MA 3231
Assignment 2

1. Find all the values of the parameter α such that the following linear program has a finite optimal solution:

$\max z = \alpha x_1 + 2x_2 - x_3$
subject to
$2x_1 - x_2 + 3x_3 \leq 4$
$-x_1 + x_2 - 2x_3 \leq 8$
$3x_1 - 3x_3 \leq 2$
$x_1,\ x_2,\ x_3 \geq 0$

Apply the simplex method for:

$$z = \alpha x_1 + 2x_2 - x_3$$

$$\begin{aligned}
w_1 &= 4 - 2x_1 + x_2 - 3x_3 \\
w_2 &= 8 + x_1 - x_2 + 2x_3 \\
w_3 &= 2 - 3x_1 + 3x_3
\end{aligned}$$

For any α, we can let x_2 enter.

$$z = 16 + (\alpha+2)x_1 + 3x_3 + 2w_2$$

$$\begin{aligned}
x_2 &= 8 + x_1 + 2x_3 - w_2 \\
w_1 &= 12 - x_1 - x_3 - w_2 \\
w_3 &= 2 - 3x_1 + 3x_3
\end{aligned}$$

For any α, we can let x_3 enter.

$$\begin{aligned}
z &= 52 + (\alpha-1)x_1 - 3w_1 - w_2 \\
x_3 &= 12 - x_1 - w_1 - w_2 \\
x_2 &= 32 - x_1 - 2w_1 - 3w_2 \\
w_3 &= 38 - 6x_1 - 3w_1 - 3w_2
\end{aligned}$$

If $\alpha \le 1$ this dictionary is optimal and we conclude the maximum value of z is $z = 52$ at $(x_1, x_2, x_3) = (0, 32, 12)$.
If $\alpha > 1$, we don't yet have an optimal dictionary and let x_1 enter.

$$z = \frac{137+19\alpha}{3} - \frac{5+\alpha}{2} w_1 - \frac{1+\alpha}{2} w_2 - \frac{\alpha-1}{6} w_3$$

$$x_1 = 19/3 - 1/2\, w_1 - 1/2\, w_2 - 1/6\, w_3$$
$$x_2 = 77/3 - 3/2\, w_1 - 5/2\, w_2 + 1/6\, w_3$$
$$x_3 = 17/3 - 1/2\, w_1 - 1/2\, w_2 + 1/6\, w_3$$

This dictionary is optimal and we conclude the maximum value of z is $(137+19\alpha)/3$ at $(x_1, x_2, x_3) = (19/3, 77/3, 17/3)$

$\therefore$ This problem has a finite solution $\forall \alpha \in \mathbb{R}$ and the maximum value of z is

$$z = \begin{cases} 52 & , \alpha \le 1 \\ \frac{137+19\alpha}{3} & , \alpha > 1 \end{cases}$$

2. Solve the following linear program using the simplex algorithm and a suitable auxiliary program:

$\max z = x_1 + 3x_2$
subject to
$-x_1 - x_2 \leq -3$
$-x_1 + x_2 \leq -1$
$x_1 + 2x_2 \leq 2$
$x_1,\ x_2 \geq 0$

Draw the region of feasible solution to this problem and indicate the solution you get at each step of the simplex algorithm (only for the original problem, i.e., Phase II).

Phase I

maximize $\quad -x_0$

subject to
$$-x_1 - x_2 - x_0 \leq -3$$
$$-x_1 + x_2 - x_0 \leq -1$$
$$x_1 + 2x_2 - x_0 \leq 2$$
$$x_0,\ x_1,\ x_2 \geq 0$$

$$\begin{aligned}
\xi &= -x_0 \\
w_1 &= -3 + x_1 + x_2 + x_0 \\
w_2 &= -1 + x_1 - x_2 + x_0 \\
w_3 &= 2 - x_1 - 2x_2 + x_0
\end{aligned}$$

