MA 595 Homework 4

May 9, 2021

Practice Problems

Reference

$$w_1 = e^{2\pi i} = 1, \quad w_2 = e^{\pi i} = -1, \quad w_3 = e^{2\pi i/3}, \quad w_4 = e^{\pi i/2} = i, \quad w_n = e^{2\pi i/n}$$

$$F_1 = [1], \quad F_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix},$$

$$(F_n)_{jk} = (w_n)^{jk} = e^{2\pi ijk/n} \quad j,k \in \{0,1,\ldots,n-1\}, \quad F_n^{-1} = \frac{1}{n}\overline{F}$$

$$F_n = \begin{bmatrix} I_m & D_m \\ I_m & -D_m \end{bmatrix} \begin{bmatrix} F_m & 0 \\ 0 & F_m \end{bmatrix} \begin{bmatrix} \text{even - odd} \\ \text{permutation} \end{bmatrix}, \quad (D_m)_{jj} = (w_m)^j, \quad j = 0,1,\ldots,m-1, \quad m = n/2$$

$$y_j = y'_j + w_n^j y''_j, \quad y_{j+m} = y'_j - w_n^j y''_j, \quad j = 0,1,\ldots,m-1$$

Linear Algebra 4th Edition Strang Problem Set 10.3

1. (Problem 10.3.8) Compute $y = F_8 c$ by the three FFT steps for $c = (1,0,1,0,1,0,1,0)$. Repeat the computation for $c = (0,1,0,1,0,1,0,1)$.

   We will introduce binary superscripts to denote even and odd splittings of c for each recursive step. For $c = (1,0,1,0,1,0,1,0)$ we have

$$c^0 = (1,1,1,1), \ c^1 = (0,0,0,0)$$
$$c^{00} = (1,1), \ c^{01} = (1,1) \quad c^{10} = (0,0), \ c^{11} = (0,0)$$
$$c^{000} = (1), \ c^{001} = (1) \qquad c^{010} = (1), \ c^{011} = (1) \qquad c^{100} = (0), \ c^{101} = (0) \qquad c^{110} = (0), \ c^{111} = (0)$$

   To calculate $y = F_8 c$ using the FFT, we need to use similar notation to track the recursive splittings of vectors based on y.

$$y_j = y_j^0 + w_8^j y_j^1, \quad y_{j+4} = y_j^0 - w_8^j y_j^1, \;\; j = 0, 1, 2, 3$$

$$\begin{aligned} y_j^0 &= y_j^{00} + w_4^j y_j^{01}, \quad y_{j+2}^0 = y_j^{00} - w_4^j y_j^{01}, \;\; j = 0, 1 \\ y_j^1 &= y_j^{10} + w_4^j y_j^{11}, \quad y_{j+2}^1 = y_j^{10} - w_4^j y_j^{11}, \;\; j = 0, 1 \end{aligned}$$

$$\begin{aligned} y_j^{00} &= y_j^{000} + w_2^j y_j^{001}, \quad y_{j+1}^{00} = y_j^{000} - w_2^j y_j^{001}, \;\; j = 0 \\ y_j^{01} &= y_j^{010} + w_2^j y_j^{011}, \quad y_{j+1}^{01} = y_j^{010} - w_2^j y_j^{011}, \;\; j = 0 \\ y_j^{10} &= y_j^{100} + w_2^j y_j^{101}, \quad y_{j+1}^{10} = y_j^{100} - w_2^j y_j^{101}, \;\; j = 0 \\ y_j^{11} &= y_j^{110} + w_2^j y_j^{111}, \quad y_{j+1}^{11} = y_j^{110} - w_2^j y_j^{111}, \;\; j = 0 \end{aligned}$$

$$y^{000} = c^{000}, \; y^{001} = c^{001}, \ldots, y^{111} = c^{111}$$

To get y, work backwards from the base case. Based on the given c, many of the terms in the expansion are zeroed out.

