§ 3.3 Reduction of Order

D'Alembert's reduction of order method.

Suppose y_1 is a solution of $y'' + p(x)y' + q(x)y = 0$. To find another solution y_2 s.t. y_1 and y_2 are linearly independent, look for solutions of the form $y = v(x)y_1$.

$$y' = v'y_1 + vy_1'$$
$$y'' = v''y_1 + v'y_1' + v'y_1' + vy_1''$$

$$\begin{aligned} 0 = y'' + p(x)y' + q(x)y &= v''y_1 + 2v'y_1' + vy_1'' + p(v'y_1 + vy_1') + qvy_1 \\ &= v(y_1'' + py_1' + qy_1) + v''y_1 + 2v'y_1' + pv'y_1 \\ &= v''y_1 + 2v'y_1' + pv'y_1 \end{aligned}$$

$0 = v'' + 2v'y_1'/y_1 + pv' = v'' + (2y_1'/y_1 + p)v'$, (on any interval s.t. $y_1(x) \neq 0$)

$0 = w' + (p + 2\,y_1'/y_1)w$, $w = v'$

$\mu = e^{\int p + 2y_1'/y_1 \, dx} = e^{\int p\,dx} e^{\ln y_1^2} = y_1^2 e^{\int p\,dx}$

$y_1^2 w e^{\int p\,dx} = c$

$w = c/y_1^2 \, e^{-\int p\,dx}$, set $c = 1$

$w(x) = 1/y_1^2 \, e^{-\int p\,dx}$

$\frac{1}{w} w' = -p - 2\,y_1'/y_1$

$w(x) = Ce^{-\int p\,dx} e^{-2\ln|y_1|}$

$w(x) = 1/y_1^2 \, e^{-\int p\,dx}$ set $c = 1$

$v(x) = \int w\,dx = \int 1/y_1^2 \, e^{-\int p\,dx}\,dx$

3.3.1 $y''-y=0$, $y_1(x)=e^x$

For Problems 1–10, a differential equation and one solution are given. Use d'Alembert's reduction of order method to find a second linearly independent solution. What is the general solution of the differential equation?

Differential equation	*Solution*
1 $y''-y=0$	$y_1(x)=e^x$
2. $y''+y=0$	$y_1(x)=\sin x$
3. $y''-4y'+4y=0$	$y_1(x)=e^{2x}$
4. $y''+y'=0$	$y_1(x)=1$
5. $xy''+y'=0$	$y_1(x)=1$
6. $xy''-2(x+1)y'+4y=0$	$y_1(x)=e^{2x}$
7. $x^2y''-6y=0$	$y_1(x)=x^3$
8. $x^2y''-xy'+y=0$	$y_1(x)=x$
9. $(x^2+1)y''-2xy'+2y=0$	$y_1(x)=x$
10. $y''+\frac{1}{x}y'+\left(1-\frac{1}{4x^2}\right)y=0$	$y_1(x)=\frac{1}{\sqrt{x}}\sin x$

Look for a solution y of the form $y=vy_1$.

$$y=vy_1$$
$$y'=v'y_1+vy_1'$$
$$y''=v''y_1+v'y_1'+v'y_1'+vy_1''$$

$$\begin{aligned}0=y''-y &= v''y_1+2v'y_1'+vy_1''-vy_1\\ &= v''y_1+2v'y_1'+v(y_1''-y_1)\\ &= v''y_1+2v'y_1'\\ &= v''e^x+2v'e^x\end{aligned}$$

$0=v''+2v'=w'+2w$, $w=v'$

$0=(we^{2x})'\rightarrow w(x)=A'e^{-2x}$

$v(x)=\int w\,dx=Ae^{-2x}+B\longrightarrow v(x)=e^{-2x}$

Only one nonzero solution is needed

Pick $A=1$, $B=0$.

$y_2(x)=v(x)y_1(x)=e^{-2x}e^x=\boxed{e^{-x}}$

General solution: $\boxed{y(x)=c_1e^x+c_2e^{-x}}$ since e^x, e^{-x} independent

3.1.2 $y''+y=0$, $y_1(x)=\sin x$

$$y=vy_1$$
$$y'=v'y_1+vy_1'$$
$$y''=v''y_1+2v'y_1'+vy_1''$$

$$\begin{aligned}0&=y''+y\\ &=v''y_1+2v'y_1'+v(y_1''+y_1)\\ &=v''y_1+2v'y_1'\end{aligned}$$

$0=v''+2v'y_1'/y_1$ (in an interval where $\sin x\neq 0$)

$0=v''+2v'\cot x$

$0=w'+2w\cot x$ $\quad\phi(x)=e^{2\int\cot x\,dx}=\sin^2 x$

$0=(w\sin^2x)'$

$A=w\sin^2 x$

$w(x)=A\sec^2 x\longrightarrow v(x)=\int w\,dx=A\tan x+B$

Pick $A=1$, $B=0$.

