Example 1

Calculate with and without using l'Hopital's rule.

$$\lim_{y\to 0} \quad \frac{\sin(y)\cos(y)}{\sin(2y)}$$

- w/o l'Hôpital

$$\lim_{y\to 0} \frac{\sin y \cos y}{\sin 2y} = \lim_{y\to 0} \frac{\sin y \cos y}{2\sin y\cos y} = \lim_{y\to 0} \frac{1}{2} = \frac{1}{2}$$

- w/ l'Hôpital

$$\lim_{y\to 0} \frac{\sin y \cos y}{\sin 2y} \overset{(L'H)}{=} \lim_{y\to 0} \frac{\cos y \cos y + \sin y(-\sin y)}{2\cos 2y}$$

$$= \lim_{y\to 0} \frac{\cos^2 y - \sin^2 y}{2\cos 2y} = \frac{1-0}{2\cdot 1} = \frac{1}{2}$$

THEOREM 10—Limits of Continuous Functions If g is continuous at the point b and $\lim_{x\to c} f(x) = b$, then

$$\lim_{x\to c} g(f(x)) = g(b) = g(\lim_{x\to c} f(x)).$$

Ex) $\lim_{x\to 0} e^{\sin x} = e^{\lim_{x\to 0} \sin x} = e^{0} = 1$

$\lim_{y\to 27} \ln\left(\frac{x+5}{x-5}\right) = \ln\left(\lim_{y\to 27} \frac{x+5}{x-5}\right) = \ln \frac{32}{22}$

$\lim_{x\to 0^+} \left(\frac{1}{x}\right)^{x} = \lim_{x\to 0^+} e^{\ln\left(\frac{1}{x}\right)^{x}}$

$= \lim_{x\to 0^+} e^{x\ln(1/x)}$

$= e^{\lim_{x\to 0^+}\left(\frac{\ln(1/x)}{1/x}\right)}$

(L'H) $= e^{\lim_{x\to 0^+} \frac{-x\cdot x^{-2}}{-x^{-2}}}$

$= e^{\lim_{x\to 0^+} x} = e^{0} = 1$

$\frac{d}{dx}\ln(1/x)$
$= \frac{1}{1/x}\cdot -1x^{-2}$
$= -x\cdot x^{-2}$

$\frac{d}{dx}\frac{1}{x} = -x^{-2}$

Example 2

Calculate the limit. Note that $\ln(8x^2)$ and $\ln(1-\cos x)$ are undefined at $x=0$ but that this limit may exist nevertheless.

$$\lim_{x\to 0} \quad \ln(8x^2) - \ln(1-\cos x)$$

$$= \lim_{x\to 0} \ln\left(\frac{8x^2}{1-\cos x}\right) = \ln\left(\lim_{x\to 0} \frac{8x^2}{1-\cos x}\right)$$

$$\overset{(L'H)}{=} \ln\left(\lim_{x\to 0} \frac{16x}{\sin x}\right) \overset{(L'H)}{=} \ln\left(\lim_{x\to 0} \frac{16}{\cos x}\right)$$

$$= \ln\left(\frac{16}{\cos 0}\right) = \ln 16 .$$

Example 3

Calculate the limit.

$$\lim_{y\to 1^+} \quad y^{\frac{1}{1-y}}$$

$$= \lim_{y\to 1^+} e^{\ln y^{\frac{1}{1-y}}} = \lim_{y\to 1^+} e^{\frac{\ln y}{1-y}} \quad \left(\text{log properties}\right)$$

$$= e^{\lim_{y\to 1^+} \frac{\ln y}{1-y}} \quad (\text{L'H}) \quad = e^{\lim_{y\to 1^+} \frac{1/y}{-1}} = e^{(1/1)/-1} = e^{-1} = \frac{1}{e}$$

Example 4

Calculate the limit or explain why the limit does not exist

$$\lim_{x\to 0} \frac{\tan(x)}{1-\cos(x)}$$ DNE

Reason: If this limit exists, then we would have,

$$\lim_{x\to 0} \frac{\tan x}{1-\cos x} \overset{(L'H)}{=} \lim_{x\to 0} \frac{\sec^2 x}{\sin x}$$

You don't have to use L'H, just makes it a bit easier

As $x \to 0$, $\sec^2 x = \left(\frac{1}{\cos x}\right)^2 \to 1$.

As $x \to 0^+$, $\frac{1}{\sin x} \to \infty$.

As $x \to 0^-$, $\frac{1}{\sin x} \to -\infty$.

Since the numerator of $\frac{\sec^2 x}{\sin x}$ approaches a finite value (1), while the denominator of $\frac{\sec^2 x}{\sin x}$ approaches ∞ if we approach $x=0$ from the right and $-\infty$ if we approach $x=0$ from the left,

$$\lim_{x\to 0^+} \frac{\sec^2 x}{\sin x} = \infty \neq -\infty = \lim_{x\to 0^-} \frac{\sec^2 x}{\sin x}.$$

Therefore $\lim_{x\to 0} \frac{\sec^2 x}{\sin x}$ DNE, so $\lim_{x\to 0} \frac{\tan x}{1-\cos x}$ DNE.

Example 5

Calculate the limit or explain why the limit does not exist

$$\lim_{x\to 0+} \quad \frac{x-1}{x^3}$$

As x approaches 0 from the right, $x-1$ approaches -1 while $\frac{1}{x^3}$ approaches ∞. This means

$$\lim_{x\to 0^+} \frac{x-1}{x^3} = -\infty$$

(since $x-1 \to -1$ as $x \to 0^+$, think of this roughly as $\frac{-1}{\varepsilon}$ for very small but positive values for ε).

Problem 1

Calculate with and without using l'Hopital's rule.

$$\lim_{y\to 2\pi} \quad \frac{\sin(6y)}{\sin(3y)}$$

Hint: $\sin(6y) = \sin(2(3y))$ and follow example 1.

Solution omitted (HW Problem).
A helpful identity is $\sin 2\theta = 2\sin\theta\cos\theta$.

Problem 2

Calculate the limit.

$$\lim_{x\to\infty} \quad (1+2x)^{\frac{1}{2\ln(x)}}$$

$$= \lim_{x\to\infty} e^{\ln(1+2x)^{\frac{1}{2\ln x}}} = e^{\lim_{x\to\infty} \frac{\ln(1+2x)}{2\ln x}}$$

$$\overset{(L'H)}{=} e^{\lim_{x\to\infty}\left(\frac{\frac{2}{1+2x}}{\frac{2}{x}}\right)} = e^{\lim_{x\to\infty} \frac{2x}{2+4x}} = e^{\frac{2}{4}} = e^{\frac{1}{2}} = \sqrt{e}$$

Problem 3

Calculate the limit.

$$\lim_{x\to 0^+} \quad \frac{1}{x^4} - \frac{1}{x^5}$$

Solution omitted (HW problem). One way to do this problem is to rewrite as $\lim_{x\to 0^+} \frac{x-1}{x^5}$.