Example 1 : N-th Term Test

Using the n-th term test, what can you say about convergence or divergence of the following series?

$$\sum_{n=1}^{\infty} \frac{n^3 - 2n + 4}{n^2 + n - 1}$$

The nth-Term Test for Divergence

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

$$a_n = \frac{n^3 - 2n + 4}{n^2 + n - 1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n - 2/n + 4/n^2}{1 + 1/n - 1/n^2}$$

$$= \infty \neq 0$$

The series $\sum_{n=1}^{\infty} \frac{n^3 - 2n + 4}{n^2 + n - 1}$ diverges.

Example 2 : N-th Term Test

Using the n-th term test, what can you say about convergence or divergence of the following series?

$$\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n}$$

The nth-Term Test for Divergence

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

$$a_n = \frac{1-(-1)^n}{n}$$

$$\frac{0}{n} \le a_n \le \frac{2}{n} \quad \text{for all } n.$$

$$\lim_{n\to\infty} \quad 0 \le \lim_{n\to\infty} a_n \le \lim_{n\to\infty} \frac{2}{n}$$

$$0 \le \lim_{n\to\infty} a_n \le 0$$

By the squeeze thm, $\lim_{n\to\infty} a_n = 0.$

The n^{th} term test does not tell us whether the series conv. or div.

Example 3 : Geometric Series

Determine if the series converges or diverges. If the series converges calculate the sum.

$$\sum_{n=1}^{\infty} \frac{(-9)^n}{3^{2n}}$$

If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ converges to $a/(1 - r)$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \qquad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

$$\sum_{n=1}^{\infty} \frac{(-9)^n}{3^{2n}} = \sum_{n=1}^{\infty} \frac{(-9)^n}{(3^2)^n} \qquad (x^p)^q = x^{pq}$$

$$= \sum_{n=1}^{\infty} \frac{(-9)^n}{9^n}$$

$$= \sum_{n=1}^{\infty} \left(\frac{-9}{9}\right)^n$$

$$= \sum_{n=1}^{\infty} (-1)^n = \sum_{n=1}^{\infty} (-1)(-1)^{n-1}$$

$|r| = |-1| = 1 \geqslant 1$. The series diverges.

Example 4 : Geometric Series

Determine if the series converges or diverges. If the series converges calculate the sum.

$$\sum_{n=2}^{\infty}(-1)^{n+1}\frac{(4)^{2n}}{3^{5n-2}}$$

If $\sum_{n=k}^{\infty} a_n$ is a geo series

with $r = a_{n+1}/a_n$. $(a_{n+1} = a_n \cdot r)$

Then,

1) If $|r| \geq 1$, the series diverges
2) If $|r| < 1$, the series converges

to $\dfrac{a_k}{1-r}$

$\sum_{n=k}^{\infty} a_n$, $a_n = \dfrac{(-1)^{n+1} 4^{2n}}{3^{5n-2}}$, $k=2$

$$r = a_{n+1}/a_n = \frac{(-1)^{n+2} 4^{2(n+1)}}{3^{5(n+1)-2}} \Big/ \frac{(-1)^{n+1} 4^{2n}}{3^{5n-2}}$$

$$r = \frac{(-1)^{n+2}\,4^{2n+2}}{3^{5n+3}} \cdot \frac{3^{5n-2}}{(-1)^{n+1}\,4^{2n}}$$

$$r = \frac{(-1)^{n+2}}{(-1)^{n+1}} \frac{4^{2n+2}}{4^{2n}} \cdot \frac{3^{5n-2}}{3^{5n+3}}$$

$$r = \frac{(-1)\,4^{2}}{3^{5}} = \frac{-16}{243}$$

$$|r| = \frac{16}{243} < 1$$

The series converges to

$$\frac{a_k}{1-r} = \frac{\left(\frac{(-1)^3\,4^4}{3^8}\right)}{1-\left(-\frac{16}{243}\right)} = \frac{-256}{6993}$$

Example 5 : Integral Test

Use the integral test to show that the series converges.

