COMPUTATIONAL FINANCE & RISK MANAGEMENT
UNIVERSITY of WASHINGTON
Department of Applied Mathematics

# CFRM 410: Probability and Statistics for Computational Finance

## Week 10 Hypothesis Testing

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Slides originally produced by Kjell Konis

# Outline

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- Natural asymmetry between the two hypotheses $\implies$ type I error worse than type II
  - For example, an email SPAM filter

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Testing a Hypothesis

Significance

$p$-Values

R Lab

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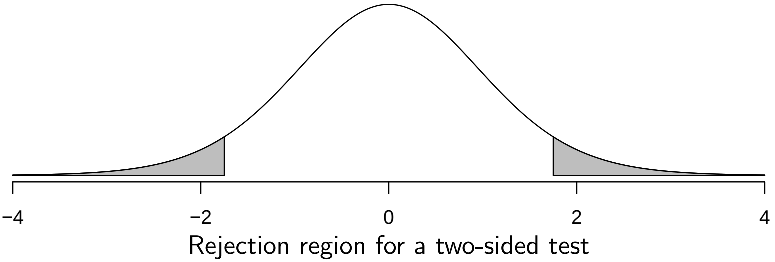
$$\mathsf{P}\big[\, T \text{ extreme} \mid H_0\big] = \mathsf{P}\big[\text{Reject } H_0 \mid H_0\big]$$

- In other words, choose the critical value so that

$$\mathsf{P}\big[\, T \text{ more extreme than critical value(s)} \mid H_0 \text{ is true}\big] = \alpha$$

# Rejection Regions for One and Two Sided Tests



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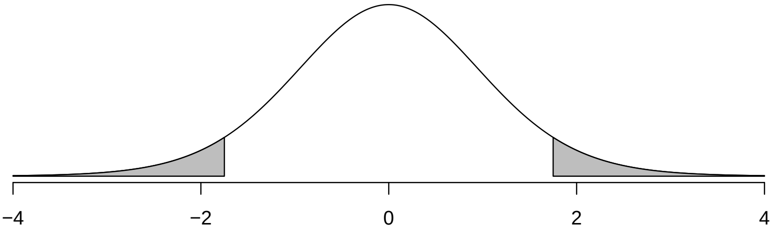
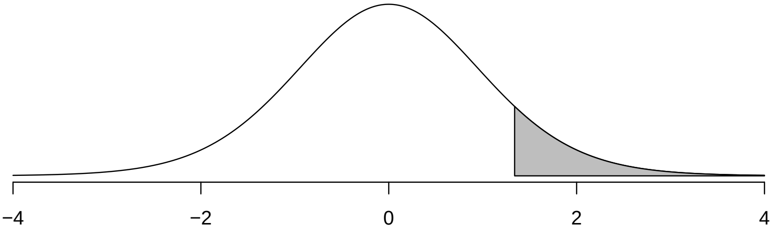



Rejection region for a two-sided test

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- $t_\gamma$ is the $\gamma$ quantile of the $t_{n-1}$ distribution

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- **Example** (Two-sided test for a normal population)

  Reject the null hypothesis $H_0: \ \mu = \mu_0$ in favor of the alternative hypothesis $H_1: \ \mu \neq \mu_0$ at significance level $\alpha$ if the observed value $t$ of $T$ satisfies $t \geq t_{1-\alpha/2}$ or $t \leq t_{\alpha/2}$

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- As a decision rule, equivalent to the critical value(s) decision rule

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  ***p* value** The probability of observing a value of $T$ as extreme or more extreme than $t$ under the null hypothesis $H_0$. Reject the null hypothesis $H_0$ in favor of the alternative hypothesis $H_1$ when $p \leq \alpha$

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- The more precise $H_1$ can be stated, the more important it is to choose an appropriate test statistic $T$

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Assumptions:

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- Normality of returns: $r_t \sim \mathcal{N}(\mu, \sigma^2)$ for all $t$
- No serial correlation: $\mathrm{Cov}(r_s, r_t) = 0$ for $t \neq s$

  $\implies$ $R_s$ and $R_t$ are independent

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**Two Sample $t$-test** The hypotheses are statements about the means of two separate normal populations, many variations:

- Variance known: $\sigma_1^2 = \sigma_2^2$ or $\sigma_1^2 \neq \sigma_2^2$
- Equal variance (unknown): $\sigma_1^2 = \sigma_2^2$ estimated by pooled $S^2$
- Different variances (unknown): $\sigma_1^2$ estimated by $S_1^2$ and $\sigma_2^2$ estimated by $S_2^2$

