Written Homework 7 Solutions

MA 4291 (Tilley)
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pg. 239: # 1,2,3,4,6
pg. 243: # 1,2,3,4
pg. 248: # 1,2,3,6,7
pg. 255: # 1,2,4,6

Part 1: Chapter 6 Exercises

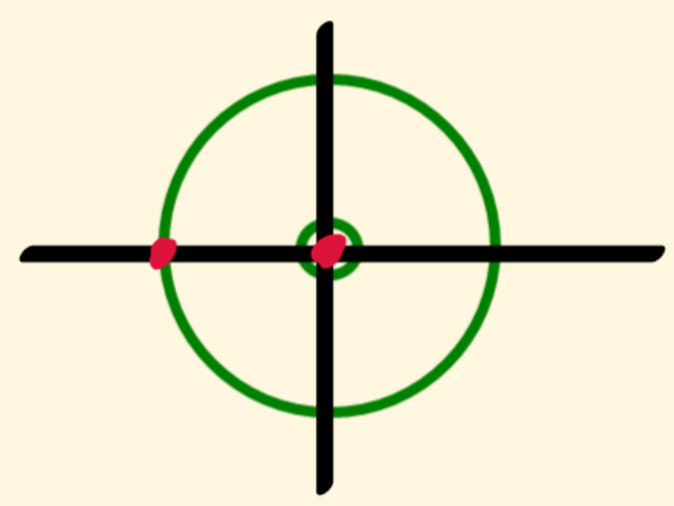
Chapter 6 Section 72: 1, 2, 3, 4, 6

1. Find the residue at $z = 0$ of the function

(a) $\dfrac{1}{z+z^2}$; *(b)* $z\cos\left(\dfrac{1}{z}\right)$; *(c)* $\dfrac{z-\sin z}{z}$; *(d)* $\dfrac{\cot z}{z^4}$; *(e)* $\dfrac{\sinh z}{z^4(1-z^2)}$.

Ans. *(a)* 1; *(b)* $-1/2$; *(c)* 0; *(d)* $-1/45$; *(e)* $7/6$.

(a) $$\frac{1}{z+z^2} = \frac{1}{z}\,\frac{1}{1+z}$$



$$= \sum_{n=0}^{\infty} (-1)^n z^{n-1} \qquad (0 < |z| < 1)$$

$$= z^{-1} - 1 + z - z^2 + \ldots$$

The residue is the coefficient of z^{-1}, which is $\boxed{1}$

(b) Using $\cos z = 1 - \frac{z^2}{2} + \frac{z^4}{4!} - \ldots \quad (|z| < \infty)$,

$$z\cos(1/z) = z - \frac{1}{2}\frac{1}{z} + \frac{1}{4!}\frac{1}{z^3} - \ldots \quad (0 < |z| < \infty)$$

The residue is the coefficient of z^{-1}, which is $\boxed{-\frac{1}{2}}$

(c) $\dfrac{z - \sin z}{z} = 1 - \dfrac{1}{z}\sin z$

$= 1 - \dfrac{1}{z}\left(z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \dots\right)$

$= \dfrac{z^2}{3!} - \dfrac{z^4}{5!} + \dots \qquad (0 < |z| < \infty)$

The residue is the coefficient of z^{-1}, which is $\boxed{0}$

(d) Using $\cot z = \dfrac{1}{z} - \dfrac{z}{3} - \dfrac{z^3}{45} - \dfrac{2}{945}z^5 - \mathcal{O}(z^7)$ for $0 < |z| < \infty$,

$\dfrac{\cot z}{z^4} = \dfrac{1}{z^3} - \dfrac{1}{3}\dfrac{1}{z^2} - \underline{\dfrac{1}{45}}\dfrac{1}{z} - \dots \qquad (0 < |z| < \infty)$

The residue is the coefficient of z^{-1}, $\boxed{-\dfrac{1}{45}}$

(e) Using $\sinh z = z + \dfrac{z^3}{3!} + \dfrac{z^5}{5!} + \dots \quad (|z| < \infty)$

$\dfrac{\sinh z}{z^4(1 - z^2)} = \left(\dfrac{1}{z^4} + \dfrac{1}{z^2} + 1 + z^2 + \dots\right)\left(z + \dfrac{z^3}{3!} + \dfrac{z^5}{5!} + \dots\right)$

$= \dfrac{1}{z^3} + \underline{\left(\dfrac{1}{3!} + 1\right)}\dfrac{1}{z} + \left(\dfrac{1}{5!} + \dfrac{1}{3!} + 1\right)z + \dots \qquad (0 < |z| < 1)^*$

The residue is the coefficient of z^{-1}, $\boxed{\dfrac{7}{6}}$

* Singular points : $z = 0, \pm 1$

2. Use Cauchy's residue theorem (Sec. 70) to evaluate the integral of each of these functions around the circle $|z| = 3$ in the positive sense:

(a) $\dfrac{\exp(-z)}{z^2}$; (b) $\dfrac{\exp(-z)}{(z-1)^2}$; (c) $z^2 \exp\left(\dfrac{1}{z}\right)$; (d) $\dfrac{z+1}{z^2-2z}$.

Ans. (a) $-2\pi i$; (b) $-2\pi i/e$; (c) $\pi i/3$; (d) $2\pi i$.

