MA 2631 Conference 4

September 22, 2021

1. Consider the random variable X with the probability mass distribution

$$\mathbb{P}[X=-1]=0.3, \qquad \mathbb{P}[X=2]=0.5, \qquad \mathbb{P}[X=5]=0.1, \qquad \mathbb{P}[X=10]=0.1.$$

Calculate the expected value and variance of X as well as the expectation of Y with $Y=e^X$.

$$\begin{aligned}
E[X] &= -1\mathbb{P}[X=-1]+2\mathbb{P}[X=2]+5\mathbb{P}[X=5]+10\mathbb{P}[X=10] \\
&= (-1)(0.3)+(2)(0.5)+5(0.1)+10(0.1) \\
&= \boxed{2.2} \\
E[X^2] &= (-1)^2\mathbb{P}[X=-1]+2^2\mathbb{P}[X=2]+5^2\mathbb{P}[X=5]+10^2\mathbb{P}[X=10] \\
&= (1)(0.3)+(4)(0.5)+25(0.1)+100(0.1) \\
&= 14.8 \\
\mathrm{Var}[X] &= E[X^2]-(E[X])^2 = 14.8-2.2^2 \\
&= \boxed{9.96} \\
E[Y] &= E[e^X] = .3e^{-1}+.5e^2+.1e^5+.1e^{10} \\
&\approx \boxed{2221.29}
\end{aligned}$$

2. Let X be a Poisson distributed random variable with parameter $\lambda = 2$.

   a) Calculate
$$\mathbb{P}\left[X \geq 2 \,|\, X \geq 1\right].$$

   b) Calculate
$$E\left[\frac{1}{X+1}\right].$$

Answer:

a)

$$\begin{aligned}
\mathbb{P}\left[X \geq 2 \,|\, X \geq 1\right] &= \frac{\mathbb{P}[\{X \geq 2\} \cap \{X \geq 1\}]}{\mathbb{P}[X \geq 1]} = \frac{\mathbb{P}[X \geq 2]}{\mathbb{P}[X \geq 1]} \\
&= \frac{1 - \mathbb{P}[X < 2]}{1 - \mathbb{P}[X < 1]} \\
&= \frac{1 - \mathbb{P}[X = 0] - \mathbb{P}[X = 1]}{1 - \mathbb{P}[X = 0]} \\
&= \frac{1 - e^{-\lambda} - e^{-\lambda}\lambda}{1 - e^{-\lambda}} \\
&= \boxed{\frac{1 - 3e^{-2}}{1 - e^{-2}} \approx 0.687}
\end{aligned}$$

b)

$$\begin{aligned}
E\left[\frac{1}{X+1}\right] &= \sum_{n=0}^{\infty} \frac{1}{1+n} e^{-\lambda} \frac{\lambda^n}{n!} \\
&= \frac{1}{\lambda} \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^{n+1}}{(n+1)!} \\
&= \frac{1}{\lambda} \sum_{m=1}^{\infty} e^{-\lambda} \frac{\lambda^m}{m!} \\
&= \frac{1}{\lambda} \left(-e^{-\lambda} + \sum_{m=0}^{\infty} e^{-\lambda} \frac{\lambda^m}{m!} \right) \\
&= \frac{1}{\lambda} (-e^{-\lambda} + 1) \\
&= \boxed{\frac{1 - e^{-2}}{2} \approx 0.432}
\end{aligned}$$

3. Let Y be a Poisson distributed random variable with parameter λ, where λ is a non-negative integer. Calculate
$$E\,[\,|Y-\lambda|\,]\,.$$

$$\begin{aligned}
E[|Y-\lambda|] &= \sum_{k=0}^{\infty} |k-\lambda| \frac{\lambda^k}{k!} e^{-\lambda} \\
&= \sum_{k=0}^{\lambda-1} (\lambda-k) \frac{\lambda^k}{k!} e^{-\lambda} + \sum_{\lambda+1}^{\infty} (k-\lambda) \frac{\lambda^k}{k!} e^{-\lambda} \\
&= \sum_{k=0}^{\lambda-1} \lambda \frac{\lambda^k}{k!} e^{-\lambda} - \sum_{k=0}^{\lambda-1} k \frac{\lambda^k}{k!} e^{-\lambda} + \sum_{\lambda+1}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} - \sum_{\lambda+1}^{\infty} \lambda \frac{\lambda^k}{k!} e^{-\lambda} \\
&= \lambda \left(\sum_{k=0}^{\lambda-1} \frac{\lambda^k}{k!} e^{-\lambda} - \sum_{k=0}^{\lambda-1} \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} + \sum_{\lambda+1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} - \sum_{\lambda+1}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} \right) \\
&= \lambda \left(\frac{\lambda^{\lambda-1}}{(\lambda-1)!} e^{-\lambda} + \frac{\lambda^{\lambda}}{\lambda!} e^{-\lambda} \right) \\
&= \lambda \left(\frac{\lambda^{\lambda-1}}{(\lambda-1)!} e^{-\lambda} + \frac{\lambda^{\lambda-1}}{(\lambda-1)!} e^{-\lambda} \right) \\
&= \boxed{\frac{2\lambda^{\lambda}}{(\lambda-1)!} e^{-\lambda}}
\end{aligned}$$