

## TOPICS FOR TODAY

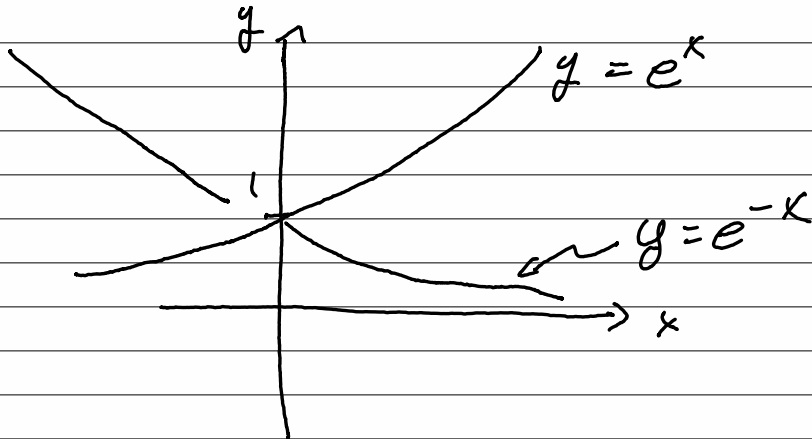
### • REVIEW OF $e^x$ AND $\ln(x)$

$e^x$ : EXPONENTIAL FUNCTION

$e \approx 2.7172 \dots$  TRANSCENDENTAL  
NUMBER

THIS FUNCTION IS SPECIAL, SINCE

$$\frac{d}{dx} \{e^x\} = e^x \quad ; \quad \text{IDENTICAL UNDER DIFFERENTIATION}$$



NOTE:  $e^x > 0$  FOR ALL  $-\infty < x < \infty$

$$\bullet e^{a+b} = e^a e^b$$

$$\bullet e^{-a} = 1/e^a$$

$$\bullet e^{na} = (e^a)^n$$

$$\bullet e^x = 1 + x + \frac{x^2}{2!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

EX: SOLVE  $\frac{dy}{dx} = e^x$ ,  $y(3) = 2$

$$\int \frac{dy}{dx} dx = \int e^x dx$$

$$\Rightarrow y = e^x + C$$

$$y(3) = e^3 + C = 2$$

$$\Rightarrow C = 2 - e^3$$

$$\therefore y(x) = (e^x - e^3) + 2$$

TRY:

$$\frac{dy}{dx} = e^{-x}, \quad y(0) = 1$$

$$\frac{dy}{dx} = e^{-4x}, \quad y(-1) = 2$$

$$\frac{dy}{dx} = 7e^x, \quad y(0) = 4$$

## NATURAL LOGARITHM

$$\ln(x) = \int_1^x \frac{1}{t} dt, \quad x > 0$$

$$\Rightarrow \frac{d \ln(x)}{dx} = \frac{1}{x}, \quad x > 0$$

WE CAN EXTEND THIS DEFINITION FOR  $x \neq 0$ :

