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Fall 2022 - A Term

MA 2631

Probability Theory

Section AL01 / AD01

Midterm Exam, 09/16/2022, 12:00–12:50

This exam consists of four (4) problems on two (2) pages. You have fifty (50) minutes for the exam. **Good luck!**

Exercise	1	2	3	4	Σ
Points					

1. Consider two urns, one with 2 red, 1 green and 1 yellow ball, and the second one with 2 red and 1 green ball. You pick first an urn at random and then a ball from there ar random.
   a) What is the probability that the drawn ball is red?
   b) Given you have drawn a red ball, what is the probability it came from the first urn?

   Denote the events

   U_1 *. . . ball drawn from urn 1*

   U_2 *. . . ball drawn from urn 2*

R ... *red ball drawn*
G ... *green ball drawn*
Y ... *yellow ball drawn*

And note that the information given is

$$\mathbb{P}[U_1] = \mathbb{P}[U_2] = \frac{1}{2},$$
$$\mathbb{P}[R\,|U_1] = \frac{2}{4} = \frac{1}{2}, \quad \mathbb{P}[G\,|U_1] = \frac{1}{4}, \quad \mathbb{P}[Y\,|U_1] = \frac{1}{4},$$
$$\mathbb{P}[R\,|U_2] = \frac{2}{3}, \quad \mathbb{P}[G\,|U_2] = \frac{1}{3}. \quad \textbf{\textit{(2.5 points)}}$$

a) By the formula of the total probability we have

$$\mathbb{P}[R] \overset{\textit{(1 pt)}}{=} \mathbb{P}[R\,|U_1]\cdot\mathbb{P}[U_1]+\mathbb{P}[R\,|U_2]\cdot\mathbb{P}[U_2] \overset{\textit{(0.5 pts)}}{=} \frac{1}{2}\cdot\frac{1}{2}+\frac{2}{3}\cdot\frac{1}{2} \overset{\textit{(1 pt)}}{=} \frac{7}{12}.$$

b) By Bayes's formula

$$\mathbb{P}[U_1\,|R] \overset{\textit{(1.5 pts)}}{=} \frac{\mathbb{P}[R\,|U_1]\cdot\mathbb{P}[U_1]}{\mathbb{P}[R\,|U_1]\cdot\mathbb{P}[U_1]+\mathbb{P}[R\,|U_2]\cdot\mathbb{P}[U_2]}$$
$$\overset{\textit{(0.5 pt)}}{=} \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{1}{2}+\frac{2}{3}\cdot\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{7}{12}} \overset{\textit{(1 pt)}}{=} \frac{3}{7}.$$

2. Suppose that A and B are independent events on a sample space Ω. Prove that A^c and B^c are independent.

We have

$$\begin{aligned}
\mathbb{P}[A^c\cap B^c] &\overset{\textit{(2 pts)}}{=} \mathbb{P}[(A\cup B)^c] \overset{\textit{(1 pt)}}{=} 1-\mathbb{P}[A\cup B] \\
&\overset{\textit{(2 pts)}}{=} 1-\Big(\mathbb{P}[A]+\mathbb{P}[B]-\mathbb{P}[A\cap B]\Big) \\
&\overset{\textit{(1 pt)}}{=} 1-\Big(\mathbb{P}[A]+\mathbb{P}[B]-\mathbb{P}[A]\cdot\mathbb{P}[B]\Big) \\
&\overset{\textit{(0.5 pts)}}{=} 1-\mathbb{P}[A]-\mathbb{P}[B]\big(1-\mathbb{P}[A]\big) \\
&\overset{\textit{(0.5 pts)}}{=} \big(1-\mathbb{P}[A]\big)\big(1-\mathbb{P}[B]\big) \overset{\textit{(1 pt)}}{=} \mathbb{P}[A^c]\cdot\mathbb{P}[B^c]
\end{aligned}$$

using DeMorgan, the property of the complement and the inclusion exclusion principle and again the property of the complement consecutively.

3. As you want to move, you pack your boxes carefully not too break a thing. In particular, you have 4 plates and 6 bubble wraps, placing them on top of each other. Note that the plates and bubble wraps, are identical, so you cannot differentiate between them.

   a) In how many ways can you arrange the plates and bubble wraps?
   b) In how many of those ways your belongings are safe, i.e., each plate is separate from the next by at least one bubble wrap?
   c) If you staple plates and bubble wrap at random in a moving box, how likely is it that you end up in an unsecured configuration in which you might brake plates (as two plates are not separated by bubble wrap)?

   Note: No need to simplify formulas.

   a) *You are choosing 4 slots for the plates out of 6 possible* ***(1 point)*** *thus* $\binom{10}{4}$ ***(2 points)****. [Alternative solution 1: You are choosing 4 slots for the plates out of 6 possible* ***(1 point)*** *thus* $\binom{10}{6}$. ***(2 points)****; Alternative solution 2: You have* 10! *ways t arrange the 10 objects, but as you can not discern the plates and wraps you have to divide this by the number of ways how to arrange them in each category,* 3! *and* 6! ***(1 point)*** *thus* $\frac{10!}{6! \cdot 4!}$. ***(2 points)****.*
   b) *Assume the bubble wraps are already assorted, then you have 5 spaces between two of them as well two at the end, hence 7 in total.* ***(1 point)****. Thus you have 7 places to choose from for the 4 plates, thus* $\binom{10}{4}$ *possibilities.* ***(2 points)***
   c) *We have to divide the stable configurations by all configurations* ***(1 point)****, so*

$$\frac{\binom{7}{4}}{\binom{10}{4}} \quad \textit{\textbf{(1 point)}}$$

4. Consider a discrete random variable Z with probability mass function

$$p_Z(0) = 0.2, \quad p_Z(2) = 0.4, \quad p_Z(5) = c, \quad p_Z(8) = 0.1,$$

   for some constant c.

   a) What is c?

b) What is $\mathbb{E}[Z]$?

c) Calculate $\mathbb{P}[Z = 5 \,|\, Z \geq 3]$.

a) As all probabilities have to sum up to 1 ***(1 point)****, we have*

$$0.2 + 0.4 + c + 0.1 = 1 \quad \textbf{\textit{(0.5 point)}}$$

and thus

$$c = 0.3 \quad \textbf{\textit{(1 point)}}$$

b) We have

$$\begin{aligned}\mathbb{E}[Z] &\overset{\textbf{\textit{(1 pt)}}}{=} 0 * p_z(0) + 2 * p_(2) + 5 * p_z(5) + 0.1 * p_Z(8) \\ &= 0 * 0.2 + 2 * 0.4 + 5 * 0.3 + 8 * 0.1 \overset{\textbf{\textit{(1 pt)}}}{=} 3.1\end{aligned}$$

c) By the definition of conditional probabilities

$$\begin{aligned}\mathbb{P}[Z = 5 \,|\, Z \geq 3] &\overset{\textbf{\textit{(1 pt)}}}{=} \frac{\mathbb{P}[\{Z = 5\} \cap \{Z \geq 3\}]}{\mathbb{P}[Z \geq 3]} \overset{\textbf{\textit{(1 pt)}}}{=} \frac{\mathbb{P}[Z = 5]}{\mathbb{P}[Z \geq 3]} \\ &\overset{\textbf{\textit{(0.5 pt)}}}{=} \frac{\mathbb{P}[Z = 5]}{\mathbb{P}[Z = 5] + \mathbb{P}[Z = 8]} = \frac{0.3}{0.3 + 0.1} \\ &\overset{\textbf{\textit{(1 pt)}}}{=} \frac{3}{4}.\end{aligned}$$