Assignment 9

1. Let X be a continuous random variable with density f, expectation $\mathbb{E}[X] = \mu$ and variance $\mathbb{V}ar[X] = \sigma^2$. Define a new random variable $Y := aX + b$ for some $a, b \in \mathbb{R}$.

   a) Calculate the standard deviation $\mathbb{S}D[Y]$.

   b) Express the moment generating function m_Y in terms of m_X.

a) $Var[Y] = Var[aX+b] = a^2 Var[X] = a^2\sigma^2$

$SD[Y] = \sqrt{Var[Y]} = \boxed{|a|\sigma}$

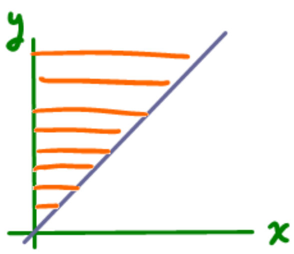
b) $m_X(t) = E[e^{tX}]$

$m_Y(t) = E[e^{tY}] = E[e^{t(aX+b)}] = e^{tb} E[e^{t(aX)}] = \boxed{e^{tb} m_X(at)}$

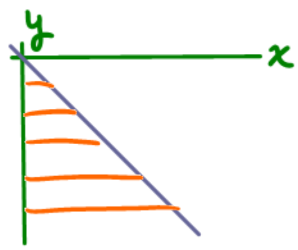
2. Prove that for an arbitrary continuous random variable X with density f we have

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}[X > x]\,dx - \int_0^\infty \mathbb{P}[X < -x]\,dx.$$

$$\int_0^\infty P[X>x]\,dx = \int_0^\infty \int_x^\infty P[X=y]\,dy\,dx$$
$$= \int_0^\infty \int_0^y P[X=y]\,dx\,dy$$
$$= \int_0^\infty y\,P[X=y]\,dy$$
$$= \int_0^\infty x\,P[X=x]\,dx$$



$$\int_0^\infty P[X<-x]\,dx = \int_0^\infty \int_{-\infty}^{-x} P[X=y]\,dy\,dx$$
$$= \int_{-\infty}^0 \int_0^{-y} P[X=y]\,dx\,dy$$
$$= \int_{-\infty}^0 -y\,P[X=y]\,dy$$
$$= -\int_{-\infty}^0 y\,P[X=y]\,dy$$
$$= -\int_{-\infty}^0 x\,P[X=x]\,dx$$



$$\int_0^\infty P[X>x]\,dx - \int_0^\infty P[X<-x]\,dx = \int_0^\infty x\,P[X=x]\,dx + \int_{-\infty}^0 x\,P[X=x]\,dx$$
$$= \int_{-\infty}^\infty x\,P[X=x]\,dx$$
$$= E[X]$$

3. Assume that $U^{0,1}$ is a uniformly distributed random variable on the unit interval. Find a real-valued function $g : [0, 1) \to \mathbb{R}$ such that $Y := g(U^{0,1})$ is an exponentially distributed random variable with parameter $\lambda > 0$.

We want Y to have cdf $F_Y(y) = 1 - e^{-\lambda y}$. Solve $x = F_Y(y)$ for x to find $F_Y^{-1}(x) = y$.

$$1 - x = e^{-\lambda y}$$

$$y = F_Y^{-1}(x) = -\frac{1}{\lambda}\ln(1-x)$$

If we generate an x_0 from $X \sim U^{0,1}$ and compute $y_0 = -\frac{1}{\lambda}\ln(1-x_0)$, this y_0 has exponential distribution.

$$\therefore\ g(x) = -\frac{1}{\lambda}\ln(1-x)$$

4. The lifetime of an electrical device (in months) is given by the continuous random variable X with density

$$f(x) = \begin{cases} cxe^{-\frac{x}{2}} & \text{if } x > 0; \\ 0 & \text{if } x \leq 0. \end{cases}$$

a) What is c?

b) What is the probability that the device functions more than 5 months?

c) What is the expected lifetime of the device?

