Sample Exam 1
MA 2051 BD01-BD03,BD05 - Differential Equations
Worcester Polytechnic Institute
Fall 2021
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You are allowed a 45 minutes to complete all aspects of the exam. There are six problems to be solved. You should have ample room on these sheets to complete your work. The total number of points is 100.

Note:

- All materials except for a pencil or pen should be put in a book bag, and that book bag needs to be completely closed.
- Please put your name and section number in the upper right-hand corner of this page.
- No calculators, phones, tablets, computers, or watches are allowed during the exam.
- The exam is closed book: no textbooks or notes of any kind are allowed on the exam.
- You have 45 minutes to complete the exam.
- You need to show your WPI identification in order to turn in your completed exam.
- This exam is subject to WPI's Academic Honesty Policy, and by taking this exam, you agree not to discuss its contents with any other WPI student without your instructor's approval.

1. (10 points) Classify the following differential equations in terms of order, linear or nonlinear. Further, if the equations are linear, state whether the equation is either homogeneous or nonhomogeneous and if the coefficients are constant or variable.

(a)

$$y^{(iv)} + 3t^2\, y'' + 8\, y - \cos t = 0\ .$$

4th order
Linear
Nonhomogeneous
Variable coefficients

(b)

$$\frac{d^2y}{dt^2} + y = \alpha\, y^3\ .$$

2nd order
Nonlinear

(c)

$$\frac{d^2y}{dt^2} + y = 0\ .$$

2nd order
Linear
Homogeneous
Constant coefficients

2. (20 points)

(a) Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{x^2 + 2\,x\,y}{x^2}$$

$$\frac{dy}{dx} - \frac{2}{x}y = 1 \quad \left\{ \mu(x) = e^{\int -2/x\,dx} = e^{-2\ln x} = \frac{1}{x^2} \right\}$$

$$\frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3}y = \frac{1}{x^2}$$

$$y/x^2 = \int 1/x^2\,dx = -\frac{1}{x} + c$$

$$\boxed{y(x) = cx^2 - x \ , \ x \neq 0}$$

(b) Solve the initial-value problem

$$\frac{dy}{dx} = \frac{x + x\,y^2}{y} \ , \ y(1) = 0 \ .$$

$$\frac{dy}{dx} = x\left(\frac{1+y^2}{y}\right)$$

$$\int \frac{y}{1+y^2}\,dy = \int x\,dx$$

$$u = 1+y^2 \ , \ du = 2y\,dy$$

$$\int \frac{\frac{1}{2}du}{u} = \int x\,dx$$

$$\ln|1+y^2| = x^2 + c$$

$$\ln(1+y^2) = x^2 + c$$

Apply $y(1)=0$

$$\ln 1 = 1 + c$$
$$0 = 1 + c$$
$$c = -1$$

$$\ln(1+y^2) = x^2 - 1$$

$$y^2 + 1 = e^{x^2-1}$$

$$\boxed{y^2 = e^{x^2-1} - 1 \ , \quad x \geq 1}$$

Note:

$y \geq 0 \Rightarrow 0 \leq \ln(1+y^2) = x^2 - 1 \rightarrow x^2 \geq 1$

$y(1) = 0 \Rightarrow x \geq 1$

3. (20 points) Find the general solution of the differential equation

$$y' + y = e^{-x} \sin x \ ,$$

$\mu(x) = e^{\int 1 dx} = e^{x}$

$e^{x} y' + e^{x} y = \sin x$

$y e^{x} = -\cos x + C$

$$\boxed{y(x) = -e^{-x} \cos x + C e^{-x}}$$

4. (20 points) If the number of bacteria in a culture is 5 million at the end of 6 hours and 8 million at the end of 9 hours, how many bacteria were present initially?

B: millions of bacteria

t: hours elapsed

$\frac{dB}{dt} = kB$, $B(6) = 5$, $B(9) = 8$, k constant

$B(t) = B_0 e^{kt}$

$\left.\begin{array}{l} 5 = B_0 e^{6k} \\ 8 = B_0 e^{9k} \end{array}\right\} \longrightarrow \frac{8}{5} = e^{3k}$ $\quad k = \frac{1}{3} \ln 8/5$

$B(t) = B_0 e^{(\frac{1}{3} \ln 8/5)t}$

$5 = B_0 e^{2 \ln 8/5}$

$5 = B_0 (8/5)^2$

$\frac{25}{64} \cdot 5 = B_0$

Initially there were $B(0) = B_0 = \boxed{\frac{125}{64} \text{ million bacteria}}$

5. (15 points) A tank with a capacity of 500 liters contains 200 liters of water with 100 grams of a material in solution. Water containing a concentration of 1 g/liter of the material enters the tank at a rate of 3 liters/min, and the well-stirred mixture leaves the tank at the same rate. Find the amount of mass of the material in the tank at any given time t.

$$\frac{dQ}{dt} = 1 \cdot 3 - \frac{Q}{200} \cdot 3 = 3 - \frac{3}{200} Q \quad \frac{\text{grams}}{\text{minute}} \qquad Q(0) = 100$$

$$\frac{dQ}{dt} + \frac{3}{200} Q = 3$$

$$Q e^{3t/200} = \int 3 e^{3t/200} \, dt = 200 e^{3t/200} + c$$

$$Q(t) = 200 + c e^{-3t/200}$$

$$100 = Q(0) = 200 + c \rightarrow c = -100$$

$$Q(t) = 200 - 100 e^{-3t/200}$$

6. (15 points) A cup of coffee has an temperature of 80° C when it is poured and has cooled to 75° C after 1 minute in a room with an ambient temperature of 25° C. Find using Newton's Law of Cooling the time $t = t_d$ when the coffee will reach a temperature of 60° C.

$T(0) = 80°C = T_0$

$T(1) = 75°C$

$m = 25°C$

$$\frac{dT}{dt} = -k(T - m) = -k(T - 25)$$

$$T(t) = 80e^{-kt} + 25(1 - e^{-kt})$$

$$75 = 80e^{-k} + 25 - 25e^{-k} = 55e^{-k} + 25$$

$$\frac{50}{55} = e^{-k}$$

$$\ln 10/11 = -k$$

$$T(t) = 80e^{(\ln 10/11)t} + 25(1 - e^{(\ln 10/11)t})$$

$$60 = T(t_d) = 55e^{t_d \ln 10/11} + 25$$

$$\frac{35}{55} = e^{\ln(10/11)t_d}$$

$$\frac{7}{11} = \left(\frac{10}{11}\right)^{t_d}$$

$$\ln 7/11 = t_d \ln 10/11$$

$$\boxed{t_d = \frac{\ln 7/11}{\ln 10/11} = \frac{\ln 11/7}{\ln 11/10} \text{ minutes}}$$