MA 3475 Exam II Review

1. Find the extremal of the functional

$$J[y,z] = \int_0^1 \left((y')^2 - (z')^2 - 8y'y - 4y^2\right)\, dx,$$

subject to the boundary conditions $y(0) = 1, y(1) = 0, z(0) = 0, z(1) = e$.

Answer

Set up and solve the system of Euler-Lagrange equations.

$$\begin{aligned} 0 &= F_y - \frac{d}{dx}F_{y'} = -8y' - 8y - \frac{d}{dx}[2y' - 8y] = -8y' - 8y - 2y'' + 8y' = -2y'' - 8y \\ 0 &= F_z - \frac{d}{dx}F_{z'} = 0 - \frac{d}{dx}[-2z'] = 2z'' \end{aligned}$$

$$\begin{aligned} 0 &= -2y'' - 8y \implies 0 = y'' + 4y \\ 0 &= r^2 + 4 \\ r &= \pm 2i \\ y(x) &= c_1\cos(2x) + c_2\sin(2x) \end{aligned}$$

$$0 = 2z'' \implies z(x) = c_3 x + c_4$$

Apply the boundary conditions.

$$\begin{aligned} 1 &= y(0) = c_1 \\ 0 &= y(1) = c_1\cos 2 + c_2 \sin 2 = \cos 2 + c_1 \sin 2 \implies c_2 = -\frac{\cos 2}{\sin 2} \end{aligned}$$

$$\begin{aligned} 0 &= z(0) = c_4 \\ e &= z(1) = c_3 + c_4 = c_3 \end{aligned}$$

$$\boxed{y(x) = \cos 2x - \frac{\cos 2}{\sin 2}\sin 2x}$$

$$\boxed{z(x) = ex}$$

2. Find the extremal of the functional

$$J[y,z] = \int_a^b \left((y')^2 + (z')^2 + yz\right)\, dx$$

Answer

Set up and solve the system of Euler-Lagrange equations.

$$\begin{aligned} 0 &= F_y - \frac{d}{dx}F_{y'} = z - \frac{d}{dx}[2y'] = z - 2y'' \\ 0 &= F_z - \frac{d}{dx}F_{z'} = y - \frac{d}{dx}[2z'] = y - 2z'' \end{aligned}$$

$$\begin{aligned} 0 &= z - 2y'' \\ 0 &= y - 2z'' \end{aligned}$$

$$\begin{aligned} 0 &= z - 2y'' \implies 2y^{(iv)} = z'' \\ \text{Then, } 0 &= y - 2z'' = y - 2(2y^{(iv)}) = y - 4y^{(iv)} \end{aligned}$$

$$\begin{aligned} 0 &= 4r^4 - 1 \quad \text{(characteristic equation)} \\ 0 &= r^4 - \frac{1}{4} = (r^2 - \frac{1}{2})(r^2 + \frac{1}{2}) \\ r &= \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}i \end{aligned}$$

$$\boxed{y(x) = c_1 \exp\left(\frac{\sqrt{2}}{2}x\right) + c_2 \exp\left(-\frac{\sqrt{2}}{2}\right) + c_3 \cos\left(\frac{\sqrt{2}}{2}x\right) + c_4 \sin\left(\frac{\sqrt{2}}{2}x\right)}$$

We can use a similar process to solve for $z(x)$ (the differential equation is $4z^{(iv)} - z = 0$) or use that $z = 2y''$ to get $z(x)$, which allows us to express $z(x)$ using the same constants used to express $y(x)$.

$$\boxed{z(x) = 2y''(x) = c_1 \exp\left(\frac{\sqrt{2}}{2}x\right) + c_2 \exp\left(-\frac{\sqrt{2}}{2}\right) - c_3 \cos\left(\frac{\sqrt{2}}{2}x\right) - c_4 \sin\left(\frac{\sqrt{2}}{2}x\right)}$$

3. Find the extremal of the functional

$$J[y] = \int_0^{\pi/2} \left((y')^2 - (y'')^2\right) \, dx$$

subject to the boundary conditions $y(0) = 0, y'(0) = 0, y\left(\frac{\pi}{2}\right) = 1, y'\left(\frac{\pi}{2}\right) = 1$.

Answer Set up and solve the Euler-Lagrange equation.

$$\begin{aligned} 0 &= F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} \\ &= 0 - \frac{d}{dx}[2y'] + \frac{d^2}{dx^2}[-2y''] \\ &= -2y'' - 2y^{(iv)} \end{aligned}$$

$$\begin{aligned} 0 &= -2r^2 - 2r^4 \quad \text{(characteristic equation)} \\ 0 &= r^2(r^2+1) \end{aligned}$$

$$r = 0 \text{ (double root)}, \; r = \pm i$$

$$\begin{aligned} y(x) &= c_0 + c_1 x + c_2 \cos x + c_3 \sin x \\ y'(x) &= c_1 - c_2 \sin x + c_3 \cos x \end{aligned}$$

Apply the boundary conditions.

$$\begin{aligned} 0 &= y(0) = c_0 + c_2 \\ 0 &= y'(0) = c_1 + c_3 \\ 1 &= y\left(\frac{\pi}{2}\right) = c_0 + c_1\frac{\pi}{2} + c_3 \\ 1 &= y'\left(\frac{\pi}{2}\right) = c_1 - c_2 \end{aligned}$$

$$c_0 = 1, c_1 = 0, c_2 = -1, c_3 = 0$$

$$\boxed{y(x) = 1 - \cos x}$$

4. Find the extremal of the functional

$$J[y] = \int_a^b \left((y''')^2 + (y'')^2\right)\, dx$$

How many boundary conditions do you need to specify at the two endpoints to determine the extremal uniquely?

