# MA 3831 Principles of Real Analysis 1: Homework 1

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## Exercise 1

Show that for all $x, y \in \mathbb{Q}$,

a) $\left|\frac{1}{x}\right| = \frac{1}{|x|}$ for $x \neq 0$.

b) $||x| - |y|| \leq |x - y| \leq |x| + |y|$

Answer:

a) In class it was established that $|ab| = |a||b|$ for all $a, b \in \mathbb{Q}$. Set $a = x \neq 0$ and $b = 1/x$ in this identity to get:
$$1 = |1| = \left|x\frac{1}{x}\right| = |x|\left|\frac{1}{x}\right| \iff \frac{1}{|x|} = \left|\frac{1}{x}\right|$$

b) In class the triangle identity was established: $|a + b| \leq |a| + |b|$ for all $a, b \in \mathbb{Q}$. Set $a = x$ and $b = -y$ in this identity to get:
$$|x - y| = |x + (-y)| \leq |x| + |-y| = |x| + |y|$$
To establish $||x| - |y|| \leq |x - y|$ write, in turn, $x = x - y + y$ and $y = y - x + x$ to get:
$$\begin{aligned} |x| &= |(x - y) + y| \leq |x - y| + |y| \implies |x| - |y| \leq |x - y| \\ |y| &= |(y - x) + x| \leq |y - x| + |x| = |x - y| + |x| \implies -|x - y| \leq |x| - |y| \end{aligned}$$
Since $-|x - y| \leq |x| - |y| \leq |x - y|$, conclude $||x| - |y|| \leq |x - y|$.

## Exercise 2

Let $T = (0, 1) \cup \{2\}$. Find, with proof, $\sup T$.

Answer: Since $t \leq 2$ for all $t \in T$, 2 is an upper bound of the set $T$ of real numbers. Therefore $\sup T$ exists. To prove that $\sup T = 2$, show that for any $\epsilon > 0$, $2 - \epsilon$ is not an upper bound of $T$: Let $\epsilon > 0$ be given. Since $2 \in T$ and $2 - \epsilon < 2$, $2 - \epsilon$ is not an upper bound of $T$. Since 2 is an upper bound of $T$ and any real number less than 2 is not an upper bound of $T$, conclude by the definition of supremum that $\sup T = 2$.

## Exercise 3

Let $S$ and $T$ be two bounded above subsets of $\mathbb{R}$. Show that $S + T$ is bounded above, where $S + T$ is defined:
$$S + T = \{s + t : s \in S, t \in T\}\ .$$
Answer: Let $M_S$ be an upper bound of $S$ and $M_T$ be an upper bound of $T$. Then $S + T$ is bounded above by $M_S + M_T$. To prove this, suppose $u \in S + T$. By the definition of $S + T$ we know $u = s + t$ for some $s \in S$ and some $t \in T$. Since $s \in S$, $s \leq M_S$. Since $t \in T$, $t \leq M_T$. Thus $u = s + t \leq M_S + M_T$. Since $u$ was an arbitrary element of $S + T$, conclude $u \leq M_S + M_T$ for all $u \in S + T$ and therefore $S + T$ is bounded above.

## Exercise 4

(Exercise 1.4.4 of Understanding Analysis 2nd Edition by Stephen Abbott) Let $a < b$ be real numbers and consider the set $T = \mathbb{Q} \cap [a, b]$. Show $\sup T = b$.

Answer: For any $t \in T$, $t \in \mathbb{Q}$ and $t \in [a, b]$. Since $t \in [a, b]$, $t \leq b$ which means that $b$ is an upper bound of $T \subset \mathbb{R}$ and therefore $\sup T$ exists. To prove that $\sup T = b$, show that for any $\epsilon > 0$, $b - \epsilon$ is not an upper bound. Using the fact that between any two real numbers there exists a rational number, there exists a $t \in \mathbb{Q}$ such that $b - \epsilon < t < b$ since $b - \epsilon$ and $b$ are both real numbers. The rational number $t$ can be chosen such that $a \leq t$ as well (if $\epsilon > b - a$ one can simply create a subinterval of $(b - \epsilon, b)$ contained within $T$). Thus $t \in T$ with $b - \epsilon < t$ which shows that $b - \epsilon$ is not an upper bound of $T$. Conclude that since $b$ is an upper bound of $T$ and any real number less than $b$ is not an upper bound of $T$ that $\sup T = b$.

## Exercise 5

Use the definition of a convergent sequence to show that $b_n = \frac{1}{\sqrt{n}}$ converges to 0.

Answer: Let $\epsilon > 0$. By the Archimedean property of $\mathbb{R}$ there is an $N \in \mathbb{N}$ such that $N > 1/\epsilon^2$. This implies $\sqrt{N} = \sqrt{|N|} > |1/\epsilon| = 1/\epsilon$ which implies $0 < 1/\sqrt{N} < \epsilon$. For any $n \geq N$ we have $1/\sqrt{n} \leq 1/\sqrt{N}$. Therefore for any $n \geq N$:

$$|b_n - 0| = \left| \frac{1}{\sqrt{n}} - 0 \right| = \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} < \epsilon \; .$$

Since $\epsilon > 0$ was arbitrary, this shows that for any $\epsilon > 0$ there exists an $N = N(\epsilon) \in \mathbb{N}$ such that

$$|b_n - 0| < \epsilon \text{ for all } n \geq N \; .$$

Conclude by the definition of a convergent sequence that $b_n$ converges to 0 as $n \to \infty$.

## Exercise 6

Use the definition of a convergent sequence to show that any constant sequence is convergent.

Answer: Assume $b_n$ is a constant sequence with $b_n = b \in \mathbb{R}$ for $n = 1, 2, 3, \ldots$. Then $b_n$ converges with limit $b$. To prove this, let $\epsilon > 0$ and take $N = 1$. Then for any integer $n \geq N$, $b_n = b$ so that $|b_n - b| = |b - b| = |0| = 0 < \epsilon$. This shows that for any $\epsilon > 0$ one can find an $N \in \mathbb{N}$ such that $|b_n - b| < \epsilon$ for all $n \geq N$ from which we conclude by the definition of a convergent sequence that $b_n$ converges to $b$. Since $b_n$ was an arbitrary constant sequence, conclude that every constant sequence of real numbers is convergent.