MA 3475 Exam II, 3/18/2021 NAME ____________________

1. (5 points) Find the extremal of the functional

$$J[y,z] = \int_0^{\pi/2} \left((y')^2 + (z')^2 + 2yz\right)\, dx$$

subject to the boundary conditions $y(0) = 1, z(0) = 1, y\left(\frac{\pi}{2}\right) = 0, z\left(\frac{\pi}{2}\right) = 0$.

Answer

Set up and solve the system of Euler-Lagrange equations.

$$\begin{aligned}
0 &= F_y - \frac{d}{dx}F_{y'} = 2z - \frac{d}{dx}[2y'] = 2z - 2y'' \\
0 &= F_z - \frac{d}{dx}F_{z'} = 2y - \frac{d}{dx}[2z'] = 2y - 2z''
\end{aligned}$$

$$\begin{aligned}
0 &= 2z - 2y'' \\
0 &= 2y - 2z''
\end{aligned}$$

$$\begin{aligned}
0 &= 2z - 2y'' \implies y^{(iv)} = z'' \\
\text{Then, } 0 &= 2y - 2z'' = 2y - 2(y^{(iv)}) = 2y - 2y^{(iv)}
\end{aligned}$$

$$\begin{aligned}
0 &= 2r^4 - 2 \quad \text{(characteristic equation)} \\
0 &= r^4 - 1 = (r^2 - 1)(r^2 + 1) \\
r &= \pm 1, \pm 1i
\end{aligned}$$

$$y(x) = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$$

We can use a similar process to solve for $z(x)$ (the differential equation is $z^{(iv)} - z = 0$) but since we want to solve for the constants instead use that $z = y''$ to express $z(x)$ using the same constants as we used to express $y(x)$.

$$z(x) = y''(x) = c_1 e^x + c_2 e^{-x} - c_3 \cos x - c_4 \sin x$$

Apply the boundary conditions.

$$
\begin{aligned}
1 = y(0) &= c_1 + c_2 + c_3 \quad (1) \\
1 = z(0) &= c_1 + c_2 - c_3 \quad (2) \\
0 = y\left(\frac{\pi}{2}\right) &= c_1 \exp\left(\frac{\pi}{2}\right) + c_2 \exp\left(-\frac{\pi}{2}\right) + c_4 \quad (3) \\
0 = z\left(\frac{\pi}{2}\right) &= c_1 \exp\left(\frac{\pi}{2}\right) + c_2 \exp\left(-\frac{\pi}{2}\right) - c_4 \quad (4)
\end{aligned}
$$

$$
\begin{aligned}
(1), (2) &\implies c_3 = 0 \\
(3), (4) &\implies c_4 = 0 \\
(1) + (2) &\implies c_1 + c_2 = 1 \quad (5) \\
(3) + (4) &\implies -c_1 e^{\pi} = c_2 \quad (6) \\
(5), (6) &\implies c_1 = \frac{1}{1 - e^{\pi}}, c_2 = \frac{e^{\pi}}{e^{\pi} - 1}
\end{aligned}
$$

$$
\boxed{y(x) = \frac{1}{1 - e^{\pi}} e^{x} + \frac{e^{\pi}}{e^{\pi} - 1} e^{-x}}
$$

$$
\boxed{z(x) = \frac{1}{1 - e^{\pi}} e^{x} + \frac{e^{\pi}}{e^{\pi} - 1} e^{-x}}
$$

2. (5 points) Find the extremal of the functional

$$J[y] = \int_a^b \left((y')^2 + (y'')^2\right)\, dx$$

How many boundary conditions do you need to specify at the two endpoints to determine the extremal uniquely?

Answer

Set up and solve the Euler-Lagrange equation.

$$\begin{aligned} 0 &= F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} \\ &= 0 - \frac{d}{dx}[2y'] + \frac{d^2}{dx^2}[2y''] \\ &= -2y'' + 2y^{(iv)} \end{aligned}$$

$$\begin{aligned} 0 &= -2r^2 + 2r^4 \\ 0 &= r^4 - r^2 = r^2(r^2 - 1) \\ r_1, r_2 &= 0, r_3 = 1, r_4 = -1 \end{aligned}$$

$$y(x) = c_0 e^{0x} + c_1 x e^{0x} + c_2 e^{x} + c_3 e^{-x}$$

$$\boxed{y(x) = c_0 + c_1 x + c_2 e^{x} + c_3 e^{-x}}$$

You'll need to specify 4 boundary conditions at the endpoints to determine the four constants c_0, c_1, c_2, and c_3 uniquely.

3. (5 points) Find the extremal of the functional

$$J[y] = \int_0^1 (y')^2 \, dx$$

subject to the conditions $y(0) = 0, y(1) = 0$ and the constraint $\int_0^1 y(x)\, dx = A$, where A is a given constant. You need to find the Lagrange multiplier as a function of A and cannot leave your answer in terms of the Lagrange multiplier.

Answer

Let $G(x, y, y') = y(x)$ so that $A = \int_0^1 y(x) \; dx = \int_0^1 G(x, y, y') \; dx$. First set up and solve the Euler-Lagrange equation for $y(x)$.

$$\begin{aligned} 0 &= F_y - \frac{d}{dx} F_{y'} + \lambda(G_y - \frac{d}{dx} G_{y'}) \\ &= 0 - \frac{d}{dx}[2y'] + \lambda(1 - \frac{d}{dx}[0]) = \lambda - 2y'' \end{aligned}$$

$$y''(x) = \frac{\lambda}{2} \implies y(x) = \frac{\lambda}{4}x^2 + c_1 x + c_2$$

Next apply the boundary conditions $y(0) = 0$ and $y(1) = 0$.

$$\begin{aligned} 0 &= y(0) = c_2 \\ 0 &= y(1) = \frac{\lambda}{4} + c_1 \implies c_1 = -\frac{\lambda}{4} \end{aligned}$$

$$y(x) = \frac{\lambda}{4}x^2 - \frac{\lambda}{4}x$$

Finally, apply the integral constraint to determine λ in terms of the given constant A.

$$\begin{aligned} A &= \int_0^1 \left(\frac{\lambda}{4}x^2 - \frac{\lambda}{4}x \right) \; dx \\ &= \frac{\lambda}{12} - \frac{\lambda}{8} = -\frac{1}{24}\lambda \\ \lambda &= -24A \end{aligned}$$

$$\boxed{y(x) = -6Ax^2 + 6Ax}$$