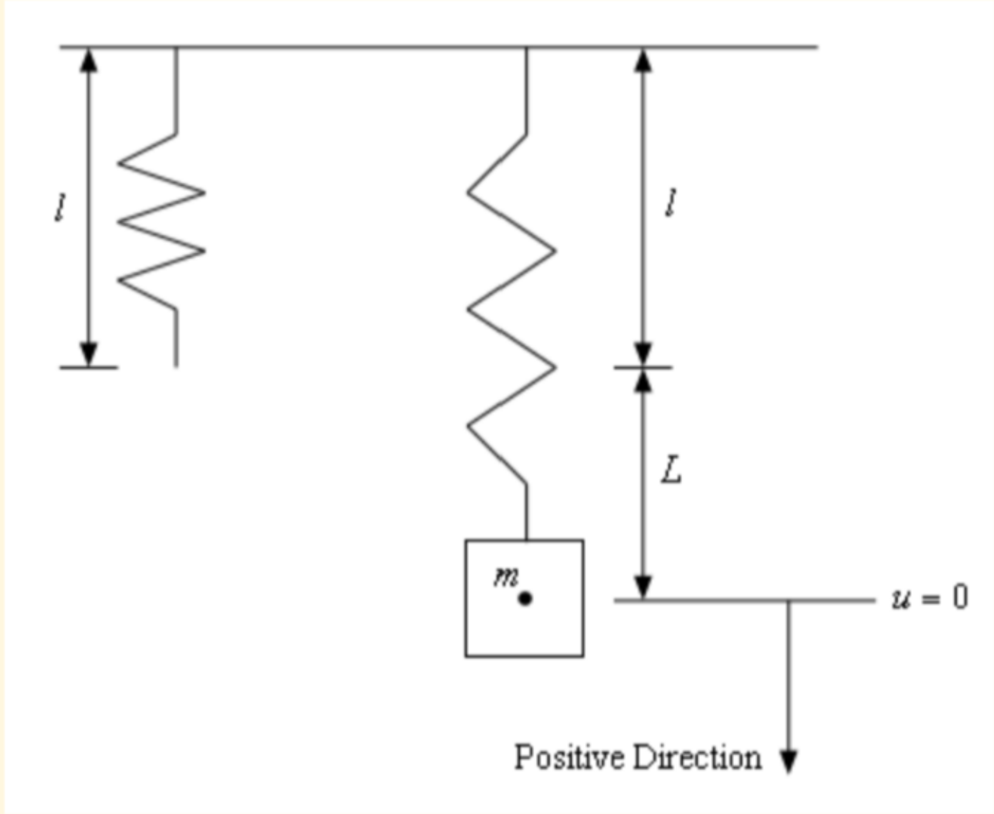
§3.11 Forced Vibrations

Background

Develop a differential equation that gives the displacement of the object at any time t
Recall Newton's 2^{nd} Law $ma = \sum_i F_i$.
In this case takes the form



$$mu'' = f(t, u, u')$$

We assume 4 forces are acting upon the object.

1. $F_g = mg$ Gravitational Force

2. $F_s = -k(L+u)$ Spring Force

   Follow's Hooke's Law - the force exerted by the spring is proportional ($k>0$) to the displacement u of the spring from its natural length L. The minus sign accounts for the assumption that this force always acts in the direction of equilibrium (where the spring is at its natural length).

3. $F_d = -\gamma u'$ Damping Force

   Counteracts the object's movement with a force proportional ($\gamma > 0$) to the object's velocity u'.

   downward movement : $u' > 0 \longleftrightarrow F_d > 0 \longleftrightarrow$ upward force
   upward movement : $u' < 0 \longleftrightarrow F_d < 0 \longleftrightarrow$ downward force

4. $F(t)$ External Force

   Any additional forces we want to include. Call $F(t)$ the forcing function, Let $f(t,u,u'')$ be the sum of these 4 forces.

$$mu'' = f(t,u,u'') = mg - k(L+u) - \gamma u' + F(t)$$

$$mu'' + \gamma u' + ku = mg - kL + F(t)$$

Consider the case where the object is at rest. That means $F(t) \equiv 0$ and $u(t) = u'(t) = u''(t) = 0$. Then $\underline{mg = kL.}$

The equation describing the object's movement is therefore

$$mu'' + \gamma u' + ku = F(t)$$
$u(0) = u_0$ initial displacement
$u'(0) = u_0'$ initial velocity