$$\sum_{n=2}^{\infty} \frac{5n^4}{(n^5+17)^2}$$

THEOREM 9—The Integral Test Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

$$f(x) = \frac{5x^4}{(x^5+17)^2} \quad , \quad x \geq 2$$

f is continuous, positive and decreasing on $x \geq 2$ and $f(n) = a_n$. ✓

$$\int_2^b \frac{5x^4}{(x^5+17)^2}\,dx \qquad \begin{aligned} u &= x^5+17 \\ du &= 5x^4\,dx \end{aligned}$$

$$= \int_{49}^{b^5+17} \frac{du}{u^2}$$

$$= -\frac{1}{u}\Bigg|_{49}^{b^5+17} = \frac{1}{49} - \frac{1}{b^5+17}$$

$$\int_2^\infty \frac{5x^4}{(x^5+17)^2}\,dx = \lim_{b\to\infty} \frac{1}{49} - \frac{1}{b^5+17}$$

$$= \frac{1}{49} \quad \text{(convergent).}$$

Since $\displaystyle\int_2^\infty \frac{5x^4}{(x^5+17)^2}\,dx$ converges,

the series $\displaystyle\sum_{n=2}^{\infty} \frac{5n^4}{(n^5+17)^2}$ also converges.

Example 6 : Integral Test

Use the integral test to show that the series diverges.

$$\sum_{n=1}^{\infty} \frac{(\ln n)^2}{\sqrt{n}}$$

THEOREM 9—The Integral Test Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\, dx$ both converge or both diverge.

$f(x) = \dfrac{(\ln x)^2}{\sqrt{x}}$, $1 \leq x < \infty$

f is continuous but not positive and not decreasing on $1 \leq x < \infty$, which you can see by graphing. However, by solving $f'(x) < 0$ (and more graphing), we see that f is continuous, positive, and decreasing on $55 \leq x < \infty$.

$$\underbrace{\int_1^\infty f(x)\,dx}_{?} = \underbrace{\int_1^{55} f(x)\,dx}_{\text{finite, conv.}} + \underbrace{\int_{55}^\infty f(x)\,dx}_{?}$$

$\int_1^\infty f(x)\,dx$ converges/diverges if

$\int_{55}^\infty f(x)\,dx$ converges/diverges.

For $55 \le x < \infty$, since $\ln x$ is an increasing function,

$$1 < \ln 55 \le \ln x$$
$$1 < (\ln 55)^2 \le (\ln x)^2$$

$$\Longrightarrow \quad \frac{(\ln 55)^2}{\sqrt{x}} \le \frac{(\ln x)^2}{\sqrt{x}}$$

$$\Longrightarrow \quad \int_{55}^\infty \frac{(\ln 55)^2}{\sqrt{x}}\,dx \le \int_{55}^\infty \frac{(\ln x)^2}{\sqrt{x}}\,dx$$

$$\Longrightarrow \lim_{b\to\infty} (\ln 55)^2\, 2\sqrt{b} - (\ln 55)^2\, 2\sqrt{55} \le \int_{55}^\infty \frac{(\ln x)^2}{\sqrt{x}}\,dx$$

$$\infty \le \int_{55}^{\infty} \frac{(\ln x)^2}{\sqrt{x}}\,dx$$

$\therefore$ $\displaystyle\int_{55}^{\infty} \frac{(\ln x)^2}{\sqrt{x}}\,dx$ diverges to ∞.

$\therefore$ $\displaystyle\int_{1}^{\infty} \frac{(\ln x)^2}{\sqrt{x}}\,dx$ diverges to ∞.

$$\sum_{n=1}^{\infty} \frac{(\ln n)^2}{\sqrt{n}} = \underbrace{\sum_{n=1}^{54} \frac{(\ln n)^2}{\sqrt{n}}}_{\text{finite}} + \underbrace{\sum_{n=55}^{\infty} \frac{(\ln n)^2}{\sqrt{n}}}_{\text{diverges to } \infty}$$

This implies that $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^2}{\sqrt{n}}$ diverges (to ∞).

