COMPUTATIONAL FINANCE & RISK MANAGEMENT

UNIVERSITY *of* WASHINGTON

Department of Applied Mathematics

CFRM 410: Probability and Statistics for Computational Finance

Week 6 Distributions

Jake Price

Instructor, Computational Finance and Risk Management

University of Washington

Slides originally produced by Kjell Konis

Outline

Expected Utility Framework

- W_0: Initial wealth (considered fixed)
- W: End of period wealth (considered random)
- $U(w)$: Utility function
- $E[U(W)]$: Expected Utility

Setup

- Consider a single period, with investment decisions made at the beginning and returns received at the end
- Key Assumption: investors seek to maximize end-of-period expected utility

Basic Properties

- Used only to rank investments
  - *Utility functions are invariant under positive affine transformations*
- If U_1 and U_2 are related by

$$U_2 = aU_1 + b \qquad a > 0$$

then U_1 and U_2 are equivalent $\implies U_1 \sim U_2$

  - Intuition: rank investments by utility, so only the order, not the absolute level, is important

Key Assumptions

1. Investors prefer more to less

2. Investors are never satisfied $\implies$ $U(w)$ is a strictly increasing function of wealth w

$$U(x) > U(y) \quad \text{when} \quad x > y$$

Play the Lotto

Lottery: an asset that has a risky payoff

- Initial wealth: W_0
- An investor with utility function $U(W)$ considers a lottery with payoffs h_i, $i = 1, 2$
- End of period wealth

$$W = \begin{cases} W_0 + h_1 & \text{with probability } p \\ W_0 + h_2 & \text{with probability } 1 - p \end{cases}$$

- Expected utility from participating in the lottery

$$E[U(W)] = pU(W_0 + h_1) + (1 - p)U(W_0 + h_2)$$

- Suppose that

$$h_1 = -|h_1| < 0 \qquad h_2 = |h_2| > 0$$

Risk Aversion

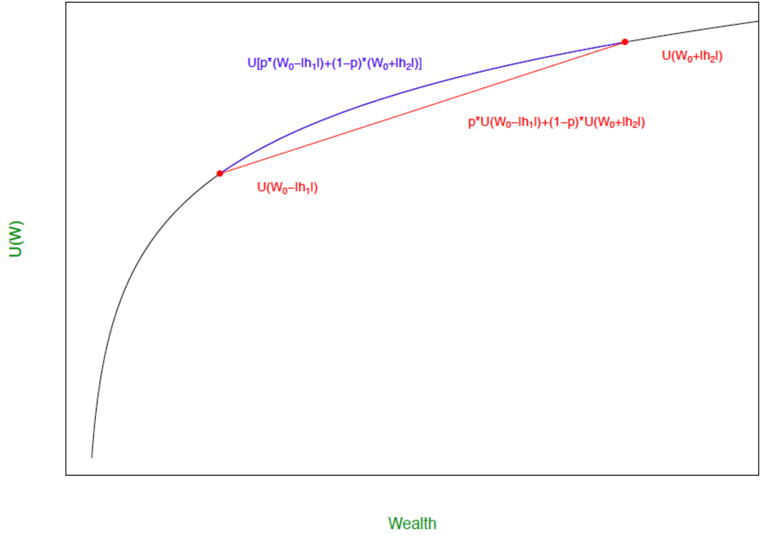
If, for any admissible h_1 and h_2, and for all $p \in (0, 1)$, the utility function satisfies

$$\begin{aligned} &U(p(W_0 - |h_1|) + (1-p)(W_0 + |h_2|)) \\ &\qquad \geq pU(W_0 - |h_1|) + (1-p)U(W_0 + |h_2|) \end{aligned}$$

then the investor is *risk averse*

- Typically "risk averse" for strict inequality $>$

Risk Aversion



Risk Aversion

- Interpretation: if the investor were offered the lottery or a fixed payment of $E[h]$, the investor would always take the fixed payment

$$U(E[W]) = U(W_0 - \underbrace{p|h_1| + (1-p)|h_2|}_{\text{expected payoff}}) \geq E[U(W)]$$

  In fact, a risk-averse investor would actually accept a payment less than the expected lottery payoff rather than participate in the lottery

- Mathematically, the utility function of a risk-averse investor is concave

- Jensen's Inequality: if $U(w)$ is a concave function and W is a random variable, then

$$U(E[W]) \geq E[U(W)]$$

Certainty Equivalent

- For a risk-averse investor

$$U(E[W]) \geq E[U(W)]$$

- Question: what fixed payoff would the investor accept such that they are indifferent between participating in the lottery and accepting the payoff?
- Solve for w_C

$$U(w_C) = E(U(W)]$$

- Certainty Equivalent: the fixed level of wealth, w_C, offered to an investor that would make them indifferent to participating in the lottery or accepting the certainty equivalent wealth

Certainty Equivalent

- A risk-averse investor's certainty equivalent satisfies:

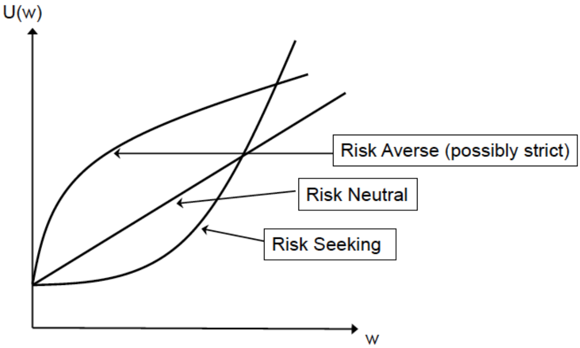
  $$w_C \leq E[W]$$

  Interpretation: a risk-averse investor would give up some expected value to avoid risk since they would be indifferent to receiving a fixed payoff less than that expected from the lottery, and actually participating in the lottery

Utility and Risk Preferences

Three Classficiations: Risk-Averse, Risk-Neutral, and Risk-Seeking

- Risk-Averse: utility function is concave
- Risk-Neutral: utility function is linear
- Risk-Seeking: utility function is strictly convex



Outline

Discrete Distributions

A discrete distribution is described by a *probability mass function*

$$f_X(x) = P(X = x)$$

and a support $S_X = \{x_1, x_2, \ldots\}$

Discrete Distributions

A discrete distribution is described by a *probability mass function*

$$f_X(x) = P(X = x)$$

and a support $S_X = \{x_1, x_2, \ldots\}$

The *expected value* (expectation) of a discrete random variable is

$$\mathrm{E}(X) = \sum_{x_i \in S_X} x_i \, f_X(x_i) = \sum_{x_i \in S_X} x_i \, P(X = x_i)$$

Discrete Distributions

A discrete distribution is described by a *probability mass function*

$$f_X(x) = P(X = x)$$

and a support $S_X = \{x_1, x_2, \ldots\}$

The *expected value* (expectation) of a discrete random variable is

$$\mathrm{E}(X) = \sum_{x_i \in S_X} x_i\, f_X(x_i) = \sum_{x_i \in S_X} x_i\, P(X = x_i)$$

The expected value of a function g of a discrete random variable is

$$\mathrm{E}\big[g(X)\big] = \sum_{x_i \in S_X} g(x_i)\, f_X(x_i) = \sum_{x_i \in S_X} g(x_i)\, P(X = x_i)$$

Discrete Distributions

A discrete distribution is described by a *probability mass function*

$$f_X(x) = P(X = x)$$

and a support $S_X = \{x_1, x_2, \ldots\}$

The *expected value* (expectation) of a discrete random variable is

$$\mathrm{E}(X) = \sum_{x_i \in S_X} x_i \, f_X(x_i) = \sum_{x_i \in S_X} x_i \, P(X = x_i)$$

The expected value of a function g of a discrete random variable is

$$\mathrm{E}\big[g(X)\big] = \sum_{x_i \in S_X} g(x_i) \, f_X(x_i) = \sum_{x_i \in S_X} g(x_i) \, P(X = x_i)$$

The *variance* of a discrete random variable is

$$\mathrm{Var}(X) = \mathrm{E}\big[(X - \mathrm{E}(X))^2\big] = \sum_{x_i \in S_X} \big[(x_i - \mathrm{E}(X))^2\big] \, f_X(x_i)$$

Bernoulli Trial

A *Bernoulli* random variable X takes values

$$X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1 - p \end{cases}$$

Bernoulli Trial

A *Bernoulli* random variable X takes values

$$X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1 - p \end{cases}$$

Notation: $X \sim \text{Bernoulli}(p)$ for $0 \leq p \leq 1$

Bernoulli Trial

A *Bernoulli* random variable X takes values

$$X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

Notation: $X \sim \text{Bernoulli}(p)$ for $0 \leq p \leq 1$

$X = 1$ is often called a success; $X = 0$ a failure

Bernoulli Trial

A *Bernoulli* random variable X takes values

$$X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

Notation: $X \sim \text{Bernoulli}(p)$ for $0 \leq p \leq 1$

$X = 1$ is often called a success; $X = 0$ a failure

Probability mass function

$$f_X(x) = P(X = x) = \begin{cases} p & x = 1 \\ 1-p & x = 0 \end{cases} \qquad x \in \{0, 1\}$$

