1. Calculate the moment generating function for an exponentially distributed random variable $X \sim \text{Exp}(\lambda)$, $\lambda > 0$ and use it to calculate its expected value and variance.

- mgf

$$m_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx = \int_0^{\infty} e^{tx} \lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_0^{\infty} e^{(t-\lambda)x}\,dx = \frac{\lambda}{t-\lambda} \lim_{m\to\infty}\left[e^{(t-\lambda)m} - 1\right]$$

$$= \frac{\lambda}{\lambda - t} \quad \text{for } t < \lambda$$

The integral diverges for $t \geq \lambda$ which means that for such t, the mgf doesn't exist.

- mean

$$E[Xe^{tX}] = m_X'(t) = \frac{\lambda}{(\lambda-t)^2}$$

$$\Rightarrow E[X] = m_X'(0) = \frac{\lambda}{(\lambda-0)^2} = \frac{1}{\lambda}$$

- variance

$$E[X^2 e^{tX}] = m_X''(t) = \frac{2\lambda}{(\lambda-t)^3}$$

$$\Rightarrow E[X^2] = m_X''(0) = \frac{2\lambda}{(\lambda-0)^3} = \frac{2}{\lambda^2}$$

$$\text{var}[X] = E[X^2] - E[X]^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

3. Let $X \sim \mathcal{N}(0,1)$ be a standard normal distributed random variable. Calculate the moment generating function $m_X(\lambda) = \mathbb{E}[e^{\lambda X}]$ for $\lambda \in \mathbb{R}$. Use the moment generating function to calculate mean and variance of X, confirming what we know already.

- mgf

$$m_X(\lambda) = \int_{-\infty}^{\infty} e^{\lambda x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2-2\lambda x)/2}\, dx$$

$$= \alpha \int_{-\infty}^{\infty} e^{-(x^2-2\lambda x+\lambda^2)/2}\, dx \qquad \left\{ \alpha := \frac{e^{\lambda^2/2}}{\sqrt{2\pi}} \right.$$

$$= \alpha \int_{-\infty}^{\infty} e^{-(x-\lambda)^2/2}\, dx$$

$$= \alpha \int_{-\infty}^{\infty} e^{-u^2/2}\, du \qquad \left\{ \begin{array}{l} u = x-\lambda \\ du = dx \\ u \to \pm\infty \\ \text{as } x \to \pm\infty \end{array} \right.$$

$$= \alpha\sqrt{2\pi} = e^{\lambda^2/2}$$

- mean

$$\mu = E[X] = m_X'(0) = \left(\lambda e^{\lambda^2/2}\right)\Big|_{\lambda=0} = 0$$

- variance

$$\sigma^2 = E[X^2] - E[X]^2 = E[X^2]$$

$$= m_X''(0) = \left(e^{\lambda^2/2} + \lambda^2 e^{\lambda^2/2}\right)\Big|_{\lambda=0} = 1$$

Assignment 8, Problem 3 Hint

Let $X \sim U^{0,1}$. We know $F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$

Define $Y := g(X)$. We want $F_Y(y) = \begin{cases} 0 & y < 0 \\ 1 - e^{-\lambda y} & y \geq 0 \end{cases}$

Idea to pick the right function g:

Solve $x = F_Y(y) = 1 - e^{-\lambda y}$ $(y \geq 0)$ for y

to get $y = F_Y^{-1}(x)$ and define $g(x) := F_Y^{-1}(x)$

Now carefully show:

$$F_Y(y) = P[Y \le y] = P[g(X) \le y] = \begin{cases} 0 & y < 0 \\ 1 - e^{-\lambda y} & y \geq 0 \end{cases}$$

If this equality holds you can conclude $Y \sim Exp(\lambda)$.