II. B Axiomatic Set Theory

Axiom I (Existence): $\exists$ a set.

Axiom II (Equality): Two sets are equal iff they contain the same elements

Ex: $\{x\} = \{1\}$ iff $x = 1$

Axiom III (specification): If A is any set and $Q(x)$ is any predicate, then $\exists$ a set B s.t.

$B = \{x \in A \mid Q(x) \text{ is true}\}$.

Note: $Q_1(x) := \neg Q(x)$, $B = \{x \in A \mid Q_1(x) \text{ is false}\}$

Thm: $\exists! \varnothing$

Pf: By Axiom I, $\exists$ some set A.

let $Q(x) =$ "$x \neq x$". By Axiom III, $\exists$ $\varnothing = \{x \in A \mid x \neq x\}$.

By Axiom II, any other set with no elements must be the same set ($\varnothing = \{\ \}$).

QED

Axiom IV (Pairing): If A and B are sets, then $\exists$ a set $\mathcal{C}$ s.t. $A \in \mathcal{C}$ and $B \in \mathcal{C}$.

Axiom V (Union): Suppose $\mathcal{A}$ is a set whose elements are sets. Then $\exists$ $\bigcup \mathcal{A} = \{x \in A \mid A \in \mathcal{A}\}$.

Axiom VI (Power Set): For any set A, $\exists$ the power set of A:

$$\mathcal{P}(A) = \{B \mid B \subseteq A\}$$

Axiom VII (Induction): $\exists$ an inductive set, i.e. a set $\mathbb{X}$ s.t. "successor"

1) $0 \in \mathbb{X}$
2) If $A \in \mathbb{X}$, then $S(A) \in \mathbb{X}$

Ex) $\mathbb{X} = \mathbb{N} = \{0, 1, 2, \dots\}$ and $S(A) = A + 1$.

Notice that we can identify numbers with sets.

$$0 \longleftrightarrow \emptyset$$

$$1 \longleftrightarrow \emptyset \cup \{\emptyset\}$$

$$2 \longleftrightarrow \emptyset \cup \{\emptyset\} \cup \{\emptyset, \{\emptyset\}\}$$

$$3 \longleftrightarrow \emptyset \cup \{\emptyset\} \cup \{\emptyset, \{\emptyset\}\} \cup \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$$

$$\vdots$$

Axiom VIII (Choice): Suppose X is an uncountable set of sets. Then it is possible to select exactly one element from each set in X.