$$y^{000} = 1, y^{001} = 1, y^{010} = 1, y^{011} = 1, y^{100} = 0, y^{101} = 0, y^{110} = 0, y^{111} = 0$$

$$\begin{aligned} y_0^{00} = 1 + 1 = 2, \quad y_1^{00} = 1 - 1 = 0, &\implies y^{00} = (2, 0) \\ y_0^{01} = 1 + 1 = 2, \quad y_1^{01} = 1 - 1 = 0, &\implies y^{01} = (2, 0) \\ & y^{10} = (0, 0) \\ & y^{11} = (0, 0) \end{aligned}$$

$$\begin{aligned} y_0^0 = 4, \;\; y_1^0 = 0, \;\; y_2^0 = 0, \;\; y_3^0 = 0 \\ \implies y^0 = (4, 0, 0, 0) \\ y^1 = (0, 0, 0, 0) \end{aligned}$$

$$y = (4, 0, 0, 0, 4, 0, 0, 0)$$

To check this result, calculate the matrix multiplication $F_8 c$, which gives $y = (4, 0, 0, 0, 4, 0, 0, 0)$.

For $c = (0, 1, 0, 1, 0, 1, 0, 1)$, we have

$$(c^{000}, c^{001}, c^{010}, c^{011}, c^{100}, c^{101}, c^{110}, c^{111}) = (0, 0, 0, 0, 1, 1, 1, 1)$$

This means all terms with 0 leading in the superscript like $y^0, y^{00}, y^{01}, y^{000}, y^{010}, y^{001}, y^{011}$ are zero. Only terms with 1 leading in the superscript are nonzero. Less detail will be shown in this calculation since it is similar to the previous calculation. Only nonzero y^b vectors written out.

$$\begin{aligned} y_0^{10} &= 1 + 1 = 2, y_1^{10} = 1 - 1 = 0 \implies y^{10} = (2, 0) \\ y_0^{11} &= 1 + 1 = 2, y_1^{11} = 1 - 1 = 0 \implies y^{11} = (2, 0) \\ y_0^1 &= 2 + 2 = 4, y_1^1 = 0, y_2^1 = 2 - 2 = 0, y_3^1 = 0 \implies y^1 = (4, 0, 0, 0) \end{aligned}$$

$$y = (4, 0, 0, 0, -4, 0, 0, 0)$$

2. (Problem 10.3.9)

If $w = e^{2\pi i/64}$ then w^2 and $\sqrt{w}$ are among the ______ and ______ roots of 1.

$w^2 = e^{2\pi i/32}$ so w^2 is among the 32nd roots of 1.
$\sqrt{w} = e^{2\pi i/128}$ so $\sqrt{w}$ is among the 128th roots of 1.

3. (Problem 10.3.13)

(a) Two eigenvectors of C are $(1,1,1,1)$ and $(1,i,i^2,i^3)$. Find the eigenvalues.

$$\begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = e_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{bmatrix} \begin{bmatrix} 1 \\ i \\ i^2 \\ i^3 \end{bmatrix} = e_2 \begin{bmatrix} 1 \\ i \\ i^2 \\ i^3 \end{bmatrix}$$

The matrix equations give the systems of equations:

$$\begin{aligned} c_0 + c_1 + c_2 + c_3 &= e_1 & \quad c_0 + ic_1 + i^2c_2 + i^3c_3 &= e_2 \\ c_3 + c_0 + c_1 + c_2 &= e_1 & \quad c_3 + c_0 i + c_1 i^2 + c_2 i^3 &= ie_2 \\ c_2 + c_3 + c_0 + c_1 &= e_1 & \quad c_2 + c_3 i + c_0 i^2 + c_1 i^3 &= i^2 e_2 \\ c_1 + c_2 + c_3 + c_0 &= e_1 & \quad c_1 + c_2 i + c_3 i^2 + c_0 i^3 &= i^3 e_2 \end{aligned}$$

The four equations in the first system of equations all agree that $e_1 = c_0 + c_1 + c_2 + c_3$. All four equations in the second system of equations are all multiples of the first equation. That is, all equations agree that $e_2 = c_0 + ic_1 + i^2c_2 + i^3c_3$.

