Sample Exam 3
MA 2051 BD01,BD02,BD03,BD05 - Differential Equations
Worcester Polytechnic Institute
Fall 2021
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You are allowed a 45 minutes to complete all aspects of the exam. There are five problems to be solved. You should have ample room on these sheets to complete your work. The total number of points is 100.

Note:

- All materials except for a pencil or pen should be put in a book bag, and that book bag needs to be completely closed.
- Please put your name and section number in the upper right-hand corner of this page.
- No calculators, phones, tablets, computers, or watches are allowed during the exam.
- The exam is closed book: no textbooks or notes of any kind are allowed on the exam.
- You have 45 minutes to complete the exam.
- You need to show photo identification in order to turn in your completed exam.
- This exam is subject to WPI's Academic Honesty Policy, and by taking this exam, you agree not to discuss its contents with any other WPI student without your instructor's approval.

Laplace Transform Table

$$F(s) = \mathcal{L}\{f\} = \int_0^\infty e^{-s\,t}\, f(t)\, dt \tag{1}$$

$$\mathcal{L}\{f'\} = s\,\mathcal{L}\{f\} - f(0) \tag{2}$$

$$\mathcal{L}\{f''\} = s^2\,\mathcal{L}\{f\} - s\,f(0) - f'(0) \tag{3}$$

$$\mathcal{L}\{1\} = \frac{1}{s}\ ,\ s > 0 \tag{4}$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}\ ,\ s > 0 \tag{5}$$

$$\mathcal{L}\left\{e^{-at}\right\} = \frac{1}{s+a}\ ,\ s > -a \tag{6}$$

$$\mathcal{L}\{\cos(at)\} = \frac{s}{(s^2+a^2)}\ ,\ s > 0 \tag{7}$$

$$\mathcal{L}\{\sin(at)\} = \frac{a}{(s^2+a^2)}\ ,\ s > 0 \tag{8}$$

$$\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\} \tag{9}$$

$$\mathcal{L}\{c\,f\} = c\,\mathcal{L}\{f\}\ ,\ c \text{ const.} \tag{10}$$

$$\mathcal{L}\left\{e^{-at}\, f(t)\right\} = F(s+a) \tag{11}$$

$$\mathcal{L}\{t\, f(t)\} = -\frac{dF}{ds} \tag{12}$$

1. (20 points) Consider the initial value problem

$$y'' + 25\,y = \sin(\omega t)\ ,\ y(0) = 0\ ,\ y'(0) = 3\ ,\omega > 0\ .$$

(a) Find the solution if $\omega = 2$.

(b) For what value or values of ω will resonance occur?

a. $y'' + 25y = \sin 2t\ ,\quad y(0) = 0\ ,\quad y'(0) = 3\ ,\quad \omega > 0$

$r^2 + 25 = 0 \rightarrow r = \pm 5i$

$y_h = c_1 \cos 5t + c_2 \sin 5t$

$y_p = A\cos 2t + B\sin 2t$

$\sin 2t = -4A\cos 2t - 4B\sin 2t + 25A\cos 2t + 25B\sin 2t$

$A = 0\ ,\quad B = 1/21$

$0 = y(0) = c_1$

$3 = y'(0) = 5c_2 + \frac{2}{21} \rightarrow 5c_2 = \frac{63}{21} - \frac{2}{21} \rightarrow 5c_2 = 61/21$

$c_2 = 61/105$

$\boxed{y(t) = \frac{61}{105}\sin 5t + \frac{1}{21}\sin 2t}$

b. $\boxed{\text{Resonance if } \omega = 5}$

2. (10 points) Using the *definition* of the Laplace transform, find $F(s) = \mathcal{L}\{te^t\}$ for $s > 1$. Show your work. You will receive no credit if you just show the answer.

$$F(s) = \mathcal{L}\{te^t\} = \int_0^\infty te^t e^{-st}\,dt$$

$$= \int_0^\infty te^{-t(s-1)}\,dt$$

$$= -\frac{1}{s-1} te^{-t(s-1)} \Big|_0^\infty + \int_0^\infty \frac{1}{s-1} e^{-t(s-1)}\,dt$$

$$= 0 - \frac{1}{(s-1)^2} e^{-t(s-1)} \Big|_0^\infty$$

$$= \frac{1}{(s-1)^2}, \quad s > 1$$

$$\boxed{F(s) = \frac{1}{(s-1)^2}, \quad s > 1}$$

3. (20 points) Find the inverse Laplace transform of the function

$$F(s) = \frac{2s+4}{s^2-1}.$$

Show your work. You will receive no credit if you just show the answer.

$$\frac{2s+4}{(s+1)(s-1)} = \frac{A}{s+1} + \frac{B}{s-1}$$

$$2s+4 = A(s-1) + B(s+1)$$

$s=1$: $6 = 2B \rightarrow B = 3$

$s=-1$: $2 = -2A \rightarrow A = -1$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = -\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= -e^{-t} + 3e^{t}$$

$$\boxed{f(t) = 3e^{t} - e^{-t}}$$

4. (20 points) Use the Laplace transform method to solve the following initial-value problem

$$y'' + 10y' + 25y = 0 \ , \ y(0) = 0 \ , \ y'(0) = 10 \ .$$

Show your work including the Laplace transform of the solution and its partial fraction decomposition if appropriate.

$$\mathcal{L}\{y''\} + 10\mathcal{L}\{y'\} + 25\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$s^2Y - sy(0) - y'(0) + 10sY - 10y(0) + 25Y = 0$$

$$s^2Y - 0 - 10 + 10sY + 25Y = 0$$

$$(s^2 + 10s + 25)Y = 10$$

$$Y(s) = \frac{10}{(s+5)^2}$$

Suppose $F(s) = \frac{10}{s^2}$

Then $Y(s) = F(s+5)$

$f(t) = \mathcal{L}^{-1}\{F(s)\} = 10\mathcal{L}^{-1}\{\frac{1}{s^2}\} = 10t$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{F(s+5)\} = e^{-5t}f(t) = 10te^{-5t}$$

$$\boxed{y(t) = 10te^{-5t}}$$

5. (20 points) For the ODE below, state the *form* of the particular solution $y_p(x)$ based on the method of undetermined coefficients. DO **NOT** DETERMINE THE VALUES OF THE COEFFICIENTS.

$$y''(x) - 2y'(x) + y(x) = 3\,e^{x} + \sin(x)\ .$$

Check homogeneous solution

$r^2 - 2r + 1 = 0$

$(r-1)^2 = 0 \quad \longrightarrow \quad y_h(x) = c_1 e^{x} + c_2 x e^{x}$

$r = 1$ repeated

To guess y_p, want $y_p = Ae^{x} + B\sin x + C\cos x$ but have to multiply by x considering the e^{x}, xe^{x} terms in y_h

$$\therefore \boxed{y_p = Ax^2e^{x} + B\sin x + C\cos x}$$