based on Lectures of Chapter 4.1–4.3

1. Consider the random variable X with the probability mass distribution

$$\mathbb{P}[X = -1] = 0.3, \qquad \mathbb{P}[X = 2] = 0.5, \qquad \mathbb{P}[X = 5] = 0.1, \qquad \mathbb{P}[X = 10] = 0.1.$$

   Calculate the expected value and variance of X as well as the expectation of Y with $Y = e^{2X}$.

2. Remember that the geometric distribution is given by the probability mass distribution

$$p(n) = P[X = n] = (1-p)^n p$$

   for non-negative integers n and some $p \in (0,1)$.

   a) Prove that the probability mass distribution describes indeed a probability, i.e. show that

$$\sum_{n=0}^{\infty} p(n) = 1.$$

   b) Calculate the probability of $\mathbb{P}[X \geq 3]$.

   c) Prove that it holds for non-negative integers n, i that

$$\mathbb{P}[X \geq n+i \,|\, X \geq n] = \mathbb{P}[X \geq i].$$

3. Let X be a Poisson distributed random variable with parameter $\lambda = 2$.

   a) Calculate

$$\mathbb{P}\big[X \geq 2 \,|\, X \geq 1\big].$$

   b) Calculate

$$\mathbb{E}\left[\frac{1}{X+1}\right].$$

4. Let Y be a Poisson distributed random variable with parameter λ, where λ is a non-negative integer. Calculate

$$\mathbb{E}\big[|Y - \lambda|\big].$$

1. Consider the random variable X with the probability mass distribution

$$\mathbb{P}[X=-1]=0.3, \qquad \mathbb{P}[X=2]=0.5, \qquad \mathbb{P}[X=5]=0.1, \qquad \mathbb{P}[X=10]=0.1.$$

Calculate the expected value and variance of X as well as the expectation of Y with $Y=e^{2X}$.

$$EX = -0.3 + 1 + 0.5 + 1 = \underline{2.2}$$

$$EX^2 = 0.3 + 2 + 2.5 + 10 = 14.8$$

$$\mathrm{Var}\,X = EX^2 - (EX)^2 = 14.8 - 4.84 = \underline{9.96}$$

$$EY = 0.3e^{-2} + 0.5e^{4} + 0.1e^{10} + 0.1e^{20} \approx \underline{4.85\times 10^{7}}$$

2. Remember that the geometric distribution is given by the probability mass distribution

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for non-negative integers n and some $p \in (0, 1)$.

a) Prove that the probability mass distribution describes indeed a probability, i.e. show that

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c) Prove that it holds for non-negative integers n, i that

$$\mathbb{P}[X \geq n + i \,|\, X \geq n] = \mathbb{P}[X \geq i].$$

a) $\sum_{n=0}^{\infty} (1-p)^n p = \frac{p}{1-(1-p)} = \frac{p}{p} = 1$

b) $P[X \geq 3] = 1 - P[X < 3] = 1 - \sum_{n=0}^{2} p(n)$

$= 1 - p\left((1-p)^0 + (1-p)^1 + (1-p)^2\right)$

$= 1 - p - p(1-p) - p(1-2p+p^2)$

$= 1 - 2p + p^2 - p + 2p^2 - p^3 = 1 - 3p + 3p^2 - p^3$

c) $P[X \geq n+i \,|\, X \geq n] = \frac{P[X \geq n \,|\, X \geq n+i]\, P[X \geq n+i]}{P[X \geq n]}$

$= \frac{P[X \geq n+i]}{P[X \geq n]} = \frac{1 - \sum_0^{n+i} p(k)}{1 - \sum_0^{n} p(k)}$

