COMPUTATIONAL FINANCE & RISK MANAGEMENT
UNIVERSITY *of* WASHINGTON
Department of Applied Mathematics

CFRM 410: Probability and Statistics for Computational Finance

Week 7 Multivariate Random Variables

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Outline

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Multivariate Random Variables

An *n-dimensional random vector* is a function from the sample space S into $\mathbb{R}^n$ (n-dimensional Euclidean space)

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- The vector (X, Y) is a *bivariate random vector*

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Discrete Case: The joint probability of X and Y is described by the *joint mass function*

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for all possible pairs $(x_i, y_j) \in S_{XY}$

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Continuous Case: The joint density of X and Y is described by the *joint density function*

$$f_{X,Y}(x, y)$$

Joint Density

The joint distribution function is

$$P(X \leq x, Y \leq y) = F_{X,Y}(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(u, v)\, du\, dv$$

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- $f_{X,Y}(x,y) = \dfrac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$
- $P(a_1 \leq X \leq b_1, a_2 \leq Y \leq b_2) = \displaystyle\int_{a_2}^{b_2} \int_{a_1}^{b_1} f_{X,Y}(x,y)\, dx\, dy$

Marginal Distribution

Let (X, Y) be a bivariate random vector with joint density (mass) function $f_{X,Y}(x, y)$, then the function

$$\text{Discrete: } f_X(x_i) = \sum_{y_j \in S_Y} f_{X,Y}(x_i, y_j)$$

$$\text{Continuous: } f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy$$

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The *marginal cumulative distribution function* of X is

$$\text{Discrete: } F_X(x) = \sum_{x_i \le x} f_X(x_i)$$

$$\text{Continuous: } F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$$

Example

Suppose the random vector (X, Y) takes values in the set

$$\{(1,2),(1,4),(2,3),(3,2),(3,4)\}$$

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What are $f_X(x)$ and $f_Y(y)$?

$$f_X(x) = \begin{cases} 0.4 & x = 1 \\ 0.2 & x = 2 \\ 0.4 & x = 3 \end{cases} \qquad f_Y(y) = \begin{cases} 0.4 & y = 2 \\ 0.2 & y = 3 \\ 0.4 & y = 4 \end{cases}$$

Expected Value

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The expected value of a function $g(X, Y)$ is

$$\mathrm{E}\big(g(X, Y)\big) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)\, f_{X,Y}(x, y)\, dx\, dy$$

Expected Value of the Sum of Two Random Variables

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Suppose (X, Y) is a bivariate random vector with joint pdf $f_{X,Y}(x, y)$

What is the expected value of $(X + Y)$?

Outline

Independence

Two discrete random variables X and Y are *independent* if

$$P(X = x_i, Y = y_j) = P(X = x_i)P(Y = y_j)$$

holds for all pairs $(x_i, y_j) \in S_{XY}$

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In general, X and Y are *independent* if

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The notation $X, Y \overset{iid}{\sim} f$ means that X and Y are *independent and identically distributed*, that is $f_X(x) = f_Y(y) = f$

Example

Recall the previous example

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$$f_X(x) = \begin{cases} 0.4 & x = 1 \\ 0.2 & x = 2 \\ 0.4 & x = 3 \end{cases} \qquad f_Y(y) = \begin{cases} 0.4 & y = 2 \\ 0.2 & y = 3 \\ 0.4 & y = 4 \end{cases}$$

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Are X and Y independent?

$$P(X = 1, Y = 4) = 0.2 \neq 0.16 = P(X = 1)P(Y = 4)$$

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If X and Y are independent

$$f_{X|Y}(x|y) = f_X(x) \qquad f_{Y|X}(y|x) = f_Y(y) \qquad \text{for all } x, y$$

Example

Let (X, Y) be a bivariate random vector with joint pdf

$$f_{X,Y}(x,y) = \begin{cases} x + y & 0 < x < 1, \ \ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

