MA 1024 Conference 4

1. Directional Derivatives
2. Tangent Lines to Level Curves
3. Critical Points and the Second Derivative Test

Example 1 (Directional Derivatives)

Find the derivative of $f(x,y) = \frac{x-y}{xy+2}$ at $P_0 = (1,-1)$ in the direction of $\mathbf{v} = \langle 12, 5 \rangle$.

1. Find $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$.
2. Calculate $(\nabla f)_{P_0}$.
3. The directional derivative is $D_{\mathbf{u}} f = (\nabla f)_{P_0} \cdot \mathbf{u}$, where $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$.

Using the quotient rule:

$$\frac{\partial f}{\partial x} = \frac{(xy+2) - (x-y)y}{(xy+2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{-(xy+2) - (x-y)x}{(xy+2)^2}$$

$$(\nabla f)_{(1,-1)} = \left\langle \frac{\partial f}{\partial x}(1,-1), \frac{\partial f}{\partial y}(1,-1) \right\rangle$$
$$= \langle 3, -3 \rangle$$

$$\vec{u} = \frac{\vec{V}}{|\vec{V}|} = \frac{\langle 12, 5 \rangle}{\sqrt{12^2+5^2}} = \langle 12/13, 5/13 \rangle$$

$$\boxed{(\nabla f)_{(1,-1)} \cdot \vec{u} = \langle 3, -3 \rangle \cdot \left\langle \frac{12}{13}, \frac{5}{13} \right\rangle = \frac{21}{13}}$$

Example 2 (Directional Derivatives)

Find the direction in which $f(x,y) = \frac{x-y}{xy+2}$ increases most rapidly at the point $P_0 = (1,-1)$. What is this maximum rate of change?

The function f increases most rapidly when $\cos\theta = 1$ or when $\theta = 0$ and $\mathbf{u}$ is the direction of ∇f. That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is

$$D_{\mathbf{u}}f = |\nabla f| \cos(0) = |\nabla f|.$$

The gradient vector at P_0 was $(\nabla f)_{(1,-1)} = \langle 3, -3 \rangle$. So f increases most rapidly in the direction of the vector $\langle 3, -3\rangle$. The standard convention is to give a unit vector in this direction:

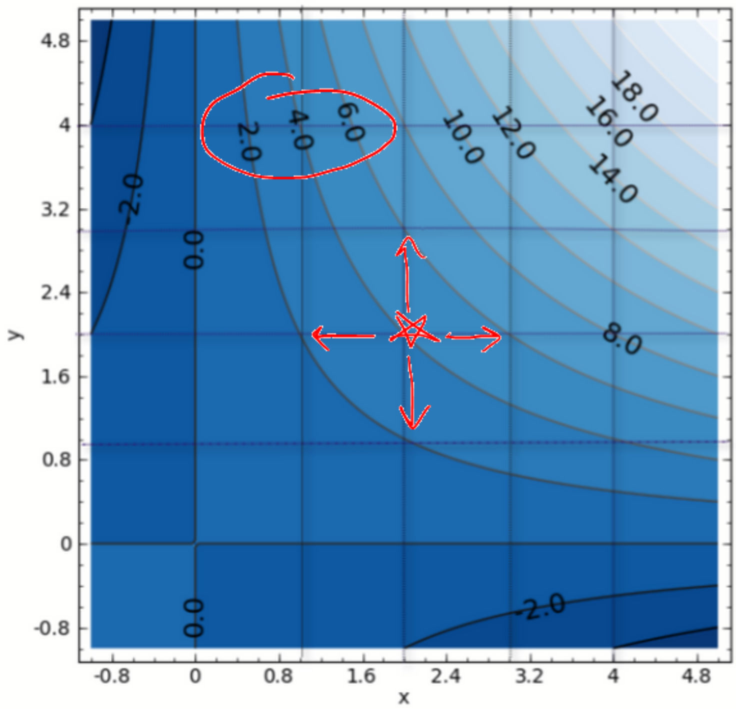
$$\frac{\langle 3, -3\rangle}{|\langle 3, -3\rangle|} = \frac{\langle 3, -3\rangle}{\sqrt{18}} = \left\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right\rangle$$

That is, $\left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$ is a unit vector pointing in the direction of steepest ascent (most rapid increase of f).

