MA 2631 Conference 7

October 5, 2022

Beware of typos but I try to be careful.

1. In a small town, there are 50 births a year. Assume that the probability that a newborn is a girl is 50%. How likely is it that in a given year, there are at least 25 and at most 27 girls born. Calculate this probability

   a) exactly.

   b) by an approximation with the normal distribution.

   Answer:

   a) Model this using a binomial distribution with $n = 50$ and success probability $p = \frac{1}{2}$. Denoting the number of newborn girls by X we get

$$\begin{aligned} P[25 \leq X \leq 27] =& \binom{50}{25}\left(\frac{1}{2}\right)^{25}\left(\frac{1}{2}\right)^{25} + \binom{50}{26}\left(\frac{1}{2}\right)^{26}\left(\frac{1}{2}\right)^{24} + \binom{50}{27}\left(\frac{1}{2}\right)^{27}\left(\frac{1}{2}\right)^{23} \\ =& \frac{1}{2^{50}}\left(\binom{50}{25} + \binom{50}{26} + \binom{50}{27}\right) \\ =& \frac{50!}{23! \cdot 25! \cdot 2^{50}}\left(\frac{1}{24 \cdot 25} + \frac{1}{24 \cdot 26} + \frac{1}{25 \cdot 26}\right) = \frac{50!}{23! \cdot 25! \cdot 2^{50} \cdot 208} \\ &\approx 31.62\% \end{aligned}$$

   b) Note that we have $\mu = E[X] = n \cdot p = 50 \cdot \frac{1}{2} = 25$ and $\sigma = \text{var}(X) = n \cdot p \cdot (1-p) = 12.5$. By approximation (with continuity correction) using a standard normal distribution Z

$$\begin{aligned} P[25 \leq X \leq 27] \approx& P\left[\frac{24.5 - 25}{\sqrt{12.5}} \leq Z \leq \frac{27.5 - 25}{\sqrt{12.5}}\right] = \Phi\left(\frac{2.5}{\sqrt{12.5}}\right) - \Phi\left(\frac{-0.5}{\sqrt{12.5}}\right) \\ =& \Phi\left(\frac{2.5}{\sqrt{12.5}}\right) + \Phi\left(\frac{0.5}{\sqrt{12.5}}\right) - 1 \approx \Phi(0.7071) + \Phi(0.1414) - 1 \\ \approx& 0.7601 + 0.5563 - 1 = 31.64\% \end{aligned}$$

   If you don't use a continuity correction (the problem statement never said you had to!), you would have used 25 in the calculation above instead of 24.5 and 27 above instead of 27.5. The result is a much rougher approximation (about .2142).

2. Consider a biased coin that shows *heads* in $\frac{2}{3}$ of all cases and *tails* only in $\frac{1}{3}$ of all cases. The coin is flipped consecutively (and independently) 200 times.

   a) What is the probability that *tails* shows up the first time at the 10th flip?

   b) Calculate the probability *heads* shows up more than 150 times (using a suitable approximation).

   Answer:

   a) Let X denote the number of *heads* before the first *tails*. Then X is a geometric random variable with success probability $\frac{1}{3}$ and we have

   $$P[X = 9] = \left(1 - \frac{1}{3}\right)^9 \cdot \frac{1}{3} = \frac{2^9}{3^{10}} \approx 0.8671\%.$$

   a) Let Y be the number of heads out of the 200 flips. Then Y is binomially distributed with 200 trials and success probability $p = \frac{2}{3}$. Note that $E[Y] = 200 \cdot \frac{2}{3} = \frac{400}{3}$ and $\mathrm{var}[Y] = 200 \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{400}{9}$. This means $\frac{Y - \frac{400}{3}}{\sqrt{\frac{400}{9}}}$ is approximately distributed as a standard normal random variable Z and

   $$P[Y > 150] \approx 0.62\%^*.$$

   You'd need to fill in the intermediate steps of this calculation using a process similar to what you see in the previous problem (the details are important!). For the calculation above a continuity correction wasn't used. [Note: alternative calculations lead to

   $$P[Y \geq 151] \approx 0.40\%,$$
   $$P[Y \geq 150.5] \approx 0.5\%,$$

   the last one incorporating the continuity correction.]

