HW 4.

1.

1.) $$a_n = \frac{(n+1)(n+2)}{n!}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{(n+2)(n+3)}{(n+1)!} \cdot \frac{n!}{(n+1)(n+2)} \right|$$

$$= \left| \frac{n!}{(n+1)!} \frac{n+3}{n+1} \right|$$

$$= \frac{1}{n+1} \cdot \frac{n+3}{n+1}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{n+3}{n+1} = 0 < 1$$

Since $L < 1$, by ratio test, Σa_n converges.

2. $a_n = \dfrac{(-2)^n}{n!\,3^n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-2)^{n+1}}{(n+1)!\,3^{n+1}} \cdot \frac{n!\,3^n}{(-2)^n}\right|$$

$$= \left|\frac{(-2)^{n+1}}{(-2)^n} \cdot \frac{n!}{(n+1)!} \cdot \frac{3^n}{3^{n+1}}\right|$$

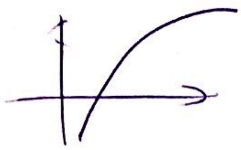
$$= \left|(-2) \cdot \frac{1}{n+1} \cdot \frac{1}{3}\right|$$

$$= \frac{2}{3} \cdot \frac{1}{n+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2}{3} \cdot \frac{1}{n+1} = 0$$

By ratio test, since $L = 0 < 1$, the series converges.

3. $\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{1}{\ln(n+1)}$



1) $u_n = \frac{1}{\ln(n+1)} > 0$

2) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0$

2) Since $n+1 \leq n+2$ and $\ln x$ is increasing.

then $\ln(n+1) \leq \ln(n+2)$. It follows that.

$$\frac{1}{\ln(n+2)} \leq \frac{1}{\ln(n+1)}$$

i.e. $u_{n+1} \leq u_n$.

By Alternating series test, $\sum (-1)^{n+1} u_n$.

converges.

4. $u_n = \dfrac{n+1}{n+2}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n+1}{n+2} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{2}{n}} = 1 \neq 0$$

By alternating series test,

$\sum a_n$ diverges.

Apply ratio test.

~~4.~~
5 $$a_n = \frac{x^n}{\sqrt{n}\, 3^n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{\sqrt{n+1}\cdot 3^{n+1}} \cdot \frac{\sqrt{n}\, 3^n}{x^n}\right|$$

$$= \left|\frac{x^{n+1}}{x^n} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} \cdot \frac{3^n}{3^{n+1}}\right|$$

$$= \left| x \cdot \sqrt{\frac{n}{n+1}} \cdot \frac{1}{3}\right|$$

$$= |x| \cdot \sqrt{\frac{n}{n+1}} \cdot \frac{1}{3}$$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x| \cdot \frac{1}{3}.$$

When $\frac{|x|}{3} < 1$, that is, $-3 < X < 3$. series converges

~~$\frac{|x|}{3} > 1$. $X > 3$ or $X < -3$ series diverges.~~

When $x=3$. $a_n = \frac{3^n}{\sqrt{n}\, 3^n} = \frac{1}{\sqrt{n}}$.

$\sum a_n$ diverges since $p = \frac{1}{2} < 1$.

When $x=-3$. $a_n = \frac{(-3)^n}{\sqrt{n}\, 3^n} = \frac{(-1)^n}{\sqrt{n}}$.

converges. by alternating series test.

In summary, $\sum a_n$ converges when

$-3 \leq x < 3$.

6.

$$a_n = \frac{n^2 x^n}{2^n (n+1)}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^2 x^{n+1}}{2^{n+1}(n+2)} \cdot \frac{2^n (n+1)}{n^2 x^n}\right|$$

$$= \left|\left(\frac{n+1}{n}\right)^2 \cdot \frac{x^{n+1}}{x^n} \cdot \frac{2^n}{2^{n+1}} \cdot \frac{n+1}{n+2}\right|$$

$$= \cancel{|}\left(\frac{n+1}{n}\right)^2\cancel{|} \cdot |x| \cdot \frac{1}{2} \cdot \frac{n+1}{n+2}$$

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \cdot |x| \cdot \frac{1}{2} \cdot \frac{n+1}{n+2}$$

$$= \frac{1}{2}|x|$$

By ratio test, when $\frac{1}{2}|x| < 1$, that is

$|x| < 2 \iff -2 < x < 2$ the power series

converges.

