MA 1971 Exercise Set 2 Answers

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1. Let $x \in (A \cap B)^c$. Then $x \notin A \cap B$, which means $x \notin A$ of $x \notin B$. Equivalently, $x \in A^c$ or B^c. Thus $x \in A^c \cup B^c$. Therefore $(A \cap B)^c \subseteq A^c \cup B^c$. Each of these steps is reversible. Therefore we have also $(A \cap B)^c \supseteq A^c \cup B^c$.

2. Let $x \in A$. Since $A \subset B$, $x \in B$. Since $B \subset C$, $x \in C$. Therefore $A \subset C$.

3. $$A_U^c = [0,3) \cup [7,10], \quad A_{\mathbb{R}}^c = (-\infty,3) \cup [7,\infty), \quad B_U^c = [0,3) \cup (3,6) \cup (6,9) \cup (9,10]$$

4. Consider the following counterexample. Let $A = \{a\}, B = \{b\}$ so that $A \cup B = \{a,b\}$.

$$\begin{aligned}
\mathcal{P}(A) &= \{\emptyset, \{a\}\} \\
\mathcal{P}(B) &= \{\emptyset, \{b\}\} \\
\mathcal{P}(A) \cup \mathcal{P}(B) &= \{\emptyset, \{a\}, \{b\}\} \\
\mathcal{P}(A \cup B) &= \{\emptyset, \{a\}, \{b\}, \{a,b\}\} \\
\mathcal{P}(A) \cup \mathcal{P}(B) &\neq \mathcal{P}(A \cup B)
\end{aligned}$$

5. (a) Base case: $\frac{1(1+1)}{2} = 1 = \sum_{i=1}^{1} i$.
Inductive step: Suppose for $k \in \mathbb{Z}^+$ that $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$. Then,

$$\sum_{i=1}^{k+1} i = k + 1 + \sum_{i=1}^{k} i = k + 1 + \frac{k(k+1)}{2} = \frac{2(k+1) + k(k+1)}{2} = \frac{(k+1)((k+1)+1}{2}.$$

The claim holds by the principle of induction.

(b) Base case: $\frac{1(1+1)(2+1)}{6} = 1 = 1^2 = \sum_{i=1}^{1} i^2$.
Inductive step: Suppose for $k \in \mathbb{Z}^+$ that $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$. Then,

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 = (k+1)^2 + \sum_{i=1}^{k} i^2 &= (k+1)^2 + \frac{k(k+1)(2k+1)}{6} \\
&= \frac{6(k+1)^2 + k(k+1)(2k+1)}{6} \\
&= \frac{(k+1)[6(k+1) + 2k^2 + k]}{6} \\
&= \frac{(k+1)[2k^2 + 7k + 6]}{6} \\
&= \frac{(k+1)[2k(k+2) + 3(k+2)]}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}
\end{aligned}$$

The claim holds by the principle of induction.

6. (a) **Claim**: For any positive integer n, $6 \mid (n^3 - n)$.
Proof): Note that $n^3 - n = (n-1)n(n+1)$ is the product of three nonnegative integers. Therefore, 3 divides one of $n-1, n, n+1$. Also, 2 divides at least one of $n-1, n, n+1$ (it is possible that it is the same one in cases like, say $n = 12$). Then either one of $n-1, n, n+1$ can be written as a multiple of both 2 and 3, and since 2 and 3 are both prime, this number can be written as a multiple of their product 6. Or, one of $n-1, n, n+1$ can be written as a multiple of 2 and another as a multiple of 3. Rewriting these numbers as $2k$ and $3j$ for integers k, j shows that $(n-1)n(n+1)$ is a multiple of 6.

Claim: For any positive integer n, $6 \mid (n^3 - n)$.
Proof (Induction with two base cases):
Base case: $6 \mid (1^3 - 1)$ since $1^3 - 1 = 0$ and $6 \mid 0$. Also, $6 \mid (2^3 - 2)$ since $2^3 - 2 = 6$ and $6 = 6 \cdot 1 \implies 6 \mid (2^3 - 2)$.
Inductive step: Suppose for some $k \in \mathbb{Z}^+$ that $6 \mid k^3 - k$. Then $k^3 - k = 6j$ for some integer j. Consider the case for $k + 2$.

$$(k+2)^3 - (k+2) = k^3 + 6k^2 + 12k + 8 - (k+2) = k^3 - k + 6k^2 + 12k + 6 = 6j + 6(k^2 + 2k + 1) .$$

Thus $(k+2)^3 - (k+2) = 6(j + k^2 + 2k + 1)$, which shows that $6 \mid [(k+2)^3 - (k+2)]$.

(b) **Lemma** For any $n \in \mathbb{Z}^+$, $6 \mid 3n(n+1)$.
Proof (Induction):
Base case: For $6 \mid 3(1)(1+1)$ since $3(1)(1+1) = 6$.
Inductive step: Suppose for some $k \in \mathbb{Z}^+$ that $6 \mid 3k(k+1)$. Then $3k(k+1) = 3k^2 + 3k = 6j$ for some integer j. Consider the case for $k + 1$.

$$3(k+1)(k+2) = 3k^2 + 9k + 6 = 3k^2 + 3k + 6k + 6 = 6(j + k + 1) .$$

The lemma holds by the principle of induction.

