

Written Homework 4 Solutions

MA 4291 (Tilley)
C-Term 2022

Chapter 4 Section 38: 3, 4 (page 121)

3. Show that if m and n are integers,

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \begin{cases} 0 & \text{when } m \neq n, \\ 2\pi & \text{when } m = n. \end{cases}$$

Just turn in the problems in **bold red font**:

pg. 121: # 3,4
pg. 125: # 5,6
pg. 135: # 1,3,6,7,10
pg. 140: # 2,5,6,8
pg. 149: # 4,5
pg. 160: # 2,4

For $m, n \in \mathbb{Z}$

If $m \neq n$,

$$\begin{aligned} \int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta &= \frac{1}{i(m-n)} e^{i(m-n)\theta} \Big|_0^{2\pi} \\ &= \frac{1}{i(m-n)} \left[e^{i(m-n)2\pi} - e^0 \right] = \frac{1}{i(m-n)} [1 - 1] = 0 \end{aligned}$$

If $m \neq n$,

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \int_0^{2\pi} e^0 d\theta = \int_0^{2\pi} d\theta = 2\pi$$

4. According to definition (2), Sec. 38, of definite integrals of complex-valued functions of a real variable,

$$\int_0^\pi e^{(1+i)x} dx = \int_0^\pi e^x \cos x dx + i \int_0^\pi e^x \sin x dx.$$

Evaluate the two integrals on the right here by evaluating the single integral on the left and then using the real and imaginary parts of the value found.

Ans. $-(1 + e^\pi)/2, (1 + e^\pi)/2$.

$$\int_0^\pi e^{(1+i)x} dx = \int_0^\pi e^x e^{ix} dx = \int_0^\pi e^x \cos x dx + i \int_0^\pi e^x \sin x dx$$

$$\begin{aligned} \int_0^\pi e^{(1+i)x} dx &= \frac{1}{1+i} \left[e^{(1+i)\pi} - 1 \right] = \frac{1}{1+i} \left[e^\pi e^{i\pi} - 1 \right] = -\frac{1}{1+i} \left[e^\pi + 1 \right] \\ &= -\frac{1-i}{2} \left[e^\pi + 1 \right] = -\frac{1}{2} \left[e^\pi + 1 \right] + \frac{i}{2} \left[e^\pi + 1 \right] \end{aligned}$$

$$\Rightarrow \int_0^\pi e^x \cos x dx = \operatorname{Re} \left[\int_0^\pi e^{(1+i)x} dx \right] = -\frac{1}{2} \left[e^\pi + 1 \right]$$

$$\int_0^\pi e^x \cos x dx = \operatorname{Im} \left[\int_0^\pi e^{(1+i)x} dx \right] = \frac{i}{2} \left[e^\pi + 1 \right]$$

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Chapter 4 Section 39: 5, 6 (page 126)

5. Suppose that a function $f(z)$ is analytic at a point $z_0 = z(t_0)$ lying on a smooth arc $z = z(t)$ ($a \leq t \leq b$). Show that if $w(t) = f[z(t)]$, then

$$w'(t) = f'[z(t)]z'(t)$$

when $t = t_0$.

Suggestion: Write $f(z) = u(x, y) + i v(x, y)$ and $z(t) = x(t) + iy(t)$, so that

$$w(t) = u[x(t), y(t)] + i v[x(t), y(t)].$$

Then apply the chain rule in calculus for functions of two real variables to write

$$w' = (u_x x' + u_y y') + i(v_x x' + v_y y'),$$

and use the Cauchy–Riemann equations.

$$z : \mathbb{R} \rightarrow \mathbb{C}, \quad z(t) = x(t) + iy(t)$$

$$f : \mathbb{C} \rightarrow \mathbb{C}, \quad f(z) = f(x+iy) = u(x, y) + iv(x, y)$$

$$w : \mathbb{R} \rightarrow \mathbb{C}, \quad w(t) = (f \circ z)(t) = f[z(t)] = u[x(t), y(t)] + iv[x(t), y(t)]$$

$$w'(t_0) = \left. \frac{d}{dt} [u[x(t), y(t)] + iv[x(t), y(t)]] \right|_{t=t_0}$$

$$= \left[\frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + i \left(\frac{\partial v}{\partial x} \frac{dx}{dt} + \frac{\partial v}{\partial y} \frac{dy}{dt} \right) \right] \Big|_{t=t_0}$$

$$= \left[u_x x' + u_y y' + i(v_x x' + v_y y') \right] \Big|_{t=t_0} \quad \begin{pmatrix} \text{At } t=t_0, \\ u_x = v_y \\ u_y = -v_x \end{pmatrix}$$

$$= \left[(u_x + iv_x)x' + (iu_x - v_x)y' \right] \Big|_{t=t_0}$$

$$= \left[(u_x + iv_x)x' + i(u_x + iv_x)y' \right] \Big|_{t=t_0}$$

$$= f'(z_0) x'(t_0) + f'(z_0) y'(t_0)$$

$$= f'(z(t_0)) z'(t_0)$$

6. Let $y(x)$ be a real-valued function defined on the interval $0 \leq x \leq 1$ by means of the equations

$$y(x) = \begin{cases} x^3 \sin(\pi/x) & \text{when } 0 < x \leq 1, \\ 0 & \text{when } x = 0. \end{cases}$$

(a) Show that the equation

$$z = x + iy(x) \quad (0 \leq x \leq 1)$$

represents an arc C that intersects the real axis at the points $z = 1/n$ ($n = 1, 2, \dots$) and $z = 0$, as shown in Fig. 38.

