MA 3231 Homework 2

1. Solve the following linear program using the simplex algorithm:

   $\max z = 10x_1 + 6x_2 + 4x_3$
   subject to
   $4x_1 + 5x_2 + 2x_3 + x_4 \leq 20$
   $3x_1 + 4x_2 - x_3 + x_4 \leq 30$
   $x_1, x_2, x_3, x_4 \geq 0$

   Answer: An initial feasible solution to this problem is $(x_1, x_2, x_3, x_4) = 0$. The problem written in an equivalent form using slack variables is:

   $\max z = 10x_1 + 6x_2 + 4x_3$
   subject to
   $w_1 = 20 - 4x_1 - 5x_2 - 2x_3 - x_4$
   $w_2 = 30 - 3x_1 - 4x_2 + x_3 - x_4$
   $x_1, x_2, x_3, x_4, w_1, w_2 \geq 0$

   Now solve by the simplex method.

$$\begin{array}{l} \underline{z = 10x_1 + 6x_2 + 4x_3} \\ w_1 = 20 - 4x_1 - 5x_2 - 2x_3 - x_4 \\ w_2 = 30 - 3x_1 - 4x_2 + x_3 - x_4 \end{array}$$

$$\begin{array}{l} z = 50 - \frac{13}{2}x_2 - x_3 - \frac{5}{4}x_4 - \frac{5}{2}w_1 \\ \hline x_1 = 5 - \frac{5}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_4 - \frac{1}{4}w_1 \\ w_2 = 15 - \frac{1}{4}x_2 + \frac{5}{2}x_3 - \frac{1}{4}x_4 + \frac{3}{4}w_1 \end{array}$$

   This dictionary is optimal. The maximum value of z in the feasible region is

$$\boxed{z = 50 \text{ at } (x_1, x_2, x_3, x_4) = (5, 0, 0, 0).}$$

2. Solve the following linear program using the simplex algorithm:

$\min z = -7x_1 - 8x_2$
subject to
$4x_1 + x_2 \leq 100$
$-2x_1 - 2x_2 \geq -160$
$x_1 \leq 40$
$x_1, x_2 \geq 0$

Answer: First rewrite this program in standard form. This shows that we should maximize $z' = -z$ in the feasible region. Afterward, negate this maximum value of z' to find the minimum value of z in the feasible region.

$-\max -z = 7x_1 + 8x_2$
subject to
$4x_1 + x_2 \leq 100$
$2x_1 + 2x_2 \leq 160$
$x_1 \leq 40$
$x_1, x_2 \geq 0$

With $(x_1, x_2) = (0, 0)$ as an initial feasible solution, solve the following problem:

$\max z' = 7x_1 + 8x_2$
subject to
$w_1 = 100 - 4x_1 - x_2$
$w_2 = 160 - 2x_1 - 2x_2$
$w_3 = 40 - x_1$
$x_1, x_2, w_1, w_2, w_3 \geq 0$

$$\begin{aligned} &\underline{z' = 7x_1 + 8x_2} \\ &w_1 = 100 - 4x_1 - x_2 \\ &w_2 = 160 - 2x_1 - 2x_2 \\ &w_3 = 40 - x_1 \end{aligned}$$

$$\begin{aligned} &\underline{z' = 640 - x_1 - 4w_2} \\ &w_1 = 20 - 3x_1 - \frac{1}{2}w_2 \\ &x_2 = 80 - x_1 - \frac{1}{2}w_2 \\ &w_3 = 40 - x_1 \end{aligned}$$

The maximum value of z' in the feasible region is $z' = 640$ at $(x_1, x_2) = (0, 80)$. For the original problem this means the minimum value of z in the feasible region is

$$\boxed{z = -640 \text{ at } (x_1, x_2) = (0, 80).}$$

3. Solve the following linear program using the simplex algorithm and a suitable auxiliary program:

$\max z = 2x_1 + 6x_2$
subject to
$-x_1 - x_2 \leq -3$
$-3x_1 + 3x_2 \leq 3$
$x_1 + 2x_2 \leq 2$
$x_1, x_2 \geq 0$

Answer:

Phase 1
$\max \xi = -x_0$
subject to
$-x_1 - x_2 - x_0 \leq -3$
$-3x_1 + 3x_2 - x_0 \leq 3$
$x_1 + 2x_2 - x_0 \leq 2$
$x_0, x_1, x_2 \geq 0$

$$\begin{array}{l}
\underline{\xi = -x_0} \\
w_1 = -3 + x_1 + x_2 + x_0 \\
w_2 = 3 + 3x_1 - 3x_2 + x_0 \\
w_3 = 2 - x_1 - 2x_2 + x_0
\end{array}$$

$$\begin{array}{l}
\underline{\xi = -3 + x_1 + x_2 - w_1} \\
x_0 = 3 - x_1 - x_2 + w_1 \\
w_2 = 6 + 2x_1 - 4x_2 + w_1 \\
w_3 = 5 - 2x_1 - 3x_2 + w_1
\end{array}$$

$$\begin{array}{l}
\xi = -\dfrac{1}{2} - \dfrac{1}{2}x_2 - \dfrac{1}{2}w_1 - \dfrac{1}{2}w_3 \\
\hline
x_0 = \dfrac{1}{2} + \dfrac{1}{2}x_1 + \dfrac{1}{2}w_1 + \dfrac{1}{2}w_3 \\
w_2 = 6 + 2x_1 - 4x_2 + w_1 \\
x_1 = \dfrac{5}{2} - \dfrac{3}{2}x_2 + \dfrac{1}{2}w_1 - \dfrac{1}{2}w_3
\end{array}$$

The optimal solution to the auxiliary problem is $\xi = -\frac{1}{2}$ with $(x_0, x_1, x_2) = (\frac{1}{2}, \frac{5}{2}, 0)$. Since $x_0 \neq 0$ for the optimal solution to the auxiliary problem, the original problem is infeasible. Do not perform Phase 2.

4. Solve the following linear program using the simplex algorithm and a suitable auxiliary program:

   $\min z = -2x_1 - 3x_2 - 4x_3$
   subject to
   $2x_2 + 3x_3 \geq 5$
   $x_1 + x_2 + 2x_3 \leq 4$
   $x_1 + 2x_2 + 3x_3 \leq 7$
   $x_1, x_2, x_3 \geq 0$

   Answer: To solve this problem, find the optimal solution to the following problem in standard form and then negate the result for z'.

   $\max z' = 2x_1 + 3x_2 + 4x_3$
   subject to
   $-2x_2 - 3x_3 \leq -5$
   $x_1 + x_2 + 2x_3 \leq 4$
   $x_1 + 2x_2 + 3x_3 \leq 7$
   $x_1, x_2, x_3 \geq 0$

   Phase 1
   $\max \xi = -x_0$
   subject to
   $-2x_2 - 3x_3 - x_0 \leq -5$
   $x_1 + x_2 + 2x_3 - x_0 \leq 4$
   $x_1 + 2x_2 + 3x_3 - x_0 \leq 7$
   $x_0, x_1, x_2, x_3 \geq 0$

$$\begin{aligned}
&\underline{\xi = -x_0} \\
&w_1 = -5 + 2x_2 + 3x_3 + x_0 \\
&w_2 = 4 - x_1 - x_2 - 2x_3 + x_0 \\
&w_3 = 7 - x_1 - 2x_2 - 3x_3 + x_0
\end{aligned}$$

$$\begin{aligned}
&\underline{\xi = -5 + 2x_2 + 3x_3 - w_1} \\
&x_0 = 5 - 2x_2 - 3x_3 + w_1 \\
&w_2 = 9 - x_1 - 3x_2 - 5x_3 + w_1 \\
&w_3 = 12 - x_1 - 4x_2 - 6x_3 + w_1
\end{aligned}$$

$$\begin{aligned}
&\underline{\xi = -x_0} \\
&x_3 = \frac{5}{3} - \frac{2}{3}x_2 + \frac{1}{3}w_1 - \frac{1}{3}x_0 \\
&w_2 = \frac{2}{3} - x_1 + \frac{1}{3}x_2 - \frac{2}{3}w_1 + \frac{5}{3}x_0 \\
&w_3 = 2 - x_1 - w_1 + 2x_0
\end{aligned}$$

   This dictionary is optimal with ξ maximized at $x_0 = 0$. Continue with Phase 2.