# Outline

```
> library(quantmod)
```

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> getSymbols("^RUT")  #Russell 2000
> getSymbols("IWL")   #Russell Top 200
```

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           RUT.Open RUT.High RUT.Low RUT.Close RUT.Adjusted
2007-01-03   788.31   796.62  779.31    787.42       787.42
2007-01-04   786.42   791.83  779.70    789.95       789.95
2007-01-05   787.70   787.70  774.55    775.87       775.87
2007-01-08   776.20   778.83  769.27    776.99       776.99
2007-01-09   777.10   778.74  768.69    778.33       778.33
2007-01-10   775.78   779.45  773.14    778.87       778.87
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> R2000 <- monthlyReturn(RUT["2013"]$RUT.Adjusted)
> RT200 <- monthlyReturn(IWL["2013"]$IWL.Adjusted)
```

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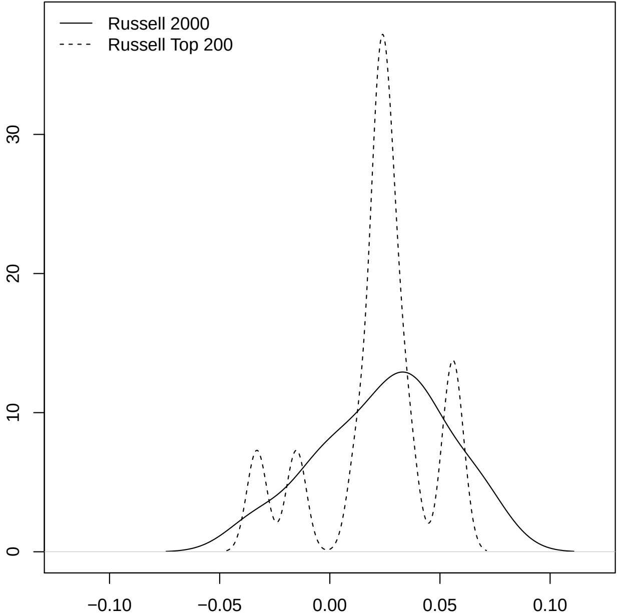
> R2000 <- monthlyReturn(RUT["2013"]$RUT.Adjusted)
> RT200 <- monthlyReturn(IWL["2013"]$IWL.Adjusted)

> plot(density(R2000))
> lines(density(RT200), lty = 2)
```


Russell 2000
Russell Top 200
0
10
20
30
−0.10
−0.05
0.00
0.05
0.10

```
> plot(density(R2000, bw = 0.015))
> lines(density(RT200, bw = 0.0175), lty = 2)
```

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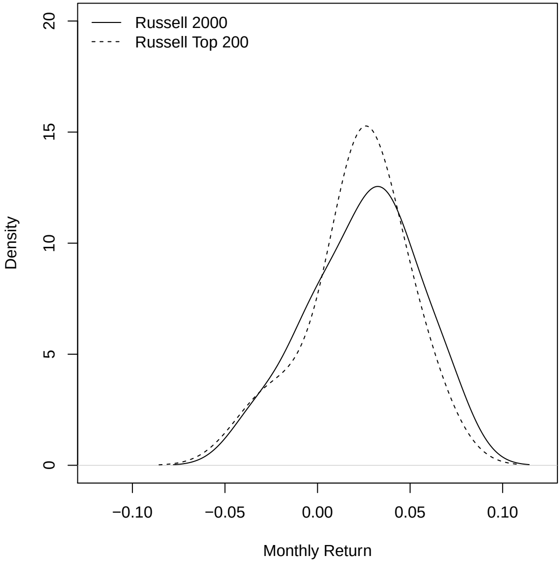

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```
> d <- drop(coredata(R2000 - RT200))
> S2 <- var(d)
```

# A Paired Two-Sample $t$ Test

- Plain English: Are small cap returns larger than large cap returns?
- Statistics: Is the mean ($\mu_s$) of the small cap returns distribution greater than the mean ($\mu_l$) of the large cap returns distribution?
- Hypothesis Test: $H_0$: $\mu_s \leq \mu_l$ vs. $H_1$: $\mu_s > \mu_l$?
- Paired Test: $H_0$: $\delta_\mu = \mu_s - \mu_l \leq 0$ vs. $H_1$: $\delta_\mu = \mu_s - \mu_l > 0$?

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> qt(0.95, df = 11)
[1] 1.795885
> 1 - pt(0.5970462, df = 11)
[1] 0.281282
```

# A Paired Two-Sample $t$ Test in R

```
> t.test(R2000, RT200, alternative = "greater", paired = TRUE)
```

# A Paired Two-Sample $t$ Test in R

```
> t.test(R2000, RT200, alternative = "greater", paired = TRUE)

Paired t-test

data:  R2000 and RT200
t = 0.597, df = 11, p-value = 0.2813
alternative hypothesis: true difference in means is
                        greater than 0
95 percent confidence interval:
 -0.006060183          Inf
sample estimates:
mean of the differences
            0.003018096
```

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