MA 1024 Conference 2

DEFINITION A function $f(x, y)$ is **continuous at the point (x_0, y_0)** if

1. f is defined at (x_0, y_0),
2. $\lim_{(x, y)\to(x_0, y_0)} f(x, y)$ exists,
3. $\lim_{(x, y)\to(x_0, y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

Continuity of Composites

If f is continuous at (x_0, y_0) and g is a single-variable function continuous at $f(x_0, y_0)$, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is continuous at (x_0, y_0).

For example, the composite functions

$$e^{x-y}, \qquad \cos\frac{xy}{x^2+1}, \qquad \ln\left(1 + x^2y^2\right)$$

are continuous at every point (x, y).

Putting these together gives at least one method for finding a multivariable function limit!

To find a limit, first check for continuity. If the function is continuous at the specified point, evaluate the function at that point to get the limit.

See Example 1 ↓

Example 1

Find the limit, if it exists.

$$\lim_{(x,y)\to(-7,1)} \frac{xy\cos(x+7y)}{e^{x^2+y^2}}$$

Let $f(x,y) = \dfrac{xy\cos(x+7y)}{e^{x^2+y^2}}$.

Since f is continuous, we have:

$\lim_{(x,y)\to(-7,1)} f(x,y) = f(-7,1)$. So,

$$\lim_{(x,y)\to(-7,1)} \frac{xy\cos(x+7y)}{e^{x^2+y^2}} = \frac{(-7)(1)\cos(-7+7)}{e^{(-7)^2+1^2}}$$

$$= \frac{-7}{e^{50}}.$$

Note: This idea generalizes to higher dimensions. For example in the case of $f(x,y,z)$

Example 2

Find the limit, if it exists.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$$

See Dr. Bill's Lecture 4, starting with

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} \cdot \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}}$$

and using:

THEOREM 1—Properties of Limits of Functions of Two Variables The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1. *Sum Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$
2. *Difference Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$
3. *Constant Multiple Rule:* $\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL$ (any number k)
4. *Product Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) \cdot g(x,y)) = L \cdot M$
5. *Quotient Rule:* $\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer, and if n is even, we assume that $L > 0$.

If the limit of a function at a given point exists, the limit is independent of the path of approach. Equivalently, we have the following test which can show that a limit does not exist.

Two-Path Test for Nonexistence of a Limit

If a function $f(x, y)$ has different limits along two different paths in the domain of f as (x, y) approaches (x_0, y_0), then $\lim_{(x, y)\to(x_0, y_0)} f(x, y)$ does not exist.

⚠ Cannot prove the existence of a limit, only the nonexistence of a limit.

- In the case that $(x_0, y_0) = (0,0)$, these are some good paths to try:

$y = 0$ (x-axis)

$x = 0$ (y-axis)

$y = x$

$y = -x$

$y = mx$, m is a real number

- <u>Note</u>: This idea generalizes to higher dimensions

Example 3

Show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

Try $y=0$:

$$\lim_{(x,0)\to(0,0)} \frac{x^2-0^2}{x^2+0^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

Try $x=0$:

$$\lim_{(0,y)\to(0,0)} \frac{0^2-y^2}{0^2+y^2} = \lim_{y\to 0} \frac{-y^2}{y^2} = -1$$

This limit does not exist

by:

Two-Path Test for Nonexistence of a Limit

If a function $f(x, y)$ has different limits along two different paths in the domain of f as (x, y) approaches (x_0, y_0), then $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.

Example 3.5

Show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{2x^2}{x^2+y^2}$$

• Try the path $y=0$ (x-axis):

$$\lim_{(x,0)\to(0,0)} \frac{2x^2}{x^2+0^2} = \lim_{x\to 0} \frac{2x^2}{x^2} = 2.$$

• Try along the path $x=0$ (y-axis)

$$\lim_{(0,y)\to(0,0)} \frac{2\cdot 0^2}{0^2+y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0.$$

• So the limit does not exist by:

Two-Path Test for Nonexistence of a Limit

If a function $f(x, y)$ has different limits along two different paths in the domain of f as (x, y) approaches (x_0, y_0), then $\lim_{(x, y)\to(x_0, y_0)} f(x, y)$ does not exist.