(b) Verify that the arc C in part (a) is, in fact, a *smooth* arc.

Suggestion: To establish the continuity of $y(x)$ at $x = 0$, observe that

$$0 \leq \left| x^3 \sin\left(\frac{\pi}{x}\right) \right| \leq x^3$$

when $x > 0$. A similar remark applies in finding $y'(0)$ and showing that $y'(x)$ is continuous at $x = 0$.

Def: A set of points $z = (x, y)$ is said to be an arc if

$$x = x(t) \quad y = y(t) \quad a \leq t \leq b$$

where $x(t), y(t) : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions.

(a) $z = x + iy(x)$ ($0 \leq x \leq 1$). Parameterize $z = z(t) = t + it^3 \sin(\pi/t)$ using

$$x = x(t) = t \text{ for } 0 \leq t \leq 1$$

$$y = y(x(t)) = \begin{cases} x^3 \sin(\pi/x) = t^3 \sin(\pi/t), & 0 < t \leq 1 \\ 0, & t = 0 \end{cases}$$

Then $x = x(t)$ and $y = y(t)$. $x(t) = t$ is continuous and $y(t)$ is continuous on $0 < t \leq 1$ as the composition / product of continuous functions. Also $\lim_{t \rightarrow 0^+} t^3 \sin(\pi/t) = 0$ (part b)

so y is continuous at $t = 0$ as well.

$\therefore z = x + iy$ represents an arc. Call this arc C .

C intersects the real axis $\Leftrightarrow y = 0$.

$y(x) = 0$ when $x = 0$ and also when $\sin(\pi/x) = 0$.

$\sin(\pi/x) = 0 \Leftrightarrow \pi/x = \pi n, n \in \mathbb{Z} \Leftrightarrow x = 1/n, n \in \mathbb{Z}$

Note that $x = 1/n \in [0, 1] \ \forall n$ (we assumed $x \in [0, 1]$)

Def: A smooth arc is an arc $z = z(t)$ ($a \leq t \leq b$) s.t. $z'(t)$ is continuous on $a \leq t \leq b$ (i.e. $z \in C^1[a, b]$) and $z'(t) \neq 0 \quad \forall t \in (a, b)$

(b) In order to be an arc (smooth or not), $z(t)$ must be continuous. Equivalently, the components $x(t)$ and $y(t)$ must be continuous. The only point where continuity is questionable is $t=0$.

Claim: $\lim_{t \rightarrow 0^+} y(t) = 0$

Proof: With $y(t)$ parameterized as in part a,

$$|y(t) - 0| = |t^3 \sin(\pi/t)| \leq |t^3| = t^3 \rightarrow 0 \text{ as } t \rightarrow 0^+ \quad \square$$

Since $\lim_{t \rightarrow 0^+} y(t) = y(0)$, $y(t)$ and thereby $z(t)$ is continuous at 0.

△ Next show $z'(t) = x'(t) + iy'(t)$ is continuous on $[0, 1]$

$$z'(t) = 1 + i(3t^2 \sin(\pi/t) - \pi t \cos(\pi/t)) \quad \text{for } 0 < t \leq 1$$

$$\begin{aligned} z'(0) &= \lim_{\Delta t \rightarrow 0} \frac{z(0 + \Delta t) - z(0)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\Delta t + i(\Delta t)^3 \sin(\pi/\Delta t)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} (1 + (\Delta t)^2 \sin(\pi/\Delta t)) = 1 + 0 = 1 \end{aligned}$$

(by reasoning similar to that used to prove $y(t)$ is continuous at 0).

Since $z'(t)$ can be written using compositions, products, and difference of continuous functions on $0 < t \leq 1$, $z'(t)$ is also continuous on $0 < t \leq 1$. To show continuity at $t=0$,

$$\begin{aligned} |z'(t) - z'(0)| &= |1 + i(3t^2 \sin(\pi/t) - \pi t \cos(\pi/t)) - 1| \\ &\leq 3t^2(|\sin(\pi/t)| + \pi t) + |\cos(\pi/t)| \\ &\leq 3t^2 + \pi t \rightarrow 0 \text{ as } t \rightarrow 0^+ \end{aligned}$$

Since $\lim_{t \rightarrow 0^+} z'(t) = z'(0)$, $z'(t)$ is continuous at $t=0$ as well.

△ For $0 < t < 1$, $0 = z'(t) = 1 + i(3t^2 \sin(\pi/t) - \pi t \cos(\pi/t))$ would require both $0 = 1$ and $0 = 3t^2 \sin(\pi/t) - \pi t \cos(\pi/t)$ by matching real and imaginary components. Clearly $0 \neq 1$ so $z'(t) \neq 0$ for $0 < t < 1$. Conclude $z(t)$ is a smooth arc.

