# MA 2631 Conference 3

September 15, 2021

1. There are two urns. In the first urn there are 6 red and 2 black balls, in the second one 3 red and 7 black. We roll a (fair) die, and if the die shows a 1 or 2 we draw at random a ball from the first urn and in the case that the die shows a 3, 4, 5, or 6 we draw a ball at random from the second urn.

   a) What is the probability that the drawn ball is red?

   b) If the drawn ball is red, what is the probability that it was from the first urn?

   Answer: Let $R$ be the event that the drawn ball is red, $U_1$ be the event that the ball was drawn from the first urn, and $U_2$ the event that the ball was drawn from the second urn.

   a) $\mathbb{P}[R] = \mathbb{P}[R|U_1]\mathbb{P}[U_1] + \mathbb{P}[R|U_2]\mathbb{P}[U_2] = \frac{6}{8}\frac{2}{6} + \frac{3}{10}\frac{4}{6} = \frac{9}{20} = 45\%$.

   b) $\mathbb{P}[U_1|R] = \frac{\mathbb{P}[R|U_1]\mathbb{P}[U_1]}{\mathbb{P}[R|U_1]\mathbb{P}[U_1]+\mathbb{P}[R|U_2]\mathbb{P}[U_2]} = \frac{5}{9} \approx 55.56\%$.

2. Two students participate in a quiz show where they are asked a true-false question. Both know, independently, the correct answer with probability $p$. Which of the following strategies is better for the team?

   i) Choose a priori one of the two who will give the answer.

   ii) Give the common answer if the answers agree, and if not flip a coin to decide which answer is given.

   Answer: Calculate the probability of getting the question right under both strategies and compare:

   i) Using this strategy, the probability that the team gets the correct answer is $p$ since the chosen representative knows the answer with probability $p$.

   ii) Let $A_1$ be the event that the first student has the correct answer, $A_2$ the event that the second student has the correct answer, $D$ the event that the students disagree, and $E$ the event that the team gets the correct answer. We want to find $\mathbb{P}[E]$. The event $E$ occurs if any of the events $A_1 \cap A_2$, $D \cap A_1$, or $D \cap A_2$ occurs, and these three events are mutually exclusive. Since $A_1$ and $A_2$ are independent, $A_1$ and $A_2^c$ are independent and $A_1^c$ and $A_2$ are independent.

$$\begin{aligned}
\mathbb{P}[A_1 \cap A_2] &= p \cdot p = p^2 \\
\mathbb{P}[D \cap A_1] &= \mathbb{P}[A_1 \cap A_2^c] = \mathbb{P}[A_1]\mathbb{P}[A_2^c] = p(1-p) \\
\mathbb{P}[D \cap A_2] &= \mathbb{P}[A_1^c \cap A_2] = \mathbb{P}[A_1^c]\mathbb{P}[A_2^c] = (1-p)p \\
\therefore \mathbb{P}[E] &= p^2 + \frac{1}{2}p(1-p) + \frac{1}{2}(1-p)p = p^2 + p - p^2 = p.
\end{aligned}$$

   The terms $p(1-p)$ and $(1-p)p$ were multiplied by $1/2$ since they flip a coin if they disagree. Conclude that for 2 students that either of these strategies will get the team a correct answer with probability $p$.

3. Let $\Omega$ be a sample space and $E$, $F$, $G$ be independent events such that $\mathbb{P}[G] > 0$. Show that the events $E$ and $F$ are independent when using instead of $\mathbb{P}$ the probability $\mathbb{Q}$ defined as $\mathbb{Q}[A] = P[A\,|G]$ for events $A$.

   Answer: Show that $\mathbb{Q}[E \cap F] = \mathbb{Q}[E]\mathbb{Q}[F]$.

$$\begin{aligned}
\mathbb{Q}[E \cap F] &= \mathbb{P}[E \cap F|G] \\
&= \frac{\mathbb{P}[(E \cap F) \cap G]}{\mathbb{P}[G]} \\
&= \frac{\mathbb{P}[E \cap F \cap G]}{\mathbb{P}[G]} \\
&= \frac{\mathbb{P}[E]\mathbb{P}[F]\mathbb{P}[G]}{\mathbb{P}[G]} \\
&= \frac{\mathbb{P}[E]\mathbb{P}[F]\mathbb{P}[G]}{\mathbb{P}[G]} \frac{\mathbb{P}[G]}{\mathbb{P}[G]} \\
&= \frac{\mathbb{P}[E]\mathbb{P}[G]}{\mathbb{P}[G]} \frac{\mathbb{P}[F]\mathbb{P}[G]}{\mathbb{P}[G]} \\
&= \frac{\mathbb{P}[E \cap G]}{\mathbb{P}[G]} \frac{\mathbb{P}[F \cap G]}{\mathbb{P}[G]} \\
&= \mathbb{P}[E|G]\mathbb{P}[F|G] \\
&= \mathbb{Q}[E]\mathbb{Q}[F].
\end{aligned}$$