The max rate of change is the derivative in the direction of $\langle 1/\sqrt{2}, -1/\sqrt{2} \rangle$, which is $|\nabla f| = \sqrt{18} = 3\sqrt{2}$

Example 3 (Directional Derivatives)

Use the contour diagram to estimate the directional derivative of $f(x, y)$ at the point $P_0 = (2, 2)$ in the direction of $\mathbf{v} = \hat{\imath}$. (Note that $\hat{\imath} = \langle 1, 0 \rangle$) is already a unit length vector).



We want to approximate $(\nabla f)_{(2,2)} \cdot \langle 1, 0 \rangle$.

Use tangent line slopes:

$$f_x(x_0, y_0) \approx \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

$$f_y(x_0, y_0) \approx \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

Using the data from the point $(3, 2)$

$$\boxed{f_x(2,2) \approx \frac{f(2+1, 2) - f(2,2)}{1} = \frac{6-4}{1} = 2}$$

We could have used the point $(1,2)$ instead to apprx. f_x like this:

$$f_x(2,2) \approx \frac{f(2-1,2)-f(2,2)}{-1} = \frac{2-4}{-1} = 2$$

We found $f_x(2,2) \approx 2$ using two different data points. You shouldn't expect this to always happen.

Using the data point $(2,3)$:

$$\boxed{f_y(2,2) \approx \frac{f(2,2+1)-f(2,2)}{1} = \frac{6-4}{1} = 2}$$

Alternatively, we could have picked $(2,1)$:

$$f_y(2,2) \approx \frac{f(2,2-1)-f(2,2)}{-1} = \frac{2-4}{-1} = 2$$

Again, it is just a coincidence that different data pts produced the same apprx of f_y. It is also just an artifact of the contour map that $f_x = f_y$ (totally coincidental as far as we know).

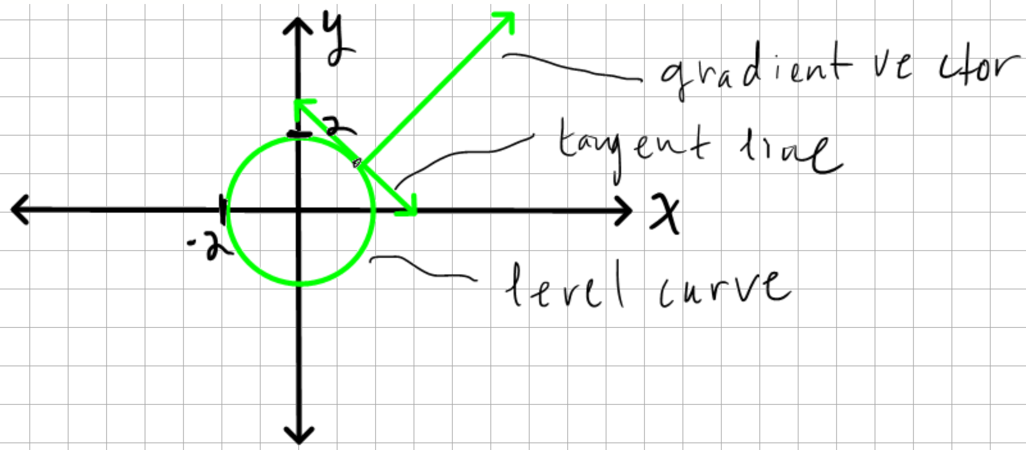
Commentary aside, the directional derivative is:

$$\boxed{(\nabla f)\big|_{(2,2)} \cdot \langle 1,0\rangle = \langle f_x(2,2), f_y(2,2)\rangle \cdot \langle 1,0\rangle \approx \langle 2,2\rangle \cdot \langle 1,0\rangle = 2}$$

Example 4 (Tangent Lines to Level Curves)

Let $f(x, y) = x^2 + y^2$. Sketch the level curve $f(x, y) = 4$ together with ∇f at the point $P_0 = (\sqrt{2}, \sqrt{2})$. Then write an equation for the tangent line.

Graphing:
Note
$\sqrt{2} \approx 1.4$



- The level curve when $f(x,y)=4$ is the circle of radius 2 with center at the origin: $\boxed{x^2+y^2=4}$.

