§3.1 Introduction to Second-order Linear Equations

3.1.17 Find the specific function among the given two parameter family that satisfies the initial condition.

$$y(x) = c_1 e^{2x} + c_2 x e^{2x} \quad , \quad y(0) = 0 \ , \ y'(0) = 2$$

$$y'(x) = 2c_1 e^{2x} + c_2 e^{2x} + 2c_2 x e^{2x}$$

$$0 = y(0) = c_1 \qquad 2 = y'(0) = c_2$$

$$\boxed{y(x) = 2xe^{2x}}$$

3.1.25 Solve $y'y'' = 1$ for $y(x)$.

Let $v = y'$, $v' = y''$.

$$\tfrac{1}{2}(v^2)' = vv' = 1$$
$$v^2 = 2x + c_1$$
$$y' = \pm\sqrt{2x + c_1}$$

$$\boxed{y(x) = \pm \tfrac{1}{3}(2x + c_1) + c_2}$$

3.1.27 Solve $y'' = 1 + (y')^2$

Let $v = y'$, $v' = y''$.

$$v' = 1 + v^2$$
$$(1+v^2)^{-1} v' = 1$$
$$\arctan v = x + c_1$$
$$y' = \tan(x + c_1)$$

$$\boxed{\begin{aligned} y(x) &= -\ln|\cos(x+c_1)| + c_2 \\ &= \ln|\sec(x+c_1)| + c_2 \end{aligned}}$$

§3.2 Fundamental Solutions of the Homogeneous Equation

DEFINITION: Linear Independence and Dependence

Two functions f and g are said to be **linearly dependent** on an interval I if there exist two constants k_1 and k_2, not both zero, that satisfy

$$k_1 f(x) + k_2 g(x) = 0 \qquad (1)$$

for all x in the interval I. Two functions f and g are said to be **linearly independent** on an interval I if they are not linearly dependent on I. That is, Eq. (1) holds for all x in I only for $k_1 = k_2 = 0$.

THEOREM 3.3: Wronskian Test for Linear Independence

Assume that the coefficients $p(x)$ and $q(x)$ in the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \qquad (5)$$

are continuous on (a,b) and that y_1 and y_2 are two given solutions. If *any one* of the following statements is true, then all of the others are also true.

- y_1 and y_2 are linearly independent solutions on (a,b).
- $W[y_1, y_2](x) \neq 0$ for all x in (a,b).
- $W[y_1, y_2](x_0) \neq 0$ for at least one x_0 in (a,b).

3.2.7 Determine whether the functions f and g are independent on $(-1,1)$. Compute the Wronskian of f and g.

$f(x) = e^x \cos x$
$g(x) = e^{2x}\cos x$

$0 \equiv k_1 f(x) + k_2 g(x) = e^x \cos x (k_1 + k_2 e^x)$

No choice of k_1, k_2 satisfies $0 \equiv e^x \cos x (k_1 + k_2 e^x)$

f and g are independent.

$$W[f,g] = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g$$

$$= e^x \cos x (2e^{2x}\cos x - e^{2x}\sin x) - (e^x \cos x - e^x \sin x)(e^{2x}\cos x)$$

$$= e^{3x}\cos x (2\cos x - \sin x - \cos x + \sin x) = \boxed{e^{3x}\cos^2 x} \not\equiv 0 \qquad \boxed{\text{Independent}}$$

For Problems 11–20, carry out the following steps.
(a) Verify that the functions y_1 and y_2 are solutions of the specified differential equation.
(b) Verify that y_1 and y_2 are linearly independent.
(c) Find the general solution of the differential equation.
(d) Find the solution of the indicated initial-value problem.

3.2.17 $y'' - y = 0, \quad y_1(x) = \sinh x \quad y_2(x) = \cosh x \quad y(0) = 0 \quad y'(0) = 1$

(a) $y_1'' - y_1 = \sinh x - \sinh x = 0$
$y_2'' - y_2 = \cosh x - \cosh x = 0$

(b) $W[y_1, y_2] = \begin{vmatrix} \sinh x & \cosh x \\ \cosh x & \sinh x \end{vmatrix}$
$= \sinh^2 x - \cosh^2 x = -1 \not\equiv 0$

(c) $y(x) = c_1 \sinh x + c_2 \cosh x$

(d) $0 = y(0) = c_2$
$1 = y'(0) = c_1$
$\longrightarrow \boxed{y(x) = \sinh x}$

3.2.19 $x^2 y'' + xy' - y = 0, \quad y_1(x) = x \quad y_2(x) = x^{-1} \quad y(1) = 0 \quad y'(1) = 0$

