§ 3.3 Reduction of Order

D'Alembert's reduction of order method.

Suppose y_1 is a solution of $y'' + p(x)y' + q(x)y = 0$. To find another solution y_2 s.t. y_1 and y_2 are linearly independent, look for solutions of the form $y = v(x)y_1$.

$$y' = v'y_1 + vy_1'$$
$$y'' = v''y_1 + v'y_1' + v'y_1' + vy_1''$$

$$\begin{aligned} 0 = y'' + p(x)y' + q(x)y &= v''y_1 + 2v'y_1' + vy_1'' + p(v'y_1 + vy_1') + qvy_1 \\ &= v(y_1'' + py_1' + qy_1) + v''y_1 + 2v'y_1' + pv'y_1 \\ &= v''y_1 + 2v'y_1' + pv'y_1 \end{aligned}$$

$0 = v'' + 2v'y_1'/y_1 + pv' = v'' + (2y_1'/y_1 + p)v'$, (on any interval s.t. $y_1(x) \neq 0$)

$0 = w' + (p + 2\, y_1'/y_1)w$, $w = v'$

$\mu = e^{\int p + 2y_1'/y_1\, dx} = e^{\int p\, dx} e^{\ln y_1^2} = y_1^2 e^{\int p\, dx}$

$y_1^2 w e^{\int p\, dx} = C$

$w = C/y_1^2\, e^{-\int p\, dx}$, set $c = 1$

$w(x) = 1/y_1^2\, e^{-\int p\, dx}$

$\frac{1}{w} w' = -p - 2\, y_1'/y_1$

$w(x) = Ce^{-\int p dx} e^{-2\ln|y_1|}$

$w(x) = 1/y_1^2\, e^{-\int p dx}$ set $c = 1$

$v(x) = \int w\, dx = \int 1/y_1^2\, e^{-\int p dx}\, dx$

$y_2 = vy_1$

$\underline{3.3.1}$ $y''-y=0$, $y_1(x)=e^x$

For Problems 1–10, a differential equation and one solution are given. Use d'Alembert's reduction of order method to find a second linearly independent solution. What is the general solution of the differential equation?

Differential equation	Solution
1 $y''-y=0$	$y_1(x)=e^x$
2. $y''+y=0$	$y_1(x)=\sin x$
3. $y''-4y'+4y=0$	$y_1(x)=e^{2x}$
4. $y''+y'=0$	$y_1(x)=1$
5. $xy''+y'=0$	$y_1(x)=1$
6. $xy''-2(x+1)y'+4y=0$	$y_1(x)=e^{2x}$
7. $x^2y''-6y=0$	$y_1(x)=x^3$
8. $x^2y''-xy'+y=0$	$y_1(x)=x$
9. $(x^2+1)y''-2xy'+2y=0$	$y_1(x)=x$
10. $y''+\frac{1}{x}y'+\left(1-\frac{1}{4x^2}\right)y=0$	$y_1(x)=\frac{1}{\sqrt{x}}\sin x$

Look for a solution y of the form $y=vy_1$.

$$y=vy_1$$
$$y'=v'y_1+vy_1'$$
$$y''=v''y_1+v'y_1'+v'y_1'+vy_1''$$

$$\begin{aligned}0=y''-y&=v''y_1+2v'y_1'+vy_1''-vy_1\\&=v''y_1+2v'y_1'+v(y_1''-y_1)\\&=v''y_1+2v'y_1'\\&=v''e^x+2v'e^x\end{aligned}$$

$$0=v''+2v'=w'+2w,\quad w=v'$$
$$0=(we^{2x})'\to w(x)=A'e^{-2x}$$

$$v(x)=\int w\,dx=Ae^{-2x}+B\to v(x)=e^{-2x}$$

Only one nonzero solution is needed
Pick $A=1$, $B=0$.

