MA 2631 Assignment 11

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1. Let X, Y be two random variables with joint cdf $F_{X,Y}$ and marginal cdfs F_X, F_Y. For $x, y \in \mathbb{R}$, express $P[X > x; Y \leq y]$ in terms of $F_{X,Y}$, F_X , and F_Y.

   Answer:

   $$\{Y \leq y\} = \Omega \cap \{Y \leq y\} = (\{X \leq x\} \cup \{X > x\}) = (\{X \leq x\} \cap \{Y \leq y\}) \cup (\{X > x\} \cap \{Y \leq y\})$$
   $$F_Y(y) = P[Y \leq y] = P[X \leq x, Y \leq y] + P[X > x, Y \leq y] = F_{X,Y}(x, y) + P[X > x, Y \leq y]$$
   $$P[X > x, Y \leq y] = F_Y(y) - F_{X,Y}(x, y)$$

2. Assume that there are 12 balls in an urn, 3 of them red, 4 white and 5 blue. Assume that you draw 2 balls of them, replacing the first ball after noting its color before drawing the second ball.

   Denote by X the number of drawn red balls and by Y the number of drawn white balls. Calculate the joint probability mass distribution of X and Y as well as their marginal distributions. Are X and Y independent?

   Answer:

   $$p_X(0) = \left(\frac{9}{12}\right)^2 = \frac{9}{16}, \quad p_X(1) = 2 \cdot \frac{3}{12}\frac{9}{12} = \frac{6}{16}, \quad p_X(2) = \left(\frac{3}{12}\right)^2 = \frac{1}{16}$$
   $$p_Y(0) = \left(\frac{8}{12}\right)^2 = \frac{4}{9}, \quad p_Y(1) = 2 \cdot \frac{4}{12}\frac{8}{12}, \quad p_Y(2) = \left(\frac{4}{12}\right)^2 = \frac{1}{9}$$

$p_{X,Y}(x,y)$	$x = 0,$	$x = 1,$	$x = 2$
$y = 0$	$\frac{25}{144}$	$\frac{30}{144}$	$\frac{9}{144}$
$y = 1$	$\frac{40}{144}$	$\frac{24}{144}$	
$y = 2$	$\frac{16}{144}$		

   X and Y are <u>not</u> independent. For instance, $p_{X,Y}(0, 0) = \frac{25}{144} \neq \frac{1}{4} = p_X(0)p_Y(0)$.

3. Assume that the joint probability mass distribution $p_{X,Y}$ of the random variables X and Y is given by

$$p_{X,Y}(1,1) = p_{X,Y}(1,2) = p_{X,Y}(1,3) = \frac{1}{12}$$
$$p_{X,Y}(2,1) = p_{X,Y}(2,2) = p_{X,Y}(2,3) = \frac{1}{4}$$

(a) Calculate the marginal probability mass distributions p_X and p_Y.

(b) Are X and Y independent?

(c) Calculate the probability mass distribution of the random variable $Z = X/Y$.

Answer:

(a) $p_X(1) = \frac{1}{4}$ $\quad p_X(2) = \frac{3}{4}$
$p_Y(1) = p_Y(1) = p_Y(3) = \frac{1}{3}$

(b) Yes X and Y are independent since $p_X(x)p_Y(y) = p_{X,Y}(x,y)$ for all $(x,y) \in \{1,2\} \times \{1,2,3\}$.

(c) $p_Z(2) = p_{X,Y}(2,1) = \frac{1}{4}$
$p_Z(1) = p_{X,Y}(1,1) + p_{X,Y}(2,2) = \frac{1}{3}$
$p_Z\left(\frac{2}{3}\right) = p_{X,Y}(2,3) = \frac{1}{4}$
$p_Z\left(\frac{1}{2}\right) = p_{X,Y}(1,2) = \frac{1}{12}$
$p_Z\left(\frac{1}{3}\right) = p_{X,Y}(1,3) = \frac{1}{12}$

4. Let X and Y be two independent standard-normal distributed random variables and define $Z = X^2 + Y^2$. Calculate the cumulative distribution function of Z. Which distribution does Z follow?