a) $1 = \int_{-\infty}^{\infty} f(x)\,dx = c\int_0^{\infty} xe^{-x/2}\,dx = c\left[-2xe^{-x/2}\Big|_0^{\infty} + 2\int_0^{\infty} e^{-x/2}\,dx\right]$

$$= c\left[0 - \left(4e^{-x/2}\right)\Big|_0^{\infty}\right] = -4c(0-1) = 4c$$

$$\boxed{c = 1/4}$$

b) $P[X>5] = \int_5^{\infty} f(x)\,dx = \frac{1}{4}\left[-2xe^{-x/2} - 4e^{-x/2}\right]\Big|_5^{\infty}$

$$= \frac{1}{4}\left[(0-0) - (-10e^{-5/2} - 4e^{-5/2})\right]$$

$$= \boxed{\frac{7}{2}e^{-5/2} \approx 0.287}$$

c) $E[X] = \int_{-\infty}^{\infty} xf(x)\,dx = \frac{1}{4}\int_0^{\infty} x^2 e^{-x/2}\,dx$

$$= \frac{1}{4}\left[-2x^2e^{-x/2}\Big|_0^{\infty} + 4\int_0^{\infty} xe^{-x/2}\,dx\right]$$

$$= \frac{1}{4}\left[0 + 4\cdot 4\right] \quad \left(\int_0^{\infty} xe^{-x/2}\,dx = 4 \text{ by part a}\right)$$

$$= \boxed{4}$$

5. Assume that X is an exponentially distributed random variable with parameter $\lambda > 1$. Calculate

a) $\mathbb{E}[X^3]$;

b) $\mathbb{E}[e^X]$.

Why did we impose the condition $\lambda > 1$ (instead of the "usual" one, $\lambda > 0$)?

a) $M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_0^\infty \lambda e^{tx} e^{-\lambda x}\,dx = \frac{\lambda}{\lambda - t} \quad (t < \lambda)$

$M_X'''(t) = \frac{3!\,\lambda}{(\lambda - t)^4}$

$E[X^3] = M_X'''(0) = \frac{6}{\lambda^3}$ (Note that $\lambda > 1 > 0 = t$)

b) $E[e^X] = \int_0^\infty \lambda e^x e^{-\lambda x}\,dx$

$= \lambda \int_0^\infty e^{(1-\lambda)x}\,dx$

$= \frac{\lambda}{1-\lambda} e^{(1-\lambda)x} \Big|_0^\infty$

$= \boxed{\frac{\lambda}{\lambda - 1} \quad \text{for} \quad \lambda > 1}$

For $\lambda \leq 1$, $\int_0^\infty \lambda e^{(1-\lambda)x}\,dx$ diverges to $+\infty$.

6. Find the cumulative distribution function F such that it has hazard rate $\lambda(t) = \frac{1}{\sqrt{t}}$ (for $t > 0$). Can you express F in terms of an exponentially distributed random variable?

Given the hazard rate $\lambda(t) = \frac{f(t)}{\bar{F}(t)} = -\frac{\bar{F}'(t)}{\bar{F}(t)} = -\left(\log \bar{F}(t)\right)'$

we have $\log \bar{F}(t) = -\int_0^t \lambda(s)\,ds + c$, $c \in \mathbb{R}$

$\log \bar{F}(t) = -\int_0^t 1/\sqrt{s}\,ds + c = -2\sqrt{t} + c$

$\bar{F}(t) = d e^{-2\sqrt{t}}$, $d \in \mathbb{R}$ $\qquad 1 = 1 - 0 = 1 - F(0) = \bar{F}(0) = d$

$F(t) = 1 - \bar{F}(t) = 1 - e^{-2\sqrt{t}}$

$f(t) = 2e^{-2\sqrt{t}}$

Let $G(x)$ be the cdf of an exponential r.v. with $\lambda = 2$.
Since $G(x) = 1 - e^{-2x}$ for $x > 0$, $F(t) = G(\sqrt{t})$ for $t > 0$. $F(t)$ must be defined for all $t \in \mathbb{R}$ since F is a cdf. Set $F(t) = 0$, $t \leq 0$.

$\therefore F(t) = \begin{cases} G(\sqrt{t}), & t > 0 \\ 0, & t \leq 0 \end{cases}$