$$\frac{dy}{dx} = \frac{1}{x}, \quad y(x_0) = y_0, \quad x_0 \neq 0$$

$$\int \frac{dy}{dx} dx = \int \frac{1}{x} dx = \ln|x| + C$$

$$y = \ln|x| + C$$

$$y(x_0) = \ln|x_0| + C = y_0$$

$$\Rightarrow C = y_0 - \ln|x_0|$$

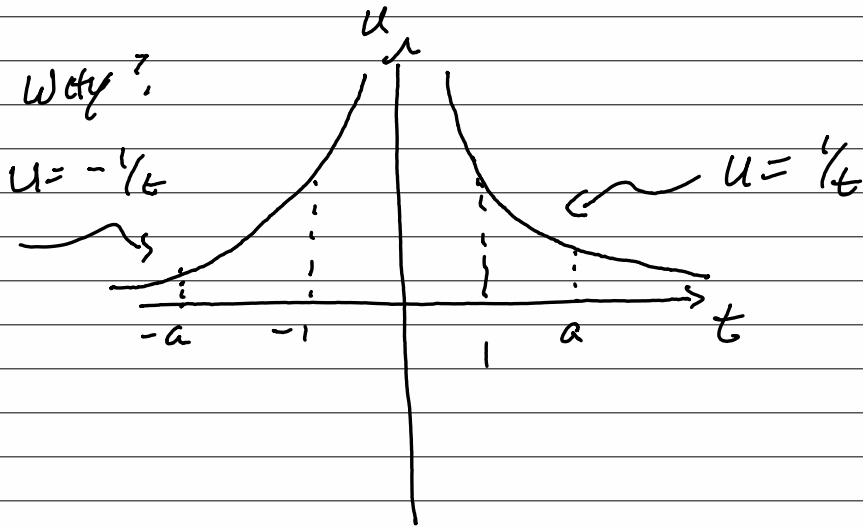
$$\begin{aligned} \therefore y &= y_0 + \ln|x| - \ln|x_0| \\ &= y_0 + \ln \left| \frac{x}{x_0} \right| \end{aligned}$$

Note : •  $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$ ,  $a > 0, b > 0$

$$\ln(1) = 0$$

$$\ln(e^x) = x \Rightarrow e^{\ln x} = x$$

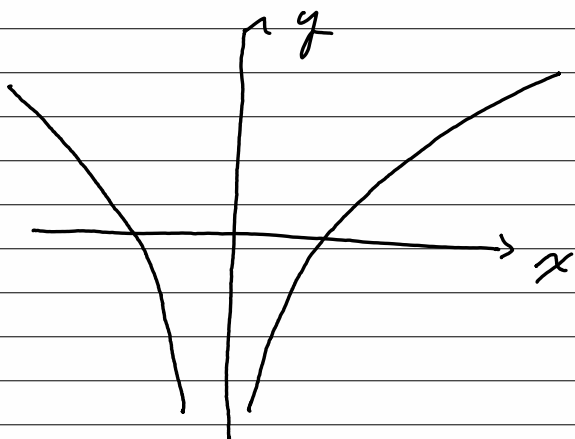
$$\therefore \ln|x| = \begin{cases} \int_1^x \frac{1}{t} dt, & x > 0 \\ \int_x^{-1} -\frac{1}{t} dt, & x < 0 \end{cases}$$



EX:  $\int_x^{-1} -\frac{1}{t} dt, x < 0$        $u = -t$   
 $du = -dt$

$$\int_x^{-1} -\frac{dt}{t} = \int_{-x}^1 \frac{du}{-u} = \int_1^{-x} \frac{du}{u}$$

$$= \int_1^{|x|} \frac{du}{u} = \underline{\underline{\ln|x|}}$$



$$y = \ln|x|$$

$\Rightarrow$  TWO BRANCHES

BRANCH DEPENDS  
ON INITIAL  
DATA

EX:  $\frac{dy}{dx} = \frac{1}{x}$ ,  $y(2) = 1$

$$\int \frac{dy}{dx} dx = \int \frac{1}{x} dx = \ln|x| + C$$

$$y = \ln|x| + C$$

$$y(2) = \ln|2| + C = 1$$

$$\therefore C = 1 - \ln|2| = 1 - \ln|2|$$

$$y(x) = 1 + \ln\left|\frac{x}{2}\right| = \underline{1 + \ln\left|\frac{x}{2}\right|}$$