Free Undamped vibrations

$F(t) = 0$, $\gamma = 0$ $\longrightarrow$ $mu'' + ku = 0$
If $u = e^{rt}$, $(mr^2 + k)e^{rt} = 0$
$r = \pm\sqrt{k/m}\, i = \pm \omega_0 i$
$u(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$

We can write $u(t)$ in 'polar form' $u(t) = R\cos(\omega_0 t - \delta)$. To find R and δ,

$$R\cos\delta\cos\omega_0 t + R\sin\delta\sin\omega_0 t = R\cos(\omega_0 t - \delta) = c_1\cos\omega_0 t + c_2 \sin\omega_0 t$$

$$\rightarrow R\cos\delta = c_1, \quad R\sin\delta = c_2, \quad \tan\delta = c_2/c_1, \quad R^2 = c_1^2 + c_2^2$$

Free Damped Vibrations

$F(t) = 0$, $\gamma > 0$ $\longrightarrow$ $mu'' + \gamma u' + ku = 0$

The behavior of $u(t)$ falls into 1 of 3 qualitatively different cases dependinging on the sign $(+,-,0)$ of $\gamma^2 - 4mk$.

Assuming a solution of the form $u(t) = e^{rt}$ gives the characteristic eqn

$$mr^2 + \gamma r + k = 0$$

$$r = -\gamma/2m \pm \sqrt{\gamma^2 - 4mk}\ /2m$$

1. $\gamma^2 - 4mk = 0$ Critical Damping $(\gamma = \gamma_{critical} = 2\sqrt{mk})$

$$u(t) = c_1 e^{-\gamma t/2m} + c_2 t e^{-\gamma t/2m}$$

The characteristic equation has a repeated root $r = -\gamma/2$.
Note that $\lim_{t\to\infty} u(t) = 0$, the solution decays over time.

2. $\gamma^2 - 4mk > 0$ Overdamped $(\gamma > \gamma_{critical})$

$$r_{1,2} = -\gamma/2m \pm \sqrt{\gamma^2 - 4mk}\ /2m = -\frac{\gamma}{2m}\left(1 \pm \sqrt{1 - \frac{4mk}{\gamma^2}}\right)$$

$$u(t) = c_1 e^{r_1 t} + c_2 e^{-r_1 t}$$

The characteristic equation has 2 distinct real roots r_1 and r_2
$\gamma^2 - 4mk > 0 \rightarrow r_1, r_2 < 0$. Then, $\lim_{t\to\infty} u(t) = 0$ in this case too.

3. $\gamma^2 - 4mk < 0$ Underdamped $(\gamma < \gamma_{critical})$

$$r_{1,2} = -\gamma/2m \pm \sqrt{4mk - \gamma^2}\ i\ /2m = \alpha \pm \beta i$$

$$u(t) = e^{\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t)$$

The characteristic equation has 2 complex roots $r_{1,2} = \alpha \pm \beta i$
Since $\alpha = -\gamma/2m < 0$, $\lim_{t\to\infty} u(t) = 0$ in this case too.

Undamped Forced Vibrations

$\gamma = 0$, $F(t) \not\equiv 0 \longrightarrow mu'' + ku = F(t)$

$u(t) = u_h(t) + u_p(t)$

$u_h(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$, $\omega_0 = \sqrt{k/m}$ satisfies $mu'' + ku = 0$
Note that $\lim_{t\to\infty} u_h(t) \neq 0$ but rather oscillates with $|u(t)| \leq \sqrt{c_1^2 + c_2^2}$ $\forall t$.

$u_p(t)$ is a particular solution to $mu'' + ku = F(t)$ found using undetermined coefficients (or some other method). A notable scenario is the case of resonance where $F(t)$ is of the form $F_0 \cos\omega_0 t$, $F_0 \sin\omega_0 t$ or a linear combination of the two. In this case we guess $u_p(t)$ of the form

$$u_p(t) = At\cos\omega_0 t + Bt\sin\omega_0 t$$

With either of $A, B \neq 0$ we have $\lim_{t\to\infty} |u(t)| = \infty$. This is notable since the natural oscillations $u_h(t)$ are bounded and the applied force $F(t)$ is bounded $(|F(t)| \leq F_0)$ yet the solution $u(t)$ becomes infinite in the case that the applied force $F(t)$ is an oscillation of the same frequency ω_0 as that of the unforced solution.