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Chapter 4 Section 42 : 1, 3, 6, 7, 10 (pages 135, 136)

For the functions f and contours C in Exercises 1 through 7, use parametric representations for C , or legs of C , to evaluate

$$\int_C f(z) dz.$$

1. $f(z) = (z+2)/z$ and C is

- (a) the semicircle $z = 2e^{i\theta}$ ($0 \leq \theta \leq \pi$);
- (b) the semicircle $z = 2e^{i\theta}$ ($\pi \leq \theta \leq 2\pi$);
- (c) the circle $z = 2e^{i\theta}$ ($0 \leq \theta \leq 2\pi$).

Ans. (a) $-4 + 2\pi i$; (b) $4 + 2\pi i$; (c) $4\pi i$.

$$\begin{aligned}
 (a) \quad \int_0^\pi f[z(\theta)] z'(\theta) d\theta &= \int_0^\pi \frac{2e^{i\theta} + 2}{2e^{i\theta}} 2ie^{i\theta} d\theta \\
 &= 2i \int_0^\pi (e^{i\theta} + 1) d\theta \\
 &= 2i \left(\frac{1}{i} e^{i\theta} + \theta \right) \Big|_0^\pi \\
 &= 2i \left(-\frac{1}{i} + \pi - \frac{1}{i} \right) = \boxed{-4 + 2\pi i}
 \end{aligned}$$

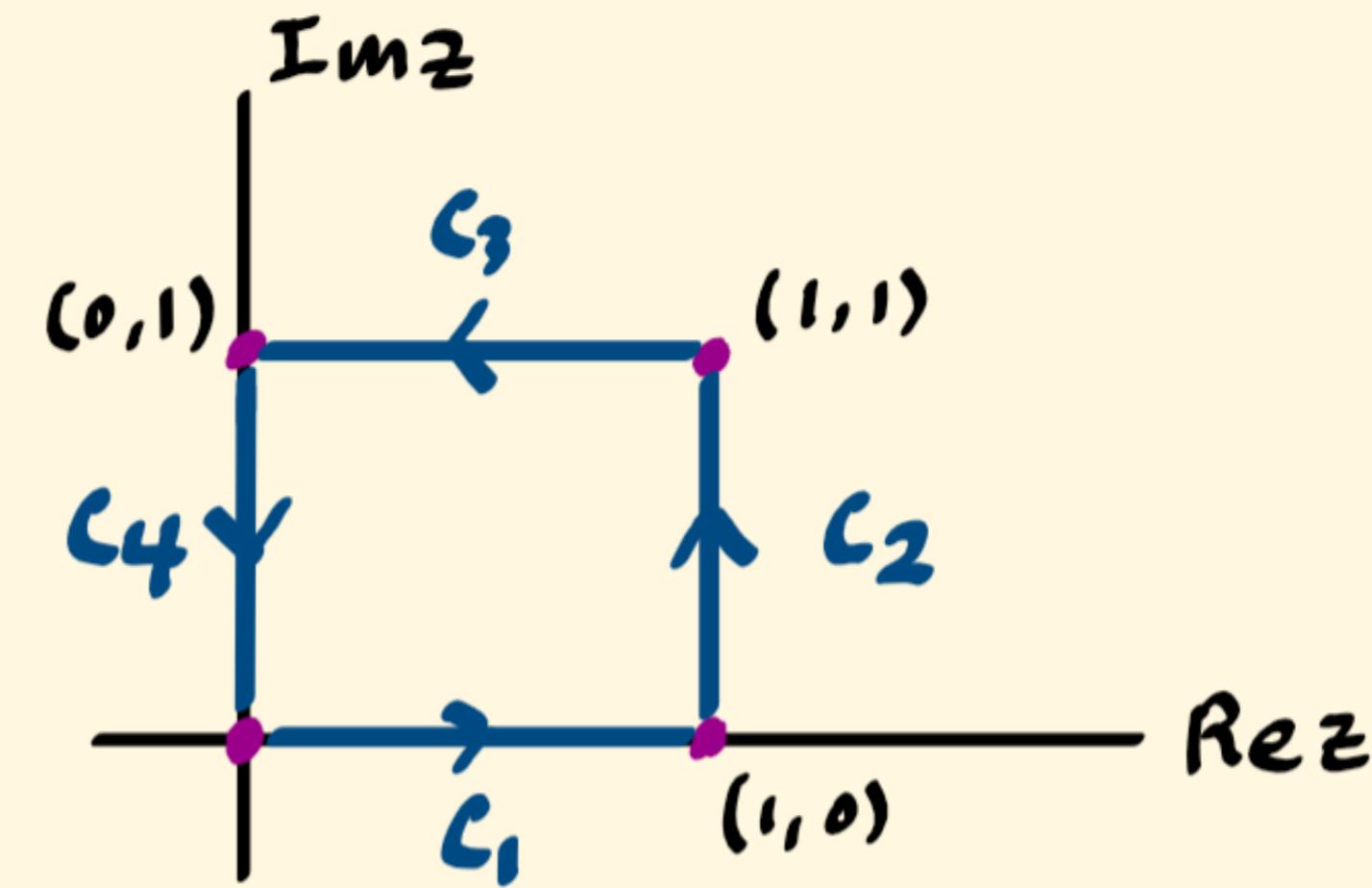
$$\begin{aligned}
 (b) \quad \int_\pi^{2\pi} f[z(\theta)] z'(\theta) d\theta &= \int_\pi^{2\pi} \frac{2e^{i\theta} + 2}{2e^{i\theta}} 2ie^{i\theta} d\theta \\
 &= 2i \left(\frac{1}{i} e^{i\theta} + \theta \right) \Big|_\pi^{2\pi} \\
 &= 2i \left(\frac{1}{i} + 2\pi + \frac{1}{i} - \pi \right) = \boxed{4 + 2\pi i}
 \end{aligned}$$