$$\begin{aligned}y_2&=vy_1\\ &=\tan x\sin x\\ &=\boxed{\cos x}\end{aligned}$$

General Solution:

$\boxed{y(x)=c_1\cos x+c_2\sin x}$

3.3.7 $x^2y''-6y=0$, $y_1(x)=x^3$

$0=y''-6x^{-2}y=v''y_1+2v'y_1'+vy_1''-6vy_1/x^2=v''y_1+2v'y_1'+v(y_1''-6y_1/x^2)$

$0=v''y_1+2v'y_1'=w'y_1+2wy_1'=x^3w'+6x^2w$

$0=w'+6/x\,w$ $\quad\phi=e^{\int 6/x\,dx}=x^6$

$0=(w\cdot x^6)'$

$w(x)=A'x^{-6}$

$v(x)=Ax^{-5}+B\rightarrow v(x)=x^{-5}$

$$\boxed{\begin{gathered}y_2(x)=x^{-2}\\ y(x)=c_1x^3+c_2x^{-2}\end{gathered}}$$

3.3.9 $(x^2+1)y'' - 2xy' + 2y = 0, \quad y_1(x) = x$

$y = vy_1$
$y' = v'y_1 + vy_1'$
$y'' = v''y_1 + 2v'y_1' + vy_1''$

$$0 = (x^2+1)(v''y_1 + 2v'y_1' + vy_1'') - 2x(v'y_1 + vy_1') + 2vy_1$$
$$= (x^2+1)(v''y_1 + 2v'y_1') - 2xv'y_1$$
$$= (x^2+1)(xv'' + 2v') - 2x^2v'$$

$$0 = xv'' + 2v' - [2x^2/(x^2+1)]v'$$

$$0 = v'' + [2/x - 2x/(x^2+1)]v' = w' + [2/x - 2x/(x^2+1)]w$$

$$\phi = x^2 e^{-\int \frac{1}{u} du} = x^2 e^{-\ln|x^2+1|} = x^2/(x^2+1)$$

$$0 = \left(\frac{x^2}{x^2+1} w\right)'$$

$$A = \frac{x^2}{x^2+1} w \quad \rightarrow \quad w(x) = A\frac{x^2+1}{x^2}, \text{ set } A = 1$$

$$v(x) = \int (1 + x^{-2})dx = x - x^{-1} + B, \text{ set } B = 0$$

$$y_2 = vy_1 = (x - 1/x)(x) = \boxed{x^2 - 1} \qquad \boxed{y(x) = c_1 x + c_2(x^2-1)}$$

3.3.11

$y = vy_1$
$y' = v'y_1 + vy_1'$
$y'' = v''y_1 + 2v'y_1' + vy_1''$

11. Reducing the General Equation One solution of the equation

$$y'' - 2by' + b^2y = 0$$

is $y_1(x) = e^{bx}$. Find a second linearly independent solution. What is the general solution?

$$0 = v''y_1 + 2v'y_1' + vy_1'' - 2b(v'y_1 + vy_1') + b^2vy_1$$
$$= v''y_1 + 2v'y_1' - 2bv'y_1$$
$$= v''e^{bx} + 2bv'e^{bx} - 2bv'e^{bx} = v''e^{bx}$$

$$0 = w' \longrightarrow v(x) = Ax + C, \text{ choose } A = 1, C = 0$$

$$\boxed{\begin{array}{l} y_2 = vy_1 = xe^{bx} \\ y(x) = c_1e^{bx} + c_2xe^{bx} \end{array}}$$

3.3.13 $y'' - y = e^x, \quad y_1(x) = e^{-x}$

$$e^x = v''y_1 + 2v'y_1' + vy_1'' - vy_1$$
$$= v''y_1 + 2v'y_1' + v(y_1'' - y_1)$$
$$= v''y_1 + 2v'y_1'$$
$$= v''e^{-x} - 2v'e^{-x}$$

$$e^{2x} = v'' - 2v' = w' - 2w$$

$$e^{2x}e^{-2x} = (we^{-2x})'$$

$$x + A = we^{-2x}$$

Set $A = 0$
$w(x) = xe^{2x} + Ae^{2x}$
$v(x) = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + B$
set $B = 0$
$v(x) = \frac{1}{2}e^{2x}(x - \frac{1}{4})$

$$y_2(x) = vy_1 = \frac{1}{2}e^x(x - 1/4)$$

Solving Nonhomogeneous from Homogeneous Ones

It is possible to use the reduction of order method to find a single solution of the nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = f(x) \qquad (16)$$

knowing a nonzero solution $y_1(x)$ of the corresponding homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \qquad (17)$$

We use the same technique and substitute $y(x) = v(x)y_1(x)$ into Eq. (16), finding the unknown function $v(x)$. For Problems 12–17, use this technique to find a solution of the given nonhomogeneous equation given the single solution $y_1(x)$ of the corresponding homogeneous equation.

