MA 3475 Exam I, 2/25/2021 NAME ______________________

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1. (5 points) Find the extremal of the functional

$$J[y] = \int_a^b 12xy + (y')^2 \, dx \, .$$

$$\begin{aligned} F(x,y,y') &= 12xy + (y')^2 \\ F_y &= 12x \\ F_{y'} &= 2y' \end{aligned}$$

$$\begin{aligned} 0 &= F_y - \frac{d}{dx}F_{y'} \quad \text{(Euler-Lagrange equation)} \\ &= 12x - \frac{d}{dx}[2y'] \\ &= 12x - 2y'' \end{aligned}$$

$$\begin{aligned} y''(x) &= 6x \\ y'(x) &= 3x^2 + c_1 \\ y(x) &= x^3 + c_1 x + c_2 \, , \quad c_1, c_2 \in \mathbb{R} \, . \end{aligned}$$

2. (5 points) Consider the functional

$$J[y] = \int_1^2 x^2 (y')^2 + 2y^2 \, dx \, .$$

- (2 points) Find the Euler-Lagrange equation of the above functional.

$$\begin{aligned} F(x, y, y') &= x^2(y')^2 + 2y^2 \\ F_y &= 4y \\ F_{y'} &= 2x^2 y' \end{aligned}$$

$$\begin{aligned} 0 &= F_y - \frac{d}{dx} F_{y'} \quad \text{(Euler-Lagrange equation)} \\ &= 4y - \frac{d}{dx}[2x^2 y'] \\ &= 4y - 2(2xy' + x^2 y'') \\ &= 4y - 4xy' - 2x^2 y'' \end{aligned}$$

Simplify:

$$0 = x^2 y''(x) + 2xy'(x) - 2y(x)$$

- (1 point) Find r such that $y(x) = x^r$ solves your Euler-Lagrange equation.
  Suppose $y(x) = x^r$ solves the ODE above for some constant r. Then $y'(x) = rx^{r-1}$ and $y''(x) = r(r-1)x^{r-2}$.

$$\begin{aligned} 0 &= x^2 r(r-1)x^{r-2} + 2xrx^{r-1} - 2x^r \\ 0 &= r(r-1)x^r + 2rx^r - 2x^r \\ 0 &= (r(r-1) + 2r - 2)x^r \\ \Longrightarrow 0 &= r^2 + r - 2 \\ 0 &= (r-1)(r+2) \\ r &= 1, -2 \end{aligned}$$

$\therefore$ $y_1(x) = x$ and $y_2(x) = x^{-2}$ solve the diferential equation. The general solution to the differential equation is

$$y(x) = c_1 x + \frac{c_2}{x^2} \, .$$

- (2 points) Find the extremal of $J[y]$ satisfying the boundary conditions $y(1) = 0$, $y(2) = -\frac{7}{4}$.

$$0 = c_1 + c_2 \implies c_2 = -c_1$$
$$-\frac{7}{4} = 2c_1 + \frac{c_2}{4} = 2c_1 - \frac{c_1}{4} = \frac{7c_1}{4}$$

$$c_1 = -1$$
$$c_2 = 1$$

$$y(x) = -x + \frac{1}{x^2}\,.$$

3. (5 points) Determine the extremal of the functional

$$J[y] = \int_0^1 (y')^2 - 2\alpha y y' - 2\beta y' \, dx$$

where α, β are nonzero constants for each of the following boundary conditions.

(a)

$$\begin{aligned} 0 &= F_y - \frac{d}{dx} F_{y'} = -2\alpha y' - \frac{d}{dx}\left[2y' - 2\alpha y - 2\beta\right] \\ &= -2\alpha y' - 2y'' + 2\alpha y' + 0 \\ &= -2y'' \\ \implies y''(x) &= 0 \\ y'(x) &= c_1 \\ y(x) &= c_1 x + c_2 \end{aligned}$$

Since $F = F(y, y')$, the Beltrami identity could be applied to get the same result:

$$\begin{aligned} F(x, y, y') &= F(y, y') = (y')^2 - 2\alpha y y' - 2\beta y' \\ F_{y'} &= 2y' - 2\alpha y - 2\beta \\ c &= F - y' F_{y'} \quad \text{(The Beltrami Identity)} \\ &= (y')^2 - 2\alpha y y' - 2\beta y' - y'(2y' - 2\alpha y - 2\beta) \\ &= (y')^2 - 2\alpha y y' - 2\beta y' - 2(y')^2 + 2\alpha y y' + 2\beta y' \\ &= -(y')^2 \\ \implies y'(x) &= \sqrt{-c} \\ y(x) &= \sqrt{-c}x + c_2 = c_1 x + c_2 \end{aligned}$$

For use in parts (c) and (d), $y'(x) = c_1$.

(b) $y(0) = 0$, $y(1) = 1$.

$$\begin{aligned} 0 &= c_1(0) + c_2 = c_2 \\ 1 &= c_1 + c_2 = c_1 \end{aligned}$$

$$y(x) = x$$

(c) $y(0) = 0$, $y(1)$ is free.

$$
\begin{aligned}
(y(0)=0): \quad & 0 = c_1(0) + c_2 = c_2 \\
(y(1) \text{ free}): \quad & 0 = F_{y'}(1, y(1), y'(1)) = 2y'(1) - 2\alpha y(1) - 2\beta \\
& \quad = 2c_1 - 2\alpha(c_1(1) + c_2) - 2\beta \\
& \quad = 2c_1 - 2\alpha c_1 - 2\beta \\
\Longrightarrow & \; c_1 = \beta/(1-\alpha)
\end{aligned}
$$

$$
y(x) = \frac{\beta}{1-\alpha} x \; .
$$

(d) $y(0)$ and $y(1)$ are both free.

$$
\begin{aligned}
(y(0) \text{ free}): \; & 0 = F_{y'}(0, y(0), y'(0)) = 2y'(0) - 2\alpha y(0) - 2\beta \\
(y(1) \text{ free}): \quad & 0 = F_{y'}(1, y(1), y'(1)) = 2y'(1) - 2\alpha y(1) - 2\beta
\end{aligned}
$$

$$
\begin{aligned}
0 &= y'(0) - \alpha y(0) - \beta = c_1 - \alpha c_2 - \beta \\
0 &= y'(1) - \alpha y(1) - \beta = c_1 - \alpha(c_1 + c_2) - \beta
\end{aligned}
$$

$$
\begin{aligned}
0 &= -\alpha c_2 + \alpha c_1 + \alpha c_2 \implies c_1 = 0 \quad (\text{since } \alpha \neq 0) \\
0 &= c_1 - \alpha c_2 - \beta = -\alpha c_2 - \beta \implies c_2 = -\beta/\alpha
\end{aligned}
$$

$$
y(x) = -\frac{\beta}{\alpha} \; .
$$