MA 2631 Assignment 10

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1. Assume that X is a normally distributed random variable with mean μ and variance σ^2. Compute the probabilities that X is not more than one, two and three standard deviations away from the mean, i.e. $P[|X-\mu| \le \sigma], P[|X-\mu| \le 2\sigma]$ and $P[|X-\mu| \le 3\sigma]$.

   Answer:

$$\begin{aligned} P[|X-\mu| \le k\sigma] &= P[\mu - k\sigma \le X \le \mu + k\sigma] \\ &= \int_{\mu-k\sigma}^{\mu+k\sigma} \frac{1}{\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}\, dx \\ &\approx \begin{cases} 68.27\% & k=1 \\ 95.45\% & k=2 \\ 99.73\% & k=3 \end{cases} \end{aligned}$$

2. Let X be a standard normal distributed random variable. Find the value of $\beta \in \mathbb{R}$ such that $P[X^2 < \beta] = 0.5$.

   Answer:

$$0.5 = P[X^2 < \beta] = P[-\sqrt{\beta} < X < \sqrt{\beta}].$$

   Look in table of cdf values for a standard normal random variable to find $\Phi(0.75)$ so that there is 25% on each side. It says $0.67 < x < 0.68$, so take $x \approx 0.675$. That is, $0.5 \approx P[-0.675 < X < 0.675]$.

   Pick $\beta = 0.675^2 = \boxed{0.455625}$ to satisfy $P[X^2 < \beta] \approx 0.5$.

3. Let $X \sim \mathcal{N}(0,1)$ be a standard normal distributed random variable. Calculate the moment generating function $m_X(\lambda) = E[e^{\lambda X}]$ for $\lambda \in \mathbb{R}$. Use the moment generating function to calculate the mean and variance of X.

Answer:

$$\begin{aligned}
m_X(\lambda) = E[e^{\lambda X}] &= \int_{-\infty}^{\infty} e^{\lambda x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2-2\lambda x)/2}\, dx \\
&= \frac{e^{\lambda^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-\lambda)^2/2}\, dx \\
&= \frac{e^{\lambda^2/2}}{\sqrt{2\pi}} \sqrt{2\pi} \\
&= \boxed{e^{\lambda^2/2}}
\end{aligned}$$

$$\begin{aligned}
m_X'(\lambda) &= \lambda e^{\lambda^2/2} \implies \mu = m_X'(0) = 0 \\
m_X''(\lambda) &= \lambda^2 e^{\lambda^2/2} + e^{\lambda^2/2} \implies \sigma^2 = m_X''(0) - \mu = 1
\end{aligned}$$

4. An Airline sold 560 tickets for an Airbus 380 flight (capacity: 555 seats) in the assumption that not all passengers that bought a ticket will arrive for the flight. Assume that the probability that a passenger will not shop up for the flight is 1%, independently for all passengers. How likely is it that more passengers show up for the flight than there are seats are available? Calculate this probability by using:

(a) a binomial distribution for the number of passengers that showed up for the flight;

(b) a normal approximation.

Answer:

(a) $\sum_{k=556}^{560} \binom{560}{k} (.99)^k (0.01)^{560-k} \approx 34.09\%$

(b) Choose one of the following three approximations. The third approximation includes the continuity correction so it is the most accurate of the three approximations.

$$\begin{aligned}
P[X \geq 556] &= P\left[\frac{X-\mu}{\sigma} \geq \frac{556-\mu}{\sigma}\right] = 1 - P\left[\frac{X-\mu}{\sigma} \leq \frac{1.6}{\sqrt{5.544}}\right] \\
&\approx 1 - \Phi\left(\frac{1.6}{\sqrt{5.544}}\right) \approx 1 - \Phi(0.680) = 1 - 0.7517 \approx 24.83\% \\
P[X > 555] &= P\left[\frac{X-\mu}{\sigma} > \frac{554-\mu}{\sigma}\right] = 1 - P\left[\frac{X-\mu}{\sigma} \leq \frac{0.6}{\sqrt{5.544}}\right] \\
&\approx 1 - \Phi\left(\frac{0.6}{\sqrt{5.544}}\right) \approx 1 - \Phi(.255) = 1 - 0.6007 = 39.03\% \\
P[X > 555.5] &= P\left[\frac{X-\mu}{\sigma} > \frac{555.5-\mu}{\sigma}\right] = 1 - P\left[\frac{X-\mu}{\sigma} \leq \frac{1.1}{\sqrt{5.544}}\right] \\
&\approx 1 - \Phi\left(\frac{1.1}{\sqrt{5.544}}\right) \approx 1 - \Phi(.467) = 1 - 0.68 = 32\%
\end{aligned}$$

5. Let X by a standard normal distributed random variable. Calculate $E[X^n]$ for an arbitrary non-negative integer n.

Answer:

Suppose n is odd.