Nonhomogeneous equation	*Homogeneous solution*
12. $y'' = 1$	$y_1(x) = 1$
13. $y'' - y = e^x$	$y_1(x) = e^{-x}$
14. $y'' + y' = e^x$	$y_1(x) = 1$
15. $y'' + y = \csc x$	$y_1(x) = \sin x$
16. $x^2y'' - xy' + y = x \quad (x \neq 0)$	$y_1(x) = x$
17. $x^2y'' + xy' - y = x \quad (x \neq 0)$	$y_1(x) = x$

The equations in Problems 18–21 are some of the most famous differential equations in physics. Use the given solution $y_1(x)$ to find a second linearly independent solution of these equations.

To find a simpler v, don't set $A = 0$ immediately.

$$v(x) = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + A/2\, e^{2x} + B, \text{ Pick } A = -1/2, B = 0$$

$$v(x) = 1/2\, xe^{2x} \rightarrow \boxed{y_2(x) = x/2\, e^x}$$

3.3.15 $\quad y'' + y = \csc x, \quad y_1(x) = \sin x$ satisfies $y'' + y = 0$

$$\begin{aligned}
\csc x &= v''y_1 + 2v'y_1' + vy_1'' + vy_1 \\
&= v''y_1 + 2v'y_1' + v(y_1'' + y_1) \\
&= v''y_1 + 2v'y_1' \\
&= \sin x\, v'' + 2\cos x\, v' \\
&= \sin x\, w' + 2\cos x\, w
\end{aligned}$$

$$\begin{aligned}
\csc^2 x &= w' + 2\cot x\, w \\
1 &= (w \sin^2 x)' \\
w(x) &= x\csc^2 x + A\csc^2 x \\
v(x) &= \int x\csc^2 x\, dx + A\int \csc^2 x\, dx \\
&= \left\{-x\cot x + \int \cot x\, dx\right\} - A\cot x + B \\
&= -x\cot x + \ln|\sin x| - A\cot x + B
\end{aligned}$$

$$\phi = e^{2\int \cot x\, dx} = \sin^2 x$$

Set $A = B = 0$

$$y_2 = \sin x \cdot v = \boxed{-x\cos x + \sin x \ln|\sin x|}$$

Notes: Error in the solution given in the textbook

$$d/dx \cot x = d/dx \frac{\cos x}{\sin x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\csc^2 x$$

3.3.17 $\quad x^2y'' + xy' - y = x, \quad x \neq 0, \quad y_1(x) = x$ satisfies $x^2y_1'' + xy_1' - y = 0$

$$\begin{aligned}
x &= x^2(v''y_1 + 2v'y_1' + y_1'') + x(v'y_1 + vy_1') - vy_1 \\
&= x^2(v''y_1 + 2v'y_1') + xv'y_1 \\
&= x^2(v''x + 2v') + x^2v' \\
&= x^3v'' + 3x^2v' = x^3w' + 3x^2w = (x^3w)'
\end{aligned}$$

$$\begin{aligned}
x^3 w &= \tfrac{1}{2}x^2 + c \qquad \text{Choose } c = 0 \\
v' = w &= \tfrac{1}{2}x^{-1} \\
v &= \tfrac{1}{2}\ln|x| + d \qquad \text{Choose } d = 1 \\
y_2 &= vy_1 = \boxed{\tfrac{1}{2}x\ln|x|}
\end{aligned}$$

Check: $\quad y_2' = \frac{1}{2}\ln|x| + \frac{1}{2}$

$\qquad\quad y_2'' = \frac{1}{2}\frac{1}{x}$

$$x^2y_2'' + xy_2' - y_2 = \tfrac{1}{2}x + \tfrac{1}{2}x\ln|x| + \tfrac{1}{2}x - \tfrac{1}{2}\ln|x| = x \;☺$$

3.3.19 $\quad$ Chebychev's Equation of Order 1

$$(1-x^2)y'' - xy' + y = 0, \quad y_1(x) = x$$

$$0 = (1-x^2)(v''y_1 + 2v'y_1' + y_1'') - x(v'y_1 + vy_1') + vy_1 = (1-x^2)(v''y_1 + 2v'y_1') - xv'y_1$$

$$0 = (1-x^2)(xv'' + 2v') - x^2v'$$

$$0 = v'' + \left(2/x - x/(1-x^2)\right)v' = w' + \left(2/x - x/(1-x^2)\right)w$$

$$\phi = e^{2\ln|x|}\, e^{-\int \frac{-1/2}{u}du} = x^2\left(e^{1/2 \ln|1-x^2|}\right) = x^2\sqrt{1-x^2}$$

$$0 = \left(x^2\sqrt{1-x^2}\, w\right)'$$

$A = x^2\sqrt{1-x^2}\, v'(x)$, set $A = 1$

$$v(x) = \int \frac{1}{x^2\sqrt{1-x^2}}\, dx \qquad \boxed{y_2(x) = x\int \frac{x}{x^2\sqrt{1-x^2}}\, dx}$$