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First, compute the marginal density of Y

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx = \int_0^1 (x + y)\, dx = \left. \frac{1}{2}x^2 + xy \right|_0^1 = y + \frac{1}{2}$$

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By the definition of conditional probability

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{x+y}{y+\frac{1}{2}} \qquad 0<x<1, \ \ 0<y<1$$

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- $\mathrm{Cov}(aX + b, cY + d) = ac\,\mathrm{Cov}(X, Y)$ $\quad$ a, b, c, d constant
- If X and Y are independent then $\mathrm{Cov}(X, Y) = 0$
- $\mathrm{Cov}(X, Y) = 0$ does NOT imply X and Y are independent

Example

Let X and Y be random variables with joint density

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$$\mathrm{E}(X) \;\; = \;\; \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_0^1 x(x+\frac{1}{2})\, dx = \frac{1}{3}x^3 + \frac{1}{4}x^2\Big|_0^1 = \frac{7}{12}$$

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\mathrm{E}(X^2) &= \int_{-\infty}^{\infty} x^2\, f_X(x)\, dx = \int_0^1 x^2(x+\frac{1}{2}) = \frac{1}{4}x^4 + \frac{1}{6}x^3\Big|_0^1 = \frac{5}{12} \\
\mathrm{Var}(X) &= \mathrm{E}(X^2) - \big(\mathrm{E}(X)\big)^2 = \frac{5}{12} - \left[\frac{7}{12}\right]^2 = \frac{11}{144}
\end{aligned}$$

Example (continued)

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\, dx\, dy$$

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\begin{aligned}
\mathrm{E}(XY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\, dx\, dy \\
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&= \int_0^1 \frac{1}{3}y + \frac{1}{2}y^2\, dy \\
&= \frac{1}{6}y^2 + \frac{1}{6}y^3 \Big|_{y=0}^{y=1} = \frac{1}{3}
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&= \int_0^1 \frac{1}{3}y + \frac{1}{2}y^2\, dy \\
&= \frac{1}{6}y^2 + \frac{1}{6}y^3 \Big|_{y=0}^{y=1} = \frac{1}{3}
\end{aligned}
$$

$$
\mathrm{Cov}(X,Y) = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y) = \frac{1}{3} - \left[\frac{7}{12}\right]\left[\frac{7}{12}\right] = -\frac{1}{144}
$$

Variance of the Sum of Two Random Variables

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$$
\begin{aligned}
\mathrm{Var}(X+Y) &= \mathrm{E}\big[(X+Y)^2\big] - \big[\mathrm{E}(X+Y)\big]^2 \\
&= \mathrm{E}(X^2 + 2XY + Y^2) - \big[\mathrm{E}(X) + \mathrm{E}(Y)\big]^2
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&= \mathrm{E}(X^2) + \mathrm{E}(2XY) + \mathrm{E}(Y^2) \\
&\qquad - \left[\mathrm{E}(X)\right]^2 - 2\mathrm{E}(X)\mathrm{E}(Y) - \left[\mathrm{E}(Y)\right]^2
\end{aligned}
$$

Variance of the Sum of Two Random Variables

Let (X, Y) be a bivariate random vector with joint pdf $f_{X,Y}(x, y)$

What is the variance of $(X + Y)$?

$$
\begin{aligned}
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&= \mathrm{E}(X^2) + \mathrm{E}(2XY) + \mathrm{E}(Y^2) \\
&\quad - \big[\mathrm{E}(X)\big]^2 - 2\mathrm{E}(X)\mathrm{E}(Y) - \big[\mathrm{E}(Y)\big]^2 \\
&= \mathrm{E}(X^2) - \big[\mathrm{E}(X)\big]^2 \\
&\quad + 2\big[\mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y)\big] \\
&\quad + \mathrm{E}(Y^2) - \big[\mathrm{E}(Y)\big]^2
\end{aligned}
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More generally,

$$\mathrm{Var}(aX + bY + c) = a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y) + 2ab\,\mathrm{Cov}(X, Y)$$

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The *correlation* is a unit-free measure of the (linear) dependence between two random variables X and Y

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- Correlation $\neq$ Causality!

