§5.3 The Inverse Laplace Transform

5.3: 5, 7, 11, 13, 17, 21, 23

1. $\frac{1}{s^3}$
2. $\frac{2}{s}+\frac{3}{s-1}+\frac{7}{s^3}$
3. $\frac{5}{s^2+3}$
4. $\frac{3}{s-3}+\frac{4}{s+3}$
5. $\frac{1}{s^2+3s}$
6. $\frac{s+1}{s^2+2s+10}$
7. $\frac{1}{s^2+4s+4}$
8. $\frac{3s+5}{s^2-6s+25}$
9. $\frac{s+1}{s^2+s-2}$
10. $\frac{5}{s^2+s-6}$
11. $\frac{2s+4}{s^2-1}$
12. $\frac{7}{(s+2)^2+3}$
13. $\frac{2s+16}{s^2+4s+13}$
14. $\frac{6}{(s+2)^4}$
15. $\frac{7s^2+23s+30}{(s-2)(s^2+2s+5)}$
16. $\frac{4}{s^2(s^2+4)}$
17. $\frac{3}{(s^2+1)(s^2+4)}$
18. $\frac{7s-1}{(s+1)(s+2)(s-3)}$
19. $\frac{s^2-2}{s^3+8s^2+7s}$
20. $\frac{s^2+9s+2}{(s-1)^2(s+3)}$

The Inverse of Derivatives

For Problems 21–22, use Property 3 from Table 5.4 to find the given inverse Laplace transforms.

21. $\mathfrak{L}^{-1}\left\{\tan^{-1}\left(\frac{1}{s}\right)\right\}$
22. $\mathfrak{L}^{-1}\left\{\ln\left(\frac{s-a}{s-b}\right)\right\}$

Division by s Equals Integration in t

For Problems 23–26, use Property 4 from Table 5.4 to find the inverse transform of the given function.

23. $F(s)=\frac{1}{s^2}$
25. $F(s)=\frac{1}{s(s^2+1)}$

5.3.5 $\mathcal{L}^{-1}\left\{\frac{1}{s^2+3s}\right\}$

$$\frac{1}{s(s+3)} = \frac{A}{s}+\frac{B}{s+3} \rightarrow 1 = A(s+3)+Bs$$

$s=0$: $1=3A \rightarrow A=1/3$

$s=-3$: $1=-3B \rightarrow B=-1/3$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+3s}\right\} = \frac{1}{3}\mathcal{L}^{-1}\{1/s\} - \frac{1}{3}\mathcal{L}\{1/s+3\} = \frac{1}{3}-\frac{1}{3}e^{-3t}$$

5.3.7 $\mathcal{L}^{-1}\{(s^2+4s+4)^{-1}\}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+4}\right\}$$

$s^2+4s+4 = (s+2)^2$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}$$

$F(s) = \frac{1}{s^2} = \frac{1!}{s^{1+1}}$

$$= \mathcal{L}^{-1}\{F(s-(-2))\}$$

$\mathcal{L}^{-1}\{F(s-c)\} = e^{ct}f(t)$

$$= e^{-2t}f(t)$$

$\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$

$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$

$$= \underline{te^{-2t}}$$

5.3.11 $\mathcal{L}^{-1}\left\{\frac{2s+4}{s^2-1}\right\}$

$$\frac{2s+4}{s^2-1} = 2\frac{s}{s^2-1} + 4\frac{1}{s^2-1}$$

$$\mathcal{L}^{-1}\left\{\frac{2s+4}{s^2-1}\right\} = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2-1^2}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s^2-1^2}\right\}$$

