Worcester Polytechnic Institute
Department of Mathematical Sciences
Faculty: Francesca Bernardi, John Goulet, Xavier Ramos Olivé, Dina Rassias, Stephan Sturm

Fall 2020 - A Term

# MA 1023
# Calculus III

# Conference 1 – Ideas
## *Covers material from Lecture 1-3 & Active Learning 1*

1. Calculate the following limits with and without using l'Hôpital's rule:

a) $\lim\limits_{x\to 1} \dfrac{x^8-1}{x-1}$ b) $\lim\limits_{y\to 0} \dfrac{\sin(y)\cos(y)}{\sin(2y)}$ c) $\lim\limits_{z\to 0} \dfrac{\tan(z)}{1-\cos(z)}$

*a) By factorizing we get*

$$\lim_{x\to 1} \frac{x^8-1}{x-1} = \lim_{x\to 1} \frac{(x^4+1)(x^2+1)(x+1)(x-1)}{x-1} = \lim_{x\to 1}(x^4+1)(x^2+1)(x+1) = 8$$

*while using L'Hôpital's rule we have*

$$\lim_{x\to 1} \frac{x^8-1}{x-1} \overset{L'H}{=} \lim_{x\to 1} \frac{8x^7}{1} = 8.$$

*We can apply the rule as at* $x=1$ *we have the indeterminate form* $\frac{0}{0}$.

*b) By the addition theorem for sine* $\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x)$ *we have*

$$\lim_{y\to 0} \frac{\sin(y)\cos(y)}{\sin(2y)} = \lim_{y\to 0} \frac{\sin(y)\cos(y)}{2\sin(y)\cos(y)} = \frac{1}{2}$$

*whereas L'Hôpital's rule (we have an indeterminate form of type* $\frac{0}{0}$*) gives*

$$\lim_{y\to 0} \frac{\sin(y)\cos(y)}{\sin(2y)} \overset{L'H}{=} \lim_{y\to 0} \frac{\cos^2(y)-\sin^2(y)}{2\cos(2y)} = \frac{1-0}{2\cdot 1} = \frac{1}{2}.$$

*c) Using the facts that* $\tan(z) = \frac{\sin(z)}{\cos(z)}$ *and that* $\sin^2(z) + \cos^2(z) = 1$ *we have*

$$\lim_{z\to 0} \frac{\tan(z)}{1-\cos(z)} = \lim_{z\to 0} \frac{\sin(z)}{\big(1-\cos(z)\big)\cos(z)} = \lim_{z\to 0} \frac{\sin(z)\big(1+\cos(z)\big)}{\big(1-\cos(z)^2\big)\cos(z)}$$
$$= \lim_{z\to 0} \frac{\sin(z)\big(1+\cos(z)\big)}{\sin^2(z)\cos(z)} = \lim_{z\to 0} \frac{1+\cos(z)}{\sin(z)\cos(z)},$$

*where we enlarged the fraction in the second step by a factor* $1+\cos(z)$*. As the nominator converges to* $2$ *and the denominator converges towards* $0$ *from different sides when* $z$ *goes to zero from different sides, the limit is not defined. With L'Hôpital's rule (indeterminate form of type* $\frac{0}{0}$*)*

$$\lim_{z\to 0} \frac{\tan(z)}{1-\cos(z)} \overset{L'H}{=} \lim_{z\to 0} \frac{\frac{1}{\cos^2(z)}}{\sin(z)} = \lim_{z\to 0} \frac{1}{\sin(z)\cos^2(z)}$$

*and again the limit is not defined as the nominator converges to* $1$ *and the denominator converges towards* $0$.

