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TOPICS FOR TODAY

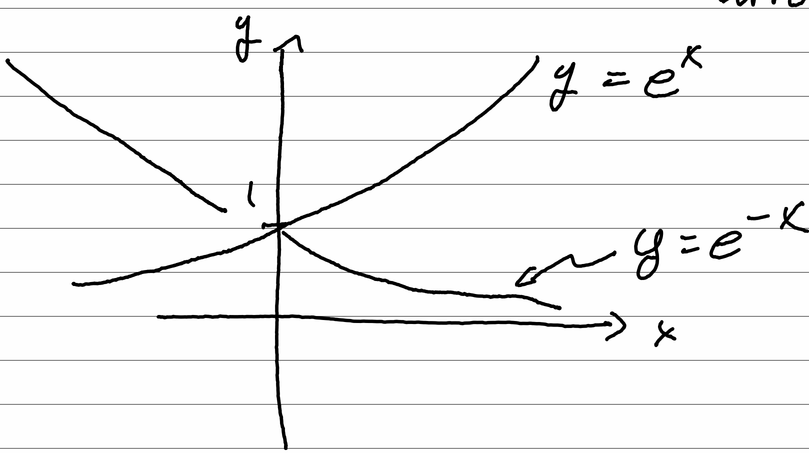
- REVIEW OF e^x AND $\ln|x|$

e^x: EXPONENTIAL FUNCTION

$e \approx 2.7172....$ TRANSCENDENTAL NUMBER

THIS FUNCTION IS SPECIAL, SINCE

$$\frac{d}{dx}\{e^x\} = e^x$$: IDENTICAL UNDER DIFFERENTIATION



NOTE:
- $e^x > 0$ FOR ALL $-\infty < x < \infty$
- $e^{a+b} = e^a e^b$
- $e^{-a} = 1/e^a$
- $e^{na} = (e^a)^n$

- $e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

EX: SOLVE $\frac{dy}{dx} = e^x, \quad y(3) = 2$

$$\int \frac{dy}{dx}\,dx = \int e^x\,dx$$

$$\Rightarrow y = e^x + C_1$$

$$y(3) = e^3 + C_1 = 2$$

$$\Rightarrow C_1 = 2 - e^3$$

$$\therefore y(x) = (e^x - e^3) + 2$$

TRY:

$$\frac{dy}{dx} = e^{-x}, \quad y(0) = 1$$

$$\frac{dy}{dx} = e^{-4x}, \quad y(-1) = 2$$

$$\frac{dy}{dx} = 7e^x, \quad y(0) = 4$$

NATURAL LOGARITHM

$$\ln(x) = \int_1^x \frac{1}{t}\,dt, \quad x > 0$$

$$\Rightarrow \frac{d[\ln(x)]}{dx} = \frac{1}{x}, \quad x > 0$$

WE CAN EXTEND THIS DEFINITION FOR $X \neq 0$:

$$\frac{dy}{dx} = \frac{1}{x}, \quad y(x_0) = y_0, \; x_0 \neq 0$$

$$\int \frac{dy}{dx}\,dx = \int \frac{1}{x}\,dx = \ln|x| + c$$

$$y = \ln|x| + c$$

$$y(x_0) = \ln|x_0| + c_1 = y_0$$

$$\Rightarrow c_1 = y_0 - \ln|x_0|$$

$$\therefore y = y_0 + \ln|x| - \ln|x_0|$$

$$= \underline{y_0 + \ln\left|\frac{x}{x_0}\right|}$$

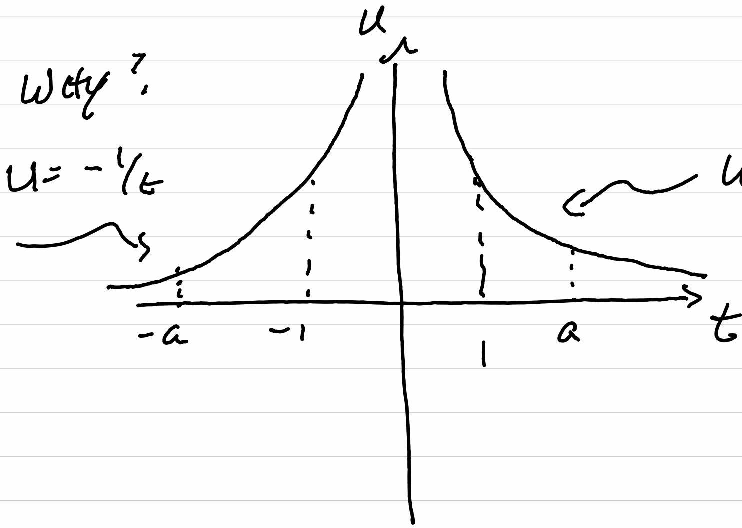
NOTE: $\bullet \ln(a/b) = \ln(a) - \ln(b), \quad a > 0, b > 0$

$$\ln(1) = 0$$

$$\ln(e^x) = x \Rightarrow e^{\ln x} = x$$

$$\therefore \ln|x| = \begin{cases} \int_1^x \frac{1}{t}\,dt, & x>0 \\ \int_x^{-1} -\frac{1}{t}\,dt, & x<0 \end{cases}$$

Why?

