MA 3831 Principles of Real Analysis 1

Discussion 1

January 20, 2022

Exercise 1

Determine the supremum and infimum (if these exist) for each of the following sets. No proofs necessary. Determine if the sets are bounded or not.

a) $A = \{(-1)^n : n = 1, 2, 3, \dots\}$

b) $B = \left\{n - \frac{2}{3+n^2} : n = 1, 2, 3, \dots\right\}$

c) $C = \{(-1)^n n : n = 1, 2, 3, \dots\}$

d) $D = \{\sqrt{n+1} - \sqrt{n} : n = 1, 2, 3, \dots\}$

Answer: Following the definitions used in lecture and the course textbook, require supremums and infimums to be real numbers (for example we do not write $\sup C = \infty$ in the case that C was not bounded above) and said that a set is bounded if it is both bounded above and bounded below.

a) $\sup A = 1$, $\inf A = -1$, bounded.

b) No supremum, $\inf B = \frac{1}{2}$, unbounded.

c) No supremum, no infimum, unbounded.

d) $\sup D = \sqrt{2} - 1$, $\inf D = 0$, bounded.

Exercise 2

True or False: If a_n and b_n are sequences such that $a_n < b_n$ for all n, then $\lim_{n\to\infty} a_n < \lim_{n\to\infty} b_n$. If this is true, try to write out a quick proof. If this is false, provide a counterexample.

Answer: This statement is false in general. Consider the counterexample of $a_n = 1/(2n)$ and $b_n = 1/n$. In this case $a_n < b_n$ for each n yet $\lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} b_n$.

Exercise 3

Show that if $A \subset B \subset \mathbb{R}$ and B is bounded above, then A is bounded above.

Answer: Since B is bounded above there exists an $M \in \mathbb{R}$ such that $b \leq M$ for all $b \in B$. Let $a \in A$. Since $A \subset B$, $a \in B$ as well. So $a \leq M$. Since a was arbitrary, this means $a \leq M$ for all $a \in A$. Therefore A is bounded above.

Exercise 4

Show that if $a \leq b$ for each $a \in A$ and each $b \in B$, then $\sup A \leq \inf B$.

Answer: First let $a \in A$ be fixed. Since $a \leq b$ for all $b \in B$, a is a lower bound of B and must be less than or equal to the greatest lower bound (infimum) of B. That is, $a \leq \inf B$. Since a was taken arbitrarily, this shows that $a \leq \inf B$ for all $a \in A$. In other words, the real number $\inf B$ is an upper bound of the set A. So $\inf B$ must be greater than or equal to the least upper bound of A. That is, $\sup A \leq \inf B$.

Exercise 5

A maximum element of a set A is an element of A which is greater than or equal to every element of A. Show that if $\sup A \in A$, then $\sup A$ is the maximum element of A.

Answer: By definition of supremum, $a \leq \sup A$ for every element $a \in A$. If $\sup A \in A$ as well then $\sup A$ is an element of A which is greater than or equal to every element of A. Therefore $\sup A$ is the maximum element of A whenever $\sup A$ is contained in [is an element of] A.

Exercise 6

Use the following lemma to prove that the limit of a convergent sequence is unique. (If there is time, use a proof by contradiction to prove the lemma as well).

Lemma If $|a| \leq \epsilon$ for all $\epsilon > 0$, then $a = 0$.

Proof (lemma): Assume $|a| \leq \epsilon$ for all $\epsilon > 0$ but $a \neq 0$. Then $|a|/2 > 0$ so we may take $\epsilon = |a|/2$. By hypothesis we have $0 < |a| \leq |a|/2$, which is impossible. Since $a \neq 0$ leads to a contradiction, conclude that if $|a| \leq \epsilon$ for all $\epsilon > 0$, it must be the case that $a = 0$.

Proof (limit uniqueness): We can also use a proof by contradiction for this statement. Assume a_n is a convergent sequence converging to limits L and M with $L \neq M$ (note: the negation of "the limit is unique" is "there are two or more distinct limits" so we only need to look at two distinct limits). By the definition of a convergent sequence:

There exists an $N_1 \in \mathbb{N}$ such that $|a_n - L| < \epsilon/2$ whenever $n \geq N_1$.

There exists an $N_2 \in \mathbb{N}$ such that $|a_n - M| < \epsilon/2$ whenever $n \geq N_2$.

Put $N \in \mathbb{N}$ such that $N \geq N_1$ and $N \geq N_2$. Then both of the inequalities above hold whenever $n \geq N$. It follows by the triangle inequality that whenever $n \geq N$:

$$|L - M| = |L - a_n + a_n - M| \leq |L - a_n| + |a_n - M| = |a_n - L| + |a_n - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\ .$$

Since $\epsilon > 0$ was arbitrary, $|L - M| < \epsilon$ for all $\epsilon > 0$ so $L - M = 0 \implies L = M$ (note: $|L - M| < \epsilon \implies |L - M| \leq \epsilon$ as the former condition is stronger). This contradicts the assumption $L \neq M$, so the assumption that a_n could converge to more than one limit is false - the limit of a convergent sequence is unique.