MA 2631 Assignment 5

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1. Show that if A and B are independent events on a sample space Ω, then also A^c and B are independent.

   Answer: Rewriting B as the union of disjoint sets $B = (A \cap B) \cup (A^c \cap B)$,

   $$P[B] = P[A \cap B] + P[A^c \cap B] = P[A]P[B] + P[A^c \cap B] \implies P[A^c \cap B] = P[B](1 - P[A]) = P[A^c]P[B].$$

   Since $P[A^c \cap B] = P[A^c]P[B]$, A^c and B are independent events.

2. Suppose that A, B and C are independent events on a sample space Ω with $P[A \cap B] \neq 0$. Prove that

   $$P[A \cap C | A \cap B] = P[C].$$

   Answer:

   $$P[A \cap C | A \cap B] = \frac{P[(A \cap C) \cap (A \cap B)]}{P[A \cap B]} = \frac{P[A \cap B \cap C]}{P[A \cap B]} = \frac{P[A]P[B]P[C]}{P[A]P[B]} = P[C].$$

3. Assume that in a family the birth of a boy and a girl is equally likely and that the family has $n \geq 2$ children. Are the events A and B independent?

   A : There is at least one boy and at least one girl in the family,
   B : There is at most one girl in the family.

   Answer: Considering all the 2^n possibilities (by gender) of n children and omitting the two cases where all children are the same gender, $P[A] = (2^n - 2)/2^n$. Considering the one case where there are no girls and the n ways to have exactly one girl out of the n children, $P[B] = (n+1)/2^n$. For $n \geq 2$, $A \cap B$ is the event that of the n children, exactly one is a girl so that $P[A \cap B] = n/2^n$. The events A and B are independent iff

   $$P[A \cap B] = P[A]P[B] \iff \frac{n}{2^n} = \frac{2^n - 2}{2^n}\frac{n+1}{2^n} \iff 2^{n-1} = n + 1.$$

   For $n = 2$, $2^{n-1} = 2 \neq 3 = n + 1$. For $n = 3$, $2^{n-1} = 4 = n + 1$. We will next show by induction that $2^{n-1} > n + 1$ for $n \geq 4$.

   For $n = 4, 2^{n-1} = 8 > 5 = n + 1$. Assume that for some integer $n \geq 4$ that $2^{n-1} > n + 1$.

   $$2^{(n-1)+1} = 2 \cdot 2^{n-1} > 2 \cdot (n+1) = n + 2 + n \geq n + 2 + 4 > n + 2 = (n+1) + 1.$$

   Conclude that for $n \geq 4$, $2^{n-1} \neq n + 1$ and therefore that $P[A \cap B] = P[A]P[B]$ iff $n = 3$. That is, A and B are independent iff there are $n = 3$ children in the family.

4. Let A, B, C be independent events on a sample space Ω with $P[A] = \frac{1}{2}$, $P[B] = \frac{2}{3}$, and $P[C] = \frac{3}{4}$. Calculate $P[A \cup (B \cap C)]$.

   Answer:

$$P[A \cup (B \cap C)] = P[A] + P[B \cap C] - P[A \cap B \cap C] = P[A] + P[B]P[C] - P[A]P[B]P[C] = \frac{1}{2} + \frac{2}{3}\frac{3}{4} - \frac{1}{2}\frac{2}{3}\frac{3}{4} = \frac{3}{4}.$$

5. Consider the probability mass distribution $P[Y = i] = c \cdot 0.1^i$ on the non-negative integers for some constant c.

   (a) Calculate c.

   (b) Calculate $P[Y = 0]$ and $P[Y > 2]$.

   (c) Calculate $P[Y \leq 5 | Y > 2]$.

   Answer:

   (a)

$$1 = \sum_{i=0}^{\infty} P[Y = i] = \sum_{i=0}^{\infty} c \cdot 0.1^i = \frac{c}{.9} \implies c = 0.9 = \frac{9}{10}.$$

   (b)

$$P[Y = 0] = c \cdot 0.1^0 = c = 0.9.$$

$$P[Y > 2] = 1 - P[Y \leq 2] = 1 - (0.9 + 0.09 + 0.009) = 0.001.$$

   (c)

$$P[Y \leq 5 | Y > 2] = \frac{P[2 < Y \leq 5]}{P[Y > 2]} = \frac{0.9 \cdot (0.1^3 + 0.1^4 + 0.1^5)}{0.001} = 0.999.$$

6. Assume you are flipping a fair coin until head appears the 5th time. Let Y denote the number of tails that occur. Calculate the probability mass distribution of Y.

   Answer: The value $P[Y = n]$ for $n = 0, 1, 2, \ldots$ should reflect the case the there are n tails and 5 heads in $n + 5$ flips. The position of the 5th heads is then 'fixed' as the last position in the sequence. Then choose n spots out of the remaining $n + 4$ spots in which the n tails' appear.

$$P[Y = n] = \binom{n+4}{n} \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^5 = \binom{n+4}{n} \frac{1}{2^{n+5}}.$$