Constructing sets

Thm: $\exists$ a set with one element.

Pf: Let $A = \emptyset$, $B = \emptyset$. Then by Axiom IV (Pairing), $\exists$ a set $\mathcal{C}$ s.t. $A \in \mathcal{C}$ and $B \in \mathcal{C}$. So $\emptyset \in \mathcal{C}$.

By Axiom III (Specification),

$$\{\emptyset\} = \{X \in \mathcal{C} \mid X = \emptyset\}.$$ QED

Thm: $\exists$ a set with two elements.

Pf: Let $A = \{\emptyset\}$, $B = \emptyset$. Then by Axiom IV (Pairing), $\exists$ a set $\mathcal{C}$ s.t. $\{\emptyset\} = A \in \mathcal{C}$ and $\emptyset = B \in \mathcal{C}$. By Axiom III, let $\{\emptyset, \{\emptyset\}\} =$

$$\{X \in \mathcal{C} \mid X = \emptyset \vee X = \{\emptyset\}\}.$$ QED

Ex) Axiom V (Unioning): Suppose

$$\mathcal{A} = \{\{1, 3\}, \{2, 7\}, \{1, -1\}\}.$$

By unioning, (Axiom V), $\exists$

$$\bigcup \mathcal{A} = \{-1, 1, 2, 3, 7\}.$$

Thm: $\exists$ a set with three elements.

Pf (Outline): Fill in the details of:

① Create the set $\{\{\{\emptyset\}\}\}$.

② Let $A = \{\emptyset, \{\emptyset\}\}$, $B = \{\{\{\emptyset\}\}\}$.

Use pairing to get

$$\{\underbrace{\{\emptyset, \{\emptyset\}\}}_{A}, \underbrace{\{\{\{\emptyset\}\}\}}_{B}\}$$

③ Use unioning.

Remark: Using these axioms, one cannot get $A = \{A\}$ or $A \in A$. This avoids Russell's Paradox and similar issues.

*ZF or ZFC

(may be Zermelo-Fraenkel
Zermelo-Fraenkel Choice)

Thm: $\sum_{j=1}^{n} j^3 = 1 + 8 + 27 + \dots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.

Pf (Induction):

① Check $P(1)$: $1 = \left(\frac{1(2)}{2}\right)^2$ ✓

② Does $P(k) \Rightarrow P(k+1)$?

Notice that $\sum_{j=1}^{k+1} j^3 = \left(\sum_{j=1}^{k} j^3\right) + (k+1)^3 = \underbrace{\left(\frac{k(k+1)}{2}\right)^2}_{P(k)} + (k+1)^3$. Thus, $\sum_{j=1}^{k+1} j^3 = \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$

$$= \left[\left(\frac{k}{2}\right)^2 + (k+1)\right](k+1)^2$$

$$= \frac{1}{4}\left[k^2 + 4k + 4\right](k+1)^2$$

$$= \frac{1}{4}(k+2)^2(k+1)^2$$

$$= \left[\frac{(k+1)(k+2)}{2}\right]^2 \quad (P(k+1))$$