# MA 422X HW #5

1. **Conjugate Gradient Method - Krylov Subspace Method**
   In class, we stated that the residual vectors $\vec{r}_k$ for $k = 1, 2, \ldots, n$ for the conjugate gradient method satisfy $< \vec{r}_k, \vec{v}_j >= 0$ for each $j = 1, 2, \ldots, k$. Prove that this is true using mathematical induction as follows:

   (a) Show that $< \vec{r}_1, \vec{v}_1 >= 0$

   (b) Assume $< \vec{r}_k, \vec{v}_j >= 0$ for each $k \leq \ell$ and $j = 1, 2, \ldots, k$, and show that this implies that $< \vec{r}_{\ell+1}, \vec{v}_j >= 0$ for each $j = 1, 2, \ldots, \ell$

   (c) Show that $< \vec{r}_{\ell+1}, \vec{v}_{\ell+1} >= 0$

2. **Characteristic polynomial & Cayley-Hamilton Theorem**:
   Let $p_A(t) = det(tI - A)$ be the characteristic monic polynomial for $A \in \mathbb{R}^{n\times n}$ (monic implies that the coefficient of the highest degree term is positive). Prove that $p_A(A) = 0$ (where the right hand side is the zero matrix). Note that you are showing that the characteristic polynomial formed from $det(tI - A)$ and then evaluated at $A$ is zero (i.e. do not just plug $A$ into $t$ in $tI - A$).

   *Hint*: Note that the roots of the polynomial $p_A(t)$ are the eigenvalues, so rewrite the characteristic polynomial in terms of factors of the eigenvalues. It might also be helpful to rewrite $A$ using the Schur Decomposition, $A = QRQ^T$ where $Q$ is orthogonal and $R$ is an upper triangular matrix where eigenvalues are on the diagonal.

   *Remark*: From Cayley-Hamilton Theorem, we are guaranteed that the characteristic polynomial corresponding to $A \in \mathbb{R}^{n\times n}$ is of degree $n$ and $p_A(A) = 0$. We say that the polynomial $p_A$ annihilates $A$ since it is equal to the zero matrix when we evaluate at $A$. For $A$ nonsingular and square, this also allows us to rewrite the inverse of a matrix $A$ in terms of powers of $A$.

3. **Minimal polynomials**
   We can define the minimal polynomial for $A$ as the unique monic polynomial $q_A(t)$ of **minimum** degree that annihilates $A$. Suppose that the degree of the minimal polynomial is $m \leq n$ where $A \in \mathbb{R}^{n\times n}$. Show that similar matrices $A$ and $B$ have the same degree minimal polynomial. That is, let $A$, $B$, $S$ $\in \mathbb{R}^{n\times n}$ where $A = SBS^{-1}$.

   **Preamble to #5**:
   **Jordan Normal Form**
   The Jordan block $J_{i_k}(\lambda_i)$ is a $k \times k$ square matrix associated with eigenvalue $\lambda_i$ where $\lambda_i$ is on the diagonal, 1's are on the sup-diagonal, and 0's are in all other entries. For example:

$$J_{i_3}(\lambda_i) = \begin{pmatrix} \lambda_i & 1 & 0 \\ 0 & \lambda_i & 1 \\ 0 & 0 & \lambda_i \end{pmatrix}$$

   where $J_{i_3}$ is a $3 \times 3$ matrix associated with e-value $\lambda_i$. The Jordan matrix is a block diagonal matrix where all blocks are Jordan blocks. For example:

$$J = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_3 & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} J_{1_2} & 0_2 & 0_2 \\ 0_1 & J_{2_1} & 0_1 \\ 0_3 & 0_3 & J_{3_3} \end{pmatrix}$$

   where on the right hand matrix, $0_i$ corresponds to a matrix of zeros of dimension $i \times i$. Note that the eigenvalues of $J$ are on the diagonal, where some are repeated. In addition, a specific eigenvalue may be repeated in a Jordan block of size 2 and one of size 1, for example. (In above example, could have $\lambda_1 = \lambda_2$)

4. **Jordan Normal Form & the Minimal Monic Polynomial**
   Show that the minimal monic polynomial that annihilates $J$ is $q_J(t) = \prod_{i=1}^{m}(t - \lambda_i)^{r_i}$ where $m$ is the total number of distinct eigenvalues $\lambda_i$ and $r_i$ is the order of the largest Jordan block of $J$ corresponding to the eigenvalue $\lambda_i$.

   **Remark**: There exists a Jordan matrix $J$ that is similar to $A$ where $J$ is unique, up to reordering of blocks on the diagonal. The idea is that one could determine $J = Q^{-1}AQ$ and then determine the minimal polynomial from $J$. This is often could be easier than fully calculating the characteristic polynomial from a large, dense matrix $A$.