MA 3831 Principles of Real Analysis 1
Homework 2 Notes

1. Prove $1/a_n \to 1/A$, $A > 0$ using $\exists N_1$ s.t. $A/2 < |a_n|$ $\forall n > N$.

   $\exists N \in \mathbb{N}$ s.t. $\forall n > N$, $0 < A/2 < |a_n| \leftrightarrow 0 < 1/|a_n| < 2/A$

   $\forall n > N$, $|1/a_n - 1/A| = |A - a_n| / (|a_n| A) < 2|a_n - A| / A^2$

3. Prove or $\boxed{\text{disprove}}$: $\lim(a_{n+1} - a_n) = 0 \Rightarrow a_n$ converges

   Counterexample: $a_n = \ln n$ $(n = 1, 2, 3, \dots)$

   $\lim(a_{n+1} - a_n) = \lim(\ln(n+1) - \ln n) = \lim \ln\left(\frac{n+1}{n}\right) = \ln 1 = 0$

   $\lim a_n = +\infty$

   $\therefore \lim(a_{n+1} - a_n) = 0 \not\Rightarrow a_n$ converges

4. $q > 0$ fixed. Show $a_n = q^n/n!$ is eventually decreasing.

   Eventually decreasing: $\exists N$ s.t. $\forall m, n \geq N$, $m > n \Rightarrow a_m \leq a_n$

   Let $N \in \mathbb{N}$ with $N \geq q - 1$.

   Base Case: $a_{N+1} = q^{N+1}/(N+1)! \leq q^N/N! \leq a_N$ since

   $q^{N+1}/(N+1)! \leq q^N/N!$ iff $q \leq N+1$ iff $q - 1 \leq N$

   Assume $a_{n+1} \leq a_n$ for some $n \geq N$.

   $a_{n+2} = q^{n+2}/(n+2)! = (q^{n+1}/(n+1)!)(q/(n+2)) = a_{n+1}\, q/(n+2) \leq a_{n+1}$

   since $0 < q \leq N+1 \leq n+1 \leq n+2 \Rightarrow 0 < q/(n+2) \leq 1$

   $\therefore \forall m, n \geq N$ with $m > n$, $a_m \leq a_{m-1} \leq \dots \leq a_n$

5. (2.6B of D & D)

$a_1 = 0$, $a_{n+1} = \sqrt{5+2a_n}$, $n \geq 1$. Show $\lim a_n = L$ exists, find L.

Prove $0 \leq a_n \leq a_{n+1} \leq 1+\sqrt{6}$ by induction.

Base Case ($n=1$): $a_1 = 0$, $a_2 = \sqrt{5+2a_1} = \sqrt{5} \leq 1+\sqrt{6}$

Assume $0 \leq a_n \leq a_{n+1} \leq 1+\sqrt{6}$

Show $0 \leq a_{n+1} \leq a_{n+2} \leq 1+\sqrt{6}$

$a_{n+1} = \sqrt{5+2a_n} \geq \sqrt{5+0} \geq 0$

$a_{n+2} = \sqrt{5+2a_{n+1}} \geq \sqrt{5+2a_n} = a_{n+1}$

$a_{n+2} = \sqrt{5+2a_{n+1}} \leq \sqrt{5+1+\sqrt{6}} = \sqrt{6+\sqrt{6}} \leq 1+\sqrt{6}$ since

$0 \leq \sqrt{6+\sqrt{6}} \leq 1+\sqrt{6}$ iff $6+\sqrt{6} \leq 1+2\sqrt{6}+6$ iff $0 \leq 1+\sqrt{6}$

$\therefore$ $0 \leq a_{n+1} \leq a_{n+2} \leq 1+\sqrt{6}$, completing induction proof.

To find L, let $f(x) = \sqrt{5+2x}$ and solve $x = f(x)$:

$x = \sqrt{5+2x} \rightarrow x^2-2x-5=0 \rightarrow x = 1 \pm \sqrt{6} \rightarrow L = 1+\sqrt{6}$

Why solve $x \leq f(x)$ as well?

6. (2.7.A of D & D)

Show $a_n = \dfrac{n\cos^n(n)}{\sqrt{n^2+n}}$ has a convergent subsequence

Bolzano-Weierstrass Theorem: Every bdd sequence of real numbers has a convergent subsequence.

$$|a_n| = \frac{|n\cos^n(n)|}{|\sqrt{n^2+n}|} = \frac{n|\cos n|^n}{\sqrt{n^2+n}} \leq \frac{n \cdot 1^n}{\sqrt{n^2}} = \frac{n}{|n|} = 1$$

Since a_n is bounded ($|a_n| \leq 1$), a_n has a convergent subsequence.

7. (2.7.6 of D&D)

Suppose $\lim x_{3n-1} = \lim x_{3n} = \lim x_{3n+1} = L \in \mathbb{R}$

Fix $\varepsilon > 0$

$\exists N_1$ s.t. if $3n-1 > N_1$, $|a_{3n-1} - L| < \varepsilon$

$\exists N_2$ s.t. if $3n > N_2$, $|a_{3n} - L| < \varepsilon$

$\exists N_3$ s.t. if $3n+1 > N_3$, $|a_{3n+1} - L| < \varepsilon$

Put $N = \max\{N_1, N_2, N_3\}$

For any $m \in \mathbb{Z}$, $\exists n \in \mathbb{Z}$ s.t. exactly one of $m = 3n-1$, $m = 3n$, or $m = 3n+1$ holds. Then for $m > N$, $|x_m - L| < \varepsilon$.

$\therefore \lim x_m = L$