Theorem. *Let C be a simple closed contour, described in the positive sense. If a function f is analytic inside and on C except for a finite number of singular points z_k $(k = 1, 2, \ldots, n)$ inside C (Fig. 87), then*

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z). \tag{1}$$

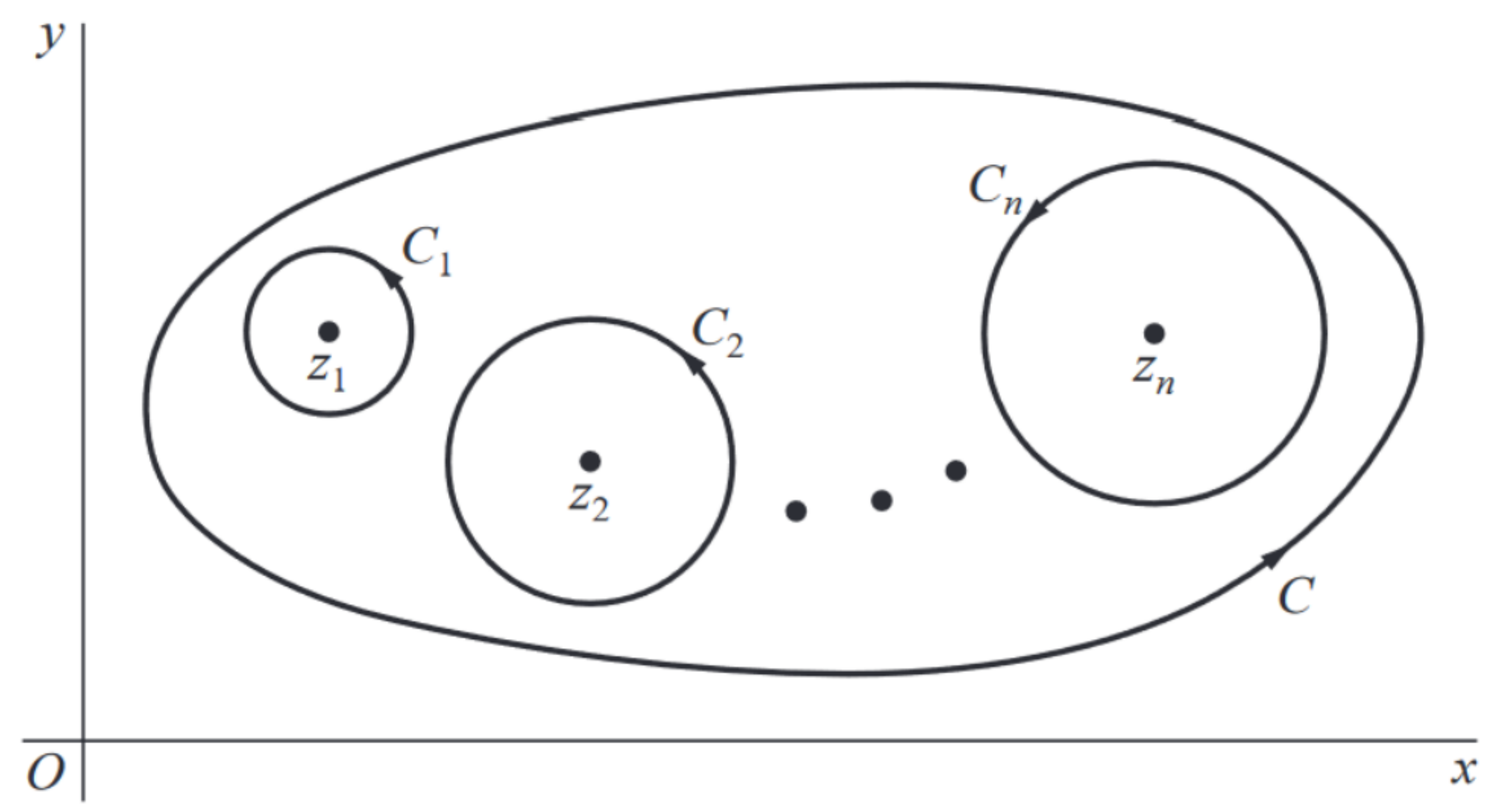


FIGURE 87

(a) The integrand has an isolated singularity at $z = 0$, which is interior to C.

$$z^{-2}\exp(-z) = z^{-2} - z^{-1} + \frac{1}{2} - \frac{1}{3!}z + \frac{1}{4!}z^2 - \ldots$$

$$\operatorname*{Res}_{z=0} f(z) = -1 \Rightarrow \int_C z^{-2}\exp(-z)\,dz = \boxed{-2\pi i}$$

(b) The integrand has an isolated singularity at $z=1$, which is interior to C.

$$f(z) = \frac{e^{-z}}{(z-1)^2} = \frac{1}{e}\frac{1}{(z-1)^2}e^{1-z}$$

$$= \frac{1}{e}\frac{1}{(z-1)^2}\left(1 + (1-z) + \frac{(1-z)^2}{2} + \frac{(1-z)^3}{3!} + \ldots\right)$$

$$= \frac{1}{e}\frac{1}{(z-1)^2} - \frac{1}{e}\frac{1}{z-1} + \frac{1}{2e} - \frac{1}{2e}(z-1) + \ldots$$

for $0<|z-1|<\infty$. The residue is the coefficient of $(z-1)^{-1}$.

$$\int_C (z-1)^{-2}e^{-z}\,dz = 2\pi i \operatorname*{Res}_{z=1} f(z) = \boxed{-2\pi i/e}$$

(c) The integrand has an isolated singularity at $z=0$, which is interior to C.

$$f(z) = z^2 \exp(1/z)$$

$$= z^2\left(1 + \frac{1}{z} + \frac{1}{2}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \frac{1}{4!}\frac{1}{z^4} + \dots\right)$$

$$= z^2 + z + \frac{1}{2} + \frac{1}{6}\frac{1}{z} + \frac{1}{24}\frac{1}{z^2} + \dots$$

for $0 < |z| < \infty$. The residue is $\frac{1}{6}$.

$$\int_C z^2 \exp(1/z)\,dz = 2\pi i \operatorname*{Res}_{z=0} f(z) = \boxed{\frac{\pi i}{3}}$$

(d) The integrand has isolated singularities at $z=0$ and $z=2$. By partial fractions:

$$f(z) = \frac{z+1}{z(z-2)} = \frac{-1/2}{z} + \frac{3/2}{z-2}$$

Since $-\frac{1}{2z}$ is a Laurent series when $0<|z|<1$ and $\frac{3/2}{z-2}$ is a Laurent series when $0<|z-2|<1$,

$$\int_C \frac{z+1}{z^2-2z}\,dz = 2\pi i\cdot\left(-\tfrac{1}{2}\right) + 2\pi i\cdot(3/2) = \boxed{2\pi i}$$

3. Use the theorem in Sec. 71, involving a single residue, to evaluate the integral of each of these functions around the circle $|z| = 2$ in the positive sense:

(a) $\dfrac{z^5}{1 - z^3}$; (b) $\dfrac{1}{1 + z^2}$; (c) $\dfrac{1}{z}$.

