Def: A sequence $\{a_n\}$ with $a_n \in \mathbb{R}$ converges to a limit L iff given any $\epsilon > 0$, $\exists N \in \mathbb{Z}^+$ such that $|a_n - L| < \epsilon$ whenever $n > N$.

Def: A sequence $\{a_n\}$ with $a_n \in \mathbb{R}$ is Cauchy iff given $\epsilon > 0$ $\exists N \in \mathbb{Z}^+$ such that

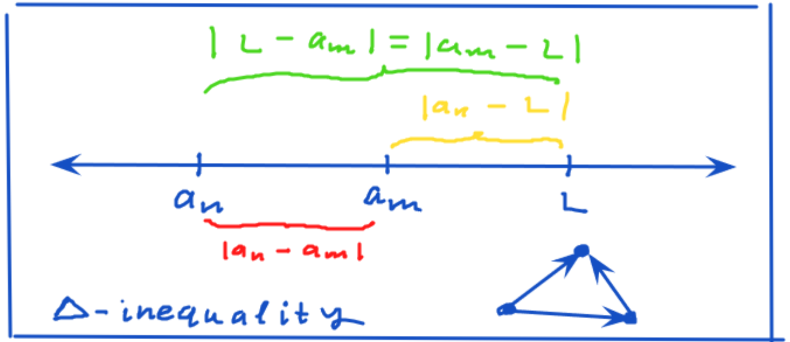
$$|a_n - a_m| < \epsilon \text{ whenever } n, m > N.$$

Thm: Because $\mathbb{R}$ is complete, a sequence is Cauchy iff it is convergent.

Pf ($\Leftarrow$): Suppose $\{a_n\}$ converges to $L \in \mathbb{R}$. Given $\epsilon > 0$, $\exists N \in \mathbb{Z}^+$ s.t. $|a_n - L| < \frac{\epsilon}{2}$ whenever $n > N$. Then by the Δ-inequality

$$|a_n - a_m| = |a_n - L + L - a_m| \le |a_n - L| + |L - a_m|$$
$$= |a_n - L| + |a_m - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

provided that $n, m > N$. QED



Ex Consider the sequence

$$\{a_n\}_{n=0} = \left\{3, \frac{31}{10}, \frac{314}{10^2}, \underset{n}{\frac{3141}{10^3}}, \frac{31415}{10^4}, \frac{314159}{10^5}, \dots\right\}$$

$m = n + k$ for $k > 0$

Notice that for $m > n$

$$|a_n - a_m| = \left| \frac{314\ldots\overbrace{00\ldots0}^{k}}{10^{n+k}} - \frac{3141\ldots xxx}{10^m} \right|$$

$$= \frac{\overbrace{xx\ldots x}^{k}}{10^m} < \frac{10^k}{10^m} = 10^{-n} = \frac{1}{10^n} < \epsilon$$

provided that $1/\epsilon < 10^n$, $\log(1/\epsilon) < n \log 10$. Thus we can define $N := \left\lceil \frac{\log 1/\epsilon}{\log 10} \right\rceil$. So $\{a_n\}$ is a cauchy sequence.

$$\left| \frac{314100}{100000} - \frac{314159}{100000} \right| = \frac{59}{10^5}$$

$$\frac{1}{10^n} < \epsilon$$
$$\frac{1}{\epsilon} < 10^n$$
$$\log(1/\epsilon) < \log(10^n)$$
$$\log(1/\epsilon) < n \log(10)$$