MA 3475, Homework 5, due midnight Sunday, 03/14/2021

Show your steps as unsupported answers may receive no credit.

1. (4 points) Find the extremals of the functional

$$J[y] = \int_0^1 ((y')^2 + x^2)\, dx$$

subject to the conditions $y(0) = 0, y(1) = 0, \int_0^1 y^2\, dx = 2$.

2. (for practice) Given two points A and B in the xy-plane, let γ be a curve joining them. Among all such curves of given length ℓ, find the one such that γ and the line joining A and B encloses the greatest area. You may assume that A is the origin is B is a point in the first quadrant for simplicity.

3. (for practice) Among all triangles with base fixed at the points $(-a, 0), (0, a), a > 0$ and a given perimeter, which one encloses the greatest area. Try to solve the problem without calculus of variations and then with calculus of variations.

4. (4 points) Among all curves joining a given point $(0, b), b > 0$ on the y-axis and a point $(a, 0), a > 0$ on the x-axis, and enclosing a given area S together with the x-axis, find the curve which generates the least surface area of solid of revolution when rotated about the x-axis. Full credit if you can write down the functional and constraint and the Euler's equation. No need to solve it.

5. (4 points) Find the extremal of the functional

$$J[y] = \int_0^1 \frac{y}{(y')^2}\, dx$$

satisfying the boundary conditions $y(0) = 1, y(1) = 4$.

(Hint. The Euler's equation is $\dfrac{yy''}{(y')^2} = \dfrac{1}{2}$. Let $q = \frac{y}{y'}$. Show that q satisfies $q' = \frac{1}{2}$. From this deduce that $y(x) = C_1(x + C_2)^2$.)

6. (4 points) According to problem 10 on page 130 of the textbook, there are two extremals to problem 5 above. Write down the second variations, $\delta^2 J[h]$, for each of these extremals. Which one of them corresponds to a weak minimum?