HW 3, Problem 2.

a) r $2 \le x < \infty$,

$0 < 1 < \ln 2 \le \ln x$, so

$0 < 1 < (\ln 2)^2 \le (\ln x)^2$, and

$$0 \le \frac{1}{(\ln x)^2} \le \frac{1}{(\ln 2)^2}$$

$$0 \leq \frac{1}{x^3(\ln x)^2} \leq \frac{1}{x^3 \underbrace{(\ln 2)^2}_{\text{constant}}}$$

$$\int_2^\infty 0 \, dx \leq \int_2^\infty \frac{1}{x^3(\ln x)^2} dx \leq \int_2^\infty \frac{1}{x^3(\ln 2)^2} dx$$

$$0 \leq \underbrace{\int_2^\infty \frac{1}{x^3(\ln x)^2} dx}_{\text{Difficult}} \leq \frac{1}{(\ln 2)^2} \underbrace{\int_2^\infty \frac{1}{x^3} dx}_{\text{Less Difficult}}$$

b) Notice that $f(x) = \frac{\ln x}{\sqrt{3x}}$ is not always positive and decreasing on $1 \leq x < \infty$. You can still use the integral test w/ some adjustments (not as bad as ex. 6). Also, use the fact that $\ln(x)$ is an increasing fn like part (a) to make integration easier.

Example 7 : Estimating a Sum with the Integral Test

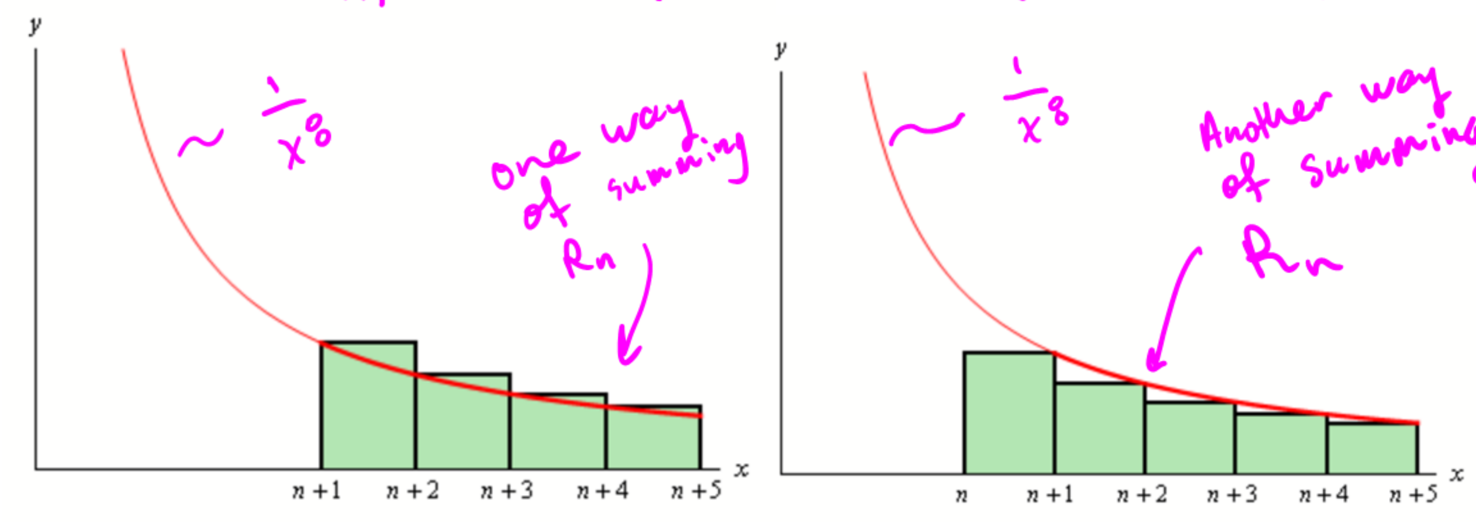
You want to calculate the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^8}.$$