2. Calculate the following limits of fractions

a) $\displaystyle\lim_{x\to 0} \frac{8x^2}{\cos(x)-1}$ b) $\displaystyle\lim_{\varphi\to\frac{\pi}{2}} \frac{1-\sin(\varphi)}{1+\cos(2\varphi)}$ c) $\displaystyle\lim_{t\to 0} \frac{t\sin(t)}{1-\cos(t)}$

*a) We have an indeterminate form of type* $\frac{0}{0}$ *and using L'Hôpital's rule gives*

$$\lim_{x\to 0} \frac{8x^2}{\cos(x)-1} \overset{L'H}{=} \lim_{x\to 0} \frac{16x}{\sin(x)} = \frac{16}{\lim_{x\to 0} \frac{\sin(x)}{x}} = 16$$

*as we know from class that* $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.

*b) Using L'Hôpital's rule twice (both times for the indeterminate form* $\frac{0}{0}$*) we get*

$$\lim_{\varphi\to\frac{\pi}{2}} \frac{1-\sin(\varphi)}{1+\cos(2\varphi)} \overset{L'H}{=} \lim_{\varphi\to\frac{\pi}{2}} \frac{-\cos(\varphi)}{-2\sin(2\varphi)} \overset{L'H}{=} \lim_{\varphi\to\frac{\pi}{2}} \frac{\sin(\varphi)}{-4\cos(2\varphi)} = \frac{1}{-4(-1)} = \frac{1}{4}.$$

*c) Enlarging the fraction with* $1+\cos(t)$*, using* $\sin^2(t)+\cos^2(t) = 1$ *as well as* $\lim_{t\to 0} \frac{\sin(t)}{t} = 1$ *we have*

$$\lim_{t\to 0} \frac{t\sin(t)}{1-\cos(t)} = \lim_{t\to 0} \frac{t\sin(t)\big(1+\cos(t)\big)}{1-\cos^2(t)} = \lim_{t\to 0} \frac{t\sin(t)\big(1+\cos(t)\big)}{\sin^2(t)}$$
$$= \lim_{t\to 0} \frac{1+\cos(t)}{\frac{\sin(t)}{t}} = \frac{2}{1} = 2.$$

3. Calculate the following limits of products, powers and differences

a) $\lim_{x\to 0} x^2 e^{-x}$ b) $\lim_{y\to 1+} y^{\frac{1}{1-y}}$ c) $\lim_{x\to\infty} (1+2x)^{\frac{1}{2\ln(x)}}$

d) $\lim_{z\to\infty} \left(\frac{z^2+1}{z+2}\right)^{\frac{1}{z}}$ e) $\lim_{y\to 0+} \left(\frac{3y+1}{y} - \frac{1}{1-\cos(y)}\right)$

*a) We have just*

$$\lim_{x\to 0} x^2 e^{-x} = \left(\lim_{x\to 0} x^2\right)\left(\lim_{x\to 0} e^{-x}\right) = 0 \cdot 1 = 0.$$

*(applying L'Hôpital's rule is of course wrong here, as this is not an indeterminate form).*

*b) This is an indeterminate form of type $1^{-\infty}$ and applying L'Hôpital's rule for the $\frac{-\infty}{-\infty}$ indeterminate form after an Exp-Log reformulation gives*

$$\lim_{y\to 1+} y^{\frac{1}{1-y}} = \lim_{y\to 1+} \exp\left(\ln\left(y^{\frac{1}{1-y}}\right)\right) = \exp\left(\lim_{y\to 1+} \frac{\ln(y)}{1-y}\right)$$
$$\overset{\text{L'H}}{=} \exp\left(\lim_{y\to 1+} \frac{\frac{1}{y}}{-1}\right) = e^{-1} = \frac{1}{e}\left(\approx 0.37\right)$$

