MA 2631 Assignment 6

Hubert J. Farnsworth

September 21, 2021

1. The picture on the last page (taken from https://www.mass.gov/info-details/covid-19-response-reporting) shows that in early September 2021, the percentage of positive Covid-tests were rising for both, higher education tests and non-higher education tests, though the overall positivity rate was actually slightly declining.

   At first sight this might look paradoxical, and you might even think that there might be an error. However, you should be able to explain this phenomenon with your knowledge about conditional probabilities and related theorems. Please do so.

   Answer: Let HE be the event that the test was a higher education test and NHE the event that test was a non-higher education test. Let P be the event that the test is positive.

   $$\mathbb{P}[P] = \mathbb{P}[P|HE]\mathbb{P}[HE] + \mathbb{P}[P|NHE]\mathbb{P}[NHE]$$

   While both $mathbbP[P|HE]$ and $\mathbb{P}[P|NHE]$ may be increasing, the fact that $\mathbb{P}[HE]$ is increasing at the expense of $\mathbb{P}[NHE]$ (as more tests are beginning to be performed at higher education institutions) and that $\mathbb{P}[P|HE]$ is much smaller than $P[P|NHE]$ causes $\mathbb{P}[P]$ to decrease overall.

2. Consider the random variable X with the probability mass distribution

   $$P[X = 1] = 0.3, \quad P[X = 4] = 0.25, \quad P[X = 7] = 0.4, \quad P[X = 10] = 0.05.$$

   Calculate the expected value of X and Y with $Y = 3X + 2$.

   Answer:

   $$E[X] = 0.3 \cdot 1 + 0.25 \cdot 4 + 0.4 \cdot 7 + 0.05 \cdot 10 = 4.6$$
   $$E[Y] = E[3X + 2] = 3E[X] + 2 = 15.8$$

3. Let X be a random variable describing the number of failures before the first success of an independently repeated experiment with success probability $p = \frac{3}{4}$.

   (a) Calculate the probability that there are not more than two failures before the first success.

   (b) Calculate $E[2^X]$.

   Answer:

   (a) $P[X = 0] + P[X = 1] + P[X = 2] = \frac{3}{4}(1 + \frac{1}{4} + \frac{1}{16}) = \frac{63}{64} \approx 98.44\%$.

   (b) $E[2^X] = \sum_{k=0}^{\infty} 2^k P[X = k] = \sum_{k=0}^{\infty} 2^k \left(\frac{1}{4}\right)^k \frac{3}{4} = \sum_{k=0}^{\infty} \frac{3}{4} \left(\frac{1}{2}\right)^k = \frac{3}{4} \cdot 2 = \frac{3}{2}$.

4. Let X be a random variable taking values in $\mathbb{N}$. Show that $E[X] = \sum_{n=1}^{\infty} P[X \geq n]$.

Answer:

$$\sum_{n=1}^{\infty} P[X \geq n] = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} P[X = k] = \sum_{k=1}^{\infty} \sum_{n=1}^{k} P[X = k] = \sum_{k=1}^{\infty} kP[X = k] = E[X].$$