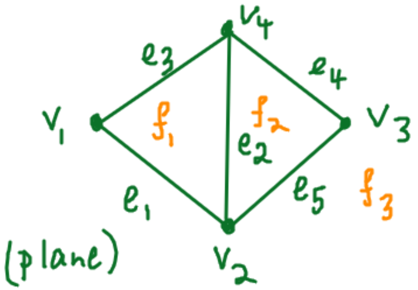
V. Graph Theory



Def: Graph : (V, E, Φ)

V : Collection of vertices

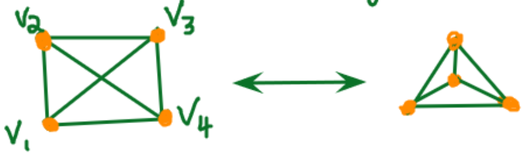
E : Collection of edges

$\Phi : E \to V \times V$ where

$\Phi(e_i) = (v_j, v_k)$ iff

e_i connects v_j and v_k, $j \leq k$

Ex K_4



plane, planar

Not plane, Complete

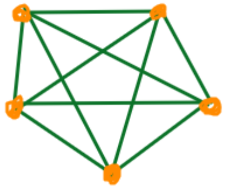
Def : Order : $|V|$ (# of vertices)

Degree(v_k) : $\deg(v_k)$ (# of edges incident on v_k).

Degree(G) : # of edges incident to each vertex.

Regular : $\deg(v_i) = \deg(v_k)$ $\forall i, k$

Ex K_5

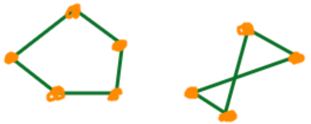


Not planar

Def : Cycle : $\{v_1, e_1, v_2, e_2, \ldots, e_{n-1}, v_n\}$

where $e_i \neq e_j$ $\forall i \neq j$ and $v_i \neq v_j$ $\forall i \neq j$

except $v_1 = v_n$



Notice that there are the same number of vertices and edges in any cycle.

Def : Tree : a connected graph containing no cycles.

Thm: For any tree: $|V| - |E| = 1$

Pf (Induction on $|E|$): Start with $|E| = 0$; then $|V| = 1$, hence the statement is true for $|E| = 0$. Now suppose the result is true for $|E| = m-1$; we must show that the statement is true for $|E| = m \geq 1$.

Since $|E| \geq 1$, every vertex must have positive degree, or the graph would be disconnected.

Suppose all vertices have degree 2 or greater. Then the graph would contain a cycles, thus not a tree. So there must be at least one vertex of degree 1. Let $\tilde{G}$ be the graph with one edge and this vertex of degree 1 deleted. Notice that $|\tilde{E}| = m-1$, so by the inductive hypothesis, $|\tilde{V}| - |\tilde{E}| = 1$. Thus $|V| - |E| = (|\tilde{V}| + 1) - (|\tilde{E}| + 1) = |\tilde{V}| - |\tilde{E}| = 1$.

QED