Calculus III, ~~Fall 2019~~

Practice Exam II, Math 1023 Name: ____________________

Show **all work clearly and in order.**

1. 1) (8 points) Find an equation of the tangent line to the given curve

$$x = t^2 + 3t, \quad y = t^3$$

at $t = -1$.

$\frac{dx}{dt} = 2t + 3$ 1 pt $\qquad \frac{dy}{dt} = 3t^2$ 1 pt $\qquad \frac{dy}{dx} = \frac{3t^2}{2t+3}$ 1 pt

$x_0 = (-1)^2 - 3 = -2$ 1 pt $\qquad y_0 = -1$ 1 pt $\qquad k = \frac{3}{-2+3} = 3$ 1 pt

$y + 1 = 3(x+2)$ 2 pts

2) (8 points) Find the value of $\frac{d^2y}{dx^2}$ at $t = -1$.

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{6t(2t+3) - 3t^2 \cdot 2}{(2t+3)^2}}{2t+3}$$

3 pts

1 pt

$$= \frac{6t^2 + 18t}{(2t+3)^3}$$

2pts.

$t = -1$

$$\frac{d^2y}{dx^2} = \frac{6 - 18}{1^3} = -12$$

2 pts

2. (14 points) Find the length of the curves

$$x = \ln t - \frac{t^2}{2}, \quad y = 1 - 2t \qquad\qquad 1 \le t \le 3$$

$$\frac{dx}{dt} = \frac{1}{t} - \frac{1}{2}\cdot 2t = \frac{1}{t} - t \qquad\qquad \frac{dy}{dt} = -2$$

3 pts　　　　3 pts

$$L = \int_1^3 \sqrt{\left(\frac{1}{t} - t\right)^2 + (-2)^2}\, dt$$

2 pts

$$= \int_1^3 \sqrt{\frac{1}{t^2} - 2 + t^2 + 4}\, dt$$

$$= \int_1^3 \sqrt{\left(\frac{1}{t} + t\right)^2}\, dt$$

2 pts

$$= \int_1^3 \frac{1}{t} + t \, dt$$

1 pt

$$= \ln t + \frac{1}{2}t^2 \Big|_1^3$$

2 pts

$$= \ln 3 + \frac{9}{2} - \left(\ln 1 + \frac{1}{2}\right) = \ln 3 + 4$$

1 pt

3. (18 points) 1) (5 points) Find the Cartesian equation of the polar curve $r = 2\cos\theta$.

Sol. $r = 2\cos\theta$

$r^2 = 2r\cos\theta$ 1 pt

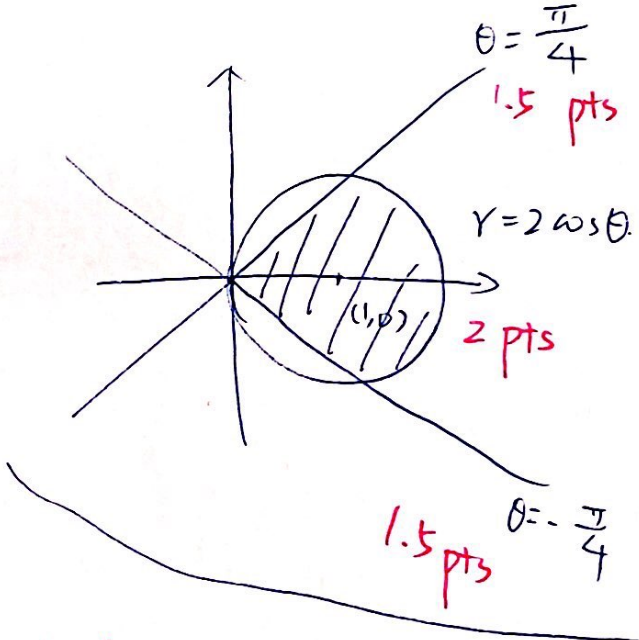
$x^2 + y^2 = 2x$ 2 pts

$x^2 - 2x + 1 + y^2 = 1$

$(x-1)^2 + y^2 = 1$ 2 pts

2) (5 points) Sketch the region bounded by this polar curve $r = 2\cos\theta$ and $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$.

Sol. $r = 2\cos\theta$ is a circle centered at $(1, 0)$ with radius 1.



2) (8 points) Find the area of the above region.

