#3b Thm: Any closed walk with an odd number of edges contains an (odd) cycle.

Thm: $\exists$ a closed walk with an odd number of edges $\Rightarrow$ $\exists$ an odd cycle.

Contrapositive:

no odd cycles $\Rightarrow$ no closed walks with odd number of edges.

· only even cycles $\Rightarrow$ only closed walks with an even number of edges.

Pf (contrapositive): Suppose a graph G has no odd cycles. Pick any vertex in G; call it v_0; color it red. Next color all vertices adjacent to v_0 blue. Next color red all vertices adjacent to the blue vertices. This coloring is possible because there are no odd cycles. Continue this process until the entire graph is colored. (This is possible because the graph is finite).

Now start a walk from v_0. If an odd number of steps are taken, one will always be on a blue vertex. Thus no closed walk can have an odd number of steps. (Thus any closed walk must have an even number of steps).

Pf (Direct): Given a graph G, suppose there is a closed walk with an odd number of edges. If all vertices are distinct (except for the first and last), then this walk is itself an odd cycle, and we are done. If not, then there is at least one vertex that repeats in the middle of the walk. Use this vertex to divide our original closed walk into 2 or more closed subwalks. Notice that at least one subwalk is odd. Pick one of the shortest odd subwalks. If are shortest subwalk is a cycle, we're done. If not, continue the process. This process must end with an odd cycle.