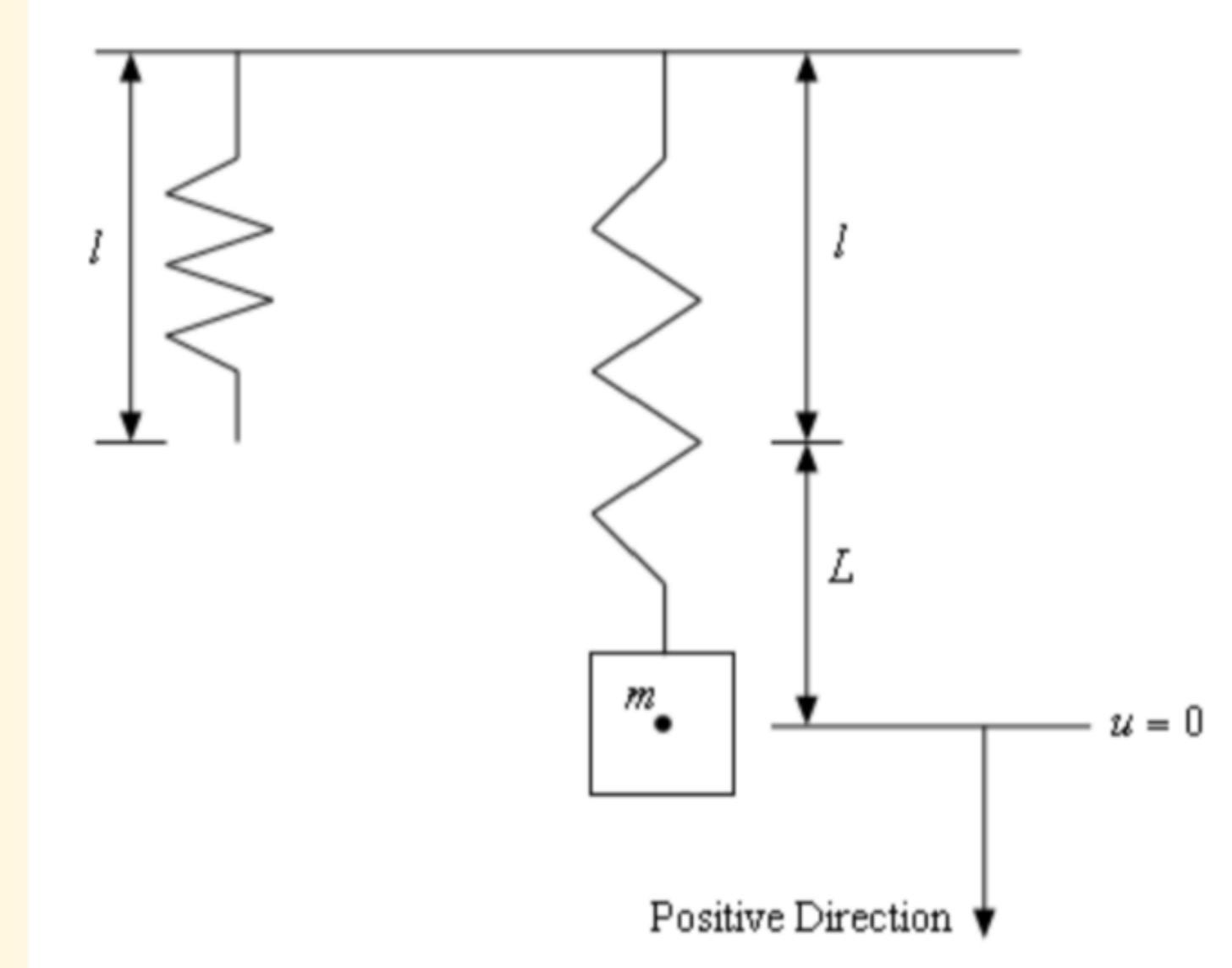


§ 3.11 Forced Vibrations

Background

Develop a differential equation that gives the displacement of the object at any time t . Recall Newton's 2nd Law $ma = \sum F_i$. In this case takes the form

$$mu'' = f(t, u, u')$$



We assume 4 forces are acting upon the object.