(b) $P = F\Lambda F^{-1}$ immediately gives $P^2 = F\Lambda^2F^{-1}$ and $P^3 = F\Lambda^3F^{-1}$. Then

$$\begin{aligned} C &= c_0 I + c_1 P + c_2 P^2 + c_3 P^3 \\ &= c_0 FIF^{-1} + c_1 F\Lambda F^{-1} + c_2 F\Lambda^2 F^{-1} + c_3 F\Lambda^3 F^{-1} \\ &= F(c_0 I + c_1\Lambda + c_2\Lambda^2 + c_3\Lambda^3)F^{-1} \\ &= FEF^{-1}\,. \end{aligned}$$

The matrix E is diagonal. It contains the eigenvalues of C.

4. (Problem 10.3.14)

Find the eigenvalues of the "periodic" $-1, 2, -1$ matrix from $E = 2I - \Lambda - \Lambda^3$, with the eigenvalues of P in Λ.

$$C = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \quad \text{has} \quad c_0 = 2, c_1 = -1, c_2 = 0, c_3 = -1\,.$$

Eigenvalues $e_l = 2 - 1 - 1 = 0$, $e_2 = 2 - i - i^3 = 2$, $e_3 = 2 - (-1) - (-1) = 4$, $e_4 = 2 - i^3 - i^9 = 2 + i - i = 2$.

5. (Problem 10.3.15)

To multiply C times a vector x, we can multiply $F(E(F^{-1}x))$ instead. The direct way uses n^2 separate multiplications. Knowing E and F, the second way uses only $n\log_2 n + n$ multiplications. How many of those come from E, how many from F, and how many from F^{-1}?

Diagonal E needs n multiplications, Fourier matrices F and F^{-1} need $\frac{1}{2}n\log_2 n$ multiplications each by the FFT.

Introduction to Applied Mathematics Strang Problem Set 5.5

1. (Problem 5.5.1)

   What are F^2 and F^4 for the 4 by 4 Fourier matrix F.

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \quad , \quad F^2 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix} \quad , \quad F^4 = 16I_{4\times 4} \; .$$

2. (Problem 5.5.5)

   Compute $y = F_4x$ by the three steps of the FFT for the even vector $x' = (2, 6, 6, 6)$ and the odd vector $x'' = (0, -2, 0, 2)$.

   Let $x^0 = x'$, $y^0 = y$, and $F = F_4$.

$$\begin{array}{cc} x^{00} = (2,6) & x^{01} = (6,6) \\ x^{000} = (2) \quad x^{001} = (6) & x^{010} = (6) \quad x^{011} = (6) \end{array}$$

$$\begin{aligned}
&y^{000} = (2) \quad y^{001} = (6) \quad y^{010} = (6) \quad y^{011} = (6) \\
&y^{00}_0 = 2 + 6 = 8 \quad y^{00}_1 = 2 - 6 = -4 \implies y^{00} = (8, -4) \\
&y^{01}_0 = 6 + 6 = 12 \quad y^{01}_1 = 6 - 6 = 0 \implies y^{01} = (12, 0) \\
&y^0_0 = 8 + 12 = 20 \quad y^0_1 = -4 + 0i = -4 \quad y^0_2 = 8 - 12 = -4 \quad y^0_3 = -4 - 0i = -4 \\
&\implies y^0 = (20, -4, -4, -4)
\end{aligned}$$

   Let $x^1 = x''$, $y^1 = y$, and $F = F_4$.

