§3.7 Solving Nonhomogeneous Equations: Undetermined Coefficients

Consider the nonhomogeneous equation and complementary equation

$$(1)\quad y'' + p(x)y' + q(x)y = f(x)$$
$$(2)\quad y'' + p(x)y' + q(x)y = 0$$

Suppose Y_1, Y_2 are solutions to (1). Then $Y_1 - Y_2$ is a solution to (2)
Suppose y_1, y_2 are a fundamental set of solutions to (2). Any solution to (2) can be written as a linear combination of y_1, y_2. This means:

$$Y_1 - Y_2 = c_1 y_1 + c_2 y_2 = y_h$$

Suppose y is the general solution to (1). Then y satisfies (1) and may take the place of Y_1. If we can find any solution y_p to (1), it may take the place of Y_2

$$y = c_1 y_1 + c_2 y_2 + Y_2 = y_h + y_p$$

This shows that in order to solve (1), find the homogeneous solution y_h of (2) and then try to guess a particular solution y_p based on $f(x)$. The table provides strategies for finding y_p based on the form of $f(x)$.

Type	$f(x)$	$y_p(x)$
1	a (constant)	$x^s \cdot A$ (constant)
2	$P_n(x) = a_0x^n + \cdots + a_{n-1}x + a_n$	$x^s \cdot (A_0x^n + \cdots + A_{n-1}x + A_n)$
3	$ae^{\alpha x}$	$x^s \cdot Ae^{\alpha x}$
4	$a\cos kx + b\sin kx$	$x^s \cdot (A\cos kx + B\sin kx)$
5	$P_n(x)e^{\alpha x}$	$x^s(A_0x^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n)e^{\alpha x}$
6	$P_n(x)\cos kx$	$x^s \cdot \big((A_0x^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n)\cos kx$
7	$P_n(x)\sin kx$	$+ (B_0x^n + B_1x^{n-1} + \cdots + B_{n-1}x + B_n)\sin kx\big)$
8	$P_n(x)e^{\alpha x}\cos kx$	$x^s \cdot \big((A_0x^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n)e^{\alpha x}\cos kx$
9	$P_n(x)e^{\alpha x}\sin kx$	$+ (B_0x^n + B_1x^{n-1} + \cdots + B_{n-1}x + B_n)e^{\alpha x}\sin kx\big)$

Note: The exponent s in x^s is the *smallest* of the integers 0, 1, or 2 (in other words, x^s will be either 1, x, or x^2) that ensures that no term in the particular solution $y_p(x)$ is also a solution of the corresponding homogeneous equation.

PROBLEMS: Section 3.7

For each differential equation in Problems 1–21, find the general solution by finding the homogeneous solution and a particular solution.

1. $y' = 1$
2. $y' + y = 1$
3. $y' + y = x$
4. $y'' = 1$
5. $y'' + 4y' = 1$
6. $y'' + 4y = 1$
7. $y'' + 4y' = x$
8. $y'' + y' - 2y = 3 - 6x$
9. $y'' + y = 6e^x + 3$
10. $y'' - y' - 2y = 6e^x$
11. $y'' + y' = 6 \sin 2x$
12. $y'' + 4y' + 5y = 2e^x$
13. $y'' + 3y' = \sin x + 2 \cos x$
14. $y'' + 4y' + 4y = xe^{-x}$
15. $y'' - y = x \sin x$
16. $y'' - 3y' + 2y = e^x \sin x$
17. $y'' - 4y' + 4y = xe^{2x}$
18. $y'' + y = 12 \cos^2 x$
19. $y'' - 4y' + 3y = 20 \cos x$
20. $y'' - y = 8xe^x$
21. $y'' - 5y' + 6y = \cosh x$

3.7.3 $y' + y = x$

Homogeneous solution $y_h(x)$ satisfies $y' + y = 0$.

Find $y_h(x) = Ce^{-x}$, C constant

Particular Solution $y_p(x)$ satisfies $y' + y = x$.

Try $y_p(x) = Ax + B$.

$x = y_p' + y = A + Ax + B$

Match coefficients.

$1x = Ax \rightarrow A = 1$, $0 = A + B \rightarrow B = -1$

$y_p(x) = x - 1$

General Solution $y(x) = y_h(x) + y_p(x)$

$y(x) = Ce^{-x} + x - 1$

3.7.5 $y'' + 4y' = 1$

$0 = y_h'' + 4y_h'$

$0 = \lambda^2 + 4\lambda = \lambda(\lambda + 4)$

$\lambda = 0, -4$

$y_h(x) = c_1 + c_2 e^{-4x}$

Try $y_p(x) = Ax$

$1 = y_p'' + 4y_p' = 0 + 4A$

$1/4 = A$

$y_p(x) = \frac{1}{4}x$

General Solution

$y(x) = c_1 + c_2 e^{-4x} + \frac{1}{4}x$

3.7.7 $y'' + 4y' = x$

$y_h(x) = c_1 + c_2 e^{-4x}$

(see above)

