MA 3475 Exam 1 Review Problems

February 16, 2021

Find the extremal(s) of the following functionals.

Problem 1

$$J[y] = \int_a^b \left(3x + \sqrt{y'(x)}\right)\, dx$$

$$\begin{aligned}
F(x,y,y') &= 3x + \sqrt{y'} \\
F_y &= 0, \quad F_{y'} = \frac{1}{2\sqrt{y'}} \\
0 &= F_y - \frac{d}{dx}F_{y'} = 0 - \frac{d}{dx}\left[\frac{1}{2\sqrt{y'}}\right] \qquad \text{(Euler-Lagrange)} \\
\implies c &= \frac{1}{2\sqrt{y'}} \\
y'(x) &= 4c^2 =: c_1 \\
y(x) &= \int_a^b y'(x)\, dx = c_1 x + c_2
\end{aligned}$$

The constants c_1, c_2 can be determined if we know $y(a), y(b)$.

Problem 2

$$J[y] = \int_a^b \left(y(x) - y(x)y'(x) + x(y'(x))^2\right)\, dx, \quad a > 0$$

$$\begin{aligned}
F(x,y,y') &= y - yy' + x(y')^2 \\
F_y &= 1 - y', \quad F_{y'} = -y + 2xy' \\
0 &= F_y - \frac{d}{dx}F_{y'} = 1 - y - \frac{d}{dx}\left[2xy' - y\right] \quad \text{(Euler-Lagrange)} \\
0 &= 1 - y' - 2y' - 2xy'' + y' \\
0 &= (-2x)y'' - 2y' + 1 \\
y'' + \frac{1}{x}y &= \frac{1}{2x} \\
\mu(x) &= \exp\left(\int \frac{1}{x}\,dx\right) = x \\
xy'' + y &= \frac{1}{2} \\
\frac{d}{dx}\left[xy'\right] &= \frac{1}{2} \\
xy'(x) &= \frac{1}{2}x + c_1 \\
y'(x) &= \frac{1}{2} + \frac{c_1}{x} \\
y(x) &= \int_a^b y'(x)\,dx = \frac{1}{2}x + c_1\ln(x) + c_2
\end{aligned}$$

The constants c_1, c_2 can be determined if we know $y(a), y(b)$.

Problem 3

$$J[y] = \int_a^b \sqrt{1 + (y'(x))^2}\,dx$$

$$\begin{aligned}
F(x,y,y') &= F(y') = \sqrt{1 + (y')^2} \\
0 &= F_y - \frac{d}{dx}F_{y'} = 0 - \frac{d}{dx}\left[\frac{y'}{\sqrt{1+(y')^2}}\right] \quad \text{(Euler-Lagrange)} \\
\implies c &= \frac{y'}{\sqrt{1+(y')^2}} \\
y'(x) &= \frac{c}{\sqrt{1-c^2}} \\
y(x) &= \frac{c}{\sqrt{1-c^2}}x + c_2 \\
y(x) &= c_1 x + c_2, \quad \left(c_1 = \frac{c}{\sqrt{1-c^2}}\right).
\end{aligned}$$

The constants c_1, c_2 can be determined if we know $y(a), y(b)$.

Problem 4

$$J[y] = \int_a^b \left[2y(x) + (y'(x))^2\right]\,dx$$

$$
\begin{aligned}
F(x,y,y') &= F(y,y') = 2y + (y')^2 \\
0 &= F_y - \frac{d}{dx} F_{y'} = 2 - \frac{d}{dx}\left[2y'\right] \quad \text{(Euler-Lagrange)} \\
\implies y''(x) &= 1 \\
y'(x) &= x + c_1 \\
y(x) &= \frac{x^2}{2} + c_1 x + c_2
\end{aligned}
$$

The constants c_1, c_2 can be determined if we know $y(a), y(b)$.

We could also have used Beltrami's identity since $F(x,y,y') = F(y,y')$.

Problem 5

$$
J[y] = \int_a^b \left[y^2(x) - 4y(x)y'(x) + 4(y'(x)^2)\right] \, dx
$$

$$
\begin{aligned}
F(x,y,y') &= F(y,y') = y^2 - 4yy' + 4(y')^2 \\
0 &= F_y - \frac{d}{dx} F_{y'} = 2y - 4y' - \frac{d}{dx}\left[-4y + 8y'\right] \quad \text{(Euler-Lagrange)} \\
0 &= 2y - 4y' + 4y' - 8y'' \\
0 &= 4y'' - y \\
\implies y(x) &= c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{x}{2}} \quad \text{(using the characteristic equation method with } 4r^2 - 1 = 0\text{)}
\end{aligned}
$$

An alternative way to write this solution:

$$
\begin{aligned}
y(x) &= c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{x}{2}} \\
&= (c_1 + c_2)\frac{e^{x/2} + e^{-x/2}}{2} + (c_1 - c_2)\frac{e^{x/2} - e^{-x/2}}{2} \\
&= (c_1 + c_2)\cosh(x/2) + (c_1 - c_2)\sinh(x/2) \\
&= c_3 \cosh(x/2) + c_4 \sinh(x/2) \ .
\end{aligned}
$$

As before, we need to know $y(a), y(b)$ in order to determine the constant coefficients.