Section 3.3

1. $y = c_1e^x + c_2e^{-x}$ 2. $y = c_1 \sin x + c_2 \cos x$ 3. $y = c_1e^{2x} + c_2xe^{2x}$ 4 $y = c_1 + c_2e^{-x}$ 5. $y = c_1 + c_2 \ln x$ 6. $y = c_1e^{2x} + c_2(2x^2 + 2x + 1)$ 7. $y = c_1x^3 + c_2x^{-2}$ 8. $y = c_1x + c_2x\ln x$ 9. $y = c_1x + c_2(x^2 - 1)$

10. $y = c_1\frac{1}{\sqrt{x}}\sin x + c_2\frac{1}{\sqrt{x}}\cos x$ 11. $y_2(x) = xe^{bx}, y(x) = c_1e^{bx} + c_2xe^{bx}$ 12. $y = \frac{x^2}{2}$ 13. $y = \frac{x}{2}e^x$

14. $y = \frac{1}{2}e^x$ 15. $y = -x\cos x - \sin x \ln|\sin x|$ 16. $y = \frac{x}{2}\ln^2 x$ 17. $y = \frac{x}{2}\ln x$

18. $y = (1 - 2x^2)\int (1 - 2x^2)^{-2} e^{-x^2}\, dx$ 19. $y = x\int \frac{1}{x^2\sqrt{1 - x^2}}\, dx$ 20. $y = (x - 1)\int \frac{xe^{-x}}{(x - 1)^2}\, dx$

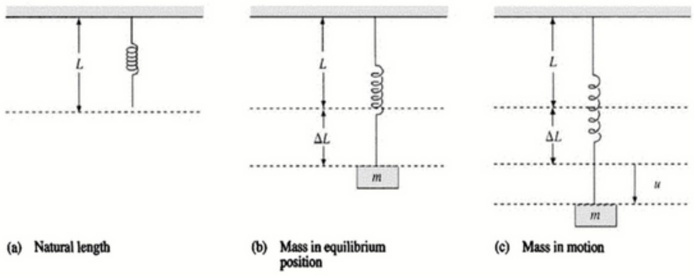
21. $y = \frac{\cos x}{\sqrt{x}}$ 22. $y'' - 3y' + 2y = 0, y_1(x) = e^x, y_2(x) = e^{2x}$ 23. $y'' - y = 0, y_1(x) = e^x, y_2(x) = e^{-x}$

25. (c) $y_2(x) = e^{-x}$ 26. (b) e^x, e^{-x} (c) $y = c_1x + c_2xe^x + c_3xe^{-x}$

§ 3.9 Mechanical Systems and Simple Harmonic Motion

Forces acting on the mass:

Figure 3.7
Simple mass-spring system



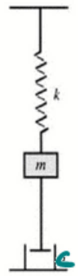
1. $F_g = mg$ gravitational force

2. $F_s = -k(\Delta L + u)$ restoring force of the linear[†] spring, $k > 0$.

3. $F_c = -c\dot{u}$ damping force is the resistance/friction the medium exerts on the mass acting in the direction opposite to the velocity, $c > 0$.[††]

4. $F_e = F(t)$ external force(s) applied upward/downward to the mass or the spring mount.

Figure 3.8
Mass-spring system with damping



By Newton's Second Law,

$$
\begin{aligned}
m\ddot{u} = F_g + F_s + F_c + F_e &= mg - k(\Delta L + u) - c\dot{u} + F(t) \\
&= mg - k\Delta L - ku - c\dot{u} + F(t) \\
&= -ku - c\dot{u} + F(t) \qquad (k\Delta L = mg)^{\dagger\dagger\dagger}
\end{aligned}
$$

$\boxed{m\ddot{u} + c\dot{u} + ku = F(t)}$ Unforced: $F(t) \equiv 0$ Undamped: $c = 0$
Forced: $F(t) \not\equiv 0$ Damped: $c > 0$

† Most springs are not actually linear but rather "hard springs" meaning that it takes increasingly more force to stretch the spring as u increases.
†† For small velocities damping is proportional to $\dot{u}$ ($F_c = -c\dot{u}$) but for larger velocities damping may be proportional to the square of velocity ($F_c = -c\dot{u}^2$) or some other function of $\dot{u}$.
††† If the mass is allowed to rest at equilibrium $u = \dot{u} = \ddot{u} = F(t) = 0 \longrightarrow 0 = mg - k\Delta L$

