MA 1024 Conference 7

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Example 1 (Triple Integral in Rectangular Coordinates, General Solids)

Evaluate $\int\int\int_E xyz\, dV$, where E is the solid for which $0 \le z \le 1$, $0 \le y \le z$, $0 \le x \le y$.

$$\int_0^1 \int_0^z \int_0^y xyz \, dx\, dy\, dz$$

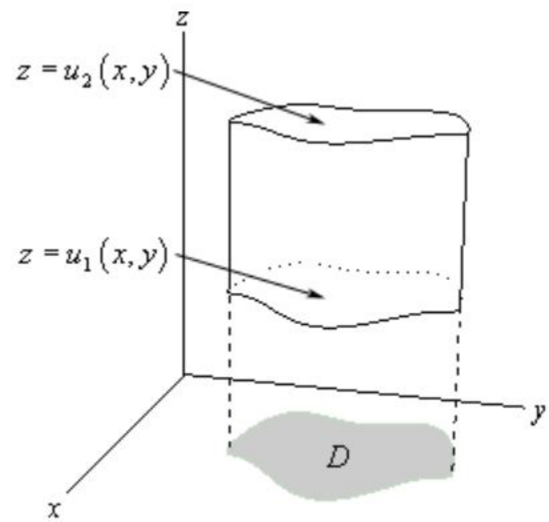
$$= \int_0^1 \int_0^z \left(\frac{x^2 yz}{2} \Big|_0^y \right) dy\, dz$$

$$= \int_0^1 \int_0^z \frac{y^3 z}{2} \, dy\, dz$$

$$= \int_0^1 \left(\frac{y^4 z}{8} \Big|_0^z \right) dz$$

$$= \int_0^1 \frac{z^5}{8} \, dz$$

$$= \frac{z^6}{48} \Big|_0^1 = 1/48$$



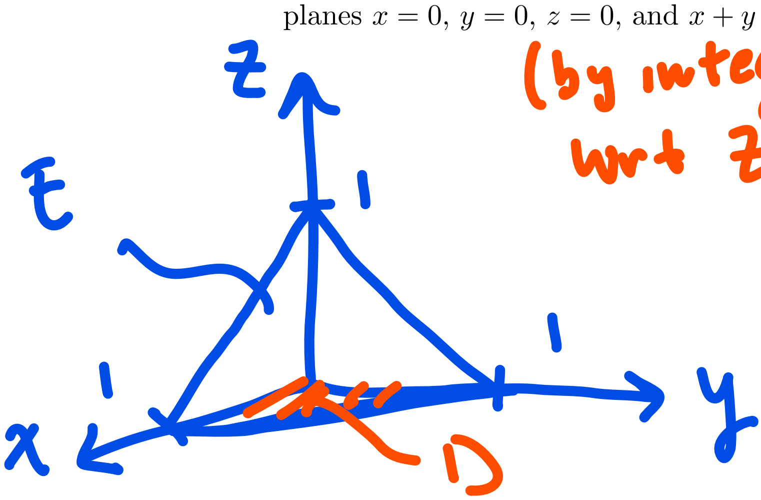
$$E = \{(x,y,z) \mid (x,y) \in D,\ \ u_1(x,y) \le z \le u_2(x,y)\}$$

$$\iiint\limits_E f(x,y,z)\, dV = \iint\limits_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz \right] dA$$

Example 2 (Triple Integral in Rectangular Coordinates, General Solids)

Evaluate $\int\int\int_E z\, dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

(by integrating wrt z first).



