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Fall 2020 - A Term

MA 1023
Calculus III

Conference 2 – Ideas
Covers material from Lecture 4-6 & Active Learning 2

1. Calculate the following improper integrals:

$$\text{a)}\quad \int_1^\infty \frac{\big(\ln(w)\big)^2}{w}\,dw \qquad\qquad \text{b)}\quad \int_{-1}^\infty \frac{1}{|z+2|^5}\,dz$$

a) Using integration by parts with $f(w) = \big(\ln(w)\big)^2$ and $g'(w) = \frac{1}{w}$ and thus$f'(w) = 2\ln(w)\frac{1}{w}$ and $g'(w) = \ln(w)$ we have

$$\begin{aligned}\int_1^\infty \frac{\big(\ln(w)\big)^2}{w}\,dw &= \lim_{b\to\infty}\int_1^b \frac{\big(\ln(w)\big)^2}{w}\,dw\\ &= \lim_{b\to\infty}\left(\Big[\big(\ln(w)\big)^3\Big]_{w=1}^b - \int_1^b 2\ln(w)\frac{1}{w}\cdot\ln(w)\,dw\right)\\ &= \lim_{b\to\infty}\Big[\big(\ln(w)\big)^3\Big]_{w=1}^b - 2\int_1^\infty \frac{\big(\ln(w)\big)^2}{w}\,dw\end{aligned}$$

Thus, rearranging the terms

$$\int_1^\infty \frac{\big(\ln(w)\big)^2}{w}\,dw = \frac{1}{3}\lim_{b\to\infty}\Big[\big(\ln(w)\big)^3\Big]_{w=1}^b = \frac{1}{3}\lim_{b\to\infty}\ln(b)\big)^3 = +\infty$$

as both, $\ln(x)$ and x^3 diverge to $+\infty$ for $x\to+\infty$.
Alternatively this can be done bu u-substitution. Setting $u = \ln(w)$ we have $w = e^u$ and $dw = e^u\,du$, whence

$$\begin{aligned}\int_1^\infty \frac{\big(\ln(w)\big)^2}{w}\,dw &= \lim_{b\to\infty}\int_1^b \frac{\big(\ln(w)\big)^2}{w}\,dw = \lim_{b\to\infty}\int_0^{\ln(b)} \frac{u^2}{e^u}\cdot e^u\,du\\ &= \lim_{b\to\infty}\int_0^{\ln(b)} u^2\,du = \lim_{b\to\infty}\frac{\big(\ln(b)\big)^3}{3} = +\infty\end{aligned}$$

b) As in the range of integration the term $z+2$ is alway positive, we have

$$\int_{-1}^{\infty} \frac{1}{|z+2|^5}\,dz = \lim_{b\to\infty}\int_{-1}^{b} \frac{1}{(z+2)^5}\,dz = \lim_{b\to\infty}\Big[-\frac{1}{4(z+2)^4}\Big]_{z=-1}^{b}$$
$$= \frac{1}{4} - \lim_{b\to\infty}\frac{1}{4(b+2)^4} = \frac{1}{4}$$

as $4(b+2)^4$ goes to $+\infty$ as b goes to ∞.

2. Do the following integrals converge or diverge? Explain your answer carefully. Note that you do not have to calculate the integrals explicitly for that.

a) $\displaystyle\int_1^{\infty} \frac{\big(\arctan(x)\big)^3}{x^2}\,dx$ b) $\displaystyle\int_1^{\infty} \frac{2+\cos(y)}{\sqrt[3]{y}}\,dy$

a) We note that $\arctan(x)$ is positive on the whole domain of the integration, and thus so is the integrand. Moreover, since $\arctan(x) \le \frac{\pi}{2}$ we have

$$\int_1^{\infty} \frac{\big(\arctan(x)\big)^3}{x^2}\,dx \le \int_1^{\infty} \frac{\left(\frac{\pi}{2}\right)^3}{x^2}\,dx = \left(\frac{\pi}{2}\right)^3 \lim_{b\to\infty}\int_1^{b}\frac{1}{x^2}\,dx$$
$$= \left(\frac{\pi}{2}\right)^3 \cdot \left(1 - \lim_{b\to\infty}\frac{1}{x}\right) = \left(\frac{\pi}{2}\right)^3.$$

Thus the integral converges.

