§ 8.2 The Simplex Method and Karmarkar's Method

8.2.1 Suppose the cost is $cx = x_1 + x_2 + x_3$ and the constraint is $x_1 + 2x_2 + 3x_3 = 12$ with $x \geq 0$. Start from $(12, 0, 0)$ and take a simplex step:

a) If x_2 goes up from 0 with $x_3 = 0$ and $x_1 + 2x_2 = 12$, find the cost in terms of x_2.

$cx = 12 - 2x_2 + x_2 = 12 - x_2$.
cost decreases as x_2 enters, $r_2 = -1$.

b) If x_3 is increased from 0 with $x_2 = 0$ and $x_1 + 3x_3 = 12$, find the cost in terms of x_3. What is the reduced cost r_3? Why should x_3 be the entering variable?

$cx = 12 - 3x_3 + x_3 = 12 - 2x_3$.
cost decreases as x_3 enters, $r_3 = -2$.
Since $r_3 < r_2$ cost decreases more per unit increase of x_3 than r_2, so x_3 should be the entering variable.

c) The new corner is $(0,0,4)$ when x_3 enters and x_1 leaves. Show that increasing x_2 from zero with $x_1 = 0$ and $2x_2 + 3x_3 = 12$ increases cost; the corner $(0,0,4)$ is optimal.

$cx = x_2 + x_3 = x_2 + 4 - \frac{2}{3}x_2 = 4 + \frac{1}{3}x_2$, $r_2 = \frac{1}{3}$.
cost increases as x_2 enters. Since moving from $(12,0,0)$ to $(0,0,4)$ reduced cost, we would not want to let x_1 enter back either. Conclude $(0,0,4)$ minimizes cost in the feasible region.

<u>8.2.2</u> With cost vector $c = [0,7,9,0]$ and constraints $x_1 + x_2 + x_3 = 6$, $x_2 + 2x_3 + x_4 = 1$, $x_i \geqslant 0$, explain directly why $x = (6,0,0,1)$ is optimal.

Since the first and fourth component of c are zero, we minimize cost by setting $x_2 = x_3 = 0$ and determine x_1, x_4 satisfying the constraints. Then $x_1 = 6$ and $x_4 = 1$ must hold so $x = (6,0,0,1)$ minimizes cost.

8.2.3 Write down the tableau T for the previous question. How do you see the stopping test is passed?

$$T = \left[\begin{array}{c|c} A & b \\ c & 0 \end{array}\right] = \left[\begin{array}{ccccc} 1 & 1 & 1 & 0 & 6 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 7 & 9 & 0 & 0 \end{array}\right]$$

Since none of the reduced costs $r_1 = 0$, $r_2 = 7$, $r_3 = 9$, $r_4 = 0$ are negative, this corner minimizes cost in the feasible region.

8.2.4 For the same problem start the simplex method at the wrong corner $x = (5, 1, 0, 0)$. Use elimination on T to reach a tableau T' in which columns 1 and 2 come directly from the identity matrix. What is r, the vector of reduced costs in the last row. What is the cost at the corner $(5, 1, 0, 0)$? Should x_3 or x_4 be the entering variable?

$$T' = \begin{bmatrix} 1 & 0 & -1 & -1 & 5 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & -5 & -7 & -7 \end{bmatrix}$$

$r = [0\ 0\ -5\ -7]$
At $(5,1,0,0)$, $cx = 7$
Since $r_4 = -7 < -5 = r_3$, x_4 should be the entering variable.

8.2.5 Instead of the tableau split A and c into

$$A = \left[\begin{array}{cc:cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right] = [B\ N] \text{ and}$$

$$c = [0\ 7 \,\vdots\, 9\ 0] = [c_B\ c_N].$$

Compute the reduced costs $r = c_N - c_B B^{-1} N$.
Should x_3 or x_4 be the entering variable?

$$r = [9\ 0] - [0\ 7]\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = [9\ 0] - [0\ 7]\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$= [9\ 0] - [14\ 7] = [-5\ -7]$$ This shows $r_3 = -5$, $r_4 = -7$
x_4 should enter since $r_4 < r_3$.