Ans. (a) $-2\pi i$; (b) 0; (c) $2\pi i$.

Theorem. *If a function f is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then*

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]. \tag{7}$$

(a) $\dfrac{1}{z^2} f\left(\dfrac{1}{z}\right) = \dfrac{1}{z^2} \dfrac{1/z^5}{1 - 1/z^3} = -\dfrac{1}{z^4} \dfrac{1}{1 - z^3}$

$= -\dfrac{1}{z^4}(1 + z^3 + z^6 + \dots)$

$= -\dfrac{1}{z^4} - \underline{\dfrac{1}{z}} - z^2 - \dots$ $(0 < |z| < 1)$

$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = 2\pi i \cdot (-1) = \boxed{-2\pi i}$

(b) $\dfrac{1}{z^2} f\left(\dfrac{1}{z}\right) = \dfrac{1}{z^2} \dfrac{1}{1 + 1/z^2} = \dfrac{1}{1 + z^2} = \dfrac{1}{1 - (-z^2)}$

$= 1 - z^2 + z^4 - z^6 + \dots$ $(0 < |z| < 1)$

$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = 2\pi i \cdot 0 = \boxed{0}$

(c) $\dfrac{1}{z^2} f\left(\dfrac{1}{z}\right) = \dfrac{1}{z}$ (Laurent series for $0 < |z| < 1$)

$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = 2\pi i \cdot 1 = \boxed{2\pi i}$

4. Let C denote the circle $|z| = 1$, taken counterclockwise, and use the following steps to show that

$$\int_C \exp\left(z + \frac{1}{z}\right) dz = 2\pi i \sum_{n=0}^{\infty} \frac{1}{n!\,(n+1)!}.$$

(a) By using the Maclaurin series for e^z and referring to Theorem 1 in Sec. 65, which justifies the term by term integration that is to be used, write the above integral as

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_C z^n \exp\left(\frac{1}{z}\right) dz.$$

(b) Apply the theorem in Sec. 70 to evaluate the integrals appearing in part *(a)* to arrive at the desired result.

(a) $\int_C \exp(z + \frac{1}{z})dz = \int_C e^{1/z} e^z \, dz$

$= \int_C e^{1/z} \left(\sum_{n=0}^{\infty} \frac{z^n}{n!} \right) dz$

$= \sum_{n=0}^{\infty} \frac{1}{n!} \int_C z^n e^{1/z} \, dz$

(b) For each n, let $f_n(z) = z^n e^{1/z}$

$f_n(z) := z^n e^{1/z} = z^n \left(1 + \frac{1}{z} + \dots + \frac{1}{(n+1)!} \frac{1}{z^{n+1}} + \dots \right)$

$= z^n + z^{n-1} + \dots \frac{1}{(n+1)!} \frac{1}{z} + \dots$

$\int_C z^n e^{1/z} dz = 2\pi i \cdot \operatorname{Res}_{z=0} f_n(z) = 2\pi i \cdot \frac{1}{(n+1)!}$

$\therefore \int_C \exp(z + \frac{1}{z})dz = \sum_{n=0}^{\infty} \frac{1}{n!} \int_C z^n e^{1/z} dz$

$= 2\pi i \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}$

6. Let the degrees of the polynomials

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \neq 0)$$

and

$$Q(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_m z^m \qquad (b_m \neq 0)$$

be such that $m \geq n + 2$. Use the theorem in Sec. 71 to show that if all of the zeros of $Q(z)$ are interior to a simple closed contour C, then

$$\int_C \frac{P(z)}{Q(z)}\, dz = 0.$$

[Compare with Exercise 3*(b)*.]

Theorem. *If a function f is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then*

$$\int_C f(z)\, dz = 2\pi i \operatorname*{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z} \right) \right]. \tag{7}$$

* * * * *

Chapter 6 Section 72 : 1,2,3,4 (pg 243)

1. In each case, write the principal part of the function at its isolated singular point and determine whether that point is a pole, a removable singular point, or an essential singular point:

(a) $z\exp\left(\frac{1}{z}\right)$; *(b)* $\frac{z^2}{1+z}$; *(c)* $\frac{\sin z}{z}$; *(d)* $\frac{\cos z}{z}$; *(e)* $\frac{1}{(2-z)^3}$.

We saw in Sec. 69 that the theory of residues is based on the fact that if f has an isolated singular point at z_0, then $f(z)$ has a Laurent series representation

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n} + \cdots \tag{1}$$

in a punctured disk $0 < |z-z_0| < R_2$. The portion

$$\frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n} + \cdots \tag{2}$$

of the series, involving negative powers of $z - z_0$, is called the *principal part* of f at z_0. We now use the principal part to identify the isolated singular point z_0 as one of three special types. This classification will aid us in the development of residue theory that appears in following sections.

(a) $f(z) = z\exp(1/z)$ has an isolated singularity at $z_0 = 0$.

$$f(z) = z\left(1 + \frac{1}{z} + \frac{1}{2}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \ldots\right)$$

$$= z + 1 + \frac{1}{2}\frac{1}{z} + \frac{1}{3!}\frac{1}{z^2} + \ldots \qquad |z| > 0$$

The principal part of $f(z)$ at $z_0 = 0$ is

$$\frac{1}{2}\frac{1}{z} + \frac{1}{3!}\frac{1}{z^2} + \ldots$$

$f(z)$ has an essential singular point at $z_0 = 0$, with the residue $b_1 = \frac{1}{2}$.