b) We note that $-1 \le \cos(y) \le 1$ and for $y \ge 1$ also $\sqrt[3]{y} \le y$. Therefore

$$0 \le \frac{1}{y} \le \frac{2+\cos(y)}{\sqrt[3]{y}}.$$

As

$$\int_1^{\infty}\frac{1}{y}\,dy = \lim_{c\to\infty}\int_1^{c}\frac{1}{y}\,dy = \lim_{c\to\infty}\Big[\ln(y)\Big]_1^{c} = \lim_{c\to\infty}\ln(c) - \ln(1) = \lim_{c\to\infty}\ln(c) = \infty$$

this integral diverges. Therefore also the integral over the larger function diverges,

$$\int_1^{\infty}\frac{2+\cos(y)}{\sqrt[3]{y}}\,dy = \infty.$$

3. For which values of p does the following integral converge?

$$\int_e^{\infty}\frac{1}{x\big(\ln(x)\big)^p}\,dx$$

We note that using the u-substitution $y = \ln(x)$ we have $x = e^y$ and $dx = e^y\,dy$ and thus

$$\int_e^\infty \frac{1}{x\big(\ln(x)\big)^p}\,dx = \lim_{b\to\infty}\int_e^b \frac{1}{x\big(\ln(x)\big)^p}\,dx = \lim_{b\to\infty}\int_1^{\ln(b)} \frac{1}{e^y y^p}e^y\,dy = \lim_{b\to\infty}\int_1^{\ln(b)} \frac{1}{y^p}\,dy$$

Thus, if $p \neq 1$ we have

$$\int_e^\infty \frac{1}{x\big(\ln(x)\big)^p}\,dx = \lim_{b\to\infty}\left[\frac{1}{(1-p)y^{p-1}}\right]_1^{\ln(b)} = \lim_{b\to\infty}\frac{\big(\ln(b)\big)^{1-p}}{1-p} - \frac{1}{(1-p)}$$

Thus, as $\ln(b)$ diverges to $+\infty$ for $b\to\infty$, the first term converges to 0 if $p > 1$ and diverges to $+\infty$ of $p < 1$. Then there is the special case $p = 1$

$$\int_e^\infty \frac{1}{x\big(\ln(x)\big)^p}\,dx = \lim_{b\to\infty}\big[\ln(y)\big]_1^{\ln(b)} = \lim_{b\to\infty}\ln\big(\ln(b)\big) = +\infty$$

as, twice consecutively, $\ln(b)$ diverges to $+\infty$ for $b\to\infty$. Thus we conclude that the integral converges for $p > 1$ and diverges otherwise.

4. Explain what it means to say that

$$\int_{-\infty}^3 f(x)\,dx, \qquad f(x) \geq 0.$$

diverges. Your answer might be a mix of text, graphics and mathematical expressions.

As $f(x) \geq 0$, the integral describes the area under the curve f between $-\infty$ and 3, see Figure 1.

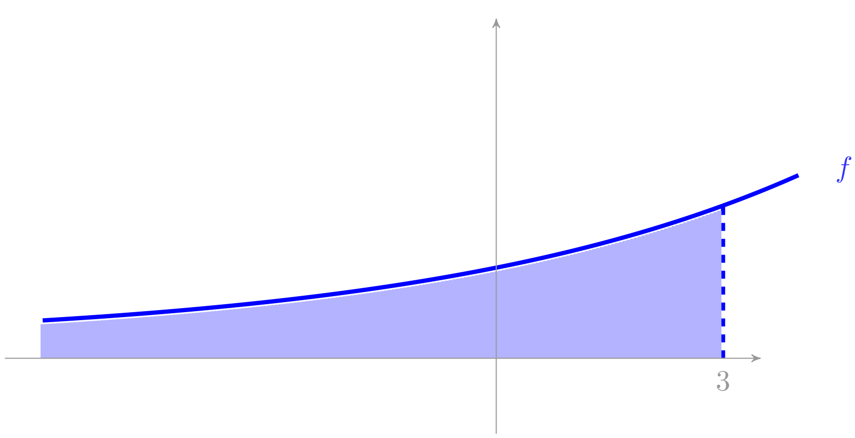

Figure 1: Area under the function f from $-\infty$ to 3.

The integral is defined as

$$\int_{-\infty}^{3} f(x)\,dx = \lim_{a\to\infty} \int_{a}^{3} f(x)\,dx$$

and to say that the integral diverges means that the limit does not exist. As f is a non-negative function, the function of which we are taking the limit is non-decreasing, and thus it means that the limit has to be $+\infty$. Thus the area under the curve f between $-\infty$ and 3 is infinite.

5. Let $P_n(x)$ be a n-th degree polynomial. What can we conclude about the improper integral

$$\int_{1}^{\infty} \frac{P_n(x)}{5x^n+1}\,dx$$

Note that a polynomial of degree n is given as $a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$ where $a_n \neq 0$. Thus we have

$$\lim_{x\to\infty} \frac{P_n(x)}{5x^n+1} = \lim_{x\to\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0}{5x^n+1} = \lim_{x\to\infty} \frac{a_n + \frac{a_{n-1}}{x} + \cdots \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}}{5 + \frac{1}{x^n}} = \frac{a_n}{5}.$$

Thus the function has a horizontal asymptote at $\frac{a_n}{5}$ which is not zero as $a_n \neq 0$. Remember from class (Lecture 5) that an integral up to $+\infty$ over any function with a horizontal asymptote other than 0 diverges, so the given integral necessarily diverges.