$$= 2\cosh t + 4\sinh t$$

$$= e^t + e^{-t} + 2e^t - 2e^{-t}$$

$$= \underline{3e^t - e^{-t}}$$

Section 5.3

1. $\frac{1}{2}t^2$ 2. $2+3e^t+\frac{7}{2}t^2$ 3. $\frac{5}{\sqrt{3}}\sin\sqrt{3}t$ 4. $3e^{3t}+4e^{-3t}$ 5. $\frac{1}{3}-\frac{1}{3}e^{-3t}$ 6. $e^{-t}\cos 3t$ 7. te^{-2t}
8. $e^{3t}\left(3\cos 4t+\frac{7}{2}\sin 4t\right)$ 9. $\frac{1}{3}e^{-2t}+\frac{2}{3}e^t$ 10. $e^{2t}-e^{-3t}$ 11. $3e^t-e^{-t}$ 12. $\frac{7}{\sqrt{3}}e^{-2t}\sin 3t$
13. $2e^{-2t}\cos 3t+4e^{-2t}\sin 3t$ 14. t^3e^{-2t} 15. $8e^{2t}-e^{-t}\cos 2t+3e^{-t}\sin 2t$ 16. $t-\frac{1}{2}\sin 2t$
17. $\sin t-\frac{1}{2}\sin 2t$ 18. $2e^{-t}-3e^{-2t}+e^{3t}$ 19. $-\frac{2}{7}+\frac{1}{6}e^{-t}+\frac{47}{42}e^{-7t}$ 20. $2e^t+3te^t-e^{-3t}$ 21. $\frac{\sin t}{t}$
22. $\frac{e^{bt}-e^{at}}{t}$ 23. t 24. e^t-1 25. $1-\cos t$ 26. $(t-1)e^t+1$ 30. $5e^{-t}-18e^{-2t}+15e^{-3t}$
31. (b) $\frac{3}{s-2}-\frac{2}{s+1}$

$\underline{5.3.13}$ $\mathcal{L}^{-1}\left\{\frac{2s+16}{s^2+4s+13}\right\}$

$$\frac{2s+16}{s^2+4s+13} = \frac{2s+16}{s^2+4s+4+9} = \frac{2s+16}{(s+2)^2+3^2} = \frac{2(s+2)+12}{(s+2)^2+3^2} = 2\,\frac{s+2}{(s+2)^2+3^2} + 4\,\frac{3}{(s+2)^2+3^2}$$

$$\mathcal{L}^{-1}\left\{\frac{2s+16}{s^2+4s+13}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+3^2}\right\} + 4\mathcal{L}^{-1}\left\{\frac{3}{(s+2)^2+3^2}\right\}$$

$$= \underline{2e^{-2t}\cos 3t + 4e^{-2t}\sin 3t}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$$

$$\mathcal{L}^{-1}\left\{\frac{b}{s^2+b^2}\right\} = \sin bt$$

$$\mathcal{L}^{-1}\left\{F(s-c)\right\} = e^{ct}f(t)$$

$\underline{5.3.17}$ $\mathcal{L}^{-1}\left\{\frac{3}{(s^2+1)(s^2+4)}\right\}$

$$3 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$

$$s=i: \quad 3 = 3Ai + 3B \longrightarrow A=0,\ B=1$$
$$s=2i: \quad 3 = -6Ci - 3D \longrightarrow C=0,\ D=-1$$

$$\mathcal{L}^{-1}\left\{\frac{3}{(s^2+1)(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} = \underline{\sin t - \tfrac{1}{2}\sin 2t}$$

$\underline{5.3.21}$ Find $\mathcal{L}^{-1}\left\{\arctan(1/s)\right\}$ using the property

$$\mathcal{L}^{-1}\left\{F(s)\right\} = \frac{(-1)^n}{t^n}\mathcal{L}^{-1}\left\{\frac{d^nF(s)}{ds^n}\right\}$$

$$\frac{d}{ds}\arctan(1/s) = \frac{1}{1+(1/s)^2}\left(-s^{-2}\right) = -\frac{1}{1+1/s^2}\,\frac{1}{s^2} = -\frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\arctan(1/s)\right\} = \frac{(-1)^1}{t^1}\mathcal{L}^{-1}\left\{-\frac{1}{s^2+1}\right\} = \frac{1}{t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \frac{\sin t}{t}$$

