MA 2631 Assignment 2

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1. 65 students are registered for a math class, which will be held in two sections.

   (a) In how many ways can the students be split into two sections?

   (b) Due to the size of the classroom at most 34 students can be in each section. In how many ways can the two sections be organized under this constraint?

   Answer:

   (a) For each of the 65 students, there are 2 choices for which class the student can be assigned to. This gives 2^{65} ways. Exclude the 2 cases in which all students are assigned to the same class, leaving the other empty. Therefore there are $2^{65} - 2 \approx 3.7 \cdot 10^{19}$ ways to split the students into two sections.

   (b) Since each section can accommodate no more than 34 students, the only allowable values for the number of students in a classroom are 34, 33, 32, and 31.

$$\binom{65}{34} + \binom{65}{33} + \binom{65}{32} + \binom{65}{31} \approx 1.4 \times 10^{19}.$$

2. Prove by induction that for all positive integers n it holds that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0.$$

Answer: First establish the base case $n = 1$.

$$\sum_{k=0}^{1}(-1)^k\binom{n}{k} = (-1)^0\binom{1}{0} + (-1)^1\binom{1}{1} = 1 - 1 = 0.$$

Assume the result holds for $n \geq 1$.

$$\begin{aligned}
\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k} &= (-1)^0\binom{n+1}{0} + \sum_{k=1}^{n}(-1)^k\binom{n+1}{k} + (-1)^{n+1}\binom{n+1}{n+1} \\
&= 1 + \sum_{k=1}^{n}(-1)^k\binom{n+1}{k} + (-1)^{n+1} \\
&= 1 + \sum_{k=1}^{n}(-1)^k\left(\binom{n}{k-1} + \binom{n}{k}\right) + (-1)^{n+1} \\
&= 1 + \sum_{k=1}^{n}(-1)^k\binom{n}{k-1} + \sum_{k=1}^{n}(-1)^k\binom{n}{k} + (-1)^{n+1} \\
&= \sum_{j=0}^{n-1}(-1)^{j+1}\binom{n}{j} + (-1)^{n+1} + 1 + \sum_{k=1}^{n}(-1)^k\binom{n}{k} \\
&= \sum_{j=0}^{n-1}(-1)^{j+1}\binom{n}{j} + (-1)^{n+1}\binom{n}{n} + (-1)^0\binom{n}{0} + \sum_{k=1}^{n}(-1)^k\binom{n}{k} \\
&= -\sum_{j=0}^{n-1}(-1)^{j}\binom{n}{j} - (-1)^{n}\binom{n}{n} + \sum_{k=0}^{n}(-1)^k\binom{n}{k} \\
&= -\sum_{j=0}^{n}(-1)^{j}\binom{n}{j} + \sum_{k=0}^{n}(-1)^k\binom{n}{k} \\
&= 0
\end{aligned}$$

3. There is an easier way to prove this statement of the previous problem, using results from the lecture. Please provide that proof.

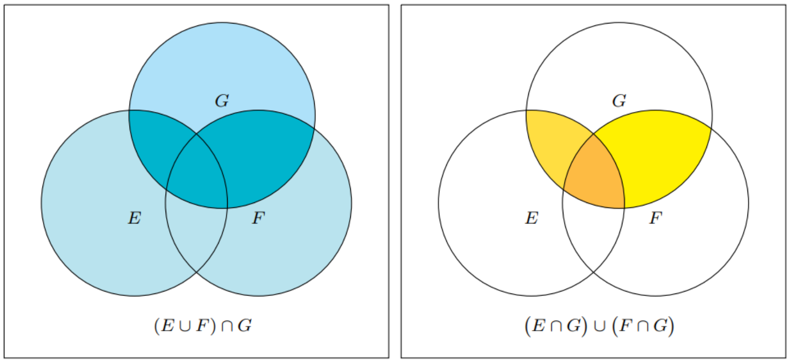
   Answer: Using the binomial formula $\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = (x+y)^n$,

$$\begin{aligned} 0 &= (1 + (-1))^n \\ &= \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (-1)^k \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k}. \end{aligned}$$

4. Let E, F, G be events on a sample space Ω. We know from class the distributive law

$$(E \cup F) \cap G = (E \cap G) \cup (F \cap G).$$

Illustrate by Venn diagrams (one diagram for the left and one for the right hand of the equation).



5. Given a family of events $E_1, E_2, \ldots, E_n, \ldots$ on some sample space Ω, construct a new family $F_1, F_2, \ldots, F_n, \ldots$ on the same sample space Ω such that the F_i are disjoint and

$$\bigcup_{k=1}^{n} F_k = \bigcup_{k=1}^{n} E_k.$$

Answer: Define the family F_n by:

$$\begin{aligned}
&F_1 := E_1\\
&F_2 := E_2 \backslash F_1 = E_2 \backslash E_1 = E_2 \cap E_1^c\\
&F_3 := E_3 \backslash (F_1 \cup F_2) = E_3 \cap (F_1 \cup F_2)^c = E_3 \cap (F_1^c \cap F_2^c)\\
&F_4 := E_4 \backslash (F_1 \cup F_2 \cup F_3) = E_4 \cap (F_1 \cup F_2 \cup F_3)^c = E_4 \cap (F_1^c \cap F_2^c \cap F_3^c)\\
&\quad \vdots\\
&F_n = E_n \backslash \bigcup_{k=1}^{n-1} F_k = E_n \cap \left(\bigcap_{k=1}^{n-1} F_k^c \right).
\end{aligned}$$