1. $F_g = mg$ Gravitational Force

2. $F_s = -k(l+u)$ Spring Force

Follow's Hooke's Law - the force exerted by the spring is proportional ($k > 0$) to the displacement u of the spring from its natural length L . The minus sign accounts for the assumption that this force always acts in the direction of equilibrium (where the spring is at its natural length).

3. $F_d = -ru'$ Damping Force

Counteracts the object's movement with a force proportional ($r > 0$) to the object's velocity u' .

downward movement : $u' > 0 \leftrightarrow F_d > 0 \leftrightarrow$ upward force

upward movement : $u' < 0 \leftrightarrow F_d < 0 \leftrightarrow$ downward force

4. $F(t)$ External Force

Any additional forces we want to include. Call $F(t)$ the forcing function.

Let $f(t, u, u'')$ be the sum of these 4 forces.

$$mu'' = f(t, u, u'') = mg - k(l+u) - \gamma u' + F(t)$$

$$mu'' + \gamma u' + ku = mg - kl + F(t)$$

Consider the case where the object is at rest. That means $F(t) \equiv 0$ and $u(t) = u'(t) = u''(t) = 0$. Then $mg = kl$.

The equation describing the object's movement is therefore

$$mu'' + \gamma u' + ku = F(t)$$

$u(0) = u_0$ initial displacement

$u'(0) = u'_0$ initial velocity

Free Undamped Vibrations

$$F(t) = 0, \gamma = 0 \rightarrow mu'' + ku = 0$$

$$\text{If } u = e^{rt}, (mr^2 + k)e^{rt} = 0$$

$$r = \pm \sqrt{k/m} i = \pm \omega_0 i$$

$$u(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

We can write $u(t)$ in 'polar form' $u(t) = R \cos(\omega_0 t - \delta)$. To find R and δ ,

$$R \cos \delta \cos \omega_0 t + R \sin \delta \sin \omega_0 t = R \cos(\omega_0 t - \delta) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

$$\rightarrow R \cos \delta = c_1, R \sin \delta = c_2, \tan \delta = c_2/c_1, R^2 = c_1^2 + c_2^2$$

Free Damped Vibrations

$$F(t) = 0, \gamma > 0 \rightarrow mu'' + \gamma u' + ku = 0$$

The behavior of $u(t)$ falls into 1 of 3 qualitatively different cases depending on the sign (+, -, 0) of $\gamma^2 - 4mk$.

Assuming a solution of the form $u(t) = e^{rt}$ gives the characteristic eqn

$$mr^2 + \gamma r + k = 0$$

$$r = -\frac{\gamma}{2m} \pm \sqrt{\frac{\gamma^2 - 4mk}{4m^2}} / 2m$$

1. $\gamma^2 - 4mk = 0$ critical Damping ($r = r_{critical} = 2\sqrt{mk}$)

$$u(t) = c_1 e^{-\frac{\gamma t}{2m}} + c_2 t e^{-\frac{\gamma t}{2m}}$$

The characteristic equation has a repeated root $r = -\frac{\gamma}{2}$.

Note that $\lim_{t \rightarrow \infty} u(t) = 0$, the solution decays over time.

2. $\gamma^2 - 4mk > 0$ Overdamped ($\gamma > r_{critical}$)

$$r_{1,2} = -\frac{\gamma}{2m} \pm \sqrt{\frac{\gamma^2 - 4mk}{4m^2}} / 2m = -\frac{\gamma}{2m} \left(1 \pm \sqrt{1 - \frac{4mk}{\gamma^2}} \right)$$

$$u(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

The characteristic equation has 2 distinct real roots r_1 and r_2

$\gamma^2 - 4mk > 0 \rightarrow r_1, r_2 < 0$. Then, $\lim_{t \rightarrow \infty} u(t) = 0$ in this case too.

