MA 1971 Exercise Set 1 Answers

Hubert J. Farnsworth

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1. (a) $A(x) =$ "$x^2 \geq 0$".
   Then "For all $x \in \mathbb{R}$, $A(x)$" is true.
   (b) $B(x) =$ "$x^2 = 2$".
   Then "For all $x \in \mathbb{R}$, $B(x)$" is false, but "There exists $x \in \mathbb{R}$ such that $B(x)$" is true.

2. (a) Hypothesis: "For $x, y, z \in \mathbb{Z}^+$, if $x + y$ is odd and y+z is odd".
   Conclusion "Then $x + z$ is odd".
   This statement is false.
   (b) Hypothesis: "If x is an integer".
   Conclusion: "Then $x^2 \geq x$.
   This statement is true.
   (c) Hypothesis: "For $x \in \mathbb{R}$, if $x^2 > 11$".
   Conclusion: "Then x is positive".
   This statement is false.
   (d) Hypothesis: "If f is a polynomial of odd degree".
   Conclusion: "Then f has at least one real root".
   This statement is true.
   (e) Hypothesis: "If x is an integer".
   Conclusion: "Then $x^3 \geq x$".
   This statement is false.

3. (a)

A	B	$A \wedge (A \implies B)$	$A \wedge (A \implies B) \implies B$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

   (b)

A	B	C	$A \implies (B \wedge C)$	$A \implies B$	$(A \implies (B \wedge C)) \implies (A \implies B)$
T	T	T	T	T	T
T	T	F	F	T	T
T	F	T	F	F	T
T	F	F	F	F	T
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	T	T

4.

A	B	$A \implies B$	$B \implies A$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

5.

A	B	$\neg(A \wedge B)$	$\neg A \wedge \neg B$	$\neg(A \vee B)$	$\neg A \vee \neg B$
T	T	F	F	F	F
T	F	T	F	F	T
F	T	T	F	F	T
F	F	T	T	T	T

6. x is less than or equal to 7.

7. (a) There exists a polynomial with both real and genuinely complex roots.
   (b) There exists an $x \in \mathbb{R}$ such that $x \geq 0$ or x is rational.
   (c) There exist $x, y, z \in \mathbb{Z}^+$ such that $x + y$ is odd or $y + z$ is odd.

8. (a) All prime numbers are even.
   (b) There is a real number x such that $x^3 > x$ or $x^3 < x$.
   (c) There is a positive integer that cannot be written as the sum of distinct powers of three.
   (d) For all positive real numbers y, there is a real number x such that $y^2 > x$ or $y^2 < x$.

9. (a) There exists an odd integer x such that x^2 is odd (True).
   (b) There exists a continuous function f that is not differentiable (True).
   (c) There exists a differentiable function f that is not continuous (False).
   (d) There exists a polynomial f with integer coefficients that has zero real roots (True).

10. (a) $27 > 5$ and $27 > 10$
    (b) $2^3 = 8 \neq 2$.
    (c) The number 2 is a prime number that is even.

11. Let $x, y, z \in \mathbb{Z}$. Suppose $x + y$ and $y + z$ are even. Then there exist integers k and j such that $x + y = 2k$ and $y + z = 2j$. This implies that $x + z = 2k - y + 2j - y = 2(k + j - y)$. Since $k + j - y$ is an integer, this shows that $x + z$ is even.

12. (a) If $x^2 \leq 0$, then $x \geq 0$.
    (b) If there does not exist a y such that $xy = 1$, then $x = 0$.
    (c) If x^2 is an odd integer, then x is an odd integer.
    (d) If $x + z$ is even, then $x + y$ is even or $y + z$ is even.
    (e) If f is a polynomial with zero real roots, then f must be of even degree.

13.

Q	$\neg Q$	P	$\neg P$	$P \wedge \neg P$	$\neg Q \implies (P \wedge \neg P)$
T	F	T	F	T	T
T	F	T	F	F	T
F	T	F	T	F	F
F	T	F	T	F	F

14. Suppose that x is an integer assume that x is both even and odd. Then there exist integers k and j such that $x = 2k$ and $x = 2j + 1$. This means $1 = 2k - 2j = 2(k - j)$. Since k and j are integers, $k - j$ is also an integer. It follows that $\frac{1}{2} = k - j$ is an integer. This is a contradiction since $\frac{1}{2} \notin \mathbb{Z}$. This contradiction arose from the assumption that x is both even and odd and so we must conclude that x cannot be both even and odd.