(1-\Phi(d_-)) - qSe^{-q(T-t)}(1-\Phi(d_+)) + Se^{-q(T-t)}\phi(d_+)\frac{\partial}{\partial t}(\sigma\sqrt{T-t}) \\
&= rKe^{-r(T-t)}(1-\Phi(d_-)) - qSe^{-q(T-t)}(1-\Phi(d_+)) - \frac{S\sigma e^{-q(T-t)}}{2\sqrt{T-t}}\phi(d_+)
\end{aligned}$$

Checking using Put-Call parity:

$$\begin{aligned}
\theta(P) &= \theta(C) - \frac{\partial}{\partial t}\left(Se^{-q(T-t)}\right) + \frac{\partial}{\partial t}\left(Ke^{-r(T-t)}\right) \\
&= -\frac{\sigma Se^{-q(T-t)}}{2\sqrt{T-t}}\phi(d_+) + qSe^{-q(T-t)}\Phi(d_+) - rKe^{-r(T-t)}\Phi(d_-) \\
&\quad -qSe^{-q(T-t)} + rKe^{-r(T-t)} \\
&= -\frac{\sigma Se^{-q(T-t)}}{2\sqrt{T-t}}\phi(d_+) + rKe^{-r(T-t)}(1-\Phi(d_-)) - qSe^{-q(T-t)}(1-\Phi(d_+))
\end{aligned}$$

Finally, we compute $\rho(P)$:

$$\begin{aligned}
\rho(P) &= \frac{\partial P}{\partial r} = \frac{\partial}{\partial r}\left(Ke^{-r(T-t)}\Phi(-d_-)\right) - \frac{\partial}{\partial r}\left(Se^{-q(T-t)}\Phi(-d_+)\right) \\
&= -(T-t)Ke^{-r(T-t)}\Phi(-d_-) - Ke^{-r(T-t)}\phi(-d_-)\frac{\partial d_-}{\partial r} + Se^{-q(T-t)}\phi(-d_+)\frac{\partial d_+}{\partial r} \\
&= -(T-t)Ke^{-r(T-t)}(1-\Phi(d_-)) + Se^{-q(T-t)}\phi(-d_+)\frac{\partial}{\partial r}(d_+ - d_-) \\
&= (T-t)Ke^{-r(T-t)}(\Phi(d_-)-1) + Se^{-q(T-t)}\phi(-d_+)\frac{\partial}{\partial r}(\sigma\sqrt{T-t}) \\
&= (T-t)Ke^{-r(T-t)}(\Phi(d_-)-1)
\end{aligned}$$

Checking with Put-Call parity:

$$\begin{aligned}
\rho(P) &= \rho(C) - \frac{\partial}{\partial r}\left(Se^{-q(T-t)}\right) + \frac{\partial}{\partial r}\left(Ke^{-r(T-t)}\right) \\
&= K(T-t)e^{-r(T-t)}\Phi(d_-) - K(T-t)e^{-r(T-t)} \\
&= K(T-t)e^{-r(T-t)}(\Phi(d_-)-1)
\end{aligned}$$