$$\begin{array}{cc} x^{10} = (0,0) & x^{11} = (-2,2) \\ x^{100} = (0) \quad x^{101} = (0) & x^{110} = (-2) \quad x^{111} = (2) \end{array}$$

$$\begin{aligned}
&y^{100} = (0) \quad y^{101} = (0) \quad y^{110} = (-2) \quad y^{111} = (2) \\
&y^{10}_0 = 0 + 0 = 0 \quad y^{10}_1 = 0 - 0 = 0 \implies y^{10} = (0, 0) \\
&y^{11}_0 = -2 + 2 = 0 \quad y^{11}_1 = -2 - 2 = -4 \implies y^{11} = (0, -4) \\
&y^1_0 = 0 + 0 = 0 \quad y^1_1 = 0 + (-4)i = -4i \quad y^1_2 = 8 - 12 = 0 - 0 = 0 \quad y^1_3 = 0 - (-4)i = 4i \\
&\implies y^1 = (0, -4i, 0, 4i)
\end{aligned}$$

3. (Problem 5.5.6)

   What is $y = F_8x$ if $x = (2, 0, 6, -2, 6, 0, 6, 2)$?

   Let $x^0 = (2, 6, 6, 6)$ be the even splitting of x and $x^1 = (0, -2, 0, 2)$ be the odd splitting of x. From the previous problem we know that $F_4x^0 = y^0 = (20, -4, -4, -4)$ and $F_4x^1 = y^1 =$

$(0, -4i, 0, 4i)$. Using the results from Problem 5.5.5 along with equation (2) on page 449,

$$y_j = y_j^0 + w_8^j y_j^1, \quad y_{j+m} = y_j^0 - w_8^j y_j^1, \quad j = 0, 1, 2, 3, \quad w_8 = e^{2\pi i/8} = e^{i\pi/4} \ .$$

$$\begin{aligned}
&y_0 = y_0^0 + w_8^0 y_0^1 = 20 + 0 = 20 \\
&y_1 = y_1^0 + w_8^1 y_1^1 = -4 + w_8^1(-4i) = -4 - 4iw_8 \\
&y_2 = y_2^0 + w_8^2 y_2^1 = -4 + w_8^2(0) = -4 \\
&y_3 = y_3^0 + w_8^3 y_3^1 = -4 + w_8^3(4i) = -4 + 4iw_8^3 \\
&y_4 = y_0^0 - w_8^0 y_0^1 20 - 0 = 20 \\
&y_5 = y_1^0 - w_8^1 y_1^1 = -4 - w_8^1(-4i) = -4 + 4iw_8 \\
&y_6 = y_2^0 - w_8^2 y_2^1 - 4 - w_8^2(0) = -4 \\
&y_7 = y_3^0 - w_8^3 y_3^1 - 4 - w_8^3(4i) = -4 - 4iw_8^3 \\
&\therefore y = (20, -4 - 4iw_8, -4, -4 + 4iw_8^3, 20, -4 + 4iw_8, -4, -4 - 4iw_8^3)
\end{aligned}$$

Assigned Problems

1. See Section 1 of HW4Script.m

   (a) Let $f(x) = -x^2 + 2\pi x$. Sample f at the $n = 4$ points $x_j = 2\pi j/4$ for $j = 0, 1, 2, 3$. Find coefficients c_0, c_1, c_2, c_3 such that $p(x) = c_0 + c_1 e^{ix} + c_2 e^{2ix} + c_3 e^{3ix}$ matches $f(x)$ at $x = 0, \pi/2, \pi, 3\pi/2$.

