MA 3475 HW 1 Solutions

February 21, 2021

Problem 1

$$
\begin{aligned}
f(x,y) &= 3y^2 - 2y^3 - 3x^2 + 6xy \\
f_x(x,y) &= -6x + 6y \quad f_{xx}(x,y) = -6 \\
f_y(x,y) &= 6y - 6y^2 + 6x \quad f_{yy}(x,y) = 6 - 12y \\
f_{xy} &= 6 \\
\text{To find critical points set } 0 &= f_{xx} = f_{yy} \\
0 &= -6x + 6y \\
0 &= 6y - 6y^2 + 6x \\
\implies \text{ critical points: } &(0,0), (2,2)
\end{aligned}
$$

Use $D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2$ in the second derivative test.

Since $D(0,0) = -72 < 0$, the critical point $(0,0)$ is a saddle point.

Since $D(2,2) = 72 > 0$ and $f_{xx}(2,2) = -6 < 0$, there is a local maximum of $8 = f(2,2)$ at the critical point $(2,2)$.

Problem 2

$$
\begin{aligned}
&\min_{(x,y)} f(x,y) = x^2y \quad \text{s.t} \quad g(x,y) = x^2 + y^2 - 3 = 0 \\
&\max_{(x,y)} f(x,y) = x^2y \quad \text{s.t} \quad g(x,y) = x^2 + y^2 - 3 = 0
\end{aligned}
$$

$$
\begin{aligned}
\nabla f &= \langle 2xy, x^2 \rangle \\
\nabla g &= \langle 2x, 2y \rangle \\
\nabla f &= \lambda \nabla g \\
\implies xy &= \lambda x \text{ and } x^2 = 2\lambda y
\end{aligned}
$$

If $x = 0$ then $y = 0$ and vice versa. So one solution to this system of equations is $(x,y) = (0,0)$. Otherwise if $(x,y) \neq (0,0)$,

$$
\begin{aligned}
\frac{xy}{x} &= \lambda = \frac{x^2}{2y} \\
y &= \frac{x^2}{2y} \\
2y^2 &= x^2 = 3 - y^2 \quad \text{(using constraint)} \\
y^2 &= 1 \\
y &= \pm 1 \\
\implies (x,y) &= (\pm\sqrt{2}, \pm 1)
\end{aligned}
$$

$$
\begin{aligned}
f(\pm\sqrt{2},1) &= 2 \quad \textbf{Maximum}\\
f(\pm\sqrt{2},-1) &= -2 \quad \textbf{Minimum}\\
f(0,0) &= 0 \quad \textbf{Neither maximum nor minimum}
\end{aligned}
$$

Problem 3

$$
\begin{aligned}
\min_{n_h} f(n_1,...,n_H) &= \min_{n_h} \sum_{h=1}^{H} \left(\frac{S_h}{n_h} - \frac{1}{N_h}\right)\left(\frac{N_h}{N}\right)^2\\
\text{subject to} \quad 0 = g(n_1,...,n_H) &= -n + \sum_{h=1}^{H} n_h\\
\text{with} \quad N = N_1 + ... + N_H, &\quad H, N_h, N, S_h, n \text{ constant and } n < N.
\end{aligned}
$$

$$
\begin{aligned}
\nabla f &= \left\langle -\frac{N_1^2 S_1}{N^2 n_1^2}, ..., -\frac{N_H^2 S_H}{N^2 n_H^2} \right\rangle\\
\lambda \nabla g &= \langle \lambda, ..., \lambda \rangle\\
\nabla f = \lambda \nabla g &\implies \frac{N_h^2 S_h}{N^2 n_h^2} = -\lambda, h = 1, ..., H\\
n_h &= \frac{N_h \sqrt{S_h}}{\sqrt{-\lambda} N}\\
n = \sum_{h=1}^{H} n_h &= \frac{1}{\sqrt{-\lambda} N} \sum_{h=1}^{H} N_h \sqrt{S_h}\\
\frac{1}{\sqrt{-\lambda}} &= \frac{nN}{\sum_{h=1}^{H} N_h \sqrt{S_h}}\\
\therefore \quad n_h &= \frac{n N_h \sqrt{S_h}}{\sum_{h=1}^{H} N_h \sqrt{S_h}}
\end{aligned}
$$

If we make the further assumption that $S_h = S$ for each $h = 1, ..., H$, then

$$
n_h = \frac{n N_h \sqrt{S}}{\sqrt{S} \sum_{h=1}^{H} N_h} = n \frac{N_h}{N} = n p_h \ ,
$$

where $p_h = N_h / N$. Then,

$$
\sum_{h=1}^{H} p_h = \frac{1}{N} \sum_{h=1}^{H} N_h = N/N = 1 \ .
$$