Expected Value and Variance (Bernoulli Trial)

Expected value of $X \sim \text{Bernoulli}(p)$

$$\mathsf{E}(X) = \sum_{x_i \in S_X} x_i f_X(x_i)$$

Expected Value and Variance (Bernoulli Trial)

Expected value of $X \sim \text{Bernoulli}(p)$

$$\mathsf{E}(X) = \sum_{x_i \in S_X} x_i f_X(x_i) = 1 \cdot f_X(1) = 0 \cdot (1-p) + 1 \cdot p = p$$

Expected Value and Variance (Bernoulli Trial)

Expected value of $X \sim \text{Bernoulli}(p)$

$$\mathsf{E}(X) = \sum_{x_i \in S_X} x_i f_X(x_i) = 1 \cdot f_X(1) = 0 \cdot (1-p) + 1 \cdot p = p$$

Need second moment to compute the variance

$$\mathsf{E}(X^2) = 0^2 \cdot f_X(0) + 1^2 \cdot f_X(1)$$

Expected Value and Variance (Bernoulli Trial)

Expected value of $X \sim \text{Bernoulli}(p)$

$$\mathsf{E}(X) = \sum_{x_i \in S_X} x_i f_X(x_i) = 1 \cdot f_X(1) = 0 \cdot (1-p) + 1 \cdot p = p$$

Need second moment to compute the variance

$$\mathsf{E}(X^2) = 0^2 \cdot f_X(0) + 1^2 \cdot f_X(1) = 0 \cdot (1-p) + 1 \cdot p = p$$

Expected Value and Variance (Bernoulli Trial)

Expected value of $X \sim \text{Bernoulli}(p)$

$$\mathrm{E}(X) = \sum_{x_i \in S_X} x_i f_X(x_i) = 1 \cdot f_X(1) = 0 \cdot (1-p) + 1 \cdot p = p$$

Need second moment to compute the variance

$$\mathrm{E}(X^2) = 0^2 \cdot f_X(0) + 1^2 \cdot f_X(1) = 0 \cdot (1-p) + 1 \cdot p = p$$

Use the relation

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$$

Expected Value and Variance (Bernoulli Trial)

Expected value of $X \sim \text{Bernoulli}(p)$

$$\mathrm{E}(X) = \sum_{x_i \in S_X} x_i f_X(x_i) = 1 \cdot f_X(1) = 0 \cdot (1-p) + 1 \cdot p = p$$

Need second moment to compute the variance

$$\mathrm{E}(X^2) = 0^2 \cdot f_X(0) + 1^2 \cdot f_X(1) = 0 \cdot (1-p) + 1 \cdot p = p$$

Use the relation

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = p - p^2 = p(1-p)$$

Binomial Distribution

A *Binomial* random variable is the sum of a sequence of n independent Bernoulli trials

Binomial Distribution

A *Binomial* random variable is the sum of a sequence of n independent Bernoulli trials

Each Bernoulli trial has the same probability parameter p

Binomial Distribution

A *Binomial* random variable is the sum of a sequence of n independent Bernoulli trials

Each Bernoulli trial has the same probability parameter p

Probability mass function

$$f_X(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \quad \text{for} \quad x = 0, 1, \ldots, n$$

zero otherwise

Binomial Distribution

A *Binomial* random variable is the sum of a sequence of n independent Bernoulli trials

Each Bernoulli trial has the same probability parameter p

Probability mass function

$$f_X(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \quad \text{for} \quad x = 0, 1, \ldots, n$$

zero otherwise

Interpret outcome as the number of successes in a sequence of n trials

Binomial Distribution

A *Binomial* random variable is the sum of a sequence of n independent Bernoulli trials

Each Bernoulli trial has the same probability parameter p

Probability mass function

$$f_X(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \quad \text{for} \quad x = 0, 1, \ldots, n$$

zero otherwise

Interpret outcome as the number of successes in a sequence of n trials

A binomial random variable X is denoted $X \sim \text{Binomial}(n, p)$

Binomial Distribution

A *Binomial* random variable is the sum of a sequence of n independent Bernoulli trials

Each Bernoulli trial has the same probability parameter p

Probability mass function

$$f_X(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \quad \text{for} \quad x = 0, 1, \ldots, n$$

zero otherwise

Interpret outcome as the number of successes in a sequence of n trials

A binomial random variable X is denoted $X \sim \text{Binomial}(n, p)$

The random variable $X \sim \text{Binomial}(1, p)$ is a Bernoulli trial

Binomial Theorem

Theorem For any two real numbers x and y and for an integer $n \geq 0$

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

Binomial Theorem

Theorem For any two real numbers x and y and for an integer $n \geq 0$

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

Proof: $$(x+y)^n = (x+y)\,(x+y)\cdot \cdots \cdot (x+y)$$

Binomial Theorem

Theorem For any two real numbers x and y and for an integer $n \geq 0$

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

Proof: $$(x+y)^n = (x+y)\,(x+y)\cdot\cdots\cdot(x+y)$$

Let $x = p$ and $y = 1 - p$ then

$$1 = (p + (1-p))^n = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$

Binomial Theorem

Theorem For any two real numbers x and y and for an integer $n \geq 0$

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

Proof: $$(x+y)^n = (x+y)\,(x+y)\cdot\cdots\cdot(x+y)$$

Let $x = p$ and $y = 1 - p$ then

$$1 = (p + (1-p))^n = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$

Also neat: let $x = y = 1$ then

$$2^n = \sum_{i=0}^{n} \binom{n}{i}$$

Expected Value (Binomial Distribution)

Expected value of a random variable $X \sim \text{Binomial}(n, p)$

$$\mathrm{E}(X) \quad = \quad \sum_{x \in S_X} x\, f_X(x) = \sum_{x=0}^{n} x\, f_X(x) = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$$

Expected Value (Binomial Distribution)

Expected value of a random variable $X \sim \text{Binomial}(n, p)$

$$
\begin{aligned}
\mathrm{E}(X) &= \sum_{x \in S_X} x\, f_X(x) = \sum_{x=0}^{n} x\, f_X(x) = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=1}^{n} x \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}
\end{aligned}
$$

Expected Value (Binomial Distribution)

Expected value of a random variable $X \sim \text{Binomial}(n, p)$

$$
\begin{aligned}
E(X) &= \sum_{x \in S_X} x\, f_X(x) = \sum_{x=0}^{n} x\, f_X(x) = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=1}^{n} x \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \\
&= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} (1-p)^{n-x} \qquad \begin{pmatrix} u = x-1 \\ m = n-1 \end{pmatrix}
\end{aligned}
$$

Expected Value (Binomial Distribution)

Expected value of a random variable $X \sim \text{Binomial}(n, p)$

$$
\begin{aligned}
\mathrm{E}(X) &= \sum_{x \in S_X} x\, f_X(x) = \sum_{x=0}^{n} x\, f_X(x) = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=1}^{n} x \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \\
&= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} (1-p)^{n-x} \qquad \begin{pmatrix} u = x-1 \\ m = n-1 \end{pmatrix} \\
&= np \sum_{u=0}^{n-1} \frac{(n-1)!}{u!\big[n-(u+1)\big]!} p^{u} (1-p)^{n-(u+1)}
\end{aligned}
$$

Expected Value (Binomial Distribution)

Expected value of a random variable $X \sim \text{Binomial}(n, p)$

$$
\begin{aligned}
\mathrm{E}(X) &= \sum_{x \in S_X} x\, f_X(x) = \sum_{x=0}^{n} x\, f_X(x) = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=1}^{n} x \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \\
&= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} (1-p)^{n-x} \qquad \begin{pmatrix} u = x-1 \\ m = n-1 \end{pmatrix} \\
&= np \sum_{u=0}^{n-1} \frac{(n-1)!}{u!\big[n-(u+1)\big]!} p^u (1-p)^{n-(u+1)} \\
&= np \left[\sum_{u=0}^{m} \frac{m!}{u!(m-u)!} p^u (1-p)^{m-u} \right]
\end{aligned}
$$

Expected Value (Binomial Distribution)