3. $\gamma^2 - 4mk < 0$ Underdamped ($\gamma < r_{critical}$)

$$r_{1,2} = -\frac{\gamma}{2m} \pm \sqrt{4mk - \gamma^2} i / 2m = \alpha \pm \beta i$$

$$u(t) = e^{\alpha t} (c_1 \cos \beta t + c_2 \sin \beta t)$$

The characteristic equation has 2 complex roots $r_{1,2} = \alpha \pm \beta i$

Since $\alpha = -\frac{\gamma}{2m} < 0$, $\lim_{t \rightarrow \infty} u(t) = 0$ in this case too.

Undamped Forced Vibrations

$$\gamma = 0, F(t) \neq 0 \rightarrow mu'' + Ku = F(t)$$

$$u(t) = u_h(t) + u_p(t)$$

$u_h(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t, \omega_0 = \sqrt{K/m}$ satisfies $mu'' + Ku = 0$

Note that $\lim_{t \rightarrow \infty} |u_h(t)| \neq 0$ but rather oscillates with $|u_h(t)| \leq \sqrt{c_1^2 + c_2^2} \ \forall t$.

$u_p(t)$ is a particular solution to $mu'' + Ku = F(t)$ found using undetermined coefficients (or some other method). A notable scenario is the case of resonance where $F(t)$ is of the form $F_0 \cos \omega_0 t, F_0 \sin \omega_0 t$ or a linear combination of the two. In this case we guess $u_p(t)$ of the form

$$u_p(t) = At \cos \omega_0 t + Bt \sin \omega_0 t$$

With either of $A, B \neq 0$ we have $\lim_{t \rightarrow \infty} |u(t)| = \infty$. This is notable since the natural oscillations $u_h(t)$ are bounded and the applied force $F(t)$ is bounded ($|F(t)| \leq F_0$) yet the solution $u(t)$ becomes infinite in the case that the applied force $F(t)$ is an oscillation of the same frequency ω_0 as that of the unforced solution.

Damped Forced Vibrations

$$\gamma > 0, F(t) \neq 0 \rightarrow mu'' + \gamma u' + Ku = F(t)$$

$$u(t) = u_h(t) + u_p(t) \quad \text{or} \quad u(t) = u_{\text{transient}}(t) + u_{\infty}(t)$$

$u_h(t)$ satisfies $mu'' + \gamma u' + Ku = 0$. From 'Free Damped Vibrations' we know there are 3 cases for the form of $u_h(t)$ based on the value $\gamma^2 - 4mK$. In any case $\lim_{t \rightarrow \infty} u_h(t) = 0$. Call $u_h(t)$ the transient solution $u_{\text{transient}}$.

$u_p(t)$ is a particular solution of $mu'' + \gamma u' + Ku = F(t)$. Since $u(t)$ approaches $u_p(t)$ asymptotically, call $u_p(t)$ the steady state solution $u_{\infty}(t)$.

3.11.3 Find the steady state solution $u_{\infty}(t) = R \cos(\omega_0 t - \delta)$

$$\ddot{u} + 4\dot{u} + 4u = \cos t$$

$$\lambda^2 + 4\lambda + 4 = 0$$

$$(\lambda + 2)^2 = 0$$

$$u_h = c_1 e^{-2t} + c_2 t e^{-2t}$$

$$u_{\infty} = A \cos t + B \sin t$$

$$\cos t = (3A + 4B) \cos t + (3B - 4A) \sin t$$

$$\begin{aligned} 3A + 4B &= 1 \\ 3B - 4A &= 0 \end{aligned}$$

$$u_{\infty} = \frac{3}{25} \cos t + \frac{4}{25} \sin t$$

$$A = \frac{1}{25} \sqrt{9+16} = \frac{1}{5}$$

$$\tan \delta = \frac{4}{3} \rightarrow \delta = \arctan \frac{4}{3} \approx 0.93$$

$$\omega_0 = 1$$

$$u_{\infty} = \frac{1}{5} \cos(t - \delta), \quad \delta = \arctan \frac{4}{3}$$