Consider the case of unforced, undamped motion $m\ddot{u} + ku = 0$, which can be written $\ddot{u} + \omega_0^2 u$, $\omega_0 = \sqrt{k/m}$. The general solution can be written:

$$
\begin{aligned}
u(t) &= c_1 \cos\omega_0 t + c_2 \sin\omega_0 t \\
&= A\left(c_1/R \cos\omega_0 t + c_2/R \sin\omega_0 t\right) \qquad A = \sqrt{c_1^2 + c_2^2} \\
&= A(\cos\delta \cos\omega_0 t + \sin\delta \sin\omega_0 t) \qquad \cos\delta = c_1/R, \ \sin\delta = c_2/R \\
&= A\cos(\omega_0 t - \delta) \qquad \cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi
\end{aligned}
$$

A: amplitude
δ: phase angle (note $\tan\delta = c_2/c_1$)
ω_0: circular frequency $\longrightarrow$ natural frequency: $f = \omega_0/2\pi \longrightarrow$ period: $T = 2\pi/\omega_0$

For Problems 1–5, find the simple harmonic motion described by the given initial-value problem. Determine the amplitude, phase angle, frequency, and period of the motion.

1. $\ddot{u} + u = 0 \quad u(0) = 1 \quad \dot{u}(0) = 0$
2. $\ddot{u} + u = 0 \quad u(0) = 0 \quad \dot{u}(0) = 1$
3. $\ddot{u} + u = 0 \quad u(0) = 1 \quad \dot{u}(0) = 1$
4. $\ddot{u} + 9u = 0 \quad u(0) = 1 \quad \dot{u}(0) = 1$
5. $\ddot{u} + 4u = 0 \quad u(0) = 1 \quad \dot{u}(0) = -2$

Section 3.9

1. $u = \cos t, \left(R = 1, \delta = 0, f = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}, T = 2\pi\right)$ **2.** $u = \sin t, \left(R = 1, \delta = \frac{\pi}{2}, f = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}, T = 2\pi\right)$
3. $u = \sqrt{2}\cos\left(t - \frac{\pi}{4}\right), \left(R = \sqrt{2}, \delta = \frac{\pi}{4}, f = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}, T = 2\pi\right)$
4. $u = \sqrt{2}\cos\left(3t - \frac{\pi}{4}\right), R = \sqrt{2}, \delta = \frac{\pi}{4}, f = \frac{\omega_0}{2\pi} = \frac{3}{2\pi}, T = \frac{2\pi}{3}\Big)$
5. $u = \sqrt{2}\cos\left(2t - \frac{7\pi}{4}\right), \left((R = \sqrt{2}, \delta = \frac{7\pi}{4}, f = \frac{\omega_0}{2\pi} = \frac{1}{\pi}, T = \pi\right)$ **6.** (a) $\ddot{u} + 64u = 0 \quad u(0) = \frac{1}{4} \quad \dot{u}(0) = 1$
(b) $u = \frac{\sqrt{3}}{8}\left(\frac{2\sqrt{3}}{5}\cos 8t + \frac{\sqrt{3}}{5}\sin 8t\right) \doteq 0.28\cos(8t - 0.46)$
(c) $R = 0.28$ ft, $\delta = 0.46, f = \frac{\omega_0}{2\pi} = \frac{4}{\pi}$ oscillations/sec, $T = \frac{\pi}{4}$ sec **7.** (a) $\ddot{u} + 64u = 0 \; u(0) = \frac{1}{3} \; \dot{u}(0) = -4$
(b) $\dot{u} + 64u = 0 \; u(0) = -\frac{1}{6} \; \dot{u}(0) = 1$
8. (a) $u = \frac{1}{6}\cos 16t$ (b) $R = \frac{1}{6}$ ft, $T = \frac{\pi}{8}$ sec, $f = \frac{8}{\pi}$ oscillations/sec (c) $\frac{\pi}{32}$ sec, $v = -\frac{8}{3}$ ft/sec
9. $u = -\frac{1}{3}\cos 8t + \frac{1}{4}\sin 8t \doteq \frac{5}{12}\cos(8t - 2.5)$ **10.** (a) $\sqrt{2}\cos\left(t - \frac{\pi}{4}\right)$ (b) $\sqrt{2}\cos\left(t - \frac{7\pi}{4}\right)$ (c) $\sqrt{2}\cos\left(t - \frac{3\pi}{4}\right)$
(d) $\sqrt{2}\cos\left(t - \frac{5\pi}{4}\right)$ **11.** $R = \sqrt{c_1^2 + c_2^2}, \sin\delta = -\frac{c_1}{R}, \cos\delta = \frac{c_2}{R}\left(\tan\delta = -\frac{c_1}{c_2}\right)$
15. Periods are the same. **16.** $T = 2\pi\sqrt{\frac{L}{g}}$ **18.** $f = \frac{1}{2\pi}\sqrt{\frac{kR^2}{mR^2 + I}}$ cycles/unit time **20.** 653 lb
21. (a) $x(t) = \sqrt{R^2 - d^2}\cos\sqrt{\frac{g}{R}}t$ (b) 2552 seconds (42 min)

