Pf (Analysis): Suppose for $x>1$, $n \le x < n+1$.

[Euler c. 1730]

Notice that $\ln(x) \le \sum_{k=1}^{n} \frac{1}{k} < \sum_{m} \frac{1}{m}$

and that $\frac{1}{m} = \frac{1}{\prod_{p<k} p^{k_p}} = \prod_{p<k} \frac{1}{p^{k_p}}$.

So $\sum_m \left(\prod_{p<k} \frac{1}{p^{k_p}} \right) \underset{\uparrow\text{ factor}}{=} \prod_{p<k} \underbrace{\left(\sum_{k=0}^{\infty} \frac{1}{p^k} \right)}_{\text{Geometric Series}}$

$= \prod_{p<x} \left(\frac{1}{1-1/p} \right) = \prod_{p<x} \left(\frac{p}{p-1} \right)$ *

[Notice that the j^{th} prime satisfies $p_j - 1 \ge j$, and that $\frac{p_j}{p_j-1} = \frac{p_j-1+1}{p_j-1} = 1 + \frac{1}{p_j-1} \le 1 + \frac{1}{j}$.]

* $\le \prod_{p_j<x} \left(1 + \frac{1}{j}\right) = \prod_{j=1}^{\pi(x)} \frac{j+1}{j} = \frac{2}{1} \cdot \frac{\cancel{3}}{\cancel{2}} \cdot \cancel{\ldots} \cdot \frac{\pi(x)+1}{\cancel{\pi(x)}} = \pi(x)+1$. (†)

Thus $\ln(x) \le \pi(x)+1 \Rightarrow \ln(x) - 1 \le \pi(x)$

and so there must be a countably infinite number of primes. QED

Ex) If $n=4$, $x=\pi+1$

Then $m \in \{1,2,3,4,6,8,9,12,16,18,24,\ldots\}$

In this case,

$\sum_m \frac{1}{m} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \ldots$

$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{2\cdot 3} + \frac{1}{2^3} + \frac{1}{3^2} + \frac{1}{2^2\cdot 3} + \ldots$

$= \underbrace{\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots\right)}_{2}\underbrace{\left(1 + \frac{1}{3} + \frac{1}{3^2} + \ldots\right)}_{3}$

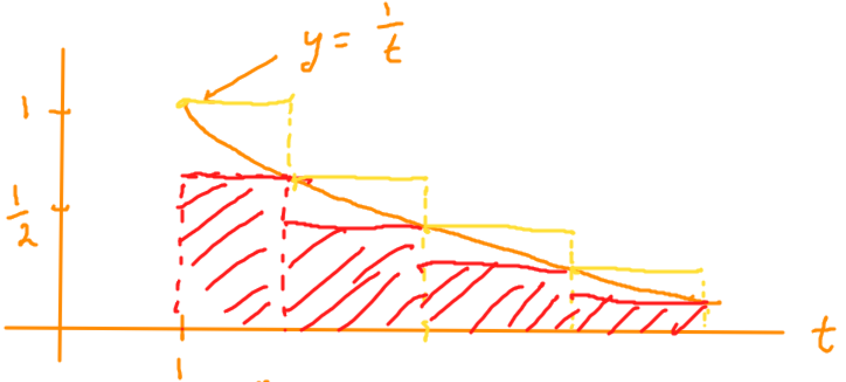
Geometric series: $\sum_{n=0}^{\infty} \left(\frac{1}{k}\right)^n = \frac{1}{1-\frac{1}{k}}$, $\frac{1}{|k|} < 1$

or $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$, $|r|<1$

(†) $\pi(x) :=$ # of primes $\le x$

Lemma: $\ln(x)$ grows without bound

Pf: Recall that $\ln(x) := \int_1^x \frac{1}{t}\,dt$, $x > 1$



$$\ln(x) \geq \sum_{k=1}^{n-1} \frac{1}{k+1} = \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{1/2} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{1/2} + \dots + \frac{1}{n}$$

$$\geq \frac{1}{2} + \frac{1}{2} + \frac{1}{2} +$$

Harmonic Series $= 1 + \frac{1}{2} + \frac{1}{3} + \dots\dots$

(Divergent)