$x+y+z=1 \rightarrow z = 1-x-y$

$$E = \{(x,y,z) \mid (x,y) \in D,\ 0 \le z \le 1-x-y\}$$

$$\iiint_E z\, dV = \iint_D \left(\int_0^{1-x-y} z \, dz \right) dA$$

$$= \iint_D \left(\frac{z^2}{2} \Big|_0^{1-x-y} \right) dA$$

$$= \iint_D \frac{1}{2}(1-x-y)^2 \, dA$$

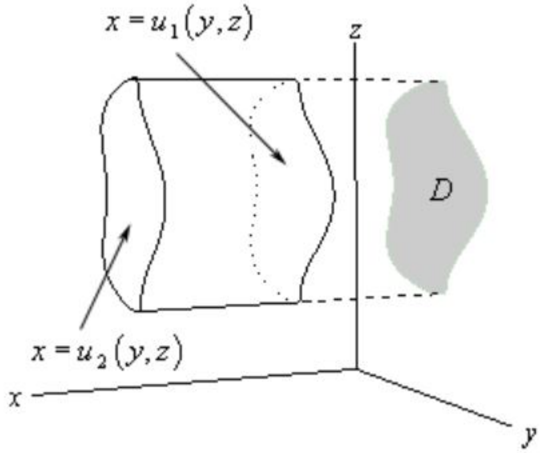
$$= \int_0^1 \int_0^{1-x} \frac{1}{2}(1-x-y)^2 \, dy\, dx$$

$$= \int_0^1 \left(\frac{1}{2}\left(-\frac{1}{3}\right)(1-x-y)^3 \Big|_0^{1-x} \right) dx$$

$$= \int_0^1 +\frac{1}{6}(1-x)^3 \, dx$$

$$= \frac{1}{6}\left(-\frac{1}{4}\right)(1-x)^4 \Big|_0^1$$

$$= \left(\frac{1}{6}\right)\left(-\frac{1}{4}\right)(-1) = 1/24$$

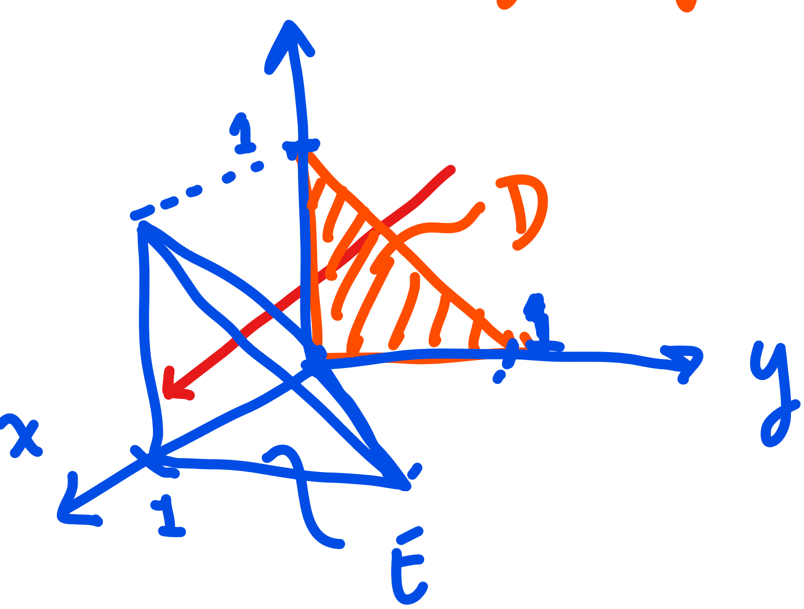


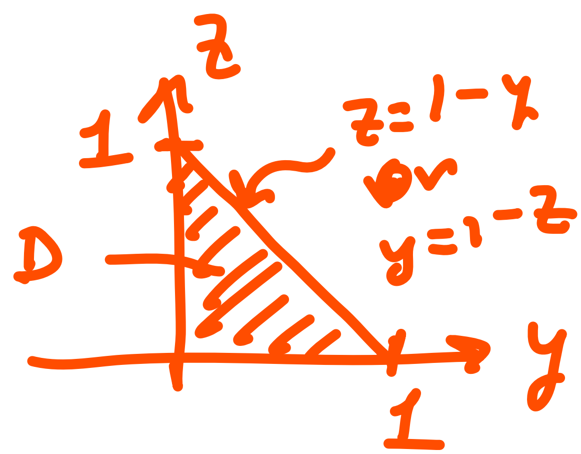
$$E = \{(x, y, z) \mid (y, z) \in D,\ \ u_1(y, z) \le x \le u_2(y, z)\}$$

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\, dx\right] dA$$

Example 3 (Triple Integral in Rectangular Coordinates, General Solids)

Set up the triple integral to find the mass of the tetrahedron with vertices $(0, 0, 0)$, $(1, 1, 0)$, $(1, 0, 0)$, and $(1, 0, 1)$ with the density function $\rho(x, y, z) = yx$. (by integrating wrt x first)



