MA 1024 Conference 5

1. Lagrange Multipliers
2. Review

The Method of Lagrange Multipliers

Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $\nabla g \neq \mathbf{0}$ when $g(x, y, z) = 0$. To find the local maximum and minimum values of f subject to the constraint $g(x, y, z) = 0$ (if these exist), find the values of x, y, z, and λ that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g(x, y, z) = 0. \tag{1}$$

For functions of two independent variables, the condition is similar, but without the variable z.

Example 1 (Lagrange Multipliers)

Find the extreme values of $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

Constraint: $g(x, y) = x^2 + y^2 = 1$

$$\nabla f = \langle 2x, 4y \rangle$$

$$\nabla g = \langle 2x, 2y \rangle$$

$$\nabla f = \lambda \nabla g$$

$$\langle 2x, 4y \rangle = \langle 2\lambda x, 2\lambda y \rangle$$

Solve the system.

$$\left\{ \begin{array}{ll} 2x = 2\lambda x & \text{(A)} \\ 4y = 2\lambda y & \text{(B)} \\ x^2 + y^2 = 1 & \text{(C)} \end{array} \right\}$$

By (A), $x=0$ or $\lambda=1$

If $x=0$, then $y=1$ or $y=-1$ by (C)

If $\lambda=1$, then $y=0$ by (B). so $x=\pm 1$ by (C)

Therefore, f has possible extreme values at $(0,1)$, $(0,-1)$, $(1,0)$, $(-1,0)$.

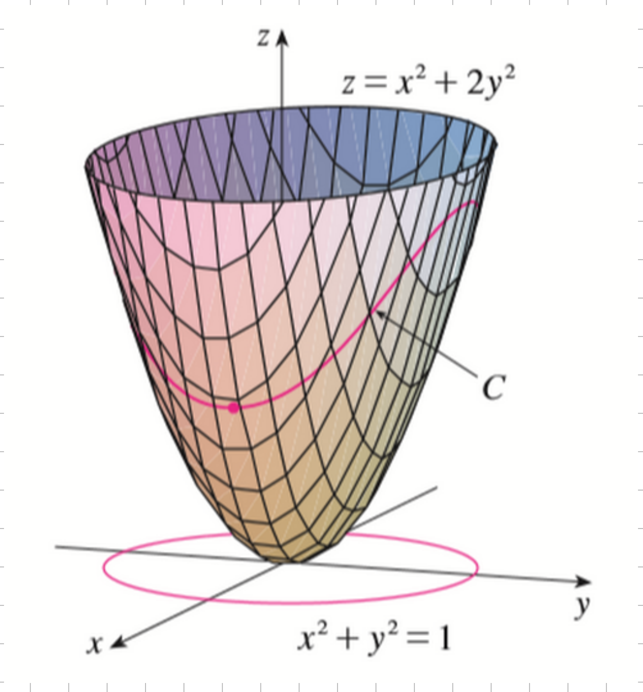
Check:

$f(0,1) = f(0,-1) = 2$

$f(1,0) = f(-1,0) = 1$

The max value of f on the circle $x^2+y^2=1$ is 2.

The min value of f on the circle $x^2+y^2=1$ is 1.



Example 2 (Limits)

Show that the following limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{3y^2 - 4x^{10}}{4x^{10} + 2x^5y - 3y^2}$$

Failed Attempts:

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$y=0$ $\quad \lim_{(x,0)\to(0,0)} \frac{-4x^{10}}{4x^{10}} = \lim_{x\to 0} -1 = -1$

$x=0$ $\quad \lim_{(0,y)\to(0,0)} \frac{3y^2}{-3y^2} = \lim_{y\to 0} -1 = -1$

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$y=mx$ $\quad \lim_{(x,mx)\to(0,0)} \frac{3m^2 - 4x^{10}}{4x^{10} + 2mx^6 - 3m^2x^2} = ?$

Try other paths through the origin. More failed attempts gives me an idea.

↓

Example 2 (Limits)

Show that the following limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{3y^2 - 4x^{10}}{4x^{10} + 2x^5y - 3y^2}$$

Try paths $y = x^5$ and $y = -x^5$, which both pass through $(0,0)$.

$$\lim_{(x,x^5)\to(0,0)} \frac{3(x^5)^2 - 4x^{10}}{4x^{10} + 2x^5x^5 - 3(x^5)^2} = \dots = -\frac{1}{3}$$

$$\lim_{(x,-x^5)\to(0,0)} \frac{3(-x^5)^2 - 4x^{10}}{4x^{10} + 2x^5(-x^5) - 3(x^5)^2} = \dots = 1$$

Since $\lim_{(x,y)\to(0,0)} \frac{3y^2 - 4x^{10}}{4x^{10} + 2x^5y - 3y^2}$ depends on the path of approach, the limit DNE.