Example

Let X and Y be random variables with joint density

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Autocovariance and Autocorrelation

Let $\{X_t, t \in \mathbb{Z}\}$ be a sequence of time-ordered random variables

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$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = \mathsf{Corr}(X_{t+h}, X_t)$$

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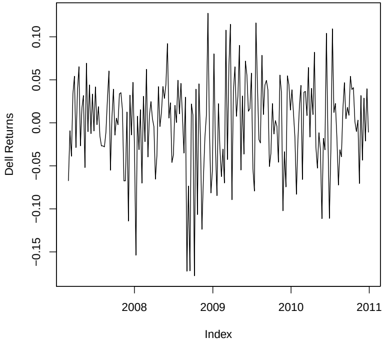
Recall: independence implies uncorrelated

Dell Returns Example



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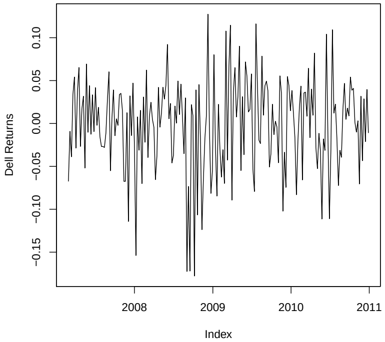

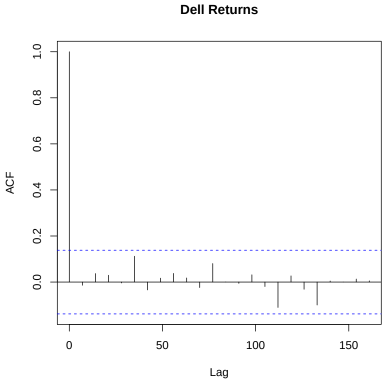

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- Consider the Dell returns squared ...

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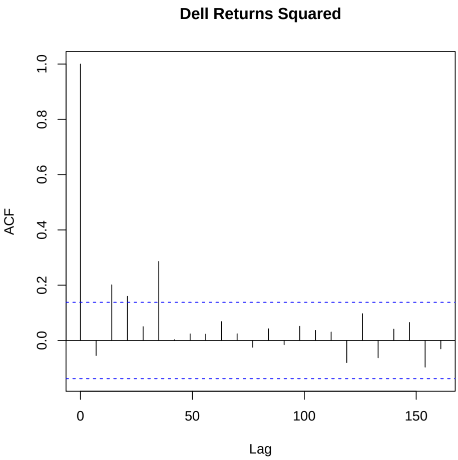

Outline

Mixture Distributions

A random variable X has a *mixture distribution* if its distribution depends on a quantity that also has a distribution

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Let ...

$$\begin{aligned} Y &\sim \text{Binomial}(1, 0.05) \\ X|Y=1 &\sim \mathcal{N}(-12, 6^2) \end{aligned}$$

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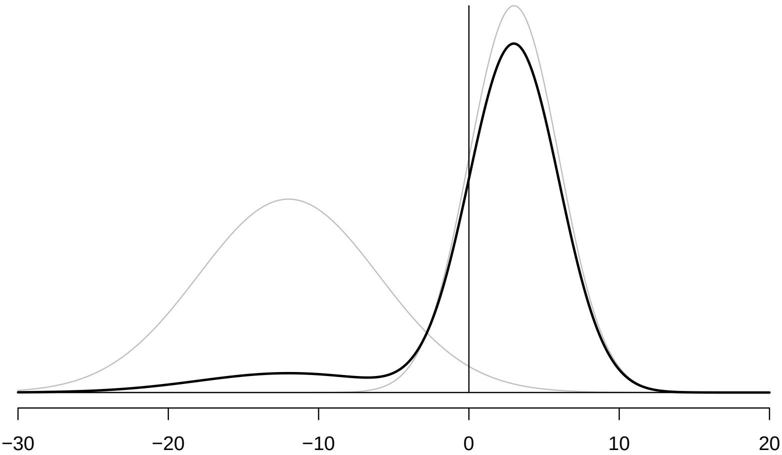
The distribution of X depends on Y and Y has a distribution