(a) $x^2 y_1'' + xy_1' - y_1 = 0 + x - x = 0$
$x^2 y_2'' + x y_2' - y_2 = 2x^2/x^3 - x/x^2 - 1/x = 0$

(b) $W[y_1, y_2] = \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix}$
$= -x^{-1} - x^{-1} = -2x^{-1} \not\equiv 0$

(c) $y(x) = c_1 x + c_2/x$

(d) $0 = y(1) = c_1 + c_2 \longrightarrow c_2 = -c_1$
$0 = y'(1) = c_1 - c_2 = 2c_1$
$\boxed{y(x) = 0}$

25. Wronskian Identically Zero or Never Zero* A useful property of the Wronskian of two solutions $y_1(x)$ and $y_2(x)$ of

$$y'' + p(x)y' + q(x)y = 0 \qquad (18)$$

is that it is either *identically* zero for all x or *never* zero. Solve the following problems to verify this important property.

(a) Since y_1, y_2 satisfy 18,

$$\begin{aligned} -y_2\,[y_1'' + p(x)y_1' + q(x)y_1 = 0] \\ y_1\,[y_2'' + p(x)y_2' + q(x)y_2 = 0] \\ \hline y_1y_2'' - y_1''y_2 + (y_1y_2' - y_1'y_2)\,p(x) = 0 \end{aligned}$$

Note also that for the Wronskian,

$$W(x) = y_1y_2' - y_1'y_2$$
$$W'(x) = y_1'y_2' + y_1y_2'' - y_1''y_2 - y_1'y_2'$$
$$= y_1y_2'' - y_1''y_2$$

This shows that $W' + p(x)W = 0$

(b) $W' + p(x)W = 0 \longrightarrow W(x) = ce^{-\int p(x)\,dx}$

Since $e^{-\int p(x)dx} > 0$, $0 = W(x)$ iff $c = 0$ and $0 \neq W(x)$ iff $c \neq 0$.

$\therefore$ If the Wronskian is zero for any x, it must be zero for all x.
If the Wronskian is nonzero for any x, it must be nonzero for all x.

§ 3.4 Homogeneous Equations with Constant Coefficients: Real Roots

For Problems 1–19, find the general solution of the given differential equation. When initial conditions are given, find the solution that satisfies the stated conditions.

1. $y'' = 0$
2. $y'' - y' = 0$
3. $y'' - 9y = 0$
4. $4y'' - y = 0$
5. $y'' - 3y' + 2y = 0$
6. $y'' - y' - 2y = 0$
7. $y'' + 2y' + y = 0$
8. $4y'' - 4y' + y = 0$
9. $2y'' - 3y' + y = 0$
10. $y'' - 6y' + 9y = 0$
11. $y'' - 8y' + 16y = 0$
12. $y'' - 25y = 0$ $\quad y(0) = 0 \quad y'(0) = 0$
13. $y'' + y' - 2y = 0$ $\quad y(0) = 1 \quad y'(0) = 0$
14. $y'' + 2y' + y = 0$ $\quad y(0) = 0 \quad y'(0) = 1$
15. $y'' - 9y = 0$ $\quad y(0) = 1 \quad y'(0) = 0$
16. $y'' - 6y' + 9y = 0$ $\quad y(0) = 1 \quad y'(0) = 0$
17. $y'' - 4y' + 4y = 0$ $\quad y(0) = 1 \quad y'(0) = 1$
18. $4y'' - 4y' - 3y = 0$ $\quad y(0) = 0 \quad y'(0) = 1$
19. $y'' - 2y' + y = 0$ $\quad y(1) = 0 \quad y'(1) = 1$

3.4.9 $2y'' - 3y' + y = 0$

$$2\lambda^2 - 3\lambda + 1 = 0$$
$$2\lambda^2 - 2\lambda - \lambda + 1 = 0$$
$$2\lambda(\lambda - 1) - 1(\lambda - 1) = 0$$
$$(2\lambda - 1)(\lambda - 1) = 0 \longrightarrow \lambda = 1/2,\ 1$$

$$\boxed{y(x) = c_1 e^{x/2} + c_2 e^{x}}$$

3.4.18 $4y'' - 4y' - 3y = 0,\ y(0) = 0 \quad y'(0) = 1$

$$\lambda = 4/8 \pm \sqrt{16 + 48}\,/\,8 = 1/2 \pm 1 = -1/2,\ 3/2$$

$$y(x) = c_1 e^{-x/2} + c_2 e^{3x/2}$$

$$0 = y(0) = c_1 + c_2 \rightarrow c_1 = -c_2$$
$$1 = y'(0) = -1/2\, c_1 + 3c_2/2 = \tfrac{1}{2}c_2 + \tfrac{3c_2}{2} = 2c_2 \rightarrow c_2 = 1/2$$

$$\boxed{y(x) = 1/2\, e^{3x/2} - 1/2\, e^{-x/2}}$$

§ 3.5 Homogeneous Equations with Constant Coefficients: Complex Roots

For Problems 1–15, determine the general solution of the given differential equation. If initial conditions are given, find the solution satisfying the stated conditions.