Use the integral test to determine how many terms you need to add in order to estimate the sum with a guaranteed accuracy of four decimal places.

$$\sum_{k=1}^{\infty} \frac{1}{k^8} = \sum_{k=1}^{n} \frac{1}{k^8} + \sum_{k=n+1}^{\infty} \frac{1}{k^8}$$

$$\underset{\text{(sum)}}{S} = \underset{\text{(partial sum)}}{S_n} + \underset{\text{(Remainder)}}{R_n}$$



Source: Paul's Online Math Notes Estimating The Value Of A Series.
Please search for this webpage for a very detailed and clear derivation.

R_n overestimates $\int_{n+1}^{\infty} \frac{1}{x^8}\,dx$

R_n underestimates $\int_{n}^{\infty} \frac{1}{x^8}\,dx$

Bounds for the Remainder in the Integral Test

Suppose $\{a_k\}$ is a sequence of positive terms with $a_k = f(k)$, where f is a continuous positive decreasing function of x for all $x \geq n$, and that Σa_n converges to S. Then the remainder $R_n = S - s_n$ satisfies the inequalities

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx. \qquad (1)$$

You want to calculate the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^8}.$$

Use the integral test to determine how many terms you need to add in order to estimate the sum with a guaranteed accuracy of four decimal places.

$$\int_{n+1}^{\infty} \frac{1}{x^8}\,dx \leq R_n \leq \int_{n}^{\infty} \frac{1}{x^8}\,dx$$

$$\lim_{\to\infty} \left(-\frac{1}{7}\right)\frac{1}{x^7}\Big|_{n+1}^{b} \leq R_n \leq \lim_{b\to\infty} \left(-\frac{1}{7}\right)\frac{1}{x^7}\Big|_{n}^{b}$$

$$\lim_{a\to\infty} \frac{1/7}{(n+1)^7} - \frac{1/7}{a^7} \leq R_n \leq \lim_{b\to\infty} \frac{1/7}{n^7} - \frac{1/7}{b^7}$$

$$0 \leq \frac{1/7}{(n+1)^7} \leq R_n \leq \frac{1/7}{n^7}$$

For 4 decimal places of accuracy, want $R_n < 0.0001$. Use

$$R_n \leq \frac{1/7}{n^7} < 0.0001$$ to guarantee this.

$$(0<)\ \frac{1/7}{n^7} < 0.0001$$

$$\frac{1}{7}\frac{1}{0.0001} < n^7$$

$$2.03 \approx \sqrt[7]{\frac{1000}{7}} < n$$

First $n > 2.03$ is $n = 3$

To check, use something like wolfram alpha to see that

$$\sum_{n=1}^{\infty} \frac{1}{n^8} - \sum_{n=1}^{3} \frac{1}{n^8} \approx 0.00001869 < 0.0001$$

Notice that

$$\sum_{n=1}^{\infty} \frac{1}{n^8} - \sum_{n=1}^{2} \frac{1}{n^8} \approx 0.0001711 > 0.0001$$

(Adding 2 terms is not accurate to 4 decimal places while 3 terms is enough)

1. **(10 points)** Explain why the series

$$\sum_{n=1}^{\infty} a_n$$

with $a_n \geq 0$ converges if and only if

$$\int_1^{\infty} f(x)\,dx$$

converges for some function $f(x)$ with $f(n) = a_n$. Your answer might be a mix of text, graphics and mathematical expressions.

- For graphing, pick an f you know of s.t. $\int_1^{\infty} f(x)\,dx = \infty$.

  Graph this along w/ a Riemann approximation for a corresponding series (should the apprx. over or underestimate the integral to show divergence?).

- Repeat ↑ with an f you know of s.t. $\int_1^{\infty} f(x)\,dx$ converges.

- Read a proof of the integral test and write what you understand in your own words. If you need a reference: https://tutorial.math.lamar.edu/classes/calcii/IntegralTest.aspx