$$\begin{aligned}
 (c) \quad \int_0^{2\pi} f[z(\theta)] z'(\theta) d\theta &= \int_0^{2\pi} \frac{2e^{i\theta} + 2}{2e^{i\theta}} 2ie^{i\theta} d\theta \\
 &= \int_0^\pi \frac{2e^{i\theta} + 2}{2e^{i\theta}} 2ie^{i\theta} d\theta + \int_\pi^{2\pi} \frac{2e^{i\theta} + 2}{2e^{i\theta}} 2ie^{i\theta} d\theta \\
 &= -4 + 2\pi i + 4 + 2\pi i = \boxed{4\pi i}
 \end{aligned}$$

3. $f(z) = \pi \exp(\pi \bar{z})$ and C is the boundary of the square with vertices at the points $0, 1, 1+i$, and i , the orientation of C being in the counterclockwise direction.

Ans. $4(e^\pi - 1)$.

$$z(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 1 + (t-1)i & 1 \leq t \leq 2 \\ 1 - (t-2) + i & 2 \leq t \leq 3 \\ i - (t-3)i & 3 \leq t \leq 4 \end{cases}$$



$$z'(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ i & 1 \leq t \leq 2 \\ -1 & 2 \leq t \leq 3 \\ -i & 3 \leq t \leq 4 \end{cases}$$

$$\begin{aligned} \int_C f dz &= \int_{C_1} f dz + \int_{C_2} f dz + \int_{C_3} f dz + \int_{C_4} f dz \\ &= \int_0^1 f(z(t)) z'(t) dt + \int_1^2 f(z(t)) z'(t) dt \\ &\quad + \int_2^3 f(z(t)) z'(t) dt + \int_3^4 f(z(t)) z'(t) dt \\ &= \int_0^1 \pi \exp[\pi t] dt + \int_1^2 i \pi \exp[\pi(1-(t-1)i)] dt \\ &\quad - \int_2^3 \pi \exp[\pi(3-t-i)] dt - \int_3^4 i \pi \exp[\pi(t-4)i] dt \\ &= (e^\pi - 1) + 2e^\pi + (e^\pi - 1) - 2 = \boxed{4e^\pi - 4} \end{aligned}$$

Integrals evaluated by Wolfram Alpha.

6. $f(z)$ is the branch

$$z^{-1+i} = \exp[(-1+i)\log z] \quad (|z| > 0, 0 < \arg z < 2\pi)$$

of the indicated power function, and C is the unit circle $z = e^{i\theta}$ ($0 \leq \theta \leq 2\pi$).

$$\text{Ans. } i(1 - e^{-2\pi}).$$

$$\begin{aligned} \int_C f dz &= \int_0^{2\pi} \exp[(-1+i)\log \exp(i\theta)] i \exp[i\theta] d\theta \\ &= \int_0^{2\pi} i \exp[-i\theta + i^2\theta] \exp[i\theta] d\theta \quad (*) \\ &= \int_0^{2\pi} i e^{-\theta} d\theta \\ &= -i e^{-\theta} \Big|_0^{2\pi} = -i(e^{-2\pi} - 1) = \boxed{i(1 - e^{-2\pi})} \end{aligned}$$

(*) $\log e^{i\theta} = i\theta$? See the end of page 101 for an explanation of why this is ok even though $0 < \arg z < 2\pi$ instead of $-\pi < \arg z < \pi$

7. $f(z)$ is the principal branch

$$z^i = \exp(i \operatorname{Log} z) \quad (|z| > 0, -\pi < \operatorname{Arg} z < \pi)$$

of this power function, and C is the semicircle $z = e^{i\theta}$ ($0 \leq \theta \leq \pi$).

$$\text{Ans. } -\frac{1 + e^{-\pi}}{2}(1 - i).$$

$$\begin{aligned} \int_C f dz &= \int_0^\pi f[z(\theta)] z'(\theta) d\theta \\ &= \int_0^\pi \exp[i \operatorname{Log} e^{i\theta}] i e^{i\theta} d\theta \\ &= \int_0^\pi i e^{i^2\theta} e^{i\theta} d\theta \\ &= \int_0^\pi i e^{(i-1)\theta} d\theta \\ &= \frac{i}{i-1} [e^{(i-1)\theta}] \Big|_0^\pi \\ &= \frac{i}{i-1} [e^{i\pi} e^{-\pi} - 1] \\ &= -\frac{i(-1-i)}{2} [e^{-\pi} + 1] = \boxed{-\frac{1+e^{-\pi}}{2}(1-i)} \end{aligned}$$

10. Let C_0 and C denote the circles

$$z = z_0 + Re^{i\theta} \quad (-\pi \leq \theta \leq \pi) \quad \text{and} \quad z = Re^{i\theta} \quad (-\pi \leq \theta \leq \pi),$$

respectively.

(a) Use these parametric representations to show that

$$\int_{C_0} f(z - z_0) dz = \int_C f(z) dz$$

when f is piecewise continuous on C .