To show that the F_k are disjoint, suppose $0 < m < n$.

$$\begin{aligned}
F_n \cap F_m &= \big(E_n \cap (\cap_{k=1}^{n-1} E_k^c)\big) \cap \big(E_m \cap (\cap_{j=1}^{m-1} E_j^c)\big)\\
&= \big(E_n \cap (\cap_{k=1}^{m-1} E_k^c) \cap E_m^c \cap (\cap_{k=m+1}^{n-1} E_k^c)\big) \cap \big(E_m \cap (\cap_{j=1}^{m-1} E_j^c)\big)\\
&= \big(E_n \cap (\cap_{k=1}^{m-1} E_k^c) \cap (\cap_{k=m+1}^{n-1} E_k^c)\big) \cap E_m^c \cap E_m \cap (\cap_{j=1}^{m-1} E_j^c)\\
&= \big(E_n \cap (\cap_{k=1}^{m-1} E_k^c) \cap (\cap_{k=m+1}^{n-1} E_k^c)\big) \cap \emptyset \cap (\cap_{j=1}^{m-1} E_j^c)\\
&= \emptyset.
\end{aligned}$$

Prove that $\bigcup_{k=1}^{n} F_k = \bigcup_{k=1}^{n} E_k$ by induction. For $n = 1$ the equality holds by the definition $F_1 := E_1$. Assume the equality holds for some $n \geq 1$.

$$
\begin{aligned}
\bigcup_{k=1}^{n+1} F_k &= F_{n+1} \cup \left(\bigcup_{k=1}^{n} F_k \right) \\
&= \left(E_{n+1} \cap \left(\bigcup_{k=1}^{n} F_k \right)^c \right) \cup \left(\bigcup_{k=1}^{n} F_k \right) \\
&= \left(E_{n+1} \cup \left(\bigcup_{k=1}^{n} F_k \right) \right) \cap \left(\left(\bigcup_{k=1}^{n} F_k \right)^c \cup \left(\bigcup_{k=1}^{n} F_k \right) \right) \\
&= \left(E_{n+1} \cup \left(\bigcup_{k=1}^{n} F_k \right) \right) \cap \Omega \\
&= \left(E_{n+1} \cup \left(\bigcup_{k=1}^{n} F_k \right) \right) \cap \Omega \\
&= \left(E_{n+1} \cup \left(\bigcup_{k=1}^{n} E_k \right) \right) \cap \Omega \\
&= \left(\bigcup_{k=1}^{n+1} E_k \right) \cap \Omega \\
&= \bigcup_{k=1}^{n+1} E_k .
\end{aligned}
$$

6. 4 dice are rolled.

   (a) Describe mathematically the sample space of this experiment.

   (b) Describe mathematically the events

$$E = \text{"exactly three dice show a six"},$$
$$F = \text{"at least two dice show a six"}.$$

   (c) Describe mathematically the events $E^c, E \cap F, E^c \cup F^c$.

Answer:

(a) $\Omega = \{(i,j,k,l) : i,j,k,l \in \{1,2,3,4,5,6\}\}$.

(b)

$$\begin{aligned} E = &\{(6,6,6,k) : k \in \{1,2,3,4,5\}\} \cup \{(6,6,k,6) : k \in \{1,2,3,4,5\}\} \\ &\cup \{(6,k,6,6) : k \in \{1,2,3,4,5\}\} \cup \{(k,6,6,k) : k \in \{1,2,3,4,5\}\} \end{aligned}$$

$$\begin{aligned} F = &\{(6,6,k,j) : k,j \in \{1,2,3,4,5,6\}\} \cup \{(6,k,6,j) : k,j \in \{1,2,3,4,5,6\}\} \\ &\cup \{(6,k,j,6) : k,j \in \{1,2,3,4,5,6\}\} \cup \{(k,6,6,j) : k,j \in \{1,2,3,4,5,6\}\} \\ &\cup \{(k,j,6,6) : k,j \in \{1,2,3,4,5,6\}\} \cup \{(k,6,j,6) : k,j \in \{1,2,3,4,5,6\}\} \end{aligned}$$

(c)

$$\begin{aligned} E^c = &\{(6,6,6,6)\} \cup \{(i,j,k,l) : i,j,k,l \in \{1,2,3,4,5\}\} \\ &\cup \{(6,6,k,j) : k,j \in \{1,2,3,4,5\}\} \cup \{(6,k,6,j) : k,j \in \{1,2,3,4,5\}\} \\ &\cup \{(6,k,j,6) : k,j \in \{1,2,3,4,5\}\} \cup \{(k,6,6,j) : k,j \in \{1,2,3,4,5\}\} \\ &\cup \{(k,j,6,6) : k,j \in \{1,2,3,4,5\}\} \cup \{(k,6,j,6) : k,j \in \{1,2,3,4,5\}\} \\ &\cup \{(6,i,j,k) : (i,j,k \in \{1,2,3,4,5\}\} \cup \{(i,6,j,k) : (i,j,k \in \{1,2,3,4,5\}\} \\ &\cup \{(i,j,6,k) : (i,j,k \in \{1,2,3,4,5\}\} \cup \{(i,j,k,6) : (i,j,k \in \{1,2,3,4,5\}\} \\ E \cap F = &E \quad \text{since } E \subseteq F. \\ E^c \cup F^c = &E^c \text{since } F^c \subseteq E^c. \end{aligned}$$