Assignment 10

1. Assume that X is a normally distributed random variable with mean μ and variance σ^2. Compute the probabilities that X is not more than one, two and three standard deviations away from the mean, i.e. $\mathbb{P}\big[|X-\mu| \le \sigma\big]$, $\mathbb{P}\big[|X-\mu| \le 2\sigma\big]$ and $\mathbb{P}\big[|X-\mu| \le 3\sigma\big]$. The other way round, how you have to choose k such that X stays with 95% in the interval $(\mu - k, \mu + k)$? How about 99%?

$$P[|X-\mu| \le k\sigma] = P[\mu - k\sigma \le X \le \mu + k\sigma]$$

$$= \int_{\mu-k\sigma}^{\mu+k\sigma} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}\,dx \approx \begin{cases} 68.27\,\%, & k=1 \\ 95.45\,\%, & k=2 \\ 99.73\,\%, & k=3 \end{cases}$$

2. Let X be a standard normal distributed random variable. How we have to choose $\beta \in \mathbb{R}$ such that $\mathbb{P}\big[X^2 < \beta\big] = 0.5$?

$$0.5 = P[X^2 < \beta] = P[-\sqrt{\beta} < X < \sqrt{\beta}]$$

Look in the table for the closest value to $\Phi(x) = 0.75$ so that we have 25% on each side. It says $0.67 < x < 0.68$, say $x \approx 0.675$. That is:

$$0.5 \approx P[-0.675 < X < 0.675]$$

Take $\beta = 0.675^2 = 0.455625$ to satisfy $P[X^2 < \beta] \approx .5$

3. Let $X \sim \mathcal{N}(0,1)$ be a standard normal distributed random variable. Calculate the moment generating function $m_X(\lambda) = \mathbb{E}[e^{\lambda X}]$ for $\lambda \in \mathbb{R}$. Use the moment generating function to calculate mean and variance of X, confirming what we know already.

$$m_X(\lambda) = E[e^{\lambda X}]$$
$$= \int_{-\infty}^{\infty} e^{\lambda x} e^{-x^2/2} \frac{1}{\sqrt{2\pi}}\,dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2 - 2\lambda x)/2}\,dx \qquad \left(\begin{aligned} &x^2 - 2\lambda x + \lambda^2 - \lambda^2 \\ &= (x-\lambda)^2 - \lambda^2 \end{aligned}\right)$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-\lambda)^2/2} e^{\lambda^2/2}$$
$$= \frac{e^{\lambda^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-\lambda)^2/2}\,dx$$
$$= \frac{e^{\lambda^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2}\,du$$
$$= \frac{e^{\lambda^2/2}}{\sqrt{2\pi}} \sqrt{2\pi} = \boxed{e^{\lambda^2/2}}$$

$$\boxed{\begin{aligned} &m_X'(\lambda) = \lambda e^{\lambda^2/2} \\ &m_X''(\lambda) = \lambda^2 e^{\lambda^2/2} + e^{\lambda^2/2} \\ &\text{mean}: \ \mu = m_X'(0) = 0 \\ &\text{Variance}: \ \sigma^2 = m_X''(0) - \mu^2 = 1 \end{aligned}}$$

4. An Airline sold 560 tickets for an Airbus 380 flight (capacity: 555 seats) in the assumption that not all passengers that bought a ticket will arrive for the flight. Assume that the probability that a passenger will not shop up for the flight is 1%, independently for all passengers. How likely is it that there are more passengers showing up for the flight than seats are available? Calculate this probability by using

a) a binomial distribution for the number of passengers that showed up for the flight;

b) a normal approximation.

a) $$\sum_{k=556}^{560} \binom{560}{k} (.99)^k (.01)^{560-k} \approx 34.09\%$$

b) $$P[X \geq 556] = P\left[\frac{X-\mu}{\sigma} \geq \frac{556-\mu}{\sigma}\right] = 1-P\left[\frac{X-\mu}{\sigma} \leq \frac{1.6}{\sqrt{5.544}}\right]$$

$$\approx 1-\Phi\left(\frac{1.6}{\sqrt{5.544}}\right) \approx 1-\Phi(0.680) = 1-0.7517 \approx 24.83\%$$