Phase 2 Using the results of Phase 1, $z' = 2x_1 + 3x_2 + 4x_3 = \frac{20}{3} + 2x_1 + \frac{1}{3}x_2 + \frac{4}{3}w_1$.

$$\begin{array}{l} z' = \dfrac{20}{3} + 2x_1 + \dfrac{1}{3}x_2 + \dfrac{4}{3}w_1 \\ \hline x_3 = \dfrac{5}{3} - \dfrac{2}{3}x_2 + \dfrac{1}{3}w_1 \\ w_2 = \dfrac{2}{3} - x_1 + \dfrac{1}{3}x_2 - \dfrac{2}{3}w_1 \\ w_3 = 2 - x_1 - w_1 \end{array}$$

$$\begin{array}{l} z' = 8 + x_2 - 2w_2 \\ \hline x_3 = \dfrac{5}{3} - \dfrac{2}{3}x_2 + \dfrac{1}{3}w_1 \\ x_1 = \dfrac{2}{3} + \dfrac{1}{3}x_2 - \dfrac{2}{3}w_1 - w_2 \\ w_3 = \dfrac{4}{3} - \dfrac{1}{3}x_2 - \dfrac{1}{3}w_1 + w_2 \end{array}$$

$$\begin{array}{l} z' = \dfrac{21}{2} - \dfrac{3}{2}x_3 + \dfrac{1}{2}w_1 - 2w_2 \\ \hline x_2 = \dfrac{5}{2} - \dfrac{3}{2}x_3 - \dfrac{1}{2}w_1 \\ x_1 = \dfrac{3}{2} - \dfrac{1}{2}x_3 - \dfrac{1}{2}w_1 - w_2 \\ w_3 = \dfrac{1}{2} + \dfrac{1}{2}x_3 - \dfrac{1}{2}w_1 + w_2 \end{array}$$

$$\begin{array}{l} z' = 11 - x_3 - w_2 - w_3 \\ \hline x_2 = 3 - x_3 + w_2 - w_3 \\ x_1 = 1 - x_3 - 2w_2 + w_3 \\ w_1 = 1 + x_3 + 2w_2 - 2w_3 \end{array}$$

The optimal solution is $z' = 11$ at $(x_1, x_2, x_3) = (1, 3, 0)$. The optimal solution to the original problem is therefore:

$$\boxed{z = -11 \text{ at } (x_1, x_2, x_3) = (1, 3, 0).}$$

5. Explain why the following dictionary cannot be the optimal dictionary for any linear programming problem in which w_1 and w_2 are the initial slack variables:

$$\begin{array}{l}
\underline{z = 4 - w_1 - 2x_2} \\
x_1 = 3 - 2x_2 \\
w_2 = 1 + w_1 - 2x_2
\end{array}$$

Answer: Suppose that to arrive at this dictionary, x_1 was the previous entering variable. Using $x_2 = \frac{3}{2} - \frac{1}{2}x_1$, the previous dictionary would have been:

$$\begin{array}{l}
\underline{z = 1 + x_1 - w_1} \\
x_2 = \dfrac{3}{2} - \dfrac{1}{2}x_1 \\
w_2 = -2 + x_1 + w_1
\end{array}$$

For $x_1 = w_1 = 0$, this does not satisfy the requirement that $w_2 \geq 0$. So this could not have been the previous dictionary. Suppose instead that w_2 was the previous entering variable. The previous dictionary would have been one of the following:

$$\begin{array}{l}
\underline{z = 3 + 2x_2 + w_2} \\
x_1 = 3 - 2x_2 \\
w_1 = 1 - 2x_2 - w_2
\end{array}$$

or

$$\begin{array}{l}
\underline{z = 3 - 2w_1 + w_2} \\
x_1 = 2 - w_1 + w_2 \\
x_2 = \dfrac{1}{2} + \dfrac{1}{2}w_1 - \dfrac{1}{2}w_2
\end{array}$$

This means it would not be possible to write the initial slack variables w_1 and w_2 as functions of x_1 and x_2. The dictionary cannot have come from a problem written in standard slack form.