$\underline{5.3.23}$ Find $\mathcal{L}^{-1}\{1/s^2\}$ using the property $\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\,d\tau$.

$$1/s^2 = \frac{1/s}{s} = \frac{F(s)}{s},\quad F(s) = 1/s$$
$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{1/s\} = 1$$

$$\longrightarrow \quad \underline{\mathcal{L}^{-1}\{1/s^2\} = \int_0^t 1\,d\tau = t}$$

§5.4 Initial Value Problems

5.4 : 3, 7, 9, 11, 13

For Problems 1–14, use the Laplace transform to solve the given initial-value problem.

1. $y' = 1$	$y(0) = 1$	
2. $y' - y = 0$	$y(0) = 1$	
3. $y' - y = e^t$	$y(0) = 1$	
4. $y' + y = e^{-t}$	$y(0) = 1$	
5. $y'' = e^t$	$y(0) = 1$	$y'(0) = 0$
6. $y'' - 3y' + 2y = 0$	$y(0) = 1$	$y'(0) = 0$
7. $y'' + 2y' = 4$	$y(0) = 1$	$y'(0) = -4$
8. $y'' + 9y = 20e^{-t}$	$y(0) = 0$	$y'(0) = 1$
9. $y'' + 9y = \cos 3t$	$y(0) = 1$	$y'(0) = -1$
10. $y'' + 4y = 4$	$y(0) = 1$	$y'(0) = 0$
11. $y'' - 2y' + 5y = 0$	$y(0) = 2$	$y'(0) = 4$
12. $y'' + 10y' + 25y = 0$	$y(0) = 0$	$y'(0) = 10$
13. $y'' + 3y' + 2y = 6$	$y(0) = 0$	$y'(0) = 2$
14. $y'' + y = \sin t$	$y(0) = 2$	$y'(0) = -1$

<u>5.4.3</u> $y' - y = e^t \quad , \; y(0) = 1$

$$\mathcal{L}\{y'\} - \mathcal{L}\{y\} = \mathcal{L}\{e^t\}$$

$$sY(s) - y(0) - Y(s) = \frac{1}{s-1} \qquad \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$(s-1)\,Y(s) = 1 + \frac{1}{s-1}$$

$$Y(s) = \frac{1}{s-1} + \frac{1}{(s-1)^2}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$\underline{y(t) = e^t + te^t} \qquad \mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$$

<u>5.4.7</u> $y'' + 2y' = 4 \; , \; y(0) = 1 \; , \; y'(0) = -4$

$$\mathcal{L}\{y'' + 2y'\} = \mathcal{L}\{4\}$$

$$s^2Y(s) - s\,y(0) - y'(0) + 2sY(s) - 2y(0) = 4/s$$

$$s^2Y - s + 4 + 2sY - 2 = 4/s$$

$$s(s+2)Y = s - 2 + 4/s$$

$$2 = A(s+2) + Bs \longrightarrow \frac{1}{s} - \frac{1}{s+2}$$
$$2 = 2A \; , \; 2 = -2B$$

$$Y = \frac{1}{s+2} - \frac{2}{s(s+2)} + \frac{4}{s^2(s+2)}$$

$$4 = As(s+2) + B(s+2) + Cs^2$$
$$4 = 2B \qquad 4 = 4C \qquad 4 = 3A + 6 + 1$$
$$2 = B \qquad 1 = C \qquad -3 = 3A$$

$$Y = \frac{1}{s+2} + \frac{1}{s+2} - \frac{1}{s} - \frac{1}{s} + \frac{2}{s^2} + \frac{1}{s+2}$$

$$Y(s) = \frac{2}{s^2} - \frac{2}{s} + \frac{3}{s+2} \longrightarrow \mathcal{L}^{-1}\{Y(s)\} = 2\mathcal{L}^{-1}\{1/s^2\} - 2\mathcal{L}^{-1}\{1/s\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$\underline{y(t) = 2t - 2 + e^{-2t}} \qquad \mathcal{L}^{-1}\{n!/s^{n+1}\} = t^n , \; \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

