COMPUTATIONAL FINANCE & RISK MANAGEMENT

UNIVERSITY *of* WASHINGTON

Department of Applied Mathematics

CFRM 410: Probability and Statistics for Computational Finance

Week 9 Estimation

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Slides originally produced by Kjell Konis

Fundamental Ideas of Statistics

Statistical Models

Point Estimation

Evaluating Point Estimators

Interval Estimators

Outline

Statistical Models

Goal: learn something about a population by looking at a subset of the population called a *sample*

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- The distribution of T depends on the density of the X_i and is called the sampling distribution of T
- $\mathrm{E}(X)$ and $\mathrm{Var}(X)$ provide partial information on the distribution of T and are particularly useful when the distribution of T can be approximated

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- The method of *moments* (easy - almost always works)
- The method of *least squares* (easy)
- The method of *maximum likelihood* (more general - optimal in many situations)

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Need one moment for each parameter in the model

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Caveat: Suppose $\theta = 1$ and that $x_1 = 0.98$, $x_2 = 0.34$, $x_3 = 0.12$, $x_4 = 0.48$ and $x_5 = 0.08$ is a realization of a random sample. The method of moments estimate of θ is

$$\hat{\theta} = 2\bar{x} = 0.8$$

which is clearly not consistent with the model!

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Method of moments estimators for μ and σ^2:

- $\hat{\mu}(X_1, \ldots, X_n) = m_1' = \frac{1}{n}\sum_{i=1}^n X_i = \bar{X}$
- $\hat{\sigma}^2(X_1, \ldots, X_n) = m_2' - (m_1')^2 = \dfrac{1}{n}\sum_{i=1}^n X_i^2 - \bar{X}^2 = \dfrac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$

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Each random variable X_i should be "*close*" to θ

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A reasonable estimator for θ would be the value minimizing

$$S(\theta) = \sum_{i=1}^{n} (X_i - \theta)^2$$

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&= \sum_{i=1}^{n} [(X_i - \bar{X}) + (\bar{X} - \theta)]^2 \\
&= \sum_{i=1}^{n} (X_i - \bar{X})^2 + 2\sum_{i=1}^{n} (X_i - \bar{X})(\bar{X} - \theta) + \sum_{i=1}^{n} (\bar{X} - \theta)^2
\end{aligned}
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Example: Method of Least Squares (continued)

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$$\Longrightarrow \quad \hat{\theta}(X_1, \ldots, X_n) = \bar{X}$$

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The joint density of a random sample from a parametric family with parameter θ is

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Definition The *maximum likelihood estimator* $\hat{\theta}_{ML}$ of a parameter θ is a value of θ giving the largest likelihood possible:

$$L(\hat{\theta}_{ML}|x_1,\ldots,x_n) \geq L(\theta|x_1,\ldots,x_n)$$

for all admissible values of the parameter θ

Calculation of $\hat{\theta}_{ML}$

Often, maximizing the log of the likelihood is easier than maximizing the likelihood itself

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General algorithm for finding $\hat{\theta}$

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1. Write down the likelihood function $L(\theta|x_1, \ldots, x_n)$
2. Define $\ell(\theta|x_1, \ldots, x_n) = \log L(\theta|x_1, \ldots, x_n)$ (the log likelihood of θ)

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3. Find $\hat{\theta}_{ML}$ such that

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$$\frac{d}{d\theta}\ell(\theta|x_1, \ldots, x_n)\bigg|_{\hat{\theta}_{ML}} = 0$$

4. Verify that $\hat{\theta}_{ML}$ is a maximum

Example: Find $\hat{p}_{ML}$ for a Sequence of Bernoulli Trials

Let $X_1, \ldots, X_n \overset{iid}{\sim}$ Bernoulli(p) be a random sample

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$$L(p|x_1, \ldots, x_n) \quad = \quad f_{X_1,\ldots,X_n}(x_1, \ldots, x_n|p)$$

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The log likelihood function is

$$\ell(p|x_1, \ldots, x_n) = \log L(p|x_1, \ldots, x_n) = \log \left[\prod_{i=1}^{n} p^{x_i}(1-p)^{(1-x_i)} \right]$$