(b) Apply the result in part (a) to integrals (5) and (6) in Sec. 42 to show that

$$\int_{C_0} (z - z_0)^{n-1} dz = 0 \quad (n = \pm 1, \pm 2, \dots) \quad \text{and} \quad \int_{C_0} \frac{dz}{z - z_0} = 2\pi i.$$

(a)

$$\int_{C_0} f(z - z_0) dz = \int_{-\pi}^{\pi} f[z_0 + Re^{i\theta} - z_0] (0 + iRe^{i\theta}) d\theta = \int_{-\pi}^{\pi} f(Re^{i\theta}) iRe^{i\theta} d\theta$$

$$\int_C f(z) dz = \int_{-\pi}^{\pi} f(Re^{i\theta}) iRe^{i\theta} d\theta$$

$$\text{Conclude } \int_{C_0} f(z - z_0) dz = \int_C f(z) dz$$

(b) From Example 2 of Sec. 42 we have

$$(5) \quad \int_C z^{n-1} dz = 0 \quad (n = \pm 1, \pm 2, \dots).$$

If a is allowed to be zero, we have

$$(6) \quad \int_C \frac{dz}{z} = \int_{-\pi}^{\pi} \frac{1}{Re^{i\theta}} iRe^{i\theta} d\theta = i \int_{-\pi}^{\pi} d\theta = 2\pi i.$$

for the same circle C parameterized by $z = Re^{i\theta}$, $-\pi \leq \theta \leq \pi$ given in this exercise.

Let $f(z) = z^{n-1}$, $n \in \mathbb{Z} \setminus \{0\}$. By part (a) and equation (5) :

$$\int_{C_0} (z - z_0)^{n-1} dz = \int_{C_0} f(z - z_0) dz = \int_C f(z) dz = \int_C z^{n-1} dz = 0 .$$

Let $f(z) = z^{-1}$. By part (a) and equation (6) :

$$\int_{C_0} \frac{dz}{z - z_0} = \int_{C_0} f(z - z_0) dz = \int_C f(z) dz = \int_C \frac{dz}{z} = 2\pi i .$$

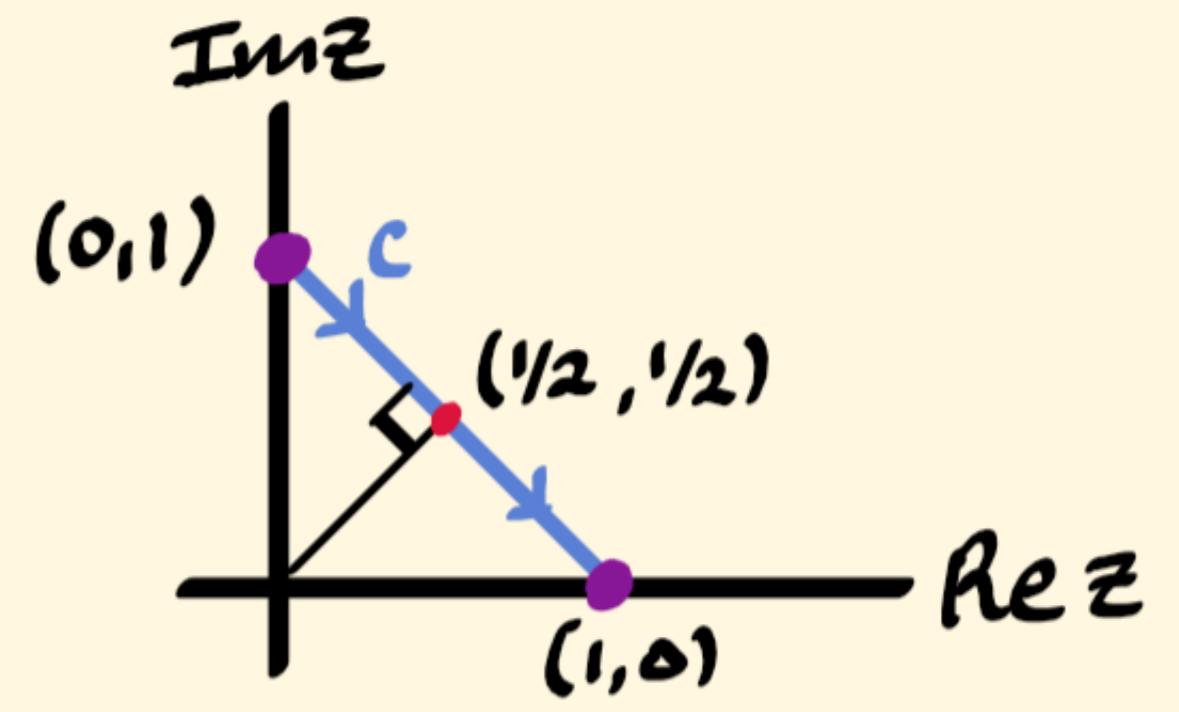
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Chapter 4 Section 43: 2, 5, 6, 8 (pages 140, 141)

2. Let C denote the line segment from $z = i$ to $z = 1$. By observing that of all the points on that line segment, the midpoint is the closest to the origin, show that

$$\left| \int_C \frac{dz}{z^4} \right| \leq 4\sqrt{2}$$

without evaluating the integral.



The shortest distance btwn the origin and C is along the line that passes through the origin and is perpendicular to C . This line intersects C at the midpoint $z_0 = 1/2 + 1/2i$.