$$y_2(x)=v(x)y_1(x)=e^{-2x}e^x=\boxed{e^{-x}}$$

General solution: $\boxed{y(x)=c_1e^x+c_2e^{-x}}$ since e^x, e^{-x} independent

$\underline{3.1.2}$ $y''+y=0$, $y_1(x)=\sin x$

$$y=vy_1$$
$$y'=v'y_1+vy_1'$$
$$y''=v''y_1+2v'y_1'+vy_1''$$

$$\begin{aligned}0&=y''+y\\&=v''y_1+2v'y_1'+v(y_1''+y_1)\\&=v''y_1+2v'y_1'\end{aligned}$$

$$0=v''+2v'y_1'/y_1\quad(\text{in an interval where }\sin x\neq 0)$$
$$0=v''+2v'\cot x$$
$$0=w'+2w\cot x\qquad \phi(x)=e^{2\int\cot x\,dx}=\sin^2x$$
$$0=(w\sin^2x)'$$
$$A=w\sin^2x$$
$$w(x)=A\sec^2x\longrightarrow v(x)=\int w\,dx=A\tan x+B$$

Pick $A=1$, $B=0$.

$$\begin{aligned}y_2&=vy_1\\&=\tan x\sin x\\&=\boxed{\cos x}\end{aligned}$$

General Solution:

$$\boxed{y(x)=c_1\cos x+c_2\sin x}$$

$\underline{3.3.7}$ $x^2y''-6y=0$, $y_1(x)=x^3$

$$0=y''-6x^{-2}y=v''y_1+2v'y_1'+vy_1''-6vy_1/x^2=v''y_1+2v'y_1'+v(y_1''-6y_1/x^2)$$

$$0=v''y_1+2v'y_1'=w'y_1+2wy_1'=x^3w'+6x^2w$$

$$0=w'+6/x\,w\qquad \phi=e^{\int 6/x\,dx}=x^6$$
$$0=(w\cdot x^6)'$$
$$w(x)=A'x^{-6}$$
$$v(x)=Ax^{-5}+B\to v(x)=x^{-5}$$

$$\boxed{\begin{aligned}&y_2(x)=x^{-2}\\&y(x)=c_1x^3+c_2x^{-2}\end{aligned}}$$

3.3.9 $(x^2+1)y'' - 2xy' + 2y = 0, \quad y_1(x) = x$

$y = vy_1$
$y' = v'y_1 + vy_1'$
$y'' = v''y_1 + 2v'y_1' + vy_1''$

$0 = (x^2+1)(v''y_1 + 2v'y_1' + vy_1'') - 2x(v'y_1 + vy_1') + 2vy_1$
$= (x^2+1)(v''y_1 + 2v'y_1') - 2xv'y_1$
$= (x^2+1)(xv'' + 2v') - 2x^2 v'$

$0 = xv'' + 2v' - [2x^2/(x^2+1)]v'$

$u = x^2+1 \quad du = 2x\,dx$

$0 = v'' + [2/x - 2x/(x^2+1)]v' = w' + [2/x - 2x/(x^2+1)]w$

$\phi = x^2 e^{-\int \frac{1}{u} du}$
$= x^2 e^{-\ln|x^2+1|} = \frac{x^2}{x^2+1}$

$0 = \left(\frac{x^2}{x^2+1} w\right)'$

$A = \frac{x^2}{x^2+1} w \rightarrow w(x) = A\frac{x^2+1}{x^2}$, set $A = 1$

$v(x) = \int (1 + x^{-2})dx = x - x^{-1} + B$, set $B = 0$

$y_2 = vy_1 = (x - 1/x)(x) = \boxed{x^2 - 1}$ $\quad \boxed{y(x) = c_1 x + c_2(x^2-1)}$

3.3.11 $y = vy_1$
$y' = v'y_1 + vy_1'$
$y'' = v''y_1 + 2v'y_1' + vy_1''$

11. Reducing the General Equation One solution of the equation

$$y'' - 2by' + b^2y = 0$$

is $y_1(x) = e^{bx}$. Find a second linearly independent solution. What is the general solution?

