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Fall 2021 - A Term

MA 2631

Probability Theory

Section AL01 / AD01

Assignment 1 – Solutions

due on Tuesday, August 31 based on Lectures of Chapter 1.1–1.3

1. Twenty workers are to be assigned to twenty different jobs. How many different assignments are possible?

   As each worker has to do one job, we have just to check in how many ways assignments can be done. Assume that the jobs are fixed and numbered. The first worker has 20 jobs to chose from. For the next, there are only 19 remaining to choose from, and so on. Thus in total there are

$$20 \cdot 19 \cdot \ldots \cdot 3 \cdot 2 \cdot 1 = 20! \approx 2.4 \cdot 10^{18}.$$

2. A tourist wants to visit six out of thirteen American cities; seven of them are on the East Coast, three on the West Coast and three in the middle of the country. In how many ways can she do that if

   i) the order of the visits does not play a role;
   ii) the order of the cities is important;
   iii) the order is not important, but she wants to visit at least three cities on the East Coast and at least two on the West Coast.

   i) We have plainly

$$\binom{7+3+3}{6} = \binom{13}{6} = \frac{13!}{7! \cdot 6!} = 1716.$$

ii) As we have for every of the $\binom{13}{6}$ *trips* $6!$ *different orderings we have*

$$\binom{13}{6} \cdot 6! = \frac{13!}{6! \cdot 7!} \cdot 6! = \frac{13!}{7!} = 1,235,520.$$

iii) Here we have add up the different possibilities:

$$\binom{7}{3} \cdot \binom{3}{2} \cdot \binom{2}{1} + \binom{7}{4} \cdot \binom{3}{2} \cdot \binom{2}{0} + \binom{7}{3} \cdot \binom{3}{3} \cdot \binom{2}{0}$$
$$= 35 \cdot 3 \cdot 2 + 35 \cdot 3 \cdot 1 + 35 \cdot 1 \cdot 1 = 35 \cdot \big(6 + 3 + 1\big) = 350.$$

3. How many words can you build from the letters

ARRANGE

if

a) you have to use all the letters?
b) you do not have to use all the letters (but every word has at least one letter)?

a) We have 7 letters, but two of them (A and R) occur two ties and we cannot distinguish between them. Thus we have two divide the $7!$ *possibilities two times by* $2!$ *to account for the identical letters,*

$$\frac{7!}{2! \cdot 2!} = 1260$$

b) This is a tedious exercise to get through all cases. Specifically,

- ***7 letter words****: 1260 as above*
- ***6 letter words****: Now we have 2 cases: The missing character can be a double character or not. In the former case we have*

$$\frac{6!}{2!} = 360$$

  in the latter

$$\frac{6!}{2! \cdot 2!} = 180$$

  As the former can happen in two ways (either A or R are missing) and the latter in $\binom{3}{2} = 3$ *(choose two from N, G or E), we have in total*

$$360 \cdot 2 + 180 \cdot 3 = 1260$$

  And so we continue...
- ***5 letter words****: 690*
- ***4 letter words****: 270*
- ***3 letter words****: 84*
- ***2 letter words****: 22*
- ***1 letter words****: 5*

So there are in total 3591 words that can be constructed.

4. Two experiments are to be performed. The first one can result in r different outcomes. If the first experiment results in outcome j, then the second experiment can result in n_j possible outcomes, $j = 1, \dots, r$. What is the total number of possibles outcomes of the two experiments?

