Thm: Every Planar graph has a vertex of degree 5 or less.

Pf (contradiction): From the Euler formula, $|V| - |E| + |F| = 2$. Because every edge is associated with two vertices,

$$\sum_{k=1}^{n} \deg(v_k) = 2|E| \quad \text{where } n = |V|.$$

Suppose the minimum degree of any vertex in our graph is six; then

$$6|V| \leq \sum_{k=1}^{n} \deg(v_k) = 2|E|$$

Thus $|V| \leq \frac{1}{3}|E|$. Next each face is bounded by at least 3 edges while each edge is associated with 2 faces. Thus $|E| \geq \frac{3}{2}|F|$, and $|F| \leq \frac{2}{3}|E|$. Therefore,

$$2 = |V| - |E| + |F| \leq \frac{1}{3}|E| - |E| + \frac{2}{3}|E| = 0. \quad \rightarrow\leftarrow$$

Corollary: The set

is an unavoidable set for the vertices in any planar graph.

Thm: Every planar graph is four colorable.

(Kemp) Pseudo Pf (Contradiction): Suppose $\exists$ a planar graph whose coloring requires 5 colors. Then there must exist a five-colorable graph G with the minimum number of vertices, and this minimal five colorable graph G must contain one of the vertices from our unavoidable set. Suppose G has a vertex of degree three or fewer. Consider the degree-three case:

Now suppose there is a vertex of degree 4:

Exercise 3b (HW) Hint