Sol

$$A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} \cdot (2\cos\theta)^2 \, d\theta. \quad \text{2 pts}$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\cos^2\theta \, d\theta. \quad \text{1 pt}$$

$2\cos^2\theta - 1 = \cos 2\theta$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 1 + \cos 2\theta \cdot d\theta, \quad \text{2 pts}$$

$$= \theta + \frac{1}{2}\sin 2\theta \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \quad \text{2 pts}$$

$$= \frac{\pi}{4} + \frac{1}{2}\sin\frac{\pi}{2} - \left(-\frac{\pi}{4} + \frac{1}{2}\sin\left(-\frac{\pi}{2}\right)\right)$$

$$= \frac{\pi}{2} + 1 \quad \text{1 pt}$$

4. (16 points) Give $\vec{u} = \langle 3, -2, 1 \rangle$, $\vec{v} = \langle -2, -1, -3 \rangle$, $\vec{w} = \langle -1, 2, a \rangle$

(1) (6 points) Find the magnitude of $\vec{u} + \vec{v}$;

$$\vec{u} + \vec{v} = \langle 3-2, -2-1, 1-3 \rangle = \langle 1, -3, -2 \rangle$$

1pt 1pt 1pt

$$|\vec{u} + \vec{v}| = \sqrt{1^2 + (-3)^2 + (-2)^2} = \sqrt{14}$$

2 pts 1 pt.

(2) (4 points) Find a unit vector parallel to the sum of $\vec{u} + \vec{v}$;

$$\vec{w} = \frac{1}{\sqrt{14}} \langle 1, -3, -2 \rangle = \left\langle \frac{1}{\sqrt{14}}, -\frac{3}{\sqrt{14}}, -\frac{2}{\sqrt{14}} \right\rangle$$

or $-\vec{w} = \left\langle \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}} \right\rangle$

(3) (6 points) Find a parametric equation of a line passing through $(1, 1, -1)$ and parallel to $\vec{v}$.

$$\vec{v} = \langle -2, -1, -3 \rangle$$

$x = 1 - 2t$ 2 pts

$y = 1 - t$ 2pts $-\infty < t < +\infty$

$z = -1 - 3t$ 2 pts

5. Given three points $P(1,-1,2)$, $Q(2,0,-1)$ and $R(0,2,1)$.

1) (16 points) Find the area of the triangle with vertices P, Q and R.

$\vec{PQ} = \langle 2-1, 0-(-1), -1-2 \rangle = \langle 1, 1, -3 \rangle$ 3 pts

$\vec{PR} = \langle 0-1, 2-(-1), 1-2 \rangle = \langle -1, 3, -1 \rangle$ 3 pts

1 pt 2 pts. 1 pt

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix} = \langle -1+9, -(-1-3), 3+1 \rangle$$

$= \langle 8, 4, 4 \rangle$ 3 pt

2 pts

$A = \frac{1}{2}\sqrt{8^2+4^2+4^2} = \frac{1}{2}\sqrt{64+32} = \frac{1}{2}\sqrt{96}$ 1 pt

2) (4 points) Find ~~the~~ an equation for plane passing through P, Q and R.

$\vec{n} = \langle 8, 4, 4 \rangle$ 2 pts

$8(x-1) + 4(y+1) + 4(z-2) = 0$ 2 pts

6. Let $\vec{r}(t) = \langle \cos 2t, \sin t, \sin(t^2) \rangle$ be the position of a particle in space.

1) (12 points) Find the particle's velocity $\vec{v}(t)$ and acceleration vector $\vec{a}(t)$ at $t = 0$.

Sol $\vec{v}(t) = (\vec{r}(t))' = \langle -2\sin 2t, \cos t, 2t\cos(t^2) \rangle$ 3 pts

$\vec{a}(t) = (\vec{v}(t))' = \langle -4\cos 2t, -\sin t, 2\cos(t^2) - 4t^2\sin(t^2) \rangle$

1pt 1pt 2pts

$\vec{v}(0) = \langle -2\cdot 0, 1, 0 \rangle = \langle 0, 1, 0 \rangle$ 2 pts

$\vec{a}(0) = \langle -4, 0, 2 \rangle$ 3 pts

2) (4 points) Determine if the above two vectors $\vec{v}(0)$ and $\vec{a}(0)$ are parallel to each other.

$$\vec{v}(0) \times \vec{a}(0) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ -4 & 0 & 2 \end{vmatrix}$$

$= \langle 2-0, -(0-0), 0-(-4) \rangle$

$= \langle 2, 0, 4 \rangle$ 2 pts

$|\vec{v}(0) \times \vec{a}(0)| = \sqrt{4+16} = \sqrt{20} \neq 0$. $\vec{v}(0)$ and $\vec{a}(0)$ are not parallel to each other. 2 pts