Try $y_p(x) = x(Ax + B)$

$x = y_p'' + 4y_p' = 2A + 8Ax + 4B$

$1 = 8A \rightarrow A = 1/8$

$0 = 2A + 4B \rightarrow B = -1/16$

General Solution

$y(x) = c_1 + c_2 e^{-4x} + \frac{1}{8}x^2 - \frac{1}{16}x$

(Error in textbook)

3.7.9 $y'' + y = 6e^x + 3$

Homogeneous Solution

$0 = y_h'' + y_h$

$y_h(x) = c_1 \cos x + c_2 \sin x$

Particular Solution

$y_p(x) = Ae^x + B$

$6e^x + 3 = Ae^x + Ae^x + B$

$y_p(x) = 3e^x + 3$

General Solution

$y(x) = c_1 \cos x + c_2 \sin x + 3e^x + 3$

3.7.12 $y'' + 4y' + 5y = 2e^x$

Homogeneous Solution

$0 = \lambda^2 + 4\lambda + 5$

$\lambda = -2 \pm i$

$y_h = e^{-2x}(c_1 \cos x + c_2 \sin x)$

Particular Solution

$y_p = Ae^x$

$2e^x = 10Ae^x$

$A = 1/5$

General Solution

$y(x) = e^{-2x}(c_1 \cos x + c_2 \sin x) + \frac{1}{5}e^x$

3.7.14 $y'' + 4y' + 4y = xe^{-x}$

$0 = \lambda^2 + 4\lambda + 4$

$0 = (\lambda + 2)^2$

$y_h = c_1 e^{-2x} + c_2 x e^{-2x}$

$y(x) = c_1 e^{-2x} + c_2 x e^{-2x} + (x-2)e^{-x}$

$y_p = (Ax + B)e^{-x}$

$y_p' = Ae^{-x} - (Ax+B)e^{-x}$

$y_p'' = -Ae^{-x} - Ae^{-x} + (Ax+B)e^{-x}$

$xe^{-x} = (A - 4A + 4A)xe^{-x} + (-2A + B + 4A - 4B + 4B)e^{-x}$

$xe^{-x} = Axe^{-x} + (2A+B)e^{-x} \to A = 1,\ B = -2$

Type	$f(x)$	$y_p(x)$
1	a (constant)	$x^s \cdot A$ (constant)
2	$P_n(x) = a_0x^n + \cdots + a_{n-1}x + a_n$	$x^s \cdot (A_0x^n + \cdots + A_{n-1}x + A_n)$
3	$ae^{\alpha x}$	$x^s \cdot Ae^{\alpha x}$
4	$a \cos kx + b \sin kx$	$x^s \cdot (A \cos kx + B \sin kx)$
5	$P_n(x)e^{\alpha x}$	$x^s (A_0x^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n)e^{\alpha x}$
6	$P_n(x) \cos kx$	$x^s \cdot \big((A_0x^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n) \cos kx$
7	$P_n(x) \sin kx$	$+ (B_0x^n + B_1x^{n-1} + \cdots + B_{n-1}x + B_n) \sin kx\big)$
8	$P_n(x)e^{\alpha x} \cos kx$	$x^s \cdot \big((A_0x^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n)e^{\alpha x} \cos kx$
9	$P_n(x)e^{\alpha x} \sin kx$	$+ (B_0x^n + B_1x^{n-1} + \cdots + B_{n-1}x + B_n)e^{\alpha x} \sin kx\big)$

Note: The exponent s in x^s is the *smallest* of the integers 0, 1, or 2 (in other words, x^s will be either 1, x, or x^2) that ensures that no term in the particular solution $y_p(x)$ is also a solution of the corresponding homogeneous equation.

3.7.13 $y'' + 3y' = \sin x + 2\cos x$

$0 = \lambda^2 + 3\lambda = \lambda(\lambda+3)$

$y_h = c_1 + c_2 e^{-3x}$

$y_p = A\sin x + B\cos x$

$\sin x + 2\cos x = -A - B + 3A - 3B$

$y_p = \frac{1}{2}\sin x - \frac{1}{2}\cos x$

General Solution

$y(x) = c_1 + c_2 e^{-3x} + \frac{1}{2}\sin x - \frac{1}{2}\cos x$

3.7.19 $y'' - 4y' + 3y = 20\cos x$

$0 = \lambda^2 - 4\lambda + 3$

$0 = (\lambda - 3)(\lambda - 1)$

$y_h = c_1 e^{3x} + c_2 e^{x}$

$y_p = A\cos x + B\sin x$

$y_p' = -A\sin x + B\cos x$

$y_p'' = -A\cos x - B\sin x$

$20 = -A - 4B + 3A = 2A - 4B$

$0 = -B + 4A + 3B = 4A + 2B$

$A = 2$, $B = -4$

General Solution $y(x) = c_1 e^{3x} + c_2 e^{x} + 2\cos x - 4\sin x$

3.7.27 $y'' + 16y = 5\sin x$, $y(0) = y'(0) = 0$.

