Ex 3 $\sqrt{x+\sqrt{x+\sqrt{x+\cdots}}}$ for $x \in [0,1]$

Sequence: $\{\underbrace{\sqrt{x}}_{f_1(x)}, \underbrace{\sqrt{x+\sqrt{x}}}_{f_2(x)}, \underbrace{\sqrt{x+\sqrt{x+\sqrt{x}}}}_{f_3(x)}, \ldots\}$

Recurrence Relation: $f_n(x) = \sqrt{x+f_{n-1}(x)}$

① Does this sequence converge?

Lemma 1: $\{f_n(x)\}$ is increasing.

Pf: $f_n(x) - f_{n-1}(x) = \sqrt{x+f_{n-1}(x)} - \sqrt{x+f_{n-2}(x)}$

$$= \frac{\left(\sqrt{x+f_{n-1}(x)} - \sqrt{x+f_{n-2}(x)}\right)\left(\sqrt{x+f_{n-1}(x)} + \sqrt{x+f_{n-2}(x)}\right)}{\sqrt{x+f_{n-1}(x)} + \sqrt{x+f_{n-2}(x)}}$$

$$= \frac{(x+f_{n-1}(x)) - (x+f_{n-2}(x))}{\sqrt{x+f_{n-1}(x)} + \sqrt{x+f_{n-2}(x)}} \qquad (x \neq 0)$$

$$= \frac{f_{n-1}(x) - f_{n-2}(x)}{\sqrt{x+f_{n-1}(x)} + \sqrt{x+f_{n-2}(x)}}$$

Since $f_n - f_{n-1}$ and $f_{n-1} - f_{n-2}$ have the same sign and since this is true in general for $f_{n-k} - f_{n-k-1}$, this sequence is increasing if $f_2(x) - f_1(x) \geq 0$. But

$$f_2(x) - f_1(x) = \frac{\left(\sqrt{x+\sqrt{x}} - \sqrt{x}\right)\left(\sqrt{x+\sqrt{x}} + \sqrt{x}\right)}{\left(\sqrt{x+\sqrt{x}} + \sqrt{x}\right)}$$

$$= \frac{\sqrt{x}}{\sqrt{x+\sqrt{x}} + \sqrt{x}} \geq 0. \quad \blacksquare$$

Lemma 2: $\{f_n(x)\}$ is bounded above by 4

Pf (Induction):

Step 1: True for $n=1 \iff f_1(x) = \sqrt{x}$ is bdd above for $x \in [0,1]$. $f_1(x) \leq 4$ ✓

Step 2 (Inductive step): Assume $f_n(x) \leq 4$ and use this to show that $f_{n+1}(x) \leq 4$.

Notice that $f_{n+1}(x) = \sqrt{x+f_n(x)} \leq \sqrt{x+4} \leq \sqrt{1+4}$
$= \sqrt{5} \leq 4. \quad \blacksquare$

② What does it converge to?

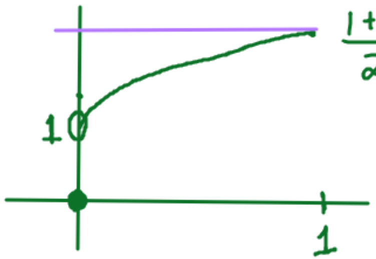
Since $f_n(x) \to L$ as $n \to +\infty$,

$$L(x) = \sqrt{x + L(x)}$$

$$L^2(x) - L(x) - x = 0$$

$$L(x) = \frac{1 \pm \sqrt{1+4x}}{2} \quad \text{for } x \in [0,1]$$

$$L(x) = \frac{1 + \sqrt{1+4x}}{2}$$



$\frac{1+\sqrt{5}}{2}$ = least upper bound

$$L(x) = \begin{cases} \frac{1+\sqrt{1+4x}}{2}, & x \in (0,1] \\ 0, & x = 0 \end{cases}$$

C. Harmonic, Geometric, and Arithmetic Means

<u>Thm</u>: Let $a_1, a_2, \ldots, a_n > 0$. Then

$$\frac{n}{1/a_1 + 1/a_2 + \cdots + 1/a_n} \le \sqrt[n]{a_1 a_2 \cdots a_n} \overset{(*)}{\le} \frac{a_1 + a_2 + \cdots + a_n}{n}$$

Harmonic Mean   Geometric mean   Arithmetic mean

with equality iff $a_1 = a_2 = \cdots = a_n = a$

Pf (Induction): Consider the second inequality $(*)$ first. Let $P(n)$ is the statement that $\sqrt[n]{a_1 a_2 \cdots a_n} \le \frac{a_1 + a_2 + \cdots a_n}{n}$.

Step 1: Is $P(1)$ true? <u>Yes</u>.

Step 2 (Inductive Step): Assume $P(n)$ is true.

A) use $P(2)$ and $P(n)$ to show $P(2n)$ is true.

B) Use $P(n)$ to show that $P(n-1)$ is true.