   As the first experiment can only result in one specific outcome, we have just to sum up the different possibilities of outcomes of the second experiments,

$$n_1 + n_2 + \ldots + n_r = \sum_{k=1}^{r} n_k.$$

5. Prove by induction that for all positive integers n it holds that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

   We check first the induction for $n = 0$ *(and* $n = 1$ *if we want)*

$$n = 0: \qquad \sum_{k=0}^{0} \binom{0}{k} = 1 = 2^0, \quad \checkmark$$

$$n = 1: \qquad \sum_{k=0}^{1} \binom{1}{k} = \binom{1}{0} + \binom{1}{1} = 1 + 1 = 2 = 2^1. \quad \checkmark$$

   Now we use the result for n *to conclude by lemma 1.9 from class and substitution*

$$\begin{aligned}
\sum_{k=0}^{n+1} \binom{n+1}{k} &= \binom{n+1}{0} + \sum_{k=1}^{n} \binom{n+1}{k} + \binom{n+1}{n+1} \\
&\overset{1.9\,Lemma}{=} 1 + \sum_{k=1}^{n} \left(\binom{n}{k-1} + \binom{n}{k} \right) + 1 \\
&= \left(\sum_{k=1}^{n} \binom{n}{k-1} + 1 \right) + \left(1 + \sum_{k=1}^{n} \binom{n}{k} \right) \\
&= \left(\sum_{k=1}^{n} \binom{n}{k-1} + \binom{n}{n} \right) + \left(\binom{0}{0} + \sum_{k=1}^{n} \binom{n}{k} \right) \\
&\overset{j=k-1}{=} \left(\sum_{j=0}^{n-1} \binom{n}{j} + \binom{n}{n} \right) + \sum_{k=0}^{n} \binom{n}{k} \\
&\overset{Ass}{=} \sum_{j=0}^{n} \binom{n}{j} + 2^n \overset{Ass}{=} 2^n + 2^n = 2^{n+1}
\end{aligned}$$

6. A soccer coach has 2 goalkeepers and 15 field players at his disposition: 5 defenders, 7 midfielders and 3 forwards.

   a) How many different soccer teams (consisting of 1 goalkeeper and 10 field players) can he build up from these players?

   b) If he wants additionally that there are at least 3 defenders, at leas 4 midfielders and at least 2 forwards in his team, how many teams he can form under this restriction?

   a) We have $\binom{2}{1} \cdot \binom{15}{10} = 2 \cdot 3003 = 6006$ *possibilities.*

   b) We remark that there are in the end three possibilities for the field players: 4 defenders, 4 midfielders an 2 forwards; 3 defenders, 5 midfielders an 2 forwards or 3 defenders, 4 midfielders an 3 forwards. Thus we have

$$
\begin{aligned}
&\binom{2}{1}\left(\binom{5}{4} \cdot \binom{7}{4} \cdot \binom{3}{2} + \binom{5}{3} \cdot \binom{7}{5} \cdot \binom{3}{2} + \binom{5}{3} \cdot \binom{7}{4} \cdot \binom{3}{3}\right) \\
=&2\big(5 \cdot 35 \cdot 3 + 10 \cdot 21 \cdot 3 + 10 \cdot 35 \cdot 1\big) \\
=&70 \cdot (15 + 18 + 10) = 70 \cdot 43 = 3010
\end{aligned}
$$

   possibilities.

8 points per problems

Additional practice problems (purely voluntary - no points, no credit, no grading):

Standard Carlton and Devore, Section 1.3: Exercises 31, 32, 36, 37, 43, 49.

Extra Remember that Pascal's triangle can be generated by filling in for each entry the number created by adding the two numbers in the line above (more formally, the k-th element in the n-th row is given as the sum of the $k-1$-st and the k-th element in the $n-1$-st row). Here all the left- and rightmost elements in each row (the 0-th and the n-th element) are 1. We try to generalize the triangle a bit by assuming that the 0-th element in each row is some (real) number a and the n-th element is some real number b (with the additional specification that the only element in the 0-th row is a). From here on, we can fill up the rest of the triangle (the interior) by the same rule as in the original Pascal's triangle (i.e., each element is the sum of the two elements above; the k-th element in the n-th row is given as the sum of the $k-1$st and the k-th element in the $n-1$st row). Can you derive an explicit (i.e., non-recursive) formula for the k-th element in the n-th row?