$\implies X$ has a mixture distribution

Density of X



Conditional Expectation

Let X and Y be any 2 random variables, then

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provided that the expectations exist

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Conditional Variance

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For the first term:

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Example (continued)

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$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}\left[\mathrm{Var}(X|Y)\right] + \mathrm{Var}\left[\mathrm{E}(X|Y)\right]\\ &= 10.85 + 10.6875 = 21.5375\end{aligned}$$

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The joint pdf $f_{U,V}(u, v)$ is positive on the set $S_{U,V}$

The map from S_{XY} to S_{UV} must be one-to-one and onto, that is

$$(u, v) = (g_1(x, y), g_2(x, y))$$

can be inverted

$$(x, y) = (h_1(u, v), h_2(u, v))$$

Bivariate Change of Variables Formula (continued)

The bivariate change of variables formula is given by

$$f_{U,V}(u,v) = f_{X,Y}\left[h_1(u,v), h_2(u,v)\right]|J|$$

Bivariate Change of Variables Formula (continued)

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The quantity J is called the Jacobian of the transformation

$$J = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

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The partial derivatives are

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Substitute expressions for h_1, h_2, and J . . .

Example (continued)

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&= \frac{1}{\sqrt{2\pi}\sqrt{2}}\exp\left[-\frac{u^2}{2(\sqrt{2})^2}\right]\ \frac{1}{\sqrt{2\pi}\sqrt{2}}\exp\left[-\frac{v^2}{2(\sqrt{2})^2}\right]
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Random variables U and V are independent if the joint density is the product of the marginal densities

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Let (X, Y) be a bivariate random vector with joint density $f_{X,Y}(x, y)$

X and Y are independent iff there exist $g(x)$ and $h(y)$ such that

$$f_{X,Y}(x, y) = g(x)\, h(y)$$

for all $x \in \mathbb{R}$ and $y \in \mathbb{R}$

Outline

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The joint density function for the bivariate normal is

$$f_{X,Y}(x,y) =$$

$$k \ \exp\left\{ -\frac{\left[\frac{x-\mu_X}{\sigma_X}\right]^2 - 2\rho\left[\frac{x-\mu_X}{\sigma_X}\right]\left[\frac{y-\mu_Y}{\sigma_Y}\right] + \left[\frac{y-\mu_Y}{\sigma_Y}\right]^2}{2(1-\rho^2)} \right\}$$

where the normalization constant

$$k = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}$$

Bivariate Normal Density

The joint density can be written compactly in matrix notation

$$f_{X,Y}(\mathbf{x}) = \frac{1}{(\sqrt{2\pi})^2 |\mathbf{\Sigma}|^{\frac{1}{2}}} \exp\left\{ -\frac{1}{2} \left[\mathbf{x} - \boldsymbol{\mu}\right]^\mathsf{T} \mathbf{\Sigma}^{-1} \left[\mathbf{x} - \boldsymbol{\mu}\right] \right\}$$

where

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \boldsymbol{\mu} = \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix} \qquad \text{and} \qquad \mathbf{\Sigma} = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_Y\sigma_X & \sigma_Y^2 \end{bmatrix}$$

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Recall ρ (correlation coefficient) is zero when X and Y independent

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hmmm ...

Sum of Normal Random Variables

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The rest seems like a good homework question

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Department of Applied Mathematics

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