

§3.7 Solving Nonhomogeneous Equations: Undetermined Coefficients

Consider the nonhomogeneous equation and complementary equation

$$(1) \quad y'' + p(x)y' + q(x)y = f(x)$$

$$(2) \quad y'' + p(x)y' + q(x)y = 0$$

Suppose Y_1, Y_2 are solutions to (1). Then $Y_1 - Y_2$ is a solution to (2)

Suppose y_1, y_2 are a fundamental set of solutions to (2). Any solution to (2) can be written as a linear combination of y_1, y_2 . This means:

$$Y_1 - Y_2 = c_1 y_1 + c_2 y_2 = y_h$$

Suppose y is the general solution to (1). Then y satisfies (1) and may take the place of Y_1 . If we can find any solution y_p to (1), it may take the place of Y_2

$$y = c_1 y_1 + c_2 y_2 + Y_2 = y_h + y_p$$

This shows that in order to solve (1), find the homogeneous solution y_h of (2) and then try to guess a particular solution y_p based on $f(x)$. The table provides strategies for finding y_p based on the form of $f(x)$.

Type	$f(x)$	$y_p(x)$
1	a (constant)	$x^s \cdot A$ (constant)
2	$P_n(x) = a_0 x^n + \dots + a_{n-1}x + a_n$	$x^s \cdot (A_0 x^n + \dots + A_{n-1}x + A_n)$
3	$a e^{\alpha x}$	$x^s \cdot A e^{\alpha x}$
4	$a \cos kx + b \sin kx$	$x^s \cdot (A \cos kx + B \sin kx)$
5	$P_n(x) e^{\alpha x}$	$x^s (A_0 x^n + A_1 x^{n-1} + \dots + A_{n-1}x + A_n) e^{\alpha x}$
6	$P_n(x) \cos kx$	$x^s \cdot \left((A_0 x^n + A_1 x^{n-1} + \dots + A_{n-1}x + A_n) \cos kx \right.$
7	$P_n(x) \sin kx$	$\left. + (B_0 x^n + B_1 x^{n-1} + \dots + B_{n-1}x + B_n) \sin kx \right)$
8	$P_n(x) e^{\alpha x} \cos kx$	$x^s \cdot \left((A_0 x^n + A_1 x^{n-1} + \dots + A_{n-1}x + A_n) e^{\alpha x} \cos kx \right.$
9	$P_n(x) e^{\alpha x} \sin kx$	$\left. + (B_0 x^n + B_1 x^{n-1} + \dots + B_{n-1}x + B_n) e^{\alpha x} \sin kx \right)$

Note: The exponent s in x^s is the *smallest* of the integers 0, 1, or 2 (in other words, x^s will be either 1, x , or x^2) that ensures that no term in the particular solution $y_p(x)$ is also a solution of the corresponding homogeneous equation.

3.7.3 $y' + y = x$

Homogeneous solution $y_h(x)$ satisfies $y' + y = 0$.

Find $y_h(x) = Ce^{-x}$, C constant

Particular Solution $y_p(x)$ satisfies $y' + y = x$.

Try $y_p(x) = Ax + B$.

$$x = y_p' + y_p = A + Ax + B$$

Match coefficients.

$$1x = Ax \rightarrow A = 1, \quad 0 = A + B \rightarrow B = -1$$

$$y_p(x) = x - 1$$

General Solution $y(x) = y_h(x) + y_p(x)$

$$y(x) = Ce^{-x} + x - 1$$

3.7.5 $y'' + 4y' = 1$

$$0 = y_h'' + 4y_h'$$

$$0 = \lambda^2 + 4\lambda = \lambda(\lambda + 4)$$

$$\lambda = 0, -4$$

$$y_h(x) = c_1 + c_2 e^{-4x}$$

Try $y_p(x) = Ax$

$$1 = y_p'' + 4y_p' = 0 + 4A$$

$$1/4 = A$$

$$y_p(x) = \frac{1}{4}x$$

General Solution

$$y(x) = c_1 + c_2 e^{-4x} + \frac{1}{4}x$$

3.7.7 $y'' + 4y' = x$

$$y_h(x) = c_1 + c_2 e^{-4x}$$

(see above)

Try $y_p(x) = x(Ax + B)$

$$x = y_p'' + 4y_p' = 2A + 8Ax + 4B$$

$$1 = 8A \rightarrow A = 1/8$$

$$0 = 2A + 4B \rightarrow B = -1/16$$

PROBLEMS: Section 3.7

For each differential equation in Problems 1–21, find the general solution by finding the homogeneous solution and a particular solution.