Expected value of a random variable $X \sim \text{Binomial}(n, p)$

$$
\begin{aligned}
\mathrm{E}(X) &= \sum_{x \in S_X} x\, f_X(x) = \sum_{x=0}^{n} x\, f_X(x) = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=1}^{n} x \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \\
&= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} (1-p)^{n-x} \qquad \begin{pmatrix} u = x-1 \\ m = n-1 \end{pmatrix} \\
&= np \sum_{u=0}^{n-1} \frac{(n-1)!}{u!\big[n-(u+1)\big]!} p^u (1-p)^{n-(u+1)} \\
&= np \left[\sum_{u=0}^{m} \frac{m!}{u!(m-u)!} p^u (1-p)^{m-u} \right] = np
\end{aligned}
$$

Variance (Binomial Distribution)

$$\mathrm{E}(X^2) = \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x}$$

Variance (Binomial Distribution)

$$
\begin{aligned}
\mathrm{E}(X^2) &= \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x} \\
&= n \sum_{x=1}^{n} x \frac{(n-1)!}{(x-1)!(n-x)!} p^x (1-p)^{n-x} \qquad (u = x - 1)
\end{aligned}
$$

Variance (Binomial Distribution)

$$
\begin{aligned}
\mathrm{E}(X^2) &= \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x} \\
&= n \sum_{x=1}^{n} x \frac{(n-1)!}{(x-1)!(n-x)!} p^x (1-p)^{n-x} && (u = x-1) \\
&= n \sum_{u=0}^{n-1} (u+1) \frac{(n-1)!}{u!(n-1-u)!} p^{u+1} (1-p)^{n-1-u} && (m = n-1)
\end{aligned}
$$

Variance (Binomial Distribution)

$$
\begin{aligned}
\mathrm{E}(X^2) &= \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x} \\
&= n \sum_{x=1}^{n} x \frac{(n-1)!}{(x-1)!(n-x)!} p^x (1-p)^{n-x} && (u = x-1) \\
&= n \sum_{u=0}^{n-1} (u+1) \frac{(n-1)!}{u!(n-1-u)!} p^{u+1} (1-p)^{n-1-u} && (m = n-1) \\
&= np \left[\sum_{u=0}^{m} u \binom{m}{u} p^u (1-p)^{m-u} \right]
\end{aligned}
$$

Variance (Binomial Distribution)

$$
\begin{aligned}
E(X^2) &= \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x} \\
&= n \sum_{x=1}^{n} x \frac{(n-1)!}{(x-1)!(n-x)!} p^x (1-p)^{n-x} && (u = x-1) \\
&= n \sum_{u=0}^{n-1} (u+1) \frac{(n-1)!}{u!(n-1-u)!} p^{u+1} (1-p)^{n-1-u} && (m = n-1) \\
&= np \left[\sum_{u=0}^{m} u \binom{m}{u} p^u (1-p)^{m-u} \right] + np \left[\sum_{u=0}^{m} \binom{m}{u} p^u (1-p)^{m-u} \right]
\end{aligned}
$$

Variance (Binomial Distribution)

$$
\begin{aligned}
\mathrm{E}(X^2) &= \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x} \\
&= n \sum_{x=1}^{n} x \frac{(n-1)!}{(x-1)!(n-x)!} p^x (1-p)^{n-x} && (u = x-1) \\
&= n \sum_{u=0}^{n-1} (u+1) \frac{(n-1)!}{u!(n-1-u)!} p^{u+1} (1-p)^{n-1-u} && (m = n-1) \\
&= np \left[\sum_{u=0}^{m} u \binom{m}{u} p^u (1-p)^{m-u} \right] + np \left[\sum_{u=0}^{m} \binom{m}{u} p^u (1-p)^{m-u} \right] \\
&= np \cdot mp + np
\end{aligned}
$$

Variance (Binomial Distribution)

$$
\begin{aligned}
\mathrm{E}(X^2) &= \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x} \\
&= n \sum_{x=1}^{n} x \frac{(n-1)!}{(x-1)!(n-x)!} p^x (1-p)^{n-x} && (u = x-1) \\
&= n \sum_{u=0}^{n-1} (u+1) \frac{(n-1)!}{u!(n-1-u)!} p^{u+1} (1-p)^{n-1-u} && (m = n-1) \\
&= np \left[\sum_{u=0}^{m} u \binom{m}{u} p^u (1-p)^{m-u} \right] + np \left[\sum_{u=0}^{m} \binom{m}{u} p^u (1-p)^{m-u} \right] \\
&= np \cdot mp + np = n(n-1)p^2 + np
\end{aligned}
$$

Variance (Binomial Distribution)

$$
\begin{aligned}
\mathrm{E}(X^2) &= \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x} \\
&= n \sum_{x=1}^{n} x \frac{(n-1)!}{(x-1)!(n-x)!} p^x (1-p)^{n-x} && (u = x-1) \\
&= n \sum_{u=0}^{n-1} (u+1) \frac{(n-1)!}{u!(n-1-u)!} p^{u+1} (1-p)^{n-1-u} && (m = n-1) \\
&= np \left[\sum_{u=0}^{m} u \binom{m}{u} p^u (1-p)^{m-u} \right] + np \left[\sum_{u=0}^{m} \binom{m}{u} p^u (1-p)^{m-u} \right] \\
&= np \cdot mp + np = n(n-1)p^2 + np
\end{aligned}
$$

$$
\operatorname{Var} X = \mathrm{E}(X^2) - \left[\mathrm{E}(X)\right]^2 = n(n-1)p^2 + np - (np)^2 = np(1-p)
$$

Poisson Distribution

The Poisson distribution is used to model the number of occurrences in a given time interval

Poisson Distribution

The Poisson distribution is used to model the number of occurrences in a given time interval

Parameter: *intensity* $\lambda \geq 0$

Poisson Distribution

The Poisson distribution is used to model the number of occurrences in a given time interval

Parameter: *intensity* $\lambda \geq 0$

Probability mass function

$$f_X(x) = P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!} \qquad \text{for } x = 0, 1, \ldots$$

Poisson Distribution

The Poisson distribution is used to model the number of occurrences in a given time interval

Parameter: *intensity* $\lambda \geq 0$

Probability mass function

$$f_X(x) = P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!} \qquad \text{for } x = 0, 1, \ldots$$

Hint: Taylor series expansion of e^x gives

$$e^{\lambda} = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}$$

Expected Value (Poisson Distribution)

Expected value of a random variable $X \sim \text{Poisson}(\lambda)$

$$\mathrm{E}(X) = \sum_{x \in S_X} x\, f_X(x) = \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!}$$

Expected Value (Poisson Distribution)

Expected value of a random variable $X \sim \text{Poisson}(\lambda)$

$$\mathsf{E}(X) \quad = \quad \sum_{x \in S_X} x\, f_X(x) = \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=1}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!}$$

Expected Value (Poisson Distribution)

Expected value of a random variable $X \sim \text{Poisson}(\lambda)$

$$
\begin{aligned}
\mathrm{E}(X) &= \sum_{x \in S_X} x\, f_X(x) = \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=1}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} \\
&= \lambda \sum_{x=1}^{\infty} \frac{\lambda^{(x-1)} e^{-\lambda}}{(x-1)!} && (u = x - 1)
\end{aligned}
$$

Expected Value (Poisson Distribution)

Expected value of a random variable $X \sim \text{Poisson}(\lambda)$

$$
\begin{aligned}
\mathrm{E}(X) &= \sum_{x \in S_X} x\, f_X(x) = \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=1}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} \\
&= \lambda \sum_{x=1}^{\infty} \frac{\lambda^{(x-1)} e^{-\lambda}}{(x-1)!} && (u = x - 1) \\
&= \lambda \sum_{u=0}^{\infty} \frac{\lambda^u e^{-\lambda}}{u!}
\end{aligned}
$$

Expected Value (Poisson Distribution)

Expected value of a random variable $X \sim \text{Poisson}(\lambda)$

$$
\begin{aligned}
\mathrm{E}(X) &= \sum_{x \in S_X} x\, f_X(x) = \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=1}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} \\
&= \lambda \sum_{x=1}^{\infty} \frac{\lambda^{(x-1)} e^{-\lambda}}{(x-1)!} && (u = x - 1) \\
&= \lambda \sum_{u=0}^{\infty} \frac{\lambda^u e^{-\lambda}}{u!} \\
&= \lambda
\end{aligned}
$$

Variance (Poisson Distribution)

$$\mathrm{E}(X^2) = \sum_{x=0}^{\infty} x^2 \frac{\lambda^x e^{-\lambda}}{x!}$$

Variance (Poisson Distribution)

$$\begin{aligned} \mathrm{E}(X^2) &= \sum_{x=0}^{\infty} x^2 \frac{\lambda^x e^{-\lambda}}{x!} \\ &= \lambda \sum_{u=0}^{\infty} (1+u) \frac{\lambda^u e^{-\lambda}}{u!} && (u = x-1) \end{aligned}$$

