MA 3831 Principles of Real Analysis 1: Homework 2

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January 26, 2022

Exercise 1

Let a_n be a sequence which converges to a positive number A. We showed in class that there is an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n > N$, $|a_n| > \frac{A}{2}$. From there, show that $\frac{1}{a_n}$ converges to $\frac{1}{A}$.

Answer: There is an $N \in \mathbb{N}$ such that $\forall n > N$, $0 < A/2 < |a_n| \iff 0 < 1/|a_n| < 2/A$. Then for all $n > N$,

$$\left| \frac{1}{a_n} - \frac{1}{A} \right| = \frac{|A - a_n|}{|a_n| A} < \frac{2}{A} |a_n - A| \ .$$

Since $|a_n - A| \to 0$, $\frac{2}{A}|a_n - A| \to 0$.

Exercise 2

Optional and omitted from the solutions.

Exercise 3

Prove or disprove:
Let a_n be a sequence of real numbers. If $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$, then a_n is convergent.

Answer: This statement is generally false. Consider the counterexample $a_n = \ln n$.

$$\lim_{n\to\infty} (a_{n+1} - a_n) = \lim_{n\to\infty} (\ln(n+1) - \ln n) = \lim_{n\to\infty} \ln \frac{n+1}{n} = \ln 1 = 0$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln n = +\infty \text{ (divergent)}$$

Exercise 4

Let q be a fixed positive number. Show that the sequence $a_n = \frac{q^n}{n!}$ is eventually decreasing.

Answer: Let $N \in \mathbb{N}$ with $N \geq q - 1$. For any $n > N$ we have $n \geq q - 1$ as well.

$$q - 1 \leq n \iff q \leq n + 1 \iff \frac{q^{n+1}}{q^n} \leq \frac{(n+1)!}{n!} \iff a_{n+1} \equiv \frac{q^{n+1}}{(n+1)!} \leq \frac{q^n}{n!} \equiv a_n$$

This shows that for $n > N$, $a_{n+1} \leq a_n$. Conclude that a_n is eventually decreasing.

Exercise 5

(2.6.B of Davidson and Donsig) Let $a_1 = 0$ and $a_{n+1} = \sqrt{5+2a_n}$ for $n \geq 1$. Show that $\lim_{n\to\infty} a_n$ exists and find the limit.

Answer: Prove by induction that $0 \leq a_n \leq a_{n+1} \leq 1+\sqrt{6}$ by induction.

Base Case ($n = 1$): $a_1 = 0$, $a_2 = \sqrt{5+2a_1} = \sqrt{5} \leq \sqrt{6} \leq 1+\sqrt{6}$. Therefore $0 \leq a_1 \leq a_2 \leq 1+\sqrt{6}$.

Assume that $0 \leq a_n \leq a_{n+1} \leq 1+\sqrt{6}$ for some $n \geq 1$.

$$\begin{aligned} a_{n+1} &= \sqrt{5+2a_n} \geq \sqrt{5+2\cdot 0} \geq 0 \\ a_{n+2} &= \sqrt{5+2a_{n+1}} \geq \sqrt{5+2a_n} = a_{n+1} \\ a_{n+2} &= \sqrt{5+2a_{n+1}} \leq \sqrt{5+2(1+\sqrt{6})} = \sqrt{7+2\sqrt{6}} \leq 1+\sqrt{6} \end{aligned}$$

To prove the last inequality $\sqrt{7+2\sqrt{6}} \leq 1+\sqrt{6}$ consider (note $\sqrt{7+2\sqrt{6}} > 0$ and $\leq 1+\sqrt{6} > 0$):

$$\sqrt{7+2\sqrt{6}} \leq 1+\sqrt{6} \iff \left(\sqrt{7+2\sqrt{6}}\right)^2 \leq (1+\sqrt{6})^2 \iff 7+2\sqrt{6} \leq 1+2\sqrt{6}+6 \iff 7+2\sqrt{6} \leq 7+2\sqrt{6}\ .$$

Therefore $0 \leq a_{n+1} \leq a_{n+2} \leq 1+\sqrt{6}$. This concludes the induction proof.
We have established that the sequence a_n is increasing and bounded above. Thus a_n must converge to some limit L and so must the subsequence a_{n+1} also converge to L.

$$\begin{gathered} L = \lim a_{n+1} = \lim \sqrt{5+2a_n} = \sqrt{5+2\lim a_n} = \sqrt{5+2L} \\ L^2 = 5+2L \implies L = 1+\sqrt{6} \text{ or } L = 1-\sqrt{6}. \end{gathered}$$

Since $1-\sqrt{6} < 0$ and $a_n \geq 0$ for all n, the only possibility is $L = 1+\sqrt{6}$.

Exercise 6

(2.7.A of Davidson and Donsig) Show that $(a_n) = \left(\frac{n\cos^n n}{\sqrt{n^2+2n}}\right)_{n=1}^{\infty}$ has a convergent subsequence.

Bolzano-Weierstrass Theorem Every bounded sequence of real numbers has a convergent subsequence.

Answer: The sequence a_n is bounded ($|a_n| \leq 1\ \forall n$). By Bolzano-Weierstrass Theorem there is a subsequence of a_n which converges

$$|a_n| = \frac{|n\cos^n n|}{|\sqrt{n^2+2n}|} = \frac{n|\cos n|^n}{\sqrt{n^2+2n}} \leq \frac{n\cdot 1^n}{\sqrt{n^2+2n}} \leq \frac{n}{\sqrt{n^2}} = \frac{n}{|n|} = \frac{n}{n} = 1\ .$$

Exercise 7

(2.7.G of Davidson and Donsig) Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Suppose there is a number L such that $L = \lim_{n\to\infty} x_{3n-1} = \lim_{n\to\infty} x_{3n} = \lim_{n\to\infty} x_{3n+1}$. Show that $\lim_{n\to\infty} x_{3n-1}$ exists and equals L.

Answer:
There is an N_1 such that $|x_{3n-1} - L| < \epsilon$ whenever $3n-1 > N_1$.

There is an N_2 such that $|x_{3n} - L| < \epsilon$ whenever $3n > N_2$.

There is an N_3 such that $|x_{3n+1} - L| < \epsilon$ whenever $3n + 1 > N_3$.

For any integer m there must exist an n such that exactly one of $m = 3n_1, m = 3n$, or $m = 3n + 1$ holds. For any $n > N := \max\{N_1, N_2, N_3\}$, $|x_m - L| < \epsilon$. Conclude that x_m converges and $\lim_{m \to \infty} x_m = L$.