# Section 2.4: Mixing Phenomena

Conference 2
MA2051 WPI
11/9/2021
Exam 1 Review

**PROBLEMS: Section 2.4**

**Mixing Problems**

1. **Standard Tank Problem** Initially, 50 pounds of salt are dissolved in a 300-gallon tank, and then a salt solution with a concentration of 2 pounds of salt per gallon of solution flows into a tank at a rate of 3 gal/min. The solution inside the tank is kept well-stirred and flows out of the tank at the same rate as that at which it flows in.
   (a) Find the amount of salt in the tank at any time.
   (b) Find the concentration of salt in the tank at any time.
   (c) Find the limiting amount of salt in the tank.
   (d) Find the limiting concentration of the salt in the tank.
2. **Standard Tank Problem** Initially, a 100-liter tank contains a salt solution that has a concentration of 0.5 kg/liter. A less concentrated salt solution with a concentration of 0.1 kg/liter flows into a tank at a rate of 4 liters/min. The solution inside the tank is kept well-stirred and flows out of the tank at the same rate as that at which it flows into the tank.
   (a) Find the amount of salt in the tank at any time.
   (b) Find the concentration of salt in the tank at any time.
   (c) Find the limiting amount of salt in the tank.
   (d) Find the limiting concentration of salt in the tank.
3. **Rate In > Rate Out** Initially, a large tank with a capacity of 100 gallons contains 50 gallons of pure water. A salt solution with a concentration of 0.1 lb/gal flows into the tank at a rate of 4 gal/min. The mixture is kept well-stirred and flows out of the tank at the rate of 2 gal/min.
   (a) Find the initial-value problem that describes the amount of salt in the tank.
   (b) Find the amount of salt in the tank until the tank overflows.
   (c) Find the concentration of salt in the tank until the tank overflows.
   (d) What is the initial-value problem that describes the amount of salt in the tank after the tank overflows?

2.(a) $\dot{Q} = 0.4 - \frac{1}{25}Q$

$Q(t) = 10 + Ce^{-t/25}$

$50 = Q(0) = 10 + C \rightarrow C = 40$

$\boxed{Q(t) = 10 + 40e^{-t/25}}$

(c) $Q_\infty(t) = \lim_{t\to\infty} Q(t) = 10 + 0$

$\boxed{Q_\infty(t) = 10}$

3.(a) $\dot{Q} = (.1)\cdot 4 - \frac{2Q}{50+(4-2)t}$

$\boxed{\dot{Q} + \frac{Q}{t+25} = 2/5, \quad Q(0) = 0}$

(b) $\mu(t) = e^{\int (t+25)^{-1}dt} = t+25$

$(t+25)\dot{Q} + Q = 2/5\, t + 10$

$\boxed{Q(t) = \frac{t^2/5 + 10t}{t+25} \quad 0 \le t \le 25}$

5. **Interesting Problem** Initially, 5 pounds of salt are dissolved in 20 gallons of water. A salt solution with a concentration of 2 lb/gal is added to the tank at a rate of 3 gal/min, and the well-stirred mixture is drained from the tank at the same rate. How long should this mixture be added in order to raise the amount of salt in the tank to 25 lb?
6. **Hmmmmm** A tank initially contains 200 gallons of fresh water, but thereafter an unknown salt concentration is poured into the tank at the rate of 2 gal/min. The mixture enters the tank and the well-stirred mixture flows out of the tank at the same rate. After 120 min the concentration of salt in the tank is 1.4 lb/gal. What is the concentration of the entering brine?
7. **Correcting a Mistake** Into a tank containing 100 gallons of fresh water, Julie was supposed to add 10 lb of salt but accidentally added 20 lb. To correct her mistake she started adding fresh water at a rate of 3 gal/min while drawing off salt solution from the tank at the same rate. How long will it take until the tank contains the correct amount of salt?

5. $\dot{Q} = 6 - 3/20\, Q, \quad Q(0) = 5$

$Q(t) = 40 - 35e^{-t/20}$

$25 = Q(t^*) = 40 - 35e^{-t^*/20}$

$\boxed{t^* = \frac{20}{3}\ln 7/3 \approx 5.65 \text{ minutes}}$

6. $\dot{Q} = 2\alpha - \frac{1}{100}Q, \quad Q(0) = 0$

$Qe^{t/100} = 200\alpha e^{t/100} + C$

$0 = Q(0)e^{0/100} + 200\alpha e^{0/100} + C$

$Q(t) = 200\alpha - 200\alpha e^{-t/100}$

$1.4 = \frac{Q(120)}{200} = \alpha(1 - e^{-120/100})$

$\boxed{\alpha = \frac{1.4}{1-e^{-6/5}} \approx 2.00342 \text{ lb/gallon}}$

7. $\dot{Q} = -3Q/100, \quad Q(0) = 20$

$Q(t) = 20\exp(-3t/100)$

$10 = Q(t^*) \rightarrow \boxed{t^* = \frac{100\ln 2}{3} \approx 23 \text{ minutes}}$

# Section 2.5: Cooling and Heating Phenomena

**3. Stephan's Law** A law that holds over greater temperature ranges than Newton's Law is **Stephan's Law,** which states that

$$\frac{dT}{dt} = -k(T^4 - M^4) \qquad (14)$$

where $T$, $M$, and $k$ are the same as defined in Newton's Law. Show that the general solution of this equation is

$$\ln\left(\frac{T+M}{T-M}\right) + 2\tan^{-1}\left(\frac{T}{M}\right) = 4M^3kt + c \qquad (15)$$

where $c$ is an arbitrary constant. *Hint*: Separate variables and then factor the fourth-order expression that occurs in the denominator into a product of two quadratic factors. Then find a partial fraction decomposition and integrate.

