Worcester Polytechnic Institute
Department of Mathematical Sciences
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Fall 2021 - A Term

MA 2631

Probability Theory

Section AL01 / AD01

Assignment 6

due on Tuesday, September 21

based on Lectures of Chapter 3.2–4.2

1. The picture on the last page (taken from https://www.mass.gov/info-details/covid-19-response-reporting) shows that in early September 2021, the percentage of positive Covid-tests were rising for both, higher education tests and non-higher education tests, though the overall positivity rate was actually slightly declining.

   At first sight this might look paradoxical, and you might even think that there might be an error. However, you should be able to explain this phenomenon with your knowledge about conditional probabilities and related theorems. Please do so.

2. Consider the random variable X with the probability mass distribution

   $$P[X = 1] = 0.3, \qquad P[X = 4] = 0.25, \qquad P[X = 7] = 0.4, \qquad P[X = 10] = 0.05.$$

   Calculate the expected value of X and Y with $Y = 3X + 2$.

3. Let X be a random variable describing the number of failures before the first success of an independently repeated experiment with success probability $p = \frac{3}{4}$.

   a) Calculate the probability that there are not more than two failures before the first success.

   b) Calculate $\mathbb{E}\left[2^X\right]$.

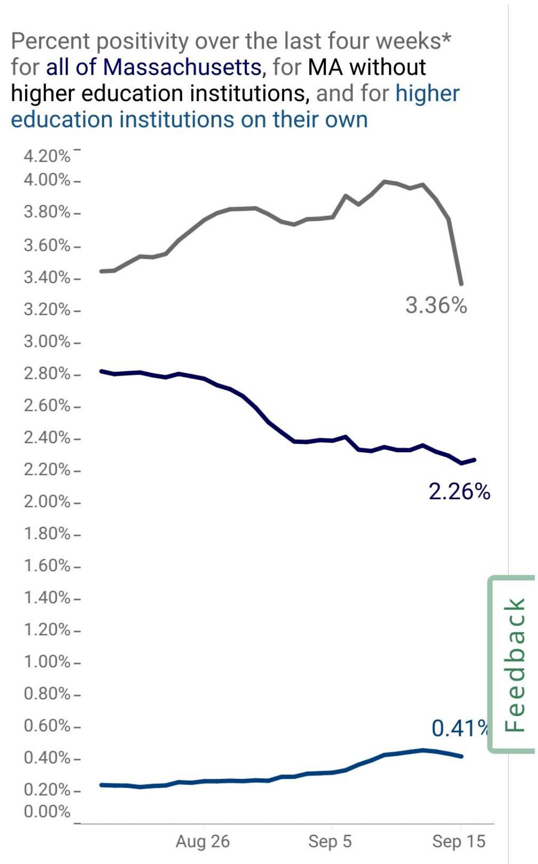
4. Let X be a random variable taking values in $\mathbb{N}$. Show that

$$\mathbb{E}[X] = \sum_{n=1}^{\infty} \mathbb{P}[X \geq n].$$

5. Fill out the midterm survey on Canvas (results are anonymous, it will be only tracked if you have taken it or not). Counts for 2 questions.

8 points per problems

Standard Carlton and Devore, Section 2.2: Exercises 11, 12, 16; Section 2.3: 30, 34, 35, 36.



MA 2631 Assignment 6 Answers

2. $E[X] = 0.3 + 0.25 \cdot 4 + 0.4 \cdot 7 + 0.05 \cdot 10 = 4.6$

   $E[Y] = E[3X+2] = 3E[X] + 2 = 15.8$

3. $P(X=n) = p^n(1-p) = \frac{3}{4}\left(\frac{1}{4}\right)^n$

   a. $P(X=0) + P(X=1) + P(X=2) = \frac{3}{4}\left(1 + \frac{1}{4} + \frac{1}{16}\right) = \frac{63}{64} \approx 98.44\%$

   b. $E[2^X] = \sum_{n=0}^{\infty} 2^n P(X=n) = \sum_{n=0}^{\infty} 2^n \cdot \frac{3}{4} \cdot \left(\frac{1}{4}\right)^n = \frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \frac{3}{2}$

4. $$\begin{aligned}\sum_{n=1}^{\infty} P[X \geq n] &= \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} P[X=k] \\ &= \sum_{k=1}^{\infty} \sum_{n=1}^{k} P[X=k] \\ &= \sum_{k=1}^{\infty} k P[X=k] \\ &= E[X]\end{aligned}$$