Homework policy: you must show your work to receive credit for these exercises. It is your responsibility to convince the grader that you understand how to solve each of these exercises and to explain precisely how you arrived at your solution.

1. Suppose that an investor holds a \$10,000 position in an asset whose daily arithmetic returns distribution (measured in percent) is standard normal. Compute the 1-day $\alpha = 0.05$ Value-at-Risk.

   If the returns distribution measured in percents is standard normal, we can use a cumulative distribution table to find that $\Phi(-1.65) \approx 0.05$, so that the 1-day $\text{VaR}(0.05) \approx (0.0165)(\$10,000) = \$165$. The losses will be \$165 or more with probability 0.05.

2. Suppose that an investor holds a \$10,000 position in an asset whose daily arithmetic returns distribution (measured in percent) is t with 5 degrees of freedom. Compute the 1-day $\alpha = 0.05$ Value-at-Risk.

   Using a t distribution calculator, we find that $0.05 = P(X < -2.015)$. Using this we have the 1-day $\text{VaR}(0.05) = (0.02015)(\$10,000) = \$201.5$

3. Use the definition $\text{Var}(U) = \mathbb{E}(U^2) - \big[\mathbb{E}(U)\big]^2$ to show that

   $$\text{Var}(aX + bY + c) = a^2\,\text{Var}(X) + b^2\,\text{Var}(Y) + 2ab\,\text{Cov}(X,Y)$$

   where a, b, and c are constants.

$$\begin{aligned}
&\text{Var}(aX + bY + c)\\
&= \mathbb{E}((aX + bY + c)^2) - \big[\mathbb{E}(aX + bY + c)\big]^2\\
&= \mathbb{E}(a^2X^2 + 2abXY + 2acX + b^2Y^2 + 2bcY + c^2) - \big[\mathbb{E}(aX) + \mathbb{E}(bY) + \mathbb{E}(c)\big]^2\\
&= a^2\mathbb{E}(X^2) + 2ab\mathbb{E}(XY) + 2ac\mathbb{E}(X) + b^2\mathbb{E}(Y^2) + 2bc\mathbb{E}(Y) + c^2 - \big[a\mathbb{E}(X) + b\mathbb{E}(Y) + c\big]^2\\
&= a^2\mathbb{E}(X^2) + 2ab\mathbb{E}(XY) + 2ac\mathbb{E}(X) + b^2\mathbb{E}(Y^2) + 2bc\mathbb{E}(Y) + c^2\\
&\qquad - a^2[\mathbb{E}(X)]^2 - 2ab\mathbb{E}(X)\mathbb{E}(Y) - 2ac\mathbb{E}(X) - b^2[\mathbb{E}(Y)]^2 - 2bc\mathbb{E}(Y) - c^2\\
&= a^2(\mathbb{E}(X^2) - [\mathbb{E}(X)]^2) + b^2(\mathbb{E}(Y^2) - [\mathbb{E}(Y)]^2) + 2ab(\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y))\\
&= a^2\,\text{Var}(X) + b^2\,\text{Var}(Y) + 2ab\,\text{Cov}(X,Y).
\end{aligned}$$

The identity $\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$ was used in the last step, along with the identity given in the problem statement.

4. Suppose that the bivariate random vector (R_1, R_2) represents the yearly rate of return on assets 1 and 2. Further, suppose that $\mathbb{E}(R_1) = 0.04$, $\mathbb{E}(R_2) = 0.06$, $\mathrm{SD}(R_1) = \sigma_1 = 0.15$, $\mathrm{SD}(R_2) = \sigma_2 = 0.30$, and $\mathrm{Cor}(R_1, R_2) = \rho_{12} = 0.6$. An investor creates a portfolio by putting \$500 in asset 1 and \$500 in asset 2.

   a) Express the portfolio's arithmetic return R_P in terms of R_1 and R_2.

      $R_P = \frac{1}{2}R_1 + \frac{1}{2}R_2$

   b) Compute $\mathbb{E}(R_P)$.

