MA 2631 Conference 6

October 7, 2021

1. In a small town, there are 50 births a year. Assume that the probability that a newborn is a girl is 50%. How likely is it that in a given year, there are at least 25 and at most 27 girls born. Calculate this probability

   a) exactly.

   b) by an approximation with the normal distribution.

   Answer:

   a) Model this using a binomial distribution with $n = 50$ and success probability $p = \frac{1}{2}$. Denoting the number of newborn girls by X we get

$$
\begin{aligned}
P[25 \leq X \leq 27] =& \binom{50}{25}\left(\frac{1}{2}\right)^{25}\left(\frac{1}{2}\right)^{25} + \binom{50}{26}\left(\frac{1}{2}\right)^{26}\left(\frac{1}{2}\right)^{24} + \binom{50}{27}\left(\frac{1}{2}\right)^{27}\left(\frac{1}{2}\right)^{23} \\
=& \frac{1}{2^{50}}\left(\binom{50}{25} + \binom{50}{26} + \binom{50}{27}\right) \\
=& \frac{50!}{23! \cdot 25! \cdot 2^{50}}\left(\frac{1}{24 \cdot 25} + \frac{1}{24 \cdot 26} + \frac{1}{25 \cdot 26}\right) = \frac{50!}{23! \cdot 25! \cdot 2^{50} \cdot 208} \\
&\approx 31.62\%
\end{aligned}
$$

   b) Note that we have $\mu = E[X] = n \cdot p = 50 \cdot \frac{1}{2} = 25$ and $\sigma = \text{var}(X) = n \cdot p \cdot (1 - p) = 12.5$. By approximation (with continuity correction) using a standard normal distribution Z

$$
\begin{aligned}
P[25 \leq X \leq 27] \approx& P\left[\frac{24.5 - 25}{\sqrt{12.5}} \leq Z \leq \frac{27.5 - 25}{\sqrt{12.5}}\right] = \Phi\left(\frac{2.5}{\sqrt{12.5}}\right) - \Phi\left(\frac{-0.5}{\sqrt{12.5}}\right) \\
=& \Phi\left(\frac{2.5}{\sqrt{12.5}}\right) + \Phi\left(\frac{-0.5}{\sqrt{12.5}}\right) - 1 \approx \Phi(0.7071) + \Phi(0.1414) - 1 \\
\approx& 0.7601 + 0.5563 - 1 = 31.64\%
\end{aligned}
$$

2. Consider a biased coin that shows *heads* in $\frac{2}{3}$ of all cases and *tails* only in $\frac{1}{3}$ of all cases. The coin is flipped consecutively (and independently) 200 times.

   a) What is the probability that *tails* shows up the first time at the 10th flip?

   b) Calculate the probability *heads* shows up more than 150 times (using a suitable approximation).

   Answer:

   a) Let X denote the number of *heads* before the first *tails*. Then X is a geometric random variable with success probability $\frac{1}{3}$ and we have

$$P[X = 9] = \left(1 - \frac{1}{3}\right)^9 \cdot \frac{1}{3} = \frac{2^9}{3^{10}} \approx 0.8671\%.$$

Let Y be the number of heads out of the 200 flips. Then Y is binomially distributed with 200 trials and success probability $p = \frac{2}{3}$. Note that $E[Y] = 200 \cdot \frac{2}{3} = \frac{400}{3}$ and $\text{var}[Y] = 200 \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{400}{9}$. This means $\frac{Y - \frac{400}{3}}{\sqrt{\frac{400}{9}}}$ is approximately distributed as a standard normal random variable Z and

$$\begin{aligned} P[Y > 150] &= P\left[\frac{Y - \frac{400}{3}}{\sqrt{\frac{400}{9}}} > \frac{150 - \frac{400}{3}}{\sqrt{\frac{400}{9}}}\right] \approx P\left[Z > \frac{\frac{50}{3}}{\frac{20}{3}}\right] = P\Big[Z > 2.5\Big] \\ &= 1 - \Phi(2.5) = 0.62\%. \end{aligned}$$

[Note: alternative calculations lead to

$$\begin{aligned} P[Y \geq 151] &\approx P\left[Z > \frac{\frac{53}{3}}{\frac{20}{3}}\right] = P\Big[Z > 2.65\Big] = 1 - \Phi(2.65) = 0.40\%, \\ P[Y \geq 150.5] &\approx P\left[Z > \frac{\frac{51.5}{3}}{\frac{20}{3}}\right] = P\Big[Z > 2.575\Big] = 1 - \Phi(2.575) = 0.5\%, \end{aligned}$$

the last one incorporating the continuity correction.]