1. $y' = 1$
2. $y' + y = 1$
3. $y' + y = x$
4. $y'' = 1$
5. $y'' + 4y' = 1$
6. $y'' + 4y = 1$
7. $y'' + 4y' = x$
8. $y'' + y' - 2y = 3 - 6x$
9. $y'' + y = 6e^x + 3$
10. $y'' - y' - 2y = 6e^x$
11. $y'' + y' = 6 \sin 2x$
12. $y'' + 4y' + 5y = 2e^x$
13. $y'' + 3y' = \sin x + 2 \cos x$
14. $y'' + 4y' + 4y = xe^{-x}$
15. $y'' - y = x \sin x$
16. $y'' - 3y' + 2y = e^x \sin x$
17. $y'' - 4y' + 4y = xe^{2x}$
18. $y'' + y = 12 \cos^2 x$
19. $y'' - 4y' + 3y = 20 \cos x$
20. $y'' - y = 8xe^x$
21. $y'' - 5y' + 6y = \cosh x$

3.7.9 $y'' + y = 6e^x + 3$

Homogeneous Solution

$$0 = y_h'' + y_h$$

$$y_h(x) = c_1 \cos x + c_2 \sin x$$

Particular Solution

$$y_p(x) = Ae^x + B$$

$$6e^x + 3 = Ae^x + Ae^x + B$$

$$y_p(x) = 3e^x + 3$$

General Solution

$$y(x) = c_1 \cos x + c_2 \sin x + 3e^x + 3$$

3.7.12 $y'' + 4y' + 5y = 2e^x$

Homogeneous Solution

$$0 = \lambda^2 + 4\lambda + 5$$

$$\lambda = -2 \pm i$$

$$y_h = e^{-2x} (c_1 \cos x + c_2 \sin x)$$

Particular Solution

$$y_p = Ae^x$$

$$2e^x = 10Ae^x$$

$$A = 1/5$$

General Solution

$$y(x) = e^{-2x} (c_1 \cos x + c_2 \sin x) + \frac{1}{5} e^x$$

3.7.14 $y'' + 4y' + 4y = xe^{-x}$

$$0 = \lambda^2 + 4\lambda + 4$$

$$0 = (\lambda + 2)^2$$

$$y_h = c_1 e^{-2x} + c_2 x e^{-2x}$$

$$y(x) = c_1 e^{-2x} + c_2 x e^{-2x} + (x-2)e^{-x}$$

$$y_p = (Ax + B)e^{-x}$$

$$y_p' = Ae^{-x} - (Ax + B)e^{-x}$$

$$y_p'' = -Ae^{-x} - Ae^{-x} + (Ax + B)e^{-x}$$

$$xe^{-x} = (A - 4A + 4A)xe^{-x} + (-2A + B + 4A - 4B + 4B)e^{-x}$$

$$xe^{-x} = Axe^{-x} + (2A + B)e^{-x} \rightarrow A = 1, B = -2$$

Type	$f(x)$	$y_p(x)$
1	a (constant)	$x^s \cdot A$ (constant)
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3	$a e^{\alpha x}$	$x^s \cdot A e^{\alpha x}$
4	$a \cos kx + b \sin kx$	$x^s \cdot (A \cos kx + B \sin kx)$
5	$P_n(x)e^{\alpha x}$	$x^s (A_0 x^n + A_1 x^{n-1} + \dots + A_{n-1}x + A_n)e^{\alpha x}$
6	$P_n(x) \cos kx$	$x^s \cdot \left((A_0 x^n + A_1 x^{n-1} + \dots + A_{n-1}x + A_n) \cos kx \right.$
7	$P_n(x) \sin kx$	$\left. + (B_0 x^n + B_1 x^{n-1} + \dots + B_{n-1}x + B_n) \sin kx \right)$
8	$P_n(x)e^{\alpha x} \cos kx$	$x^s \cdot \left((A_0 x^n + A_1 x^{n-1} + \dots + A_{n-1}x + A_n)e^{\alpha x} \cos kx \right.$
9	$P_n(x)e^{\alpha x} \sin kx$	$\left. + (B_0 x^n + B_1 x^{n-1} + \dots + B_{n-1}x + B_n)e^{\alpha x} \sin kx \right)$

Note: The exponent s in x^s is the *smallest* of the integers 0, 1, or 2 (in other words, x^s will be either 1, x , or x^2) that ensures that no term in the particular solution $y_p(x)$ is also a solution of the corresponding homogeneous equation.

3.7.13 $y'' + 3y' = \sin x + 2\cos x$

$$0 = \lambda^2 + 3\lambda = \lambda(\lambda + 3)$$

$$y_h = c_1 + c_2 e^{-3x}$$

$$y_p = A \sin x + B \cos x$$

$$\sin x + 2\cos x = -A - B + 3A - 3B$$

$$y_p = \frac{1}{2} \sin x - \frac{1}{2} \cos x$$

General Solution

$$y(x) = c_1 + c_2 e^{-3x} + \frac{1}{2} \sin x - \frac{1}{2} \cos x$$

3.7.19 $y'' - 4y' + 3y = 20 \cos x$

$$0 = \lambda^2 - 4\lambda + 3$$

$$0 = (\lambda - 3)(\lambda - 1)$$

$$y_h = c_1 e^{3x} + c_2 e^x$$

$$y_p = A \cos x + B \sin x$$

$$y_p' = -A \sin x + B \cos x$$

$$y_p'' = -A \cos x - B \sin x$$

$$20 = -A - 4B + 3A = 2A - 4B$$

$$0 = -B + 4A + 3B = 4A + 2B$$

$$A = 2, B = -4$$

General Solution $y(x) = c_1 e^{3x} + c_2 e^x + 2 \cos x - 4 \sin x$

3.7.27 $y'' + 16y = 5 \sin x, y(0) = y'(0) = 0.$

$$y_h = c_1 \cos 4x + c_2 \sin 4x$$

$$y_p = A \cos x + B \sin x$$

$$5 \sin x = -A \cos x - B \sin x + 16A \cos x + 16B \sin x \rightarrow A = 0, B = 1/3$$