TRY:  $\frac{dy}{dx} = \frac{2}{x}$ ,  $y(-3) = 9$

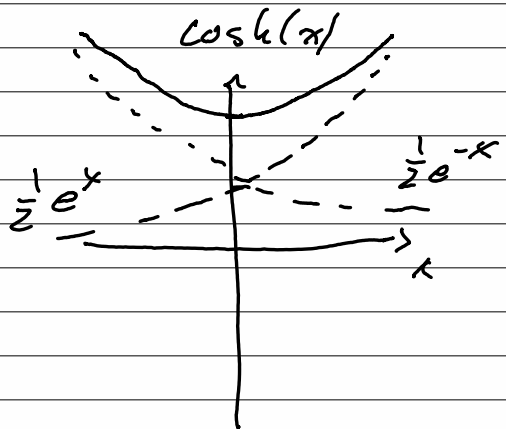
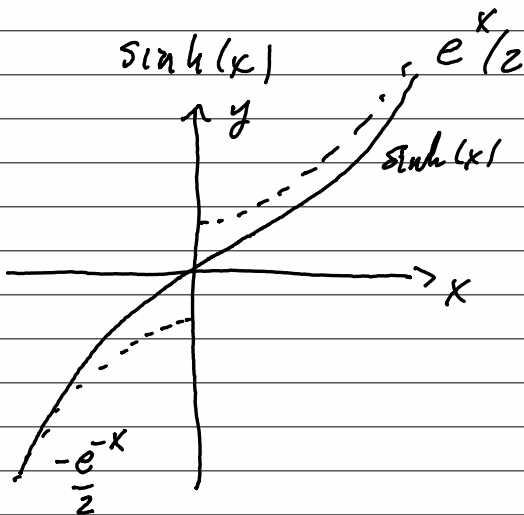
$\frac{dy}{dx} = -\frac{1}{x}$ ,  $y(4) = 2$

IF THERE IS TIME ...

## Hyperbolic Functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad (\text{Pronounced "sinh"})$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad (\text{Pronounced "cosh"})$$



$$\text{As } x \rightarrow \infty : \sinh(x) \sim \frac{1}{2}e^x$$

$$\cosh(x) \sim \frac{1}{2}e^x$$

$$x \rightarrow -\infty \quad \sinh(x) = -\frac{1}{2}e^{-x}$$

$$\cosh(x) = \frac{1}{2}e^{-x}$$

$$\frac{d}{dx} [\sinh x] = \frac{d}{dx} \left( \frac{1}{2}e^x - \frac{e^{-x}}{2} \right) = \frac{e^x}{2} + \frac{e^{-x}}{2} = \cosh(x)$$

$$\frac{d}{dx} [\cosh(x)] = \frac{d}{dx} \left[ \frac{e^x}{2} + \frac{e^{-x}}{2} \right] = \frac{e^x}{2} - \frac{e^{-x}}{2}$$

$$= \sinh(x)$$

$$\therefore \frac{d}{dx} [\sinh x] = \cosh(x)$$

$$\frac{d}{dx} [\cosh x] = \sinh(x)$$

Ans

$$[\sinh x]^2 = \left[ \frac{1}{2}e^x - \frac{1}{2}e^{-x} \right]^2$$

$$= \frac{1}{4}e^{2x} - \frac{1}{4} - \frac{1}{4} + \frac{1}{4}e^{-2x}$$

$$= \frac{1}{4} [e^{2x} + e^{-2x}] - \frac{1}{2}$$

$$[\cosh x]^2 = \left[ \frac{1}{2}e^x + \frac{1}{2}e^{-x} \right]^2$$

$$= \frac{1}{4}e^{2x} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}e^{-2x}$$

$$= \frac{1}{4} [e^{2x} + e^{-2x}] + \frac{1}{2}$$

$$\text{So } \boxed{|\sinh^2(x) - \cosh^2(x)| = 1}$$

## Taylor Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$$

$$\therefore \sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$$

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$

Looks a lot like  $\sin(x)$ ,  $\cos(x)$ ...

Try :

$$\bullet \frac{dy}{dx} = \cosh(x), \quad y(0) = 1$$

$$\bullet \frac{dy}{dx} = \sinh(x), \quad y(2) = \cosh(2)$$

$$\bullet \frac{dy}{dx} = e^{-x} + 2 \sinh x, \quad y(0) = 2$$