Thm: Let $a_1, a_2, \dots, a_n > 0$. Then,

$$\underbrace{\frac{n}{1/a_1 + 1/a_2 + \dots + 1/a_n}}_{\text{Harmonic Mean}} \le \underbrace{\sqrt[n]{a_1 a_2 \cdots a_n}}_{\text{Geometric Mean}} \le \underbrace{\frac{a_1 + a_2 + \dots + a_n}{n}}_{\text{Arithmetic Mean}}$$

with equality in both cases iff $a_1 = a_2 = \dots = a_n = a$.

Pf (Induction): Let $P(n)$ be the statement that "$\sqrt[n]{a_1 a_2 \cdots a_n} \le \frac{a_1 + a_2 + \dots + a_n}{n}$". Notice that $P(1)$ is true.

Lemma 1: $P(2)$ is true.

Pf: Notice that

$$(a_1 - a_2)^2 = a_1^2 - 2a_1a_2 + a_2^2 \geq 0$$

$$+4a_1a_2 \qquad +4a_1a_2$$

$$\iff a_1^2 + 2a_1a_2 + a_2^2 \geq 4a_1a_2$$

$$\iff \frac{(a_1+a_2)^2}{4} \geq \cancel{4}a_1a_2$$

$$\iff \frac{a_1+a_2}{2} \geq \sqrt{a_1a_2}$$

$\iff P(2)$ is true. ▲

$P(2)$: $\left(\frac{a_1+a_2}{2}\right)^2 \geq a_1a_2$

$P(n)$: $\left(\frac{a_1+\dots+a_n}{n}\right)^n \geq a_1 \cdots a_n$

Lemma 2: $P(2)$ & $P(n) \Rightarrow P(2n)$

Pf: Consider

$$a_1a_2\cdots a_{2n} = \prod_{i=1}^{2n} a_i = \left(\prod_{i=1}^{n} a_i\right)\left(\prod_{i=n+1}^{2n} a_i\right)$$

(By $P(n)$ we have) $\le \left(\sum_{i=1}^{n} \frac{a_i}{n}\right)^n \left(\sum_{i=n+1}^{2n} \frac{a_i}{n}\right)^n$

$$= \Big[\underbrace{\Big(\sum_{i=1}^{n} \frac{a_i}{n}\Big)}_{A_1}\underbrace{\Big(\sum_{i=n+1}^{2n} \frac{a_i}{n}\Big)}_{A_2}\Big]^n = [A_1A_2]^n \le \left[\frac{A_1+A_2}{2}\right]^{2n}$$ (By $P(2)$).

$$= \left[\frac{\sum_{i=1}^{n} a_i + \sum_{i=n+1}^{2n} a_i}{2n}\right]^{2n} = \left(\frac{\sum_{i=1}^{2n} a_i}{2n}\right)^{2n}.$$

Thus $P(2n)$ is true ▲

Lemma 3: $P(n) \Rightarrow P(n-1)$

Pf: Consider the product of n positives

$$\left(\prod_{i=1}^{n-1} a_i\right) A \overset{P(n)}{\le} \left(\frac{\sum_{i=1}^{n-1} a_i + A}{n}\right)^n$$

Now let $A = \sum_{k=1}^{n-1} \frac{a_k}{n-1}$. Thus

$$\left(\prod_{i=1}^{n-1} a_i\right)\cancel{A} \le \left(\frac{\sum_{i=1}^{n-1} a_i + \sum_{k=1}^{n-1} \frac{a_k}{n-1}}{n}\right)^n$$

$$= \left[\frac{(n-1)\overbrace{\sum_{i=1}^{n-1} \frac{a_i}{n-1}}^{A} + \overbrace{\sum_{k=1}^{n-1} \frac{a_k}{n-1}}^{A}}{n}\right]^n$$

$$= \left[\frac{(n-1)A + A}{n}\right]^n, \quad A = \sum_{i=1}^{n-1} \frac{a_i}{n-1}$$

$$= A^{n-1} \Rightarrow \prod_{i=1}^{n-1} a_i \le \left(\sum_{i=1}^{n-1} \frac{a_i}{n-1}\right)^{n-1} \Rightarrow P(n-1) \text{ is true.}$$