(b) $f(z) = \frac{z^2}{1+z}$ has an isolated singular point at $z_0 = -1$. $f(z)$ can be written in the form

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n \qquad 0 < |z-z_0| < R_0$$

$$f(z) = \frac{z^2-1}{1+z} + \frac{1}{1+z} = z-1+\frac{1}{1+z}$$

$$= -2 + z-(-1) + \frac{1}{1+z}$$

$$= -2\cdot(z-(-1))^0 + 1\cdot(z-(-1)) + \frac{1}{z-(-1)} \qquad 0<|z+1|<\infty$$

The principal part of $f(z)$ at $z_0=-1$ is $\frac{1}{z+1}$.

This shows that $f(z)$ has a simple pole at $z_0=-1$. Its residue there is $c_{-1}=1$.

(c) $f(z) = \frac{\sin z}{z} = 1 - \frac{1}{3!}z^2 + \frac{1}{5!}z^4 + \ldots \quad (0<|z|<\infty)$

The principal part of $f(z)$ at $z_0=0$ is 0 and $\operatorname{Res}_{z=0} f(z) = 0$.

$f(z)$ has a removable singularity at $z_0=0$.

(d) $f(z) = \frac{\cos z}{z} = \frac{1}{z} - \frac{1}{2} z + \frac{1}{4!} z^3 - \ldots$

The principal part of $f(z)$ at $z_0 = 0$ is $\frac{1}{z}$, with residue $b_1 = 1$.

$f(z)$ has a simple pole at $z_0 = 0$.

(e) $f(z) = \frac{1}{(2-z)^3} = -\frac{1}{(z-2)^3}$

$f(z)$ has a pole of order 3 at $z_0 = 2$ and $\underset{z=2}{\text{Res}}\, f(z) = b_1 = 0$.

The principal part of $f(z)$ at $z_0 = 2$ is just $f(z)$ itself.

2. Show that the singular point of each of the following functions is a pole. Determine the order m of that pole and the corresponding residue B.

(a) $\dfrac{1-\cosh z}{z^3}$; (b) $\dfrac{1-\exp(2z)}{z^4}$; (c) $\dfrac{\exp(2z)}{(z-1)^2}$.

Ans. (a) $m = 1$, $B = -1/2$; (b) $m = 3$, $B = -4/3$; (c) $m = 2$, $B = 2e^2$.

(a) $f(z) = \dfrac{1-\cosh z}{z^3}$ has a singular pt. $z_0 = 0$.

$$f(z) = \frac{1}{z^3}\left(1 - 1 - \frac{1}{2}z^2 - \frac{1}{4!}z^4 - \frac{1}{6!}z^6 - \dots\right) \quad (|z|>0)$$

$$= -\frac{1}{2}\frac{1}{z} - \frac{1}{4!}z - \dots \qquad \boxed{\begin{array}{l} m=1 \\ B=-1/2 \end{array}}$$

(b) $g(z) = \dfrac{1-e^{2z}}{z^4}$ has a singular pt. $z_0 = 0$.

$$g(z) = z^{-4}\left(1 - 1 - 2z - \frac{1}{2}(2z)^2 - \frac{1}{3!}(2z)^3 - \dots\right)$$

$$= -2z^{-3} - 2z^{-2} - \frac{4}{3}z^{-1} - \frac{2}{3}z^0 - \dots \qquad (|z|>0)$$

$\boxed{m=3}$ $\boxed{B = -4/3}$

(c) $h(z) = \dfrac{e^{2z}}{(z-1)^2} = \dfrac{e^2 e^{2(z-1)}}{(z-1)^2}$

$$= \frac{e^2}{(z-1)^2}\left(1 + 2(z-1) + \frac{1}{2}2^2(z-1)^2 + \dots\right), \quad (0<|z-1|<\infty)$$

$$= \frac{e^2}{(z-1)^2} + \frac{2e^2}{(z-1)} + 2e^2 + \dots \qquad \boxed{\begin{array}{l} m=2 \\ B=2e^2 \end{array}}$$

3. Suppose that a function f is analytic at z_0, and write $g(z) = f(z)/(z - z_0)$. Show that

(a) if $f(z_0) \neq 0$, then z_0 is a simple pole of g, with residue $f(z_0)$;

(b) if $f(z_0) = 0$, then z_0 is a removable singular point of g.

Suggestion: As pointed out in Sec. 57, there is a Taylor series for $f(z)$ about z_0 since f is analytic there. Start each part of this exercise by writing out a few terms of that series.

Theorem. *Suppose that a function f is analytic throughout a disk $|z - z_0| < R_0$, centered at z_0 and with radius R_0* (Fig. 74). *Then $f(z)$ has the power series representation*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (|z - z_0| < R_0), \tag{1}$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!} \qquad (n = 0, 1, 2, \ldots). \tag{2}$$

That is, series (1) *converges to $f(z)$ when z lies in the stated open disk.*

Since $f(z)$ is analytic at z_0, $\exists$ a disk $|z-z_0|<R_0$ s.t. $f(z)$ is analytic throughout the disk.

