MA 2631 Assignment 7

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1. Calculate the variance of X and the variance of $Y = 3X + 2$ for the random variable X with the probability mass distribution

$$P[X = 1] = 0.3, \quad P[X = 4] = 0.25, \quad P[X = 7] = 0.4, \quad P[X = 10] = 0.05.$$

Answer:

$$\begin{aligned}
E[X] &= 4.6 \\
E[X^2] &= .3 + 4 + \frac{49 \cdot 2}{5} + 5 = 28.9 \\
\mathrm{Var}[X] &= E[X^2] - E[X]^2 = 28.9 - 21.16 = \boxed{7.74} \\
\mathrm{Var}[Y] &= \mathrm{Var}[3X + 2] = 3^2\mathrm{Var}[X] = \boxed{69.66}
\end{aligned}$$

2. Suppose we pick a month at random from a non leap-year calendar and let X be the number of days in that month. Find the mean and the variance of X.

   Answer:

$$P[X = 28] = \frac{1}{12}, \quad P[X = 30] = \frac{4}{12}, \quad P[X = 31] = \frac{7}{12}.$$

$$\begin{aligned} E[X] &= \frac{28 + 120 + 217}{12} = \boxed{\frac{365}{12} = 30.41\overline{6}} \\ E[X^2] &= \frac{28^2 + 30^2 \cdot 4 + 31^2 \cdot 7}{12} = \frac{784 + 3600 + 6727}{12} = \frac{11,111}{12} = 925.91\overline{6} \\ \mathrm{Var}[X] = E[X^2] - E[X]^2 &= \frac{11,111}{12} - \frac{133,225}{144} = \boxed{\frac{107}{144} = .7430\overline{5}} \end{aligned}$$

3. Let Y be a binomial distributed random variable with n trials of success probability p. Show that $\mathrm{Var}[Y] = np(1-p)$.

   Answer: Y is characterized by $P[Y = k] = \binom{n}{k} p^k (1-p)^{n-k}$ for nonnegative integers k and has mean $E[Y] = np$.

$$
\begin{aligned}
E[Y^2] &= \sum_{k=0}^{n} k^2 P[Y = k] = \sum_{k=0}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{k=1}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{k=1}^{n} kn \binom{n-1}{k-1} p^k (1-p)^{n-k} \qquad \text{identity: } k\binom{n}{k} = n\binom{n-1}{k-1} \\
&= np \sum_{k=1}^{n} k \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} \\
&= np \sum_{j=0}^{n-1} (j+1) \binom{n-1}{j} p^j (1-p)^{n-j-1} \\
&= npE[X+1] \quad X \sim \text{Binomial}(n-1, p) \\
&= np(E[X]+1) \\
&= np((n-1)p+1)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}[Y] &= E[Y^2] - E[Y]^2 = np((n-1)p+1) - (np)^2 \\
&= (np)^2 - np^2 + np - (np)^2 \\
&= np(1-p).
\end{aligned}
$$

4. Let Z be a geometric distributed random variable with success probability p. Calculate $\text{Var}[Z]$.

Answer:

$P[Z = k] = (1-p)^k p, k = 0, 1, 2, \ldots$

$$
\begin{aligned}
E[Z] &= \frac{1-p}{p} \\
E[Z^2] &= \sum_{k=0}^{\infty} k^2 (1-p)^k p \\
&= p \sum_{k=1}^{\infty} ((k-1)+1)^2 (1-p)^k \\
&= p \sum_{k=1}^{\infty} (k-1)^2 (1-p)^k + p \sum_{k=1}^{\infty} 2(k-1)(1-p)^k + p \sum_{k=1}^{\infty} (1-p)^k \\
&= (1-p) \sum_{j=0}^{\infty} j^2 p (1-p)^j + 2(1-p) \sum_{j=0}^{\infty} j p (1-p)^j + (1-p) \\
&= (1-p) E[Z^2] + 2(1-p) E[Z] + (1-p) \\
E[Z^2](1-(1-p)) &= 2(1-p)\frac{1-p}{p} + 1 - p \\
E[Z^2] &= \frac{1-p}{p}\left[\frac{2-2p}{p} + \frac{p}{p}\right] = \frac{(1-p)(2-p)}{p^2}
\end{aligned}
$$

$$
\begin{aligned}
\text{Var}[Z] &= \frac{(1-p)(2-p)}{p^2} - \left(\frac{1-p}{p}\right)^2 = \frac{1-p}{p^2}(2-p-(1-p)) \\
&= \boxed{\frac{1-p}{p^2}}
\end{aligned}
$$

5. Assume X is a random variable that only takes on nonnegative integer values and satisfies

$$P[X \geq n + i | X \geq n] = P[X \geq i], \quad n, i \geq 0.$$

Show that X is a geometric distributed random variable.

Answer: Let $p := P[X = 0]$.

Lemma: $P[X = n] = pP[X \geq n]$ for any nonnegative integer n.

Proof. By the assumption, $P[X \geq i] = P[X \geq n + i | X \geq n] = P[X \geq n + i]/P[X \geq n]$. In particular for $i = 1$,

$$\begin{aligned} 1 - P[X = 0] = P[X \geq 1] = \frac{P[X \geq n + 1]}{P[X \geq n]} &= 1 - \frac{P[X = n]}{P[X \geq n]} \\ pP[X \geq n] = P[X = 0]P[X \geq n] &= P[X = n] \end{aligned}$$

Claim $P[X = n] = (1 - p)^n p$, $n = 0, 1, 2, \ldots$

Proof. For the base case $n = 0$, $P[X = 0] = pP[X \geq 0] = p = p(1 - p)^0$ using the lemma. Assume the claim holds for some integer $n \geq 0$.

$$\begin{aligned} P[X = n + 1] &= pP[X \geq n + 1] \quad \text{(Lemma)} \\ &= p(P[X \geq n] - P[X = n]) \\ &= p\left(\frac{P[X = n]}{p} - P[X = n]\right) \quad \text{(Lemma)} \\ &= p((1 - p)^n - p(1 - p)^n) \quad \text{(Inductive Hypothesis)} \\ &= p(1 - p)^n(1 - p) \\ &= (1 - p)^{n+1}p \end{aligned}$$

This proves the claim, which shows that X has the probability mass function as a geometrically distributed random variable.

6. The number of errors on a book page follow a Poisson distribution. It has been determined that on 10% of the pages there is at least one error.

   (a) Determine the parameter of the Poisson distribution.

   (b) What is the expected number of errors on a page?

   Answer: $P[X = k] = e^{-\lambda} \frac{\lambda^k}{k!}$, $k = 0, 1, 2, \dots$

   (a) Let $P[X = k]$ be the probability that there are k errors on a page. The given assumption that on 10% of the pages there is a least one error means $P[X > 0] = 0.1$.

   $$e^{-\lambda} \frac{\lambda^0}{0!} = P[X = 0] = P[X \leq 0] = 1 - P[X > 0] = 0.9 \implies \lambda = \boxed{\ln 10/9 \approx .10536}$$

   (b) $E[X] = \sum_{k=0}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!} = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} = \lambda e^{-\lambda} e^{\lambda} = \lambda = \boxed{\ln 10/9 \approx .10536}$