Sample Exam 2
MA 2051 BD01,BD02,BD03,BD05 - Differential Equations
Worcester Polytechnic Institute
Fall 2021
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You are allowed a 45 minutes to complete all aspects of the exam. There are five problems to be solved. You should have ample room on these sheets to complete your work. The total number of points is 100.

Note:

- All materials except for a pencil or pen should be put in a book bag, and that book bag needs to be completely closed.
- Please put your name and section number in the upper right-hand corner of this page.
- No calculators, phones, tablets, computers, or watches are allowed during the exam.
- The exam is closed book: no textbooks or notes of any kind are allowed on the exam.
- You have 45 minutes to complete the exam.
- You need to show photo identification in order to turn in your completed exam.
- This exam is subject to WPI's Academic Honesty Policy, and by taking this exam, you agree not to discuss its contents with any other WPI student without your instructor's approval.

1. (20 points) Given that $y_1(x) = x$ and $y_2(x) = x^{-2}$ are solutions of the differential equation $x^2 y'' + 2xy' - 2y = 0$ on the interval $(0, \infty)$.

(a) Verify that y_1, y_2 are linearly independent by computing their Wronskian.

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & 1/x^2 \\ 1 & -2/x^3 \end{vmatrix}$$

$$= -2/x^2 - 1/x^2 = -3/x^2 \neq 0 \text{ on } (0, \infty)$$

$\therefore$ y_1, y_2 are independent

(b) Write down the general solution of the differential equation.

$$\boxed{y(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 x + c_2/x^2}$$

(c) Find the solution that satisfies the initial conditions $y(1) = 0$, $y'(1) = 3$.

$$y'(x) = c_1 - 2c_2 x^{-3}$$

$$0 = y(1) = c_1 + c_2$$

$$-(3 = y'(1) = c_1 - 2c_2)$$

$$-3 = 3c_2$$

$$-1 = c_2$$

$$0 = c_1 + c_2 = c_1 - 1 \rightarrow c_1 = 1$$

$$\boxed{y(x) = x - 1/x^2}$$

2. (20 points) Solve the initial value problem

$$y''(t) + 49\,y(t) = 0\ ,\ y(0) = 4\ ,\ y'(0) = 0\ .$$

Assume y of the form $y = e^{rt}$

$\rightarrow\ r^2 + 49 = 0$

$r = \pm 7i$

$y(x) = c_1 \cos 7t + c_2 \sin 7t$

$y'(x) = -7c_1 \sin 7t + 7c_2 \cos 7t$

$4 = y(0) = c_1$

$0 = y'(0) = 7c_2$

$\boxed{y(x) = 4\cos 7t}$

3. (20 points) Consider the following ordinary differential equation

$$y''(x) + \lambda\, y'(x) + 9\, y(x) = 0\ .$$

Find the value of λ so that the system would be described as critically damped, and write the fundamental solutions for this case.

$r^2 + \lambda r + 9 = 0$

$r = \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 - 36}}{2}$

Critically damped if $\lambda^2 = 36$. Since $\lambda > 0$ (damping is always nonnegative), $\boxed{\lambda = 6}$.

For $\lambda = 6$, $r = -3$

$$\boxed{y(x) = c_1 e^{-3x} + c_2 x e^{-3x}}$$

4. (20 points) Find the simple harmonic motion described by the following initial value problem: $\ddot{u} + 4u = 0$, $u(0) = 1$, $\dot{u}(0) = -2$. Determine the amplitude R, phase angle δ, frequency f, and period T of the motion.

$u = c_1 \cos 2t + c_2 \sin 2t$

$\dot{u} = -2c_1 \sin 2t + 2c_2 \cos 2t$

$1 = u(0) = c_1$

$-2 = \dot{u}(0) = 2c_2$

$u = \cos 2t - \sin 2t$

$$\boxed{\begin{array}{l} u = \sqrt{2}\cos(2t + \pi/4) \\ R = \sqrt{2} \\ \delta = -\pi/4 \\ T = \pi \\ f = 1/\pi \end{array}}$$

Rewrite in 'polar form':

$u = R\cos(\omega_0 t - \delta)$

$R = \sqrt{c_1^2 + c_2^2} = \sqrt{1^2 + (-1)^2}$

$\delta = \arctan(c_2/c_1) = \arctan(-1/1) = -\pi/4$

$u = \sqrt{2}\cos(2t + \pi/4)$

$\omega_0 = 2\pi/T$

$2 = 2\pi/T \rightarrow T = \pi$

$f = 1/T = 1/\pi$

5. (20 points) A 1 kg mass is attached to a spring with a spring constant of 16 N/m and the spring is then attached to a rigid lid. The entire vertical system is then submerged in a liquid which imparts a damping force with a damping coefficient of 10 N-s/m^2. Determine the equations of motion if the weight is released from rest 1 m below the equilibrium position.

General IVP:

$m\ddot{u} + d\dot{u} + Ku = F(t)$, $u(t_0) = u_0$, $\dot{u}(t_0) = \dot{u}_0$

Given Problem:

$\ddot{u} + 10\dot{u} + 16u = 0$, $u(0) = 1$, $\dot{u}(0) = 0$

$r^2 + 10r + 16 = 0$

$r = -5 \pm \sqrt{100-64}/2 = -5 \pm 3 = -8, -2$

$u(t) = c_1 e^{-2t} + c_2 e^{-8t}$

$\dot{u}(t) = -2c_1 e^{-2t} - 8c_2 e^{-8t}$

$1 = c_1 + c_2 = -4c_2 + c_2 = -3c_2 \rightarrow c_2 = -1/3$

$0 = -2c_1 - 8c_2 \rightarrow -4c_2 = c_1$ $\qquad c_1 = 4/3$

$$\boxed{u = 4/3\, e^{-2t} - \tfrac{1}{3} e^{-8t}}$$

Problem	Points
1	
2	
3	
4	
5	
Total	