Variance (Poisson Distribution)

$$
\begin{aligned}
\mathrm{E}(X^2) &= \sum_{x=0}^{\infty} x^2 \frac{\lambda^x e^{-\lambda}}{x!} \\
&= \lambda \sum_{u=0}^{\infty} (1+u) \frac{\lambda^u e^{-\lambda}}{u!} && (u = x-1) \\
&= \lambda \left[\sum_{u=0}^{\infty} \frac{\lambda^u e^{-\lambda}}{u!} + \sum_{u=0}^{\infty} u \frac{\lambda^u e^{-\lambda}}{u!} \right]
\end{aligned}
$$

Variance (Poisson Distribution)

$$\begin{aligned}
\mathrm{E}(X^2) &= \sum_{x=0}^{\infty} x^2 \frac{\lambda^x e^{-\lambda}}{x!} \\
&= \lambda \sum_{u=0}^{\infty} (1+u) \frac{\lambda^u e^{-\lambda}}{u!} && (u = x-1) \\
&= \lambda \left[\sum_{u=0}^{\infty} \frac{\lambda^u e^{-\lambda}}{u!} + \sum_{u=0}^{\infty} u \frac{\lambda^u e^{-\lambda}}{u!} \right] \\
&= \lambda \left[1 + \lambda \right]
\end{aligned}$$

Variance (Poisson Distribution)

$$
\begin{aligned}
\mathrm{E}(X^2) &= \sum_{x=0}^{\infty} x^2 \frac{\lambda^x e^{-\lambda}}{x!} \\
&= \lambda \sum_{u=0}^{\infty} (1+u) \frac{\lambda^u e^{-\lambda}}{u!} && (u = x - 1) \\
&= \lambda \left[\sum_{u=0}^{\infty} \frac{\lambda^u e^{-\lambda}}{u!} + \sum_{u=0}^{\infty} u \frac{\lambda^u e^{-\lambda}}{u!} \right] \\
&= \lambda \left[1 + \lambda \right]
\end{aligned}
$$

$$
\operatorname{Var} X = \mathrm{E}(X^2) - \left[\mathrm{E}(X) \right]^2 = \lambda \left[1 + \lambda \right] - (\lambda)^2 = \lambda
$$

Outline

Continuous Distributions

Described by a *probability density function* $f_X(x)$ satisfying

$$P(X \leq x) = F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$$

Continuous Distributions

Described by a *probability density function* $f_X(x)$ satisfying

$$P(X \leq x) = F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$$

The support S_X is the set of points X satisfying $f_X(x) > 0$

Continuous Distributions

Described by a *probability density function* $f_X(x)$ satisfying

$$P(X \leq x) = F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$$

The support S_X is the set of points X satisfying $f_X(x) > 0$

The *expected value* (expectation) of a continuous random variable is

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_{S_X} x\, f_X(x)\, dx$$

Continuous Distributions

Described by a *probability density function* $f_X(x)$ satisfying

$$P(X \leq x) = F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

The support S_X is the set of points X satisfying $f_X(x) > 0$

The *expected value* (expectation) of a continuous random variable is

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x\, f_X(x)\,dx = \int_{S_X} x\, f_X(x)\,dx$$

The expected value of a function of a continuous random variable is

$$\mathrm{E}\big[g(X)\big] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\,dx = \int_{S_X} g(x)\, f_X(x)\,dx$$

Continuous Distributions

Described by a *probability density function* $f_X(x)$ satisfying

$$P(X \le x) = F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

The support S_X is the set of points X satisfying $f_X(x) > 0$

The *expected value* (expectation) of a continuous random variable is

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x\, f_X(x)\,dx = \int_{S_X} x\, f_X(x)\,dx$$

The expected value of a function of a continuous random variable is

$$\mathrm{E}\big[g(X)\big] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\,dx = \int_{S_X} g(x)\, f_X(x)\,dx$$

The *variance* of a continuous random variable is

$$\mathrm{Var}(X) = \mathrm{E}\big[(X - \mathrm{E}(X))^2\big] = \int_{-\infty}^{\infty} \big[x - \mathrm{E}(X)\big]^2 f_X(x)\,dx$$

Uniform Distribution

A uniform random variable distributes the mass of the distribution evenly over an interval

Uniform Distribution

A uniform random variable distributes the mass of the distribution evenly over an interval

Notation: $X \sim \mathit{uniform}(a, b)$

Uniform Distribution

A uniform random variable distributes the mass of the distribution evenly over an interval

Notation: $X \sim \mathit{uniform}(a, b)$

$\hookrightarrow$ distributes the mass uniformly over the interval $[a, b]$

Uniform Distribution

A uniform random variable distributes the mass of the distribution evenly over an interval

Notation: $X \sim \textit{uniform}(a, b)$

$\hookrightarrow$ distributes the mass uniformly over the interval $[a, b]$

Probability density function

$$f_X(x|a, b) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise.} \end{cases}$$

Uniform Distribution

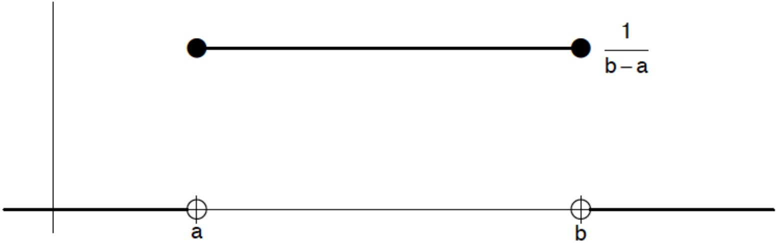
A uniform random variable distributes the mass of the distribution evenly over an interval

Notation: $X \sim uniform(a, b)$

$\hookrightarrow$ distributes the mass uniformly over the interval $[a, b]$

Probability density function

$$f_X(x|a, b) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise.} \end{cases}$$



Uniform Distribution (cumulative distribution function)

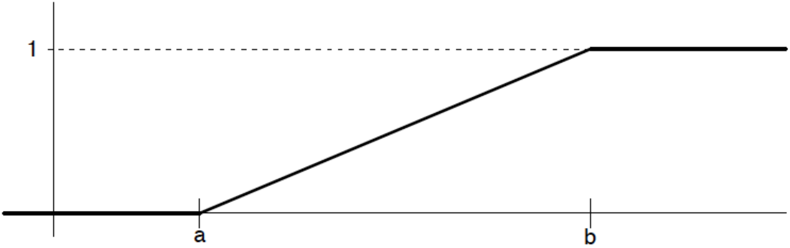
Cumulative distribution function of a uniform(a, b) random variable

$$F_X(x|a,b) = \int_{-\infty}^{x} f_X(t|a,b)\,dt = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$$

Uniform Distribution (cumulative distribution function)

Cumulative distribution function of a uniform(a, b) random variable

$$F_X(x|a,b) = \int_{-\infty}^{x} f_X(t|a,b)\,dt = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$$



Expected Value (Uniform Distribution)

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x\, f_X(x|a,b)\, dx$$

Expected Value (Uniform Distribution)

$$\begin{aligned}
\mathrm{E}(X) &= \int_{-\infty}^{\infty} x\, f_X(x|a, b)\, dx \\
&= \int_a^b \frac{x}{b-a}\, dx
\end{aligned}$$

Expected Value (Uniform Distribution)

$$
\begin{aligned}
\mathrm{E}(X) &= \int_{-\infty}^{\infty} x\, f_X(x|a, b)\, dx \\
&= \int_a^b \frac{x}{b-a}\, dx \\
&= \left. \frac{x^2}{2(b-a)} \right|_a^b
\end{aligned}
$$

Expected Value (Uniform Distribution)

$$
\begin{aligned}
\mathrm{E}(X) &= \int_{-\infty}^{\infty} x\, f_X(x|a,b)\, dx \\
&= \int_a^b \frac{x}{b-a}\, dx \\
&= \left. \frac{x^2}{2(b-a)} \right|_a^b \\
&= \frac{b^2-a^2}{2(b-a)}
\end{aligned}
$$

Expected Value (Uniform Distribution)

$$
\begin{aligned}
\mathrm{E}(X) &= \int_{-\infty}^{\infty} x\, f_X(x|a,b)\, dx \\
&= \int_a^b \frac{x}{b-a}\, dx \\
&= \left. \frac{x^2}{2(b-a)} \right|_a^b \\
&= \frac{b^2-a^2}{2(b-a)} \\
&= \frac{(b+a)(b-a)}{2(b-a)}
\end{aligned}
$$

Expected Value (Uniform Distribution)

$$\begin{aligned}
\mathrm{E}(X) &= \int_{-\infty}^{\infty} x\, f_X(x|a,b)\, dx \\
&= \int_a^b \frac{x}{b-a}\, dx \\
&= \left. \frac{x^2}{2(b-a)} \right|_a^b \\
&= \frac{b^2-a^2}{2(b-a)} \\
&= \frac{(b+a)(b-a)}{2(b-a)} \\
&= \frac{b+a}{2}
\end{aligned}$$