If z is a pt on C then $|z| \geq |z_0| = 1/\sqrt{2}$. Then $\forall z$ on C :

$$|f(z)| = \left| \frac{1}{z^4} \right| = \frac{1}{|z|^4} \leq \frac{1}{(1/\sqrt{2})^4} = \frac{1}{\sqrt{2}^4} = \frac{1}{4} =: M$$

The length of C is $\sqrt{2} =: L$. By the Theorem on page 138,

$$\left| \int_C f dz \right| = \left| \int_C \frac{dz}{z^4} \right| \leq ML = \sqrt{2} \cdot \frac{1}{4} = \frac{\sqrt{2}}{4}$$

5. Let C_R be the circle $|z| = R$ ($R > 1$), described in the counterclockwise direction. Show that

$$\left| \int_{C_R} \frac{\log z}{z^2} dz \right| < 2\pi \left(\frac{\pi + \ln R}{R} \right),$$

and then use l'Hospital's rule to show that the value of this integral tends to zero as R tends to infinity.

$$\log z = \ln R + i \operatorname{Arg} z, -\pi < \operatorname{Arg} z < \pi \Rightarrow |\log z| < \ln R + \pi$$

$$|z| = R > 1 \Rightarrow |z|^2 > R^2 \Rightarrow \frac{1}{|z|^2} < \frac{1}{R^2}$$

$$|f(z)| = \left| \frac{\log z}{z^2} \right| = \frac{|\log z|}{|z|^2} < \frac{\ln R + \pi}{R^2} =: M \quad \begin{bmatrix} \exists \varepsilon > 0 \text{ s.t. } |f(z)| \leq M - \varepsilon \\ \text{to handle the '}' in the theorem } \end{bmatrix}$$

The length of the circle is $L = 2\pi R$

$$\therefore \left| \int_{C_R} f dz \right| = \left| \int_{C_R} \frac{\log z}{z^2} dz \right| \leq L(M - \varepsilon) < LM = 2\pi \frac{\ln R + \pi}{R}$$

$$0 \leq \lim_{R \rightarrow \infty} \left| \int_{C_R} f dz \right| \leq \lim_{R \rightarrow \infty} 2\pi \frac{\ln R + \pi}{R} = \lim_{R \rightarrow \infty} 2\pi/R = 0$$

$$\Rightarrow \lim_{R \rightarrow \infty} \left| \int_{C_R} f dz \right| = 0 \Rightarrow \lim_{R \rightarrow \infty} \int_{C_R} f dz = 0$$

6. Let C_ρ denote a circle $|z| = \rho$ ($0 < \rho < 1$), oriented in the counterclockwise direction, and suppose that $f(z)$ is analytic in the disk $|z| \leq 1$. Show that if $z^{-1/2}$ represents any particular branch of that power of z , then there is a nonnegative constant M , independent of ρ , such that

$$\left| \int_{C_\rho} z^{-1/2} f(z) dz \right| \leq 2\pi M \sqrt{\rho}.$$

Thus show that the value of the integral here approaches 0 as ρ tends to 0.

Suggestion: Note that since $f(z)$ is analytic, and therefore continuous, throughout the disk $|z| \leq 1$, it is bounded there (Sec. 18).

- Since $f(z)$ is continuous on the compact set $\{z : |z| \leq 1\}$, $f(z)$ is bounded on this set: $\exists M > 0$ s.t. $|f(z)| \leq M \quad \forall z$ s.t. $|z| \leq 1$
- $|z^{-1/2}| = |\exp[-\frac{1}{2}\log z]| = |\exp[-\frac{1}{2}(\ln \rho + i(\operatorname{Arg} z + 2n\pi))]|$
 $= |\exp[\ln \rho^{-1/2}]||\exp[i(\operatorname{Arg} z + 2n\pi)]| = \rho^{-1/2} \cdot 1 = \frac{1}{\sqrt{\rho}}$
- $g(z) := z^{-1/2} f(z)$
 $|g(z)| = |z^{-1/2} f(z)| = |z^{-1/2}| |f(z)| \leq M / \sqrt{\rho}$
- The length of the circle C_ρ is the circumference: $L = 2\pi\rho$.
- By the theorem on page 138 (using $L := 2\pi\rho$, $m := M/\sqrt{\rho}$),
 $\left| \int_{C_\rho} w(z) dz \right| = \left| \int_{C_\rho} z^{-1/2} f(z) dz \right| \leq (2\pi\rho)(M/\sqrt{\rho}) = 2\pi M \sqrt{\rho}$
- Since $0 \leq \left| \int_{C_\rho} z^{-1/2} f(z) dz \right| \leq 2\pi M \sqrt{\rho}$ for each $\rho \in (0, 1)$,
 $0 \leq \lim_{\rho \rightarrow 0^+} \left| \int_{C_\rho} z^{-1/2} f(z) dz \right| \leq \lim_{\rho \rightarrow 0^+} 2\pi M \sqrt{\rho} = 0$
 $\Rightarrow \lim_{\rho \rightarrow 0^+} \left| \int_{C_\rho} z^{-1/2} f(z) dz \right| = 0$
 $\Rightarrow \lim_{\rho \rightarrow 0^+} \int_{C_\rho} z^{-1/2} f(z) dz = 0$

8. Let C_N denote the boundary of the square formed by the lines

$$x = \pm \left(N + \frac{1}{2}\right) \pi \quad \text{and} \quad y = \pm \left(N + \frac{1}{2}\right) \pi,$$

where N is a positive integer and the orientation of C_N is counterclockwise.