3.11.5 Find the steady state solution $u_{\infty}(t) = R \cos(\omega_0 t - \delta)$

$$\ddot{u} + u + u = 4 \cos 3t$$

$$u_h = e^{-t/2} \left(c_1 \cos \frac{\sqrt{3}}{2} t + c_2 \sin \frac{\sqrt{3}}{2} t \right)$$

$$A = \frac{1}{73} \sqrt{32^2 + 12^2} = 4\sqrt{3}/73$$

$$\delta = \arctan(-12/32) \approx -0.359$$

$$\omega_0 = 3$$

$$u_{\infty} = A \cos 3t + B \sin 3t$$

$$4 \cos 3t = (-9A + 3B + A) \cos 3t + (-9B - 3A + B) \sin 3t$$

$$4 = 3B - 8A$$

$$0 = -8B - 3A$$

$$A = -32/73$$

$$B = 12/73$$

$$u_{\infty} = \frac{4\sqrt{73}}{73} \cos(3t - \delta), \quad \delta = \arctan(-3/8)$$

3.11.9 Find the steady state solution of a mass that vibrates according to $\ddot{u} + 8\dot{u} + 36u = 72 \cos 6t$.

First check for resonance $\omega = \omega_0 = 6$ using the characteristic eqn.
 $6^2 + 8 \cdot 6 + 36 \neq 0 \rightarrow \text{No resonance.}$

$$u_{\infty} = C \sin 6t + D \cos 6t$$

$$72 \cos 6t = (-36C - 48D + 36C) \sin 6t + (-36D + 48C + 36D) \cos 6t \rightarrow \begin{aligned} 0 &= -48D, \quad 72 = 48C \\ 0 &= D, \quad 3/2 = C \end{aligned}$$

$$u_{\infty} = \frac{3}{2} \sin 6t$$

(Error #5 $\omega_0 = 3$ not 1)

1. $-2 \sin t \sin 2t$ 2. $2 \sin t \cos 2t$ 3. $u_{\infty} = \frac{1}{5} \cos(t - \delta), \delta = \tan^{-1}(1.33) \approx 0.93 \text{ rad}$

4. $u_{\infty} = \frac{2\sqrt{3}}{5} \cos(t - \delta), \delta = \tan^{-1}(2) \approx 1.1 \text{ rad}$

5. $u_{\infty} = \frac{4}{\sqrt{73}} \cos(t - \delta), \delta = \tan^{-1}(-3/8) = \pi - \tan^{-1}(3/8) \approx 2.78 \text{ rad}$

6. (a) $\omega = 2\sqrt{3} \text{ rad/sec}$ (b) $u = \frac{2}{3} \sin(2\sqrt{2}t) - \frac{4\sqrt{3}}{3} t \cos(2\sqrt{3}t)$

8. (a) $h(t) = \frac{49}{80\pi^2} \cos\left(\frac{2\pi t}{7}\right) + 3$ (buoy above the water line) (b) No

9. $u = \frac{3}{2} \sin 6t$ 10. $u = -\frac{1}{4} e^{-2t} (3 \sin 4t + 4 \cos 4t) + \frac{1}{2} (\sin 2t + 2 \cos 2t)$

11. Kick with the same frequency as the swing.

§ 5.1 Definition of the Laplace Transform

Definition The Laplace transform of a piecewise continuous function $f(t)$ is

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

Theorem The Laplace transform is a linear operator :

$$\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\} \text{ and } \mathcal{L}\{cf\} = c\mathcal{L}\{f\}$$

Theorem If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α ($\exists M, T$ s.t. $|f(t)| \leq M e^{\alpha t} \forall t > T$) then the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ exists $\forall s > \alpha$.

$$\underline{5.1.1} \quad f(t) = 5$$

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \int_0^\infty 5e^{-st} dt \\ &= -5/s e^{-st} \Big|_0^\infty \\ &= 5/s, \quad s > 0\end{aligned}$$

$$\underline{5.1.3} \quad f(t) = e^{2t}$$

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st} e^{2t} dt \\ &= \int_0^\infty e^{(2-s)t} dt \\ &= \frac{1}{2-s} e^{(2-s)t} \Big|_0^\infty \\ &= \frac{1}{s-2}, \quad s > 2\end{aligned}$$