Variance (Uniform Distribution)

$$\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2 \, f_X(x|a,b)\, dx = \int_a^b \frac{x^2}{b-a}\, dx$$

Variance (Uniform Distribution)

$$\begin{aligned}
\mathrm{E}(X^2) &= \int_{-\infty}^{\infty} x^2 \, f_X(x|a,b)\,dx = \int_a^b \frac{x^2}{b-a}\,dx \\
&= \left.\frac{x^3}{3(b-a)}\right|_a^b = \frac{b^3-a^3}{3(b-a)} = \frac{(b-a)(b^2+ab+a^2)}{3(b-a)}
\end{aligned}$$

Variance (Uniform Distribution)

$$\begin{aligned} \mathrm{E}(X^2) &= \int_{-\infty}^{\infty} x^2\, f_X(x|a,b)\, dx = \int_a^b \frac{x^2}{b-a}\, dx \\ &= \left.\frac{x^3}{3(b-a)}\right|_a^b = \frac{b^3-a^3}{3(b-a)} = \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} \end{aligned}$$

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \left[\mathrm{E}(X)\right]^2 = \frac{(b^2+ab+a^2)}{3} - \frac{(b+a)^2}{4}$$

Variance (Uniform Distribution)

$$\begin{aligned} \mathrm{E}(X^2) &= \int_{-\infty}^{\infty} x^2 \, f_X(x|a,b)\,dx = \int_a^b \frac{x^2}{b-a}\,dx \\ &= \left.\frac{x^3}{3(b-a)}\right|_a^b = \frac{b^3-a^3}{3(b-a)} = \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(X) &= \mathrm{E}(X^2) - \big[\mathrm{E}(X)\big]^2 = \frac{(b^2+ab+a^2)}{3} - \frac{(b+a)^2}{4} \\ &= \frac{4b^2+4ab+4a^2-3b^2-6ab-3a^2}{12} \end{aligned}$$

Variance (Uniform Distribution)

$$
\begin{aligned}
\mathrm{E}(X^2) &= \int_{-\infty}^{\infty} x^2 \, f_X(x|a,b)\, dx = \int_a^b \frac{x^2}{b-a}\, dx \\
&= \left. \frac{x^3}{3(b-a)} \right|_a^b = \frac{b^3 - a^3}{3(b-a)} = \frac{(b-a)(b^2+ab+a^2)}{3(b-a)}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}(X^2) - \left[\mathrm{E}(X)\right]^2 = \frac{(b^2+ab+a^2)}{3} - \frac{(b+a)^2}{4} \\
&= \frac{4b^2 + 4ab + 4a^2 - 3b^2 - 6ab - 3a^2}{12} \\
&= \frac{b^2 - 2ab + a^2}{12} = \frac{(b-a)^2}{12}
\end{aligned}
$$

Normal Distribution (probability density function)

A random variable $X \sim \mathcal{N}(\mu, \sigma^2)$ has probability density function

$$f_X(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ \frac{-(x-\mu)^2}{2\sigma^2} \right\} \qquad -\infty < x < \infty$$

Normal Distribution (probability density function)

A random variable $X \sim \mathcal{N}(\mu, \sigma^2)$ has probability density function

$$f_X(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{\frac{-(x-\mu)^2}{2\sigma^2}\right\} \qquad -\infty < x < \infty$$

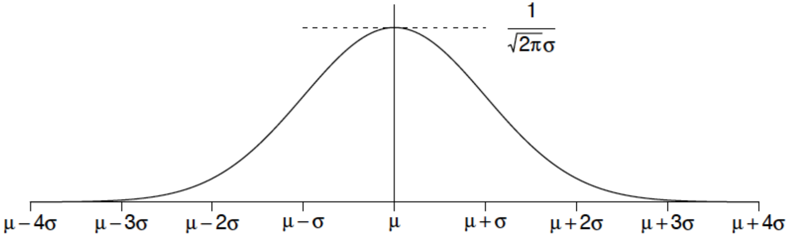
Parameterized by the mean $-\infty < \mu < \infty$ and variance $\sigma^2 > 0$

Normal Distribution (probability density function)

A random variable $X \sim \mathcal{N}(\mu, \sigma^2)$ has probability density function

$$f_X(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ \frac{-(x-\mu)^2}{2\sigma^2} \right\} \qquad -\infty < x < \infty$$

Parameterized by the mean $-\infty < \mu < \infty$ and variance $\sigma^2 > 0$



Cumulative Distribution Function (Normal Distribution)

Normal cumulative distribution function

$$F_X(x|\mu, \sigma^2) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ \frac{-(x-\mu)^2}{2\sigma^2} \right\} dx$$

Cumulative Distribution Function (Normal Distribution)

Normal cumulative distribution function

$$F_X(x|\mu,\sigma^2) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{\frac{-(x-\mu)^2}{2\sigma^2}\right\} dx$$

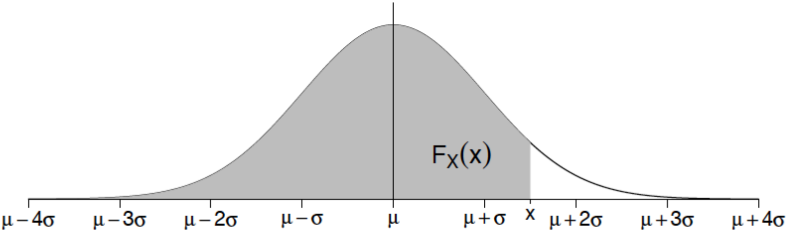
No closed-form expression for $F_X(x|\mu,\sigma^2)$

Cumulative Distribution Function (Normal Distribution)

Normal cumulative distribution function

$$F_X(x|\mu,\sigma^2) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{\frac{-(x-\mu)^2}{2\sigma^2}\right\} dx$$

No closed-form expression for $F_X(x|\mu,\sigma^2)$



Location-Scale Property (Normal Distribution)

The normal distribution is a location-scale distribution

Location-Scale Property (Normal Distribution)

The normal distribution is a location-scale distribution

- The mean μ is a *location* parameter

Location-Scale Property (Normal Distribution)

The normal distribution is a location-scale distribution

- The mean μ is a *location* parameter
- The standard deviation σ is a *scale* parameter

Location-Scale Property (Normal Distribution)

The normal distribution is a location-scale distribution

- The mean μ is a *location* parameter
- The standard deviation σ is a *scale* parameter

Let $X \sim \mathcal{N}(\mu, \sigma^2)$

$$Z = \frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$$

where $\mathcal{N}(0, 1)$ is called the *standard normal* distribution

Location-Scale Property (Normal Distribution)

The normal distribution is a location-scale distribution

- The mean μ is a *location* parameter
- The standard deviation σ is a *scale* parameter

Let $X \sim \mathcal{N}(\mu, \sigma^2)$

$$Z = \frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$$

where $\mathcal{N}(0, 1)$ is called the *standard normal* distribution

Probability that $X \sim \mathcal{N}(\mu, \sigma^2)$ falls in the interval $[a, b]$

$$P(a \leq X \leq b) = P\left(\frac{a - \mu}{\sigma} \leq Z \leq \frac{b - \mu}{\sigma}\right) = \Phi\left(\frac{b - \mu}{\sigma}\right) - \Phi\left(\frac{a - \mu}{\sigma}\right)$$

where $\Phi(x) = F_Z(z \,|\, 0, 1)$ is the *standard normal* cdf

Expected Value (Normal Distribution)

$$\mathrm{E}(X) \;=\; \int_{-\infty}^{\infty} x \, \frac{1}{\sqrt{2\pi}\sigma} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right] dx \qquad \left(\text{let } z = \frac{x-\mu}{\sigma}\right)$$

Expected Value (Normal Distribution)

$$\begin{aligned} \mathrm{E}(X) &= \int_{-\infty}^{\infty} x \, \frac{1}{\sqrt{2\pi}\sigma} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right] dx \qquad \left(\text{let } z = \frac{x-\mu}{\sigma}\right) \\ &= \int_{-\infty}^{\infty} (\mu + \sigma z) \, \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-z^2}{2}\right] dz \end{aligned}$$

Expected Value (Normal Distribution)

$$
\begin{aligned}
\mathrm{E}(X) &= \int_{-\infty}^{\infty} x \, \frac{1}{\sqrt{2\pi}\sigma} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right] dx \qquad \left(\text{let } z = \frac{x-\mu}{\sigma}\right) \\
&= \int_{-\infty}^{\infty} (\mu + \sigma z) \, \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-z^2}{2}\right] dz \\
&= \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-z^2}{2}\right] dz + \int_{-\infty}^{\infty} \frac{\sigma z}{\sqrt{2\pi}} \exp\left[\frac{-z^2}{2}\right] dz
\end{aligned}
$$

