Assignment 8

1. Assume that you have three four-sided dice with number 1, 2, 3, and 4 on the four sides and let denote by X the sum of the numbers shown on their bottom side. Write down and sketch the probability mass function and the cumulative distribution function of X.

$P[X=3] = 1/64$

$P[X=4] = 3/64$

$P[X=5] = 6/64$

$P[X=6] = 10/64$

$P[X=7] = 12/64$

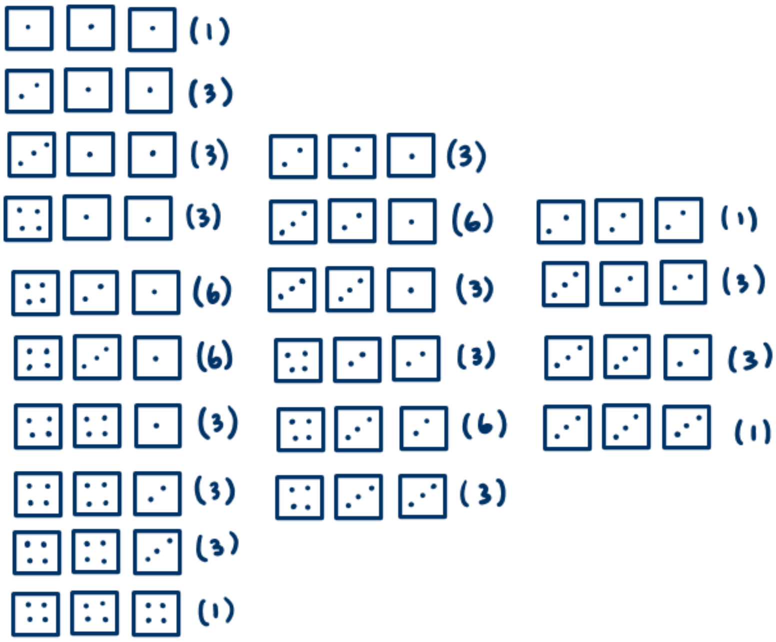
$P[X=8] = 12/64$

$P[X=9] = 10/64$

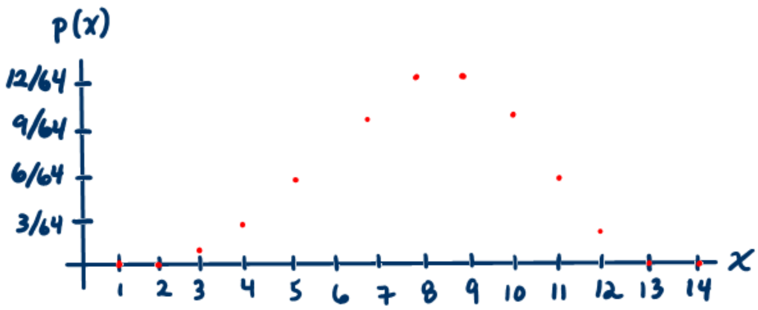
$P[X=10] = 6/64$

$P[X=11] = 3/64$

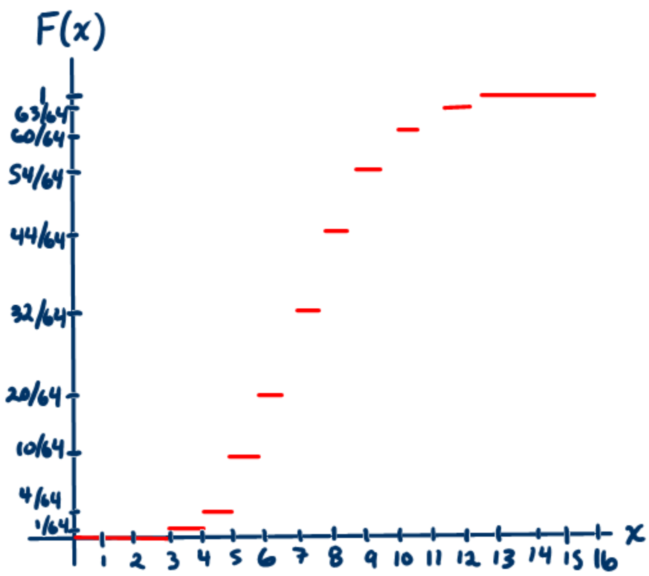
$P[X=12] = 1/64$



Probability mass Function



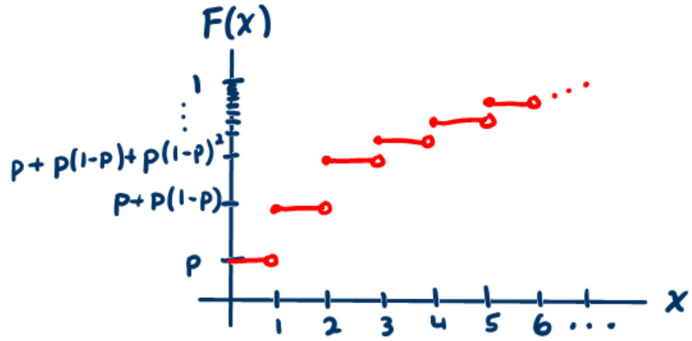
Cumulative Distribution Function



All steps of finite length are of the form since the CDF is right continuous.

2. Calculate and sketch the cumulative distribution function (cdf) of a geometric random variable (i.e., the pmf is given by $p(n) = (1-p)^n p$ for non-negative integers n and some parameter $p \in (0,1)$).

$$F(x) = \sum_{n=0}^{\lfloor x \rfloor} (1-p)^n p = 1 - (1-p)^{\lfloor x \rfloor + 1}$$



Infinitely many steps each of length 1 approaching an altitude of 1 as $x \to \infty$.

3. You arrive at a random time at a bus stop, and you know that the bus is arriving every 30 minutes. Denote with Y the random variable describing your waiting time.