3. Assume that the joint probability mass distribution $p_{X,Y}$ of the random variable X and Y is given by

$$\begin{aligned} p_{X,Y}(0,0) =& \frac{1}{3} \qquad p_{X,Y}(0,1) = \frac{1}{4}; \\ p_{X,Y}(1,0) =& \frac{1}{4} \qquad p_{X,Y}(1,1) = \frac{1}{6}. \end{aligned}$$

a) Calculate the marginal probability mass distributions p_X and p_Y.

b) What is the probability mass distribution of the random variable $Z = X^2 + Y$?

Answer:

a)

$$\begin{aligned} p_X(0) =& p_{X,Y}(0,0) + p_{X,Y}(0,1) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} \\ p_X(1) =& p_{X,Y}(1,0) + p_{X,Y}(1,1) = \frac{1}{4} + \frac{1}{6} = \frac{5}{12} \\ p_Y(0) =& p_{X,Y}(0,0) + p_{X,Y}(1,0) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} \\ p_Y(1) =& p_{X,Y}(0,1) + p_{X,Y}(1,1) = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}. \end{aligned}$$

b) The possible values that Z could take on are 0,1, and 2.

$$\begin{aligned} &P[Z=0] = P[X^2+Y=0] = P[X=0,Y=0] = p_{X,Y}(0,0) = \frac{1}{3} \\ &P[Z=1] = P[X^2+Y=1] = P[X=1,Y=0] + P[X=0,Y=1] = p_{X,Y}(1,0) + p_{X,Y}(0,1) = \frac{1}{2} \\ &P[Z=2] = P[X^2+Y=2] = P[X=1,Y=1] = p_{X,Y}(1,1) = \frac{1}{6} \end{aligned}$$

4. Assume that X and Y are jointly distributed random variables with joint density

$$f_{X,Y}(x,y) = \begin{cases} cye^{-x} & \text{if } 0 \le x < \infty,\ 0 \le y \le 1; \\ 0 & \text{else.} \end{cases}$$

for some $c \in \mathbb{R}$.

a) Calculate c.

b) Calculate the marginal probability density functions f_X and f_Y.

c) What is the probability $P[3X + Y^2 > 4]$?

Answer:

a) As the density has to integrate up to one, we have

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dxdy \\ &= \int_0^1 \int_0^{\infty} cye^{-x}\, dxdy = c\Big(\int_0^{\infty} e^{-x}\, dx\Big)\Big(\int_0^1 ydy\Big) \\ &= c\Big[-e^{-x}\Big]_0^{\infty} \Big[\frac{y^2}{2}\Big]_0^1 = c \cdot 1 \cdot \frac{1}{2} = \frac{c}{2}, \end{aligned}$$

$$\boxed{c = 2}$$

b)

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \begin{cases} \int_0^1 2ye^{-x}\, dy = 2e^{-x}\big[\frac{y^2}{2}\big]_0^1 = e^{-x} & \text{if } x \ge 0; \\ 0 & \text{if } x < 0. \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx = \begin{cases} \int_0^{\infty} 2ye^{-x}\, dx = 2y\big[-e^{-x}\big]_0^{\infty} = 2y & \text{if } 0 \le y \le 1; \\ 0 & \text{else.} \end{cases}$$

c)

$$\begin{aligned} P[3X + Y^2 > 4] &= \iint_{\{(x,y)\in[0,\infty)\times[0,1]\,:\,3x+y^2>4)\}} f_{X,Y}(x,y)\, dxdy \\ &= \iint_{\{(x,y)\in[0,\infty)\times[0,1]\,:\,x>\frac{4-y^2}{3}\}} f_{X,Y}(x,y)\, dxdy \\ &= \int_0^1 \int_{\frac{4-y^2}{3}}^{\infty} 2ye^{-x}\, dxdy = \int_0^1 2y\Big[-e^{-x}\Big]_{\frac{4-y^2}{3}}^{\infty} dy \\ &= \int_0^1 2ye^{-\frac{4-y^2}{3}}\, dy = 3e^{-\frac{4}{3}} \int_0^1 \frac{2y}{3} e^{\frac{y^2}{3}}\, dy = 3e^{-\frac{4}{3}} \Big[e^{\frac{y^2}{3}}\Big]_0^1 \\ &= 3e^{-\frac{4}{3}}\Big(e^{\frac{1}{3}} - 1\Big) = 3e^{-1} - 3e^{-\frac{4}{3}} \approx 31.28\%. \end{aligned}$$