$f(z)$ then has a power series representation as written in the theorem.

$$g(z) = (z-z_0)^{-1} f(z)$$

$$= (z-z_0)^{-1}\left[f(z_0) + f'(z_0)(z-z_0) + \tfrac{1}{2}f''(z_0)(z-z_0)^2 + \ldots\right]$$

$$= \frac{f(z_0)}{z-z_0} + f'(z_0) + \tfrac{1}{2}f''(z_0)(z-z_0) + \ldots \quad (|z-z_0|<R_0)$$

(a) If $f(z_0) \neq 0$, the first term doesn't vanish $g(z)$ has a simple pole at z_0 and $\operatorname*{Res}_{z=0} g(z) = f(z_0)$

(b) If $f(z_0) = 0$, the first term vanishes so

$$g(z) = f'(z_0) + \tfrac{1}{2}f''(z_0)(z-z_0) + \ldots$$

Then z_0 is a removable singularity of g

4. Use the fact (see Sec. 29) that $e^z = -1$ when

$$z = (2n+1)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$$

to show that $e^{1/z}$ assumes the value -1 an infinite number of times in each neighborhood of the origin. More precisely, show that $e^{1/z} = -1$ when

$$z = -\frac{i}{(2n+1)\pi} \quad (n = 0, \pm 1, \pm 2, \ldots);$$

then note that if n is large enough, such points lie in any given ε neighborhood of the origin. Zero is evidently the exceptional value in Picard's theorem, stated in Example 5, Sec. 72.

Let $D_\varepsilon = \{z : |z| < \varepsilon\}$ be a neighborhood of the origin. We want to show there exist infinitely many $z \in D_\varepsilon$ s.t. $e^{1/z} = -1$. For $n \geq 0$,

$$\lim_{n\to\infty} \left| -\frac{i}{(2n+1)\pi} \right| = \lim_{n\to\infty} \frac{1}{(2n+1)\pi} = 0$$

$$\Rightarrow \exists N \text{ s.t. } \left| -\frac{i}{(2n+1)\pi} \right| < \varepsilon \quad \forall n \geq N.$$

(Similar for $n < 0$, $n \to -\infty$ although $n < 0$ not really needed to prove the statement)

Let $z_n := -\frac{i}{(2n+1)\pi}$ for $n = 0, \pm 1, \pm 2, \ldots$

For $|n| \geq N$ we have $|z_n| < \varepsilon \Rightarrow z_n \in D_\varepsilon$ and

$$e^{1/z_n} = e^{-(2n+1)\pi/i} = e^{i(2n+1)\pi} = -1$$

$\therefore$ For any neighborhood of the origin, $e^{1/z} = -1$ for infinitely many z in the neighborhood.

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Chapter 6 Section 74: 1, 2, 3, 6, 7 (pg 248)

1. In each case, show that any singular point of the function is a pole. Determine the order m of each pole, and find the corresponding residue B.

(a) $\dfrac{z^2+2}{z-1}$; *(b)* $\left(\dfrac{z}{2z+1}\right)^3$; *(c)* $\dfrac{\exp z}{z^2+\pi^2}$.

Ans. (a) $m = 1$, $B = 3$; (b) $m = 3$, $B = -3/16$; (c) $m = 1$, $B = \pm i/2\pi$.

Theorem. *An isolated singular point z_0 of a function f is a pole of order m if and only if $f(z)$ can be written in the form*

(1) $$f(z) = \frac{\phi(z)}{(z-z_0)^m},$$

where $\phi(z)$ is analytic and nonzero at z_0. Moreover,

(2) $$\operatorname*{Res}_{z=z_0} f(z) = \phi(z_0) \quad \text{if } m = 1$$

and

(3) $$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \quad \text{if } m \geq 2.$$

(a) $f(z) = \dfrac{z^2+2}{z-1} = \dfrac{\phi(z)}{z-z_0}$

$\phi(z) = z^2+2$ is analytic and nonzero at $z_0 = 1$

By the theorem, $m=1$ and $B = \phi(1) = 3$

(b) $f(z) = \dfrac{z^3/8}{(z-(-1/2))^3} = \dfrac{\phi(z)}{(z-z_0)^3}$

$\phi(z) = z^3/8$ is analytic and nonzero at $z_0 = -1/2$

By the theorem, $m = 3$ and $B = \frac{1}{2!}\phi''(-\frac{1}{2})$

$B = \operatorname*{Res}_{z=-1/2} f(z) = \frac{1}{2}\cdot 3 \cdot 2 \cdot (-1/2) \cdot \frac{1}{8} = -\frac{3}{16}$

(c) $f(z) = \dfrac{e^z}{z^2+\pi^2}$ has isolated singularities at $z_0 = \pm i\pi$.

- $z_0 = i\pi$:

$$f(z) = \frac{e^z}{(z-i\pi)(z+i\pi)} = \frac{e^z/(z+i\pi)}{z-i\pi} = \frac{\phi(z)}{z-z_0}$$

$\phi(z) = \dfrac{e^z}{z+i\pi}$ is analytic and nonzero at z_0

$$m=1 \quad , \quad B = \phi(z_0) = \frac{e^{i\pi}}{2i\pi} = \frac{i}{2\pi}$$

- $z_0 = -i\pi$:

$$f(z) = \frac{e^z}{(z-i\pi)(z+i\pi)} = \frac{e^z/(z-i\pi)}{z-(-i\pi)} = \frac{\phi(z)}{z-z_0}$$

$\phi(z) = \dfrac{e^z}{z-i\pi}$ is analytic and nonzero at z_0

$$m=1 \quad , \quad B = \phi(z_0) = \frac{e^{-i\pi}}{-2i\pi} = -\frac{i}{2\pi}$$

2. Show that

(a) $\operatorname*{Res}_{z=-1} \dfrac{z^{1/4}}{z+1} = \dfrac{1+i}{\sqrt{2}} \quad (|z| > 0, 0 < \arg z < 2\pi);$

(b) $\operatorname*{Res}_{z=i} \dfrac{\operatorname{Log} z}{(z^2+1)^2} = \dfrac{\pi + 2i}{8};$

(c) $\operatorname*{Res}_{z=i} \dfrac{z^{1/2}}{(z^2+1)^2} = \dfrac{1-i}{8\sqrt{2}} \quad (|z| > 0, 0 < \arg z < 2\pi).$

(a) $z^{1/4} = \sqrt[4]{|z|} \exp\left[\frac{i}{4} \operatorname{Arg} z + k\pi/2\right]$, $k = 0, 1, 2, 3$

For a single-valued fn, let

$$\phi(z) = \sqrt[4]{|z|} \exp\left[\frac{i}{4} \operatorname{Arg} z\right] \qquad (k=0)$$

$$\operatorname*{Res}_{z=-1} \frac{z^{1/4}}{z+1} = \phi(-1) = e^{i\pi/4} = \boxed{\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}}$$

(b) $\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$, $z \neq 0$

$$f(z) = \frac{\operatorname{Log} z}{(z^2+1)^2} = \frac{(\operatorname{Log} z)/(z+i)^2}{(z-i)^2} = \frac{\phi(z)}{(z-i)^2}$$

$\phi(z) = (\operatorname{Log} z)/(z+i)^2$ is analytic and non-zero at the order $m=2$ pole at $z_0 = i$.

