MA 1024 Conference 3

1. Higher Order Partial Derivatives
2. Multivariable Chain Rule
3. Implicit Function Theorem

Example 1 (Higher Order Partial Derivatives)

Verify that the function $f(x,y,z) = 1/\sqrt{x^2+y^2+z^2}$ is a solution to the Laplace equation $f_{xx} + f_{yy} + f_{zz} = 0$.

$$f_x = -\frac{x}{(x^2+y^2+z^2)^{3/2}}$$

(use quotient rule)

$$f_{xx} = \frac{-(x^2+y^2+z^2)^{3/2} + 3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}$$

Since f is "symmetric" wrt x, y, z, we will get (the change is pink).

$$f_{yy} = \frac{-(x^2+y^2+z^2)^{3/2} + 3y^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}$$

$$f_{zz} = \frac{-(x^2+y^2+z^2)^{3/2} + 3z^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}$$

Check: $f_{xx} + f_{yy} + f_{zz} = 0$?

$$f_{xx} + f_{yy} + f_{zz} = \frac{-3(x^2+y^2+z^2)^{3/2} + 3(x^2+y^2+z^2)(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}$$

$$\boxed{f_{xx} + f_{yy} + f_{zz} = \frac{-3(x^2+y^2+z^2)^{3/2} + 3(x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)^3} = 0. \checkmark}$$

Example 2 (Higher Order Partial Derivatives)

Verify that the function $u(t,x) = e^{-k^2 t}\sin(kx)$ is a solution to the heat conduction equation $u_t = u_{xx}$.

$$u_t = \frac{\partial u}{\partial t} = -k^2 e^{-k^2 t} \sin(kx)$$

$$u_x = \frac{\partial u}{\partial x} = k e^{-k^2 t} \cos(kx)$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = -k^2 e^{-k^2 t} \sin(kx)$$

Check: $u_t = u_{xx}$?

$$\boxed{u_t = -k^2 e^{-k^2 t} \sin(kx) = u_{xx} \quad \checkmark}$$

Example 3 (Multivariable Chain Rule)

Use the chain rule to find $\frac{dz}{dt}$.

$$z = \arctan(y/x), \quad x = e^t, \quad y = e^{-t}.$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$e^{-t}/e^{t} = e^{-2t}$

$$\frac{\partial z}{\partial x} = \frac{-y}{x^2+y^2}, \quad \frac{dx}{dt} = e^t$$

$$\frac{\partial z}{\partial y} = \frac{1}{1+(y/x)^2}\cdot\frac{1}{x}, \quad \frac{dy}{dt} = -e^{-t}$$

$$\frac{dz}{dt} = \left(\frac{-y}{x^2+y^2}\right)e^t - \left(\frac{1}{1+(y/x)^2}\frac{1}{x}\right)e^{-t}$$

$$\frac{dz}{dt} = \left(\frac{-e^{-t}}{e^{2t}+e^{-2t}}\right)e^t - \left(\frac{1}{1+e^{-4t}}\frac{1}{e^t}\right)e^{-t}$$

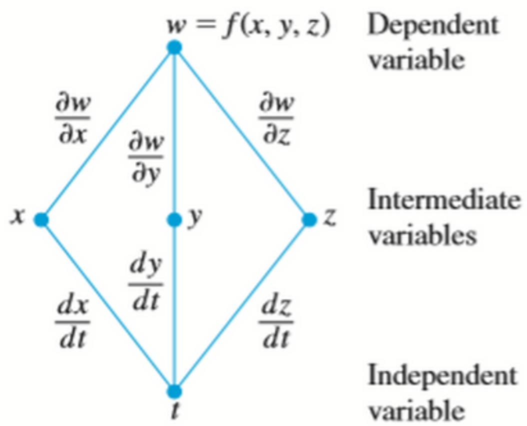
$$\boxed{\frac{dz}{dt} =} \frac{-1}{e^{2t}+e^{-2t}} - \frac{1}{e^{2t}+e^{-2t}} = \boxed{\frac{-2}{e^{2t}+e^{-2t}}}.$$

Example 4 (Multivariable Chain Rule)

Use the chain rule to find $\frac{dw}{dt}$.

$$w = xe^{yz}, \quad x = t^2, \quad y = 1 - t, \quad z = 1 + 2t\ .$$

Chain Rule



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

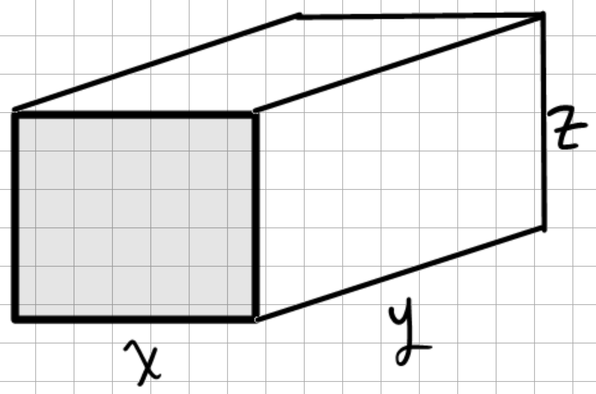
$$\frac{\partial w}{\partial x} = e^{yz}, \quad \frac{\partial w}{\partial y} = xze^{yz}, \quad \frac{\partial w}{\partial z} = xye^{yz}$$