a) What is the probability that you will have to wait longer then 10 minutes.

b) Assume that you waited already 10 minutes, what is the probability that the bus will arrive in the next 10 minutes?

Describe every of the statements of a) and b) in terms of Y and calculate the probabilities explicitly using the density f_Y.

a) $$f_Y(t) = \begin{cases} 1/30, & t \in (0,30) \\ 0, & t \notin (0,30) \end{cases}$$

$$P[Y \geq 10] = \int_{10}^{\infty} f_Y(t)\,dt = \int_{10}^{30} 1/30\,dt = 2/3$$

b) $$P[Y \leq 20 \mid Y > 10] = \frac{P[10 < Y \leq 20]}{P[Y > 10]} = \frac{\int_{10}^{20} f_Y(t)\,dt}{\int_{10}^{\infty} f_Y(t)\,dt} = \frac{\int_{10}^{20} 1/30\,dt}{\int_{10}^{30} 1/30\,dt} = \frac{1}{2}$$

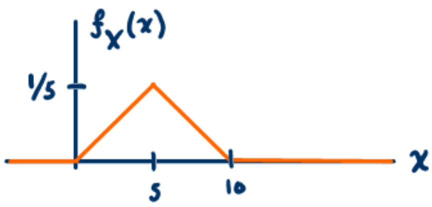
4. Assume that a random variable X has a density of the form $f_X(x) = cg(x)$ for some real constant c and

$$g(x) = \begin{cases} 0 & x < 0, \\ x & 0 \leq x < 5, \\ 10 - x & 5 \leq x < 10, \\ 0 & x \geq 10. \end{cases}$$

a) Determine the value of the constant c, sketch the function f_X.

b) What is $\mathbb{P}[3 \leq X \leq 8]$?

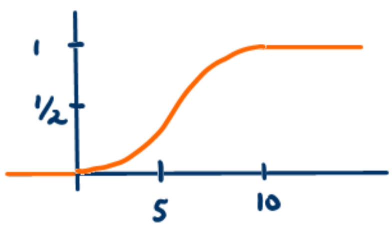
c) Calculate an sketch F_X the cumulative distribution function of X.



