MA 3475 HW 2 Solutions

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Find the extremals of the following functionals.

Problem 1

$$J[y] = \int_a^b \left(y^2(x) - (y'(x))^2\right)\, dx$$

$$\begin{aligned}
F(x,y,y') = y^2 - (y')^2, &\quad F_y = 2y, \quad F_{y'} = -2y' \\
0 = F_y - \frac{d}{dx}F_{y'} &= 2y - \frac{d}{dx}[-2y'] = 2y + 2y'' \quad \text{(Euler-Lagrange)} \\
0 &= y'' + y \\
\implies y(x) &= c_1\cos(x) + c_2\sin(x)
\end{aligned}$$

To solve the ODE $0 = y'' + y$, we used the corresponding characteristic equation $0 = r^2 + 1$. The roots of this equation are $r = 0 \pm 1i = \pm i$.

The constants c_1, c_2 can be determined if we know $y(a), y(b)$.

Alternative Solution: Since $F(x,y,y') = F(y,y')$ we can also use the Beltrami identity to solve the problem (although it will be more work in this case).

$$\begin{aligned}
F - y'F_{y'} &= c \in \mathbb{R} \\
y^2 - (y')^2 - y'(-2y') &= c \\
y^2_{(}y')^2 &= c \\
\frac{1}{\sqrt{c - y^2}}\frac{dy}{dx} &= \frac{y'}{\sqrt{c-y^2}} = 1 \\
\implies \int \frac{1}{\sqrt{c-y^2}}\, dy &= \int dx \\
\int \frac{\sqrt{c}\cos(u)}{\sqrt{c - c\sin^2(u)}}\, du &= x + c_1 \quad \text{(by the substitution } y = \sqrt{c}\sin(u)) \\
u &= x + c_1 \\
\sin(u) &= \sin(x + c_1) \\
\frac{y}{\sqrt{c}} &= \sin(x + c_1) \\
y(x) &= \sqrt{c}\sin(x + c_1)\ .
\end{aligned}$$

The constants c, c_1 can be determined if we know $y(a), y(b)$. The constant c_1 from using the Beltrami identity is not the same constant c_1 from using the Euler-Lagrange equation. Finally, note that making the substitution $y = \sqrt{c}\cos(u)$ instead of $y = \sqrt{c}\sin(u)$ is also possible. This will give a final answer in terms of the cosine function as $y(x) = \sqrt{c}\cos(-x - c_1) = \sqrt{c}\cos(x + c_1)$. However, this is equivalent to the answer given above.

Problem 2

$$J[y] = \int_a^b \left(xy'(x) + (y'(x))^2\right)\, dx$$

$$\begin{aligned}
F(x,y,y') &= xy' + (y')^2, \quad F_y = 0, \quad F_{y'} = x + 2y' \\
0 &= F_y - \frac{d}{dx}F_{y'} = 0 - \frac{d}{dx}\left[x + 2y'\right] \quad \text{(Euler-Lagrange)} \\
\implies c &= F_{y'} = x + 2y' \quad (c \in \mathbb{R}) \\
y' &= c/2 - x/2 \\
y' &= c_1 - x/2 \quad (c_1 := c/2) \\
y(x) &= c_1 x - x^2/4 + c_2
\end{aligned}$$

The constants c_1, c_2 can be determined if we know $y(a), y(b)$.

Problem 3

$$J[y] = \int_1^2 \left(\frac{\sqrt{1 + (y'(x))^2}}{x}\right)\, dx \quad y(1) = 0, y(2) = 1$$

$$\begin{aligned}
F(x,y,y') = F(x,y') &= \frac{\sqrt{1 + (y'(x))^2}}{x} \\
F_y = 0, \quad F_{y'} &= \frac{y'}{x\sqrt{1 + (y')^2}} \\
0 &= F_y - \frac{d}{dx}F_{y'} = 0 - \frac{d}{dx}\left[\frac{y'}{x\sqrt{1 + (y')^2}}\right] \quad \text{(Euler-Lagrange)} \\
\implies c &= F_{y'} = \frac{y'}{x\sqrt{1 + (y')^2}} \\
y'(x) &= \frac{cx}{\sqrt{1 - c^2x^2}} \\
y(x) &= \int y'(x)\, dx = c_1 - \frac{1}{c}\sqrt{1 - c^2x^2} \\
(y - c_1)^2 + x^2 &= \frac{1}{c^2} \\
y(1) = 0, y(2) = 1 &\implies c = 1/\sqrt{5}, c_1 = 2 \\
(y - 2)^2 + x^2 &= \sqrt{5}^2
\end{aligned}$$

Remember that we want y as a function of x for an extremal. So from this circle extract

$$y_+(x) = 2 + \sqrt{5 - x^2}, \quad y_-(x) = 2 - \sqrt{5 - x^2}\,.$$

Since $(y_+')^2 = (y_-')^2$, it follows that $J[y_+] = J[y_-]$. However, $y_-(1) = 0, y_-(2) = 1$ while $y_+(1) = 4 \neq 0, y_+(2) = 3 \neq 1$ so y_+ doesn't meet the boundary conditions and is therefore not an admissible function.