Expected Value (Normal Distribution)

$$
\begin{aligned}
\mathrm{E}(X) &= \int_{-\infty}^{\infty} x \, \frac{1}{\sqrt{2\pi}\sigma} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right] dx && \left(\text{let } z = \frac{x-\mu}{\sigma}\right) \\
&= \int_{-\infty}^{\infty} (\mu + \sigma z) \, \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-z^2}{2}\right] dz \\
&= \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-z^2}{2}\right] dz + \int_{-\infty}^{\infty} \frac{\sigma z}{\sqrt{2\pi}} \exp\left[\frac{-z^2}{2}\right] dz \\
&= \mu + \int_{-\infty}^{\infty} \frac{\sigma z}{\sqrt{2\pi}} \exp\left[\frac{-z^2}{2}\right] dz && (\text{let } v = -z^2/2)
\end{aligned}
$$

Expected Value (Normal Distribution)

$$
\begin{aligned}
\mathrm{E}(X) &= \int_{-\infty}^{\infty} x \, \frac{1}{\sqrt{2\pi}\sigma} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right] dx && \left(\text{let } z = \frac{x-\mu}{\sigma}\right) \\
&= \int_{-\infty}^{\infty} (\mu + \sigma z) \, \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-z^2}{2}\right] dz \\
&= \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-z^2}{2}\right] dz + \int_{-\infty}^{\infty} \frac{\sigma z}{\sqrt{2\pi}} \exp\left[\frac{-z^2}{2}\right] dz \\
&= \mu + \int_{-\infty}^{\infty} \frac{\sigma z}{\sqrt{2\pi}} \exp\left[\frac{-z^2}{2}\right] dz && (\text{let } v = -z^2/2) \\
&= \mu + \int_{z=-\infty}^{z=\infty} \frac{-1}{z} \frac{z}{\sqrt{2\pi}} e^v \, dv
\end{aligned}
$$

Normal Distribution (expected value)

$$= \quad \mu + \frac{-1}{\sqrt{2\pi}} \int_{z=-\infty}^{z=\infty} e^{v} \, dv$$

Normal Distribution (expected value)

$$
\begin{aligned}
&= \mu + \frac{-1}{\sqrt{2\pi}} \int_{z=-\infty}^{z=\infty} e^{v}\, dv \\
&= \mu + \frac{-1}{\sqrt{2\pi}}\; e^{v} \Big|_{z=-\infty}^{z=\infty}
\end{aligned}
$$

Normal Distribution (expected value)

$$
\begin{aligned}
&= \mu + \frac{-1}{\sqrt{2\pi}} \int_{z=-\infty}^{z=\infty} e^{v}\, dv \\
&= \mu + \frac{-1}{\sqrt{2\pi}}\, e^{v} \Big|_{z=-\infty}^{z=\infty} \\
&= \mu + \frac{-1}{\sqrt{2\pi}}\, e^{\frac{-z^2}{2}} \Big|_{z=-\infty}^{z=\infty}
\end{aligned}
$$

Normal Distribution (expected value)

$$
\begin{aligned}
&= \mu + \frac{-1}{\sqrt{2\pi}} \int_{z=-\infty}^{z=\infty} e^{v}\, dv \\
&= \mu + \frac{-1}{\sqrt{2\pi}}\, e^{v} \Big|_{z=-\infty}^{z=\infty} \\
&= \mu + \frac{-1}{\sqrt{2\pi}}\, e^{\frac{-z^2}{2}} \Big|_{z=-\infty}^{z=\infty} \\
&= \mu + \frac{-1}{\sqrt{2\pi}}\, (0 - 0)
\end{aligned}
$$

Normal Distribution (expected value)

$$
\begin{aligned}
&= \mu + \frac{-1}{\sqrt{2\pi}} \int_{z=-\infty}^{z=\infty} e^{v}\, dv \\
&= \mu + \frac{-1}{\sqrt{2\pi}}\, e^{v} \Big|_{z=-\infty}^{z=\infty} \\
&= \mu + \frac{-1}{\sqrt{2\pi}}\, e^{\frac{-z^2}{2}} \Big|_{z=-\infty}^{z=\infty} \\
&= \mu + \frac{-1}{\sqrt{2\pi}}\, (0 - 0) \\
&= \mu
\end{aligned}
$$

Variance (Normal Distribution)

$$\mathsf{E}(X)^2 \quad = \quad \int_{-\infty}^{\infty} x^2 \, \frac{1}{\sqrt{2\pi}\sigma} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right] dx \qquad \left(\text{let } z = \frac{x-\mu}{\sigma}\right)$$

Variance (Normal Distribution)

$$\begin{aligned}
\mathrm{E}(X)^2 &= \int_{-\infty}^{\infty} x^2 \, \frac{1}{\sqrt{2\pi}\sigma} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right] \, dx \qquad \left(\text{let } z = \frac{x-\mu}{\sigma}\right) \\
&= \int_{-\infty}^{\infty} (\mu + \sigma z)^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz
\end{aligned}$$

Variance (Normal Distribution)

$$
\begin{aligned}
\mathrm{E}(X)^2 &= \int_{-\infty}^{\infty} x^2 \, \frac{1}{\sqrt{2\pi}\sigma} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right] dx \qquad \left(\text{let } z = \frac{x-\mu}{\sigma}\right) \\
&= \int_{-\infty}^{\infty} (\mu + \sigma z)^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\
&= \int_{-\infty}^{\infty} (\mu^2 + 2\mu\sigma z + \sigma^2 z^2) \, \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz
\end{aligned}
$$

Variance (Normal Distribution)

$$
\begin{aligned}
\mathrm{E}(X)^2 &= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}\sigma} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right] dx \qquad \left(\text{let } z = \frac{x-\mu}{\sigma}\right) \\
&= \int_{-\infty}^{\infty} (\mu + \sigma z)^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\
&= \int_{-\infty}^{\infty} (\mu^2 + 2\mu\sigma z + \sigma^2 z^2) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\
&= \mu^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz + \mu\sigma \int_{-\infty}^{\infty} \frac{z}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\
&\quad + \sigma^2 \int_{-\infty}^{\infty} \frac{z^2}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz
\end{aligned}
$$

Variance (Normal Distribution)

$$
\begin{aligned}
\mathrm{E}(X)^2 &= \int_{-\infty}^{\infty} x^2 \, \frac{1}{\sqrt{2\pi}\sigma} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right] \, dx \qquad \left(\text{let } z = \frac{x-\mu}{\sigma}\right) \\
&= \int_{-\infty}^{\infty} (\mu + \sigma z)^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz \\
&= \int_{-\infty}^{\infty} (\mu^2 + 2\mu\sigma z + \sigma^2 z^2) \, \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz \\
&= \mu^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz + \mu\sigma \int_{-\infty}^{\infty} \frac{z}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz \\
&\qquad + \sigma^2 \int_{-\infty}^{\infty} \frac{z^2}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz \\
&= \mu^2 + \mu\sigma \, \mathrm{E}(Z) + \sigma^2 \int_{-\infty}^{\infty} \frac{z^2}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz
\end{aligned}
$$

Integration By Parts

$$= \quad \mu^2 + \sigma^2 \int_{-\infty}^{\infty} \frac{z^2}{\sqrt{2\pi}} \, e^{-\frac{z^2}{2}} \, dz$$

Integration By Parts

$$= \quad \mu^2 + \sigma^2 \int_{-\infty}^{\infty} \frac{z^2}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

Integration by parts:

$$\int u \; dv = uv - \int v \; du$$

Integration By Parts

$$= \quad \mu^2 + \sigma^2 \int_{-\infty}^{\infty} \frac{z^2}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2}}\, dz$$

Integration by parts:

$$\int u\ dv = uv - \int v\ du$$

- Let $u = \dfrac{z}{\sqrt{2\pi}} \qquad \rightarrow \qquad du = \dfrac{1}{\sqrt{2\pi}}\, dz$
- Let $v = -e^{-\frac{z^2}{2}} \qquad \rightarrow \qquad dv = z\, e^{-\frac{z^2}{2}}\ dz$

Variance (continued)

$$= \quad \mu^2 + \sigma^2 \left[\int_{-\infty}^{\infty} \frac{-z}{\sqrt{2\pi}} \left(-ze^{-\frac{z^2}{2}} \right) dz \right]$$

Variance (continued)

$$
\begin{aligned}
&= \mu^2 + \sigma^2 \left[\int_{-\infty}^{\infty} \frac{-z}{\sqrt{2\pi}} \left(-z e^{-\frac{z^2}{2}} \right) dz \right] \\
&= \mu^2 + \sigma^2 \left[\frac{-z}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \right]
\end{aligned}
$$

