B. Continued Fractions

Ex 1 $$\frac{1}{2+\underbrace{\frac{1}{2-\frac{1}{2-\frac{1}{2-\cdots}}}}_{x}}$$ Notation

Def: The value of this continued fraction is the limit of the sequence based on the fraction.

sequence: $\left\{ \frac{1}{2}, \frac{1}{2+1/2}, \frac{1}{2+\frac{1}{1-1/2}}, \dots \right\}$

$= \left\{ \frac{1}{2}, \frac{2}{5}, \frac{3}{8}, \frac{4}{11}, \dots \right\} = \{a_n\}$

Notice that

(1) $a_n > 0 \quad \forall n \in \mathbb{Z}^+$

(2) $a_n \searrow \quad \forall n \in \mathbb{Z}^+$

Axiom VII of $\mathbb{R}$ (completeness)

Every decreasing sequence in $\mathbb{R}$ that is bounded below must converge to a real number.

Ex 1a) $\mathbb{Q}$ is not complete

$\left\{ 3, \frac{31}{10}, \frac{314}{100}, \frac{3,141}{1,000}, \frac{31,415}{10,000}, \dots \right\}$

Since our sequence $\{a_n\}$ must converge, we can find its limit as follows:

Notice that because of the ...

$$x = \frac{1}{2-\frac{1}{2-\frac{1}{2-\cdots}}} = \frac{1}{2-x}$$ (Perfect square)

So $x^2 - 2x + 1 = 0$, $(x-1)^2 = 0 \Rightarrow x = 1$

Hence $\{a_n\} \to L = \frac{1}{2+x} = \frac{1}{2+1} = \frac{1}{3}$.

Def: A sequence $\{a_n\}$, where $a_n \in \mathbb{R}$ converges to a limit $L \in \mathbb{R}$ iff given any $\epsilon > 0$, $\exists N \in \mathbb{Z}^+$ s.t. $|a_n - L| < \epsilon$ whenever $n > N$.

Can we verify this definition?

Notice that in this case, $a_n = \frac{n}{3n-1}$.

Thus $\left|\frac{n}{3n-1} - \frac{1}{3}\right| = \left|\frac{3n-(3n-1)}{3(3n-1)}\right| = \left|\frac{1}{3(3n-1)}\right| < \epsilon$

if $n > N := \left\lceil \frac{1}{3} + \frac{1}{9\epsilon} \right\rceil$

Scratch work:

$\frac{1}{3(3n-1)} < \epsilon$

$\frac{1}{3\epsilon} < 3n-1$

$1 + \frac{1}{3\epsilon} < 3n$

$\frac{1}{3} + \frac{1}{9\epsilon} < n$

Thus let $N := \left\lceil \frac{1}{3} + \frac{1}{9\epsilon} \right\rceil \in \mathbb{Z}^+$

So if $n > N$, then

$\left|\frac{n}{3n-1} - \frac{1}{3}\right| < \epsilon$

Ex 2

$$\cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}$$

(Discussed Thursday)