MA 3475 HW 3 Solutions

February 24, 2021

The extra credit problems 3 and 5 are omitted from these solutions. For each of the following problems, there were several reasonable alternative methods for arriving at the correct solution but I only include one solution method for each.

Problem 1

$$J[y] = \int_0^a \frac{\sqrt{1+(y')^2}}{\sqrt{2g(b-y)}}\,.$$

(a) Let $u(x) = b - y(x)$ so that $u'(x) = -y'(x)$.

$$\begin{aligned}
F(x,u,u') = F(u,u') &= \frac{\sqrt{1+(-u')^2}}{\sqrt{2gu}} = \frac{\sqrt{1+(u')^2}}{\sqrt{2gu}} \\
c &= F - u'F_{u'} \quad \text{(The Beltrami Identity)} \\
&= \frac{\sqrt{1+(u')^2}}{\sqrt{2gu}} - u'\left(\frac{u'}{\sqrt{2gu}\sqrt{1+(u')^2}}\right) \\
&= \frac{\sqrt{1+(u')^2}}{\sqrt{2gu}} - \frac{(u')^2}{\sqrt{2gu}\sqrt{1+(u')^2}}
\end{aligned}$$

$$\begin{aligned}
c\sqrt{2gu}\sqrt{1+(u')^2} &= 1 + (u')^2 - (u')^2 = 1 \\
2c^2gu(1+(u')^2) &= 1 \\
u(1+(u')^2) &= c_1 \quad (c_1 := 1/(2c^2g)) \\
u + u(u')^2 &= c_1
\end{aligned}$$

The result above is reasonable or solve explicitly for u':

$$u' = \pm\sqrt{\frac{c_1 - u}{u}}\,.$$

(b)

$$\begin{aligned}
x(\theta) &= \frac{C}{2}(\theta - \sin\theta), \quad \frac{dx}{d\theta} = \frac{C}{2} - \frac{C}{2}\cos\theta \\
y(\theta) &= b - \frac{C}{2} + \frac{C}{2}\cos\theta, \quad \frac{dy}{d\theta} = -\frac{C}{2}\sin\theta \\
y'(x) &= \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-\sin\theta}{1-\cos\theta}
\end{aligned}$$

I found that it is relatively straightforward to verify that $u + u(u')^2$ is constant instead of working from the explicit form for u'.

$$\begin{aligned}
u+u(u')^2 &= b-y+(b-y)(-y')^2 \\
&= b-y+(b-y)(y')^2 \\
&= b-b+\frac{C}{2}-\frac{C}{2}\cos\theta+(b-b+\frac{C}{2}-\frac{C}{2}\cos\theta)\frac{\sin^2\theta}{(1-\cos\theta)^2} \\
&= \frac{C}{2}(1-\cos\theta)+\frac{C}{2}(1-\cos\theta)\frac{\sin^2\theta}{(1-\cos\theta)^2} \\
&= \frac{C}{2}(1-\cos\theta)+\frac{C}{2}\frac{\sin^2\theta}{1-\cos\theta} \\
&= \frac{C}{2}\left(\frac{(1-\cos\theta)^2}{1-\cos\theta}+\frac{\sin^2\theta}{1-\cos\theta}\right) \\
&= \frac{C}{2}\left(\frac{1-2\cos\theta+\cos^2\theta+1-\cos^2\theta}{1-\cos\theta}\right) \\
&= \frac{C}{2}\frac{2-2\cos\theta}{1-\cos\theta} \\
&= C\ .
\end{aligned}$$

That is, we have confirmed that $u+u(u')^2 = b-y+y(y')^2$ is constant, which is consistent with the Euler-Lagrange equation.

Problem 2

$$J[y] = \int_0^1 \left(\frac{1}{2}(y')^2+yy'+y'+y\right)\,dx$$

The values of $y(x)$ are not specified at the endpoints. That is, this is a variable endpoint problem (see Chapter 1 Section 6 of Gelfand and Fomin).

$$\begin{aligned}
F(x,y,y') = F(y,y') &= \frac{1}{2}(y')^2+yy'+y'+y \\
F_y &= y'+1 \\
F_{y'} &= y'+y+1
\end{aligned}$$

Since $F=F(y,y')$, you can use the Beltrami identity again but using the Euler-Lagrange equation appears a bit easier this time.

$$\begin{aligned}
0 = F_y - \frac{d}{dx}F_{y'} &= y'+1-(y''+y'+0) \\
&= -y''+1 \\
y''(x) &= 1 \\
y'(x) &= x+c_1 \\
y(x) &= \frac{x^2}{2}+c_1x+c_2
\end{aligned}$$

Since y is not specified at the endpoints, we know that the following equations must be satisfied:

$$
\begin{aligned}
0 &= F_{y'}(0, y(0), y'(0)) = y'(0) + y(0) + 1 = (0 + c_1) + (0 + 0 + c_2) + 1 = c_1 + c_2 + 1 \\
0 &= F_{y'}(1, y(1), y'(1)) = y'(1) + y(1) + 1 = (1 + c_1) + (\frac{1}{2} + c_1 + c_2) + 1 = 2c_1 + c_2 + \frac{5}{2}
\end{aligned}
$$

$$
\Longrightarrow c_1 = -\frac{3}{2}, \quad c_2 = \frac{1}{2}
$$

$$
\therefore \quad y(x) = \frac{x^2}{2} - \frac{3}{2}x + \frac{1}{2} \, .
$$

Problem 4

Find the extremal(s) of the following functional

$$
J[y] = \int_0^b \left(\sqrt{1 - k^2 + (y')^2} - ky' \right) \, dx
$$

in the class of $\mathcal{D}_2[0, b]$ functions with $y(0) = 0$, $y(b)$ free and $0 < k < 1$.

$$
\begin{aligned}
F(x, y, y') &= \sqrt{1 - k^2 + (y')^2} - ky' \\
0 &= F_y - \frac{d}{dx} F_{y'} = 0 - \frac{d}{dx} \left[\frac{y'}{\sqrt{1 - k^2 + (y')^2}} - k \right] \\
\Longrightarrow c_1 &= \frac{y'}{\sqrt{1 - k^2 + (y')^2}} - k \\
(k + c_1)^2 (1 - k^2 + (y')^2) &= (y')^2 \\
c_2 - c_2 k^2 + c_2 (y')^2 &= (y')^2 \quad (c_2 = k + c_1) \\
(y')^2 (1 - c_2) &= c_2 (1 - k^2) \\
y'(x) &= c_2 (1 - k^2)/(1 - c_2) =: c_3 \\
y(x) &= c_3 x + c_4
\end{aligned}
$$

$$
\begin{aligned}
0 &= y(0) = 0 + c_4 \implies c_4 = 0 \, . \\
0 &= F_{y'}(b, y(b), y'(b)) \\
&= \frac{y'(b)}{\sqrt{1 - k^2 + (y'(b))^2}} - k = \frac{c_3}{\sqrt{1 - k^2 + c_3}} - k \\
c_3 &= k^2 (1 - k^2 + c_3) \\
c_3 &= k^2 - k^4 + k^2 c_3 \\
c_3 (1 - k^2) &= k^2 (1 - k^2) \\
c_3 &= k^2 \quad (0 < k < 1 \implies 1 - k^2 \neq 0) \\
c_3 &= \pm k \, .
\end{aligned}
$$

$$
y(x) = \pm kx \, .
$$