(a) With the aid of the inequalities

$$|\sin z| \geq |\sin x| \quad \text{and} \quad |\sin z| \geq |\sinh y|,$$

obtained in Exercises 8(a) and 9(a) of Sec. 34, show that $|\sin z| \geq 1$ on the vertical sides of the square and that $|\sin z| > \sinh(\pi/2)$ on the horizontal sides. Thus show that there is a positive constant A , independent of N , such that $|\sin z| \geq A$ for all points z lying on the contour C_N .

(b) Using the final result in part (a), show that

$$\left| \int_{C_N} \frac{dz}{z^2 \sin z} \right| \leq \frac{16}{(2N+1)\pi A}$$

and hence that the value of this integral tends to zero as N tends to infinity.

(a)

\triangle For $N \in \{1, 2, 3, \dots\}$

$$\sin[(N + \frac{1}{2})\pi] = (-1)^N$$

$$\sin[-(N + \frac{1}{2})\pi] = -\sin[(N + \frac{1}{2})\pi] = (-1)^{N+1}$$

$$\Rightarrow |\sin[(N + \frac{1}{2})\pi]| = |\sin[-(N + \frac{1}{2})\pi]| = 1$$

On the vertical sides of the square $x = \pm(N + \frac{1}{2})\pi$:

$$|\sin z| \geq |\sin x| = 1$$

\triangle Since $\sinh y$ is strictly increasing and $N + \frac{1}{2} > \frac{1}{2}$,

$$\sinh[(N + \frac{1}{2})\pi] > \sinh[\pi/2]$$

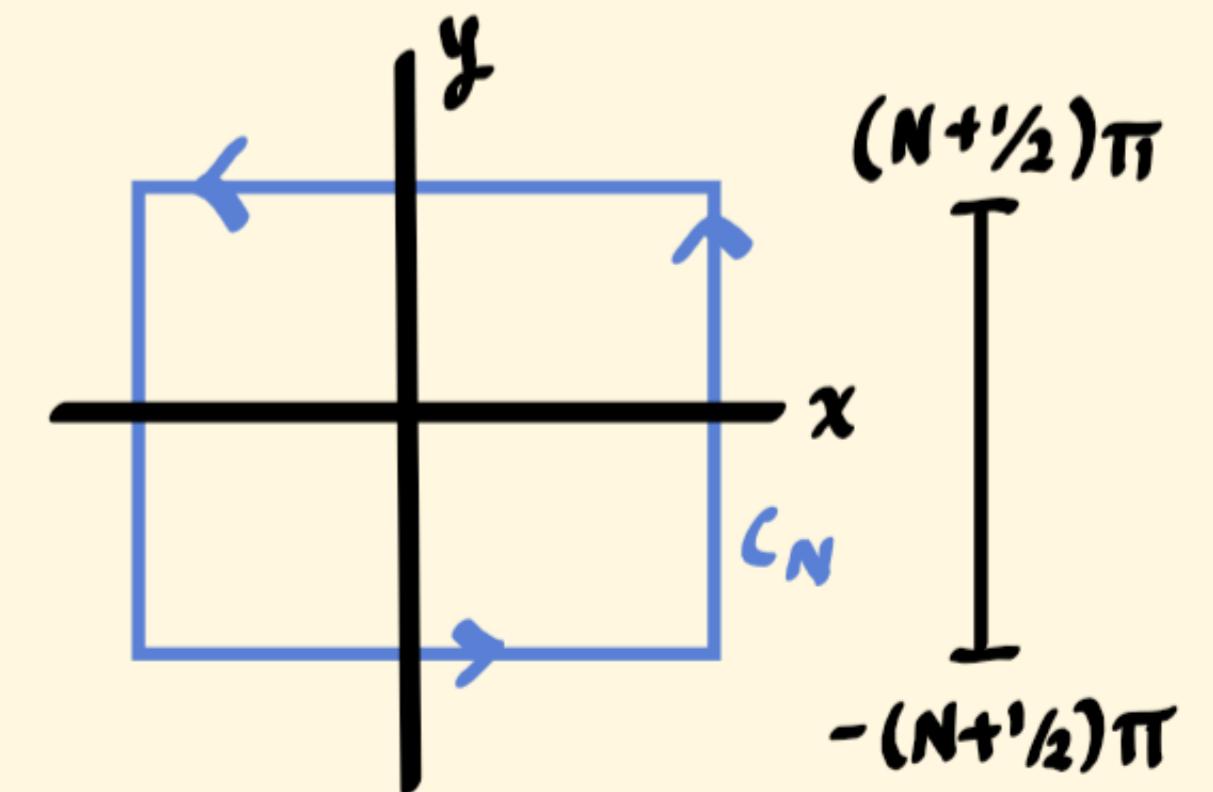
$$\Rightarrow |\sinh[(N + \frac{1}{2})\pi]| > |\sinh[\pi/2]| = \sinh[\pi/2]$$