$$\begin{aligned}
\xi &= -3 + x_1 + x_2 - w_1 \\
x_0 &= 3 - x_1 - x_2 + w_1 \\
w_2 &= 2 - 2x_2 + w_1 \\
w_3 &= 5 - 2x_1 - 3x_2 + w_1
\end{aligned}$$

$$\begin{aligned}
\xi &= -1/2 - 1/2\, x_2 - 1/2\, w_1 - 1/2\, w_3 \\
x_1 &= 5/2 - 3/2\, x_2 + 1/2\, w_1 - 1/2\, w_3 \\
x_0 &= 1/2 + 1/2\, x_2 + 1/2\, w_1 + 1/2\, w_3 \\
w_2 &= 2 - 2x_2 + w_1
\end{aligned}$$

This dictionary is optimal but with $x_0 \neq 0$. The original problem is infeasible. No Phase 2.

3. Solve the following linear program using the simplex algorithm and a suitable auxiliary program:

$\max z = 2x_1 + 3x_2 + 4x_3$
subject to
$-2x_2 - 3x_3 \leq -5$
$x_1 + x_2 + 2x_3 \leq 4$
$x_1 + 2x_2 + 3x_3 \leq 7$
$x_1,\ x_2,\ x_3 \geq 0$

Phase I

maximize $\quad -x_0$

subject to
$$-2x_2 - 3x_3 - x_0 \leq -5$$
$$x_1 + x_2 + 2x_3 - x_0 \leq 4$$
$$x_1 + 2x_2 + 3x_3 \leq 7$$
$$x_0, x_1, x_2, x_3 \geq 0$$

$$\begin{aligned}
\xi &= -x_0 \\
w_1 &= -5 + 2x_2 + 3x_3 + x_0 \\
w_2 &= 4 - x_1 - x_2 - 2x_3 + x_0 \\
w_3 &= 7 - x_1 - 2x_2 - 3x_3 + x_0
\end{aligned}$$

$$\begin{aligned}
\xi &= -5 + 2x_2 + 3x_3 - w_1 \\
x_0 &= 5 - 2x_2 - 3x_3 + w_1 \\
w_2 &= 9 - x_1 - 3x_2 - 5x_3 + w_1 \\
w_3 &= 12 - x_1 - 4x_2 - 6x_3 + w_1
\end{aligned}$$

x_0 entering
w_1 leaving

$$\begin{aligned}
\xi &= -x_0 \\
x_3 &= 5/3 - 2/3\, x_2 - 1/3\, x_0 + 1/3\, w_1 \\
w_2 &= 2/3 - x_1 + 1/3\, x_2 + 5/3\, x_0 - 2/3\, w_1 \\
w_3 &= 2 - x_1 + 2x_0 - w_1
\end{aligned}$$

x_3 entering
x_0 leaving

This dictionary is optimal with ξ maximized by $x_0 = 0$.

Phase 2

$$z = 20/3 + 2x_1 + 1/3\, x_2 + 4/3\, w_1$$
$$x_3 = 5/3 - 2/3\, x_2 + 1/3\, w_1$$
$$w_2 = 2/3 - x_1 + 1/3\, x_2 - 2/3\, w_1$$
$$w_3 = 2 - x_1 - w_1$$

$$z = 8 + x_2 - 2w_2$$
$$x_1 = 2/3 + 1/3\, x_2 - 2/3\, w_1 - w_2$$
$$x_3 = 5/3 - 2/3\, x_2 + 1/3\, w_1$$
$$w_3 = 4/3 - 1/3\, x_2 - 1/3\, w_1 + w_2$$

x_1 entering
w_2 leaving

$$z = 21/2 - 3/2\, x_3 + 1/2\, w_1 - 2w_2$$
$$x_2 = 5/2 - 3/2\, x_3 + 1/2\, w_1$$
$$x_1 = 3/2 - 1/2\, x_3 - 1/2\, w_1 - w_2$$
$$w_3 = 1/2 + 1/2\, x_3 - 1/2\, w_1 + w_2$$