General solution

$$y(x) = c_1 \cos 4x + c_2 \sin 4x + \frac{1}{3} \sin x$$

Apply I.C.'s

$$0 = y(0) = c_1$$

$$0 = y'(0) = 4c_2 + \frac{1}{3} \rightarrow c_2 = -1/12$$

$$y(x) = \frac{1}{3} \sin x - \frac{1}{12} \sin 4x$$

3.7.39 $y'' - 2y' + y = 2 \sin x$

$$z'' - 2z' + z = 2e^{ix} = 2(\cos x + i \sin x)$$

$$z_p = A e^{ix}$$

$$2e^{ix} = A i^2 e^{ix} - 2A i e^{ix} + A e^{ix}$$

$$2 = -A - 2Ai + A = -2Ai$$

$$A = i$$

$$z_p = i e^{ix} = i \cos x - \sin x$$

$$y_p = \text{Im}(z_p) = \cos x$$

3.7.41 $y'' + 25y = 20 \sin 5x$

$$z'' + 25z = 20e^{5ix} = 20(\cos 5x + i \sin 5x)$$

$$z_p = A x e^{5ix}$$

For Problems 22–28, find the solution to the initial-value problem.

22. $y' - y = 1$	$y(0) = 0$	
23. $y'' + y = 2x$	$y(0) = 1$	$y'(0) = 2$
24. $y'' + y' - 2y = 2x$	$y(0) = 0$	$y'(0) = 1$
25. $y'' - 5y' + 6y = e^x(2x - 3)$	$y(0) = 1$	$y'(0) = 3$
26. $y'' - 4y' + 4y = e^{2x}$	$y(0) = 0$	$y'(0) = 0$
27. $y'' + 16y = 5 \sin x$	$y(0) = 0$	$y'(0) = 0$
28. $y'' + 3y' + 2y = 20 \cos 2x$	$y(0) = -1$	$y'(0) = 6$

Section 3.7

- $y = x + c$
- $y = ce^{-x} + 1$
- $y = ce^{-x} + x - 1$
- $y = c_1 + c_2 x + \frac{1}{2}x^2$
- $y = c_1 + c_2 e^{-4x} + \frac{x}{4}$
- $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4}$
- $y = c_1 + c_2 e^{-4x} + \frac{1}{8}x^2 - \frac{1}{16}$
- $y = c_1 e^x + c_2 e^{-2x} + 3x$
- $y = c_1 \cos x + c_2 \sin x + 3e^x + 3$
- $y = c_1 e^{2x} + c_2 e^{-x} - 3e^x$
- $y = c_1 + c_2 e^{-x} - \frac{6}{5} \sin 2x - \frac{3}{5} \cos 2x$
- $y = e^{-2x}(c_1 \cos x + c_2 \sin x) + \frac{1}{5}e^x$
- $y = c_1 + c_2 e^{-3x} + \frac{1}{2}(\sin x - \cos x)$
- $y = c_1 e^{-2x} + c_2 x e^{-2x} + x e^{-x} - 2e^{-x}$
- $y = c_1 e^x + c_2 e^{-x} - \frac{1}{2}(x \sin x + \cos x)$
- $y = c_1 e^x + c_2 e^{2x} + \frac{1}{2}e^x(\cos x - \sin x)$
- $y = c_1 e^{2x} + c_2 x e^{2x} + \frac{1}{6}x^3 e^{2x}$
- $y = c_1 \cos x + c_2 \sin x - 2 \cos 2x + 6$
- $y = c_1 e^x + c_2 e^{3x} + 2 \cos x - 4 \sin x$
- $y = c_1 e^{-x} + e^x(c_2 - 2x + 2x^2)$
- $y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{24}(7 \cosh x + 5 \sinh x)$
- $y = e^x - 1$
- $y = \cos x + 2x$
- $y = e^x - \frac{1}{2}e^{-2x} - x - \frac{1}{2}$
- $y = e^{2x} + x e^x$
- $y = \frac{1}{2}x^2 e^{2x}$
- $y = \frac{1}{3} \sin x - \frac{1}{12} \sin 4x$
- $y = -\cos 2x + 3 \sin 2x$
- $y_p = e^x(Ax^3 + Bx^2 + Cx)$

$$39. y = c_1 e^x + c_2 x e^x + \cos x$$

$$41. y = c_1 \cos 5x + c_2 \sin 5x - 2x \cos 5x$$