5.4.9 $\quad y'' + 9y = \cos 3t \,, \quad y(0) = 1 \,, \quad y'(0) = -1$

$$s^2 Y - s\,y(0) - y'(0) + 9\,Y(s) = \frac{s}{s^2+3^2}$$

$$(s^2+9)Y - s + 1 = \frac{s}{s^2+9}$$

$$Y = -\frac{1}{s^2+9} + \frac{s}{s^2+9} + \frac{s}{(s^2+9)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$$
$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$$
$$\mathcal{L}^{-1}\left\{\frac{2as}{(s^2+a^2)^2}\right\} = t\sin(at)$$

$$\mathcal{L}^{-1}\{Y(s)\} = -\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{2\cdot 3\cdot s}{(s^2+3^2)^2}\right\}$$

$$\underline{y(t) = -\frac{1}{3}\sin 3t + \cos 3t + \frac{1}{6}t\sin 3t}$$

5.4.11 $\quad y'' - 2y' + 5y = 0 \,, \quad y(0) = 2 \,, \quad y'(0) = 4$

$$s^2 Y - s\,y(0) - y'(0) - 2(sY - y(0)) + 5Y = 0$$

$$s^2 Y - 2s - 4 - 2sY + 4 + 5Y = 0$$

$$(s^2 - 2s + 5)Y = 2s$$

$$Y(s) = \frac{2s}{s^2-2s+5} = 2\,\frac{s}{(s-1)^2+2^2}$$

$$Y(s) = 2\frac{s-1}{(s-1)^2+2^2} + \frac{2}{(s-1)^2+2^2}$$

$$\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos bt$$
$$\mathcal{L}^{-1}\left\{\frac{b}{(s-a)^2+b^2}\right\} = e^{at}\sin bt$$

$$\underline{y(t) = 2e^t\cos 2t + e^t\sin 2t}$$

5.4.13 $\quad y'' + 3y' + 2y = 6 \,, \quad y(0) = 0 \,, \quad y'(0) = 2$

$$s^2 Y - 0 - 2 + 3sY - 0 + 2Y = 6/s$$
$$(s^2+3s+2)Y = 2 + 6/s$$

$$2s+6 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1)$$
$$6 = 2A \,,\quad 4 = -B \,,\quad 2 = 2C$$
$$3 = A \,,\quad -4 = B \,,\quad 1 = C$$

$$Y(s) = \frac{2s+6}{s(s+1)(s+2)} = \frac{3}{s} - \frac{4}{s+1} + \frac{1}{s+2}$$

$$\underline{y(t) = 3 - 4e^{-t} + e^{-2t}}$$

Section 5.4

1. $t+1$ **2.** e^t **3.** $(t+1)e^t$ **4.** $(t+1)e^{-t}$ **5.** $e^t - t$ **6.** $2e^t - e^{2t}$ **7.** $3e^{-2t} + 2t - 2$
8. $2e^{-t} + \sin 3t - 2\cos 3t$ **9.** $\cos 3t - \frac{1}{3}\sin 3t + \frac{1}{6}t\sin 3t$ **10.** $y(t) \equiv 1$ **11.** $e^t(2\cos 2t + \sin 2t)$
12. $10te^{-5t}$ **13.** $3 - 4e^{-t} + e^{-2t}$ **14.** $-\frac{1}{2}\sin t + 2\cos t - \frac{1}{2}t\cos t$ **15.** $\frac{3}{4}e^t - \frac{3}{2}te^t + \frac{3}{2}t^2e^t - \frac{3}{4}e^{-t}$
16. $e^{-2t} + \cos t + 1$ **17.** $\cos t$ **23.** $Y' + s^2Y = 1$ **24.** $c_1\sin t + c_2\cos t + 1$
25. The solution of $y' = ky$ grows faster ($e^t > \cosh kt$).