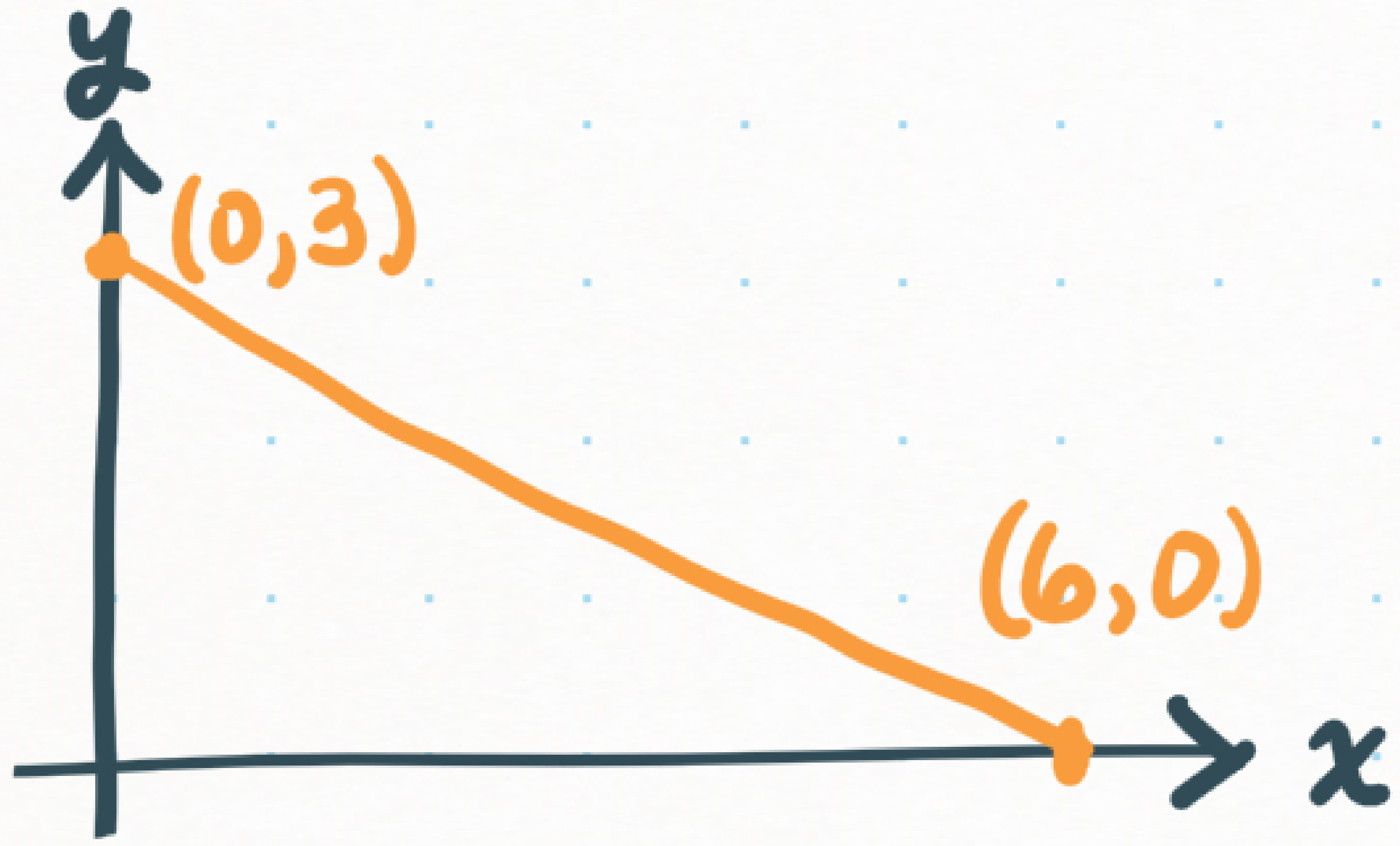
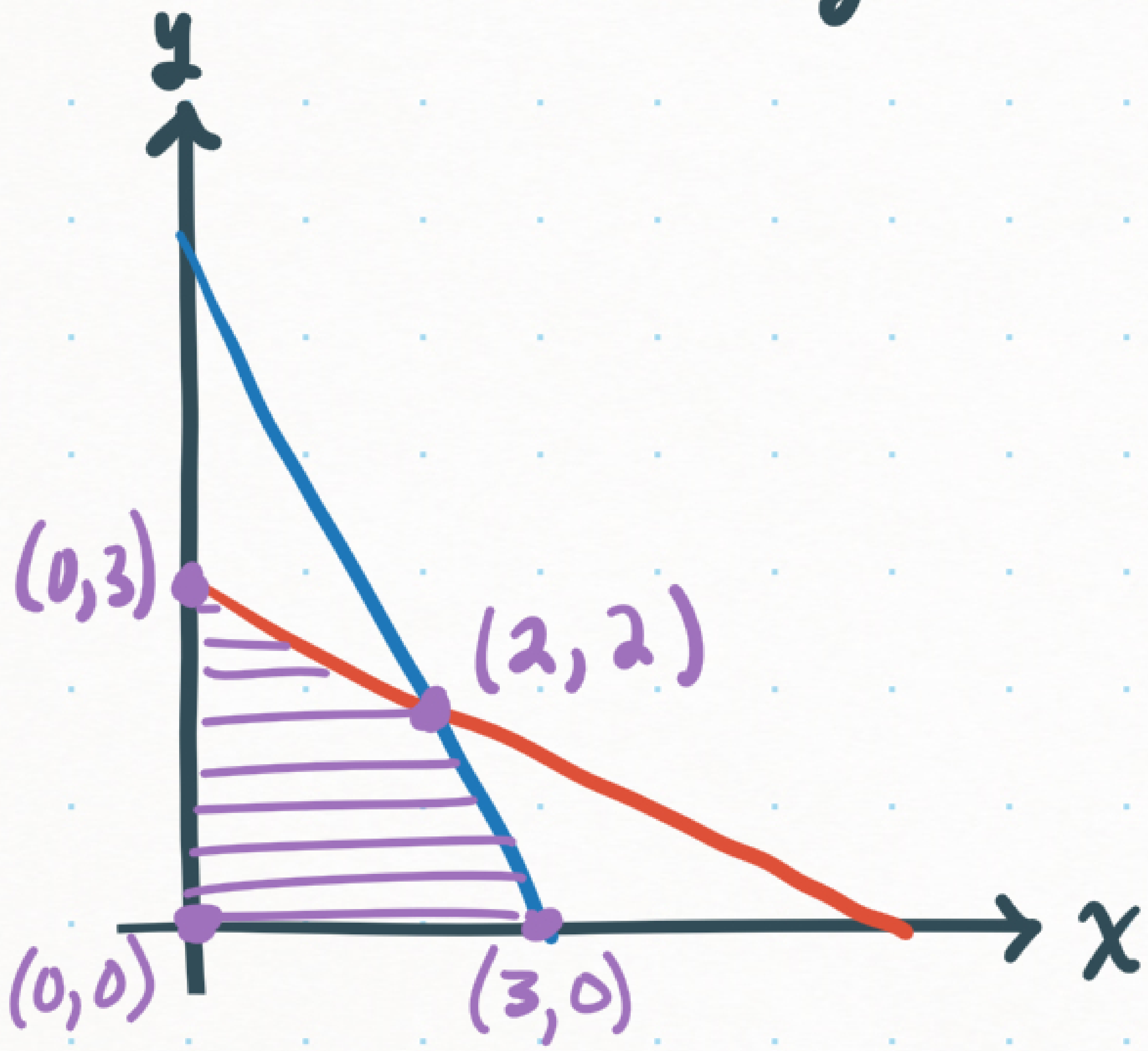
Introduction to Linear Algebra 4th Edition, Strang
Problem Set 8.4