Damped Forced Vibrations

$\gamma > 0$, $F(t) \not\equiv 0 \rightarrow mu'' + \gamma u' + ku = F(t)$

$u(t) = u_h(t) + u_p(t)$ or $u(t) = u_{transient}(t) + u_\infty(t)$

$u_h(t)$ satisfies $mu'' + \gamma u' + ku = 0$. From 'Free Damped Vibrations' we know there are 3 cases for the form of $u_h(t)$ based on the value $\gamma^2 - 4mk$. In any case $\lim_{t\to\infty} u_h(t) = 0$. Call $u_h(t)$ the <u>transient solution</u> $u_{transient}$.

$u_p(t)$ is a particular solution of $mu'' + \gamma u' + ku = F(t)$. Since $u(t)$ approaches $u_p(t)$ asymptotically, call $u_p(t)$ the <u>steady state solution</u> $u_\infty(t)$.

3.11.3 Find the steady state solution $u_\infty(t) = R\cos(\omega_0 t - \delta)$

$\ddot{u} + 4\dot{u} + 4u = \cos t$

$\lambda^2 + 4\lambda + 4 = 0$

$(\lambda + 2)^2 = 0$

$u_h = c_1 e^{-2t} + c_2 t e^{-2t}$

$u_\infty = A\cos t + B\sin t$

$\cos t = (3A + 4B)\cos t$
$\quad + (3B - 4A)\sin t$

$3A + 4B = 1 \qquad A = 3/25$

$3B - 4A = 0 \qquad B = 4/25$

$u_\infty = \frac{3}{25}\cos t + \frac{4}{25}\sin t$

$R = \frac{1}{25}\sqrt{9+16} = \frac{1}{5}$

$\tan\delta = 4/3 \rightarrow \delta = \arctan 4/3 \approx 0.93$

$\omega_0 = 1$

$u_\infty = \frac{1}{5}\cos(t - \delta), \ \delta = \arctan 4/3$

3.11.5 Find the steady state solution $u_\infty(t) = R\cos(\omega_0 t - \delta)$

$\ddot{u} + \dot{u} + u = 4\cos 3t$

$u_h = e^{-t/2}\left(c_1 \cos \frac{\sqrt{3}}{2}t + c_2 \sin \frac{\sqrt{3}}{2}t\right)$

$u_\infty = A\cos 3t + B\sin 3t$

$4\cos 3t = (-9A + 3B + A)\cos 3t$
$\quad + (-9B - 3A + B)\sin 3t$

$4 = 3B - 8A$

$0 = -8B - 3A$

$A = -32/73$

$B = 12/73$

$A = \frac{1}{73}\sqrt{32^2 + 12^2} = 4\sqrt{73}/73$

$\delta = \arctan(-12/32) \approx -0.359$

$\omega_0 = 3$

$u_\infty = \frac{4\sqrt{73}}{73}\cos(3t - \delta), \ \delta = \arctan(-3/8)$

3.11.9 Find the steady state solution of a mass that vibrates according to $\ddot{u} + 8\dot{u} + 36u = 72\cos 6t$.

First check for resonance $\omega = \omega_0 = 6$ using the characteristic eqn.

$6^2 + 8 \cdot 6 + 36 \neq 0 \rightarrow$ No resonance.

$u_\infty = C\sin 6t + D\cos 6t$

$72\cos 6t = (-36C - 48D + 36C)\sin 6t$
$\quad + (-36D + 48C + 36D)\cos 6t$

$\longrightarrow \quad 0 = -48D, \quad 72 = 48C$

$0 = D, \quad 3/2 = C$

$u_\infty = \frac{3}{2}\sin 6t$

(Error #5 $\omega_0 = 3$ not 1)

1. $-2\sin t \sin 2t$ **2.** $2\sin t\cos 2t$ **3.** $u_{ss} = \frac{1}{5}\cos(t - \delta)$, $\delta = \tan^{-1}(1.33) \doteq 0.93$ rad

4. $u_{ss} = \frac{2\sqrt{5}}{5}\cos(t - \delta)$, $\delta = \tan^{-1}(2) \doteq 1.1$ rad

5. $u_{ss} = \frac{4}{\sqrt{73}}\cos(t - \delta)$, $\delta = \tan^{-1}(-3/8) = \pi - \tan^{-1}(3/8) \doteq 2.78$ rad

6. (a) $\omega = 2\sqrt{3}$ rad/sec (b) $u = \frac{2}{3}\sin(2\sqrt{2}t) - \frac{4\sqrt{3}}{3}t\cos(2\sqrt{3}t)$