Find $\hat{p}_{ML}$ for a Sequence of Bernoulli Trials (continued)

$$= \sum_{i=1}^{n} \log \left[p^{x_i} (1-p)^{(1-x_i)} \right]$$

Find $\hat{p}_{ML}$ for a Sequence of Bernoulli Trials (continued)

$$
\begin{aligned}
&= \sum_{i=1}^{n} \log\left[p^{x_i}(1-p)^{(1-x_i)}\right] \\
&= \log(p)\sum_{i=1}^{n} x_i + \log(1-p)\sum_{i=1}^{n}(1-x_i)
\end{aligned}
$$

Find $\hat{p}_{ML}$ for a Sequence of Bernoulli Trials (continued)

$$
\begin{aligned}
&= \sum_{i=1}^{n} \log\left[p^{x_i}(1-p)^{(1-x_i)}\right] \\
&= \log(p)\sum_{i=1}^{n} x_i + \log(1-p)\sum_{i=1}^{n}(1-x_i) \\
&= \log(p)\sum_{i=1}^{n} x_i + \log(1-p)\left[n - \sum_{i=1}^{n} x_i\right]
\end{aligned}
$$

Find $\hat{p}_{ML}$ for a Sequence of Bernoulli Trials (continued)

$$
\begin{aligned}
&= \sum_{i=1}^{n} \log\left[p^{x_i}(1-p)^{(1-x_i)}\right] \\
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&= \log(p)\sum_{i=1}^{n} x_i + \log(1-p)\left[n - \sum_{i=1}^{n} x_i\right]
\end{aligned}
$$

Take the derivative of the log likelihood wrt the parameter

$$
\frac{d}{dp}\ell(p|x_1,\ldots,x_n) = \frac{1}{p}\sum_{i=1}^{n} x_i - \frac{1}{1-p}\left[n - \sum_{i=1}^{n} x_i\right]
$$

Find $\hat{p}_{ML}$ for a Sequence of Bernoulli Trials (continued)

Set the derivative equal to 0 and solve for $\hat{p}_{ML}$

$$\frac{1}{\hat{p}_{ML}} \sum_{i=1}^{n} x_i - \frac{1}{1 - \hat{p}_{ML}} \left[n - \sum_{i=1}^{n} x_i \right] \quad \stackrel{\text{set}}{=} \quad 0$$

Find $\hat{p}_{ML}$ for a Sequence of Bernoulli Trials (continued)

Set the derivative equal to 0 and solve for $\hat{p}_{ML}$

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\begin{aligned}
\frac{1}{\hat{p}_{ML}} \sum_{i=1}^{n} x_i - \frac{1}{1-\hat{p}_{ML}} \left[n - \sum_{i=1}^{n} x_i \right] &\stackrel{\text{set}}{=} 0 \\
\frac{1}{\hat{p}_{ML}} \sum_{i=1}^{n} x_i &= \frac{1}{1-\hat{p}_{ML}} \left[n - \sum_{i=1}^{n} x_i \right]
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\frac{1}{\hat{p}_{ML}} \sum_{i=1}^{n} x_i - \frac{1}{1-\hat{p}_{ML}} \left[n - \sum_{i=1}^{n} x_i \right] &\overset{\text{set}}{=} 0 \\
\frac{1}{\hat{p}_{ML}} \sum_{i=1}^{n} x_i &= \frac{1}{1-\hat{p}_{ML}} \left[n - \sum_{i=1}^{n} x_i \right] \\
(1-\hat{p}_{ML}) \sum_{i=1}^{n} x_i &= \hat{p}_{ML} \left[n - \sum_{i=1}^{n} x_i \right]
\end{aligned}
$$

Find $\hat{p}_{ML}$ for a Sequence of Bernoulli Trials (continued)

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(1-\hat{p}_{ML}) \sum_{i=1}^{n} x_i &= \hat{p}_{ML} \left[n - \sum_{i=1}^{n} x_i \right] \\
\sum_{i=1}^{n} x_i - \hat{p}_{ML} \sum_{i=1}^{n} x_i &= n\hat{p}_{ML} - \hat{p}_{ML} \sum_{i=1}^{n} x_i
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Find $\hat{p}_{ML}$ for a Sequence of Bernoulli Trials (continued)

