4° Further properties

Thm 1 (Test for divergence). Let $\sum_{n=1}^{+\infty} a_n$ be a series. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{+\infty} a_n$ is divergent.

Thm 2. If $\sum_{n=1}^{+\infty} a_n$ and $\sum_{n=1}^{+\infty} b_n$ are convergent, then

1) $$\sum_{n=1}^{+\infty} a_n \pm b_n = \sum_{n=1}^{+\infty} a_n \pm \sum_{n=1}^{+\infty} b_n$$

2) $$\sum_{n=1}^{+\infty} c a_n = c \sum_{n=1}^{+\infty} a_n.$$

Ex. $\sum_{n=1}^{+\infty} \frac{2n}{n+3}$. $\sum_{n=1}^{+\infty} \frac{3}{n} - \frac{3}{n+1} + \frac{1}{2^n}$

Remark. 1. If Σa_n is divergent and $k \neq 0$, then $\Sigma k a_n$ is divergent.

2. If Σa_n converges and Σb_n diverges, then $\Sigma a_n \pm$

Section 3. Other tests for series

(p-series)

Theorem 1. Given a series $\sum_{n=1}^{+\infty} \frac{1}{n^p}$ ..

If $p > 1$, the series converges;

If $p \leq 1$, the series diverges.

Ex 1) $\sum_{n=1}^{+\infty} \frac{4}{n^{\frac{3}{2}}}$ 2) $\sum_{n=1}^{+\infty} 2n^{-\frac{1}{2}}$ $\sum_{n=1}^{+\infty} \frac{1}{n^2} + \frac{1}{n^{\frac{1}{3}}}$

Theorem 2 (Limit comparison test) Given two series $\sum a_n$ and $\sum b_n$ with $a_n > 0$, $b_n > 0$.

If $\lim_{n \to \infty} \frac{a_n}{b_n} = L^{>0}$, then either $\sum_{n=1}^{+\infty} a_n$ and $\sum_{n=1}^{+\infty} b_n$ both converge or both diverge.

Ex

1) $\displaystyle\sum_{n=1}^{+\infty} \frac{2n}{n^2+1}$

4) $\displaystyle\sum_{n=1}^{+\infty} \frac{n^2}{n^3+n^{\frac{3}{2}}}$

3) $\displaystyle\sum_{n=1}^{+\infty} \frac{n+1}{2n^4+n^3+n^2}$

2) $\displaystyle\sum_{n=1}^{+\infty} \frac{n^2+1}{n^{\frac{5}{2}}+2}$

$$\frac{n+1}{2n^4+n^2+1}$$

Sol 1) $a_n = \dfrac{2n}{n^2+1}$. Define. $\displaystyle\sum_{n=1}^{+\infty} b_n$ with.

$b_n = \dfrac{n}{n^2} = \dfrac{1}{n}$, Since $p=1$, then Σb_n diverges by. p-series test.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n}{n^2+1} \cdot n = \lim_{n\to\infty} \frac{2n^2}{2n^2+2}$$

$$= \lim_{n\to\infty} \frac{2}{1+\frac{1}{n^2}} = 2 > 0.$$

Since $L = 1 > 0$ and Σb_n diverges.
By. limit comperison. test, Σa_n diverges.

Theorem. 3 (Ratio test) Given a series $\sum a$

If $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ and

1) $L<1$, then $\sum a_n$ converges;

2) $L>1$, then $\sum a_n$ diverges;
(including $L=+\infty$)

3) $L=1$, this test is inconclusive.

Ex. $\sum\limits_{n=1}^{+\infty} \frac{3}{2^n+1}$

$a_n = \frac{3}{2^n+1}$

$b_n = \frac{3}{2^n}$ $\quad \sum\limits_{n=1}^{+\infty} b_n$ converges. $r=\frac{1}{2}$

$\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{3}{2^n+1} \cdot \frac{2^n}{3} = \lim\limits_{n\to\infty} \frac{2^n}{2^n+1}$

$= 1 > 0.$

$\sum\limits_{n=1}^{+\infty} \frac{2^n+3^n}{4^n+5^2}$

$a_n = \frac{2^n+3^n}{4^n+5^2}$

$b_n = \frac{3^n}{4^n}$.

$\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty}$

$\cdot \frac{4^n}{4^n+5^2} =$

Ex. (Ratio Test)

1) $\sum_{n=1}^{+\infty} \frac{4^{n+1}}{n 3^{n-1}}$ 2) $\sum_{n=1}^{+\infty} \frac{n^3}{(-3)^n}$ 3) $\sum_{n=1}^{+\infty} \frac{(n+1)!}{(n+2)^2}$

Sol 1) $a_n = \frac{4^{n+1}}{n 3^{n-1}}$ $\left|\frac{a_{n+1}}{a_n}\right| = \frac{4^{n+2}}{(n+1) 3^n} \cdot \frac{n 3^{n-1}}{4^{n+1}}$

$$= 4 \cdot \frac{1}{3} \cdot \frac{n}{n+1}$$

$$\lim_{n\to\infty} \frac{4}{3} \cdot \frac{n}{n+1} = \frac{4}{3} \lim_{n\to\infty} \frac{n}{n+1} = \frac{4}{3} > 1$$

By Ratio test, $\sum a_n$ diverges.

2) $a_n = \frac{(-3)^n}{n^3}$ $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-3)^{n+1}}{(n+1)^3} \cdot \frac{n^3}{(-3)^n}\right|$

$$= \left|(-3) \cdot \left(\frac{n}{n+1}\right)^3\right| = 3 \cdot \left(\frac{n}{n+1}\right)^3$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 3 > 1.$$