Assignment 11

1. Let X, Y be two random variables with joint cdf $F_{X,Y}$ and marginal cdfs F_X, F_Y. For x, $y \in \mathbb{R}$, express
$$\mathbb{P}[X > x; Y \leq y]$$
in terms of $F_{X,Y}$ and F_X, F_Y

$\{Y \leq y\} = \Omega \cap \{Y \leq y\} = (\{X \leq x\} \cup \{X > x\}) \cap \{Y \leq y\} = (\{X \leq x\} \cap \{Y \leq y\}) \cup (\{X > x\} \cap \{Y \leq y\})$

$F_Y(y) = P[Y \leq y] = P[X \leq x, Y \leq y] + P[X > x, Y \leq y] = F_{X,Y}(x,y) + P[X > x, Y \leq y]$

$P[X > x, Y \leq y] = F_Y(y) - F_{X,Y}(x,y)$

2. Assume that there are 12 balls in an urn, 3 of them red, 4 white and 5 blue. Assume that you draw 2 balls of them, replacing any drawn ball by a ball of the same color. Denote by X the number of drawn red balls and by Y the number of drawn white balls. Calculate the joint probability mass distribution of X and Y as well as the marginal distributions. Are X and Y independent?

$P_X(0) = \left(\frac{9}{12}\right)^2 = \frac{9}{16}$ $\quad P_X(1) = 2 \cdot \frac{3}{12} \cdot \frac{9}{12} = \frac{6}{16}$ $\quad P_X(2) = \left(\frac{3}{12}\right)^2 = \frac{1}{16}$

$P_Y(0) = \left(\frac{8}{12}\right)^2 = \frac{4}{9}$ $\quad P_Y(1) = 2 \cdot \frac{4}{12} \frac{8}{12} = \frac{4}{9}$ $\quad P_Y(2) = \left(\frac{4}{12}\right)^2 = \frac{1}{9}$

$P_{X,Y}(x,y)$	$x=0$	$x=1$	$x=2$
$y=0$	$\frac{25}{144}$	$\frac{30}{144}$	$\frac{9}{144}$
$y=1$	$\frac{40}{144}$	$\frac{24}{144}$	
$y=2$	$\frac{16}{144}$		

$P_X(0) P_Y(0) = \frac{1}{4} \neq \frac{25}{144} = P_{X,Y}(0,0)$

X and Y are not independent.

3. Assume that the joint probability mass distribution $p_{X,Y}$ of the random variable X and Y is given by
$$p_{X,Y}(1,1) = p_{X,Y}(1,2) = p_{X,Y}(1,3) = \frac{1}{12};$$
$$p_{X,Y}(2,1) = p_{X,Y}(2,2) = p_{X,Y}(2,3) = \frac{1}{4}.$$

a) Calculate the marginal probability mass distributions p_X and p_Y.
b) Are X and Y independent?
c) What is the probability mass distribution of the random variable $Z = \frac{X}{Y}$?

a) $P_X(1) = 1/4$ $\quad P_X(2) = 3/4$
$P_Y(1) = 1/3$ $\quad P_Y(2) = 1/3$ $\quad P_Y(3) = 1/3$

b) Yes since $P_{X,Y}(x,y) = P_X(x) P_Y(y) \quad \forall (x,y) \in \{1,2\} \times \{1,2,3\}$

c) $P_Z(2) = P_{X,Y}(2,1) = 1/4$ $\quad P_Z(1/2) = P_{X,Y}(1,2) = 1/12$
$P_Z(1) = P_{X,Y}(1,1) + P_{X,Y}(2,2) = 1/3$ $\quad P_Z(1/3) = P_{X,Y}(1,3) = 1/12$
$P_Z(2/3) = P_{X,Y}(2,3) = 1/4$

4. Let X and Y be two independent standard-normal distributed random variables and define $Z = X^2 + Y^2$. Calculate the cumulative distribution function of Z. Which distribution follows Z?

For $z \geq 0$,

$$F_Z(z) = P[Z \leq z] = P[X^2 + Y^2 \leq z] = \iint_{\{x^2+y^2 \leq z\}} f_X(x) f_Y(y)\, dA$$

$$= \iint_{\{x^2+y^2 \leq z\}} e^{-x^2/2} e^{-y^2/2} \left(1/\sqrt{2\pi}\right)^2 dA = \frac{1}{2\pi} \iint_{\{x^2+y^2 \leq z\}} e^{-(x^2+y^2)/2}\, dA$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\sqrt{z}} r e^{-r^2/2}\, dr\, d\theta = \frac{2\pi}{2\pi} \left(-e^{-r^2/2}\right)\Big|_0^{\sqrt{z}} = 1 - e^{-z/2}$$

For $z < 0$, $F_Z(z) = P[X^2 + Y^2 \leq z] = 0$

Recall that if W is exponentially distributed with parameter λ,

$$F_W(w) = \begin{cases} 0 & , w < 0 \\ 1 - e^{-\lambda w} & , w \geq 0 \end{cases}$$

Z is exponentially distributed with $\lambda = 1/2$.

5. Let X, Y be two jointly distributed random variables with joint density

$$f_{X,Y}(x, y) = \begin{cases} cxy & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1; \\ 0 & \text{else,} \end{cases}$$

for some constant c.

a) What is the value of c?

b) Are X and Y independent?

c) Calculate $\mathbb{E}[X]$.

a) $1/c = \int_0^1 \int_0^1 xy\, dx\, dy = \left(\frac{x^2}{2}\right)\Big|_0^1 \left(\frac{y^2}{2}\right)\Big|_0^1 = 1/4 \Rightarrow \boxed{c = 4}$

b)

$$f_X(x) = \begin{cases} \int_0^1 4xy\, dy = 2x & , 0 \leq x \leq 1 \\ 0 & , \text{else} \end{cases}$$

$$f_Y(y) = \begin{cases} \int_0^1 4xy\, dx = 2y & , 0 \leq y \leq 1 \\ 0 & , \text{else} \end{cases}$$

Since $f_{X,Y}(x,y) = f_X(x) f_Y(y)$ $\forall (x,y) \in \mathbb{R}^2$, X and Y are independent.

c) $E[X] = \int_0^1 2x^2\, dx = 2/3$

6. Let $X_1, \dots, X_n$ be independent and identically distributed random variables with density f and cumulative distribution function F. Calculate density and cumulative distribution function of

$$Y = \min\{X_1, X_2, \dots X_n\}, \qquad Z = \max\{X_1, X_2, \dots X_n\}$$

in terms of f and F.

$$F_Y(y) = P[Y \le y] = P[\min\{X_1, \dots, X_n\} \le y]$$
$$= P[\{X_1 \le y\} \cup \dots \cup \{X_n \le y\}]$$
$$= 1 - P[X_1 > y, X_2 > y, \dots, X_n > y]$$
$$= 1 - P[X_1 > y]P[X_2 > y] \dots P[X_n > x]$$
$$= \boxed{1 - (1 - F(y))^n}$$

$$f_Y(y) = F_Y'(y) = \boxed{n(1 - F(y))^{n-1} f(y)}$$

$$F_Z(z) = P[Z \le z] = P[\max\{X_1, \dots, X_n\} \le z]$$
$$= P[X_1 \le z, \dots, X_n \le z] = \boxed{(F(z))^n}$$

$$f_Z(z) = F_Z'(z) = \boxed{n(F(z))^{n-1} f(z)}$$