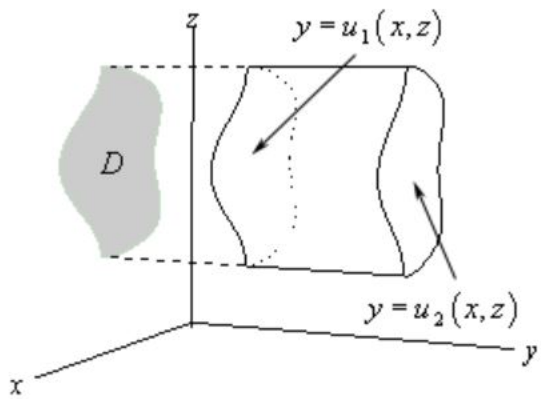


$$E = \{(x, y, z) \mid (x, y) \in D,\ 1 - y - z \leq x \leq 1\}$$

$$\text{Mass} = \iiint_E \rho(x, y, z)\, dV$$

$$= \iint_D \left(\int_{1-y-z}^{1} yx \, dz \right) dA$$

$$= \int_0^1 \int_0^{1-y} \int_{1-y-z}^{1} yx \, dx \, dz \, dy$$

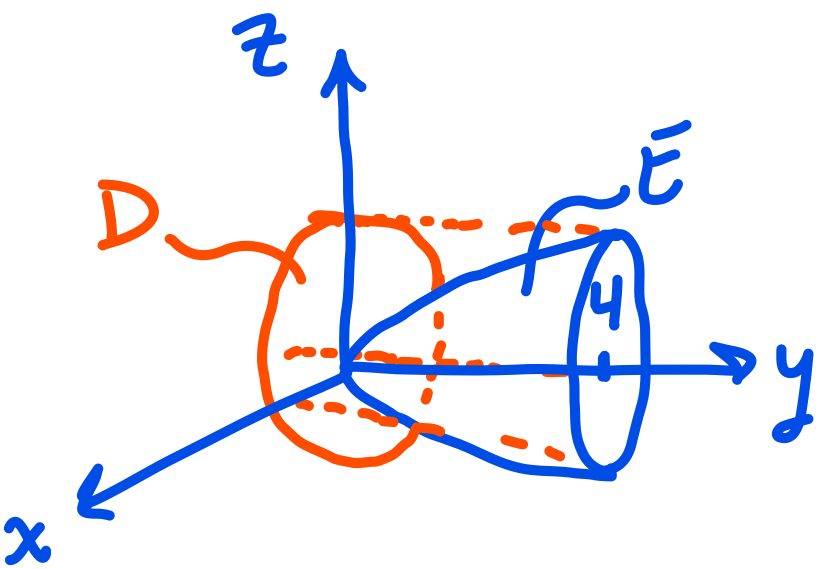


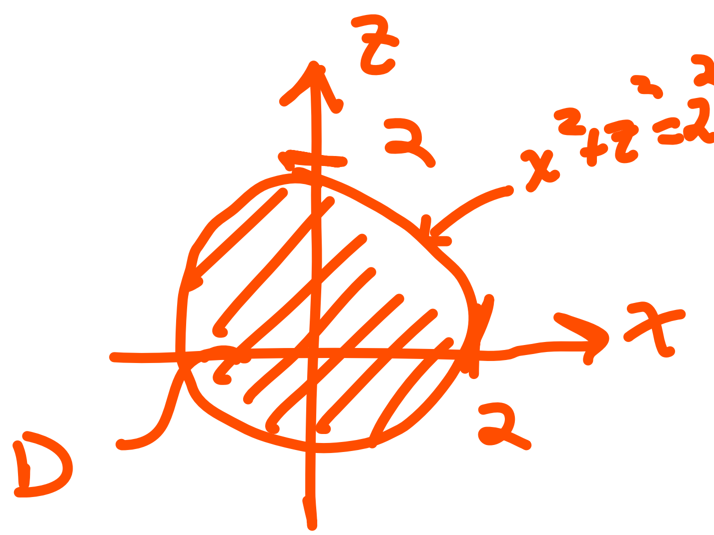
$$E = \{(x, y, z) \mid (x, z) \in D,\ u_1(x, z) \le y \le u_2(x, z)\}$$

$$\iiint\limits_E f(x, y, z)\, dV = \iint\limits_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy\right] dA$$

