MA 3475 HW 5 Solutions

March 16, 2021

Problem 1 (4 points)

$$
\begin{aligned}
0 &= F_y - \frac{d}{dx}F_{y'} + \lambda(G_y - \frac{d}{dx}G_{y'}) \\
&= 0 - \frac{d}{dx}[2y'] + \lambda(2y - \frac{d}{dx}[0]) \\
&= -2y'' + 2\lambda y
\end{aligned}
$$

$$
\begin{aligned}
0 &= -2r^2 + 2\lambda \\
r^2 &= \pm\sqrt{\lambda} \\
y(x) &= c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}
\end{aligned}
$$

Apply the boundary condition $y(0) = 0$ to get $c_2 = -c_1$. Then $y(x) = c_1(e^{\sqrt{\lambda}x} - e^{-\sqrt{\lambda}x})$. Next consider the other boundary condition $0 = y(1) = c_1(e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}})$. If $\lambda \geq 0$, the boundary condition implies either $c_1 = 0$ of $\lambda = 0$. In either case this means $y \equiv 0$. But $y \equiv 0$ does not satisfy the constraint $\int_0^1 y^2\, dx = 2$. Therefore, we must have $\lambda < 0$.

$$
\begin{aligned}
y(x) &= c_1\left(e^{\sqrt{\lambda}x} - e^{-\sqrt{\lambda}x}\right) \\
&= c_1\left(e^{\sqrt{-\lambda}ix} - e^{-\sqrt{-\lambda}ix}\right) \\
&= c_1\left[(\cos\sqrt{-\lambda}x + i\sin\sqrt{-\lambda}x) - (\cos-\sqrt{-\lambda}x + i\sin-\sqrt{-\lambda}x)\right] \\
&= c_1\left[(\cos\sqrt{-\lambda}x + i\sin\sqrt{-\lambda}x) - (\cos\sqrt{-\lambda}x - i\sin\sqrt{-\lambda}x)\right] \\
&= 2c_1 i\sin x \\
&= \beta\sin\sqrt{-\lambda}x, \quad \beta := 2c_1 i, \\
0 = y(1) &= \beta\sin\sqrt{-\lambda} \implies \sqrt{-\lambda} = \pi n, n \in \mathbb{Z} \\
y(x) &= \beta\sin n\pi x
\end{aligned}
$$

If $n = 0$, we again have $y \equiv 0$, so we must exclude $n = 0$. Apply the constraint to determine β.

$$
2 = \beta^2\int_0^1 \sin^2(n\pi x)\, dx = \beta^2\left[\frac{x}{2} - \frac{1}{4\pi n}\sin 2\pi n x\right]\Bigg|_0^1 = \beta^2/2 \implies \beta = \pm 2\ .
$$

So we have $y(x) = \pm 2\sin n\pi x$, $n \in \mathbb{Z}\backslash\{0\}$. Is $J[y]$ extremized for any choice of $n \in \mathbb{Z}\backslash\{0\}$? Use $y'(x) = \pm 2n\pi\cos(n\pi x)$ in order to evaluate $J[y]$.

$$
J[y] = \int_0^1 \left[(y')^2 + x^2\right]\, dx = \int_0^1 \left[4n^2\pi^2\cos^2(n\pi x) + x^2\right]\, dx = 2n^2\pi^2 + \frac{1}{3}\ .
$$

There is no n that maximizes the functional. The functional is minimized by taking $n = \pm 1$ (since $n = 0$ is not allowed).

$$
\boxed{y(x) = \pm 2\sin(\pm\pi x)}
$$

Problem 4 (4 points)

$$J[y] = \int_0^a 2\pi y\sqrt{1+(y')^2}\;dx \quad \text{subject to} \quad \int_0^a y\;dx = S\;.$$

$$\begin{aligned}
c &= F - y'F_{y'} + \lambda(G - y'G_{y'}) \quad \text{((The Beltrami Identity)} \\
&= 2\pi y\sqrt{1+(y')^2} - y'\frac{2\pi y y'}{\sqrt{1+(y')^2}} + \lambda(y - y'\cdot 0) \\
&= 2\pi y\sqrt{1+(y')^2} - \frac{2\pi y (y')^2}{\sqrt{1+(y')^2}} + \lambda y \\
&= 2\pi y\left(\sqrt{1+(y')^2} - \frac{(y')^2}{\sqrt{1+(y')^2}}\right) + \lambda y \\
&= \frac{2\pi y}{\sqrt{1+(y')^2}} + \lambda y
\end{aligned}$$

$$\frac{c-\lambda y}{2\pi y} = \frac{1}{\sqrt{1+(y')^2}} \implies \boxed{y'(x) = \pm\left(\frac{2\pi y}{c-\lambda y}\right)^2 - 1}\;.$$

