MA 2631

Probability Theory

Section AL01 / AD01

Midterm Exam
Friday, 09/17, 12:00-12:50pm

1. A cupboard contains 4 red, 3 blue and 2 yellow cups. In how many ways you can line up the cups if
   a) you cannot differentiate between cups of the same color?
   b) you can differentiate between cups of the same color?

a) $\frac{(4+3+2)!}{4!3!2!} = \boxed{\frac{9!}{4!3!2!}}$

b) $\boxed{9!}$

2. Suppose that A, B and C are independent events on a sample space Ω. Prove that

$$\mathbb{P}[A^c \cap B^c \cap C^c] = \mathbb{P}[A^c]\mathbb{P}[B^c]\mathbb{P}[C^c].$$

$$\begin{aligned}
P[A^c \cap B^c \cap C^c] &= 1 - P[A \cup B \cup C] \\
&= 1 - P[A] - P[B] - P[C] + P[A \cap B] + P[A \cap C] \\
&\quad + P[B \cap C] - P[A \cap B \cap C] \\
&= 1 - P[A] - P[B] - P[C] + P[A]P[B] + P[A]P[C] \\
&\quad + P[B]P[C] - P[A]P[B]P[C] \\
&= P[A]\left(P[B] + P[C] - P[B]P[C] - 1\right) \\
&\quad + P[B]\left(P[C] - 1\right) - P[C] + 1 \\
&= P[A]\left(P[B]\left(1 - P[C]\right) + P[C] - 1\right) \\
&\quad - P[B]P[C^c] + P[C^c] \\
&= P[A]\left(P[C^c]\left(P[B] - 1\right)\right) + P[C^c]\left(1 - P[B]\right) \\
&= -P[A]P[B^c]P[C^c] + P[B^c]P[C^c] \\
&= P[A^c]P[B^c]P[C^c].
\end{aligned}$$

3. 10% of WPI students who take a probability class take MA 2631 while 90% take MA 2621. You know that one quarter of the students who take MA 2631 will graduate with distinction, while only one fifth of those taking MA 2621 do so. If you meet a students who took some probability class and graduated with distinction, how likely is it that they attended MA 2631?

D : graduated with distinction

M : attended MA 2631

$$P[M|D] = \frac{P[D|M]\,P[M]}{P[D|M]\,P[M] + P[D|M^c]\,P[M^c]}$$

$$= \frac{1/4 \cdot 1/10}{1/4 \cdot 1/10 + 1/5 \cdot 9/10}$$

$$= \frac{1/40}{1/40 + 9/50}$$

$$= \frac{1/4}{1/4 + 9/5}$$

$$= \frac{5/20}{(5+36)/20}$$

$$= \boxed{5/41}$$

4. Consider a discrete random variable Z taking values on the non-negative integers with probability mass distribution

$$\mathbb{P}[Z = n] = c \cdot \frac{1}{2^n}$$

for some constant c.

a) What is c?

b) Calculate $\mathbb{P}[Z = 2 \,|\, Z \geq 1]$.

a) $1 = \sum_{n=0}^{\infty} \mathbb{P}[Z=n] = c\left(1 + \frac{1}{2} + \frac{1}{4} + \ldots\right) = c\,\frac{1}{1-1/2} = 2c$

$\boxed{c = 1/2}$

b) $\mathbb{P}[Z=2 \,|\, Z \geqslant 1] = \dfrac{\mathbb{P}[Z \geqslant 1 \,|\, Z=2]\,\mathbb{P}[Z=2]}{\mathbb{P}[Z \geqslant 1]}$

$$= \frac{1 \cdot \frac{1}{2}\,\frac{1}{2^2}}{\frac{1}{2}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right)}$$

$$= \frac{1/4}{1/2 + 1/4 + 1/8 + \ldots}$$

$$= \frac{1/4}{1}$$

$$= \boxed{1/4}$$