1. How many ways can 3 sci-fi books, 4 math books, and 1 cooking book be arranged on a shelf if

a) there are no restrictions on the arrangement?

Since there are 8 books, there are $8! = 40,320$ arrangements

b) all the sci-fi are together and all the math books are together?

Grouping by subject gives 3 'blocks' to arrange. Then arrange within the blocks: $3!\,3!\,4!\,1! = 864$ arrangements

c) only the math books must be stored together?

Now there are 5 blocks to arrange and then one block (math books) to arrange within: $5!\,4! = 2,880$ arrangements.

2. Mike has 9 friends but only room in his apartment to invite 6 over.

a) How many choices does he have in making a guest list?

He has $\binom{9}{6} = 84$ ways to choose who's on the list.

b) If 2 friends are feuding and cannot both be in attendance, how many choices does he have?

He can either exclude them both or pick one to invite. So he has $\binom{2}{0}\binom{7}{6} + \binom{2}{1}\binom{7}{5} = 7 + 2 \cdot 21 = 49$ choices.

c) If 2 friends will only attend if both are invited, how many choices does he have?

He can either exclude both or invite both. So he has $\binom{2}{0}\binom{7}{6} + \binom{2}{2}\binom{7}{4} = 7 + 35 = 42$ choices.

3. A coin is tossed repeatedly until the first time "heads" appears

a) Describe mathematically the sample space of this experiment.

Let H be the event that the toss lands heads up and T the event that the toss lands tails up. The sample space is

$$\Omega = \{ H, (T,H), (T,T,H), (T,T,T,H), \dots \}$$

b) Describe mathematically the events:

E = "There are no more than four tails"
F = "There are at least two tails"

$E = \{ H, (T,H), (T,T,H), (T,T,T,H), (T,T,T,T,H) \}$
$F = \Omega \setminus \{ H, (T,H) \} = \{ (T,T,H), (T,T,T,H), \dots \}$

c) Describe mathematically the events $E \cap F$ and $E \cup F^c$

$E \cap F = \{ (T,T,H), (T,T,T,H), (T,T,T,T,H) \}$
$E \cup F^c = E \cup \{ H, (T,H) \} = E$ since $F^c \subset E$

4. Given a family $\{E_1, E_2, E_3, \ldots, E_n, \ldots\}$ of sets on some sample space Ω, construct a new family $\{F_1, F_2, \ldots, F_n, \ldots\}$ such that the events F_i are monotone ($F_m \subset F_n$ whenever $m \le n$) and

$$\bigcup_{k=1}^{n} E_k = \bigcup_{k=1}^{n} F_k \quad \forall n \in \mathbb{Z}^+ .$$

Prove that the constructed family satisfies these properties.

Let $F_1 := E_1$

$F_2 := E_2 \cup F_1 = E_2 \cup E_1$

$F_3 := E_3 \cup F_2 = E_3 \cup E_2 \cup E_1$

$\vdots$

$F_n := E_n \cup F_{n-1} = E_n \cup \left(\bigcup_{k=1}^{n-1} E_k\right) = \bigcup_{k=1}^{n} E_k$

To prove monotonicity, suppose $m<n$ (if $m=n$ the result is immediate).

$$F_n = \bigcup_{k=1}^{n} E_k = \left(\bigcup_{k=1}^{m} E_k\right) \cup \left(\bigcup_{j=m+1}^{n} E_j\right) = F_m \cup \left(\bigcup_{j=m+1}^{n} E_j\right) \supseteq F_m .$$

Prove $\bigcup_{k=1}^{n} E_k = \bigcup_{k=1}^{n} F_k$ by induction. $F_1 := E_1$ establishes the base case. Assume the equality holds.

$$\bigcup_{k=1}^{n+1} F_k = F_{n+1} \cup \left(\bigcup_{k=1}^{n} F_k\right) = F_{n+1} \cup \left(\bigcup_{k=1}^{n} E_k\right) = \left(\bigcup_{k=1}^{n+1} E_k\right) \cup \left(\bigcup_{k=1}^{n} E_k\right) = E_{n+1} \cup \left(\bigcup_{k=1}^{n} E_k\right) \cup \left(\bigcup_{k=1}^{n} E_k\right) = \bigcup_{k=1}^{n+1} E_k .$$