$= \left[1 - p\frac{1-(1-p)^{n+i+1}}{1-(1-p)}\right] \Big/ \left[1 - p\frac{1-(1-p)^{n+1}}{1-(1-p)}\right]$

$$= \left[1 - p\,\frac{1-(1-p)^{n+i+1}}{p}\right] \Big/ \left[1 - p\,\frac{1-(1-p)^{n+1}}{p}\right]$$

$$= \frac{(1-p)^{n+i+1}}{(1-p)^{n+1}} = (1-p)^{i} = 1 - p\,\frac{1-(1-p)^{i}}{1-(1-p)}$$

$$= 1 - p\sum_{k=0}^{i-1}(1-p)^{k} = 1 - \sum_{k=0}^{i-1} p(k)$$

$$= 1 - P[X < i] = P[X \geq i]$$

3. Let X be a Poisson distributed random variable with parameter $\lambda = 2$.

a) Calculate

$$\mathbb{P}\big[X \geq 2 \,|\, X \geq 1\big].$$

b) Calculate

$$\mathbb{E}\left[\frac{1}{X+1}\right].$$

$$X \sim \text{Poisson}(\lambda) \quad , \quad P[X=k] =: p(k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

a) $$P[X \geq 2 | X \geq 1] = P[X \geq 2] / P[X \geq 1]$$

$$= (1 - P[X<2]) / (1 - P[X=0])$$

$$= \frac{1 - p(0) - p(1)}{1 - p(0)} = \frac{1 - e^{-\lambda} - \lambda e^{-\lambda}}{1 - e^{-\lambda}}$$

$(\lambda = 2)$

$$= \frac{1 - e^{-2} - 2e^{-2}}{1 - e^{-2}} = \frac{1 - 3e^{-2}}{1 - e^{-2}} \approx 0.69$$

b) $$E[(X+1)^{-1}] = \sum_{k=0}^{\infty} \frac{1}{k+1} \frac{2^k e^{-2}}{k!} = \frac{1}{2} \sum_{k=0}^{\infty} \frac{2^{k+1} e^{-2}}{(k+1)!}$$

$$= \frac{1}{2} \sum_{j=1}^{\infty} \frac{2^j e^{-2}}{j!} = \frac{1}{2}\left(- \frac{2^0 e^{-2}}{0!} + \sum_{j=0}^{\infty} \frac{2^j e^{-2}}{j!} \right)$$

$$= \frac{1}{2}\left(-e^{-2} + 1\right) = \frac{1 - e^{-2}}{2} \approx 0.43$$

4. Let Y be a Poisson distributed random variable with parameter λ, where λ is a non-negative integer. Calculate

$$\mathbb{E}\big[|Y-\lambda|\big].$$

$$E\big[|Y-\lambda|\big] = \sum_{k=0}^{\lambda-1}(\lambda-k)p_k + \sum_{k=\lambda+1}^{\infty}(k-\lambda)p_k$$

$$= \sum_{k=0}^{\lambda-1}\lambda p_k - \sum_{k=0}^{\lambda-1} k\, p_k + \sum_{k=\lambda+1}^{\infty} k\, p_k - \sum_{k=\lambda+1}^{\infty}\lambda p_k$$

$$= \sum_{k=0}^{\lambda-1}\lambda\frac{\lambda^k e^{-\lambda}}{k!} - \sum_{k=0}^{\lambda-1} k\frac{\lambda^k e^{-\lambda}}{k!}$$

$$+ \sum_{\lambda+1}^{\infty} k\frac{\lambda^k e^{-\lambda}}{k!} - \sum_{\lambda+1}^{\infty}\lambda\frac{\lambda^k e^{-\lambda}}{k!}$$

$$= \lambda\left[\sum_{0}^{\lambda-1}\frac{\lambda^k e^{-\lambda}}{k!} - \sum_{0}^{\lambda-2}\frac{\lambda^j e^{-\lambda}}{j!} + \sum_{\lambda}^{\infty}\frac{\lambda^j e^{-\lambda}}{j!} - \sum_{\lambda+1}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}\right]$$

$$= \lambda\left[\frac{\lambda^{\lambda-1}e^{-\lambda}}{(\lambda-1)!} + \frac{\lambda^{\lambda}e^{-\lambda}}{\lambda!}\right] = 2\lambda e^{-\lambda}\frac{\lambda^{\lambda-1}}{(\lambda-1)!}$$

$$= 2e^{-\lambda}\frac{\lambda^{\lambda}}{(\lambda-1)!}$$