MA 3231 Assignment 4

1. Solve the following linear program using

   a) the perturbation (lexicographic) method

   b) Bland's rule

$\max z = -\frac{3}{4}x_1 + 150x_2 - \frac{1}{50}x_3 + 6x_4$
subject to
$\frac{1}{4}x_1 + 150x_2 - \frac{1}{25}x_3 + 9x_4 \leq 0$
$\frac{1}{2}x_1 - 90x_2 - \frac{1}{50}x_3 + 3x_4 \leq 0$
$x_3 \leq 1$
$x_1, x_2, x_3, x_4 \geq 0.$

a) $z = -3/4\, x_1 + 150x_2 - 1/50\, x_3 + 6x_4$
$w_1 = \epsilon_1 - 1/4\, x_1 - 150x_2 + 1/25\, x_3 - 9x_4$ $\quad (\epsilon_1 >> \epsilon_2 >> \epsilon_3)$
$w_2 = \epsilon_2 - 1/4\, x_1 + 90x_2 + 1/50\, x_3 - 3x_4$
$w_3 = 1 + \epsilon_3 - x_3$

x_2 enters, w_1 leaves

$z = \epsilon_1 - x_1 + 1/50\, x_3 - 3x_4 - w_1$
$x_2 = \epsilon_1/150 - 1/600\, x_1 + 1/3750\, x_3 - 3/50\, x_4 - 1/150\, w_1$
$w_2 = 3/5\,\epsilon_1 + \epsilon_2 + 1/10\, x_1 + 11/250\, x_3 - 42/5\, x_4 - 3/5\, w_1$
$w_3 = 1 + \epsilon_3 - x_3$

x_3 enters, w_3 leaves

$z = 1/50 + \epsilon_1 + \epsilon_3/50 - x_1 - 3x_4 - w_1 - 1/50\, w_3$
$x_3 = 1 + \epsilon_3 - w_3$
$x_2 = 1/3750 - 1/600\, x_1 - 3/50\, x_4 - 1/150\, w_1 - 1/3750\, w_3 + \epsilon_1/150 + \epsilon_3/3750$
$w_2 = 11/250 + 3/5\,\epsilon_1 + \epsilon_2 - 11\epsilon_3/25 + 1/10\, x_1 - 42/5\, x_4 - 3/5\, w_1 - 11/250\, x_3$

The optimal solution to the unperturbed program is $z = 1/50$ at $(x_1, x_2, x_3, x_4) = (0, 1/3750, 1, 0)$.

b) $z = -3/4\, x_1 + 150x_2 - 1/50\, x_3 + 6x_4$
$w_1 = -1/4\, x_1 - 150x_2 + 1/25\, x_3 - 9x_4$
$w_2 = -1/4\, x_1 + 90x_2 + 1/50\, x_3 - 3x_4$
$w_3 = 1 - x_3$

x_2 enters, w_1 leaves

$z = -x_1 + 1/50\, x_3 - 3x_4 - w_1$
$x_2 = -1/600\, x_1 + 1/3750\, x_3 - 3/50\, x_4 - 1/150\, w_1$
$w_2 = 1/10\, x_1 + 11/250\, x_3 - 42/5\, x_4 - 3/5\, w_1$
$w_3 = 1 - x_3$

x_3 enters, w_3 leaves

$z = 1/50 - x_1 - 3x_4 - w_1 - 1/50\, w_3$
$x_3 = 1 - w_3$
$x_2 = 1/3750 - 1/600\, x_1 - 3/50\, x_4 - 1/150\, w_1 - 1/3750\, w_3$
$w_2 = 11/250 + 1/10\, x_1 - 42/5\, x_4 - 3/5\, w_1 - 11/250\, w_3$

The optimal solution to the linear program is $z = 1/50$ at $(x_1, x_2, x_3, x_4) = (0, 1/3750, 1, 0)$.

2. Consider the following linear programming problem:

$\max z = x_1 + 2x_2$
subject to
$-2x_1 - x_2 + x_3 \leq 1$
$x_1 + x_2 \leq 2$
$x_1 + x_3 \leq 3$
$x_1,\ x_2,\ x_3 \geq 0$

a) Solve the linear program.
b) Find the dual program.
c) Solve the dual program.
d) Compare the solutions of primal and dual program.

a) $z = x_1 + 2x_2$
$w_1 = 1 + 2x_1 + x_2 - x_3$
$w_2 = 2 - x_1 - x_2$
$w_3 = 3 - x_1 - x_3$

$z = 4 - x_1 - 2w_2$
$x_2 = 2 - x_1 - w_2$
$w_1 = 3 + x_1 - x_3 - w_2$
$w_3 = 3 - x_1 - x_3$

The optimal solution to the primal linear program is $z = 4$ at $(x_1, x_2, x_3) = (0, 2, 0)$.

b) minimize $u = y_1 + 2y_2 + 3y_3$
subject to $-2y_1 + y_2 + y_3 \geq 1$
$-y_1 + y_2 \geq 2$
$y_1 + y_3 \geq 0$
$y_1, y_2, y_3 \geq 0$

c) To solve the dual problem, convert to standard form.
-maximize $-u = -y_1 - 2y_2 - 3y_3$
subject to $2y_1 - y_2 - y_3 \leq -1$
$y_1 - y_2 \leq -2$
$-y_1 - y_3 \leq 0$
$y_1, y_2, y_3 \geq 0$

Phase I
maximize $\xi = -y_0$
subject to $2y_1 - y_2 - y_3 - y_0 \leq -1$
$y_1 - y_2 - y_0 \leq -2$
$-y_1 - y_3 - y_0 \leq 0$
$y_0, y_1, y_2, y_3 \geq 0$

$\xi = -y_0$
$w_1 = -1 - 2y_1 + y_2 + y_3 + y_0$
$w_2 = -2 - y_1 + y_2 + y_0$ (least feasible constraint)
$w_3 = y_1 + y_3 + y_0$

$\xi = -2 - y_1 + y_2 - w_2$
$y_0 = 2 + y_1 - y_2 + w_2$
$w_1 = 1 - y_1 + y_3 + w_2$
$w_3 = 2 + 2y_1 - y_2 + y_3 + w_2$

$\xi = -y_0$
$y_2 = 2 + y_1 + w_2 - y_0$
$w_1 = 1 - y_1 + y_3 + w_2$
$w_3 = y_1 + y_3 + y_0$

The auxiliary problem has the optimal solution $\xi = -y_0 = 0$ at $(y_1, y_2, y_3) = (0, 2, 0)$

The optimal solution to the auxiliary problem is $\xi = -y_0 = 0$ at $(y_1, y_2, y_3) = (0, 2, 0)$.

Phase 2

maximize $-u = -y_1 - 2y_2 - 3y_3 = -4 - 2y_1 - 2w_2$

subject to
$$y_2 = 2 + y_1 + w_2$$
$$w_1 = 1 - y_1 + y_3 + w_2$$
$$w_3 = y_1 + y_3$$
$$y_1, y_2, y_3, w_1, w_2, w_3 \geq 0$$

This dictionary is optimal and $-u$ is maximized at $-u = -4$, $(y_1, y_2, y_3) = (0, 2, 0)$. The optimal solution to the dual problem is $u = 4$ at $(y_1, y_2, y_3) = (0, 2, 0)$.

d) The maximum value of z in the primal problem equals the minimum value of u in the dual problem. In this case the decision variables also matched up. we cannot always expect the decision variables to match since the primal and dual problem may not have the same number of decision variables.