8.2.6 If x_4 enters the basis then x_1 or x_2 must leave. Keeping $x_3 = 0$ and $Ax = b$ show that $x_1 = 5 + x_4$ and $x_2 = 1 - x_4$. Which of these is first to reach zero as x_4 increases. What is x at this new corner?

$$6 = x_1 + x_2 + x_3 = x_1 + x_2 \longrightarrow x_1 = 6 - x_2 = 5 + x_4$$
$$1 = x_2 + 2x_3 + x_4 = x_2 + x_4 \qquad x_2 = 1 - x_4$$

x_2 reaches 0 first as x_4 increases (x_1 actually increases as x_4 increases). This occurs when $x_4 = 1$. The new corner is $x = (6, 0, 0, 1)$ and the cost is $cx = 0$. This is the optimal corner we started with in 8.2.2.

8.2.8 In equation 12, $B_{new}^{-1} = \begin{bmatrix} 1 & & v_1 \\ & \ddots & \vdots \\ & & v_m \end{bmatrix}^{-1} B_{old}^{-1}$, what is $\begin{bmatrix} 1 & & v_1 \\ & \ddots & \vdots \\ & & v_m \end{bmatrix}^{-1}$?

$$\begin{bmatrix} 1 & & & v_1 \\ & 1 & & \vdots \\ & & \ddots & \vdots \\ & & & v_m \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & & -v_1/v_m \\ & 1 & & \vdots \\ & & \ddots & -v_{m-1}/v_m \\ & & & 1/v_m \end{bmatrix}$$

The last column has entries: $-v_1/v_m, -v_2/v_m, \ldots, -v_{m-1}/v_m, 1/v_m$. Otherwise identity.

8.2.9

If we wanted to maximize cx (still with $Ax = b$ and $x \geq 0$), what would be the stopping test on r and what rules would choose the entering variable x_i and leaving variable x_j?

We would stop if $r \leq 0$ (no reduced costs positive). The entering variable x_j is chosen by finding the most positive r_j. The leaving variable x_i is the first x_i to reach 0 as x_j is increased.

8.2.13

a) Why does ordinary multiplication of an m by n matrix by an n by p matrix lead to mnp individual multiplications?

Suppose A is m by n, B is n by p and $C = AB$. The element c_{ij} of C is the product of row i of A, which is length n, with column j of B, also length n. Since there are mp such c_{ij}, mnp multiplications are needed.

b) To compute ABC, where A is m by n, B is n by p, and C is p by q, how many multiplications are used in the order $(AB)C$? How many in the order $A(BC)$?

For $(AB)C$, first mnp to compute AB. Since AB is m by p, multiplying by C requires mpq multiplications. Altogether, $mnp + mpq$.

For $A(BC)$, first npq to compute BC. Then mnq to multiply A by the n by q BC. Altogether, $npq + mnq$.

c) If B is square $(n=p)$ which order is better?

$(AB)C$ requires $mn^2 + mnq$ and $A(BC)$ requires $n^2q + mnq$. If $m<q$, $(AB)C$ is better and $A(BC)$ is better if $q<m$.

8.2.14 In Karmarkar's method with $A = [1\ 1\ 1]$, what is the projection matrix P? Verify that P is positive semidefinite and $AP = 0$.

$$P = I - A^T(AA^T)^{-1}A$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\left([1\ 1\ 1]\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)^{-1} [1\ 1\ 1]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$AP = \frac{1}{3}\left[(2-1-1)\ \ (-1+2-1)\ \ (-1-1+2)\right] = [0\ 0\ 0]$$

$$P \sim \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & -3/2 & 3/2 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & 0 & 0 \end{bmatrix}$$

The pivots 2, 3/2, and 0 are all nonnegative $\Longleftrightarrow$ P is positive semidefinite.