OR

$$P[X > 555] = P\left[\frac{X-\mu}{\sigma} > \frac{555-\mu}{\sigma}\right] = 1-P\left[\frac{X-\mu}{\sigma} \leq \frac{0.6}{\sqrt{5.554}}\right]$$

$$\approx 1-\Phi\left(\frac{0.6}{\sqrt{5.554}}\right) \approx 1-\Phi(0.255) = 1-0.6007 = 39.03\%$$

OR

$$P[X > 555.5] = P\left[\frac{X-\mu}{\sigma} > \frac{555.5-\mu}{\sigma}\right] = 1-P\left[\frac{X-\mu}{\sigma} \leq \frac{1.1}{\sqrt{5.554}}\right]$$

$$\approx 1-\Phi\left(\frac{1.1}{\sqrt{5.554}}\right) \approx 1-\Phi(0.467) = 1-0.68 = 32\%$$

5. Let X by a standard normal distributed random variable. Calculate $\mathbb{E}[X^n]$ for an arbitrary non-negative integer n.

$$E[X^{n+2}] = \int_{-\infty}^{\infty} x^{n+2} \alpha e^{-x^2/2}\, dx \qquad \alpha = 1/\sqrt{2\pi}$$

$$= \alpha\left(\int_{-\infty}^{\infty} x^{n+1} x e^{-x^2/2}\, dx\right)$$

$$= \alpha\left\{ -x^{n+1} e^{-x^2/2}\Big|_{-\infty}^{\infty} + (n+1)\int_{-\infty}^{\infty} x^n e^{-x^2/2}\, dx \right\}$$

$$= \alpha\left\{(n+1)\int_{-\infty}^{\infty} x^n e^{-x^2/2}\, dx\right\}$$

$$= \alpha(n+1)\left\{ -x^{n-1} e^{-x^2/2}\Big|_{-\infty}^{\infty} + (n-1)\int_{-\infty}^{\infty} x^{n-2} e^{-x^2/2}\, dx \right\}$$

$$= \alpha(n+1)(n-1)\int_{-\infty}^{\infty} x^{n-2} e^{-x^2/2}\, dx$$

$$\vdots$$

$$= \begin{cases} \alpha(n+1)(n-1)\dots(3)\int_{-\infty}^{\infty} x^2 e^{-x^2/2}\, dx, & n \text{ even} \\ \alpha(n+1)(n-1)\dots(2)\int_{-\infty}^{\infty} x e^{-x^2/2}\, dx, & n \text{ odd} \end{cases}$$

$$= \begin{cases} (n+1)(n-1)\dots(3), & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

<u>or</u> Suppose n is odd.

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)\,dx$$

$$= \int_{-\infty}^{0} x^n f_X(x)\,dx + \int_0^{\infty} x^n f_X(x)\,dx$$

$$= \int_{\infty}^{0} (-y)^n f_X(-y)(-dy) + \int_0^{\infty} x^n f_X(x)\,dx$$

$$= -\int_0^{\infty} y^n f_X(y)\,dy + \int_0^{\infty} x^n f_X(x)\,dx$$

$$= 0.$$

Suppose n is even and $n \geq 2$.

$$E[X^n] = \int_{-\infty}^{\infty} x^n \alpha e^{-x^2/2}\,dx \qquad \alpha = 1/\sqrt{2\pi}$$

$$= \alpha \int_{-\infty}^{\infty} x^{n-1} x e^{-x^2/2}\,dx$$

$$= \alpha \left\{ -x^{n-1} e^{-x^2/2} \Big|_{-\infty}^{\infty} + (n-1)\int_{-\infty}^{\infty} x^{n-2} e^{-x^2/2}\,dx \right\}$$