      $\mathbb{E}(R_P) = \frac{1}{2}\mathbb{E}(R_1) + \frac{1}{2}\mathbb{E}(R_2) = 0.05$

   c) Compute $\mathrm{Var}(R_P)$.

      CORRECTION : THERE WAS A COMPUTATION ERROR

$$\begin{aligned}
\mathrm{Var}(R_P) &= \mathrm{Var}(\frac{1}{2}R_1 + \frac{1}{2}R_2) \\
&= \frac{1}{4}\mathrm{Var}(R_1) + \frac{1}{4}\mathrm{Var}(R_2) + 2\frac{1}{4}\mathrm{Cov}(X,Y) \\
&= \frac{1}{4}\sigma_1^2 + \frac{1}{4}\sigma_2^2 + \frac{1}{2}\sigma_1\sigma_2\mathrm{Cor}(R_1,R_2) \\
&= \frac{1}{4}(0.15)^2 + \frac{1}{4}(0.3)^2 + \frac{1}{2}(0.15)(0.3)(0.6) \\
&= 0.0416
\end{aligned}$$

Next, suppose the investor decides to invest $\$x$ in asset 1 and $\$(1000 - x)$ in asset 2.

   d) Express the portfolio's arithmetic return R_{MV} in terms of R_1, R_2, and x.

      $R_{MV} = \frac{x}{1000}R_1 + \frac{1000-x}{1000}R_2$

   e) Compute $\mathbb{E}(R_{MV})$.

      $\mathbb{E}(R_{MV}) = \frac{x}{1000}\mathbb{E}(R_1) + \frac{1000-x}{1000}\mathbb{E}(R_2) = \frac{0.04x+60-0.06x}{1000} = \frac{60-0.02x}{1000}$

   f) Compute $\mathrm{Var}(R_{MV})$.

$$\begin{aligned}
\mathrm{Var}(R_{MV}) &= \frac{x^2}{1000^2}\sigma_1^2 + \frac{(1000-x)^2}{1000^2}\sigma_2^2 + 2\frac{1000x - x^2}{1000^2}\mathrm{Cov}(R_1,R_2) \\
&= \frac{1}{1000^2}[\sigma_1^2x^2 + \sigma_2^2(1000^2 - 2000x + x^2) + (2000x - 2x^2)\mathrm{Cov}(R_1,R_2)] \\
&= \frac{1}{1000^2}[(\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\mathrm{Cor}(R_1,R_2))x^2 + (2000\sigma_1\sigma_2\mathrm{Cor}(R_1,R_2) - 2000\sigma_2^2)x + \sigma_2^2(1000^2)] \\
&= \frac{1}{1000^2}[0.0585x^2 - 126x + 90000]
\end{aligned}$$

g) How much should the investor put into each asset to minimize the risk (i.e., variance) of the portfolio?

$0 = (2)(0.0585)x - 126 \implies x = 1076.9$

This means the investor should put \$1000 in the first asset and \$0 in the second. This answer seems incorrect but I cannot find my error.

5. Let (X, Y) be a bivariate random vector uniformly distributed on the region R, a diamond with corners at $(\pm 1, 0)$, $(0, \pm 1)$.

(a) What are S_X, S_Y, and S_{XY}?

$S_X = [-1, 1]$, $S_Y = [-1, 1]$, $S_{XY} = \{(x, y) : |x| + |y| \leq 1\}$

(b) What is the joint probability density function (pdf) $f_{XY}(x, y)$ of (X, Y)?

$f_{XY}(x, y) = c$ for some constant c. Use integration over R to determine c:

$$1 = 2\int_0^1 \int_{-1+x}^{1-x} c\, dydx = 2c\int_0^1 [(1-x)-(-1+x)]\, dx = 4c\int_0^1 (1-x)\, dx = 4c[1-\frac{1}{2}] = 2c$$

$$\implies c = \frac{1}{2}$$

We could have also used the fact that R is a square to avoid integration, but followed the more general procedure in this case.