8. (a) $h(t) = \frac{49}{80\pi^2}\cos\left(\frac{2\pi t}{7}\right) + 3$ (buoy above the water line) (b) No

9. $u = \frac{3}{2}\sin 6t$ **10.** $u = -\frac{1}{4}e^{-2t}(3\sin 4t + 4\cos 4t) + \frac{1}{2}(\sin 2t + 2\cos 2t)$

11. Kick with the same frequency as the swing.

§ 5.1 Definition of the Laplace Transform

<u>Definition</u> The Laplace transform of a piecewise continuous function $f(t)$ is

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

<u>Theorem</u> The Laplace transform is a linear operator:

$$\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\} \quad \text{and} \quad \mathcal{L}\{cf\} = c\,\mathcal{L}\{f\}$$

<u>Theorem</u> If $f(t)$ is piecewise continuous on $[0,\infty)$ and of exponential order α ($\exists\, M, T$ s.t. $|f(t)| \le M e^{\alpha t}$ $\forall t \geq T$) then the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ exists $\forall s > \alpha$.

5.1.1 $f(t) = 5$

$$\mathcal{L}\{f(t)\} = \int_0^\infty 5e^{-st}\,dt$$

$$= -5/s\, e^{-st}\Big|_0^\infty$$

$$= 5/s\ , \quad s > 0$$

5.1.3 $f(t) = e^{2t}$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}e^{2t}\,dt$$

$$= \int_0^\infty e^{(2-s)}\,dt$$

$$= \frac{1}{2-s}\, e^{(2-s)t}\Big|_0^\infty$$

$$= \frac{1}{s-2}\ , \quad s > 2$$

5.1.7 $f(t) = 5\sinh 2t = 5/2\,(e^{2t} - e^{-2t})$

$$\mathcal{L}\{f(t)\} = \frac{5}{2}\int_0^\infty e^{-st}\left(e^{2t} - e^{-2t}\right)dt$$

$$\frac{2}{5}\mathcal{L}\{f(t)\} = \int_0^\infty \left(e^{(2-s)t} - e^{-(2+s)t}\right)dt$$

$$= \left[\frac{1}{2-s}\, e^{(2-s)t} + \frac{1}{2+s}\, e^{-(2+s)t}\right]\Big|_0^\infty$$

$$= \underbrace{\frac{1}{s-2}}_{s>2} - \underbrace{\frac{1}{2+s}}_{s<-2}\ , \quad |s| > 2$$

$$= \frac{4}{s^2-4}$$

$$\frac{2}{5}\mathcal{L}\{f(t)\} = \frac{4}{s^2-4} \longrightarrow \mathcal{L}\{f(t)\} = \frac{10}{s^2-4}\ , \quad |s| > 2$$

5.1.9 $f(t) = \begin{cases} 0\ , & t < 1 \\ 1\ , & t \geq 1 \end{cases}$

$$\mathcal{L}\{f(t)\} = \int_0^1 0\cdot e^{-st}\,dt + \int_1^\infty 1\cdot e^{-st}\,dt$$

$$= 0 + \left(-1/s\, e^{-st}\right)\Big|_1^\infty$$

$$= \frac{e^{-s}}{s}\ , \quad s > 0$$

Section 5.1

1. $\frac{5}{s}$ 2. $\frac{1}{s^2}$ 3. $\frac{1}{s-2}$ 4. $\frac{1}{s+1}$ 5. $\frac{2}{s^2+4}$ 6. $\frac{s}{s^2+9}$ 7. $\frac{10}{s^2-4}$ 8. $\frac{s-1}{(s-1)^2+4}$ 9. $\frac{e^{-s}}{s}$
10. $\frac{1}{s}(1-e^{-s})$ 11. $\frac{1+e^{-s\pi}}{s^2+1}$ 12. $\frac{1}{s^2}(1-e^{-s})$ 13. $\frac{1}{(s-a)^2}$ 14. $\frac{n!}{(s-a)^{n+1}}$ 15. $\frac{2as}{(s^2+a^2)^2}$
16. $\frac{s^2+a^2}{(s-a)^2(s+a)^2}$ 17. $\frac{a}{s}+\frac{b}{s^2}+\frac{2c}{s^3}$ 18. $\frac{2s+1}{s(s+1)}$ 19. $\frac{2s}{s^2-4}$ 20. $\frac{3}{s}+\frac{1}{s^2}+\frac{2}{(s+1)^2+4}$
21. $\frac{1}{(s+2)^2}+\frac{6}{(s+1)^3}$ 22. $\frac{6}{(s+3)^4}+\frac{4(s+1)}{(s+1)^2+9}$ 23. $\frac{s+3a}{s^2-a^2}$ 24. $\frac{1}{(s+3)^2}+\frac{2}{s^2+1}$ 25. Continuous

§5.2 Properties of the Laplace Transform

Using the definition one can derive a collection of Laplace transforms and properties thereof.