Answer Set up and solve the Euler-Lagrange equation.

$$\begin{aligned} 0 &= F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} - \frac{d^3}{dx^3}F_{y'''} \\ &= 0 - \frac{d}{dx}[0] + \frac{d^2}{dx^2}[2y''] - \frac{d^3}{dx^3}[2y'''] \\ &= 2y^{(iv)} - 2y^{(vi)} \end{aligned}$$

$$\begin{aligned} 0 &= 2r^4 - 2r^6 \quad \text{(characteristic equation)} \\ 0 &= r^4(r^2-1) \implies r = 0 \text{ (root with multiplicity 4) or } r = \pm 1 \end{aligned}$$

$$\boxed{y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 e^x + c_5 e^{-x}}$$

You need to have 6 boundary conditions specified in order to determine the extremal uniquely.

5. Find the extremal of the functional

$$J[y] = \int_0^1 \left(x^2 + (y')^2\right)\, dx$$

subject to the conditions $y(0) = 0, y(1) = 0$ and the constraint $\int_0^1 (x^2 + 2y(x))\, dx = 1$.

Answer

Let $G(x, y, y') = x^2 + 2y(x)$ so that $1 = \int_0^1 (x^2 + 2y(x))\, dx = \int_0^1 G(x, y, y')\, dx$. First set up and solve the Euler-Lagrange equation for $y(x)$.

$$\begin{aligned}
0 &= F_y - \frac{d}{dx}F_{y'} + \lambda(G_y - \frac{d}{dx}G_{y'}) \\
&= 0 - \frac{d}{dx}[2y'] + \lambda(2 - \frac{d}{dx}[0]) \\
&= -2y'' + 2\lambda \\
y''(x) &= \lambda \\
y(x) &= \frac{\lambda}{2}x^2 + Ax + B
\end{aligned}$$

Next apply the boundary conditions.

$$\begin{aligned}
0 = y(0) &= B \\
0 = y(1) &= \frac{\lambda}{2} + A \implies A = -\frac{\lambda}{2} \\
y(x) &= \frac{\lambda}{2}x^2 - \frac{\lambda}{2}x
\end{aligned}$$

Finally, apply the integral constraint.

$$\begin{aligned}
1 &= \int_0^1 \left(x^2 + \lambda x^2 - \lambda x\right)\, dx \\
&= \frac{1}{3} + \frac{\lambda}{3} - \frac{\lambda}{2} = \frac{1}{3} - \frac{\lambda}{6} \\
\lambda &= -4
\end{aligned}$$

$$\boxed{y(x) = -2x^2 + 2x}$$

6. Find the extremal of the functional

$$J[y] = \int_0^1 \frac{1}{2}(y')^2 \, dx$$

subject to the conditions $y(0) = 0, y(1) = 1$ and the constraint $\int_0^1 \frac{1}{2}y^2 \, dx = 1$. For this problem you may stop after expressing $y(x)$ in terms of the multiplier λ but this should be the only unknown constant. For a challenge, determine λ to finish solving the problem.

Answer:

$$\begin{aligned} 0 &= 0 - \frac{d}{dx}[y'] + \lambda y - \frac{d}{dx}[0] \\ 0 &= y'' - \lambda y \end{aligned}$$

$$\begin{aligned} 0 &= r^2 - \lambda \\ r &= \pm\sqrt{\lambda} \end{aligned}$$

$$y(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$$

Apply the boundary conditions.

$$\begin{aligned} 0 = y(0) &= c_1 + c_2 \implies c_2 = -c_1 \\ 1 = y(1) &= c_1\left(e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}}\right) = 2c_1 \sinh\sqrt{\lambda} \implies c_1 = \frac{1}{2\sinh\sqrt{\lambda}} \\ y(x) &= \frac{1}{2\sinh\sqrt{\lambda}}\left(e^{\sqrt{\lambda}x} - e^{-\sqrt{\lambda}x}\right) = \frac{2\sinh\sqrt{\lambda}x}{2\sinh\sqrt{\lambda}} = \frac{\sinh\sqrt{\lambda}x}{\sinh\sqrt{\lambda}} \end{aligned}$$

Apply the integral constraint to determine λ.

$$1 = \int_0^1 \frac{1}{2}\left(\frac{\sinh\sqrt{\lambda}x}{\sinh\sqrt{\lambda}}\right)^2 dx$$