- $\nabla f = \langle f_x, f_y \rangle = \langle 2x, 2y \rangle$

$$\boxed{(\nabla f)_{(\sqrt{2},\sqrt{2})} = \langle 2\sqrt{2}, 2\sqrt{2} \rangle}$$

$$|\langle 2\sqrt{2}, 2\sqrt{2} \rangle| = \sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2} = 4$$

So the gradient vector is at a $45°$ and (x-comp = y-comp) and length 4.

- **Tangent Line to a Level Curve**

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0 \qquad (6)$$

$$f_x(x_0, y_0) = 2\sqrt{2}, \quad f_y(x_0, y_0) = 2\sqrt{2}$$

Tangent line equation:

$$\boxed{2\sqrt{2}(x-\sqrt{2}) + 2\sqrt{2}(y-\sqrt{2}) = 0}$$

For visualization, we might prefer the form:

$$y - \sqrt{2} = \frac{-2\sqrt{2}(x-\sqrt{2})}{2\sqrt{2}}$$

$$y = -(x-\sqrt{2}) + \sqrt{2}$$

$$y = -x + 2\sqrt{2} \approx -x + 2.8$$

Example 5 (Tangent Lines to Level Curves)

Determine if the level curves of $f(x, y) = 2x + 4y$ and $g(x, y) = 4x - 2y$ intersect at right angles.

> At every point (x_0, y_0) in the domain of a differentiable function $f(x, y)$, the gradient of f is normal to the level curve through (x_0, y_0) (Figure 14.31).

Strategy: Since gradient vectors are normal (here read normal = perpendicular) to level curves, the level curves are perpendicular if ∇f and ∇g are perpendicular. Recall that the dot product of 2 vectors is 0 when the vectors are perpendicular. So,

1) Find ∇f, ∇g
2) Level curves intersect at right angles if $\nabla f \cdot \nabla g = 0$ (no matter our choice of level curve).

$$\nabla f \cdot \nabla g = \langle 2, 4 \rangle \cdot \langle 4, -2 \rangle = 8 - 8 = 0$$

Yes, the level curves of f and g intersect at right angles.

Example 6 (Critical Points and the Second Derivative Test)

Find the local maxima, minima, and saddle points of the function $f(x, y) = x^2 + xy + y^2 + 3x - 3y + 4$.

THEOREM 11—Second Derivative Test for Local Extreme Values Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then

i) f has a **local maximum** at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).

ii) f has a **local minimum** at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).

iii) f has a **saddle point** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b).

iv) **the test is inconclusive** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b). In this case, we must find some other way to determine the behavior of f at (a, b).

- Set $f_x = 0$ & $f_y = 0$

$$f_x(x,y) = 2x + y + 3 = 0$$
$$f_y(x,y) = x + 2y - 3 = 0$$
$$\Rightarrow \quad x = -3, \quad y = 3$$

- $f_{xx}(x,y) = 2$, $f_{yy}(x,y) = 2$, $f_{xy}(x,y) = 1$

Since f, the first derivatives of f, and the second derivatives of f are all continuous, the theorem applies (also f_x & f_y are defined everywhere so $(-3,3)$ is the only critical point).

$f_{xx}(-3,3) = 2 > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 = 3 > 0$

$\Rightarrow$ $(-3,3)$ is a local min. of f

How to start HW 3 Question 30

(1 point) A company operates two plants which manufacture the same item and whose total cost functions are

$$C_1 = 10 + 0.04q_1^2 \quad \text{and} \quad C_2 = 3 + 0.05q_2^2,$$

where q_1 and q_2 are the quantities produced by each plant. The total quantity demanded, $q = q_1 + q_2$, is related to the price, p, by

$$p = 60 - 0.05q.$$

How much should each plant produce in order to maximize the company's profit?

Profit = Revenue - Costs
Revenue = (Unit Price) x (Quantity of Units Sold)

So, in brief use the general formula

Profit = Price x Quantity - Costs

Let profit P be a fn of q_1 and q_2

$$P(q_1, q_2) = pq - (C_1 + C_2)$$

$$P(q_1, q_2) = (60 - 0.05q)q - (10 + 0.04q_1^2 + 3 + 0.05q_2^2)$$

$$P(q_1, q_2) = [60 - 0.05(q_1 + q_2)][q_1 + q_2] - (10 + 0.04q_1^2 + 3 + 0.05q_2^2)$$

- To find critical points, simplify and set $P_{q_1} = 0$ and $P_{q_2} = 0$.

- Test for a max using the second derivative test.