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = -1, \quad \frac{dz}{dt} = 2$$

$$\boxed{\frac{dw}{dt} = e^{(1-t)(1+2t)} \cdot 2t - t^2(1+2t)e^{(1-t)(1+2t)} + t^2(1-t)e^{(1-t)(1+2t)} \cdot 2}$$

Example 5 (Multivariable Chain Rule)

The length x, width y, and height z of a box change with time. At a certain instant the dimensions are and $x = 1$ m, and $y = z = 2$ m. At the same time, x and y are increasing at a rate of 2 m/s while z is increasing at a rate of 3 m/s. At this instance, find the rate at which the volume of the box is changing.



$$V = xyz$$

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

t: seconds

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dz}{dt}$$

$$= (yz)\cdot 2 + (xz)\cdot 2 + (xy)\cdot 3$$

$$\left.\frac{dV}{dt}\right|_{(1,2,2)} = 2\cdot 2\cdot 2 + 1\cdot 2\cdot 2 + 1\cdot 2\cdot 3$$

$$\boxed{= 18 \text{ m}^3/\text{s}}$$

Example 6 (Multivariable Chain Rule)

Use the chain rule to find $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial s}$.

$$z = x^2 y^3, \quad x = s\ \cos t, \quad y = s\ \sin(t)\ .$$

$$\begin{tikzcd}[column sep=small]
& & z \arrow[dl, dash] \arrow[dr, dash] & & \\
& x \arrow[dl, dash] \arrow[d, dash] & & y \arrow[d, dash] \arrow[dr, dash] & \\
s & t & & s & t
\end{tikzcd}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial x} = 2xy^3 \quad , \frac{\partial x}{\partial t} = -s \cdot \sin t$$

$$\frac{\partial z}{\partial y} = 3x^2y^2 \quad , \frac{\partial y}{\partial t} = s \cdot \cos t$$

$$\boxed{\begin{aligned}\frac{\partial z}{\partial t} &= 2(s \cdot \cos t)(s \cdot \sin t)^3(-s \cdot \sin t) \\ &\quad + 3(s \cdot \cos t)^2(s \cdot \sin t)^2(s \cdot \cos t)\end{aligned}}$$

$$\frac{\partial z}{\partial x} = 2xy^3 \quad , \frac{\partial x}{\partial s} = \cos t$$

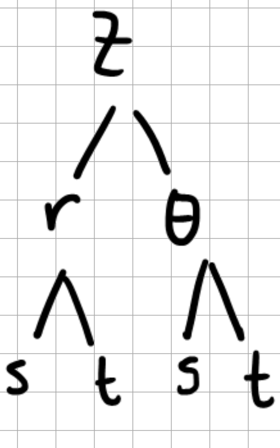
$$\frac{\partial z}{\partial y} = 3x^2y^2 \quad , \frac{\partial y}{\partial s} = \sin t$$

$$\boxed{\frac{\partial z}{\partial s} = 2(s \cdot \cos t)(s \cdot \sin t)^3(\cos t) + 3(s\cos t)^2(s \cdot \sin t)^2(\sin t)}$$

Example 7 (Multivariable Chain Rule)

Use the chain rule to find $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial s}$.

$$z = e^r \cos(\theta), \quad r = st, \quad \theta = \sqrt{s^2 + t^2}\,.$$



$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s}$$

$$\frac{\partial z}{\partial r} = e^r \cos\theta \quad , \quad \frac{\partial z}{\partial \theta} = -e^r \sin\theta$$

$$\frac{\partial r}{\partial t} = s \quad , \quad \frac{\partial \theta}{\partial t} = \frac{t}{\sqrt{s^2+t^2}}$$

$$\frac{\partial r}{\partial s} = t \quad , \quad \frac{\partial \theta}{\partial s} = \frac{s}{\sqrt{s^2+t^2}}$$

$$\boxed{\begin{aligned} \frac{\partial z}{\partial t} &= s e^r \cos\sqrt{s^2+t^2} - e^r \sin\sqrt{s^2+t^2}\,\frac{t}{\sqrt{s^2+t^2}} \\ \frac{\partial z}{\partial s} &= t e^r \cos\sqrt{s^2+t^2} - e^r \sin\sqrt{s^2+t^2}\,\frac{s}{\sqrt{s^2+t^2}} \end{aligned}}$$

Example 8 (Implicit Function Theorem)

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz = 1$.

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

F meets the conditions of the IFT (check lecture notes for the deets).

$$F_x = 3x^2 + 0 + 0 + 6yz$$

$$F_y = 0 + 3y^2 + 0 + 6xz$$

$$F_z = 0 + 0 + 3z^2 + 6xy$$

$$\boxed{\begin{aligned} \frac{\partial z}{\partial x} &= -\frac{F_x}{F_z} = -\frac{3x^2+6yz}{3z^2+6xy} = -\frac{x^2+2yz}{z^2+2xy} \\ \frac{\partial z}{\partial y} &= -\frac{F_y}{F_z} = -\frac{3y^2+6xz}{3z^2+6xy} = -\frac{y^2+2xz}{z^2+2xy} \end{aligned}}$$