MA 2631 Assignment 9

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1. Let X be a continuous random variable with density f, expectation $E[X] = \mu$, and variance $\mathrm{Var}[X] = \sigma^2$. Define a new random variable $Y := aX + b$ for some $a, b \in \mathbb{R}$.

   (a) Calculate the standard deviation $\mathrm{SD}[Y]$.

   (b) Express the moment generating function m_Y in terms of m_X.

   Answer:

   (a)
   $$\mathrm{Var}[Y] = \mathrm{Var}[aX + b] = a^2\mathrm{Var}[X] = a^2\sigma^2 \implies \mathrm{SD}[Y] = \sqrt{\mathrm{Var}[X]} = \boxed{a\sigma}$$

   (b)
   $$m_X(t) = E[e^{tx}]$$
   $$m_Y(t) = E[e^{ty}] = E[e^{t(aX+b)}] = e^{tb}E[e^{t(aX)}] = \boxed{e^{tb}m_X(at)}$$

2. Prove that for an arbitrary continuous random variable X with density f we have

$$E[X] = \int_0^\infty P[X > x]dx - \int_0^\infty P[X < -x]dx.$$

Answer:

$$\begin{aligned}
\int_0^\infty P[X > x]dx &= \int_0^\infty \int_x^\infty P[X = y]dydx \\
&= \int_0^\infty \int_0^y P[X = y]dxdy \\
&= \int_0^\infty yP[X = y]dy \\
&= \int_0^\infty xP[X = x]dx
\end{aligned}$$

$$\begin{aligned}
\int_0^\infty P[X < -x]dx &= \int_0^\infty \int_{-\infty}^{-x} P[X = y]dydx \\
&= \int_{-\infty}^0 \int_0^{-y} P[X = y]dxdy \\
&= \int_{-\infty}^0 -yP[X = y]dy \\
&= -\int_{-\infty}^0 xP[X = x]dx
\end{aligned}$$

$$\begin{aligned}
\int_0^\infty P[X > x]dx \; - \int_0^\infty P[X < -x]dx &= \int_0^\infty xP[X = x]dx \; + \int_{-\infty}^0 xP[X = x]dx \\
&= \int_{-\infty}^\infty xP[X = x]dx \equiv E[X]
\end{aligned}$$

3. Assume that $U^{0,1}$ is a uniformly distributed random variable on the unit interval. Find a real-valued function $g : [0, 1) \to \mathbb{R}$ such that $Y := g(U^{0,1})$ is an exponentially distributed random variable with parameter $\lambda > 0$.

   Answer: We want Y to have cdf $F_Y(y) = 1 - e^{-\lambda y}$. Solve $x = F_Y(y)$ for x to find $F_Y^{-1}(x) = y$.

$$1 - x = e^{-\lambda y} \implies F_Y^{-1}(x) = y = -\frac{1}{\lambda} \ln(1 - x)$$

   If we generate an x_0 from $X \sim U^{0,1}$ and compute $y_0 = -\frac{1}{\lambda} \ln(1 - x_0)$, this y_0 follows the exponential distribution with parameter λ.

$$\boxed{\therefore \quad g(x) = -\frac{1}{\lambda} \ln(1 - x)}$$

4. The lifetime of an electrical device (in months) is given by the continuous variable X with density

$$f(x) = \begin{cases} cxe^{-\frac{x}{2}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

(a) Calculate c.

(b) What is the probability the device functions more than 5 months?

(c) What is the expected lifetime of the device?

Answer:

(a)

$$\frac{1}{c} = \int_0^\infty xe^{-\frac{x}{2}}dx = \left[-2xe^{-\frac{x}{2}} + 2\int e^{-\frac{x}{2}}dx\right]\Bigg|_0^\infty = -4e^{-\frac{x}{2}}\Bigg|_0^\infty = -4(0-1) = 4 \implies \boxed{c = \frac{1}{4}}$$

(b)

$$P[X > 5] = \int_5^\infty f(x)dx = \frac{1}{4}\left[-2xe^{-\frac{x}{2}} - 4e^{-\frac{x}{2}}\right]\Bigg|_5^\infty = \frac{1}{4}[(0-0)-(-10e^{-\frac{5}{2}}-4e^{-\frac{5}{2}})] = \boxed{\frac{7}{2}e^{-\frac{5}{2}} \approx 0.287}$$

(c)

$$E[X] = \frac{1}{4}\int_0^\infty x^2e^{-\frac{x}{2}}dx = \frac{1}{4}\left[-2x^2e^{-\frac{x}{2}} + 4\int xe^{-\frac{x}{2}}dx\right]\Bigg|_0^\infty = \frac{1}{4}[0+4\cdot 4] = \boxed{4}$$

5. Assume that X is an exponentially distributed random variable with parameter $\lambda > 1$. Calculate:

(a) $E[X^3]$

(b) $E[e^X]$

Why did we impose the condition $\lambda > 1$ instead of the "usual" condition $\lambda > 0$?

Answer:

(a)

$$m_X(t) = E[e^{tX}] = \int_{-\infty}^\infty e^{tx}f(x)dx = \int_0^\infty \lambda e^{tx}e^{-\lambda x}dx = \frac{\lambda}{\lambda - t} \quad (t < \lambda)$$

$$m_X'''(t) = \frac{3!\lambda}{(\lambda-t)^4}$$

$$E[X^3] = m_X'''(0) = \boxed{\frac{6}{\lambda^3}} \quad (\text{Note } \lambda > 1 > 0 = t)$$

(b)

$$E[e^X] = \int_0^\infty \lambda e^x e^{-\lambda x}dx = \lambda\int_0^\infty e^{(1-\lambda)x}dx = \frac{\lambda}{1-\lambda}e^{(1-\lambda)x}\Bigg|_0^\infty = \boxed{\frac{\lambda}{\lambda-1} \quad (\lambda > 1)}$$

It's necessary to require $\lambda > 1$ since $\int_0^\infty \lambda e^{(1-\lambda)x}dx$ diverges to $+\infty$ for $\lambda \leq 1$.

6. Find the cumulative distribution function F such that F has hazard rate $\lambda(t) = \frac{1}{\sqrt{t}}$ for $t > 0$. If possible, express F in terms of an exponentially distributed random variable.

Answer: Given the hazard rate $\lambda(t) = \frac{f(t)}{\overline{F}(t)} = -\frac{\overline{F}'(t)}{\overline{F}(T)} = -(\log \overline{F}(t))'$ we have $\log \overline{F}(t) = -\int_0^t \lambda(s)ds + c$ for some $c \in \mathbb{R}$.

$$\begin{aligned}
\log \overline{F}(t) &= -\int_0^t \frac{1}{\sqrt{s}} dx + c = -2\sqrt{t} + c \\
\overline{F}(t) &= e^c e^{-2\sqrt{t}} \\
1 = 1 - 0 &= 1 - F(0) = \overline{F}(0) = e^c \\
\overline{F}(t) &= e^{-2\sqrt{t}} \\
F(t) &= 1 - \overline{F}(t) = 1 - e^{-2\sqrt{t}}, \quad t > 0 \\
f(t) &= 2e^{-2\sqrt{t}}, \quad t > 0
\end{aligned}$$

The cdf of an exponentially distributed random variable X with parameter $\lambda = 2$ is $G(x) = 1 - e^{-2x}$ for $x \geq 0$ and $G(x) = 0$ for $x < 0$. By setting $F(t) = 0$ for $t < 0$ we have

$$F(t) = \begin{cases} G(\sqrt{t}) & t \geq 0 \\ 0 & t < 0 \end{cases}$$