*c) Here we have an indeterminate form of type $\infty^0$ that becomes after an Exp-Log transformation an indeterminate form of type $\frac{+\infty}{+\infty}$, thus we can apply L'Hôpital's rule to get*

$$\lim_{x\to\infty} (1+2x)^{\frac{1}{2\ln(x)}} = \lim_{x\to\infty} \exp\left(\ln\left((1+2x)^{\frac{1}{2\ln(x)}}\right)\right) = \exp\left(\lim_{x\to\infty} \frac{\ln(1+2x)}{2\ln(x)}\right)$$
$$\overset{\text{L'H}}{=} \exp\left(\lim_{x\to\infty} \frac{\frac{2}{1+2x}}{\frac{2}{x}}\right) = \exp\left(\lim_{x\to\infty} \frac{x}{1+2x}\right) = \exp\left(\lim_{x\to\infty} \frac{1}{\frac{1}{x}+2}\right)$$
$$= e^{\frac{1}{2}} = \sqrt{e}\left(\approx 1.65\right)$$

*where we enlarged the fraction by a factor $\frac{1}{x}$ to be able to calculate the limit.*

*d) Here we have an indeterminate form of type $+\infty^0$ as $\lim_{z\to\infty} \frac{z^2+1}{z+2} \lim_{z\to\infty} \frac{z+\frac{1}{z}}{1+\frac{2}{z}} = +\infty$.*

$$\lim_{z\to\infty} \left(\frac{z^2+1}{z+2}\right)^{\frac{1}{z}} = \exp\left(\lim_{z\to\infty} \ln\left(\left(\frac{z^2+1}{z+2}\right)^{\frac{1}{z}}\right)\right) = \exp\left(\lim_{z\to\infty} \frac{\ln\left(\frac{z^2+1}{z+2}\right)}{z}\right)$$
$$\overset{\text{L'H}}{=} \exp\left(\lim_{z\to\infty} \frac{\frac{z^2+4z-1}{z^3+3z+2}}{1}\right) = \exp\left(\lim_{z\to\infty} \frac{1+\frac{4}{z}-\frac{1}{z^2}}{z+\frac{3}{z}+\frac{2}{z^2}}\right) = e^0 = 1$$

*as by chain and quotient rules for derivatives*

$$\left(\ln\left(\frac{z^2+1}{z+2}\right)\right)' = \frac{z+2}{z^2+1} \cdot \frac{2z\cdot(z+2) - (z^2+1)\cdot 1}{(z+2)^2} = \frac{z^2+4z-1}{z^3+3z+2}$$

e) *This is an indeterminate form of type $\infty - \infty$ and we can bring the fractions on a common denominator,*

$$\begin{aligned}
\lim_{y\to 0+}\Big(\frac{3y+1}{y} - \frac{1}{1-\cos(y)}\Big) &= \lim_{y\to 0+} \frac{(3y+1)\big(1-\cos(y)\big) - 1\cdot y}{y\big(1-\cos(y)\big)} \\
&= \lim_{y\to 0+} \frac{2y+1-\cos(y)-3\cos(y)}{y - y\cos(y)} \\
&= \lim_{y\to 0+} \frac{2y+1-\cos(y)-3y\cos(y)}{y - y\cos(y)} \\
&\overset{L'H}{=} \lim_{y\to 0+} \frac{2+\sin(y)-3\cos(y)+3y\sin(y)}{1-\cos(y)+y\sin(y)} = -\infty
\end{aligned}$$

*as the denominator converges to $0$ from the right and the nominator to $-1$. Specifically we note that for the denominator we have that the function $1-\cos(y)+y\sin(y)$ is $0$ at $0$ and has as difference quotient from the right*

$$\frac{\big(1-\cos(y)+y\sin(y)\big) - \big(1-\cos(0)+0\sin(0)\big)}{y-0} = \frac{1-\cos(y)}{y} + \sin(y) > 0$$

*for $y$ positive and close enough to zero. Thus as the difference quotient is positive, the function $1-\cos(y)+y\sin(y)$ is approaching zero from the right (it is increasing right of zero).*

4. Explain what it means to say that

$$\lim_{x\to 0+}\Big(\frac{1}{x}\Big)^x = 1.$$

Why this makes sense even though the base is infinity zero for $x = 0$, the exponent is zero and $\infty^0$ is not defined? Your answer might be a mix of text, graphics and mathematical expressions.