$$\underline{5.1.7} \quad f(t) = 5 \sinh 2t = \frac{5}{2}(e^{2t} - e^{-2t})$$

$$\mathcal{L}\{f(t)\} = \frac{5}{2} \int_0^\infty e^{-st} (e^{2t} - e^{-2t}) dt$$

$$\begin{aligned}&\frac{2}{5} \mathcal{L}\{f(t)\} = \int_0^\infty (e^{(2-s)t} - e^{-(2+s)t}) dt \\ &= \left[\frac{1}{2-s} e^{(2-s)t} + \frac{1}{2+s} e^{-(2+s)t} \right] \Big|_0^\infty \\ &= \frac{1}{s-2} - \frac{1}{2+s}, \quad |s| > 2 \\ &\quad \underbrace{}_{s>2} \quad \underbrace{}_{s<-2} \\ &= \frac{4}{s^2-4}\end{aligned}$$

$$\frac{2}{5} \mathcal{L}\{f(t)\} = \frac{4}{s^2-4} \rightarrow \mathcal{L}\{f(t)\} = \frac{10}{s^2-4}, \quad |s| > 2$$

$$\underline{5.1.9} \quad f(t) = \begin{cases} 0, & t < 1 \\ 1, & t \geq 1 \end{cases}$$

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \int_0^1 0 \cdot e^{-st} dt + \int_1^\infty 1 \cdot e^{-st} dt \\ &= 0 + (-1/s e^{-st}) \Big|_1^\infty \\ &= \frac{e^{-s}}{s}, \quad s > 0\end{aligned}$$

Section 5.1									
1. $\frac{5}{s}$	2. $\frac{1}{s^2}$	3. $\frac{1}{s-2}$	4. $\frac{1}{s+1}$	5. $\frac{2}{s^2+4}$	6. $\frac{s}{s^2+9}$	7. $\frac{10}{s^2-4}$	8. $\frac{s-1}{(s-1)^2+4}$	9. $\frac{e^{-t}}{s}$	
10. $\frac{1}{s}(1-e^{-s})$	11. $\frac{1+e^{-2s}}{s^2+1}$	12. $\frac{1}{s^2}(1-e^{-s})$	13. $\frac{1}{(s-a)^2}$	14. $\frac{n!}{(s-a)^{n+1}}$	15. $\frac{2as}{(s^2+a^2)^2}$				
16. $\frac{s^2+a^2}{(s-a)^2(s+a)^2}$	17. $\frac{a}{s} + \frac{b}{s^2} + \frac{2c}{s^3}$	18. $\frac{2s+1}{s(s+1)}$	19. $\frac{2s}{s^2-4}$	20. $\frac{3}{s} + \frac{1}{s^2} + \frac{2}{(s+1)^2+4}$					
21. $\frac{1}{(s+2)^2} + \frac{6}{(s+1)^3}$	22. $\frac{6}{(s+3)^4} + \frac{4(s+1)}{(s+1)^2+9}$	23. $\frac{s+3a}{s^2-a^2}$	24. $\frac{1}{(s+3)^2} + \frac{2}{s^2+1}$	25. Continuous					

§5.2 Properties of the Laplace Transform

Using the definition one can derive a collection of Laplace transforms and properties thereof.

$$5.2.3 \quad y(t) = (t-9)^2$$

$$Y(t) = \mathcal{L}\{y(t)\}$$

$$= \mathcal{L}\{t^2\} - 18\mathcal{L}\{t\} + 81\mathcal{L}\{1\} \quad (\text{Properties 1, 2})$$

$$= \frac{2!}{s^{2+1}} - 18 \frac{1!}{s^{1+1}} + 81 \frac{1}{s} \quad (\text{Table 5.1: 1, 2})$$

$$= \frac{2}{s^3} - \frac{18}{s^2} + \frac{81}{s} = \frac{2-18s+81s^2}{s^3}$$

$$5.2.7 \quad y(t) = t^2 \sin 2t$$

$$\text{Let } f(t) = \sin 2t$$

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{\sin 2t\} = \frac{2}{s^2 + 4} \quad (\text{Table 5.1: 6})$$