Claim: For any positive integer n, $6 \mid (n^3 - n)$.
Proof (Induction): Base case: $6 \mid (1^3 - 1)$ since $1^3 - 1 = 0$ and $6 \mid 0$.
Inductive step: Suppose for some $k \in \mathbb{Z}^+$ that $6 \mid (k^3 - k)$. Then $k^3 - k = 6j$ for some integer j. Consider the case for $k + 1$.

$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - (k+1) = k^3 - k + 3k^2 + 3k = 6j + 6l = 6(j + l) ,$$

where we used the inductive hypothesis to get $k^3 - k = 6j$ and the lemma to get $3k^2 + 3k = 6l$ for some integer l. This shows that $6 \mid [(k+1)^3 - (k+1)]$ and so the claim holds by the principle of induction.

7. Let A and B be sets such that $A \subset B$ and $Q(x)$ be the predicate $Q(x) = x \notin A$. By Axiom III (specification), there exists a set S such that

$$S = \{x \in B \mid Q(x) \text{ is true}\} = \{x \in B \mid x \notin A\} =: A^c_B .$$

8. Proof: Suppose there exists a set S such that S contains all sets. We have seen that $|\mathcal{P}(S)| = 2^{|S|} > |S|$. But since $\mathcal{P}(S)$ is also a set, this would imply that $|\mathcal{P}(S)| \leq |S|$. Thus we have the contradiction, $|\mathcal{P}(S)| < |\mathcal{P}(S)|$.

Proof: Suppose there exists a set S that contains all sets. By Axiom III, there exists a set $B = \{s \in S \mid s \notin s\}$. Since S contains all sets, $B \in S$. If $B \in B$, then by definition of B, $B \notin B$. But if $B \notin B$, then since $B \in S$ and $B \notin B$, we have $B \in B$. Therefore, if there exists a set S containing all sets, there must exists a set B such that $B \in B$ and $B \notin B$. Since $B \in B$ and $B \notin B$ is impossible, there cannot exist a set S containing all sets.

9. No, $\emptyset \neq \{\emptyset\}$. By Axiom II, two sets are equal iff they contain the same elements. However, $|\emptyset| = 0$ while $|\{\emptyset\}| = 1$. Since $\emptyset$ and $\{\emptyset\}$ do not even contain the same number of elements, it is impossible for these two sets to contain the same elements.

10. In lecture, we used Axioms I,II, and III to prove that there exists uniquely the empty set $\emptyset$. Taking $A = B = \emptyset$ in Axiom IV (pairing), there exists a set $\mathcal{C}$ such that $\emptyset \in \mathcal{C}$. Using Axiom III, we obtain the existence of the set $\{\emptyset\} = \{X \in \mathcal{C} \mid X = \emptyset\}$. Thus we have proved the existence of the sets $\emptyset$ and $\{\emptyset\}$. Taking $A = \emptyset$ and $B = \{\emptyset\}$ in Axiom IV, there exists a set $\mathcal{C}$ such that $\emptyset, \{\emptyset\} \in \mathcal{C}$. Using Axiom III, we obtain $\{\emptyset, \{\emptyset\}\} = \{X \in \mathcal{C} \mid X = \emptyset \vee X = \{\emptyset\}\}$. Taking $A = B = \{\emptyset\}$ in Axiom IV, there exists a set $\mathcal{C}$ such that $\{\emptyset\} \in \mathcal{C}$. Then by Axiom III there exists the set $\{\{\emptyset\}\} = \{X \in \mathcal{C} \mid X = \{\emptyset\}\}$. Then similarly, by applying Axiom IV and III as we just did but with $A = B = \{\{\emptyset\}\}$ we obtain the existence of the set $\{\{\{\emptyset\}\}\}$.

Take $A = \{\emptyset, \{\emptyset\}\}$ and $B = \{\{\{\emptyset\}\}\}$. By Axiom IV there exists a set $\mathcal{C}$ such that $\{\emptyset, \{\emptyset\}\} \in \mathcal{C}$ and $\{\{\{\emptyset\}\}\} \in \mathcal{C}$. By Axiom III, there exists a set

$$\mathcal{A} = \{\ \{\emptyset, \{\emptyset\}\}, \{\{\{\emptyset\}\}\}\ \} = \{\ X \in \mathcal{C} \mid X = \{\emptyset, \{\emptyset\}\} \vee X = \{\{\{\emptyset\}\}\}\ \}$$

By Axiom V (unioning), there exists a set $\cup\mathcal{A}$ such that,

$$\cup\mathcal{A} = \{x \in A \mid A \in \mathcal{A}\} = \{\ \emptyset, \{\emptyset\}, \{\{\emptyset\}\}\ \}\ .$$