MA 2631 Conference 5

September 29, 2021

The exponential random variable with parameter $\lambda > 0$:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

1. Let X be an exponential random variable with parameter λ. Calculate $\mathrm{Var}[X]$ in two ways:

   (a) By looking up $E[X]$ in the lecture notes, calculating $E[X^2]$ directly using the definition of expectation, and the formula $\mathrm{Var}[X] = E[X^2] - (E[X])^2$;

   (b) By deriving the moment generating function $M_X(t) = E[e^{tX}]$ (a challenge problem).

   Answer: The work for (b) will be given:

$$\begin{aligned} M_X(t) = E[e^{tX}] &= \int_{-\infty}^{\infty} e^{tx} f(x) dx \\ &= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx \\ &= \lambda \int_0^{\infty} e^{(t-\lambda)x} dx \\ &= \lambda \cdot \lim_{b \to \infty} \left[\frac{1}{t-\lambda} e^{(t-\lambda)x} \right]_0^b \\ &= \frac{\lambda}{\lambda - t}, \quad \lambda > t \end{aligned}$$

   For this calculation to be correct, we needed to assume that $\lambda > t$ (to avoid division by zero and to avoid a divergent integral). This assumption is ok since $\lambda > 0$ and we will be using $t = 0$ later on.

   Using $E[X^n] = M_X^{(n)}(0)$ (the n^{th} derivative of the moment generating function evaluated at $t = 0$),

$$M_X'(t) = \frac{\lambda}{(\lambda - t)^2} \implies E[X] = M_X'(0) = \frac{1}{\lambda}$$

$$M_X''(t) = \frac{2\lambda}{(\lambda - t)^3} \implies E[X^2] = \frac{2}{\lambda^2}$$

$$\mathrm{Var}(X) = E[X^2] - E[X]^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}.$$

   In general, $E[X^n] = \frac{n!}{\lambda^n}$. You could notice the pattern by taking a few more derivatives or derive this using infinite series:

$$
\begin{aligned}
M_X(t) &= \frac{\lambda}{\lambda - t}, \quad \lambda > t \\
&= \frac{1}{1 - t/\lambda} \\
&= \sum_{n=0}^{\infty} \frac{1}{\lambda^n} t^n \quad \text{geometric series with ratio } t/\lambda, |t/\lambda| < 1.
\end{aligned}
$$

But now compare $M_X(t) = \sum_{n=0}^{\infty} \frac{1}{\lambda^n} t^n$ with the Maclaurin series for $M_X(t)$:

$$
M_X(t) = \sum_{n=0}^{\infty} \frac{M_X^{(n)}(0)}{n!} t^n = \sum_{n=0}^{\infty} \frac{E[X^n]}{n!} t^n.
$$

Equating terms shows that $E[X^n]/n! = \frac{1}{\lambda^n} \implies E[X^n] = \frac{n!}{\lambda^n}$. In particular, $E[X] = \frac{1}{\lambda}$ and $E[X^2] = \frac{2}{\lambda^2}$, from which you can calculate the variance.

2. Suppose that the length of a phone call in minutes is an exponential random variable with parameter $\lambda = \frac{1}{10}$. If someone arrives immediately ahead of you at public telephone booth, find the probability you will have to wait

   (a) more than 10 minutes;

   (b) not more than one standard deviation away from the mean.

   Answer: Let X be your waiting time in minutes. The cumulative distribution function $F(x) = \int_{-\infty}^{x} f(y)dy$ was calculated in the lecture notes

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases} = \begin{cases} 0 & x < 0 \\ 1 - e^{-x/10} & x \geq 0 \end{cases}$$

   (a) $P[X > 10] = 1 - P[X \leq 10] = 1 - F(10) = 1 - (1 - e^{-\frac{1}{10}\cdot 10}) = e^{-1} \approx 0.368$

   (b) The mean μ is the expected value of X

$$\mu = E[X] = \frac{1}{\lambda} = 10.$$

   The standard deviation σ is the square root of the variance

$$\sigma = \sqrt{\text{Var}[X]} = \sqrt{1/\lambda^2} = 10.$$

$$P[\mu - \sigma \leq X \leq \mu + \sigma] = P[0 \leq X \leq 20] = F(20) - F(0) = 1 - e^{-\frac{20}{10}} - 0 = 1 - e^{-2} \approx 0.865$$

3. We say that a nonnegative random variable X is memoryless if

$$P[X > s+t | X > t] = P[X > s] \quad \text{for all } s, t \geq 0.$$

Show that an exponential random variable X with parameter λ is memoryless.

Answer: Let $s, t \geq 0$ be arbitrary.

$$\begin{aligned}
P[X > s+t | X > t] &= \frac{P[\{X > s+t\} \cap \{X > t\}]}{P[X > t]} \\
&= \frac{P[X > s+t]}{P[X > t]} \\
&= \frac{1 - P[X \leq s+t]}{1 - P[X \leq t]} \\
&= \frac{1 - F(s+t)}{1 - F(t)} \\
&= \frac{1 - (1 - e^{-\lambda(s+t)})}{1 - (1 - e^{-\lambda t})} \\
&= \frac{e^{-\lambda s} e^{-\lambda t}}{e^{-\lambda t}} \\
&= e^{-\lambda s} \\
&= 1 - (1 - e^{-\lambda s}) \\
&= 1 - F(s) \\
&= 1 - P[X \leq s] \\
&= P[X > s].
\end{aligned}$$

4. Suppose that the number of miles that a car can run before its battery wears out is exponentially distributed with an average value of $10,000$ miles. If a person desires to take a 5,000 mile trip, what is the probability that he or she will be able to complete the trip without having to replace the car battery?

Answer: Let X be the remaining number of miles (in thousands) before the car battery expires. Since $1/\lambda = E[X] = 10$, $\lambda = 1/10$. The car battery may have been in use for some $m \geq 0$ miles before starting the trip. Considering the result from problem 3, it doesn't matter what m is.

$$P[X > m+5 | X > m] = P[X > 5] = 1 - P[X \leq 5] = 1 - (1 - e^{-\frac{1}{10} \cdot 5}) = e^{-\frac{1}{2}} \approx .607.$$