• Additionally, we see that along the path $y=mx$.

$$\lim_{(x,mx)\to(0,0)} \frac{2x^2}{x^2+m^2x^2} = \lim_{x\to 0} \frac{2}{1+m^2} = \frac{2}{1+m^2}$$

Example 4

Show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

- Try $y=0$ (x-axis)

$$\lim_{x\to 0} \frac{x\cdot 0}{x^2+0^2} = \lim_{x\to 0} \frac{0}{x^2} = 0.$$

- Try $x=0$ (y-axis)

$$\lim_{y\to 0} \frac{0\cdot y}{0^2+y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0.$$

⚠ This does not prove that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = 0$.

- Try a third path, say along $y=x$.

$\lim_{(x,x)\to(0,0)} \frac{x\cdot x}{x^2+x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$. So the limit does not exist by:

Two-Path Test for Nonexistence of a Limit

If a function $f(x, y)$ has different limits along two different paths in the domain of f as (x, y) approaches (x_0, y_0), then $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.

Example 5

Find the first partial derivatives of $f(x, y, z) = x \ \arctan(yz)$

We will use $\frac{d}{dw} \arctan(w) = \frac{1}{1+w^2}$
and the chain rule
$\frac{d}{dw} h(g(w)) = h'(g(w))g'(w)$.

$$\frac{\partial f}{\partial x} = f_x(x,y,z) = \arctan(yz).$$

$$\frac{\partial f}{\partial y} = f_y(x,y,z) = x \cdot \frac{1}{1+(yz)^2} \cdot z$$

$$= \frac{xz}{1+y^2z^2}.$$

$$\frac{\partial f}{\partial z} = f_z(x,y,z) = x \cdot \frac{1}{1+(yz)^2} \cdot y$$

$$= \frac{xy}{1+y^2z^2}.$$

Example 6

Given,

$$f(x,y) = \int_y^x \cos(-8t^2 - 2t + 2)\,dt$$

Find, $f_x(x,y), f_y(x,y)$

First part [edit]

This part is sometimes referred to as the *first fundamental theorem of calculus*.[7]

Let *f* be a continuous real-valued function defined on a closed interval [*a*, *b*]. Let *F* be the function defined, for all *x* in [*a*, *b*], by

$$F(x) = \int_a^x f(t)\,dt.$$

Then *F* is uniformly continuous on [*a*, *b*] and differentiable on the open interval (*a*, *b*), and

$$F'(x) = f(x)$$

for all *x* in (*a*, *b*).

$$\frac{\partial f}{\partial x} = f_x(x,y) = \cos(-8x^2 - 2x + 2).$$

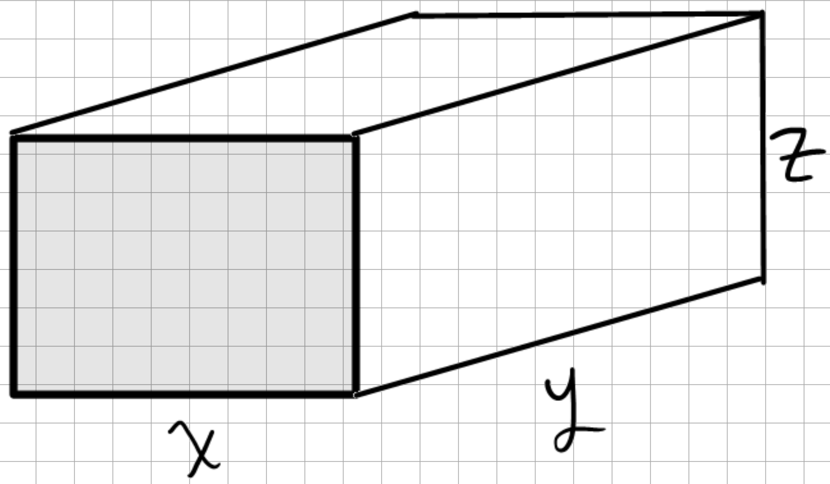
To find $f_y(x,y)$, first note that

$$f(x,y) = -\int_x^y \cos(-8t^2 - 2t + 2)\,dt$$

and work similarly to how $f_x(x,y)$ was found.

Example 7

A rectangular box has dimensions length, width, and height dimensions of x, y, and z respectively. Find the rate of change of the surface area of the box with respect to each dimension.



The surface area, S, is given by

$$S(x,y,z) = 2xz + 2yz + 2xy.$$

$$\frac{\partial S}{\partial x} = 2z + 0 + 2y = 2z + 2y.$$

$$\frac{\partial S}{\partial y} = 0 + 2z + 2x = 2z + 2x.$$

$$\frac{\partial S}{\partial z} = 2x + 2y + 0 = 2x + 2y.$$