*While $\infty^0$ is indeed an indeterminate form, we can just not plug in the value $0$ for $x$ into $\left(\frac{1}{x}\right)^x$, but we have to analyze the limit more carefully. Specifically, doing an exponential-logarithmic transformation (i.e., using the fact that $\ln(x)$ is the inverse function of $e^x$ we get*

$$\lim_{x\to 0+}\Big(\frac{1}{x}\Big)^x = \lim_{x\to 0+}\exp\left(\ln\left(\Big(\frac{1}{x}\Big)^x\right)\right) = \exp\left(\lim_{x\to 0+} x\ln\Big(\frac{1}{x}\Big)\right) = \exp\left(\lim_{x\to 0+}\frac{\ln\big(\frac{1}{x}\big)}{\frac{1}{x}}\right).$$

*Thus the actual problem is to study the indeterminate form*

$$\lim_{x\to 0+}\frac{\ln\big(\frac{1}{x}\big)}{\frac{1}{x}}$$

*which is of type $\frac{+\infty}{+\infty}$. Thus we can apply L'Hôpital's rule to get*

$$\lim_{x\to 0+}\frac{\ln\big(\frac{1}{x}\big)}{\frac{1}{x}} \overset{L'H}{=} \lim_{x\to 0+}\frac{\frac{1}{\frac{1}{x}}\cdot\big(-\frac{1}{x^2}\big)}{-\frac{1}{x^2}} = \lim_{x\to 0+} x = 0$$

*Graphically, nominator and denominator are displayed in Figure 1, it is clear to see that the functions diverge to* $+\infty$ *as* $x$ *approaches* $0$ *from the right. Plugging this result in yields the final result*

$$\lim_{x\to 0+}\left(\frac{1}{x}\right)^x = \exp\left(\lim_{x\to 0+}\frac{\ln\left(\frac{1}{x}\right)}{\frac{1}{x}}\right) = e^0 = 1.$$

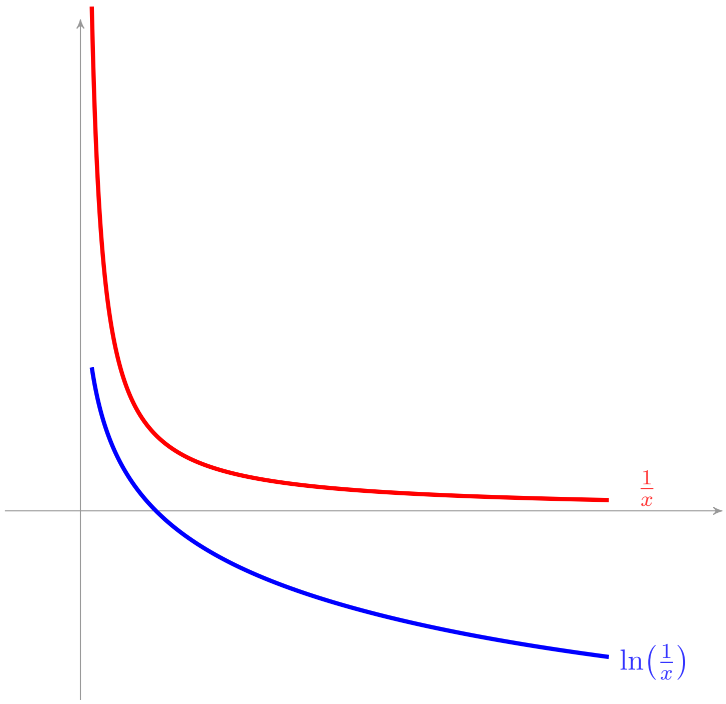


Figure 1: Function plot of nominator and denominator of the fraction $\frac{\ln(\frac{1}{x})}{\frac{1}{x}}$.