(c) What are the marginal probability density functions $f_X(x)$ and $f_Y(y)$?

$$f_X(x) = \int_{-1+|x|}^{1-|x|} \frac{1}{2}\, dy = \frac{1}{2}[(1 - |x|) - (-1 + |x|)] = 1 - |x|\,.$$

$$f_Y(y) = \int_{-1+|y|}^{1-|y|} \frac{1}{2}\, dx = \frac{1}{2}[(1 - |y|) - (-1 + |y|)] = 1 - |y|\,.$$

(d) What is the conditional probability density function $f_{Y|X}(y|X = x)$?

$$f_{Y|X}(y|X = x) = \frac{f_{X,Y}(x, y)}{f_X(x)} = \frac{\frac{1}{2}}{1 - |x|} = \frac{1}{2 - 2|x|}, \quad -1 + |x| \leq y \leq 1 - |x|$$

(e) Are X and Y independent? Explain your answer.

No. For example $f_{X,Y}(\frac{3}{4}, \frac{3}{4}) = 0$ while $f_X(\frac{3}{4})f_Y(\frac{3}{4}) = (\frac{1}{4})^2 \neq 0$.

(f) Compute $\mathrm{E}(X)$, $\mathrm{E}(Y)$, $\mathrm{Var}(X)$, $\mathrm{Var}(Y)$, $\mathrm{Cov}(X,Y)$.

$$\mathrm{E}(X) = \int_{-1}^{0}\int_{-1-x}^{1+x}\frac{1}{2}x\,dydx+\int_{0}^{1}\int_{-1+x}^{1-x}\frac{1}{2}x\,dydx = \int_{-1}^{0}(x+x^2)\,dx+\int_{0}^{1}(x-x^2)\,dx = -\frac{1}{6}+\frac{1}{6} = 0\,.$$

$$\mathrm{E}(Y) = \int_{-1}^{0}\int_{-1-y}^{1+y}\frac{1}{2}y\,dxdy+\int_{0}^{1}\int_{-1+y}^{1-y}\frac{1}{2}y\,dxdy = \int_{-1}^{0}(y+y^2)\,dy+\int_{0}^{1}(y-y^2)\,dy = -\frac{1}{6}+\frac{1}{6} = 0\,.$$

$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 \\
&= \mathrm{E}(X^2) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f_{X,Y}\,dxdy \\
&= \frac{1}{2}\int_{-1}^{0}\int_{-1-x}^{1+x} x^2\,dydx + \frac{1}{2}\int_{0}^{1}\int_{-1+x}^{1-x} x^2\,dydx \\
&= \int_{-1}^{0} x^2(1+x)\,dx + \int_{0}^{1} x^2(1-x)\,dx \\
&= \frac{1}{12} + \frac{1}{12} = \frac{1}{6}\,.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(Y) &= \mathrm{E}(Y^2) - [\mathrm{E}(Y)]^2 \\
&= \mathrm{E}(Y^2) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 f_{X,Y}\,dxdy \\
&= \frac{1}{2}\int_{-1}^{0}\int_{-1-y}^{1+y} y^2\,dxdy + \frac{1}{2}\int_{0}^{1}\int_{-1+y}^{1-y} y^2\,dxdy \\
&= \int_{-1}^{0} y^2(1+y)\,dy + \int_{0}^{1} y^2(1-y)\,dy \\
&= \frac{1}{12} + \frac{1}{12} = \frac{1}{6}\,.
\end{aligned}$$

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\begin{aligned}
\operatorname{Cov}(X,Y) &= \mathrm{E}[(X-\mathrm{E}(X))(Y-\mathrm{E}(Y))] \\
&= \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y) \\
&= \mathrm{E}(XY) - (0)(0) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\,dxdy \\
&= \frac{1}{2}\int_{-1}^{0}\int_{-1-x}^{1+x} xy\,dydx + \frac{1}{2}\int_{0}^{1}\int_{-1+x}^{1-x} xy\,dydx \\
&= \frac{1}{2}\int_{-1}^{0} 0\,dx + \frac{1}{2}\int_{0}^{1} 0\,dx \\
&= 0\;.
\end{aligned}
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