We will need $\phi'(z)$:

$$\phi'(z) = \frac{(z+i)^2/z - 2(z+i)\operatorname{Log} z}{(z+i)^4}$$

$$\operatorname*{Res}_{z=i} f(z) = \phi'(z_0) = \frac{(2i)^2/i - 4i\,\mathrm{Log}\,i}{(2i)^4}$$

$$= \frac{4i - 4i^2\pi/2}{16} = \boxed{\frac{2i+\pi}{16}}$$

(c) $f(z) = \dfrac{z^{1/2}/(z+i)^2}{(z-i)^2} = \dfrac{\phi(z)}{(z-i)^2}$

where we take the principal root of $z^{1/2} = \sqrt[2]{|z|}\exp(i/2\cdot\mathrm{Arg}\,z + k\pi)$ $k=0,1$. That is, use $k=0$ so that $\phi(z)$ is single valued on $|z|>0$, $0<\arg z<2\pi$.

$\phi(z)$ is analytic and nonzero at $z_0 = i$.

$$\phi'(z) = \frac{\tfrac{1}{2}z^{-1/2}(z+i)^2 - 2z^{1/2}(z+i)}{(z+i)^4}$$

$$\operatorname*{Res}_{z=i} f(z) = \phi'(i) = \frac{\dfrac{(2i)^2\sqrt{2}}{2+2i} - \dfrac{(2+2i)}{\sqrt{2}}(2i)}{16}$$

$$= \frac{1}{16}\left[\frac{-4\cdot 2\cdot(2-2i)}{8\sqrt{2}} - \frac{(2+2i)\cdot 16i}{8\sqrt{2}}\right]$$

$$= \frac{1}{16}\left[\frac{-16+16i-32i+32}{8\sqrt{2}}\right] = \boxed{\frac{1-i}{8\sqrt{2}}}$$

3. Find the value of the integral

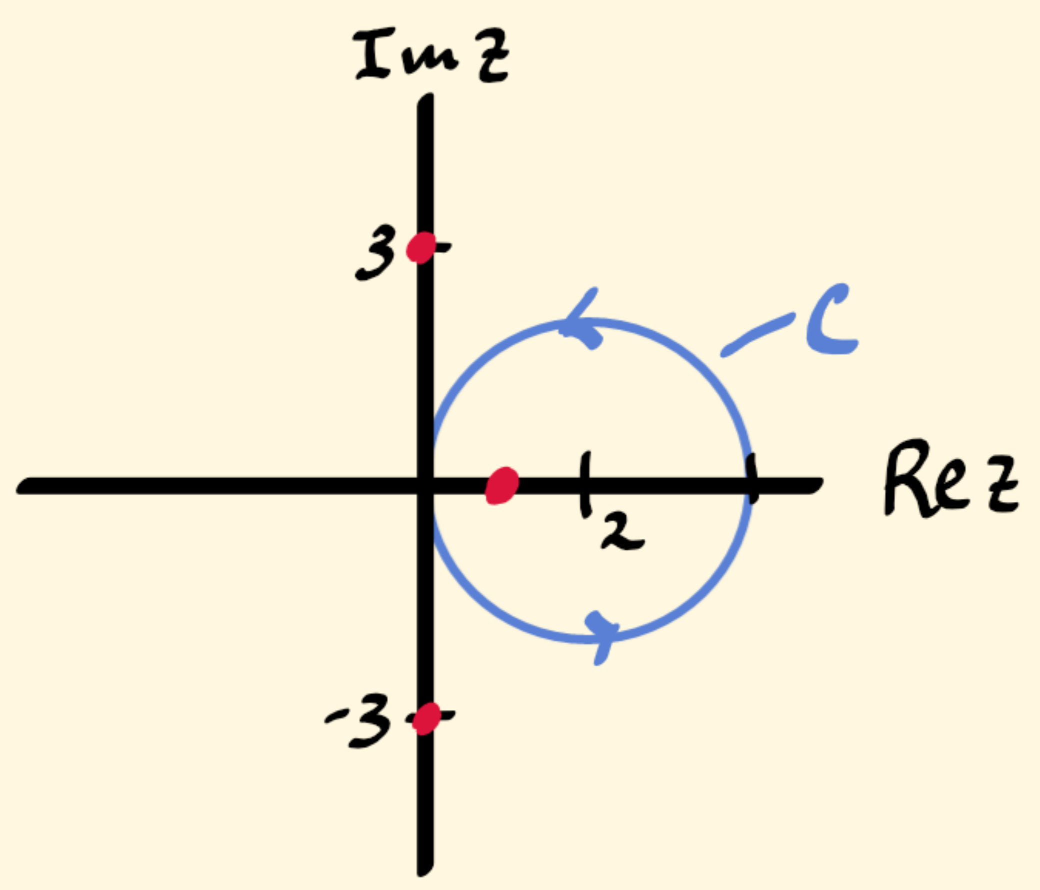
$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}\,dz,$$

taken counterclockwise around the circle *(a)* $|z-2|=2$; *(b)* $|z|=4$.

Ans. *(a)* πi; *(b)* $6\pi i$.

