MA 4222 – Mini Project 2 – Krylov Subspace Methods

Part 1

1. We will work with the Matlab m file, `CodeHarwellBoeingMatrix.m` in your MiniProject2 folder. This code loads one of the Harwell-Boeing test matrices that describes the connections in a model of a diffraction column in a chemical plant. You will look at solving the system $A\vec{x} = \vec{b}$.

   (a) Run the m file. What happens for standard conjugate gradient and biconjugate gradient (using `cgs, bicg`)? A plot of the residual or error should appear after running the m file. Why does this occur? You should look at the structure of the matrix by plotting a display of the nonzero entries in a matrix using the `spy` command. Do you meet the criteria for these methods? You should be able to see if you meet the criteria from the structure of the matrix as well as using other Matlab commands to get logical statements about whether criteria are met. Please include this graph and a commentary/discussion of how you verified whether the criteria were met.

   (b) Add the BiCGStab and the GMRES method to the comparison by modifying the provided code. Do these methods converge? Try to alter the maximum number of iterations and the restart value of the GMRES method (use `gmres(A,b,restart,tol,maxit)`). Please provide graphs and discuss results.

2. A matrix with constant entries along the diagonal is called a Toeplitz matrix. Toeplitz matrices arise in certain types of numerical solutions for differential equations, spline functions, integral equations, time series analysis, as well as in Markov chains and queueing theory.

   (a) Create a main file that solves the system $A\vec{x} = \vec{b}$ where A and $\vec{b}$ are initialized in Matlab as:

   ```
   A=toeplitz([2, -1, zeros(1, n-2)])*(n+1)^2;
   z=(1:n)'/(n+1);
   b=z.*sin(z);
   ```

   (b) Solve the system for $n = 5$, 20, 100, 500, 1000 with a tolerance 10^{-4} and the maximum number of iterations to 10000. Try solving the system using Matlab's built in commands for `cgs` and bicg (or you are welcome to write your own version), and comparing to using the standard backslash command. (You can modify the code in the first problem to do this problem).

   (c) Compare the number of iterations and the time it takes to solve each method (for each of the different cases for n). To compare computational time, you can use the `tic tic` command. Make a table and comment/discuss your results.

   (d) Make a plot of the residual as a function of iteration number for a few different matrix sizes and dimensions for different methods. What is the convergence rate? What do you observe? Again, include the plot in your report with comments. the number of iterations used and a vector with the approximate errors after each iteration from the routines when using the `iter` and `relres` return values.

   (e) Preconditioners are often used to speed up convergence. Cholesky decomposition or Cholesky factorization is a decomposition of a real valued SPD matrix into the product of a lower triangular matrix and its transpose. An incomplete Cholesky factorization is a sparse approximation of the Cholesky factorization. In Matlab, you can use `ichol(A)` for no fill-in or the `ichol(A,opts)` where you can set the type of incomplete cholesky factorization to `ict`, where the t stands for threshold dropping and you can set `droptol` to a nonnegative scalar used as a drop tolerance when performing ICT. Elements which are smaller in magnitude than a local drop tolerance are dropped from the resulting factor except for the diagonal element which is never dropped with the drop tolerance t small (less than 0.1) for more fill-in. Try a few different incomplete Cholesky factorizations and input these as preconditions to the different methods that allow for a preconditioner. Describe your numerical experiments with a short discussion as well as a table or graph in your report. In addition, also comment on the condition number of the new system. When you solve with a preconditioned P, you are solving $P^{-1}A\vec{x} = P^{-1}\vec{b}$, where for Cholesky, $P = LL^T$ where L is a lower triangular matrix.

Part 2

Find a research article that describes an application of Krylov Subspace Methods (e.g. conjugate gradient methods (CG), generalized minimal residual (GMRES), biconjugate gradient (bcg), preconditioned iterative methods). Write a few paragraphs describing how it is used in an application, what the application is about, and how these techniques are needed/help for the application. Make sure to properly cite your article as well as highlight some of the mathematical or computational complexities. If the standard method has been modified, please detail what has been done.