When $x = +2$. $\sum \frac{n^2 x^n}{2^n (n+1)} = \sum \frac{n^2 2^n}{2^n (n+1)}$

$$= \sum \frac{n^2}{n+1}$$

$\lim_{n\to\infty} \frac{n^2}{n+1} \to +\infty$ By test for divergence,

$\sum \frac{n^2}{n+1}$ diverges.

When $x = -2$ $\sum \frac{n^2 x^n}{2^n (n+1)} = \sum \frac{n^2 (-2)^n}{2^n (n+1)}$

$$= \sum \frac{n^2 \cdot (-1)^n \cdot 2^n}{2^n (n+1)}$$

$$= \sum \frac{n^2}{n+1} \cdot (-1)^n.$$

$\lim_{n\to\infty} \frac{n^2}{n+1} \to +\infty$ By alternating series

test. $\sum \frac{n^2}{n+1} (-1)^n$ diverges.

7. 1) $f(x) = x^2 - 2x + 4$ $\quad a = 2.$

$f'(x) = 2x - 2$ $\quad f''(x) = 2$ $\quad f'''(x) = 0.$

$f^{(n)}(x) = 0$ for $n \geq 3.$

$f(2) = 2^2 - 2\cdot 2 + 4 = 4$

$f'(2) = 2\cdot 2 - 2 = 2$, $\quad f''(2) = 2.$

The Taylor series generated by f at 2

is

$$\sum_{k=0}^{+\infty} \frac{f^{(k)}(2)}{k!}(x-2)^k.$$

$$= f(a) + f'(a)(x-a) + f''(a)(x-a)^2 + \cdots$$

$$= 4 + 2(x-2) + 2(x-2)^2$$

2). $f(x) = e^{2x}$ $\qquad a = 0.$

$f'(x) = 2e^{2x}$, $\quad f''(x) = 2 \cdot 2e^{2x} = 2^2 e^{2x}$, ...

$f^{(k)}(x) = 2^k e^{2x}$.

$f(0) = e^0 = 1$, $\quad f^{(k)}(0) = 2^k e^0 = 2^k$.

Taylor series. is

$$\sum_{k=0}^{+\infty} \frac{f^{(k)}(0)}{k!}(x-0)^k$$

$$= \sum_{k=0}^{+\infty} \frac{2^k}{k!} x^k.$$

$a = 0.$

8. 1). $f(x) = \ln(1+x)$

$f(0) = \ln 1 = 0$

$f'(x) = \frac{1}{1+x}$

$f'(0) = \frac{1}{1+0} = 1$

$f''(x) = -\frac{1}{(1+x)^2}$

$f''(0) = -\frac{1}{(1+0)^2} = -1$

$P_0(x) = f(0) = 0$

$P_1(x) = f(0) + f'(0)(x-0) = 0 + 1 \cdot x = x$

$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2$

$= x + \frac{-1}{2}x^2 = x - \frac{1}{2}x^2.$

$a = 2.$

2). $f(x) = \frac{1}{x}.$ $\quad f(2) = \frac{1}{2}.$

$f'(x) = -x^{-2} = -\frac{1}{x^2}$ $\quad f'(2) = -\frac{1}{4}$

$f''(x) = \frac{2}{x^3}$ $\quad f''(2) = \frac{2}{2^3} = \frac{1}{4}.$

$P_0(x) = f(2) = \frac{1}{2}$

$P_1(x) = \frac{1}{2} - \frac{1}{4}(x-2)$

$P_2(x) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{4} \cdot \frac{1}{2!}(x-2)^2.$

$= \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2.$