$$Y(s) = \mathcal{L}\{y(t)\} = \mathcal{L}\{t^2 f(t)\}$$

$$= (-1)^2 \frac{d^2}{ds^2} F(s) = \frac{d^2}{ds^2} \frac{2}{s^2 + 4} \quad (\text{Property 7})$$

$$= \frac{d}{ds} \frac{-2}{(s^2 + 4)^2} \cdot 2s = \frac{d}{ds} \frac{-4s}{(s^2 + 4)^2}$$

$$= \frac{-4(s^2 + 4)^2 + 4s(s^2 + 4)4s}{(s^2 + 4)^4} \quad (\text{Quotient Rule})$$

$$= \frac{-4(s^2 + 4) + 16s^2}{(s^2 + 4)^3} = \frac{4(3s^2 - 4)}{(s^2 + 4)^3} \quad (\text{Error in text-book solution})$$

$$5.2.9 \quad y(t) = 5e^{5t} \cos 2t$$

$$Y(s) = \mathcal{L}\{y(t)\} = 5\mathcal{L}\{e^{5t} \cos 2t\} \quad (\text{Property 2})$$

$$= 5 \frac{s-5}{(s-5)^2 + 2^2} \quad (\text{Table 5.1: 11})$$

$$= \frac{5(s-5)}{s^2 - 10s + 29}$$

For Problems 1–20, find the Laplace transform of the given function, using Table 5.1 from Section 5.1 and properties of the Laplace transform given in Table 5.2.

- | | |
|-----------------------|--|
| 1. $at^2 + bt + c$ | 13. $\sin 2t \sinh 2t$ |
| 2. $t^2 + e^{2t} - 2$ | 14. $\sin^2 t$ |
| 3. $(t-9)^2$ | 15. $\sinh 3t$ |
| 4. e^{2t-1} | 16. $\cos^3 t$ |
| 5. $(1+e^t)^2$ | 17. $t \sin^2 t$ |
| 6. $3t \sin t$ | 18. $\cos mt \sin nt \quad (m \neq n)$ |
| 7. $t^2 \sin 2t$ | 19. $\int_0^t \cos \tau d\tau$ |
| 8. $e^{-2t} \sin 3t$ | 20. $\int_0^t \sin 3\tau d\tau$ |
| 9. $5e^{5t} \cos 2t$ | |
| 10. $t^2 e^{-3t}$ | |
| 11. $te^t \cos t$ | |
| 12. $t^2 e^t \sin t$ | |

$f(t)$	$F(s) = \mathcal{L}\{f\}$	Domain of $F(s)$
1	$\frac{1}{s}$	$s > 0$
t^n (n positive integer)	$\frac{n!}{s^{n+1}}$	$s > 0$
$t^p \quad (p > -1)$	$\frac{\Gamma(p+1)}{s^{p+1}}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$e^{at} t^n, n = 1, 2, \dots$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$\sin bt$	$\frac{b}{s^2 + b^2}$	$s > 0$
$\cos bt$	$\frac{s}{s^2 + b^2}$	$s > 0$
$\sinh bt$	$\frac{b}{s^2 - b^2}$	$s > b $
$\cosh bt$	$\frac{s}{s^2 - b^2}$	$s > b $
$e^{at} \sin bt$	$\frac{b}{(s-a)^2 + b^2}$	$s > a$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2 + b^2}$	$s > a$
$u(t-c)$	$\frac{e^{-ct}}{s}$	$s > 0$
$u(t-c)f(t-c)$	$e^{-cs} F(s)$	
$\int_0^t f(t-\tau)g(\tau) d\tau$	$F(s)G(s)$	
$\delta(t-c)$	e^{-cs}	
$\frac{d^n}{dt^n} f(t)$	$s^n F(s) - s^{n-1} f(0) - \dots - f^{(n-1)}(0)$	
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	
1. $\mathcal{L}\{f+g\}$	$= \mathcal{L}\{f\} + \mathcal{L}\{g\}$	
2. $\mathcal{L}\{cf\}$	$= c\mathcal{L}\{f\}$	
3. $\mathcal{L}\{f'\}$	$= s\mathcal{L}\{f\} - f(0)$	
4. $\mathcal{L}\{f''\}$	$= s^2 \mathcal{L}\{f\} - sf(0) - f'(0)$	
5. $\mathcal{L}\{f^{(n)}\}$	$= s^n \mathcal{L}\{f\} - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$	
6. $\mathcal{L}\{e^{at} f(t)\}$	$= F(s-a)$	
7. $\mathcal{L}\{t^n f(t)\}$	$= (-1)^n \frac{d^n}{ds^n} F(s)$	
8. $\mathcal{L}\{f(at)\}$	$= \frac{1}{a} F\left(\frac{s}{a}\right)$	
9. $\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\}$	$= \frac{1}{s} F(s)$	
10. $\mathcal{L}\left\{\frac{f(t)}{t}\right\}$	$= \int_s^\infty F(\xi) d\xi$	
11. $\lim_{s \rightarrow \infty} sF(s) = f(0)$	\quad (initial-value theorem)	
12. $\lim_{s \rightarrow 0} sF(s) = f(\infty)^*$	\quad (final-value theorem)	