<u>3.9.1</u> $\ddot{u} + u = 0, \; u(0) = 1, \; \dot{u}(0) = 0$

$\lambda^2 + 1 = 0 \longrightarrow \lambda = \pm i$
$u(t) = c_1\cos t + c_2\sin t$
$1 = u(0) = c_1$
$0 = \dot{u}(0) = c_2$
$\boxed{u(t) = \cos t} = 1\cdot\cos(1\cdot t - 0)$

$\boxed{\begin{array}{l} A = 1 \\ \delta = 0 \\ \omega_0 = 1 \\ f = 1/2\pi \\ T = 2\pi \end{array}}$

<u>3.9.5</u> $\ddot{u} + 4u = 0, \; u(0) = 1, \; \dot{u}(0) = -2$

$\lambda^2 + 4 = 0 \rightarrow \lambda = \pm 2i$
$u(t) = c_1\cos 2t + c_2\sin 2t$
$1 = u(0) = c_1$
$-2 = \dot{u}(0) = 2c_2$

$\boxed{\begin{aligned} u(t) &= \cos t - \sin t \\ &= \sqrt{1^2 + (-1)^2}\cos\left(\sqrt{4/1}\, t - \arctan(-1/1)\right) \\ &= \sqrt{2}\cos(2t + \pi/4) \end{aligned}}$

$\boxed{\begin{array}{l} A = \sqrt{2} \\ \delta = \pi/4 \\ \omega_0 = 2 \\ f = 1/\pi \\ T = \pi \end{array}}$

<u>3.9.7</u> $m = w/g = 16\,\text{lbs}/32\,\text{ft/s}^2 = 1/2$ slug
$w = mg = k\,\Delta L \rightarrow k = w/\Delta L = 16\,\text{lbs}/\tfrac{1}{2}\,\text{ft} = 32\,\text{lbs/ft}$
$c = 0$ (no friction)
$F(t) = 0$ (no forcing)

7. Finding the Initial-Value Problem A 16-lb weight is attached to a frictionless spring, which in turn is suspended from the ceiling. The weight stretches the spring 6 inches and comes to rest in its equilibrium position. Find the initial-value problem that describes the motion of the weight under the following conditions.
(a) The weight is pulled down 4 inches below its equilibrium position and released with an upward initial velocity of 4 ft/sec.
(b) The weight is pushed up 2 inches and released with a downward velocity of 1 ft/sec.

$0 = m\ddot{u} + c\dot{u} + ku = \tfrac{1}{2}\ddot{u} + 32u$

a) $\boxed{0 = \ddot{u} + 64u, \; u(0) = 1/3, \; \dot{u}(0) = -4}$
b) $\boxed{0 = \ddot{u} + 64u, \; u(0) = -1/6, \; \dot{u}(0) = 1}$

<u>3.9.9</u> $m = w/g = 12/32 = 3/8, \quad k = w/\Delta L = 12/\tfrac{1}{2} = 24$

ODE : $0 = \ddot{u} + 8u$ (from $0 = 3/8\,\ddot{u} + 24u$)
IC : $u(0) = -1/3, \; \dot{u}(0) = -2$

9. Finding Simple Harmonic Motion A 12-lb weight is attached to a frictionless spring, which in turn is attached to the ceiling. The weight stretches the spring 6 inches before coming to equilibrium. Find the equation of motion of the weight if it is initially pushed upward to a position 4 inches above equilibrium and given an upward velocity of 2 ft per sec.

$0 = \lambda^2 + 64 \longrightarrow \lambda = \pm 8i$
$u(t) = c_1\cos 8t + c_2\sin 8t$
$-1/3 = u(0) = c_1, \; -2 = \dot{u}(0) = 8c_2 \longrightarrow \boxed{u(t) = -\tfrac{1}{3}\cos 8t - \tfrac{1}{4}\sin 8t}$

10. Simple Harmonic Motion in Polar Form Rewrite the following simple harmonic motions in the new polar form $R\cos(\omega_0 t - \delta)$.
(a) $\cos t + \sin t$
(b) $\cos t - \sin t$
(c) $-\cos t + \sin t$
(d) $-\cos t - \sin t$

<u>3.9.13</u> In the case of no friction ($c = 0$), the spring-mass system is conservative.

