Exercise 4

Show that if $a \le b$ for each $a \in A$ and each $b \in B$, then $\sup A \le \inf B$.

Fix $a \in A$. Since $a \le b$ for all $b \in B$, a is a lower bound of B. Thus $a \le \inf B$. But since $a \in A$ was arbitrary this implies $a \le \inf B$ for all $a \in A$. This means $\inf B$ is an upper bound of A. Therefore, $\sup A \le \inf B$.

Exercise 5

A maximum element of a set A is an element of A which is greater than or equal to every element of A. Show that if $\sup A \in A$, then $\sup A$ is the maximum element of A.

Assume $\sup A \in A$. By definition of supremum, $x \le \sup A$ for all $x \in A$. Since $\sup A \in A$ and $\sup A$ is greater than or equal to every element of A, $\sup A$ is the maximum element of A.

Exercise 6

Use the following lemma to prove that the limit of a convergent sequence is unique. (If there is time, use a proof by contradiction to prove the lemma as well).

Lemma If $|a| \le \epsilon$ for all $\epsilon > 0$, then $a = 0$.

Proof of the lemma: Suppose $|a| \le \varepsilon \; \forall \varepsilon > 0$ but $a \neq 0$. Then $|a|/2 > 0$. Take $\varepsilon = |a|/2$. By hypothesis this means $0 < |a| \le |a|/2$, which is not possible. Since $a \neq 0$ leads to a contradiction it must be the case that $a = 0$.

Proof that the limit of a convergent sequence is unique: Assume a_n is a convergent sequence with limits L and M, $L \neq M$. Let $\varepsilon > 0$. There exist positive integers N_1 and N_2 such that:

$$|a_n - L| < \varepsilon/2 \;\; \forall n \geq N_1 \quad \text{and} \quad |a_n - M| < \varepsilon/2 \;\; \forall n \geq N_2$$

Take $N \in \mathbb{N}$ s.t. $N_1, N_2 \le N$. Then for all $n \geq N$,

$$\begin{aligned} |L - M| &= |L - a_n + a_n - M| \le |L - a_n| + |a_n - M| \\ &= |a_n - L| + |a_n - M| < \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

By the lemma, $L - M = 0 \iff L = M$. This is a contradiction. Since the assumption that a_n can converge to more than one limit led to a contradiction, conclude the limit is unique.

MA 3831 Discussion 1 Notes

Exercise 1

Determine the supremum and infimum (if these exist) for each of the following sets. No proofs necessary. Determine if the sets are bounded or not.

a) $A = \{(-1)^n : n = 1, 2, 3, \dots\}$

b) $B = \{n - \frac{2}{3+n^2} : n = 1, 2, 3, \dots\}$

c) $C = \{(-1)^n n | n = 1, 2, 3, \dots\}$ $C = \{(-1)^n n : n = 1, 2, 3, \dots\}$

d) $D = \{\sqrt{n+1} - \sqrt{n} : n = 1, 2, 3, \dots\}$

set	supremum	infimum	bounded
A	1	-1	Yes
B	X	1/2	No
C	X	X	No
D	$\sqrt{2} - 1$	0	Yes

Exercise 2

True or False: If a_n and b_n are sequences such that $a_n < b_n$ for all n, then $\lim_{n\to\infty} a_n < \lim_{n\to\infty} b_n$. If this is true, try to write out a quick proof. If this is false, provide a counterexample.

False. Consider $a_n = 1/2n$, $b_n = 1/n$.

Then $a_n < b_n \ \forall n$ but

$\lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} b_n$

Exercise 3

Show that if $A \subset B \subset \mathbb{R}$ and B is bounded above, then A is bounded above.

Assume B is bounded above by $M \in \mathbb{R}$.

Let $a \in A$. Since $A \subset B$, $a \in B$ as well. This means $a \leq M$. Since a was arbitrary, $a \leq M$ for all $a \in A$.

$\therefore$ A is bounded above (M is an upper bound of A).