* The notation $f(\infty)$ means the limiting value of $f(t)$ as $t \rightarrow \infty$.

Section 5.2
1. $\frac{2a}{s^2} + \frac{b}{s^3} + \frac{c}{s^4}$
7. $\frac{2(3s^2 - 4)}{(s^2 + 4)^3}$
13. $\frac{8s}{s^4 + 64}$
18. $\frac{m+n}{2(s^2 + (m+n)^2)} + \frac{n-m}{[s^2 + (n-m)^2]}$
23. $\frac{1}{(s-2)^2}$
32. (b) $3\left(\frac{1}{(s-2)^4} + \frac{1}{(s+2)^4}\right)$ (c) $\frac{1}{2} \left(\frac{1}{(s-1)^2 + 1} + \frac{1}{(s+1)^2 + 1} \right)$
38. $\frac{1}{(s-1)(s+1)^2}$
42. $\frac{1}{s^2(s+1)^2}$
2. $\frac{2}{s^3} + \frac{1}{s-2} - \frac{2}{s}$
8. $\frac{3}{(s-2)^2 + 9}$
14. $\frac{1}{2s} - \frac{s}{2(s^2 + 4)}$
15. $\frac{3}{s^2 - 9}$
16. $\frac{3s}{4(s^2 + 1)} + \frac{s}{4(s^2 + 9)}$
19. $\frac{1}{s^2 + 1}$
20. $\frac{3}{s(s^2 + 9)}$
24. $\frac{2}{(s+1)^2}$
25. $\frac{2bs}{(s^2 + b^2)^2}$
26. $\frac{2(3s^2 - 4)}{(s^2 + 4)^3}$
27. $\frac{1}{(s+2)^2}$
28. $\frac{2}{(s-1)^3}$
29. $\frac{b}{(s-a)^2 + b^2}$
3. $\frac{2}{s^3} - \frac{18}{s^5} + \frac{81}{s^7}$
9. $\frac{5(s-5)}{(s-5)^2 + 4}$
10. $\frac{2}{(s+3)^2}$
11. $\frac{2(s-2)}{(s^2 - 2s + 2)^2}$
17. $\frac{1}{2s^2 + 2(s^2 + 4)^2}$
21. $\frac{b}{(s+1)^2 + b^2}$
22. $\frac{2}{(s+2)^3}$
33. (b) $\frac{1}{(s-3)^2} - \frac{1}{(s+3)^2}$
35. $\frac{2}{s^2 + 4}$
36. (a) $\frac{1}{s(s^2 + 1)}$ (b) $\frac{1}{s(s-2)}$
37. $\cot^{-1} s$
38. $\frac{2(s+2)}{(s^2 + 4s + 20)^2}$
39. $\frac{2(s+2)}{(s^2 + 4s + 20)^2}$
40. $\frac{2(s+1)}{s(s^2 + 2s + 2)^2}$
41. $\frac{2(3s^2 + 6s + 2)}{s(s^2 + 2s + 2)^3}$
42. $\frac{1}{s^2(s+1)^2}$
43. $\frac{1}{s(s-1)}$