   Let $f_j := f(x_j)$ and $w := e^{2\pi i/4}$. The goal is to solve for c in the following equation.

$$F_4 c = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 \\ 1 & w^2 & w^4 & w^6 \\ 1 & w^3 & w^6 & w^9 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = f$$

   Since $F_4 \overline{F}_4 = 4I$, $F_4^{-1} = \frac{1}{4}\overline{F}_4$. Therefore $c = \frac{1}{4}\overline{F}_4 f$. To solve for the coefficients component-wise, use the formula:

$$c_k = \frac{1}{4}\sum_{j=0}^{3} f_j \overline{w}^{jk} = \frac{1}{4}\sum_{j=0}^{3} f_j e^{-ijk2\pi/4} = \frac{1}{4}\sum_{j=0}^{3} f_j e^{-ikx_j}.$$

   Using matrices:

$$\begin{aligned}
\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} &= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \overline{w} & \overline{w}^2 & \overline{w}^3 \\ 1 & \overline{w}^2 & \overline{w}^4 & \overline{w}^6 \\ 1 & \overline{w}^3 & \overline{w}^6 & \overline{w}^9 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{-\pi i/2} & e^{-\pi i} & e^{-3\pi i/2} \\ 1 & e^{-\pi i} & e^{-2\pi i} & e^{-3\pi i} \\ 1 & e^{-3\pi i/2} & e^{-3\pi i} & e^{-9\pi i/2} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{3\pi^2}{4} \\ \pi^2 \\ \frac{3\pi^2}{4} \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 0 \\ \frac{3\pi^2}{4} \\ \pi^2 \\ \frac{3\pi^2}{4} \end{bmatrix} \\
&= \begin{bmatrix} \frac{5\pi^2}{8} \\ -\frac{\pi^2}{4} \\ -\frac{\pi^2}{8} \\ -\frac{\pi^2}{4} \end{bmatrix}
\end{aligned}$$

$$\boxed{p(x) = \frac{5\pi^2}{8} - \frac{\pi^2}{4}e^{ix} - \frac{\pi^2}{8}e^{2ix} - \frac{\pi^2}{4}e^{3ix}}$$

(b) Find coefficients c_{-2}, c_{-1}, c_0, c_1 such that $p(x) = c_{-2}e^{-2ix} + c_{-1}e^{-ix} + c_0 + c_1e^{ix}$ matches $f(x)$ at $x = 0, \pi/2, \pi, 3\pi/2$. This amounts to solving for c in the following matrix equation.

$$F_4 c = \begin{bmatrix} 1 & 1 & 1 & 1 \\ w^{-2} & w^{-1} & 1 & w^1 \\ w^{-4} & w^{-2} & 1 & w^2 \\ w^{-6} & w^{-3} & 1 & w^3 \end{bmatrix} \begin{bmatrix} c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = f$$

While f is the same here as in part a, F_4 and c are not the same as in part a. However, the columns of this version of F_4 are still orthogonal with length 4. Again we have $F_4\overline{F}_4 = 4I$ so that $c = \frac{1}{4}\overline{F}_4 f$.

$$\begin{bmatrix} c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ \overline{w}^{-2} & \overline{w}^{-1} & 1 & \overline{w}^1 \\ \overline{w}^{-4} & \overline{w}^{-2} & 1 & \overline{w}^2 \\ \overline{w}^{-6} & \overline{w}^{-3} & 1 & \overline{w}^3 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ e^{-\pi i} & e^{-\pi i/2} & 1 & e^{\pi i/2} \\ e^{-2\pi i} & e^{-\pi i} & 1 & e^{\pi i} \\ e^{-3\pi i} & e^{-3\pi i/2} & 1 & e^{3\pi i/2} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{3\pi^2}{4} \\ \pi^2 \\ \frac{3\pi^2}{4} \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -i & 1 & i \\ 1 & -1 & 1 & -1 \\ -1 & i & 1 & -i \end{bmatrix} \begin{bmatrix} 0 \\ \frac{3\pi^2}{4} \\ \pi^2 \\ \frac{3\pi^2}{4} \end{bmatrix}$$

$$= \begin{bmatrix} c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \end{bmatrix}$$

$$\boxed{p(x) = \frac{5\pi^2}{8}e^{-2ix} + \frac{\pi^2}{4}e^{-ix} - \frac{\pi^2}{8} + \frac{\pi^2}{4}e^{ix}}$$

2. See Section 2 of HW4Script.m