The integrand $f(z) = \dfrac{3z^3+2}{(z-1)(z^2+9)}$ has isolated singularities at $z_0 = 1, \pm 3i$ (all simple poles)

(a) The singular pt. $z_0 = 1$ is enclosed by C



$\phi(z) = (3z^3+2)/(z^2+9)$ is nonzero and analytic at $z_0 = 1$.

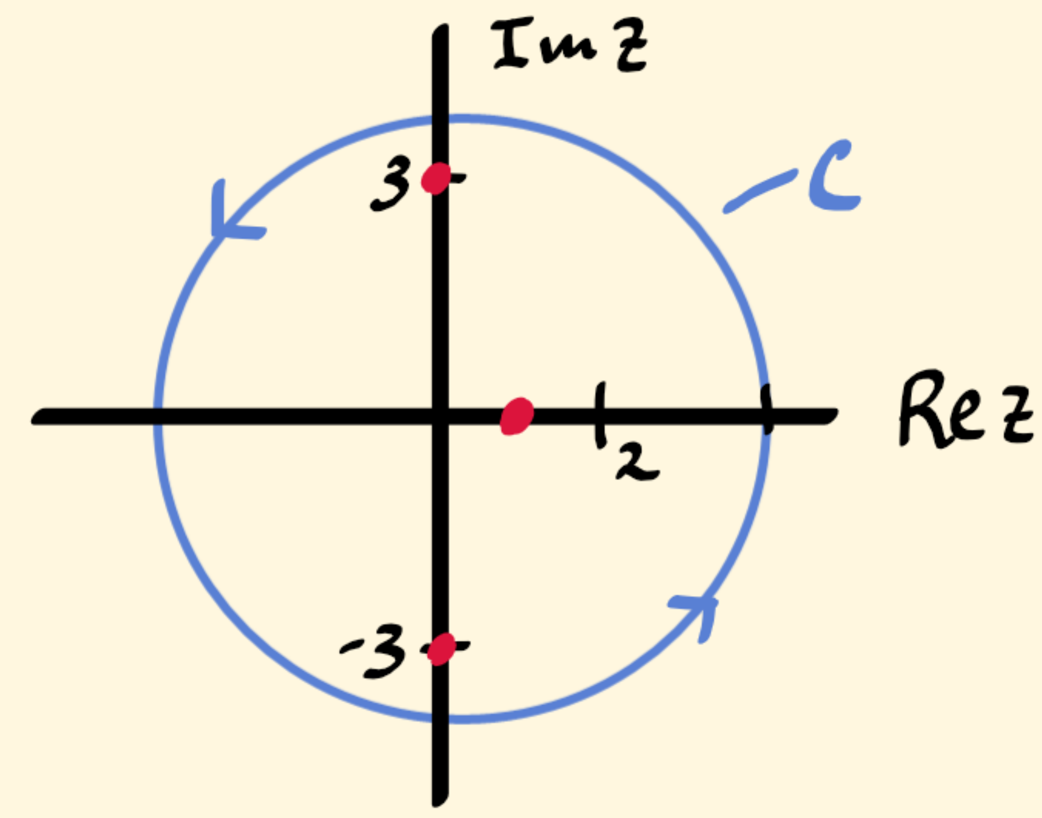
Use Cauchy's Residue Theorem :

Theorem. *Let C be a simple closed contour, described in the positive sense. If a function f is analytic inside and on C except for a finite number of singular points z_k $(k = 1, 2, \ldots, n)$ inside C* (Fig. 87), *then*

(1) $$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

$$\boxed{\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=1} f(z) = 2\pi i\,\phi(1) = 2\pi i \cdot \frac{5}{10} = \pi i}$$

(b) All three singular points are enclosed by C.



To calculate $\int_C f(z)\,dz$, find the residues at each singular pt. and apply Cauchy's Residue Theorem

- $\operatorname*{Res}_{z=-1} f(z) = \frac{1}{2}$ by part a.

- For $z_0 = 3i$, let $\phi(z) = \dfrac{3z^3+2}{(z-1)(z+3i)}$

Since $\phi(z)$ is nonzero and analytic at $3i$

$$\operatorname*{Res}_{z=3i} f(z) = \phi(3i) = \frac{3(3i)^3+2}{(3i-1)(6i)} = \frac{1}{12}(15+49i)$$

- For $z_0 = -3i$, let $\phi(z) = \dfrac{3z^3+2}{(z-1)(z-3i)}$

Since $\phi(z)$ is nonzero and analytic at $-3i$

$$\operatorname*{Res}_{z=-3i} f(z) = \phi(-3i) = \frac{3(-3i)^3+2}{(-3i-1)(-6i)} = -\frac{1}{12}(-15+49i)$$

$$\boxed{\begin{aligned}\int_C f(z)\,dz &= 2\pi i\left[\tfrac{1}{2}+\tfrac{1}{12}(15+49i)-\tfrac{1}{12}(-15+49i)\right]\\ &= 2\pi i\left[\tfrac{1}{2}+\tfrac{5}{2}\right] = 6\pi i\end{aligned}}$$

6. Use the theorem in Sec. 71, involving a single residue, to evaluate the integral of $f(z)$ around the positively oriented circle $|z| = 3$ when

(a) $f(z) = \dfrac{(3z+2)^2}{z(z-1)(2z+5)}$; *(b)* $f(z) = \dfrac{z^3(1-3z)}{(1+z)(1+2z^4)}$; *(c)* $f(z) = \dfrac{z^3 e^{1/z}}{1+z^3}$.

Ans. (a) $9\pi i$; (b) $-3\pi i$; (c) $2\pi i$.