Example 4 (Triple Integral in Rectangular Coordinates, General Solids)

Evaluate $\int\int\int_E \sqrt{x^2 + z^2}\, dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.





$$E = \{(x, y, z) \mid (x, z) \in D,\ x^2 + z^2 \le y \le 4\}$$

$$\iiint_E \sqrt{x^2+z^2}\,dV = \iint_D \left(\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,dy\right) dA$$

$$= \iint_D \left(y\sqrt{x^2+z^2} \Big|_{x^2+z^2}^{4} \right) dA$$

$$= \iint_D (4-x^2-z^2)\sqrt{x^2+z^2}\,dA$$

Convert to a polar coordinate system by $x = r\cos\theta$, $z = r\sin\theta$, $r^2 = x^2+z^2$, $dA = r\,dr\,d\theta$.

$$= \int_0^{2\pi}\int_0^2 (4-r^2)\sqrt{r^2}\; r\,dr\,d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\int_0^2 (4r^2-r^4)\,dr$$

$$= (2\pi)\left(\frac{4}{3}r^3 - \frac{1}{5}r^5\right)\Big|_0^2 = \frac{128\pi}{15}.$$

Ex 4 is suggestive of a more general strategy for triple integrals:

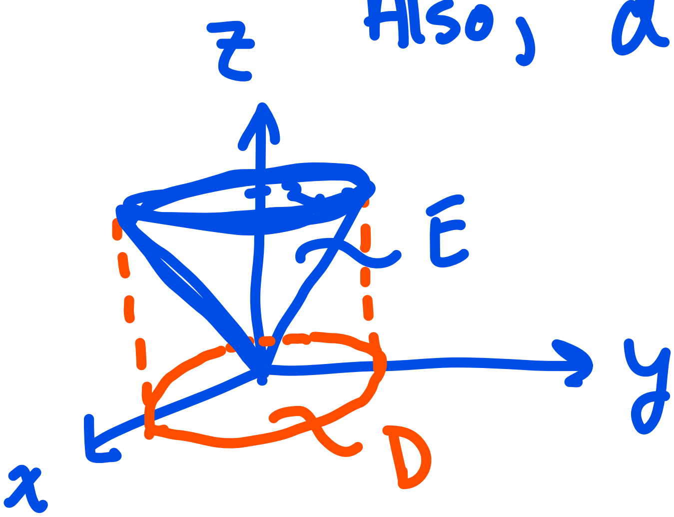
Example 5 (Triple Integral in Cylindrical Coordinates)

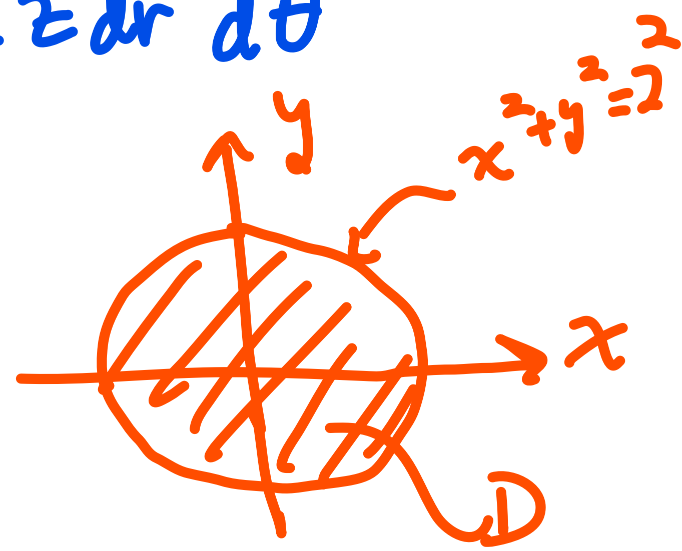
Evaluate $\int\int\int_E x^2+y^2\,dV$, where $E = \{(x,y,z) \mid -2 \le x \le 2, -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}, \sqrt{x^2+y^2} \le z \le 2\}$.

