

## MA 3231 Homework 2

1. Solve the following linear program using the simplex algorithm:

$$\begin{aligned} \max z &= 10x_1 + 6x_2 + 4x_3 \\ \text{subject to} \\ 4x_1 + 5x_2 + 2x_3 + x_4 &\leq 20 \\ 3x_1 + 4x_2 - x_3 + x_4 &\leq 30 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Answer: An initial feasible solution to this problem is  $(x_1, x_2, x_3, x_4) = 0$ . The problem written in an equivalent form using slack variables is:

$$\begin{aligned} \max z &= 10x_1 + 6x_2 + 4x_3 \\ \text{subject to} \\ w_1 &= 20 - 4x_1 - 5x_2 - 2x_3 - x_4 \\ w_2 &= 30 - 3x_1 - 4x_2 + x_3 - x_4 \\ x_1, x_2, x_3, x_4, w_1, w_2 &\geq 0 \end{aligned}$$

Now solve by the simplex method.

$$\begin{aligned} z &= 10x_1 + 6x_2 + 4x_3 \\ w_1 &= 20 - 4x_1 - 5x_2 - 2x_3 - x_4 \\ w_2 &= 30 - 3x_1 - 4x_2 + x_3 - x_4 \end{aligned}$$


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$$\begin{aligned} z &= 50 - \frac{13}{2}x_2 - x_3 - \frac{5}{4}x_4 - \frac{5}{2}w_1 \\ x_1 &= 5 - \frac{5}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_4 - \frac{1}{4}w_1 \\ w_2 &= 15 + 12x_1 - \frac{1}{4}x_2 + \frac{5}{2}x_3 - \frac{1}{4}x_4 + \frac{3}{4}w_1 \end{aligned}$$

This dictionary is optimal. The maximum value of  $z$  in the feasible region is

$z = 50 \text{ at } (x_1, x_2, x_3, x_4) = (5, 0, 0, 0).$

2. Solve the following linear program using the simplex algorithm:

$$\begin{aligned} \min z &= -7x_1 - 8x_2 \\ \text{subject to} \\ 4x_1 + x_2 &\leq 100 \\ -2x_1 - 2x_2 &\geq -160 \\ x_1 &\leq 40 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Answer: First rewrite this program in standard form. This shows that we should maximize  $z' = -z$  in the feasible region. Afterward, negate this maximum value of  $z'$  to find the minimum value of  $z$  in the feasible region.

$$\begin{aligned} -\max -z &= 7x_1 + 8x_2 \\ \text{subject to} \\ 4x_1 + x_2 &\leq 100 \\ 2x_1 + 2x_2 &\leq 160 \\ x_1 &\leq 40 \\ x_1, x_2 &\geq 0 \end{aligned}$$

With  $(x_1, x_2) = (0, 0)$  as an initial feasible solution, solve the following problem:

$$\begin{aligned} \max z' &= 7x_1 + 8x_2 \\ \text{subject to} \\ w_1 &= 100 - 4x_1 - x_2 \\ w_2 &= 160 - 2x_1 - 2x_2 \\ w_3 &= 40 - x_1 \\ x_1, x_2, w_1, w_2, w_3 &\geq 0 \end{aligned}$$

$$\begin{aligned} z' &= 7x_1 + 8x_2 \\ w_1 &= 100 - 4x_1 - x_2 \\ w_2 &= 160 - 2x_1 - 2x_2 \\ w_3 &= 40 - x_1 \end{aligned}$$

$$\begin{aligned} z' &= 640 - x_1 - 4w_2 \\ w_1 &= 20 - 3x_1 - \frac{1}{2}w_2 \\ x_2 &= 80 - x_1 - \frac{1}{2}w_2 \\ w_3 &= 40 - x_1 \end{aligned}$$

The maximum value of  $z'$  in the feasible region is  $z' = 640$  at  $(x_1, x_2) = (0, 80)$ . For the original problem this means the minimum value of  $z$  in the feasible region is

$z = -640 \text{ at } (x_1, x_2) = (0, 80).$
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3. Solve the following linear program using the simplex algorithm and a suitable auxiliary program:

$$\begin{aligned} \max z &= 2x_1 + 6x_2 \\ \text{subject to} \\ -x_1 - x_2 &\leq -3 \\ -3x_1 + 3x_2 &\leq 3 \\ x_1 + 2x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Answer:

Phase 1

$$\begin{aligned} \max \xi &= -x_0 \\ \text{subject to} \\ -x_1 - x_2 - x_0 &\leq -3 \\ -3x_1 + 3x_2 - x_0 &\leq 3 \\ x_1 + 2x_2 - x_0 &\leq 2 \\ x_0, x_1, x_2 &\geq 0 \end{aligned}$$

$$\begin{aligned} \underline{\xi = -x_0} \\ w_1 &= -3 + x_1 + x_2 + x_0 \\ w_2 &= 3 + 3x_1 - 3x_2 + x_0 \\ w_3 &= 2 - x_1 - 2x_2 + x_0 \end{aligned}$$

$$\begin{aligned} \underline{\xi = -3 + x_1 + x_2 - w_1} \\ x_0 &= 3 - x_1 - x_2 + w_1 \\ w_2 &= 6 + 2x_1 - 4x_2 + w_1 \\ w_3 &= 5 - 2x_1 - 3x_2 + w_1 \end{aligned}$$

$$\begin{aligned} \underline{\xi = -\frac{1}{2} - \frac{1}{2}x_2 - \frac{1}{2}w_1 - \frac{1}{2}w_3} \\ x_0 &= \frac{1}{2} + \frac{1}{2}x_1 + \frac{1}{2}w_1 + \frac{1}{2}w_3 \\ w_2 &= 6 + 2x_1 - 4x_2 + w_1 \\ x_1 &= \frac{5}{2} - \frac{3}{2}x_2 + \frac{1}{2}w_1 - \frac{1}{2}w_3 \end{aligned}$$

The optimal solution to the auxiliary problem is  $\xi = -\frac{1}{2}$  with  $(x_0, x_1, x_2) = (\frac{1}{2}, \frac{5}{2}, 0)$ . Since  $x_0 \neq 0$  for the optimal solution to the auxiliary problem, the original problem is infeasible. Do not perform Phase 2.

4. Solve the following linear program using the simplex algorithm and a suitable auxiliary program:

$$\begin{aligned} \min z &= -2x_1 - 3x_2 - 4x_3 \\ \text{subject to} \\ 2x_2 + 3x_3 &\geq 5 \\ x_1 + x_2 + 2x_3 &\leq 4 \\ x_1 + 2x_2 + 3x_3 &\leq 7 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Answer: To solve this problem, find the optimal solution to the following problem in standard form and then negate the result for  $z'$ .

$$\begin{aligned} \max z' &= 2x_1 + 3x_2 + 4x_3 \\ \text{subject to} \\ -2x_2 - 3x_3 &\leq -5 \\ x_1 + x_2 + 2x_3 &\leq 4 \\ x_1 + 2x_2 + 3x_3 &\leq 7 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

#### Phase 1

$$\begin{aligned} \max \xi &= -x_0 \\ \text{subject to} \\ -2x_2 - 3x_3 - x_0 &\leq -5 \\ x_1 + x_2 + 2x_3 - x_0 &\leq 4 \\ x_1 + 2x_2 + 3x_3 - x_0 &\leq 7 \\ x_0, x_1, x_2, x_3 &\geq 0 \end{aligned}$$

$$\begin{aligned} \underline{\xi = -x_0} \\ w_1 &= -5 + 2x_2 + 3x_3 + x_0 \\ w_2 &= 4 - x_1 - x_2 - 2x_3 + x_0 \\ w_3 &= 7 - x_1 - 2x_2 - 3x_3 + x_0 \end{aligned}$$

$$\begin{aligned} \underline{\xi = -5 + 2x_2 + 3x_3 - w_1} \\ x_0 &= 5 - 2x_2 - 3x_3 + w_1 \\ w_2 &= 9 - x_1 - 3x_2 - 5x_3 + w_1 \\ w_3 &= 12 - x_1 - 4x_2 - 6x_3 + w_1 \end{aligned}$$

$$\begin{aligned} \underline{\xi = -x_0} \\ x_3 &= \frac{5}{3} - \frac{2}{3}x_2 + \frac{1}{3}w_1 - \frac{1}{3}x_0 \\ w_2 &= \frac{2}{3} - x_1 + \frac{1}{3}x_2 - \frac{2}{3}w_1 + \frac{5}{3}x_0 \\ w_3 &= 2 - x_1 - w_1 + 2x_0 \end{aligned}$$

This dictionary is optimal with  $\xi$  maximized at  $x_0 = 0$ . Continue with Phase 2.

