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MA 2631

Probability Theory

Section AL01 / AD01

Conference 3 Ideas

based on Lectures of Chapter 3.1–4.1

1. There are two urns. In the first urn there are 6 red and 2 black balls, in the second one 3 red and 7 black. We roll a (fair) die, and if the die shows a 1 or 2 we draw at random a ball from the first urn and in the case that the die shows a 3, 4, 5, or 6 we draw a ball at random from the second urn.

   a) What is the probability that the drawn ball is red?

   b) If the drawn ball is red, what is the probability that it was from the first urn?

   We denote the following events:

   $$\begin{aligned} R &\ldots \textit{red ball drawn} \\ F_1 &\ldots \textit{ball drawn from first urn} \\ F_2 &\ldots \textit{ball drawn from second urn} \end{aligned}$$

   and note that we have

   $$\mathbb{P}[F_1] = \frac{1}{3}, \quad \mathbb{P}[F_2] = \frac{2}{3}, \quad \mathbb{P}[R \,|\, F_1] = \frac{3}{4}, \quad \mathbb{P}[R \,|\, F_2] = \frac{3}{10}.$$

   i) By the law of the total probability we have

   $$\mathbb{P}[R] = \mathbb{P}[R \,|F_1] \cdot \mathbb{P}[F_1] + \mathbb{P}[R \,|F_2] \cdot \mathbb{P}[F_2] = \frac{3}{4} \cdot \frac{1}{3} + \frac{3}{10} \cdot \frac{2}{3} = \frac{1}{4} + \frac{1}{5} = \frac{9}{20} = 45\%.$$

   ii) From Bayes's formula it follows that

   $$\mathbb{P}[F_1 \,|\, R] = \frac{\mathbb{P}[R \,|F_1] \cdot \mathbb{P}[F_1]}{\mathbb{P}[R \,|F_1] \cdot \mathbb{P}[F_1] + \mathbb{P}[R \,|F_2] \cdot \mathbb{P}[F_2]} = \frac{\frac{1}{4}}{\frac{9}{20}} = \frac{5}{9} \approx 55.56\%.$$

2. Let A, B and C be three independent events in a sample space Ω and define $E = A \cup B$. Show that also the events E and C are independent.

   a) *By the definition of the conditional probability and the distributive law and the inclusion-exclusion principle we have*

$$\begin{aligned}
\mathbb{P}[A \cup B \,|\, C] &= \frac{\mathbb{P}[(A \cup B) \cap C]}{\mathbb{P}[C]} = \frac{\mathbb{P}[(A \cap C) \cup (B \cap C)]}{\mathbb{P}[C]} \\
&= \frac{\mathbb{P}[A \cap C] + \mathbb{P}[B \cap C] - \mathbb{P}[(A \cap C) \cap (B \cap C)]}{\mathbb{P}[C]} \\
&= \frac{\mathbb{P}[A \cap C]}{\mathbb{P}[C]} + \frac{\mathbb{P}[B \cap C]}{\mathbb{P}[C]} - \frac{\mathbb{P}[(A \cap B) \cap C]}{\mathbb{P}[C]} \\
&= \mathbb{P}[A \,|\, C] + \mathbb{P}[B \,|\, C] - \mathbb{P}[A \cap B \,|\, C].
\end{aligned}$$

   b) i) *By the definition of the conditional probability and the distributive law we have*

$$\begin{aligned}
\mathbb{P}\big[A \cup B \,|\, A \cap B\big] &= \frac{\mathbb{P}\big[(A \cup B) \cap (A \cap B)\big]}{\mathbb{P}[A \cap B]} = \frac{\mathbb{P}\big[\big(A \cap (A \cap B)\big) \cup \big(B \cap (A \cap B)\big)\big]}{\mathbb{P}[A \cap B]} \\
&= \frac{\mathbb{P}\big[(A \cap B) \cup (A \cap B)\big]}{\mathbb{P}[A \cap B]} = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[A \cap B]} = 1.
\end{aligned}$$

   ii) *As* $\mathbb{P}[A \cap B] > 0$ *and* $A \cup B \supseteq A \cap B$, *it follows that* $\mathbb{P}[A \cup B] > 0$ *and we can define a new probability measure* $\mathbb{Q}[\,\cdot\,] = \mathbb{P}\big[\,\cdot \,|\, A \cup B\big]$ *and get by the inclusion-exclusion principle*