$$\sinh[-(N + \frac{1}{2})\pi] = -\sinh[(N + \frac{1}{2})\pi] < -\sinh[\pi/2]$$

$$\Rightarrow |\sinh[-(N + \frac{1}{2})\pi]| > \sinh[\pi/2]$$

On the horizontal sides of the square $x = \pm(N + \frac{1}{2})\pi$:

$$|\sin z| \geq |\sinh y| > \sinh[\pi/2]$$



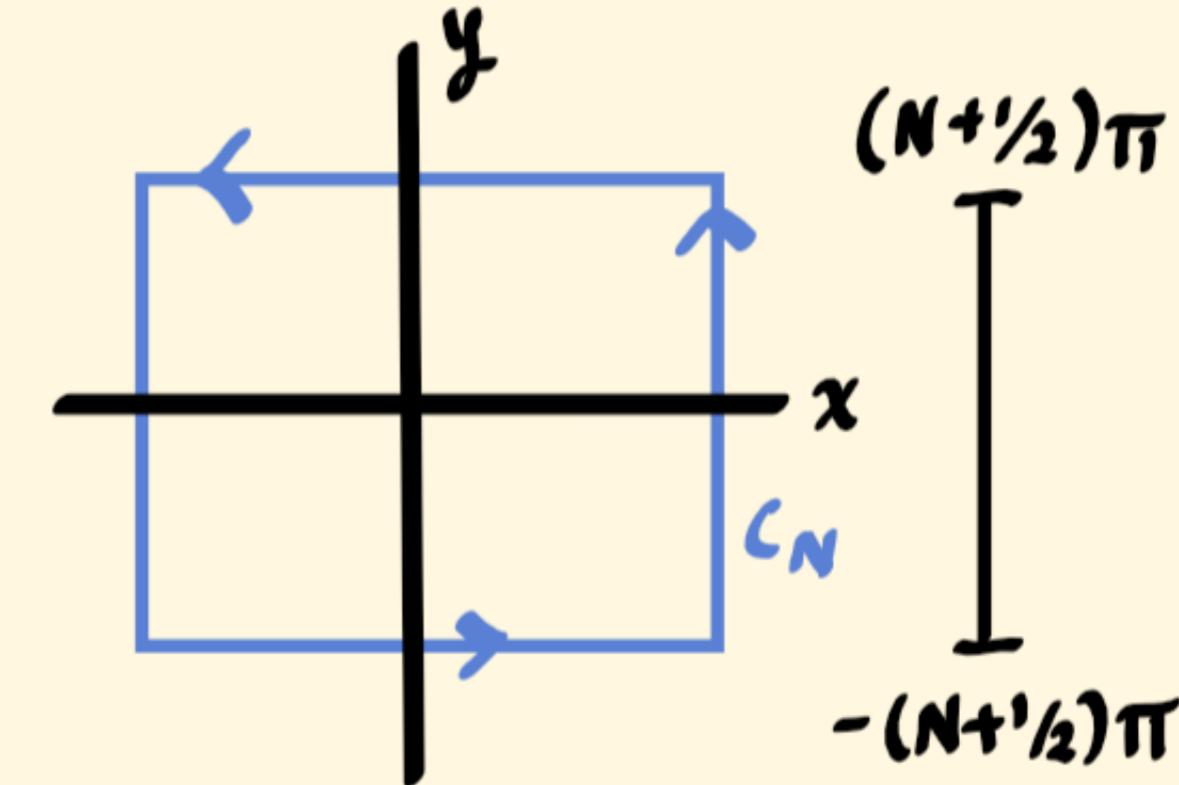
△ For any $N \in \{1, 2, \dots\}$, if z is on the contour C_N :

$$2|\sin z| > 1 + \sinh[\pi/2]$$

$$|\sin z| > A := (1 + \sinh[\pi/2])/2$$

$$|\sin z| > A \Rightarrow |\sin z| > A$$

(b) The length of the square C_N
is the perimeter $4(2N+1)\pi$



For any z on C_N , $|z| > (N+1/2)\pi$. Using this and (a),

$$|z^2 \sin z| = |z|^2 |\sin z| > [(N+1/2)\pi]^2 A$$

$$\Rightarrow \left| \frac{1}{z^2 \sin z} \right| = \frac{1}{|z^2 \sin z|} \leq \frac{1}{[(N+1/2)\pi]^2 A}$$

By the theorem on page 138,

$$\begin{aligned} \left| \int_{C_N} \frac{dz}{z^2 \sin z} \right| &\leq 4(2N+1)\pi \frac{1}{[(N+1/2)\pi]^2 A} \\ &= \frac{4(2N+1)\pi}{1/4(2N+1)^2 \pi^2 A} = \frac{16}{(2N+1)\pi A} \end{aligned}$$

$$\text{Since } 0 \leq \left| \int_{C_N} \frac{dz}{z^2 \sin z} \right| \leq \frac{16}{(2N+1)\pi A}, \quad N=1, 2, 3, \dots$$

and $\frac{16}{(2N+1)\pi A} \rightarrow 0 \text{ as } N \rightarrow \infty$

$$\lim_{N \rightarrow \infty} \left| \int_{C_N} \frac{dz}{z^2 \sin z} \right| = 0 \Rightarrow \int_{C_N} \frac{dz}{z^2 \sin z} = 0$$

* * * * *

Chapter 4 Section 45: 4, 5 (page 149)