Worcester Polytechnic Institute
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Fall 2021 - A Term

MA 2631

Probability Theory

Section AL01 / AD01

Assignment 8

due on Friday, October 1

based on Lectures of Chapter 5.2–5.3

1. Let X be a continuous random variable with density f, expectation $\mathbb{E}[X] = \mu$ and variance $\mathbb{V}ar[X] = \sigma^2$. Define a new random variable $Y := aX + b$ for some a, $b \in \mathbb{R}$.

   a) Calculate the standard deviation $\mathbb{S}D[Y]$.

   b) Express the moment generating function m_Y in terms of m_X.

2. Prove that for an arbitrary continuous random variable X with density f we have

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}[X > x]\,dx - \int_0^\infty \mathbb{P}[X < -x]\,dx.$$

3. Assume that $U^{0,1}$ is a uniformly distributed random variable on the unit interval. Find a real-valued function $g : [0,1) \to \mathbb{R}$ such that $Y := g(U^{0,1})$ is an exponentially distributed random variable with parameter $\lambda > 0$.

   Note: This approach is very important for the simulation of distributions using a computer. The built-in (pseudo-)random number generator produces standard uniform distributed random variables. The transformation in thi example shows how the random number generator can be used to generated random numbers following an exponential distribution. This can be generalized to other distribution and is often referred to as "inverse transform sampling".

4. The lifetime of an electrical device (in months) is given by the continuous random variable X with density
$$f(x) = \begin{cases} cxe^{-\frac{x}{2}} & \text{if } x > 0; \\ 0 & \text{if } x \leq 0. \end{cases}$$

   a) What is c?

   b) What is the probability that the device functions more than 5 months?

   c) What is the expected lifetime of the device?

5. Assume that X is an exponentially distributed random variable with parameter $\lambda > 1$. Calculate

   a) $\mathbb{E}[X^3]$;

   b) $\mathbb{E}[e^X]$.

   Why did we impose the condition $\lambda > 1$ (instead of the "usual" one, $\lambda > 0$)?

6. Find the cumulative distribution function F such that it has hazard rate $\lambda(t) = \frac{1}{\sqrt{t}}$ (for $t > 0$). Can you express F in terms of an exponentially distributed random variable?

8 points per problems

Additional practice problems (purely voluntary - no points, no credit, no grading):

Standard Carlton and Devore, Section 3.1: Exercises 12, 17; Section 3.2: Exercises 25, 26, 34, 37; Section 3.4: Exercises 71, 73, 79, 82;

Extra Show that the exponential distribution is the only distribution taking values on $[0, \infty)$ that is memoryless. Compare this to Homework 7, Problem 2.