x_2 entering
x_3 leaving

$$z = 11 - x_3 - w_2 - w_3$$
$$w_1 = 1 + x_3 + 2w_2 - 2w_3$$
$$x_2 = 3 - x_3 + w_2 - w_3$$
$$x_1 = 1 - x_3 - 2w_2 + w_3$$

w_1 entering
w_3 leaving

Optimal solution:

$$z = 11$$
$$(x_1, x_2, x_3) = (1, 3, 0)$$

4. Show that the following dictionary cannot be the optimal dictionary for any linear programming problem in which w_1 and w_2 are the initial slack variables:

$$\begin{array}{rcrrr} z & = & 4 & -w_1 & -2x_2 \\ \hline x_1 & = & 3 & & -2x_2 \\ w_2 & = & 1 & +w_1 & -2x_2 \end{array}$$

Hint: If it could, what was the original problem from whence it came?

Suppose that to arrive at this dictionary, x_1 was the previous entering variable. Using $x_2 = 3/2 - 1/2\, x_1$, the previous dictionary would have been

$$\begin{aligned} z &= 1 + x_1 - w_1 \\ x_2 &= 3/2 - 1/2\, x_1 \\ w_2 &= -2 + x_1 + w_1 \end{aligned}$$

This violates $w_2 \geq 0$ when $x_1 = w_1 = 0$.

Suppose w_2 was the previous entering variable. The previous dictionary would have been

$$\begin{aligned} z &= 3 + 2x_2 + w_2 \\ x_1 &= 3 - 2x_2 \\ w_1 &= 1 - 2x_2 - w_2 \end{aligned}$$

or

$$\begin{aligned} z &= 3 - 2w_1 + w_2 \\ x_1 &= 2 - w_1 + w_2 \\ x_2 &= 1/2 + 1/2\, w_1 - 1/2\, w_2 \end{aligned}$$

We would not be able to isolate w_1 and w_2 as functions of x_1, x_2.

6. Reconsider the degenerate dictionary of Lecture 2.8 that led to the cycling issue:

$$
\begin{array}{rcrrrrrrrrr}
z & = & & & x_1 & - & 2x_2 & & & - & 2x_4 \\
\hline
w_1 & = & & - & 0.5x_1 & + & 3.5x_2 & + & 2x_3 & - & 4x_4 \\
w_2 & = & & - & 0.5x_1 & + & x_2 & + & 0.5x_3 & - & 0.5x_4 \\
w_3 & = & 1 & - & x_1 & & & & & &
\end{array}
$$

Write down the corresponding linear programming problem and find the optimal solution by using

a) the lexicographic method (comment on the choice of the entering variable).

b) Bland's rule

maximize $z = x_1 - 2x_2 - 2x_4$

subject to
$$0.5x_1 - 3.5x_2 - 2x_3 + 4x_4 \leq 0$$
$$0.5x_1 - x_2 - 0.5x_3 + 0.5x_4 \leq 0$$
$$x_1 \leq 1$$
$$x_1, x_2, x_3, x_4 \geq 0$$

a)
$$z = x_1 - 2x_2 - 2x_4$$
$$w_1 = \epsilon_1 - 0.5x_1 + 3.5x_2 + 2x_3 - 4x_4$$
$$w_2 = \epsilon_2 - 0.5x_1 + x_2 + 0.5x_3 - 0.5x_4$$
$$w_3 = 1 + \epsilon_3 - x_1$$

