MA1971 Bridge to Higher Math **Name:** Solutions

Final **D Term, 2021**

Show all work needed to reach your answers.

1. (10 points) If $A = \{2, 4, 8\}$, then the power set of A is

(-1) For each missing subset, or for an extra sets.

$\mathcal{P}(A) = \{\emptyset, \{2\}, \{4\}, \{8\}, \{2,4\}, \{4,8\}, \{2,8\}, A\}$

2. (20 points) Consider the implication $A \Rightarrow B$ where A and B are themselves statements or predicates. For this implication, please state the following: 5 pts each

(a) contrapositive: $\neg B \Rightarrow \neg A$ (+1) (+1) (+1) (+1) (+1)

(b) converse: $B \Rightarrow A$ (+2) (+1) (+2)

(c) negation: $A \wedge \neg B$ (+2) (+1) (+1) (+1)

(d) inverse: $\neg A \Rightarrow \neg B$ (+1) (+1) (+1) (+1) (+1)

3. (10 points) Please give (a) the contrapositive and then (b) the negation of the following statement: "If xy is an irrational number, then $y > 6$ but $x < 0$." Please avoid the use of the words "not" and "no". 5pt each

(a) Contrapositive:

If $y \le 6$ or $x \ge 0$, then xy is a rational number. (+1) (+1) (+1) (+2)

(b) Negation:

xy is an irrational number, and $y \le 6$ or $x \ge 0$. (+1) (+1) (+1) (+1) (+1)

4. (20 points) Please show that $\sqrt{5}$ is irrational.

Pf (Contradiction): +3 Suppose $\sqrt{5} \in \mathbb{Q}$. Then $\exists\, p, q \in \mathbb{Z}^+$ such that $\sqrt{5} = p/q$ +2 where p and q have no common +2 divisors, and hence $p^2 = 5q^2$ +2. Thus $5 \mid p^2 \Rightarrow 5 \mid p$ +2 $\Rightarrow p = 5k$ for some $k \in \mathbb{Z}^+$. It then follows that $p^2 = (5k)^2 = 25k^2 = 5q^2$ +2 $\Rightarrow 5k^2 = q^2$ +1. So $5 \mid q^2 \Rightarrow 5 \mid q$ +2. This means that 5 divides +2 both p and q which contradicts that p and q have no common divisors +2. $(\rightarrow\leftarrow)$

5. (10 points) Consider a sequence $\{a_n\}$ where $a_n = p_n/q_n$ and $0 < p_n < q_n$ (so each element of the sequence is a fraction). Suppose that a_n is increasing. Does $\{a_n\}$ necessarily converge? Please either explain why it converges, or give a counterexample to show that such a sequence might diverge.

Because $a_n = p_n/q_n$ and $0 < p_n < q_n$, +3 $a_n < 1 \;\forall n \in \mathbb{Z}^+$. Since $\{a_n\}$ is an increasing +2 sequence that is bounded +2 above, it must converge +2 by the least upper bound +1 property of the real numbers.

Could be the LUB Axiom

6. (20 points) Please explain why $i^2 + j^2$ is never equal to 3 (mod 4), that is, $i^2 + j^2 \neq 4k + 3$ for any $i, j, k \in \mathbb{Z}$.

Hint: Consider the cases where i and j are each either even or odd; what do these imply?

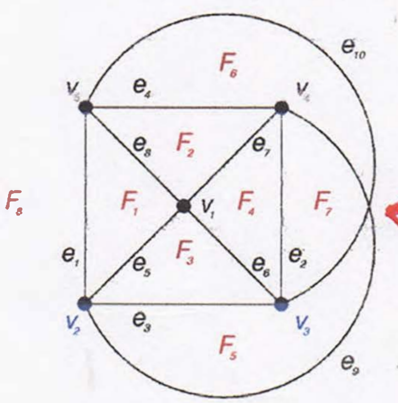
If i is even (+1), then $i = 2n$ (+1) for some $n \in \mathbb{Z}$, so $i^2 = 4n^2$ (+3) $\Leftrightarrow i^2 \equiv 0 \pmod 4$.

If j is odd (+1), then $i = 2n+1$ (+1) for some $n \in \mathbb{Z}$, so $i^2 = (2n+1)^2$

$= 4n^2 + 4n + 1 \equiv 1 \pmod 4$ (+3). The same is true for (+4) j and j^2.

Thus $i^2 + j^2$ must equal 0, 1 or 2 (mod 4) (+4) $\Rightarrow i^2 + j^2 \not\equiv 3 \pmod 4$ (+2)

7. (10 points) Consider the graph below; it is one possible drawing of K_5, the complete graph on five vertices. Recall that by the Euler formula, one might expect that $|V| - |E| + |F| = 2$. But for this graph, it seems that $|V| = 5$, $|E| = 10$ and $|F| = 8$, meaning that the Euler formula is not satisfied. Please explain what is wrong here.



K_5 is not planar, thus the Euler formula does not apply. (+10)

Alternately, one could point to this edge crossing (which makes this graph nonplanar). One could add a vertex here to make this graph planar and then the Euler formula applies, with two additional vertices.