Set the derivative equal to 0 and solve for $\hat{p}_{ML}$

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(1-\hat{p}_{ML})\sum_{i=1}^{n} x_i &= \hat{p}_{ML}\left[n - \sum_{i=1}^{n} x_i\right] \\
\sum_{i=1}^{n} x_i - \hat{p}_{ML}\sum_{i=1}^{n} x_i &= n\hat{p}_{ML} - \hat{p}_{ML}\sum_{i=1}^{n} x_i \\
\hat{p}_{ML} &= \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}
\end{aligned}$$

Find $\hat{p}_{ML}$ for a Sequence of Bernoulli Trials (continued)

Finally, verify that $\hat{p}_{ML}$ is indeed a maximum

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$$\frac{d^2}{dp^2}\ell(p|x_1,\ldots,x_n) = -\frac{1}{p^2}\sum_{i=1}^{n} x_i - \frac{1}{(1-p)^2}\left[n - \sum_{i=1}^{n} x_i\right]$$

Find $\hat{p}_{ML}$ for a Sequence of Bernoulli Trials (continued)

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$$
\begin{aligned}
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&< 0 \quad \text{for all } p \in (0,1)
\end{aligned}
$$

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Have shown that for every possible $(X_1 = x_1, \ldots, X_n = x_n)$

$\hat{p}_{ML} = \bar{x}$

Find $\hat{p}_{ML}$ for a Sequence of Bernoulli Trials (continued)

Finally, verify that $\hat{p}_{ML}$ is indeed a maximum

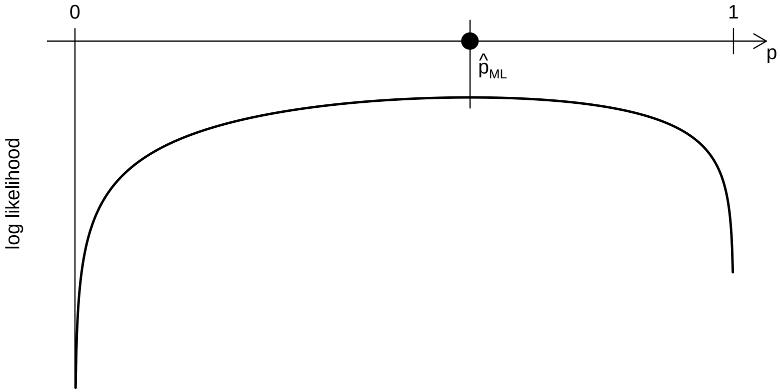
$$\begin{aligned}\frac{d^2}{dp^2}\ell(p|x_1,\ldots,x_n) &= -\frac{1}{p^2}\sum_{i=1}^{n} x_i - \frac{1}{(1-p)^2}\left[n - \sum_{i=1}^{n} x_i\right] \\ &< 0 \quad \text{for all } p \in (0,1)\end{aligned}$$

$\implies$ $\hat{p}_{ML}$ is indeed a maximum of the log likelihood

Have shown that for every possible $(X_1 = x_1, \ldots, X_n = x_n)$

$$\hat{p}_{ML} = \bar{x} \quad \implies \quad \hat{p}_{ML} = \bar{X}$$

Find $\hat{p}_{ML}$ for a Sequence of Bernoulli Trials (illustration)



Example: Find $\hat{\mu}$ for a Normal Population

Let $X_1, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$

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The log likelihood function is

$$\ell(\mu, \sigma^2 | x_1, \ldots, x_n) = \log L(\mu, \sigma^2 | x_1, \ldots, x_n)$$

