MA 1024 Conference 6

Example 1 (Double Integrals Over Rectangles)

Find $\int\int_R f(x,y)\,dA$, where $f(x,y) = (2x+3y)^{-2}$ and $R = [0,1] \times [1,2]$.

THEOREM 1—Fubini's Theorem (First Form) If $f(x, y)$ is continuous throughout the rectangular region R: $a \leq x \leq b$, $c \leq y \leq d$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

.

Example 2 (Double Integrals Over General Regions)

Find $\int\int_R f(x,y)\,dA$, where $f(x,y) = e^{\frac{x}{y}}$ and R is the region given by the set $R = \{(x,y) \mid 1 \le y \le 2,\ y \le x \le y^3\}$.

THEOREM 2—Fubini's Theorem (Stronger Form) Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

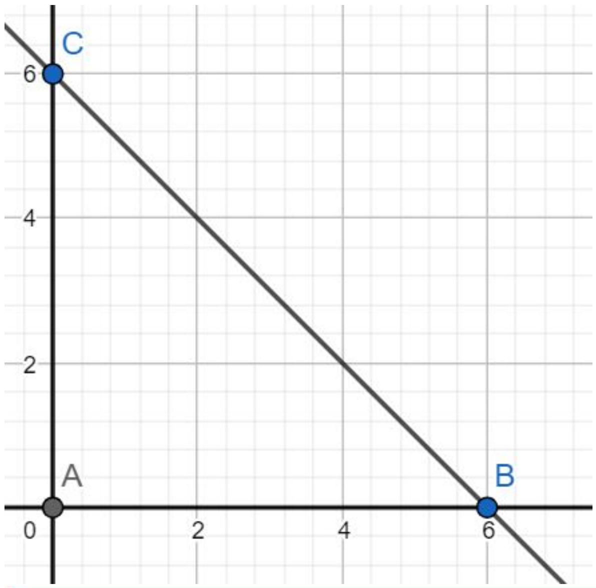
2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

.

Example 3 (Double Integrals Over General Regions)

Find $\int\int_R xy$, where R is the triangular region in the xy plane with vertices $(0,0)$, $(6,0)$, and $(0,6)$.



THEOREM 2—Fubini's Theorem (Stronger Form) Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

(Problem repeated): Find $\int\int_R xy$, where R is the triangular region in the xy plane with vertices $(0,0)$, $(6,0)$, and $(0,6)$.

Example 4 (Reversing the Order of Integration)

Evaluate the integral $I = \int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\,dx\,dy$ by reversing the order of integration (using Fubini's Theorem).

.

Example 5 (Double Integrals as Volume)

Set up an integral to find the volume of the solid bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

Example 6 (Properties of Double Integrals)

Using the maxima of the function over the region R, produce an upper estimate of the integral

$$I = \int\int_R e^{x^2+y^2}\, dA$$

Where R is the disk in the xy plane with $x^2 + y^2 \leq 1$.

If $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R, then the following properties hold.

1. *Constant Multiple:* $\displaystyle\iint_R cf(x, y)\, dA = c \iint_R f(x, y)\, dA$ (any number c)

2. *Sum and Difference:*

$$\iint_R (f(x, y) \pm g(x, y))\, dA = \iint_R f(x, y)\, dA \pm \iint_R g(x, y)\, dA$$

3. *Domination:*

(a) $\displaystyle\iint_R f(x, y)\, dA \geq 0$ if $f(x, y) \geq 0$ on R

(b) $\displaystyle\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA$ if $f(x, y) \geq g(x, y)$ on R

4. *Additivity:* $\displaystyle\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$

if R is the union of two nonoverlapping regions R_1 and R_2

DEFINITION The **area** of a closed, bounded plane region R is

$$A = \iint_R dA.$$