$$\frac{1}{T^4-m^4}\frac{dT}{dt} = -k$$

$$\int \left\{ \frac{1}{4m^3}\frac{1}{T-m} + \frac{1}{4m^3}\frac{1}{T+m} + \frac{1}{2m^2}\frac{1}{T^2+m^2} \right\} dT = \int -k\,dt$$

$$\int \left\{ -\frac{1}{T-m} + \frac{1}{T+m} + 2m\frac{1}{T^2+m^2} \right\} dT = 4m^3k\int dt$$

$$\ln\left|\frac{T+m}{T-m}\right| + 2\arctan(T/m) = 4m^3k + c$$

## Partial Fractions:

$$\frac{1}{T^4-m^4} = \frac{1}{(T-m)(T+m)(T^2+m^2)}$$

$$= \frac{A}{T-m} + \frac{B}{T+m} + \frac{CT+D}{T^2+m^2}$$

$$1 = A(T+m)(T^2+m^2) + B(T-m)(T^2+m^2) + (CT+D)(T-m)(T+m)$$

1. $T = m \rightarrow A = 1/4m^3$
2. $T = -m \rightarrow B = -1/4m^3$
3. $T = 0 \rightarrow D = -1/2m^2$
4. $T = 2m \rightarrow C = 0$

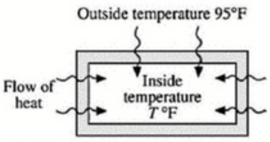


***Figure 2.25*** The heating of Professor Snarf's office

**5. Turning Off the Furnace** Professor Snarf has a very poorly insulated house and in winter keeps the furnace running constantly so that the temperature is kept at 70°F. However, the furnace breaks down at midnight with the outside temperature at 10°F, and after 30 minutes the inside temperature drops to 50°F.

(a) What is the temperature in Professor Snarf's house after 1 hour?

(b) How long will it take for the inside temperature to drop to 15°F?

**6. Temperature Inside Your Refrigerator** Here's how you can find the temperature in your refrigerator *without* actually putting a thermometer in the refrigerator. Take a can of soda from your refrigerator and let it warm for 0.5 hour and then record its temperature. Then let it warm for another 0.5 hour and take a second reading. Assume that you know the room temperature and it is 70°F. What is the temperature in the refrigerator?

**7. Warm or Cold Beer?** A cold beer with an initial temperature of 35°F warms up to 40°F in 10 minutes while sitting in a room of temperature 70°F. What will be the temperature of the beer after $t$ minutes? After 20 minutes?

**8. Case of the Cooling Corpse** In a murder investigation a corpse was found by Inspector Tousteau at exactly 8:00 P.M. Being alert, he measures the temperature of the body and finds it to be 70°F. Two hours later, Inspector Tousteau again measures the temperature of the corpse and finds it to be 60°F. If the room temperature is 50°F, when did the murder occur? See Figure 2.26.

**9. Professor Snarf's Coffee** Professor Snarf always has a cup of coffee before his 8:00 A.M. class. Suppose the temperature of the coffee is 200°F when it is freshly poured at 7:30 A.M. and 15 minutes later it cools to 120°F in a room whose temperature is 70°F. However, Professor Snarf never drinks his coffee until it cools to 90°F. When will Professor Snarf be able to drink his coffee?

**10. The Famous Coffee and Cream Problem** John and Mary are having dinner, and each orders a cup of coffee. John cools his coffee with some cream. They wait 10 minutes, and then Mary cools her coffee with the same amount of cream. The two then begin to drink. Who drinks the hotter coffee?

5.(a) $\dot{T} + kT = 10k$

$T(t) = 10 + Ce^{-kt}$

$70 = T(0) \rightarrow C = 60$

$50 = T(30) \rightarrow k = \frac{1}{30}\ln 3/2$

$T(60) = 10 + 60e^{-2\ln 3/2} = \boxed{\frac{110}{3} = 36.\overline{6}\,°F}$

(b) $15 = T(t^*)$

$$\boxed{\begin{aligned} t^* &= \frac{30\ln 12}{\ln 3/2} \\ &\approx 183.856 \text{ min} \\ &\approx 3.06 \text{ hrs} \end{aligned}}$$

7. $\dot{T} = -k(T-m) = -k(T-70)$, $T(0) = 35$, $T(10) = 40$

$T(t) = T_0 e^{-kt} + m(1-e^{-kt}) = 70 - 35e^{-kt}$

$40 = T(10) = 70 - 35e^{-10k} \longrightarrow -k = \frac{1}{10} \ln 6/7$

$$\boxed{T(t) = 70 - 35e^{-t/10 \ln 7/6} \qquad T(20) = 70 - 35e^{-2\ln 7/6} \approx 44.29\,°F}$$

9. $\dot{T} = -k(T-70)$, $T(0) = 200$

T : temperature in °F
t : minutes since 7:30 AM

$\dot{T} + kT = 70k$

$Te^{kt} = 70e^{kt} + C$

$T(t) = 70 + Ce^{-kt}$

$200 = T(0) = 70 + C \rightarrow C = 130$

$120 = T(15) = 70 + 130e^{-15k}$

$\frac{1}{15} \ln 5/13 = -k$

$T(t) = 70 + 130\exp\left(-\frac{1}{15}\ln(13/5)\, t\right)$

$90 = T(t^*)$

$\ln(2/13) = -\frac{1}{15} \ln(13/5)\, t^*$

$$\boxed{t^* = \frac{15 \ln 13/2}{\ln 13/5} \approx 29.384298}$$

**Approximately 7:59:23 AM**