$0 = v''y_1 + 2v'y_1' + vy_1'' - 2b(v'y_1 + vy_1') + b^2 vy_1$
$= v''y_1 + 2v'y_1' - 2bv'y_1$
$= v''e^{bx} + 2bv'e^{bx} - 2bv'e^{bx} = v''e^{bx}$

$\boxed{\begin{array}{l} y_2 = vy_1 = xe^{bx} \\ y(x) = c_1e^{bx} + c_2xe^{bx} \end{array}}$

$0 = w' \longrightarrow v(x) = Ax + C$, choose $A = 1$, $C = 0$

3.3.13 $y'' - y = e^x, \quad y_1(x) = e^{-x}$

$e^x = v''y_1 + 2v'y_1' + vy_1'' - vy_1$
$= v''y_1 + 2v'y_1' + v(y_1'' - y_1)$
$= v''y_1 + 2v'y_1'$
$= v''e^{-x} - 2v'e^{-x}$

$e^{2x} = v'' - 2v' = w' - 2w$

$e^{2x}e^{-2x} = (we^{-2x})'$

$x + A = we^{-2x}$

Set $A = 0$
$w(x) = xe^{2x} + Ae^{2x}$
$v(x) = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + B$
set $B = 0$
$v(x) = \frac{1}{2}e^{2x}(x - \frac{1}{4})$

$y_2(x) = vy_1 = \frac{1}{2}e^x(x - 1/4)$

Solving Nonhomogeneous from Homogeneous Ones

It is possible to use the reduction of order method to find a single solution of the nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = f(x) \qquad (16)$$

knowing a nonzero solution $y_1(x)$ of the corresponding homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \qquad (17)$$

We use the same technique and substitute $y(x) = v(x)y_1(x)$ into Eq. (16), finding the unknown function $v(x)$. For Problems 12–17, use this technique to find a solution of the given nonhomogeneous equation given the single solution $y_1(x)$ of the corresponding homogeneous equation.

Nonhomogeneous equation	*Homogeneous solution*
12. $y'' = 1$	$y_1(x) = 1$
13. $y'' - y = e^x$	$y_1(x) = e^{-x}$
14. $y'' + y' = e^x$	$y_1(x) = 1$
15. $y'' + y = \csc x$	$y_1(x) = \sin x$
16. $x^2y'' - xy' + y = x \quad (x \neq 0)$	$y_1(x) = x$
17. $x^2y'' + xy' - y = x \quad (x \neq 0)$	$y_1(x) = x$

The equations in Problems 18–21 are some of the most famous differential equations in physics. Use the given solution $y_1(x)$ to find a second linearly independent solution of these equations.

To find a simpler v, don't set $A = 0$ immediately.
$v(x) = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + A/2\, e^{2x} + B$, Pick $A = -1/2$, $B = 0$

$v(x) = 1/2\, xe^{2x} \rightarrow \boxed{y_2(x) = x/2\, e^x}$

3.3.19 Chebychev's Equation of Order 1

$$(1-x^2)y''-xy'+y=0\ ,\ y_1(x)=x$$

$$0=(1-x^2)(v''y_1+2v'y_1'+y_1'')-x(v'y_1+vy_1')+vy_1=(1-x^2)(v''y_1+2v'y_1')-xv'y_1$$

$$0=(1-x^2)(xv''+2v')-x^2v'$$

$$0=v''+\left(2/x-x/1-x^2\right)v'=w'+\left(2/x-x/1-x^2\right)w$$

$$\phi=e^{2\ln|x|}e^{-\int\frac{-1/2}{u}du}$$
$$=x^2\left(e^{1/2\ln|1-x^2|}\right)$$
$$=x^2\sqrt{1-x^2}$$

$$0=\left(x^2\sqrt{1-x^2}\,w\right)'$$

$A=x^2\sqrt{1-x^2}\,v'(x)$, set $A=1$

$$v(x)=\int\frac{1}{x^2\sqrt{1-x^2}}dx\qquad\boxed{y_2(x)=x\int\frac{x}{x^2\sqrt{1-x^2}}dx}$$