Problem Set 9.3

2 If λ is an eigenvalue of A, then $1-\lambda$ is an eigenvalue of $B = I - A$. The real eigenvalues of B have absolute value less than 1 if the real eigenvalues of A lie between 0 and 2.

If $Ax = \lambda x$ for an eigenpair (λ, x) of A, then $(I-A)x = x - \lambda x = (1-\lambda)x$.

$$|1-\lambda| < 1$$
$$-1 < 1-\lambda < 1$$
$$0 < \lambda < 2$$

6 Change the 2's to 3's and find the eigenvalues of $S^{-1}T$ for Jacobi's method:

$$Sx_{k+1} = Tx_k + b \quad \text{is} \quad \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} x_{k+1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x_k + b.$$

$$S^{-1}T = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix}, \quad \lambda = \frac{1}{3}, \frac{1}{3}.$$

$\rho = |\lambda_{max}| = 1/3 < 1$, so Jacobi's method converges.

7 Find the eigenvalues of $S^{-1}T$ for the Gauss-Seidel method applied to Problem 6:

$$\begin{bmatrix} 3 & 0 \\ -1 & 3 \end{bmatrix} x_{k+1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_k + b.$$

Does $|\lambda|_{\max}$ for Gauss-Seidel equal $|\lambda|^2_{\max}$ for Jacobi?

$$B = S^{-1}T = \begin{bmatrix} 1/3 & 0 \\ 1/9 & 1/3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 \\ 0 & 1/9 \end{bmatrix}, \quad \lambda = 0, \frac{1}{9}$$

$$\rho_{G.S.} = \frac{1}{9} = \left(\frac{1}{3}\right)^2 = \rho^2_{Jacobi}$$

9 The best ω produces two equal eigenvalues for $S^{-1}T$ in the **SOR** method. Those eigenvalues are $\omega - 1$ because the determinant is $(\omega-1)^2$. Set the trace in equation (10) equal to $(\omega-1)+(\omega-1)$ and find this optimal ω.

$$\textbf{SOR iteration matrix} \qquad S^{-1}T = \begin{bmatrix} 1-\omega & \frac{1}{2}\omega \\ \frac{1}{2}\omega(1-\omega) & 1-\omega+\frac{1}{4}\omega^2 \end{bmatrix}. \qquad (10)$$

$$1-\omega + 1-\omega + \frac{1}{4}\omega^2 = \omega - 1 + \omega - 1$$

$$\frac{1}{4}\omega^2 - 4\omega + 4 = 0$$

$$\omega^2 - 16\omega + 16 = 0$$

$$\omega = \frac{16 \pm \sqrt{16^2 - 4\cdot 16}}{2}$$

$$= \frac{16 \pm 4\sqrt{12}}{2} = 8 \pm 4\sqrt{3}$$

To minimize $\rho = |\lambda_{max}| = |\omega - 1|$,
take $\omega_{optimal} = 8 - 4\sqrt{3} \approx 1.07$.
Then $\lambda_1 = \lambda_2 = \omega_{optimal} - 1 \approx 0.07$.

15 The tridiagonal matrix of size $n-1$ with diagonals $-1, 2, -1$ has eigenvalues $\lambda_j = 2 - 2\cos(j\pi/n)$. Why are the smallest eigenvalues approximately $(j\pi/n)^2$? The inverse power method converges at the speed $\lambda_1/\lambda_2 \approx 1/4$.

Using the Taylor Series for cosine,

$$\lambda_j = 2 - 2\cos(j\pi/n) \approx 2 - 2\left[1 - \left(\frac{j\pi}{n}\right)^2\right] = \left(\frac{j\pi}{n}\right)^2.$$

The closer $j\pi/n$ is to 0, the more accurate is this approximation. That is, for smaller j.

$$\lambda_1/\lambda_2 \approx (\pi/n)^2/(2\pi/n)^2 = \frac{1}{4}.$$

21 When $A = A^{\mathrm{T}}$, the "*Lanczos method*" finds a's and b's and orthonormal $\boldsymbol{q}$'s so that $A\boldsymbol{q}_j = b_{j-1}\boldsymbol{q}_{j-1} + a_j\boldsymbol{q}_j + b_j\boldsymbol{q}_{j+1}$ (with $\boldsymbol{q}_0 = \boldsymbol{0}$). Multiply by $\boldsymbol{q}_j^{\mathrm{T}}$ to find a formula for a_j. The equation says that $AQ = QT$ where T is a tridiagonal matrix.

$$q_j^T A q_j = b_{j-1} q_j^T q_{j-1} + a_j q_j^T q_j + b_j q_j^T q_{j+1}$$

$$q_j^T A q_j = 0 + a_j q_j^T q_j + 0$$

$$a_j = q_j^T A q_j / \|q_j\|^2 = q_j^T A q_j$$

since the q_i are orthonormal.

$$A \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} \begin{bmatrix} a_1 & b_1 & & 0 \\ b_1 & a_2 & b_2 & \\ 0 & b_2 & a_3 & b_3 \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

$$AQ = QT$$

23 Suppose A is *tridiagonal and symmetric in the QR method.* From $A_1 = Q^{-1}AQ$ show that A_1 is symmetric. Write $A_1 = RAR^{-1}$ to show that A_1 is also tridiagonal. (If the lower part of A_1 is proved tridiagonal then by symmetry the upper part is too.)

Symmetric tridiagonal matrices are the best way to start in the QR method.

$$A_1^T = Q^T A^T (Q^{-1})^T = Q^{-1} A (Q^T)^T = Q^{-1} A Q = A_1$$

since $Q^T = Q^{-1}$.

$A_1 = RQ = RQRR^{-1} = RAR^{-1}$. Since R is upper triangular, R^{-1} is upper triangular. Since A is tridiagonal, A_1 cannot have nonzero entries below the first sub-diagonal. Since A_1 is symmetric this means A_1 cannot have nonzero entries above the first superdiagonal.