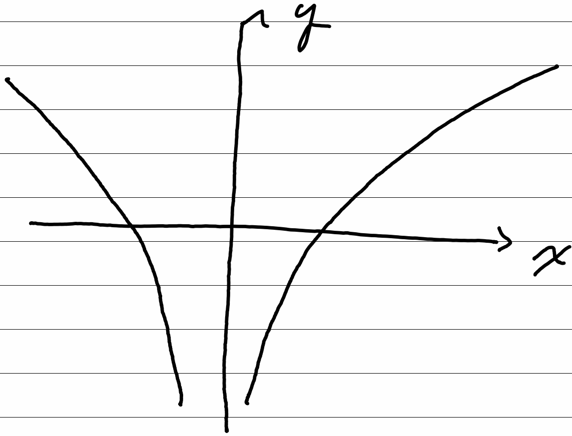


ex: $\int_x^{-1} \frac{-1}{t}\,dt, \quad x<0$

$u = -t$
$du = -dt$

$$\int_x^{-1} \frac{-dt}{t} = \int_{-x}^{1} \frac{du}{-u} = \int_1^{-x} \frac{du}{u}$$

$$= \int_1^{|x|} \frac{du}{u} = \underline{\ln|x|}$$



$y = \ln|x|$

$\Rightarrow$ TWO BRANCHES

BRANCH DEPENDS ON INITIAL DATA

<u>EX</u>: $\frac{dy}{dx} = \frac{1}{x}$, $y(2) = 1$

$$\int \frac{dy}{dx}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$$

$$y = \ln|x| + C$$

$$y(2) = \ln|2| + C = 1$$

$$\therefore C = 1 - \ln|2| = 1 - \ln(2)$$

$$y(x) = 1 + \ln\left|\frac{x}{2}\right| = \underline{1 + \ln\left|\frac{x}{2}\right|}$$

<u>TRY</u>:

- $\frac{dy}{dx} = \frac{2}{x}$, $y(-3) = 9$

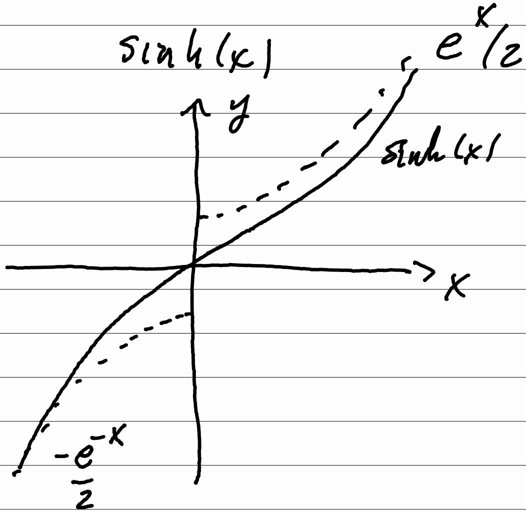
- $\frac{dy}{dx} = -\frac{1}{x}$, $y(4) = 2$

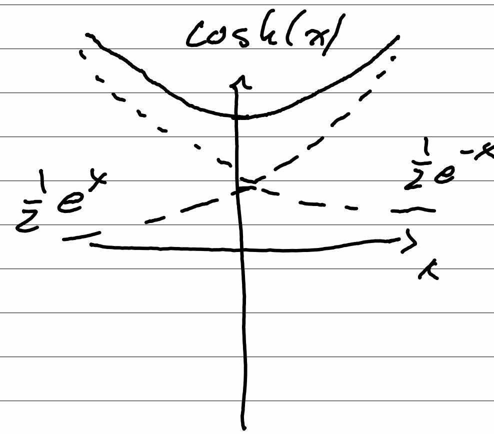
IF THERE IS TIME ...

HYPERBOLIC FUNCTIONS

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \text{(PRONOUNCED "sinch")}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad \text{(PRONOUNCED "cosh")}$$





As $x \to \infty$: $\sinh(x) \sim \frac{1}{2}e^x$

$\cosh(x) \sim \frac{1}{2}e^x$

$x \to -\infty$ $\sinh(x) = -\frac{1}{2}e^{-x}$

$\cosh(x) = \frac{1}{2}e^{-x}$

$$\frac{d}{dx}[\sinh x] = \frac{d}{dx}\left(\frac{1}{2}e^x - \frac{e^{-x}}{2}\right) = \frac{e^x}{2} + \frac{e^{-x}}{2} = \cosh(x)$$

$$\frac{d}{dx}[\cosh(x)] = \frac{d}{dx}\left[\frac{e^{x}}{2} + \frac{e^{x}}{2}\right] = \frac{e^{x}}{2} - \frac{e^{x}}{2}$$

$$= \sinh(x)$$

$$\therefore \frac{d}{dx}[\sinh x] = \cosh(x)$$

$$\frac{d}{dx}[\cosh x] = \sinh(x)$$

Ans

$$[\sinh x]^2 = \left[\frac{1}{2}e^{x} - \frac{e^{-x}}{2}\right]^2$$

$$= \frac{1}{4}e^{2x} - \frac{1}{4} - \frac{1}{4} + \frac{1}{4}e^{2x}$$

$$= \frac{1}{4}[e^{2x} + e^{-2x}] - \frac{1}{2}$$

$$[\cosh x]^2 = \left[\frac{1}{2}e^{x} + \frac{1}{2}e^{-x}\right]^2$$

$$= \frac{1}{4}e^{2x} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}e^{-2x}$$

$$= \frac{1}{4}(e^{2x} + e^{-2x}] + \frac{1}{2}$$

So $\boxed{\sinh^2(x) - \cosh^2(x) = -1 \quad \text{or} \quad \cosh^2 x - \sinh^2 x = 1}$

TAYLOR SERIES

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$\therefore \sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

LOOKS A LOT LIKE $\sin(x)$, $\cos(x)$ ...

TRY :
- $\frac{dy}{dx} = \cosh(x)$, $y(0) = 1$
- $\frac{dy}{dx} = \sinh(x)$, $y(2) = \cosh(2)$
- $\frac{dy}{dx} = e^{-x} + 2\sinh x$, $y(0) = 2$