MA 3475, Homework 3 due midnight 02/21/2021

Show your work as supported answers may receive no credit.

1. (5 points) (Brachistochrone problem) In Lecture 11, we derived the functional

$$J[y] = \int_0^a \frac{\sqrt{1+(y')^2}}{\sqrt{2g(b-y)}}\,dx$$

which represents the time it takes for a particle traveling on the graph of the function $y(x)$, starting from the point $(0,b)$ to arrive at the point $(a,0)$ under the gravitational force. (a) Using the Beltrami's identity, find the Euler-Lagrange equation of this functional, (b) Show that

$$\begin{aligned} x(\theta) &= \frac{C}{2}(\theta - \sin\theta), \\ y(\theta) &= b - \frac{C}{2}(1-\cos\theta) \end{aligned}$$

solves your Euler-Lagrange equation. Note that I am not asking you to solve the Euler-Lagrange equation, but just to show that the above is a solution. The function $(x(\theta), y(\theta))$ is a parametric curve in the plane for $\theta \in [0, \hat{\theta}]$. Note that $(x(0), y(0)) = (0, b)$ is automatically satisfied. Use the boundary condition at the other endpoint to obtain the conditions: $x(\hat{\theta}) = a, y(\hat{\theta}) = 0$ to find the constants C and $\hat{\theta}$. You don't need to do this but assume this is done numerically.

2. (5 points) Which curve minimizes the integral

$$\int_0^1 \frac{1}{2}(y')^2 + yy' + y' + y\,dx\,.$$

The values of $y(x)$ are not specified at the endpoints.

3. (Bonus, 5 points) A river with parallel straight banks b units apart has stream velocity given by $v(x,y) = v(x)\mathbf{j}$, where $\mathbf{j}$ is the unit vector in the y direction. Assume that one side of the bank is the y-axis ($x = 0$) and the point $(0,0)$ is the point of departure. What route should a boat take to reach the opposite bank ($x = b$) in the shortest possible time? Assume that the speed of the boat in still water is c, where $c > v$. (Hint.

$$J[y] = \int_0^b \frac{\sqrt{c^2(1+(y')^2) - v^2} - vy'}{c^2 - v^2}\,dx$$

I will derive this functional in class.)

4. (5 points) Find the extremal(s) of the functional

$$J[y] = \int_0^b (\sqrt{1 - k^2 + (y')^2} - ky')\, dx$$

in the class of $\mathcal{D}_2[0, b]$ functions with $y(0) = 0$ and $y(b)$ free. Here, $0 < k < 1$.

5. (Bonus, 5 points) Find the extremal of the functional

$$J[y] = \int \sqrt{x^2 + y^2}\,\sqrt{1 + (y')^2}\, dx$$

(Hint. Use polar coordinates.)