Variance (continued)

$$
\begin{aligned}
&= \mu^2 + \sigma^2 \left[\int_{-\infty}^{\infty} \frac{-z}{\sqrt{2\pi}} \left(-z e^{-\frac{z^2}{2}} \right) dz \right] \\
&= \mu^2 + \sigma^2 \left[\frac{-z}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz \right] \\
&= \mu^2 + \sigma^2 \left[0 + 1 \right]
\end{aligned}
$$

Variance (continued)

$$
\begin{aligned}
&= \mu^2 + \sigma^2 \left[\int_{-\infty}^{\infty} \frac{-z}{\sqrt{2\pi}} \left(-z e^{-\frac{z^2}{2}} \right) dz \right] \\
&= \mu^2 + \sigma^2 \left[\frac{-z}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \right] \\
&= \mu^2 + \sigma^2 [0 + 1] \\
&= \mu^2 + \sigma^2
\end{aligned}
$$

Variance (continued)

$$
\begin{aligned}
&= \mu^2 + \sigma^2 \left[\int_{-\infty}^{\infty} \frac{-z}{\sqrt{2\pi}} \left(-ze^{-\frac{z^2}{2}} \right) dz \right] \\
&= \mu^2 + \sigma^2 \left[\left. \frac{-z}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \right|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \right] \\
&= \mu^2 + \sigma^2 [0 + 1] \\
&= \mu^2 + \sigma^2
\end{aligned}
$$

$$
\mathrm{Var}(X) = \mathrm{E}(X^2) - \left[\mathrm{E}(X)\right]^2 = \mu^2 + \sigma^2 - (\mu)^2
$$

Variance (continued)

$$\begin{aligned}
&= \mu^2 + \sigma^2 \left[\int_{-\infty}^{\infty} \frac{-z}{\sqrt{2\pi}} \left(-ze^{-\frac{z^2}{2}} \right) dz \right] \\
&= \mu^2 + \sigma^2 \left[\left. \frac{-z}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \right|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \right] \\
&= \mu^2 + \sigma^2 [0 + 1] \\
&= \mu^2 + \sigma^2
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}(X^2) - \left[\mathrm{E}(X) \right]^2 = \mu^2 + \sigma^2 - (\mu)^2 \\
&= \sigma^2
\end{aligned}$$

Higher Moments (Normal Distribution)

First 4 moments and central moments

k	k^{th} moment	k^{th} central moment
1	μ	0
2	$\mu^2 + \sigma^2$	σ^2
3	$\mu^3 + 3\mu\sigma^2$	0
4	$\mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$	$3\sigma^4$

Higher Moments (Normal Distribution)

First 4 moments and central moments

k	k^{th} moment	k^{th} central moment
1	μ	0
2	$\mu^2 + \sigma^2$	σ^2
3	$\mu^3 + 3\mu\sigma^2$	0
4	$\mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$	$3\sigma^4$

Coefficient of skewness: $\alpha_3 = \dfrac{\mu_3}{(\mu_2)^{3/2}} = \dfrac{0}{\sigma^3} = 0$

Higher Moments (Normal Distribution)

First 4 moments and central moments

k	k^{th} moment	k^{th} central moment
1	μ	0
2	$\mu^2 + \sigma^2$	σ^2
3	$\mu^3 + 3\mu\sigma^2$	0
4	$\mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$	$3\sigma^4$

Coefficient of skewness: $\alpha_3 = \dfrac{\mu_3}{(\mu_2)^{3/2}} = \dfrac{0}{\sigma^3} = 0$

Coefficient of kurtosis: $\alpha_4 = \dfrac{\mu_4}{(\mu_2)^2} = \dfrac{3\sigma^4}{\sigma^4} = 3$

χ^2 Distribution

Let $Z_i \sim \mathcal{N}(0, 1)$ for $i = 1, 2, \ldots p$

χ^2 Distribution

Let $Z_i \sim \mathcal{N}(0, 1)$ for $i = 1, 2, \dots p$

The $\{Z_i\}$ must be independent

χ^2 Distribution

Let $Z_i \sim \mathcal{N}(0,1)$ for $i = 1, 2, \ldots p$

The $\{Z_i\}$ must be independent

The random variable

$$X = \sum_{i=1}^{p} Z_i^2$$

has a χ_p^2 distribution (read: chi-squared on p degrees of freedom)

χ^2 Distribution

Let $Z_i \sim \mathcal{N}(0,1)$ for $i = 1, 2, \ldots p$

The $\{Z_i\}$ must be independent

The random variable

$$X = \sum_{i=1}^{p} Z_i^2$$

has a χ_p^2 distribution (read: chi-squared on p degrees of freedom)

Hint: The Γ (gamma) function is defined to be

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} \, dt$$

and has the property that for integer $n > 0$

$$\Gamma(n) = (n-1)!$$

Probability Density Function (χ^2 Distribution)

A random variable $X \sim \chi^2_p$ has probability density function

$$f_X(x|p) = \frac{1}{\Gamma(\frac{p}{2})\,2^{\frac{p}{2}}} x^{(\frac{p}{2})-1} e^{-\frac{x}{2}} \qquad 0 \leq x < \infty$$

Probability Density Function (χ^2 Distribution)

A random variable $X \sim \chi^2_p$ has probability density function

$$f_X(x|p) = \frac{1}{\Gamma(\frac{p}{2})\,2^{\frac{p}{2}}} x^{(\frac{p}{2})-1} e^{-\frac{x}{2}} \qquad 0 \leq x < \infty$$

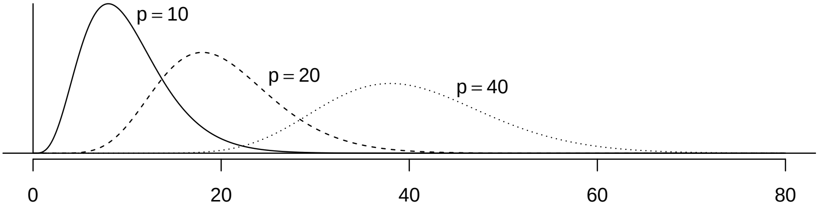
Parameterized by the degrees of freedom $p > 0$

Probability Density Function (χ^2 Distribution)

A random variable $X \sim \chi^2_p$ has probability density function

$$f_X(x|p) = \frac{1}{\Gamma(\frac{p}{2})\, 2^{\frac{p}{2}}} x^{(\frac{p}{2})-1} e^{-\frac{x}{2}} \qquad 0 \leq x < \infty$$

Parameterized by the degrees of freedom $p > 0$



χ^2 Distribution (expected value and variance)

Let $X \sim \chi^2_p$

χ^2 Distribution (expected value and variance)

Let $X \sim \chi^2_p$

- The expected value of X is $\mathrm{E}(X) = p$

χ^2 Distribution (expected value and variance)

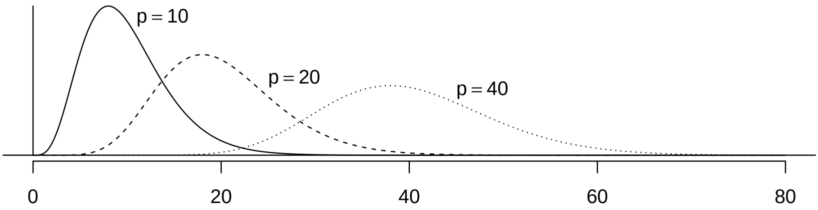
Let $X \sim \chi^2_p$

- The expected value of X is $\mathrm{E}(X) = p$
- The variance of X is $\mathrm{Var}(X) = 2p$

χ^2 Distribution (expected value and variance)

Let $X \sim \chi^2_p$

- The expected value of X is $\mathrm{E}(X) = p$
- The variance of X is $\mathrm{Var}(X) = 2p$



t Distribution

Let $X \sim \mathcal{N}(0, 1)$

t Distribution

Let $X \sim \mathcal{N}(0, 1)$

Let $Y \sim \chi^2_p$

t Distribution

Let $X \sim \mathcal{N}(0, 1)$

Let $Y \sim \chi^2_p$

Then the random variable

$$T = \frac{X}{Y}$$

has a t distribution with p degrees of freedom

Probability Density Function (t Distribution)

A random variable $X \sim t_p$ has probability density function

$$f_X(x|p) = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} \frac{1}{\sqrt{p\pi}} \frac{1}{\left(1 + \left(\frac{x^2}{p}\right)\right)^{(p+1)/2}} \qquad -\infty < x < \infty$$

Probability Density Function (t Distribution)

