Preliminaries

- Natural exponential function $f(x) = e^x$ ($e = 2.718281828...$) has the following property:

  - $f(0) = e^0 = 1$;
  - $\lim\limits_{x \to -\infty} e^x = 0$ and $\lim\limits_{x \to \infty} e^x$ diverges to ∞;
  - Derivative of $f'(x) = (e^x)' = e^x$;
  - Indefinite integral $\int e^x \; dx = e^x + C$. Let k be a constant, then
  $$\int e^{kx} \; dx = \frac{1}{k} e^{kx} + C;$$
  - The domain is all real numbers (input x can be any number), the range is the positive real numbers (output $f(x)$ is always positive);
  -
  $$\frac{1}{e^x} = e^{-x},$$
  $$e^x e^y = e^{x+y}.$$

- Natural logarithms function $g(x) = \ln x$;

  - $g(1) = \ln 1 = 0$;
  - $\lim\limits_{x \to 0} \ln x = -\infty$ and $\lim\limits_{x \to \infty} \ln x$ diverges to ∞;
  - Derivative of $g'(x) = (\ln x)' = \frac{1}{x}$;
  - Indefinite integral $\int \frac{1}{x} \; dx = \ln |x| + C$;
  - The domain is the positive real numbers (input x must be positive), the range is all real numbers (output $g(x)$ can be any real number).
  -
  $$\ln(xy) = \ln x + \ln y;$$
  $$\ln(\frac{x}{y}) = \ln x - \ln y;$$
  $$\ln(x^y) = y \ln x.$$

- Natural exponential function $f(x) = e^x$ and natural logarithms function $g(x) = \ln x$ are inverse function to each other, this means
$$f(g(x)) = e^{\ln x} = x \qquad \text{and} \qquad g(f(x)) = \ln(e^x) = x;$$

- Fundamental Theorem of Calculus (for computing definite integral):

$$\int_a^b f(x) = F(b) - F(a), \qquad F \text{ is the antiderivative of } f \text{ satisfying} \quad F'(x) = f(x);$$

- Integration by parts formula for definite integrals

$$\int_a^b u dv = uv|_a^b - \int_a^b v du;$$

- Derivatives of some functions:

$$(x^n)' = nx^{n-1}, \quad (\sin x)' = \cos x, \quad (\cos x)' = -\sin x, \quad (a^x)' = a^x \ln a; \quad (a \text{ is a constant})$$

- Antiderivatives (Indefinite integrals) for some functions (constants $+C$ are omitted here):

$$\int x^n = \frac{1}{n+1} x^{n+1}, \quad \int \sin x = -\cos x, \quad \int \cos x = \sin x;$$

Let a be a constant.

$$\int \frac{1}{1+x^2}\, dx = \arctan(x), \quad \int \frac{1}{1+(ax)^2}\, dx = \frac{1}{a} \arctan(ax), \quad \int \frac{1}{a^2+x^2}\, dx = \frac{1}{a} \arctan(\frac{x}{a}),$$

$$\int e^{ax}\, dx = \frac{1}{a} e^{ax}, \quad \int a^x = \frac{1}{\ln a} a^x.$$