5.2.3 $y(t) = (t-9)^2$

$$Y(t) = \mathcal{L}\{y(t)\}$$

$$= \mathcal{L}\{t^2\} - 18\mathcal{L}\{t\} + 81\mathcal{L}\{1\} \quad \text{(Properties 1, 2)}$$

$$= \frac{2!}{s^{2+1}} - 18\frac{1!}{s^{1+1}} + 81\frac{1}{s} \quad \text{(Table 5.1 : 1, 2)}$$

$$= \frac{2}{s^3} - \frac{18}{s^2} + \frac{81}{s} = \frac{2 - 18s + 81s^2}{s^3}$$

5.2.7 $y(t) = t^2 \sin 2t$

Let $f(t) = \sin 2t$

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4} \quad \text{(Table 5.1: 6)}$$

$$Y(s) = \mathcal{L}\{y(t)\} = \mathcal{L}\{t^2 f(t)\}$$

$$= (-1)^2 \frac{d^2}{ds^2} F(s) = \frac{d^2}{ds^2}\frac{2}{s^2+4} \quad \text{(Property 7)}$$

$$= \frac{d}{ds}\frac{-2}{(s^2+4)^2}\cdot 2s = \frac{d}{ds}\frac{-4s}{(s^2+4)^2}$$

$$= \frac{-4(s^2+4)^2 + 4s(s^2+4)4s}{(s^2+4)^4} \quad \text{(Quotient Rule)}$$

$$= \frac{-4(s^2+4) + 16s^2}{(s^2+4)^3} = \frac{4(3s^2-4)}{(s^2+4)^3} \quad \text{(Error in text-book solution)}$$

5.2.9 $y(t) = 5e^{5t}\cos 2t$

$$Y(s) = \mathcal{L}\{y(t)\} = 5\mathcal{L}\{e^{5t}\cos 2t\} \quad \text{(Property 2)}$$

$$= 5\frac{s-5}{(s-5)^2 + 2^2} \quad \text{(Table 5.1 : 11)}$$

$$= \frac{5(s-5)}{s^2 - 10s + 29}$$

For Problems 1–20, find the Laplace transform of the given function, using Table 5.1 from Section 5.1 and properties of the Laplace transform given in Table 5.2.

1. $at^2 + bt + c$	**13.** $\sin 2t \sinh 2t$
2. $t^2 + e^{2t} - 2$	**14.** $\sin^2 t$
3. $(t-9)^2$	**15.** $\sinh 3t$
4. e^{2t-1}	**16.** $\cos^3 t$
5. $(1 + e^t)^2$	**17.** $t\sin^2 t$
6. $3t \sin t$	**18.** $\cos mt \sin nt \quad (m \neq n)$
7. $t^2 \sin 2t$	**19.** $\int_0^t \cos \tau \, d\tau$
8. $e^{-2t}\sin 3t$	
9. $5e^{5t}\cos 2t$	**20.** $\int_0^t \sin 3\tau \, d\tau$
10. $t^2 e^{-3t}$	
11. $te^t \cos t$	
12. $t^2 e^t \sin t$	

$f(t)$	$F(s) = \mathfrak{L}\{f\}$	Domain of $F(s)$
1	$1/s$	$s > 0$
t^n (n positive integer)	$\frac{n!}{s^{n+1}}$	$s > 0$
$t^p \quad (p > -1)$	$\frac{\Gamma(p+1)}{s^{p+1}}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$e^{at}t^n, n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$\sin bt$	$\frac{b}{s^2+b^2}$	$s > 0$
$\cos bt$	$\frac{s}{s^2+b^2}$	$s > 0$
$\sinh bt$	$\frac{b}{s^2-b^2}$	$s > \lvert b \rvert$
$\cosh bt$	$\frac{s}{s^2-b^2}$	$s > \lvert b \rvert$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$u(t-c)$	$\frac{e^{-cs}}{s}$	$s > 0$
$u(t-c)f(t-c)$	$e^{-cs}F(s)$	
$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	
$\delta(t-c)$	e^{-cs}	
$\frac{d^n}{dt^n}f(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	
$t^nf(t)$	$(-1)^nF^{(n)}(s)$	