For Problems 22–28, find the solution to the initial-value problem.

22. $y' - y = 1$	$y(0) = 0$	
23. $y'' + y = 2x$	$y(0) = 1$	$y'(0) = 2$
24. $y'' + y' - 2y = 2x$	$y(0) = 0$	$y'(0) = 1$
25. $y'' - 5y' + 6y = e^x(2x - 3)$	$y(0) = 1$	$y'(0) = 3$
26. $y'' - 4y' + 4y = e^{2x}$	$y(0) = 0$	$y'(0) = 0$
27. $y'' + 16y = 5\sin x$	$y(0) = 0$	$y'(0) = 0$
28. $y'' + 3y' + 2y = 20\cos 2x$	$y(0) = -1$	$y'(0) = 6$

$y_h = c_1 \cos 4x + c_2 \sin 4x$

$y_p = A\cos x + B\sin x$

$5\sin x = -A\cos x - B\sin x + 16A\cos x + 16B\sin x \longrightarrow A = 0,\ B = 1/3$

General Solution

$y(x) = c_1\cos 4x + c_2 \sin 4x + \frac{1}{3}\sin x$

Apply I.C.'s

$0 = y(0) = c_1$

$0 = y'(0) = 4c_2 + \frac{1}{3} \to c_2 = -1/12$

$y(x) = \frac{1}{3}\sin x - \frac{1}{12}\sin 4x$

3.7.39 $y'' - 2y' + y = 2\sin x$

$z'' - 2z' + z = 2e^{ix} = 2(\cos x + i\sin x)$

$z_p = Ae^{ix}$

$2e^{ix} = Ai^2 e^{ix} - 2Aie^{ix} + Ae^{ix}$

$2 = -A - 2Ai + A = -2Ai$

$A = i$

$z_p = ie^{ix} = i\cos x - \sin x$

$y_p = \mathrm{Im}(z_p) = \cos x$

3.7.41 $y'' + 25y = 20\sin 5x$

$z'' + 25z = 20e^{5ix} = 20(\cos 5x + i\sin 5x)$

$z_p = Axe^{5ix}$

Section 3.7

1. $y = x + c$ **2.** $y = ce^{-x} + 1$ **3.** $y = ce^{-x} + x - 1$ **4.** $y = c_1 + c_2 x + \frac{1}{2}x^2$ **5.** $y = c_1 + c_2 e^{-4x} + \frac{x}{4}$

6. $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4}$ **7.** $y = c_1 + c_2 e^{-4x} + \frac{1}{8}x^2 - \frac{1}{16}$ **8.** $y = c_1 e^x + c_2 e^{-2x} + 3x$

9. $y = c_1 \cos x + c_2 \sin x + 3e^x + 3$ **10.** $y = c_1 e^{2x} + c_2 e^{-x} - 3e^x$ **11.** $y = c_1 + c_2 e^{-x} - \frac{6}{5}\sin 2x - \frac{3}{5}\cos 2x$

12. $y = e^{-2x}(c_1 \cos x + c_2 \sin x) + \frac{1}{5}e^x$ **13.** $y = c_1 + c_2 e^{-3x} + \frac{1}{2}(\sin x - \cos x)$

14. $y = c_1 e^{-2x} + c_2 x e^{-2x} + xe^{-x} - 2e^{-x}$ **15.** $y = c_1 e^x + c_2 e^{-x} - \frac{1}{2}(x \sin x + \cos x)$

16. $y = c_1 e^x + c_2 e^{2x} + \frac{1}{2}e^x(\cos x - \sin x)$ **17.** $y = c_1 e^{2x} + c_2 x e^{2x} + \frac{1}{6}x^3 e^{2x}$

18. $y = c_1 \cos x + c_2 \sin x - 2\cos 2x + 6$ **19.** $y = c_1 e^x + c_2 e^{3x} + 2\cos x - 4\sin x$

20. $y = c_1 e^{-x} + e^x(c_2 - 2x + 2x^2)$ **21.** $y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{24}(7\cosh x + 5\sinh x)$ **22.** $y = e^x - 1$

23. $y = \cos x + 2x$ **24.** $y = e^x - \frac{1}{2}e^{-2x} - x - \frac{1}{2}$ **25.** $y = e^{2x} + xe^x$ **26.** $y = \frac{1}{2}x^2 e^{2x}$

27. $y = \frac{1}{3}\sin x - \frac{1}{12}\sin 4x$ **28.** $y = -\cos 2x + 3\sin 2x$ **29.** $y_p = e^x(Ax^3 + Bx^2 + Cx)$

39. $y = c_1 e^x + c_2 x e^x + \cos x$

41. $y = c_1 \cos 5x + c_2 \sin 5x - 2x\cos 5x$