3. Assume that the joint probability mass distribution $p_{X,Y}$ of the random variable X and Y is given by

$$\begin{aligned} p_{X,Y}(0,0) =& \frac{1}{3} \qquad p_{X,Y}(0,1) = \frac{1}{4}; \\ p_{X,Y}(1,0) =& \frac{1}{4} \qquad p_{X,Y}(1,1) = \frac{1}{6}. \end{aligned}$$

a) Calculate the marginal probability mass distributions p_X and p_Y.

b) What is the probability mass distribution of the random variable $Z = X^2 + Y$?

Answer:

a)

$$\begin{aligned} p_X(0) =& p_{X,Y}(0,0) + p_{X,Y}(0,1) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} \\ p_X(1) =& p_{X,Y}(1,0) + p_{X,Y}(1,1) = \frac{1}{4} + \frac{1}{6} = \frac{5}{12} \\ p_Y(0) =& p_{X,Y}(0,0) + p_{X,Y}(1,0) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} \\ p_Y(1) =& p_{X,Y}(0,1) + p_{X,Y}(1,1) = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}. \end{aligned}$$

b) The possible values that Z could take on are 0,1, and 2.

$$\begin{aligned} &P[Z=0] = P[X^2+Y=0] = P[X=0, Y=0] = p_{X,Y}(0,0) = \frac{1}{3} \\ &P[Z=1] = P[X^2+Y=1] = P[X=1, Y=0] + P[X=0, Y=1] = p_{X,Y}(1,0) + p_{X,Y}(0,1) = \frac{1}{2} \\ &P[Z=2] = P[X^2+Y=2] = P[X=1, Y=1] = p_{X,Y}(1,1) = \frac{1}{6} \end{aligned}$$

4. Assume that X and Y are jointly distributed random variables with joint density

$$f_{X,Y}(x,y) = \begin{cases} cye^{-x} & \text{if } 0 \le x < \infty,\ 0 \le y \le 1; \\ 0 & \text{else.} \end{cases}$$

for some $c \in \mathbb{R}$.

a) Calculate c.
b) Calculate the marginal probability density functions f_X and f_Y.
c) What is the probability $P[3X + Y^2 > 4]$?

Answer:

a) As the density has to integrate up to one, we have

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dxdy \\ &= \int_0^1\int_0^\infty cye^{-x}\,dxdy = c\Big(\int_0^\infty e^{-x}\,dx\Big)\Big(\int_0^1 ydy\Big) \\ &= c\Big[-e^{-x}\Big]_0^\infty\Big[\frac{y^2}{2}\Big]_0^1 = c\cdot 1\cdot\frac{1}{2} = \frac{c}{2}, \end{aligned}$$

$$\boxed{c = 2}$$

b)

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \begin{cases} \int_0^1 2ye^{-x}\,dy = 2e^{-x}\big[\frac{y^2}{2}\big]_0^1 = e^{-x} & \text{if } x \ge 0; \\ 0 & \text{if } x < 0. \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \begin{cases} \int_0^\infty 2ye^{-x}\,dx = 2y\big[-e^{-x}\big]_0^\infty = 2y & \text{if } 0 \le y \le 1; \\ 0 & \text{else.} \end{cases}$$

c)

$$\begin{aligned} P[3X + Y^2 > 4] &= \iint_{\{(x,y)\in[0,\infty)\times[0,1]\,:\,3x+y^2>4)\}} f_{X,Y}(x,y)\,dxdy \\ &= \iint_{\{(x,y)\in[0,\infty)\times[0,1]\,:\,x>\frac{4-y^2}{3}\}} f_{X,Y}(x,y)\,dxdy \\ &= \int_0^1\int_{\frac{4-y^2}{3}}^\infty 2ye^{-x}\,dxdy = \int_0^1 2y\Big[-e^{-x}\Big]_{\frac{4-y^2}{3}}^\infty dy \\ &= \int_0^1 2ye^{-\frac{4-y^2}{3}}\,dy = 3e^{-\frac{4}{3}}\int_0^1 \frac{2y}{3}e^{\frac{y^2}{3}}\,dy = 3e^{-\frac{4}{3}}\Big[e^{\frac{y^2}{3}}\Big]_0^1 \\ &= 3e^{-\frac{4}{3}}\Big(e^{\frac{1}{3}} - 1\Big) = 3e^{-1} - 3e^{-\frac{4}{3}} \approx 31.28\%. \end{aligned}$$