Note: We didn't really need to check along $y = -x^5$ since the path $y = x^5$ gave a different result than the paths $y=0$ and $x=0$, so we could have stopped there.

Example 3 (Critical Points)

Find the local maximum and minimum values and saddle point(s) of the function $f(x, y) = (x^2 + y^2)e^{y^2-x^2}$.

$$f_x(x,y) = 2x e^{y^2-x^2} + (x^2+y^2)e^{y^2-x^2}(-2x)$$

$$f_x(x,y) = -2x e^{y^2-x^2}(-1+x^2+y^2)$$

$$f_y(x,y) = 2ye^{y^2-x^2} + (x^2+y^2)e^{y^2-x^2}(2y)$$

$$f_y(x,y) = 2ye^{y^2-x^2}(1+x^2+y^2)$$

Set $f_x(x,y) = 0$, $f_y(x,y) = 0$

(A) $0 = -2xe^{y^2-x^2}(x^2+y^2-1)$

(B) $0 = 2ye^{y^2-x^2}(x^2+y^2+1)$

- By (A), $x = 0$ or $x^2+y^2-1=0$

If $x = 0$, $0 = 2ye^{y^2}(y^2+1) \Rightarrow y = 0$ by (B)

If $x^2+y^2-1=0$, then $x^2+y^2=1$. By (B),

$0 = 2ye^{y^2-x^2}(1+1) \Rightarrow y = 0$.

Plug $y=0$ back into $x^2+y^2=1 \Rightarrow x=\pm 1$

- By (B) $0 = 2ye^{y^2-x^2}(x^2+y^2+1) \Rightarrow y=0.$

  But we have already covered this case.

The critical pts of f are:

$(0,0)$, $(1,0)$, $(-1,0)$

To use the $2^{\underline{nd}}$ derivative test on these pts, we need f_{xx}, f_{yy}, & f_{xy}.

The work for getting f_{xx} is shown:

use $\frac{d}{dx} g(x)\cdot h(x)\cdot j(x) = g'hj + gh'j + ghj'$ for

$$f_x(x,y) = -2xe^{y^2-x^2}(x^2+y^2-1)$$

or "distribute $-2x$"

$$f_x(x,y) = e^{y^2-x^2}(-2x^3-2xy^2+2x)$$

I did the first way in conference, here I'll use the second approach

$$f_{xx}(x,y) = -2xe^{y^2-x^2}(-2x^3-2xy^2+2x) + e^{y^2-x^2}(-6x^2-2y^2+2)$$

$$f_{xx}(x,y) = e^{y^2-x^2}\left(4x^4+4x^2y^2-4x^2-6x^2-2y^2+2\right)$$

$$f_{xx}(x,y) = e^{y^2-x^2}\left(4x^4+4x^2y^2-10x^2-2y^2+2\right)$$

"Similarly",

$$f_{yy}(x,y) = e^{y^2-x^2}\left(4y^4+4x^2y^2+10y^2+2x^2+2\right)$$

$$f_{xy}(x,y) = -4xy\,e^{y^2-x^2}\left(x^2+y^2\right)$$

- To classify the critical pts, we will need to calculate

$$D(x,y) = f_{xx}(x,y)\,f_{yy}(x,y) - \left[f_{xy}(x,y)\right]^2$$

for $(0,0)$, $(1,0)$, $(-1,0)$

- I will show the work for $D(1,0)$:

$$D(1,0) = \underbrace{\left[e^{-1}(4+0-10-0+2)\right]}^{f_{xx}(1,0)}\underbrace{\left[e^{-1}(0+0+0+2+2)\right]}^{f_{yy}(1,0)}$$

$$-\overbrace{\left[0e^{-1}(1+0)\right]^2}^{[f_{xy}(1,0)]^2}$$

$$D(1,0) = -16e^{-2} = \frac{-16}{e^2} < 0.$$

By ↓ , $(1,0)$ is a saddle point of f.

THEOREM 11—Second Derivative Test for Local Extreme Values Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then

i) f has a **local maximum** at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).

ii) f has a **local minimum** at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).

iii) f has a **saddle point** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b).

iv) **the test is inconclusive** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b). In this case, we must find some other way to determine the behavior of f at (a, b).

Calculating $D(-1,0)$ and $D(0,0)$ and considering $f_{xx}(0,0) = 2 > 0$ will show that $(-1,0)$ is a saddle point and $(0,0)$ is a minimum of f. Graphs / Level curves will show that $(0,0)$ is only a local (but not global) minimum.