A random variable $X \sim t_p$ has probability density function

$$f_X(x|p) = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} \frac{1}{\sqrt{p\pi}} \frac{1}{\left(1 + \left(\frac{x^2}{p}\right)\right)^{(p+1)/2}} \qquad -\infty < x < \infty$$

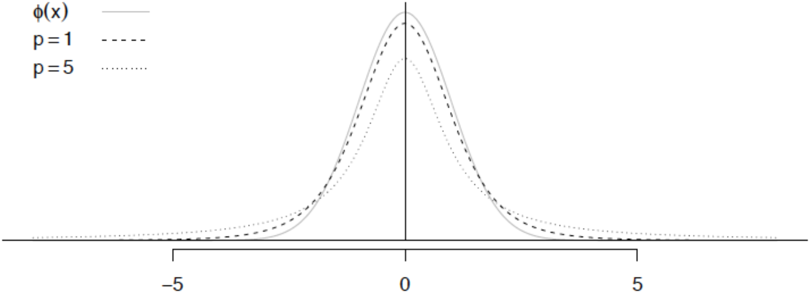
Parameterized by the degrees of freedom $p > 0$

Probability Density Function (t Distribution)

A random variable $X \sim t_p$ has probability density function

$$f_X(x|p) = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} \frac{1}{\sqrt{p\pi}} \frac{1}{\left(1 + \left(\frac{x^2}{p}\right)\right)^{(p+1)/2}} \qquad -\infty < x < \infty$$

Parameterized by the degrees of freedom $p > 0$



Expected Value and Variance (t Distribution)

Let $X \sim t_p$

Expected Value and Variance (t Distribution)

Let $X \sim t_p$

- the expected value of X is $\mathrm{E}(X) = 0$ when $p > 1$

Expected Value and Variance (t Distribution)

Let $X \sim t_p$

- the expected value of X is $\mathrm{E}(X) = 0$ when $p > 1$
- the variance of X is $\mathrm{Var}(X) = \frac{p}{p-2}$ when $p > 2$

Expected Value and Variance (t Distribution)

Let $X \sim t_p$

- the expected value of X is $\mathrm{E}(X) = 0$ when $p > 1$
- the variance of X is $\mathrm{Var}(X) = \frac{p}{p-2}$ when $p > 2$

Variations:

- Noncentral t distribution

$$t_p \approx \frac{\text{normal}(0,\ 1)}{\sqrt{\chi_p^2/p}} \qquad \text{noncentral } t_p \approx \frac{\text{normal}(\mu, 1)}{\sqrt{\chi_p^2/p}}$$

has noncentrality parameter $\delta = \sqrt{\mu^2}$

Expected Value and Variance (t Distribution)

Let $X \sim t_p$

- the expected value of X is $\mathrm{E}(X) = 0$ when $p > 1$
- the variance of X is $\mathrm{Var}(X) = \frac{p}{p-2}$ when $p > 2$

Variations:

- Noncentral t distribution

$$t_p \approx \frac{\text{normal}(0,\ 1)}{\sqrt{\chi^2_p/p}} \qquad \text{noncentral } t_p \approx \frac{\text{normal}(\mu, 1)}{\sqrt{\chi^2_p/p}}$$

  has noncentrality parameter $\delta = \sqrt{\mu^2}$

- Skewed t Distribution

Outline

Absolute Risk Measures

- A widely used absolute risk measure is Value-at-Risk (VaR)
- Developed after the stock market crash of 1987. JP Morgan published the methodology in 1994
- In 1997 US SEC ruled that public corporations must disclose quantitative information about their derivatives activity
- Major banks and dealers chose to use VaR in their financial statements to implement the rule
- VaR is the preferred measure of market risk in Basel II

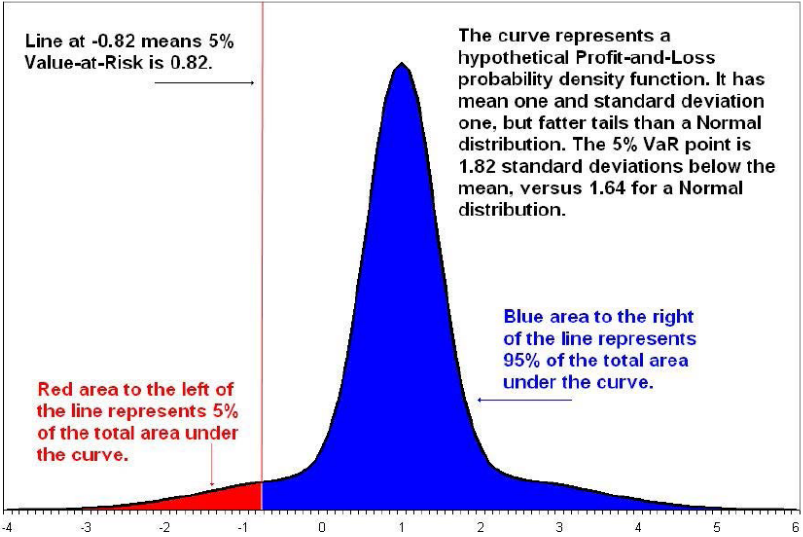
VaR

- Value-at-Risk is an effort to enable risk managers to make a statement of the form:
  - *we are* $1-\alpha$ *percent certain that we will not lose more than V dollars in the next N days*
- V is the VaR of the asset associated with a horizon of N days and certainty level of $(1-\alpha)\times 100\%$
- It is the loss level over N days that we are $(1-\alpha)\times 100\%$ certain will not be exceeded
- In general, VaR is the loss corresponding to the α^{th} percentile of the distribution of the change in value of the portfolio over the next N days

VaR

Value-at-Risk (VaR): a threshold value of loss over a given period, often stated with a probability that the loss will exceed this threshold.
Example:

- An asset with "a one-day $\alpha = 0.05$ VaR of \$ 1 million" means that there is a 5% probability (1 in 20 chance) that the asset's loss will exceed \$ 1 million over a one-day period
- Note that the actual loss could be much more that \$ 1 million; in this sense, VaR is not aptly named


Line at -0.82 means 5%
Value-at-Risk is 0.82.
The curve represents a hypothetical Profit-and-Loss probability density function. It has mean one and standard deviation one, but fatter tails than a Normal distribution. The 5% VaR point is 1.82 standard deviations below the mean, versus 1.64 for a Normal distribution.
Blue area to the right of the line represents 95% of the total area under the curve.
Red area to the left of the line represents 5% of the total area under the curve.
-4 -3 -2 -1 0 1 2 3 4 5 6

Statistical Definition of VaR

$\Delta W = W - W_0$: the 1-period change in the asset value $1-\alpha$: VaR

certainty level q_α: the α-quantile of the 1-period change in the asset

value

$$P(\Delta W \leq q_\alpha) = \int_{-\infty}^{q_\alpha} f(x)dx = F(q_\alpha) = \alpha$$

$$\implies q_\alpha = F^{-1}(\alpha)$$

The VaR is:

$$\text{VaR}(\alpha) = -q_\alpha$$

Interpretation: the asset/portfolio \$ losses will be q_α or larger with probability α

Statistical Definition of VaR

Equivalently, we can think of VaR in terms of the distribution of the asset's 1-period arithmetic rate of return

$$
\begin{aligned}
\alpha = P(\Delta W \leq q_\alpha) &= P\left(W_0\left(\frac{W - W_0}{W_0}\right) \leq q_\alpha\right) \\
&= P(W_0 r \leq q_\alpha) = P\left(r \leq \frac{q_\alpha}{W_0}\right) \\
&= P(r \leq q_\alpha^{(r)}
\end{aligned}
$$

where

$$\alpha - \text{quantile of r :} \qquad q_\alpha^{(r)} = \frac{q_\alpha}{W_0}$$

The 1-period VaR is

$$\text{VaR}(\alpha) = -q_\alpha^{(r)} W_0$$

where W_0 is the initial value of the asset

Example

$W_0 = \$1,000,000$: asset price today $\alpha = 0.05 \implies$ 95% VaR certainty

level $q_\alpha = -\$1,000$: the $\alpha-$quantile for the 1-day asset price change

$$\text{VaR}(\alpha) = -q_\alpha = \$1,000$$

VaR

Why is VaR in widespread use?

- Distills risk-management to a single number
- Easy to implement using a model or empirical distribution

Serious limitation of VaR:

- It is a quantile measure; no information about how large losses are beyond VaR!
- Easy to misunderstand and potentially catastrophic if it creates a false sense of security

Interesting New York Times article on VaR:
`http://www.nytimes.com/2009/01/04/magazine/04risk-t.html?dlbk=&pagewanted=all&_r=0`

COMPUTATIONAL FINANCE & RISK MANAGEMENT
UNIVERSITY *of* WASHINGTON
Department of Applied Mathematics

http://computational-finance.uw.edu