Answer: For $z < 0$, $F_Z(z) = P[Z \leq z] = P[X^2 + Y^2 \leq z] = 0$ since $X^2 + Y^2 \geq 0$.

For $z \geq 0$,

$$\begin{aligned} F_Z(z) = P[X^2+Y^2 \leq z] &= \iint_{\{x^2+y^2\leq z\}} f_{X,Y}(x,y)\, dA = \iint_{\{x^2+y^2\leq z\}} f_X(x) f_Y(y)\, dA \\ &= \iint_{\{x^2+y^2\leq z\}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dA = \frac{1}{2\pi} \iint_{\{x^2+y^2\leq z\}} e^{-(x^2+y^2)/2}\, dA \\ &= \int_0^{2\pi} \int_0^{\sqrt{z}} r e^{-r^2/2}\, dr d\theta = \frac{2\pi}{2\pi} \left(-e^{-r^2/2}\right)\Big|_0^{\sqrt{z}} = 1 - e^{-z/2} \end{aligned}$$

$$\therefore \quad F_Z(z) = \begin{cases} 0 & z < 0 \\ 1 - e^{-z/2} & z \geq 0 \end{cases}$$

Conclude that Z is exponentially distributed with parameter $\lambda = 2$.

5. Let X, Y be two jointly distributed random variables with joint density

$$f_{X,Y}(x,y) = \begin{cases} cxy & 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{else} \end{cases}$$

(a) Determine the value of the constant c.

(b) Are X and Y independent?

(c) Calculate $E[X]$.

Answer:

(a) $\frac{1}{c} = \int_0^1 \int_0^1 xy\ dxdy = \left(\frac{x^2}{2}\right)\Big|_0^1 \left(\frac{y^2}{2}\right)\Big|_0^1 = \frac{1}{4} \implies \boxed{c = 4}$

(b) $f_X(x) = \begin{cases} \int_0^1 4xy\ dy & 0 \le x \le 1 \\ 0 & \text{else} \end{cases} = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$

$f_Y(y) = \begin{cases} \int_0^1 4xy\ dx & 0 \le y \le 1 \\ 0 & \text{else} \end{cases} = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{else} \end{cases}$

Since $f_{X,Y}(x,y) = 4xy = (2x)(2y) = f_X(x)f_Y(y)$ for $(x,y) \in [0,1]^2$ and $f_{X,Y}(x,y) = 0 = f_X(x)f_Y(y)$ for $(x,y) \notin [0,1]^2$, $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ for all $(x,y) \in \mathbb{R}^2$. Conclude that X and Y are independent.

(c) $E[X] = \int_0^1 2x^2\ dx = \frac{2}{3}$

6. Let $X_1, \ldots, X_n$ be iid random variables with density f and cumulative distribution function F. Calculate in terms of f and F the density and cumulative distribution functions of the random variables

(a) $Y = \min\{X_1, \ldots, X_n\}$,

(b) $Z = \max\{X_1, \ldots, X_n\}$.

Answer:

(a)

$$\begin{aligned} F_Y(y) &= P[Y \le y] = P[\min\{X_1, \ldots, X_n\} \le y] = P[\{X_1 \le y\} \cup \cdots \cup \{X_n \le y\}] \\ &= 1 - P[X_1 > y, \ldots, X_n > y] = 1 - P[X_1 > y]P[X_2 > y] \ldots P[X_n > y] \\ &= 1 - (1 - F(y))(1 - F(y)) \ldots (1 - F(y)) \\ &= \boxed{1 - (1 - F(y))^n} \\ f_Y(y) &= F_Y'(y) = \boxed{n(1 - F(y))^{n-1} f(y)} \end{aligned}$$

(b)

$$\begin{aligned} F_Z(z) &= P[Z \le z] = P[\max\{X_1, \ldots, X_n\} \le z] \\ &= P[\{X_1 \le z\} \cap \cdots \cap \{X_n \le z\}] = P[X_1 \le z, X_2 \le z, \ldots, X_n \le z] \\ &= \boxed{(F(z))^n} \\ f_Z(z) &= F_Z'(z) = \boxed{n(F(z))^{n-1} f(z)} \end{aligned}$$