2. Solve the following linear program using Bland's rule to resolve degeneracy:

$$\max z = 10x_1 - 57x_2 - 9x_3 - 24x_4$$

subject to

$$0.5x_1 - 5.5x_2 - 2.5x_3 + 9x_4 \leq 0$$
$$0.5x_1 - 1.5x_2 - 0.5x_3 + x_4 \leq 0$$
$$x_1 \leq 1$$
$$x_1, x_2, x_3, x_4 \geq 0.$$

$$z = 10x_1 - 57x_2 - 9x_3 - 24x_4$$
$$w_1 = -0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4$$
$$w_2 = -0.5x_1 + 1.5x_2 + 0.5x_3 - x_4$$
$$w_3 = 1 - x_1$$

Let x_1 enter, w_1 leave.

$$z = 53x_2 + 41x_3 - 204x_4 - 20w_1$$
$$x_1 = 11x_2 + 5x_3 - 18x_4 - 2w_1$$
$$w_2 = -4x_2 - 2x_3 + 8x_4 + w_1$$
$$w_3 = 1 - 11x_2 - 5x_3 + 18x_4 + 2w_1$$

Let x_2 enter, w_2 leave.

$$z = 29/2\, x_3 - 98x_4 - 27/4\, w_1 - 53/4\, w_2$$
$$x_2 = -1/2\, x_3 + 2x_4 + 1/4\, w_1 - 1/4\, w_2$$
$$x_1 = -1/2\, x_3 + 4x_4 + 3/4\, w_1 - 11/4\, w_2$$
$$w_3 = 1 + 1/2\, x_3 - 4x_4 - 3/4\, w_1 + 11/4\, w_2$$

Let x_3 enter, x_1 leave.

$$z = -29x_1 + 18x_4 + 15w_1 - 93w_2$$
$$x_3 = -2x_1 + 8x_4 + 3/2\, w_1 - 11/2\, w_2$$
$$x_2 = x_1 - 2x_4 - 1/2\, w_1 + 5/2\, w_2$$
$$w_3 = 1 - x_1$$

Let x_4 enter, x_2 leave.

$$z = -20x_1 - 9x_2 + 21/2\, w_1 - 141/2\, w_2$$
$$x_4 = 1/2\, x_1 - 1/2\, x_2 - 1/4\, w_1 + 5/4\, w_2$$
$$x_3 = 2x_1 - 4x_2 - 1/2\, w_1 + 9/2\, w_2$$
$$w_3 = 1 - x_1$$

Let w_1 enter, x_3 leave.

$$z = 22x_1 - 93x_2 - 21x_3 + 24w_2$$
$$w_1 = 4x_1 - 8x_2 - 2x_3 + 9w_2$$
$$x_4 = -\tfrac{1}{2}x_1 + \tfrac{3}{2}x_2 + \tfrac{1}{2}x_3 - w_2$$
$$w_3 = 1 - x_1$$

Let x_1 enter, x_4 leave.

$$z = -27x_2 + x_3 - 44x_4 - 20w_2$$
$$x_1 = 3x_2 + x_3 - 2x_4 - 2w_2$$
$$w_1 = 4x_2 + 2x_3 - 8x_4 + w_2$$
$$w_3 = 1 - 3x_2 - x_3 + 2x_4 + 2w_2$$

Let x_3 enter, w_3 leave.

$$z = 1 - 30x_2 - 42x_4 - 18w_2 - w_3$$
$$x_3 = 1 - 3x_2 + 2x_4 + 2w_2 - w_3$$
$$x_1 = 1 - w_3$$
$$w_1 = 2 - 2x_2 - 4x_4 + 5w_2 - 2w_3$$

The maximum value of z subject to the given constraints is $z = 1$ at $(x_1, x_2, x_3, x_4) = (1, 0, 1, 0)$.

3. Consider the linear program

$\max z = 3x_1 + 5x_2$
subject to
$x_1 + 2x_2 \leq 5$
$x_1 \leq 3$
$x_2 \leq 2$
$x_1,\ x_2 \geq 0$

Compare the efficiency of the implementation of the simplex algorithm if

a) You are using the largest positive coefficient for the entering variable (and the lexicographic method for the leaving)

b) You are using Bland's rule

a) maximize $z = 3x_1 + 5x_2$

subject to $w_1 = 5 + \epsilon_1 - x_1 - 2x_2$

$w_2 = 3 + \epsilon_2 - x_1$

$w_3 = 2 + \epsilon_3 - x_2$

$\epsilon_1 >> \epsilon_2 >> \epsilon_3 >> 0$

Let x_2 enter, w_3 leave.

$z = 10 + 5\epsilon_3 + 3x_1 - 5w_3$
$x_2 = 2 + \epsilon_3 - w_3$
$w_1 = 1 + \epsilon_1 - 2\epsilon_3 - x_1 + 2w_3$
$w_2 = 3 + \epsilon_2 - x_1$

Let x_1 enter, w_1 leave.

