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Fall 2021 - A Term

# MA 2631

# Probability Theory

Section AL01 / AD01

## Assignment 11 – last assignment

due on Friday, October 8

based on Lectures of Chapter 6.1–6.2

1. Let $X$, $Y$ be two random variables with joint cdf $F_{X,Y}$ and marginal cdfs $F_X$, $F_Y$. For $x$, $y \in \mathbb{R}$, express
$$\mathbb{P}[X > x; Y \leq y]$$
in terms of $F_{X,Y}$ and $F_X$, $F_Y$

2. Assume that there are 12 balls in an urn, 3 of them red, 4 white and 5 blue. Assume that you draw 2 balls of them, replacing any drawn ball by a ball of the same color. Denote by $X$ the number of drawn red balls and by $Y$ the number of drawn white balls. Calculate the joint probability mass distribution of $X$ and $Y$ as well as the marginal distributions. Are $X$ and $Y$ independent?

3. Assume that the joint probability mass distribution $p_{X,Y}$ of the random variable $X$ and $Y$ is given by
$$\begin{aligned} p_{X,Y}(1,1) =& p_{X,Y}(1,2) = p_{X,Y}(1,3) = \frac{1}{12}; \\ p_{X,Y}(2,1) =& p_{X,Y}(2,2) = p_{X,Y}(2,3) = \frac{1}{4}. \end{aligned}$$
   a) Calculate the marginal probability mass distributions $p_X$ and $p_Y$.
   b) Are $X$ and $Y$ independent?
   c) What is the probability mass distribution of the random variable $Z = \frac{X}{Y}$?

4. et $X$ and $Y$ be two independent standard-normal distributed random variables and define $Z = X^2 + Y^2$. Calculate the cumulative distribution function of $Z$. Which distribution follows $Z$?

5. Let $X$, $Y$ be two jointly distributed random variables with joint density

$$f_{X,Y}(x,y) = \begin{cases} cxy & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1; \\ 0 & \text{else,} \end{cases}$$

for some constant $c$.

   a) What is the value of $c$?
   b) Are $X$ and $Y$ independent?
   c) Calculate $\mathbb{E}[X]$.

6. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with density $f$ and cumulative distribution function $F$. Calculate density and cumulative distribution function of

$$Y = \min\{X_1, X_2, \ldots X_n\}, \qquad Z = \max\{X_1, X_2, \ldots X_n\}$$

in terms of $f$ and $F$.

8 points per problems

**Additional practice problems (purely voluntary - no points, no credit, no grading)**:

**Standard** Carlton and Devore, Section 4.1: Exercises 1, 3, 4, 8, 11, 13, 14, 19 ; Section 4.2: Exercises 23, 24, 29

**Hard** Prove that for independent random variables $X \sim \mathcal{N}\left(\mu_X, \sigma_X^2\right)$ and $Y \sim \mathcal{N}\left(\mu_Y, \sigma_Y^2\right)$ we have

$$X + Y \sim \mathcal{N}\left(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2\right)$$

**Challenging** Let $E_1, \ldots, E_n, \ldots$ be independent, exponentially distributed random variables with parameter $\lambda > 0$ and set

$$Z_n = \max_{\{1 \le n \le N\}} E_n - \frac{\log N}{\lambda}.$$

Calculate the limiting distribution of $Z_N$ for $N \to \infty$ by calculating the limiting cumulative distribution function

$$F(x) = \lim_{N \to \infty} \mathbb{P}[Z_n \le x].$$