$\epsilon_1 \gg \epsilon_2 \gg \epsilon_3$

$$z = 2\epsilon_2 + x_3 - 3x_4 - 2w_2$$
$$x_1 = 2\epsilon_2 + 2x_2 + x_3 - x_4 - 2w_2$$
$$w_1 = \epsilon_1 - \epsilon_2 + 2.5x_2 + 1.5x_3 - 3.5x_4 + w_2$$
$$w_3 = 1 + \epsilon_3 - 2\epsilon_2 - 2x_2 - x_3 + x_4 + 2w_2$$

$$z = 1 + \epsilon_3 - 2x_2 - 2x_4 - w_3$$
$$x_3 = 1 + \epsilon_3 - 2\epsilon_2 - 2x_2 + x_4 + 2w_2 - w_3$$
$$x_1 = 1 + \epsilon_3 - w_3$$
$$w_1 = 1.5 + \epsilon_1 - 4\epsilon_2 + 1.5\epsilon_3 - 0.5x_2 - 2x_4 + 4w_2 - 1.5w_3$$

This dictionary is optimal. Drop ϵ_i's to get the get the optimal solution to the unperturbed problem:

maximum z is $z=1$ at $(x_1, x_2, x_3, x_4) = (1,0,1,0)$

b)
$$\begin{aligned}
z &= x_1 - 2x_2 - 2x_4\\
w_1 &= -0.5x_1 + 3.5x_2 + 2x_3 - 4x_4\\
w_2 &= -0.5x_1 + x_2 + 0.5x_3 - 0.5x_4\\
w_3 &= 1 - x_1
\end{aligned}$$

$$\begin{aligned}
z &= 5x_2 + 4x_3 - 10x_4 - 2w_1\\
x_1 &= 7x_2 + 4x_3 - 8x_4 - 2w_1\\
w_2 &= -2.5x_2 - 1.5x_3 + 3.5x_4 + w_1\\
w_3 &= 1 - 7x_2 - 4x_3 + 8x_4 + 2w_1
\end{aligned}$$

$$\begin{aligned}
z &= x_3 - 3x_4 - 2w_2\\
x_2 &= -0.6x_3 + 1.4x_4 + 0.4w_1 - 0.4w_2\\
x_1 &= -0.2x_3 + 1.8x_4 + 0.8w_1 - 2.8w_2\\
w_3 &= 1 + 0.2x_3 - 1.8x_4 - 0.8w_1 + 2.8w_2
\end{aligned}$$

$$\begin{aligned}
z &= -5x_1 + 6x_4 + 4w_1 - 16w_2\\
x_3 &= -5x_1 + 9x_4 + 4w_1 - 14w_2\\
x_2 &= 3x_1 - 4x_4 - 2w_1 + 8w_2\\
w_3 &= 1 - x_1
\end{aligned}$$

$$\begin{aligned}
z &= -\tfrac{1}{2}x_1 - \tfrac{3}{2}x_2 + w_1 - 4w_2\\
x_4 &= \tfrac{3}{4}x_1 - \tfrac{1}{4}x_2 - \tfrac{1}{2}w_1 + 2w_2\\
x_3 &= \tfrac{7}{4}x_1 - \tfrac{9}{4}x_2 - \tfrac{1}{2}w_1 + 4w_2\\
w_3 &= 1 - x_1
\end{aligned}$$

$$z = 3x_1 - 6x_2 - 2x_3 + 4w_2$$
$$w_1 = 7/2\,x_1 - 9/2\,x_4 - 2x_3 + 8w_2$$
$$x_4 = -x_1 + 2x_2 + x_3 - 2w_2$$
$$w_3 = 1 - x_1$$

$$z = x_3 - 3x_4 - 2w_2$$
$$x_1 = 2x_2 + x_3 - x_4 - 2w_2$$
$$w_1 = 5/2\,x_2 + 3/2\,x_3 - 7/2\,x_4 + w_2$$
$$w_3 = 1 - 2x_2 - x_3 + x_4 + 2w_2$$

$$z = 1 - 2x_2 - 2x_4 - w_3$$
$$x_3 = 1 - 2x_2 + x_4 + 2w_2 - w_3$$
$$x_1 = 1$$
$$w_1 = 3/2 - 1/2\,x_2 - 2x_4 + 4w_2 - 3/2\,w_3$$

Optimal solution $z = 1$ at
$(x_1, x_2, x_3, x_4) = (1, 0, 1, 0)$