$z = 13 + 3\epsilon_1 - \epsilon_3 - 3w_1 + w_3$
$x_1 = 1 + \epsilon_1 - 2\epsilon_3 - w_1 + 2w_3$
$x_2 = 2 + \epsilon_3 - w_3$
$w_2 = 2 - \epsilon_1 + \epsilon_2 + 2\epsilon_3 + w_1 - 2w_3$

Let w_3 enter, w_2 leave.

$$z = 14 + \frac{5}{2}\epsilon_1 + \frac{1}{2}\epsilon_2 - \frac{5}{2}w_1 - \frac{1}{2}w_2$$
$$w_3 = 1 - \frac{1}{2}\epsilon_1 + \frac{1}{2}\epsilon_2 + \epsilon_3 + \frac{1}{2}w_1 - \frac{1}{2}w_2$$
$$x_1 = 3 + \epsilon_2 - w_2$$
$$x_2 = 1 + \frac{1}{2}\epsilon_1 - \frac{1}{2}\epsilon_2 - \frac{1}{2}w_1 + \frac{1}{2}w_2$$

This dictionary is optimal. Drop all ϵ_i's to find solution to the unperturbed problem. Maximum z is $z = 14$ at $(3,1)$. This method took 3 simplex steps.

b)
$$z = 3x_1 + 5x_2$$
$$w_1 = 5 - x_1 - 2x_2$$
$$w_2 = 3 - x_1$$
$$w_3 = 2 - x_2$$

Let x_1 enter, w_2 leave

$$z = 9 + 5x_2 - 3w_2$$
$$x_1 = 3 - w_2$$
$$w_1 = 2 - 2x_2 + w_2$$
$$w_3 = 2 - x_2$$

Let x_2 enter, w_1 leave

$$z = 14 - \frac{5}{2}w_1 - \frac{1}{2}w_2$$
$$x_2 = 1 - \frac{1}{2}w_1 + \frac{1}{2}w_2$$
$$x_1 = 3 - w_2$$
$$w_3 = 1 + \frac{1}{2}w_1 - \frac{1}{2}w_2$$

This dictionary is optimal. The maximum value of z is $z = 14$ at $(3,1)$. Bland's rule took only 2 simplex steps.

4. Consider the following linear programming problem:

$\max z = 2x_1 - 3x_2 + 2x_3 + 12x_4$
subject to
$4x_1 + 5x_2 + 2x_3 \leq 10$
$2x_1 - x_3 + x_4 \leq 30$
$4x_2 + 2x_3 + x_4 \leq 20$
$x_1,\ x_2,\ x_3,\ x_4 \geq 0$

Find the dual program to this linear program.

Given a linear programming problem in standard form,

$$\begin{aligned} &\text{maximize} && \sum_{j=1}^{n} c_j x_j && \\ &\text{subject to} && \sum_{j=1}^{n} a_{ij} x_j \leq b_i && i = 1, 2, \ldots, m \\ & && x_j \geq 0 && j = 1, 2, \ldots, n, \end{aligned} \tag{5.1}$$

the associated *dual linear program* is given by

$$\begin{aligned} &\text{minimize} && \sum_{i=1}^{m} b_i y_i && \\ &\text{subject to} && \sum_{i=1}^{m} y_i a_{ij} \geq c_j && j = 1, 2, \ldots, n \\ & && y_i \geq 0 && i = 1, 2, \ldots, m. \end{aligned}$$

minimize $W = 10y_1 + 30y_2 + 20y_3$

subject to $4y_1 + 2y_2 \geqslant 2$

$5y_1 + 4y_3 \geqslant -3$

$2y_1 - y_2 + 2y_3 \geqslant 2$

$y_2 + y_3 \geqslant 12$

$y_1, y_2, y_3 \geqslant 0$

5. Consider the following dual linear programming problem:

$\min \xi = 3y_1 + y_2 + 2y_3$
subject to
$2y_1 + 3y_2 - y_3 \geq 5$
$-y_1 + 4y_3 \geq 10$
$y_1 + 2y_2 \geq 7$
$3y_1 - 2y_3 \geq 7$
$y_1, y_2, y_3 \geq 0$

a) Rewrite the problem in standard form.

b) What is the primal problem corresponding to this dual linear program.

a) maximize $\xi' = -3y_1 - y_2 - 2y_3$

subject to $-2y_1 - 3y_2 + y_3 \leq -5$

$(\xi' = -\xi)$ $y_1 - 4y_3 \leq -10$

$-y_1 - 2y_2 \leq -7$

$-3y_1 + 2y_3 \leq -7$

$y_1, y_2, y_3 \geq 0$

b) maximize $z = 5x_1 + 10x_2 + 7x_3 + 7x_4$

subject to $2x_1 - x_2 + x_3 + 3x_4 \leq 3$

$3x_1 + 2x_3 \leq 1$

$-x_1 + 4x_2 - 2x_4 \leq 2$

$x_1, x_2, x_3, x_4 \geq 0$