B. Thm (Euler): For any connected planar graph $|V| - |E| + |F| = 2$.

Pf (Induction on $|E|$): Suppose $|E| = 0$.

Then $|V| = 1$ and $|F| = 1$; thus $|V| - |E| + |F| = 2$. ✓

Now suppose the result holds for $|E| = m-1$; we must show it must be true for $|E| = m$. Supposed G is any planar connected graph with $|E| = m$, and suppose first G has no cycles. Then G is a tree, $|V| - |E| = 1$ and $|F| = 1$ and thus Euler's formula holds.

Next suppose G has at least one cycle. Let $\tilde{G} = G - \{e_k\}$ where e_k is any edge in a cycle. So $|\tilde{E}| = m-1$ and hence Euler holds for $\tilde{G}$. Now $|\tilde{V}| = |V|$ and $|\tilde{F}| = |F| - 1$. Thus,

$$|V| - |E| + |F| = |\tilde{V}| - (|\tilde{E}| + 1) + |\tilde{F}| + 1$$

$$= |\tilde{V}| - |\tilde{E}| + |\tilde{F}| = 2$$

Thus Euler's formula holds for G. ■

C. The Four Color Problem

Began in 1852 with Guthrie.

"Proof" given in 1879 by Kempe. In 1890, Heawood found a flaw in the Kempe proof.