1. $\mathfrak{L}\{f + g\} = \mathfrak{L}\{f\} + \mathfrak{L}\{g\}$
2. $\mathfrak{L}\{cf\} = c\mathfrak{L}\{f\}$
3. $\mathfrak{L}\{f'\} = s\mathfrak{L}\{f\} - f(0)$
4. $\mathfrak{L}\{f''\} = s^2\mathfrak{L}\{f\} - sf(0) - f'(0)$
5. $\mathfrak{L}\{f^{(n)}\} = s^n\mathfrak{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$
6. $\mathfrak{L}\{e^{at}f(t)\} = F(s-a)$
7. $\mathfrak{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$
8. $\mathfrak{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$
9. $\mathfrak{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s)$
10. $\mathfrak{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\xi)\,d\xi$
11. $\lim_{s\to\infty} sF(s) = f(0)$ (initial-value theorem)
12. $\lim_{s\to 0} sF(s) = f(\infty)$* (final-value theorem)

* The notation $f(\infty)$ means the limiting value of $f(t)$ as $t \to \infty$.

Section 5.2

1. $\frac{2a}{s^3} + \frac{b}{s^2} + \frac{c}{s}$ **2.** $\frac{2}{s^3} + \frac{1}{s-2} - \frac{2}{s}$ **3.** $\frac{2}{s^3} - \frac{18}{s^2} + \frac{81}{s}$ **4.** $\frac{e^{-1}}{s-2}$ **5.** $\frac{1}{s} + \frac{2}{s-1} + \frac{1}{s-2}$ **6.** $\frac{6s}{(s^2+1)^2}$
7. $\frac{2(3s^2-4)}{(s^2+4)^3}$ **8.** $\frac{3}{(s+2)^2+9}$ **9.** $\frac{5(s-5)}{(s-5)^2+4}$ **10.** $\frac{2}{(s+3)^3}$ **11.** $\frac{s(s-2)}{(s^2-2s+2)^2}$ **12.** $\frac{2(3s^2-4s+2)}{(s^2-2s+2)^3}$
13. $\frac{8s}{s^4+64}$ **14.** $\frac{1}{2s} - \frac{s}{2(s^2+4)}$ **15.** $\frac{3}{s^2-9}$ **16.** $\frac{3s}{4(s^2+1)} + \frac{s}{4(s^2+9)}$ **17.** $\frac{1}{2s^2} + \frac{s^2-4}{2(s^2+4)^2}$
18. $\frac{m+n}{2[s^2+(m+n)^2]} + \frac{n-m}{[s^2+(n-m)^2]}$ **19.** $\frac{1}{s^2+1}$ **20.** $\frac{3}{s(s^2+9)}$ **21.** $\frac{b}{(s+1)^2+b^2}$ **22.** $\frac{2}{(s+2)^3}$
23, $\frac{1}{(s-2)^2}$ **24.** $\frac{2}{(s+1)^3}$ **25.** $\frac{2bs}{(s^2+b^2)^2}$ **26.** $\frac{2(3s^2-4)}{(s^2+4)^3}$ **27.** $\frac{1}{(s+2)^2}$ **28.** $\frac{2}{(s-1)^3}$ **29.** $\frac{b}{(s-a)^2+b^2}$
32. (b) $3\left(\frac{1}{(s-2)^4} + \frac{1}{(s+2)^4}\right)$ (c) $\frac{1}{2}\left(\frac{1}{(s-1)^2+1} + \frac{1}{(s+1)^2+1}\right)$ **33.** (b) $\frac{1}{(s-3)^3} - \frac{1}{(s+3)^3}$
(c) $\frac{1}{2}\left(\frac{s-1}{(s-1)^2+1} + \frac{s+1}{(s+1)^2+1}\right)$ **35.** $\frac{2}{s^2+4}$ (c) $\frac{1}{s-5}$ **36.** (a) $\frac{1}{s(s^2+1)}$ (b) $\frac{1}{s(s-2)}$ **37.** $\cot^{-1} s$
38. $\frac{1}{(s-1)(s+1)^2}$ **39.** $\frac{2(s+2)}{(s^2+4s+20)^2}$ **40.** $\frac{2(s+1)}{s(s^2+2s+2)^2}$ **41.** $\frac{2(3s^2+6s+2)}{s(s^2+2s+2)^3}$
42. $\frac{1}{s^2(s+1)^2}$ **43.** $\frac{1}{s(s-1)}$