Typo: $KE = \tfrac{1}{2}m\dot{u}^2$

13. Simple Harmonic Motion from Conservation of Energy The conservation of energy says that if no energy is lost through friction, then in any mechanical system the sum of the kinetic and potential energies remains constant over time. Use the fact that the kinetic energy of a vibrating spring is $KE = 1/2m\ddot{u}^2$ and the potential energy is given by $PE = 1/2ku^2$ to derive the differential equation

$$m\ddot{u} + ku = 0 \tag{15}$$

Hint: Differentiate the equation $KE + PE =$ constant with respect to time.

Total Energy $= E(t) = KE + PE = \tfrac{1}{2}m\dot{u}^2 + \tfrac{1}{2}ku^2 =$ constant

$\dot{E} = m\dot{u}\ddot{u} + ku\dot{u} = 0 \rightarrow \dot{u}(m\ddot{u} + ku) = 0$. This holds for all $t \geq 0$
Either $\dot{u} \equiv 0$ or $m\ddot{u} + ku \equiv 0$. The first case is just equilibrium.
$\therefore m\ddot{u} + ku = 0$.

§3.10 Unforced Damped Vibrations

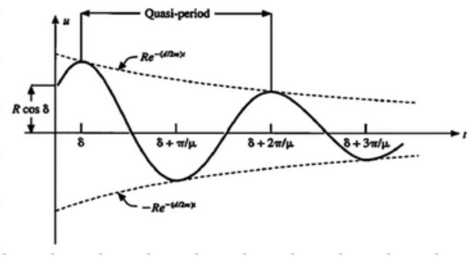
In this section damping proportional to the first power of velocity is studied. Under this assumption the motion of a mass attached to a damped vibrating string follows the equation $m\ddot{u} + c\dot{u} + Ku = 0$. The correspond characteristic equation has roots:

$$\lambda = -c/2m \pm \sqrt{c^2 - 4mK}\,/\,2m$$

1. $c^2 - 4mK < 0$ The 'underdamped' case produces damped oscillatory motion:

$$u(t) = e^{-(c/2m)t}\left[c_1 \cos\omega t + c_2 \sin\omega t\right],$$

$$= Ae^{-(c/2m)t}\cos(\omega t - \delta),\ A = \sqrt{c_1^2 + c_2^2},\ \tan\delta = c_2/c_1$$



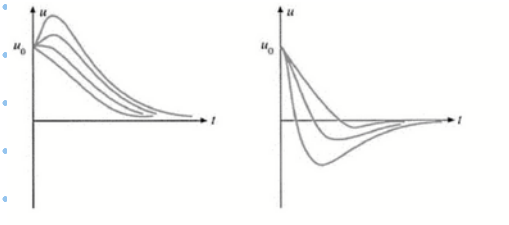
$\delta = \arctan c_2/c_1$: phase angle
$\omega = \sqrt{4mK - c^2}\,/2m$: damped circular frequency
$f = \omega/2\pi$: damped natural frequency
$T = 2\pi/\omega = 4\pi m/\sqrt{4mK - c^2}$: damped period

2. $c^2 - 4mK = 0$ The 'critically damped' case occurs when the characteristic equation has a repeated root and produces critically damped motion:

$$u(t) = (c_1 + c_2 t)e^{-(c/2m)t}$$

To show that u has at most one extremum,

$$0 = \dot{u}(t) = c_2 e^{-c/2m\,t} - c/2m\,(c_1 + c_2 t)e^{-c/2m\,t}$$
$$0 = c_2 - cc_1/2m - cc_2/2m\ t$$
$$t = (c_2 - cc_1/2m)(-2m/cc_2) = c_1/c_2 - 2m/c$$



(a) If the initial slope is positive or not too negative, the curve will *not* cross the $u = 0$ line but only approach it.

(b) If the initial slope is sufficiently negative, the curve will cross the $u = 0$ line once.

$\therefore$ $u(t)$ has one critical point $t = c_1/c_2 - 2m/c$ and so at most one extremum with $t > 0$ (depending on the values of c_1, c_2, m, and c).

3. $c^2 - 4mK > 0$ The 'overdamped' case occurs when the resistance c is large relative to the spring constant K. There are no oscillations with overdamped motion:

$$u(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t},\quad \lambda_1 = -c/2m + \sqrt{c^2 - 4mK}\,/2m,\ \lambda_2 = -c/2m - \sqrt{c^2 - 4mK}\,/2m$$

Note that $\lambda_2 < 0$ and also since $c^2 - 4mK < c^2$, $\lambda_1 < -c/2m + \sqrt{c^2}/2m = 0$ as well. Since both $\lambda_1, \lambda_2 < 0$, $u(t) \to 0$ as $t \to \infty$. Setting $\dot{u}(t) = 0$ shows that $u(t)$ has at most one extremum with $t > 0$. $\therefore$ The qualitative possibilities match case 2.

