Test for convergence / divergence of a series.

$$\sum_{n=1}^{+\infty} a_n$$

1) By definition, find $\{S_n\}$ and check $\lim_{n\to\infty} S_n$.

2) Geometric series. $\frac{a_{n+1}}{a_n} = r$. a_1 first term.

$\sum_{n=1}^{+\infty} a_1 r^{n-1}$

1) $-1 < r < 1$, $\sum a_n = \frac{a_1}{1-r}$.

2) $r \geq 1$ or $r \leq -1$, $\sum a_n$ diverges.

p-series.

$\sum \frac{1}{n^p}$.

1) $p > 1$. converges

2) $p \leq 1$ diverges.

3) Test for divergence. $\lim_{n\to\infty} a_n \neq 0$. diverges

4) limit comparison.

5) Ratio. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$

6) Alternating series test. $\sum_{n=1}^{+\infty} (-1)^n u_n$.

Section 4 Power series.

1° Definition. Let x be a variable and c_0, c_1, c_2 ... be numbers. A power series about $x=0$ (centered at 0) is a series of the form. $\sum_{n=0}^{+\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$

or $\sum_{n=1}^{+\infty} c_n x^n = c_1 x + c_2 x^2 + \cdots + c_n x^n$.

(A power series about (centered at) $x=a$ is a series of the form

$$\sum_{n=0}^{+\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$$

$c_0, c_1, c_2, \ldots, c_n \ldots$ are called coefficients.)

Question: for which value of x is the power series convergent?

$\sum_{n=1}^{+\infty} \frac{1}{\sqrt{n}} x^n$

$\sum_{n=1}^{+\infty} n^2 (x-1)^n$

For $x=0$, $\sum c_n x^n$ converges. For $x \neq 0$,

key idea: 1) apply ratio test to determine an inequality of x;

2) Check the endpoints of the inequality.

all

Ex Find the values of x such that the following power series converge.

1) $\sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^n}{n}$ 4) $\sum_{n=1}^{+\infty} \frac{(x-1)^n}{5^n n^5}$

2) $\sum_{n=1}^{+\infty} \frac{x^n}{n!}$ 3) $\sum_{n=1}^{+\infty} n!\, x^n$.

4) $\frac{(1+n)x^n}{4^n n^2}$

Sol. 1) $a_n = (-1)^{n-1} \frac{x^n}{n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| (-1)^n \frac{x^{n+1}}{n+1} \cdot \frac{n}{(-1)^{n-1} x^n} \right|$$

$$= \left| (-1)^{n-n+1} \frac{x^{n+1}}{x^n} \cdot \frac{n}{n+1} \right|$$

$$= \left| x \cdot \frac{-n}{n+1} \right| = |x| \cdot \left| \frac{-n}{n+1} \right| = |x| \frac{n}{n+1}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |x| \lim_{n\to\infty} \frac{n}{n+1} = |x|$$

By ratio test, when $L = |x| < 1$, that is, $-1 < x < 1$

the series converges.

When $x=1$, $a_n = (-1)^{n-1}\frac{1}{n}$ $\quad \sum (-1)^{n-1}\frac{1}{n}$.

Since $u_n = \frac{1}{n} > 0$, $\lim_{n\to\infty} u_n = 0$ and. $u_{n+1} \le u_n$.

$\sum a_n$ converges by alternating series test.

When $x=-1$, $a_n = (-1)^{n-1}\frac{(-1)^n}{n} = \frac{(-1)^{2n}}{(-1)} \cdot \frac{1}{n}$.

$= -\frac{1}{n}$ $\quad -\sum_{n=1}^{+\infty} \frac{1}{n}$

Since $p=1 \le 1$, $\sum a_n$ diverges. by p-series test.

In summery, $\sum (-1)^{n-1}\frac{x^n}{n}$. converges when

$-1 < x \le 1$.

Thm (Alternating Series Test)

$$\sum_{n=1}^{+\infty} a_n = \sum_{n=1}^{+\infty} (-1)^{n+1} u_n \quad \text{or} \quad \sum_{n=1}^{+\infty} (-1)^n u_n \quad \text{with } u_n >$$

I) If $\lim_{n\to\infty} u_n \neq 0$, then the alternating series diverges;

II) If ① $\lim_{n\to\infty} u_n = 0$ and

② $u_{n+1} \leq u_n$ for all n

then $\sum a_n$ converges.

$$\sum_{n=1}^{+\infty} \frac{(1+n)x^n}{4^n n^2} \qquad a_n = \frac{(1+n)x^n}{4^n n^2}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{(n+2)\, x^{n+1}}{4^{n+1}(n+1)^2} \cdot \frac{4^n n^2}{(n+1)x^n} \right|$$

$$= \frac{n+2}{n+1} \cdot |x| \cdot \frac{1}{4} \cdot \left(\frac{n}{n+1}\right)^2$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{4}|x|$$

When $\frac{1}{4}|x| < 1$, that is, $-4 < x < 4$. the series converges.

When $x = -4$. $a_n = \frac{n+1}{n^2} \cdot (-1)^n. = (-1)^n \cdot (\frac{1}{n} + \frac{1}{n^2})$

by alternating . , Σ converges.

When $x = 4$, $a_n = \frac{n+1}{n^2}$, compare. with $\Sigma \frac{1}{n}$.

by limit comparison, Σa_n diverges.

$$f^{(n)}(x) = n\cdot(n-1)\cdot(n-2)\cdot\cdots 1\cdot C_n + \text{a sum of terms}$$

with $(x-a)$ as a factor.

$$f^n(a) = n!\, C_n \quad \Rightarrow \quad C_n = \frac{f^n(a)}{n!}$$

The Taylor series generated by a function f

at a point a_0 is

$$\sum_{n=0}^{+\infty} C_n (x-a)^n = \sum_{n=0}^{+\infty} \frac{f^n(a)}{n!} (x-a)^n$$

$$= f(a) + f'(a)(x-a) + f''(a)(x-a)^2.$$

$$+ \cdots$$

If $a=0$, the Taylor series is also called a

Maclaurin series, that is

$$\sum_{n=0}^{+\infty} \frac{f^n(0)}{n!} x^n$$

The Taylor polynomial of order n. generated by f at a is

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

generated by f

When the Taylor series. (as a power series) converges, it converges to $f(x)$.