1. Draw the region in the xy plane where $x+2y=6$ and $x \geq 0$ and $y \geq 0$. Which point in this feasible minimizes the cost $c = x+3y$? Which point gives maximum cost? Those points are at corners.



At $(0,3)$, $c=9$ (maximum cost)
At $(6,0)$, $c=6$ (minimum cost)

2. Draw the region in the xy plane where $x+2y\le 6$, $2x+y\le 6$, $x\ge 0$, and $y\ge 0$. Which corner minimizes the cost $c=2x-y$?



$c(0,0)=0$
$c(0,3)=-3$
$c(2,2)=2$
$c(3,0)=6$

$(0,3)$ minimizes $c(x,y)$ and $c^*=-3$.

Note: The solution given by the textbook states that the feasibility region includes the corners $(0,0)$, $(2,2)$, $(0,6)$ and $(6,0)$ with c minized at $(6,0)$. This is clearly incorrect since $2\cdot 6+0=12 \nleq 6$ so this is not a valid cost and is also not even the minimum among those 4 suggested points.

3. What are the corners of the set $x_1 + 2x_2 - x_3 = 4$ with $x_1, x_2, x_3 \geq 0$? Show that the cost $c = x_1 + 2x_3$ can be very positive[†] in this feasible set.

There are 2 bounded corners $(4,0,0)$ and $(0,2,0)$. Since $-x_3 = 4$ cannot be satisfied with $x_3 \geq 0$, the third 'corner' takes $x_3 \to \infty$. For any $M > 4$ we can make $c = M$ by taking $x_3 = (M-4)/3$, $x_2 = 0$, $x_1 = 4 + (M-4)/3$.

Then $x_1, x_2, x_3 \geq 0$, $c = 4 + (M-4)/3 + 2(M-4)/3 = M$ and $x_1 + 2x_2 - x_3 = 4 + (M-4)/3 + 0 - (M-4)/3 = 4$. Since $M > 4$ was arbitrary, c is unbounded above in the feasibility region.

† The textbook has 'negative' here instead of 'positive'. But since $x_1, x_3 \geq 0$, $x_1 + 2x_3 < 0$ is clearly impossible. I made this change so that the question could make sense. One could also make a change like $c = x_1 - 2x_3$ and keep the word 'negative'.

4. Start at $x=(0,0,2)$ where the machine solves all 4 problems for \$16. Move to $x=(0,1,?)$ to find the reduced cost r (savings per hour) for work by the student. Find r for the PhD by moving to $x=(1,0,?)$ with 1 hour of PhD work.

The feasibility region is $x_1+x_2+2x_3=4$ with $x_1, x_2, x_3 \geq 0$

The cost is $c=5x_1+3x_2+8x_3$.

Moving to 1 hr of student work makes $x=(0,1,3/2)$ and $c=\$15$, so $r=-\$1$.

Moving to 1 hr of PhD work makes $x=(1,0,3/2)$ and $c=\$17$, so $r=\$1$ (cost increases).

5. Start Example 1 from the PhD corner $(4,0,0)$ with c changed to $c=[5,3,7]$. Show that r is better for the machine even when the total cost is lower for the student.

From $x=(4,0,0)$ to $x=(2,0,1)$ makes the cost \$17 and $r=-\$3$. From $(4,0,0)$ to $(3,1,0)$ makes the cost \$18 and $r=\$-2$. So for the first step we choose $x=(2,0,1)$. Then from $x=(2,0,1)$ to $x=(0,0,2)$ makes the cost \$14 so $r=-\$3$. From $x=(2,0,1)$ to $x=(1,1,1)$ makes the cost \$15 so $r=-\$2$. So for step 2 choose $x=(0,0,2)$. From $x=(0,0,2)$ to $x=(0,1,3/2)$ makes the cost \$13.5 and $r=-\$0.50$ so we would choose $x=(0,1,3/2)$. Then from $x=(0,1,3/2)$ to $x=(0,2,1)$ costs \$13 and $r=-\$0.50$. So we choose $x=(0,2,1)$. Two more steps just like this give $x^*=(0,4,0)$. Even though the optimal solution is to have the student do all the work, the simplex method chooses the machine on the first step since r is most negative by doing so at that point.

8. These problems are also dual. Prove weak duality, that $y^Tb \le c^Tx$ for

Primal Problem: Minimize c^Tx with $Ax \geq b$ and $x \geq 0$.

Dual Problem: Maximize y^Tb with $A^Ty \le c$ and $y \geq 0$.

Using the stated inequalities we have:

$$y^Tb \le y^TAx = (A^Ty)^T x \le c^Tx.$$