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Fall 2021 - A Term

MA 2631

Probability Theory

Section AL01 / AD01

Assignment 4

due on Tuesday, September 14

based on Lectures of Chapter 3.1–3.2

1. We are given a die and six fair coins. First we roll the die and then we flip exactly the number of coins the die shows. What is the probability that we get exactly two "heads"?

2. A microchip for a cellphone is produced by the factories A, B and C. Factory A produces 30% of all microchips, B 50% and C 20%. A microchip produced by A is defective in 2% of all cases, a chip produced in B is defective in 3% and a chip produced in C is defective in 0.5% of all cases. Assume you get a cell phone with a defective microchip. How likely is it that it was produced in factory A?

3. a) Let A, B, C be three events in a sample space Ω with a probability $\mathbb{P}$ satisfying $\mathbb{P}[C] > 0$. Show that
$$P[A \cup B \,|\, C] = P[A \,|\, C] + P[B \,|\, C] - P[A \cap B \,|\, C].$$
   b) Let $A_1, \dots, A_n$ be mutually exclusive events in a sample space Ω and C an event in Ω with $\mathbb{P}[C] > 0$. Show that
$$P\Big[\bigcup_{i=1}^{n} A_i \,|\, C\Big] = \sum_{i=1}^{n} P[A_i \,|\, C].$$

4. An AIDS test can detect an HIV infection with 99% accuracy, whereas it can accurately identify the absence of an infection with 98% probability. Noting that approximately 0.6% of the US population is infected by HIV, how likely is it that a positive AIDS test indicates indeed a HIV infection?

5. We are given 5 coins,two of them have *heads* on both sides, one *tail* on both sides and two are regular with *heads* on one side and *tails* on the other. We pick a coin at random and throw it. What is the probability that

   i) the bottom side shows *heads* while we can't see the upper side?

   ii) the bottom side shows also *heads* if the upper side shows *heads*?

6. (Continuation from Assignment 3, problem 2) A person picks 13 cards out of a standard deck of 52. Assume now that while picking one card is visible and the player recognizes the ace of hearts among the 13 cards.

   iii) What is now the probability that he has at least one ace in his hand?

   iv) What is now the probability that he has exactly one ace in his hand?

8 points per problems

Additional practice problems (purely voluntary - no points, no credit, no grading):

Standard Carlton and Devore, Section 1.4: Exercises 50, 51, 56, 57, 62, 72, 74

Hard Ann and Clara want to meet at the cafeteria between 1 p.m. and 2 p.m. The first that arrives waits for 20 minutes and, if the other one does not show up, leaves the cafeteria.

   i) Assuming that they arrive independently at a random time point between 1 p.m. and 2 p.m., what is the probability that they will actually meet?

   ii) Given that they met, what is the probability that the first of them waited for more than 10 minutes?

Extra *The Monty Hall Problem*: As WPI has a strong connection to goats, here a classical brain teaser about the game show *Let's Make a Deal*:

"Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?"

While there are many heuristics for the solution, can you set up a rigorous mathematical formulation that, using conditional probabilities, allows you to calculate explicitly the probabilities for winning given you switch or keep your choice? Try to spell out as explicitly as possible which modeling assumptions you make.

For more information see https://en.wikipedia.org/wiki/Monty_Hall_problem.