Equations Relating Rectangular (x, y, z) and Cylindrical (r, θ, z) Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$
$$r^2 = x^2 + y^2, \qquad \tan\theta = y/x$$

Also, $dV = r\,dz\,dr\,d\theta$





In rectangular coordinates, integration is more difficult:

$$\iiint_E (x^2+y^2)\,dV = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\,dz\,dy\,dx$$

Converting to cylindrical is easier:

$$\iiint_E (x^2+y^2)\,dV = \int_0^{2\pi}\int_0^2\int_{\sqrt{r^2}}^2 r^2\, r\,dz\,dr\,d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\int_0^2\int_r^2 r^3\,dz\,dr$$

$$= (2\pi)\int_0^2 \left(zr^3 \Big|_r^2 \right) dr$$

$$= 2\pi\int_0^2 (2r^3 - r^4)\,dr$$

$$= 2\pi\left(\frac{2}{4}r^4 - \frac{1}{5}r^5 \Big|_0^2\right)$$

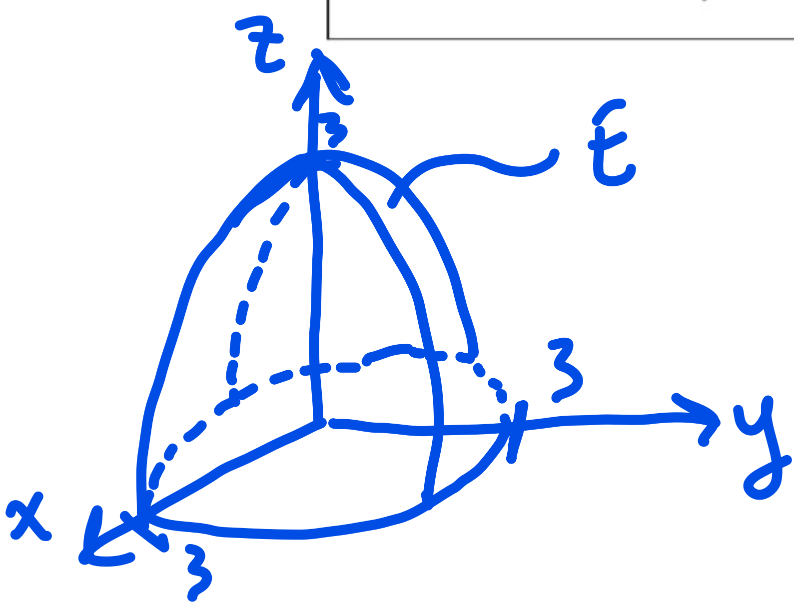
$$= 2\pi\left(8 - 32/5\right) = \boxed{\frac{16\pi}{5}}$$

Example 6 (Triple Integral in Spherical Coordinates)

Use spherical coordinates to evaluate $\int\int\int_E \frac{e^{-(x^2+y^2+z^2)}}{\sqrt{x^2+y^2+z^2}}\, dV$, where E is the the hemisphere bounded by $z = \sqrt{9-x^2-y^2}$ and $z = 0$.

Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates

$$r = \rho \sin\phi, \qquad x = r\cos\theta = \rho\sin\phi\cos\theta,$$
$$z = \rho\cos\phi, \qquad y = r\sin\theta = \rho\sin\phi\sin\theta, \tag{1}$$
$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2}.$$



(Also, $dV = \rho^2 \sin\phi$)

ρ: distance of a pt in E from origin

ϕ: angle btwn (+)z-axis and a line segment out to a pt in E

θ: angle btwn (+) x-axis and a line segment out to a pt in E.

For any pt in E, we have

$$0 \le \rho \le 3, \quad 0 \le \phi \le \frac{\pi}{2}, \quad 0 \le \theta \le 2\pi$$

$$\iiint_E \frac{e^{-(x^2+y^2+z^2)}}{\sqrt{x^2+y^2+z^2}}\, dV$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^3 \frac{e^{-\rho^2}}{\sqrt{\rho^2}}\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi/2} \sin\phi\, d\phi\right)\left(\int_0^3 e^{-\rho^2}\rho\, d\rho\right)$$

$$= (2\pi)\left(-\cos\phi \Big|_0^{\pi/2}\right)\left(\int_0^9 e^{-u}\tfrac{1}{2}\, du\right)$$

$$= (2\pi)(1)\left(-\tfrac{1}{2}e^{-u}\Big|_0^9\right) = \pi\left(1 - e^{-9}\right)$$ ◡̈