a) $$1 = \int_{-\infty}^{\infty} f_X(x) = \int_{-\infty}^{\infty} c\,g(x)\,dx = c\left[\int_0^5 x\,dx + \int_5^{10} (10-x)\,dx\right] = c\,(25/2 + 25/2) = 25c \Rightarrow \boxed{c = \frac{1}{25}}$$

b) $$P[3 \leq X \leq 8] = \int_3^8 f_X(x)\,dx = \int_3^5 x/25\,dx + \int_5^8 (10-x)/25\,dx = 8/25 + 21/50 = \boxed{\frac{37}{50}}$$

c) $$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \begin{cases} 0, & x \leq 0 \\ x^2/50, & 0 < x \leq 5 \\ -x^2/50 + 2x/5 - 1, & 5 < x \leq 10 \\ 1, & x \geq 10 \end{cases}$$



5. Assume that a random variable Y has a density of the form $f_X(x) = cg(x)$ for some real constant c and

$$g(x) = \begin{cases} 0 & x < 0, \\ \sin(x) & 0 \le x < \pi, \\ 0 & x \ge \pi. \end{cases}$$

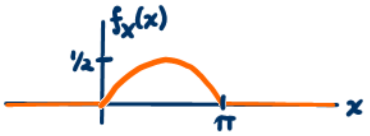
a) Determine the value of the constant c, sketch the function f_X.

b) What is $\mathbb{P}[X \ge \frac{\pi}{6} \mid X \le \frac{2\pi}{3}]$?

c) Calculate an sketch F_X the cumulative distribution function of X.

a) $1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^{\pi} c \sin x\,dx = c(\cos(0) - \cos(\pi)) = 2c$

$\boxed{c = 1/2}$



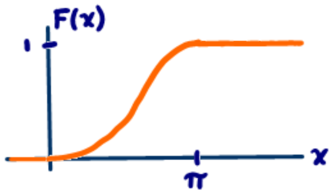
b) $$P[X \ge \pi/6 \mid X \le 2\pi/3] = \frac{P[\pi/6 \le X \le 2\pi/3]}{P[X \le 2\pi/3]}$$

$$= \frac{\int_{\pi/6}^{2\pi/3} \frac{1}{2}\sin x\,dx}{\int_0^{2\pi/3} \frac{1}{2}\sin x\,dx}$$

$$= \frac{\cos \pi/6 - \cos 2\pi/3}{\cos(0) - \cos(2\pi/3)}$$

$$= \frac{\sqrt{3}/2 + 1/2}{1 + 1/2} = \boxed{\frac{\sqrt{3} + 1}{3} \approx 0.91068}$$

c) $$F(x) = \int_{-\infty}^{x} f_X(t)\,dt = \begin{cases} 0 & , x \le 0 \\ (1-\cos x)/2 & , 0 < x < \pi \\ 1 & , x \ge \pi \end{cases}$$



6. Let X be a continuous random variable with density

$$f(x) = \begin{cases} cx^2e^{-x} & \text{if } x \in [0, 1] \\ 0 & \text{else.} \end{cases}$$

a) Determine the value of the constant c

b) Calculate the expectation of X.

a) $1 = \int_{-\infty}^{\infty} f(x)\,dx = c\int_0^1 x^2 e^{-x}\,dx$

$\int x^2 e^{-x}\,dx = -x^2e^{-x} + 2\int xe^{-x}\,dx = -x^2e^{-x} + 2(-xe^{-x} + \int e^{-x}dx)$

$\int x^2 e^{-x}\,dx = -e^{-x}(x^2 + 2x + 2)$

$c\int_0^1 x^2e^{-x}dx = c(2 - 5/e) \Rightarrow \boxed{c = \frac{1}{2 - 5/e} = \frac{e}{2e - 5}}$

b) $E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = c\int_0^1 x^3 e^{-x}\,dx$

$= c\left[-x^3e^{-x} + 3\int x^2 e^{-x}dx\right]_0^1 = c\left[-x^3e^{-x} + 3(-x^2e^{-x} + 2\int xe^{-x}dx)\right]_0^1$

$= c\left[-x^3e^{-x} - 3x^2e^{-x} - 6xe^{-x} - 6e^{-x}\right]_0^1 = c\left[6 - 16/e\right]$

$= \boxed{\frac{6 - 16/e}{2 - 5/e} = \frac{6e - 16}{2e - 5}}$