Problem 5 (4 points)

$$
\begin{aligned}
0 = F_y - \frac{d}{dx}F_{y'} &= \frac{1}{(y')^2} - \frac{d}{dx}\frac{-2y}{(y')^3} \\
&= \frac{1}{(y')^2} + \frac{d}{dx}2y(y')^{-3} \\
&= \frac{1}{(y')^2} + (2y'(y')^{-3} - 6y(y')^{-4}y'')
\end{aligned}
$$

$$
\begin{aligned}
\frac{6yy''}{(y')^4} &= \frac{3}{(y')^2} \\
\frac{yy''}{(y')^2} &= \frac{1}{2} \\
\text{Let } q(x) &= \frac{y}{y'} \\
q'(x) &= \frac{(y')^2 - yy''}{(y')^2} = 1 - \frac{yy''}{(y')^2} = \frac{1}{2} \\
q'(x) &= \frac{1}{2} \implies q(x) = \frac{x}{2} + c_1 \\
\frac{y}{y'} &= q = \frac{x}{2} + c_1 \\
\frac{1}{y}\frac{dy}{dx} &= \frac{2}{x + c_1} \quad \text{(reassigning) } 2c_1 = c_1 \\
y(x) &= c_2(x + c_1)^2
\end{aligned}
$$

$$
\begin{aligned}
1 &= y(0) = c_2 c_1^2, \\
4 &= y(1) = c_2(1 + c_1)^2 \\
(c_1, c_2) &= (1, 1) \text{ or } (-1/3, 9)
\end{aligned}
$$

$$
\boxed{y(x) = (x+1)^2, \quad y(x) = 9\left(x - \frac{1}{3}\right)^2 = (3x-1)^2}
$$

Problem 6 (4 points)

In Chapter 5, section 24 of the Gelfand and Fomin text we saw that for functionals of the form

$$J[y] = \int_a^b F(x, y, y')\, dx,$$

defined for curves $y = y(x)$ with fixed end points $y(a) = A, y(b) = B$, the second variation $\delta^2 J[h]$ (where h represents any admissible test function) can be written as

$$\delta^2 J[h] = \int_a^b (P(h')^2 + Qh^2)\, dx$$
$$P = P(x) = \frac{1}{2} F_{y'y'}, \quad Q = Q(x) = \frac{1}{2}\left(F_{yy} - \frac{d}{dx} F_{yy'}\right).$$

I believe there was an error in the textbook where we're given $Q(x) = \frac{1}{2}F_{yy'} - \frac{1}{2}\frac{d}{dx}F_{yy'}$. This does not agree with previous lines in the derivation and does not agree with the results from lecture. Also, in lecture the factor of $\frac{1}{2}$ was omitted from both P and Q. So no points deducted if your answers are equal to the answers below multiplied by 2.

From problem 5, $F(x, y, y') = \frac{y}{(y')^2}$.

$$F_{y'} = -\frac{2y}{(y')^3}, \quad F_{y'y'} = \frac{6y}{(y')^4}, \quad F_y = \frac{1}{(y')^2}, \quad F_{yy} = 0, \quad F_{yy'} = -\frac{2}{(y')^3}$$
$$P(x) = \frac{1}{2} F_{y'y'} = \frac{3y}{(y')^4}$$
$$Q(x) = \frac{1}{2}\left(F_{yy} - \frac{d}{dx}F_{yy'}\right) = \frac{1}{2}\left(0 - \frac{d}{dx}\left[-\frac{2}{(y')^3}\right]\right) = \frac{1}{2}\frac{d}{dx}\left[\frac{2}{(y')^3}\right] = -\frac{3y''}{(y')^4}$$

For $y(x) = (x+1)^2$, $y'(x) = 2(x+1) = 2x+2$, $y''(x) = 2$:

$$\begin{aligned}
\delta^2 J[h] &= \int_0^1 (Ph'^2 + Qh^2)\, dx \\
&= \int_0^1 \left[\frac{3y}{(y')^4}h'^2 - \frac{3y''}{(y')^4}h^2\right] dx \\
&= \int_0^1 \left[\frac{3(x+1)^2}{(2(x+1))^4}h'^2 - \frac{3\cdot 2}{(2(x+1))^4}h^2\right] dx \\
&= \int_0^1 \left[\frac{3}{16(x+1)^2}h'^2 - \frac{3}{8(x+1)^4}h^2\right] dx
\end{aligned}$$

For $y(x) = (3x-1)^2$, $y'(x) = 2(3x-1)(3) = 6(3x-1) = 18x - 6$, $y''(x) = 18$:

$$\begin{aligned}
\delta^2 J[h] &= \int_0^1 (Ph'^2 + Qh^2)\, dx \\
&= \int_0^1 \left[\frac{3y}{(y')^4}h'^2 - \frac{3y''}{(y')^4}h^2\right] dx \\
&= \int_0^1 \left[\frac{3(3x-1)^2}{6^4(3x-1)^4}h'^2 - \frac{3\cdot 18}{6^4(3x-1)^4}h^2\right] dx \\
&= \int_0^1 \left[\frac{3}{1296(3x-1)^2}h'^2 - \frac{54}{6^4(3x-1)^4}h^2\right] dx \\
&= \int_0^1 \left[\frac{1}{432(3x-1)^2}h'^2 - \frac{1}{24(3x-1)^4}h^2\right] dx
\end{aligned}$$