$$\begin{aligned} E[X^n] &= \int_{-\infty}^{\infty} x^n f_X(x)dx = \int_{-\infty}^{0} x^n f_X(x)dx + \int_{0}^{\infty} x^n f_X(x)dx \\ &= \int_{\infty}^{0} (-y)^n f_X(-y)(-dy) + \int_{0}^{\infty} x^n f_X(x)dx \\ &= -\int_{0}^{\infty} y^n f_X(y)dy + \int_{0}^{\infty} x^n f_X(x)dx = 0. \end{aligned}$$

Suppose n is even and $n \geq 2$ (we already know $E[X^0] = E[1] = 1$). Let $a = 1/\sqrt{2\pi}$.

$$\begin{aligned} E[X^n] &= a\int_{-\infty}^{\infty} x^n e^{-x^2/2}dx = a\int_{-\infty}^{\infty} x^{n-1}\left(xe^{-x^2/2}\right)dx \\ &= a\left[-x^{n-1}e^{-x^2/2}\Big| + (n-1)\int_{-\infty}^{\infty} x^{n-2}e^{-x^2/2}dx\right] \\ &= (n-1)\int_{-\infty}^{\infty} x^{n-2}ae^{-x^2/2}dx \\ &= (n-1)E[X^{n-2}]. \end{aligned}$$

$$\begin{aligned} E[X^0] &= 1 \\ E[X^2] &= (2-1)E[X^0] = 1 \\ E[X^4] &= (4-1)E[X^2] = 3\cdot 1 \\ E[X^6] &= (6-1)E[X^4] = 5\cdot 3\cdot 1 \\ &\vdots \\ E[X^n] &= (n-1)\cdot(n-3)\cdot\dots\cdot 3\cdot 1. \end{aligned}$$

Prove this last claim by induction. The base case has already been shown. Assume the identity holds for an even integer $n \geq 2$. Replacing n with $n+2$ in the integration above, $E[X^{n+2}] = (n+1)E[X^n] = (n+1)\cdot(n-1)\cdot\dots\cdot 1$. Using the formula $1\cdot 3\cdot\dots\cdot k = (2k)!/(2^k k!)$ for odd k conclude:

$$E[X^n] = \begin{cases} 0 & n \text{ odd} \\ \frac{n!}{(2^{n/2}(n/2)!} & n \text{ even} \end{cases}$$

6. A random variable X is called log-normal with parameters μ and σ if $X = e^Y$ where $Y \sim \mathcal{N}(\mu, \sigma^2)$.

(a) Express the cdf F_X and the density f_X of X in terms of density ϕ and cdf Φ of a standard normal variable.

(b) What are expectation and variance of X?

(c) Let $\mu = 0$ and $\sigma = 1$. Calculate $P[X > 2]$ and find α such that $P[X \le \alpha] = 99\%$.

Answer:

(a)

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\begin{aligned}
&F_X(x) = P[X \le x] = P[e^Y \le x] = \begin{cases} 0 & x \le 0 \\ P[Y \le \ln x] & x > 0 \end{cases} \\
&Y \sim \mathcal{N}(\mu, \sigma^2) \implies \frac{Y-\mu}{\sigma} \sim \mathcal{N}(0,1) \\
&P[Y \le \ln x] = P\left[\frac{Y-\mu}{\sigma} \le \frac{\ln x - \mu}{\sigma}\right] = \Phi\left(\frac{\ln x - \mu}{\sigma}\right) \\
&F_X(x) = \begin{cases} 0 & x \le 0 \\ \Phi\left(\frac{\ln x - \mu}{\sigma}\right) & x > 0 \end{cases} \\
&f_X(x) = F_X'(x) = \begin{cases} 0 & x \le 0 \\ \phi\left(\frac{\ln x - \mu}{\sigma}\right)\frac{1}{\sigma x} & x > 0 \end{cases}
\end{aligned}
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(b)

$$
\begin{aligned}
E[X] &= \int_0^\infty xa\exp\left[-\left(\frac{\ln x - \mu}{\sqrt{2}\sigma}\right)^2\right]\frac{1}{\sigma x}dx, \quad a = \frac{1}{\sqrt{2\pi}} \\
&= \frac{a}{\sigma}\int_0^\infty \exp\left[-\left(\frac{\ln x - \mu}{\sqrt{2}\sigma}\right)^2\right]dx \\
&= a\int_{-\infty}^\infty \exp\left[\sigma w + \mu - w^2/2\right]dw, \quad w = \frac{\ln x - \mu}{\sigma} \\
&= e^{\mu+\sigma^2/2}\int_{-\infty}^\infty ae^{-(w-\sigma)^2/2}dw \\
&= \boxed{e^{\mu+\sigma^2/2}} \\
E[X^2] &= \boxed{e^{2\sigma^2+2\mu}} \text{ by a similar calculation.}
\end{aligned}
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(c)

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\begin{aligned}
P[X > 2] = 1 - P[X \le 2] = 1 - \Phi(\ln 2) &\approx \boxed{24.24\%} \\
0.99 = P[X \le \alpha] &= \Phi(\ln\alpha) \\
\ln\alpha &= \Phi^{-1}(0.99) \\
\alpha &= e^{\Phi^{-1}(0.99)} \approx \boxed{10.25}
\end{aligned}
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