Worcester Polytechnic Institute
Department of Mathematical Sciences
Professor: Stephan Sturm
Teaching Assistant: Dane Johnson

Fall 2021 - A Term

MA 2631

Probability Theory

Section AL01 / AD01

Assignment 5

due on Friday, September 17

based on Lectures of Chapter 3.3–4.1

1. Show that if A and B are independent events on a sample space Ω, then also A^c and B are independent.

2. Suppose that A, B and C are independent events on a sample space Ω with $\mathbb{P}[A \cap B] \neq 0$. Prove that
$$\mathbb{P}[A \cap C \mid A \cap B] = \mathbb{P}[C].$$

3. Assume that in a family the birth of a boy and a girl is equally likely and that the family has $n \geq 2$ children. Are the events

   A … There is at least one boy and at least one girl in the family

   B … There is at most one girl in the family

   independent?

   Hint: Note that the answer depends on n and you might want to prove a general statement, e.g., by induction.

4. Let A, B, C be *independent* events on a sample space Ω with $\mathbb{P}[A] = \frac{1}{2}$, $\mathbb{P}[B] = \frac{2}{3}$ and $\mathbb{P}[C] = \frac{3}{4}$. Calculate
$$\mathbb{P}\big[A \cup (B \cap C)\big].$$

5. Consider the probability mass distribution $\mathbb{P}[Y = i] = c \cdot 0.1^i$ on the non-negative integers for some constant c.
   a) Calculate c.
   b) Calculate $\mathbb{P}[Y = 0]$ and $\mathbb{P}[Y > 2]$.
   c) Calculate $\mathbb{P}[Y \leq 5 \,|\, Y > 2]$.

6. Assume you are flipping a fair coin until head appears the 5-th time. Let Y denote the number of tails that occur. Calculate the probability mass distribution of Y.

8 points per problems

Additional practice problems (purely voluntary - no points, no credit, no grading):

Standard Carlton and Devore, Section 1.5: 84, 85, 87, 91, 92; Section 2.2: 11, 12, 17

Extra In the early days of the Covid-19 pandemic it was often reported that the chance of an infection is only relatively likely if one stays close to an infected person for (more than) 15 minutes[1]. A clever WPI student was pondering the idea if this would mean that if they meets three friends for 5 minutes each, the chance of an infection would be lower than meeting one friend for 15 minutes and thus the risk would be manageable. Are they right? To answer the question you can assume that

i) the 15 minutes estimates comes from the idea that with increasing time spent, the infection gets likelier, and the 15 minutes mark a threshold amount, a probability p.

ii) the probability of an infection is equally likely in each of the 15 minutes.

iii) that an eventual Covid-19 infection of the friends is independent.

Note: practically there are good arguments against assumptions ii) and iii). More about this and more reasonable assumptions later in class.

[1] E.g., "Based on our current knowledge, a close contact is someone who was within 6 feet of an infected person for at least 15 minutes starting from 48 hours before illness onset until the time the patient is isolated." https://www.cdc.gov/coronavirus/2019-ncov/php/principles-contact-tracing.html

MA 2631 Assignment 5 Answers

1. $P(A\cap B) = P(A)P(B)$ since A, B are independent.
   $B = (A\cap B)\cup(A^c\cap B)$ is a disjoint union.
   $P(B) = P(A\cap B) + P(A^c\cap B) = P(A)P(B) + P(A^c\cap B)$
   $P(A^c\cap B) = P(B)(1-P(A)) = P(A^c)P(B).$

2. $P(C) = P(A\cap C\,|\,A\cap B)$?
   $$P(A\cap C\,|\,A\cap B) = \frac{P((A\cap C)\cap(A\cap B))}{P(A\cap B)} = \frac{P(A\cap B\cap C)}{P(A)P(B)} = \frac{P(A)P(B)P(C)}{P(A)P(B)} = P(C)$$

3. $P(A\cap B) = P(A)P(B)$?
   $$P(A) = \frac{2^n-2}{n}, \quad P(B) = \frac{1+n}{2^n}, \quad P(A\cap B) = \frac{n}{2^n}$$
   $P(A\cap B) = P(A)P(B)$ iff $n+1 = 2^{n-1}$
   For $n=2$, $n+1 = 3 \neq 2 = 2^{n-1}$. For $n=3$, $n+1 = 4 = 2^{n-1}$
   For $n=4$, $n+1 = 5 < 8 = 2^{n-1}$
   Assume for some $n \geq 4$, $n+1 < 2^{n-1}$.
   $$2^{n-1+1} = 2^{n-1}\cdot 2 > (n+1)\cdot 2 = n+2+n \geq n+2+4 > (n+1)+1$$
   Conclude that $2^{n-1} \neq n+1$ for all $n\geq 4$ and therefore A and B are independent iff $n=3$.

4. $$\begin{aligned} P(A\cup(B\cap C)) &= P(A) + P(B\cap C) - P(A\cap B\cap C) \\ &= P(A) + P(B)P(C) - P(A)P(B)P(C) \\ &= \tfrac{1}{2} + \tfrac{2}{3}\tfrac{3}{4} - \tfrac{1}{2}\tfrac{2}{3}\tfrac{3}{4} = \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{3}{4} \end{aligned}$$

5. $P(Y=i) = c\cdot 0.1^i, \quad i = 0, 1, 2, \ldots$
   a) $1 = c(1 + 0.1 + 0.1^2 + 0.1^3 + \ldots) = \dfrac{c}{1-0.1} = \dfrac{c}{.9} \Rightarrow c = 0.9$
   b) $P(Y=0) = 1$, $P(Y>2) = 1 - P(Y\leq 2) = 1 - .9 - .09 - .009 = .001$
   c) $P(Y\leq 5\,|\,Y>2) = \dfrac{P(2<Y\leq 5)}{P(Y>2)} = \dfrac{.9(.1^3 + .1^4 + .1^5)}{.001} = .999$

6. $P(Y=n) = \binom{n+4}{n}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^5 = \binom{n+4}{n}\left(\frac{1}{2}\right)^{n+5}$