1. $y'' + 9y = 0$
2. $y'' + y' + y = 0$
3. $y'' - 4y' + 5y = 0$
4. $y'' + 2y' + 8y = 0$
5. $y'' + 2y' + 4y = 0$
6. $y'' - 4y' + 7y = 0$
7. $y'' - 10y' + 26y = 0$
8. $3y'' + 4y' + 9y = 0$
9. $y'' - y' + y = 0$
10. $y'' + y' + 2y = 0$
11. $y'' + 4y = 0$ $\quad y(0) = 1 \quad y'(0) = -1$
12. $y'' - 4y' + 13y = 0$ $\quad y(0) = 1 \quad y'(0) = 0$
13. $y'' + 2y' + 2y = 0$ $\quad y(0) = 1 \quad y'(0) = 0$
14. $y'' - y' + y = 0$ $\quad y(0) = 1 \quad y'(0) = 0$
15. $y'' - 4y' + 7y = 0$ $\quad y(0) = -1 \quad y'(0) = 0$

3.5.11 $\quad y'' + 4y = 0, \quad y(0) = 1, \quad y'(0) = -1$

$$\lambda^2 + 4 = 0 \longrightarrow \lambda = \pm 2i$$
$$y(x) = c_1 \cos 2x + c_2 \sin 2x$$

$$1 = y(0) = c_1$$
$$-1 = y'(0) = 2c_2$$

$$\boxed{y(x) = \cos 2x - \tfrac{1}{2}\sin 2x}$$

3.5.13 $\quad y'' + 2y' + 2y = 0, \quad y(0) = 1 \quad y'(0) = 0$

$$\lambda^2 + 2\lambda + 2 = 0$$

$$\lambda = -1 \pm \sqrt{4-8}/2 = -1 \pm i$$

$$y(x) = e^{-x}(c_1 \cos x + c_2 \sin x)$$

$$y'(x) = -e^{-x}(c_1 \cos x + c_2 \sin x) + e^{-x}(-c_1 \sin x + c_2 \cos x)$$

$$1 = y(0) = c_1$$
$$0 = y'(0) = -c_1 + c_2$$

$$\boxed{y(x) = e^{-x}(\cos x + \sin x)}$$

3.5.17 $\quad$ Solve $y''(x) = y(-x)$

Differentiate twice: $\quad y''(x) = y(-x) \rightarrow y^{(4)}(x) = (-1)^2 y''(-x) = y''(-x)$

By reassigning $x := -x$, this is equivalent to $y^{(4)}(-x) = y''(x)$

Substitute into the original equation

$$y(-x) = y''(x) = y^{(4)}(-x)$$

$$y(t) = y^{(4)}(t), \quad x = -t$$

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$$
$$y''(t) = c_1 e^t + c_2 e^{-t} - c_3 \cos t - c_4 \sin t$$

Replace t by $-x$ in $y(t)$ and t by x in $y''(t)$ to get

$$y(-x) = c_1 e^{-x} + c_2 e^{x} + c_3 \cos x - c_4 \sin x$$
$$y''(x) = c_1 e^{x} + c_2 e^{-x} - c_3 \cos x - c_4 \sin x$$

$$0 = y''(x) - y(-x) = e^{x}(c_1 - c_2) - e^{-x}(c_1 - c_2) - 2c_3 \cos x = 2(c_1 - c_2)\sinh x - 2c_3 \cos x \quad (\forall x)$$

$$\left.\begin{aligned} x = \pi/2 &: \quad 0 = (c_1 - c_2)\sinh \pi/2 \rightarrow c_1 = c_2 \\ x = 0 &: \quad 0 = 0 - 2c_3 \cos 0 \rightarrow c_3 = 0 \end{aligned}\right\} \longrightarrow \begin{aligned} y(-x) &= 2c_1 \cosh x - c_4 \sin x \\ y(x) &= 2c_1 \cosh x + c_4 \sin x \\ y''(x) &= 2c_1 \cosh x - c_4 \sin x \end{aligned}$$

$$\therefore \boxed{y(x) = A\cosh x + B\sin x}$$