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&= \sum_{i=1}^{n} \log \frac{1}{\sqrt{2\pi}\sigma} + \sum_{i=1}^{n} \left(-\frac{(x_i - \mu)^2}{2\sigma^2} \right)
\end{aligned}
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&= \sum_{i=1}^{n} \log \frac{1}{\sqrt{2\pi}\sigma} + \sum_{i=1}^{n} \left(-\frac{(x_i - \mu)^2}{2\sigma^2} \right) \\
&= -n \log(\sqrt{2\pi}\sigma) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2
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&= \sum_{i=1}^{n} \log \frac{1}{\sqrt{2\pi}\sigma} + \sum_{i=1}^{n} \left(-\frac{(x_i - \mu)^2}{2\sigma^2} \right) \\
&= -n \log(\sqrt{2\pi}\sigma) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2 \\
&\propto -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2
\end{aligned}
$$

Example: Find $\hat{\mu}$ for a Normal Population

Take the derivative of the log likelihood

$$\frac{d}{d\mu}\ell(\mu, \sigma^2 | x_1, \ldots, x_n) \quad = \quad \frac{d}{d\mu}\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right]$$

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&= \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \mu)
\end{aligned}
$$

Set the derivative equal to 0 and solve for $\hat{\mu}_{ML}$

$$
\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \hat{\mu}_{ML}) \stackrel{\text{set}}{=} 0
$$

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&= \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \mu)
\end{aligned}$$

Set the derivative equal to 0 and solve for $\hat{\mu}_{ML}$

$$\begin{aligned}
\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \hat{\mu}_{ML}) &\overset{\text{set}}{=} 0 \\
\sum_{i=1}^{n} x_i &= n\hat{\mu}_{ML}
\end{aligned}$$

Example: Find $\hat{\mu}$ for a Normal Population

Take the derivative of the log likelihood

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Set the derivative equal to 0 and solve for $\hat{\mu}_{ML}$

$$\begin{aligned} \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\hat{\mu}_{ML}) &\overset{\text{set}}{=} 0 \\ \sum_{i=1}^{n}x_i &= n\hat{\mu}_{ML} \quad \Longrightarrow \quad \hat{\mu}_{ML} = \frac{1}{n}\sum_{i=1}^{n}x_i = \bar{x} \end{aligned}$$

Example: Find $\hat{\mu}$ for a Normal Population

Finally, need to verify that $\hat{\mu}_{ML}$ is indeed a maximum

$$\frac{d^2}{d\mu^2}\ell(\mu, \sigma^2 | x_1, \ldots, x_n) \quad = \quad \frac{d}{d\mu}\left[\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \mu)\right]$$

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&= -\frac{n}{\sigma^2}
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$\implies$ $\hat{\mu}_{ML}$ is a local maximum of the log likelihood

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&< 0
\end{aligned}
$$

$\implies$ $\hat{\mu}_{ML}$ is a local maximum of the log likelihood

The maximum likelihood estimator of μ is

$\hat{\mu}_{ML} = \bar{X}$

Outline

Statistical Models

Point Estimation

Evaluating Point Estimators

Interval Estimators

Bias

The *bias* of an estimator $\hat{\theta}$ is defined to be

$$\mathsf{Bias}_{\hat{\theta}}(\theta) = \mathsf{E}(\hat{\theta}) - \theta$$

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- If $\text{Bias}_{\hat{\theta}}(\theta) < 0$ then $\hat{\theta}$ tends to underestimate θ

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- If $\text{Bias}_{\hat{\theta}}(\theta) > 0$ then $\hat{\theta}$ tends to overestimate θ
- If $\text{Bias}_{\hat{\theta}}(\theta) \equiv 0$ then $\hat{\theta}$ is an *unbiased* estimator of θ

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- If $\text{Bias}_{\hat{\theta}}(\theta) \equiv 0$ then $\hat{\theta}$ is an *unbiased* estimator of θ

A qualitative measure of an estimator $\hat{\theta}$ is the absence of a systematic relationship between $\hat{\theta}$ and θ

Bias

The *bias* of an estimator $\hat{\theta}$ is defined to be

$$\mathrm{Bias}_{\hat{\theta}}(\theta) = \mathrm{E}(\hat{\theta}) - \theta$$

Interpretation of bias:

- If $\mathrm{Bias}_{\hat{\theta}}(\theta) < 0$ then $\hat{\theta}$ tends to underestimate θ
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Later: better measure is given by the average size of $(\hat{\theta} - \theta)^2$

Example: Bias

Let $X_1, \dots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$

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\text{Bias}_{\hat{\mu}}(\mu) &= \text{E}(\bar{X}) - \mu \\
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&= 0 \implies \bar{X} \text{ is an unbiased estimator of } \mu
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Example: Bias (continued)

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&= \frac{n-1}{n}\text{E}(S^2) - \sigma^2 \\
&= \frac{n-1}{n}\sigma^2 - \sigma^2 \\
&= \frac{-\sigma^2}{n} < 0 \quad \Rightarrow \quad \hat{\sigma}^2 \text{ tends to underestimate } \sigma^2
\end{aligned}$$

Mean Squared Error

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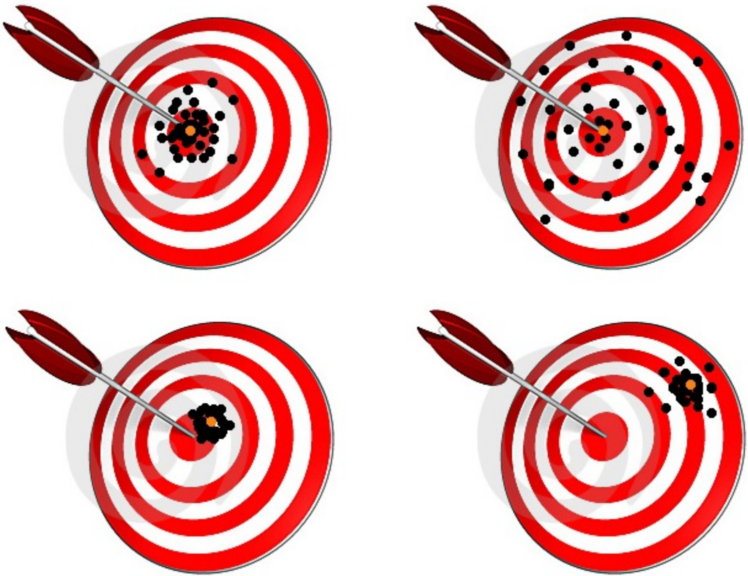
&= \mathsf{Var}(\hat{\theta}) + [\mathsf{Bias}_{\hat{\theta}}(\theta)]^2
\end{aligned}$$

Bias-Variance Tradeoff (illustration)



Efficiency

Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be two unbiased estimators of same parameter θ

- $\text{MSE}(\hat{\theta}_1) = \text{Var}(\hat{\theta}_1) + [\text{Bias}_{\hat{\theta}_1}(\theta)]^2 = \text{Var}(\hat{\theta}_1)$
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- Which estimator of μ is preferable, $\bar{X}$ or M?

Consistency

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A sequence of estimators $W_n = W_n(X_1, \dots, X_n)$ is a *consistent sequence of estimators* of the parameter θ if for every $\epsilon > 0$ and every θ

$$\lim_{n \to \infty} \mathrm{P}(|W_n - \theta| < \epsilon) = 1$$

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$\Longrightarrow$ $\bar{X}_n$ is a consistent sequence of estimators of θ

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- $\mathrm{E}\big(g(\bar{X}_n)\big) = \mathrm{E}\big(a\bar{X}_n + b\big) = a\,\mathrm{E}(\bar{X}_n) + b = a\mu + b$

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Since $g'(p) = 1 - 2p$

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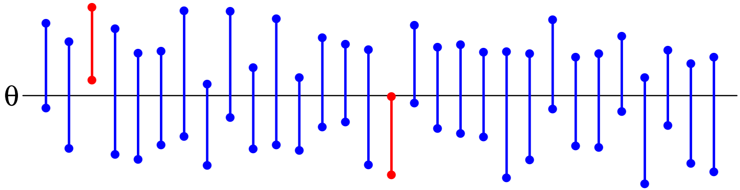

Figure: 30 repetitions of the confidence interval calculation

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- Let $X_1, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, 1^2)$
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Department of Applied Mathematics

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