Pf: **Lemma:** $\prod_{k=0}^{n-1} F_k = F_n - 2$ ← Recurrence Formula

Pf: (See previous lecture)

⋮

$$\prod_{k=0}^{n} F_k = \dots = \left(2^{2^n} - 1\right)\left(2^{2^n} + 1\right)$$

$$= \left(2^{2^n}\right)^2 - 1$$

$$= \left(2^{2^{n+1}} + 1\right) - 2$$

$$= F_{n+1} - 2$$

Return to main proof:

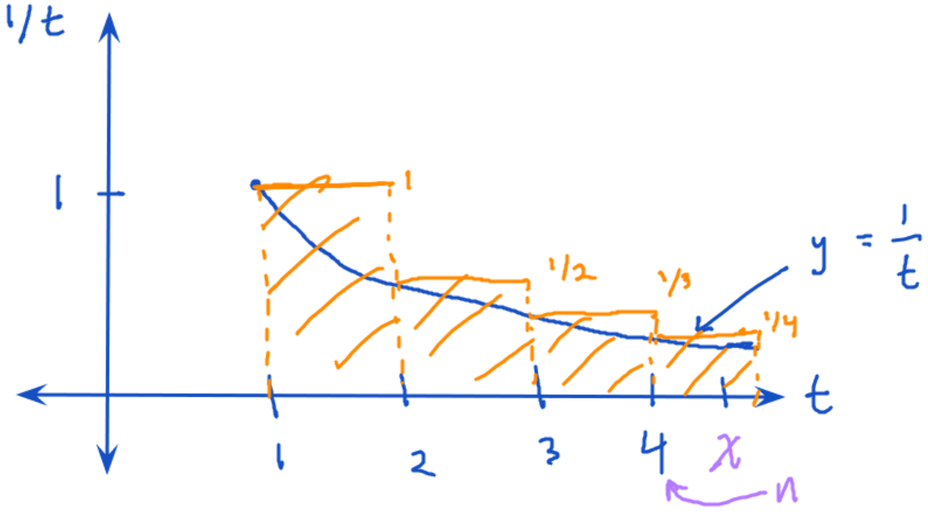
spse $\exists\ p$ s.t. $p \mid F_n$ and $p \mid F_k$ for some k, $0 \le k \le n-1$. Then p must be 2. But all Fermat numbers are odd; thus F_n and F_k can have no common prime divisors. Now since there countably infinitely many Fermat numbers, each with at least one unique prime factor, there must be countably infinitely primes.

QED

4th Pf (Analysis): Suppose $n \le x < n+1$,

(p.4) and recall that

$$\ln(x) = \int_1^x \frac{1}{t}\,dt, \quad \forall\, x \geq 1$$



$m \in \{1, \underline{2}, \underline{3}, 4, 6, 8, 9, \cancel{10}, 12, 14, \cancel{15} \ldots\}$

Notice $\ln(x) = \int_1^x \frac{dt}{t} \le \sum_{k=1} 1/k \le \sum_m \frac{1}{m}$

where m is any integer that has only prime factors between 2 and n.

So each $m = \prod_{p<n} p^{k_p}$ and notice

$\frac{1}{m} = \prod_{p<n} \frac{1}{p^{k_p}}$. Next notice that

$$\sum_m \frac{1}{m} = \sum_m \prod_{p<n} \frac{1}{p^{k_p}} = \prod_{p<x} \left(\sum_{p \le x} \frac{1}{p^{k_p}} \right)$$