$$\begin{aligned}
\mathbb{P}\big[A \cap B \,|\, A \cup B\big] &= \mathbb{Q}[A \cap B] = \mathbb{Q}[A] + \mathbb{Q}[B] - \mathbb{Q}[A \cap B] \\
&= \mathbb{P}\big[A \,|\, A \cup B\big] + \mathbb{P}\big[B \,|\, A \cup B\big] - \mathbb{P}\big[A \cup B \,|\, A \cup B\big] \\
&= \mathbb{P}\big[A \,|\, A \cup B\big] + \mathbb{P}\big[B \,|\, A \cup B\big] - \frac{\mathbb{P}\big[(A \cup B) \cap (A \cup B)\big]}{\mathbb{P}[A \cup B]} \\
&= \mathbb{P}\big[A \,|\, A \cup B\big] + \mathbb{P}\big[B \,|\, A \cup B\big] - \frac{\mathbb{P}[A \cup B]}{\mathbb{P}[A \cup B]} \\
&= \mathbb{P}\big[A \,|\, A \cup B\big] + \mathbb{P}\big[B \,|\, A \cup B\big] - 1.
\end{aligned}$$

3. Two students participate in a quiz show where they are asked a true-false question. Both know, independently, the correct answer with probability p. Which of the following strategies is better for the team?

   i) Choose a priori one of the two who will give the answer.

   ii) Give the common answer if the answers agree, and if not flip a coin to decide which answer is given.

Denote the events A_1 *the answer of the first student is correct and* A_2 *the answer of the second student s correct. We know* $\mathbb{P}[A_1] = \mathbb{P}[A_2] = p$ *Therefore we have for i)*

$$\mathbb{P}[A_1] = \mathbb{P}[A_2] = p$$

and for ii) we note that the event that they disagree is

$$D = (A_1^c \cap A_2) \cup (A_1 \cap A_2^c)$$

and note $A_1 \cap D = A_1 \cap A_2^c$ *and likewise* $A_2 \cap D = A_1^c \cap A_2$*. Moreover,*

$$\begin{aligned}
\mathbb{P}[D \cap A_2] &= \mathbb{P}[A_1^c \cap A_2] = \mathbb{P}[A_1^c] \cdot \mathbb{P}[A_2] = (1 - \mathbb{P}[A_1])\mathbb{P}[A_2] = p(1-p) \\
\mathbb{P}[A_1 \cap D] &= \mathbb{P}[A_1 \cap A_2^c] = \mathbb{P}[A_1] \cdot \mathbb{P}[A_2^c] = \mathbb{P}[A_1](1 - \mathbb{P}[A_2^]) = p(1-p).
\end{aligned}$$

Denote now the event that the coin toss chooses the first student by H_1 *and the second student by* H_2*. We have then*

$$\begin{aligned}
&\mathbb{P}[A_1 \cap A_2] + \mathbb{P}[H_1 \cap A_1 \cap D] + \mathbb{P}[H_2 \cap A_2 \cap D] \\
&= \mathbb{P}[A_1] \cdot \mathbb{P}[A_2] + \mathbb{P}[H_1] \cdot \mathbb{P}[A_1 \cap D] + \mathbb{P}[H_2] \cdot \mathbb{P}[A_2 \cap D] \\
&= p^2 + \frac{1}{2}p + (1-p)\frac{1}{2}p(1-p) = p.
\end{aligned}$$

Thus, the success probability of both strategies is the same.

4. Let Ω be a sample space and E, F, G be independent events such that $\mathbb{P}[G] > 0$. Show that the events E and F are independent when using instead of $\mathbb{P}$ the probability $\mathbb{Q}$ defined as $\mathbb{Q}[A] = \mathbb{P}[A \,|G]$ for events A.

   We have, using the definitions of $\mathbb{Q}$ *and the conditional probability as well as the independence under* $\mathbb{P}$*,*

$$\begin{aligned}
\mathbb{Q}[E \cap F] &= \mathbb{P}[E \cap F \mid G] = \frac{\mathbb{P}[E \cap F \cap G]}{\mathbb{P}[G]} = \frac{\mathbb{P}[E] \cdot \mathbb{P}[F] \cdot \mathbb{P}[G]}{\mathbb{P}[G]} = \mathbb{P}[E] \cdot \mathbb{P}[F] \\
&= \frac{\mathbb{P}[E] \cdot \mathbb{P}[G]}{\mathbb{P}[G]} \cdot \frac{\mathbb{P}[F] \cdot \mathbb{P}[G]}{\mathbb{P}[G]} = \mathbb{P}[E \,|G] \cdot \mathbb{P}[F \mid G] = \mathbb{Q}[E] \cdot \mathbb{Q}[F].
\end{aligned}$$

5. Assume that you have three four-sided dice with number 1, 2, 3, and 4 on the four sides and let denote by X the sum of the numbers shown on their bottom side. Write down and sketch the probability mass function of X.