MA 3831 Principles of Real Analysis 1

Discussion 2

January 26, 2022

Exercise 1

Show that the sequence converges using the definition of a convergent sequence.

a) $a_n = 3 - \frac{4}{n}$

b) $b_n = \frac{2n+3}{3n+2}$

Answer:

a) It appears that a_n converges to the limit $L = 3$. To prove this using the definition show that:

For any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ with $n > N$, $|a_n - 3| < \epsilon$.

Let $\epsilon > 0$. Choose N to be an integer larger than $4/\epsilon$. For any integer $n > N$ it follows that $4/n < \epsilon$ so that

$$|a_n - 3| = \left| \left(3 - \frac{4}{n}\right) - 3 \right| = \frac{4}{n} < \epsilon \ .$$

Conclude that the sequence a_n converges to 3.

b) Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > \frac{5}{9\epsilon}$. For any $n > N$, $\frac{5}{9n} < \epsilon$ so that

$$\left| b_n - \frac{2}{3} \right| = \left| \frac{2n+3}{3n+2} - \frac{2}{3} \right| = \left| \frac{5}{9n+6} \right| = \frac{5}{9n+6} < \frac{5}{9n} < \epsilon.$$

Exercise 2

Let $A = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$. Show using the definition of supremum that $\sup A = 1$.

Answer: Since $1/2 \in A$, $A \neq \emptyset$. Since $1 - 1/n < 1$ for any positive integer, $a \leq 1$ for all $a \in A$. Therefore A is a nonempty set of real numbers bounded above. Thus $\sup A$ exists. To prove that $\sup A = 1$, show that:

(i) 1 is an upper bound of A

(ii) If $M \in \mathbb{R}$ is any upper bound of A then $1 \leq M$.

We have already established item (i). Note that item (ii) is equivalent (use the contrapositive) to the statement "Any real number strictly less than 1 is not an upper bound". Using more symbolic notation: $\forall \epsilon > 0$, $\exists\, a \in A$ such that $1 - \epsilon < a$ (if you show that such an a exists you have shown that $1 - \epsilon$ is not an upper bound of A). By the Archimedean property there is an integer $n_\epsilon > 1/\epsilon$. This implies $1 - \epsilon < 1 - 1/n_\epsilon$. Since $1 - 1/n_\epsilon \in A$, $1 - \epsilon$ is not an upper bound of A. Conclude $\sup A = 1$.