PROBLEMS: Section 3.10

Properties of Damped Oscillations

For Problems 1–4, determine the damped amplitude, the damped natural frequency, the damped period, and the time constant. Sketch the graphs of the functions.

1. $u(t) = 2e^{-t}\cos\left(2t - \pi\right)$

2. $u(t) = 3e^{-2t}\cos\left(\sqrt{3}t - \frac{\pi}{3}\right)$

3. $u(t) = 5e^{-0.25t}\cos\left(t + \pi\right)$

4. $u(t) = e^{-2t}\cos\left(4t - \pi\right)$

Underdamped, Critically Damped, or Overdamped?

For Problems 5–11, determine whether the motion described by the given differential equation is underdamped, critically damped, or overdamped.

5. $\ddot{u} + 4u = 0$
6. $\ddot{u} - 2\dot{u} + u = 0$
7. $\ddot{u} + 4\dot{u} + 4u = 0$
8. $\ddot{u} + 2d\dot{u} + d^2u = 0 \quad (d > 0)$
9. $\ddot{u} + 2d\dot{u} + k^2u = 0 \quad (d > 0 \text{ and } d^2 = k^2)$
10. $\ddot{u} + 2d\dot{u} + ku = 0 \quad (d^2 > k \text{ and } k < 0)$
11. $\ddot{u} + d\dot{u} + u = 0 \quad (0 < d < 2)$

12. Unforced Damped Vibrations A 32-lb weight is attached to the lower end of a coil spring, which in turn is suspended from the ceiling. The weight stretches the spring 2 ft in the process. The weight is then pulled down 6 inches below its equilibrium position and released. The resistance of the medium is given as 4 lb/(ft/sec).
(a) Determine the motion of the weight.
(b) Determine the damped amplitude, damped frequency, damped period, and time constant of the subsequent motion.
(c) Sketch the graph of the motion.

13. Unforced Damped Vibrations A 10-lb weight is attached to a spring suspended from the ceiling. When the weight comes to rest, the spring is stretched by 1.5 inches. The damping constant for the system is $5\sqrt{2}$ lbs/(ft/sec). The weight is pulled down 3 inches from the equilibrium position and released.
(a) Determine the motion of the weight.
(b) Determine the damped amplitude, damped natural frequency, and damped period of the subsequent motion.
(c) Sketch the graph of the motion.

14. Maximum Displacement An 8-lb weight stretches a spring 6 ft, thereby reaching its equilibrium position. Assuming a damping constant for the system of 4 lb/(ft/sec), the weight is pulled down 3 inches below its equilibrium position and given a downward velocity of 2 ft/sec. When will the mass attain its maximum displacement below equilibrium? What is the maximum displacement? Sketch the graph of the motion.

15. Finding the Differential Equation for Observations John starts in motion a vibrating mass attached to a spring. He observes that after 10 seconds the damping amplitude has decreased by 75% and that the damped period is 2 seconds. How does John determine the differential equation of motion from these two observations?

16. Interesting Phenomena Show that if the motion of a mass that is described by

$$m\ddot{u} + d\dot{u} + ku = 0 \tag{21}$$

is critically damped or overdamped, then the mass can pass through the equilibrium position *at most once*, regardless of the initial conditions.

17. Decrease in the Velocity For an underdamped mass-spring system, show that if v is the velocity of the mass at *any time*, then the velocity one (damping) period later is $ve^{-(d/2m)T_d}$.

18. System Identification of the Damping Constant d It is generally almost impossible to measure the damping constant d directly. One of the important uses of differential equations is to find system parameters such as this *indirectly*. Starting with the equation for underdamped motion

$$u(t) = Re^{-(d/2m)t}\cos(\mu t - \delta) \tag{22}$$

verify the following mathematical steps that will allow one to estimate d for underdamped motion.
(a) Show that the ratio of two successive maximum displacements at time t and $t + T_d$ is $e^{(d/2m)T_d}$. Thus successive maximum displacements form a geometric sequence with ratio $e^{-(d/2m)T_d}$.
(b) Show that the natural logarithm of this ratio, called the **logarithmic decrement,** is given by

$$\Delta = \frac{d}{2m}T_d = \frac{\pi d}{m\mu} \tag{23}$$

(c) How would you use the logarithmic decrement to find the damping constant d?

19. Need New Shocks? The vibrations of an automobile weighing 1600 lb (50 slugs) are controlled by a spring-shock system in which the shock absorbers provide the damping. Suppose the spring constant is $k = 4800$ lb/ft and the dashpot mechanism provides a damping constant of $c = 200$ lb-slug/ft. Find the speed (in miles per hour) at which resonance vibrations will occur if the car is driven on a washboard road surface with an amplitude of 2 inches and a wavelength of $L = 30$ ft. See Figure 3.20.