Phase 2 Using the results of Phase 1,  $z' = 2x_1 + 3x_2 + 4x_3 = \frac{20}{3} + 2x_1 + \frac{1}{3}x_2 + \frac{4}{3}w_1$ .

$$\underline{z' = \frac{20}{3} + 2x_1 + \frac{1}{3}x_2 + \frac{4}{3}w_1}$$

$$x_3 = \frac{5}{3} - \frac{2}{3}x_2 + \frac{1}{3}w_1$$

$$w_2 = \frac{2}{3} - x_1 + \frac{1}{3}x_2 - \frac{2}{3}w_1$$

$$w_3 = 2 - x_1 - w_1$$

$$\underline{z' = 8 + x_2 - 2w_2}$$

$$x_3 = \frac{5}{3} - \frac{2}{3}x_2 + \frac{1}{3}w_1$$

$$x_1 = \frac{2}{3} + \frac{1}{3}x_2 - \frac{2}{3}w_1 - w_2$$

$$w_3 = \frac{4}{3} - \frac{1}{3}x_2 - \frac{1}{3}w_1 + w_2$$

$$\underline{z' = \frac{21}{2} - \frac{3}{2}x_3 + \frac{1}{2}w_1 - 2w_2}$$

$$x_2 = \frac{5}{2} - \frac{3}{2}x_3 - \frac{1}{2}w_1$$

$$x_1 = \frac{3}{2} - \frac{1}{2}x_3 - \frac{1}{2}w_1 - w_2$$

$$w_3 = \frac{1}{2} + \frac{1}{2}x_3 - \frac{1}{2}w_1 + w_2$$

$$\underline{z' = 11 - x_3 - w_2 - w_3}$$

$$x_2 = 3 - x_3 + w_2 - w_3$$

$$x_1 = 1 - x_3 - 2w_2 + w_3$$

$$w_1 = 1 + x_3 + 2w_2 - 2w_3$$

The optimal solution is  $z' = 11$  at  $(x_1, x_2, x_3) = (1, 3, 0)$ . The optimal solution to the original problem is therefore:

$z = -11$ at $(x_1, x_2, x_3) = (1, 3, 0)$ .
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5. Explain why the following dictionary cannot be the optimal dictionary for any linear programming problem in which  $w_1$  and  $w_2$  are the initial slack variables:

$$\begin{aligned} z &= 4 - w_1 - 2x_2 \\ x_1 &= 3 - 2x_2 \\ w_2 &= 1 + w_1 - 2x_2 \end{aligned}$$

Answer: Suppose that to arrive at this dictionary,  $x_1$  was the previous entering variable. Using  $x_2 = \frac{3}{2} - \frac{1}{2}x_1$ , the previous dictionary would have been:

$$\begin{aligned} z &= 1 + x_1 - w_1 \\ x_2 &= \frac{3}{2} - \frac{1}{2}x_1 \\ w_2 &= -2 + x_1 + w_1 \end{aligned}$$

For  $x_1 = w_1 = 0$ , this does not satisfy the requirement that  $w_2 \geq 0$ . So this could not have been the previous dictionary. Suppose instead that  $w_2$  was the previous entering variable. The previous dictionary would have been one of the following:

$$\begin{aligned} z &= 3 + 2x_2 + w_2 \\ x_1 &= 3 - 2x_2 \\ w_1 &= 1 - 2x_2 - w_2 \end{aligned}$$

or

$$\begin{aligned} z &= 3 - 2w_1 + w_2 \\ x_1 &= 2 - w_1 + w_2 \\ x_2 &= \frac{1}{2} + \frac{1}{2}w_1 - \frac{1}{2}w_2 \end{aligned}$$

This means it would not be possible to write the initial slack variables  $w_1$  and  $w_2$  as functions of  $x_1$  and  $x_2$ . The dictionary cannot have come from a problem written in standard slack form.