$$= (n-1)\int_{-\infty}^{\infty} x^{n-2} \alpha e^{-x^2/2}\,dx$$

$$= (n-1) E[X^{n-2}]$$

Using this and that $E[X^0] = 1$,

$$E[X^2] = (2-1)E[X^0] = 1$$

$$E[X^4] = (4-1)E[X^2] = 3 \cdot 1$$

$$E[X^6] = (6-1)E[X^4] = 5 \cdot 3 \cdot 1$$

$$\vdots$$

$$E[X^n] = (n-1) \cdot \ldots \cdot 3 \cdot 1$$

Prove this by induction. We know $E[X^2] = 1$ so $E[X^2] = 1 = (2-1)E[X^0]$ establishes the base. Assume $E[X^n] = (n-1)\ldots \cdot 3 \cdot 1$, $n \geq 2$. $E[X^{n+2}] = (n+1)E[X^n]$ by the integration above. Using the inductive hypothesis $E[X^{n+2}] = (n+1)(n-1)\ldots 3 \cdot 1$. Using the formula $1 \cdot 3 \cdot \ldots \cdot n = (2n)!/2^n n!$ conclude

$$\boxed{E[X^n] = \begin{cases} 0, & n \text{ odd} \\ n!/2^{n/2}(n/2)!, & n \text{ even} \end{cases}}$$

6. A random variable X is called *log-normal* with parameters μ and σ, if $X = e^Y$ where $Y \sim \mathcal{N}(\mu, \sigma^2)$.

a) Express the cdf F_X and the density f_X of X in terms of density φ and cdf Φ of a standard normal variable.

b) What are expectation and variance of X?

c) Let now $\mu = 0$ and $\sigma = 1$. Calculate $\mathbb{P}[X > 2]$ and find α such that $\mathbb{P}[X \leq \alpha] = 99\%$.

a) $F_X(x) = P[X \leq x] = P[e^Y \leq x] = \begin{cases} 0, & x \leq 0 \\ P[Y \leq \ln x], & x > 0 \end{cases}$

Since $Y \sim \mathcal{N}(\mu, \sigma^2)$, $\frac{Y-\mu}{\sigma} \sim \mathcal{N}(0,1)$.

$P[Y \leq \ln x] = P\left[\frac{Y-\mu}{\sigma} \leq \frac{\ln x - \mu}{\sigma}\right] = \Phi\left(\frac{\ln x - \mu}{\sigma}\right)$.

$$F_X(x) = \begin{cases} 0, & x \leq 0 \\ \Phi\left(\frac{\ln x - \mu}{\sigma}\right), & x > 0 \end{cases}$$

$$f_X(x) = F_X'(x) = \begin{cases} 0, & x \leq 0 \\ \varphi\left(\frac{\ln x - \mu}{\sigma}\right)\frac{1}{\sigma x}, & x > 0 \end{cases}$$

b) $E[X] = \int_0^\infty x \alpha \exp\left[-\left(\frac{\ln x - \mu}{\sqrt{2}\sigma}\right)^2\right] \frac{1}{\sigma x}\, dx$, $\alpha = 1/\sqrt{2\pi}$

$= \alpha/\sigma \int_0^\infty \exp\left[-\left(\frac{\ln x - \mu}{\sqrt{2}\sigma}\right)^2\right] dx$ $\quad w = (\ln x - \mu)/\sigma \leftrightarrow x = e^{\sigma w + \mu}$, $dw = \frac{1}{x\sigma} dx$

$= \alpha \int_{-\infty}^\infty \exp(\sigma w + \mu - w^2/2)\, dw$

$= \alpha \int_{-\infty}^\infty \exp\left[-(w-\sigma)^2/2\right] \exp(\mu + \sigma^2/2)\, dw$

$= \exp(\mu + \sigma^2/2) \int_{-\infty}^\infty \alpha \exp\left[-(w-\sigma)^2/2\right] dw$

$= \exp(\mu + \sigma^2/2)$

$E[X^2] = e^{2\sigma^2 + 2\mu}$ by a similar calculation.

c) $P[X > 2] = 1 - P[X \leq 2] = 1 - \Phi(\ln 2) \approx 24.24\%$

$.99 = P[X \leq \alpha] = \Phi(\ln \alpha)$

$\ln \alpha = \Phi^{-1}(.99)$

$\alpha = e^{\Phi^{-1}(.99)} \approx 10.25$