Theorem. *If a function f is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then*

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]. \tag{7}$$

(a) $$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} \cdot \frac{(3/z+2)^2}{1/z\,(1/z-1)(2/z+5)} \cdot \left(\frac{z^2}{z^2}\right)$$

$$= \frac{(3+2z)^2}{z(1-z)(2+5z)}$$

$$\phi(z) = \frac{(3+2z)^2}{(1-z)(2+5z)} \Rightarrow \frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{\phi(z)}{z}$$

Since $\phi(z)$ is analytic and nonzero at $z_0 = 0$,

$$\operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = \phi(0) = \frac{3^2}{1 \cdot 2} = \frac{9}{2}$$

$$\int_C f(z)\,dz = 2\pi i \cdot \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = \boxed{9\pi i}$$

(b) $$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} \cdot \frac{1/z^3\,(1-3/z)}{(1+1/z)(1+2/z^4)} \cdot \frac{z^4}{z^4}$$

$$= \frac{z^3-3}{z(z+1)(z^4+2)}$$

$$\phi(z) = \frac{z^3-3}{(z+1)(z^4+2)} \Rightarrow \frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{\phi(z)}{z}$$

Since $\phi(z)$ is analytic and nonzero at $z_0 = 0$,

$$\int_C f(z)\,dz = 2\pi i \cdot \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = 2\pi i\,\phi(0)$$

$$= 2\pi i \cdot \frac{-3}{1\cdot 2} = \boxed{-3\pi i}$$

(c) $$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} \cdot \frac{z^{-3} e^z}{1+z^{-3}} \cdot \frac{z^3}{z^3} = \frac{e^z}{z^2(z^3+1)}$$

$$\phi(z) = \frac{e^z}{z^3+1} \Rightarrow \frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{\phi(z)}{z^2}$$

$$\phi'(z) = \frac{(z^3+1)e^z - 3z^2 e^z}{(z^3+1)^2}$$ $\left\{\text{Needed since } 0 \text{ is an order 2 pole}\right.$

$$\int_C f(z)\,dz = 2\pi i \cdot \phi'(0) = 2\pi i \cdot \frac{1-0}{1^2} = \boxed{2\pi i}$$

7. Let z_0 be an isolated singular point of a function f and suppose that

$$f(z) = \frac{\phi(z)}{(z - z_0)^m},$$

where m is a positive integer and $\phi(z)$ is analytic and nonzero at z_0. By applying the extended form (6), Sec. 51, of the Cauchy integral formula to the function $\phi(z)$, show that

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!},$$

as stated in the theorem of Sec. 73.

Suggestion: Since there is a neighborhood $|z - z_0| < \varepsilon$ throughout which $\phi(z)$ is analytic (see Sec. 24), the contour used in the extended Cauchy integral formula can be the positively oriented circle $|z - z_0| = \varepsilon/2$.

* * * * *

Chapter 6 Section 77: 1, 2, 4, 6 (pg 255)

1. Show that the point $z = 0$ is a simple pole of the function

$$f(z) = \csc z = \frac{1}{\sin z}$$

and that the residue there is unity by appealing to

(a) Theorem 2 in Sec. 76;

(b) the Laurent series for $\csc z$ that was found in Exercise 2, Sec. 67.

(a)

$$p(z) = 1 \qquad q(z) = \sin z \quad \Rightarrow \quad \begin{cases} p, q \text{ analytic at } 0 \\ p(0) \neq 0 \\ q(0) = 0 \\ q'(0) = \cos 0 = 1 \neq 0 \end{cases}$$

Theorem 2. *Let two functions p and q be analytic at a point z_0. If*

$$p(z_0) \neq 0, \quad q(z_0) = 0, \quad \textit{and} \quad q'(z_0) \neq 0,$$

then z_0 is a simple pole of the quotient $p(z)/q(z)$ and

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}. \tag{2}$$

By Thm 2, 0 is a simple pole of $f(z) = \dfrac{p(z)}{q(z)}$ and

$$\operatorname*{Res}_{z=0} f(z) = \operatorname*{Res}_{z=0} \frac{p(z)}{q(z)} = \frac{p(0)}{q'(0)} = \frac{1}{1} = 1$$

(b)

2. By writing $\csc z = 1/\sin z$ and then using division, show that

$$\csc z = \frac{1}{z} + \frac{1}{3!}z + \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right]z^3 + \cdots \qquad (0 < |z| < \pi).$$

(Section 67 pg 225)

Since $f(z)$ has the form $f(z) = \frac{b_1}{z} + \sum_{n=0}^{\infty} a_n z^n$,

$z_0 = 0$ is a pole of order 1 (simple). The residue is $b_1 = 1$

2. Show that

(a) $\operatorname*{Res}_{z=\pi i} \dfrac{z - \sinh z}{z^2 \sinh z} = \dfrac{i}{\pi}$;

(b) $\operatorname*{Res}_{z=\pi i} \dfrac{\exp(zt)}{\sinh z} + \operatorname*{Res}_{z=-\pi i} \dfrac{\exp(zt)}{\sinh z} = -2\cos(\pi t)$.

(a)

$$p(z) = \frac{z - \sinh z}{z^2}, \quad q(z) = \sinh z \quad \Rightarrow \quad \begin{cases} p, q \text{ analytic at } i\pi \\ p(i\pi) \neq 0 \\ q(i\pi) = 0 \\ q'(i\pi) = \cosh i\pi = -1 \neq 0 \end{cases}$$

By Thm 2 (Pg 253),

$$\operatorname*{Res}_{z=i\pi} \frac{(z - \sinh z)/z^2}{\sinh z} = \frac{p(i\pi)}{q'(i\pi)}$$

$$= \frac{i\pi/(i\pi)^2}{-1} = \underline{\frac{i}{\pi}}$$

(b)

$$p(z) = \exp(zt), \quad q(z) = \sinh z \quad \Rightarrow \quad \begin{cases} p, q \text{ analytic at } \pm i\pi \\ p(\pm i\pi) \neq 0 \\ q(\pm i\pi) = 0 \\ q'(\pm i\pi) = \cosh[\pm i\pi] = -1 \neq 0 \end{cases}$$

By Thm 2 (Pg 253),

$$\operatorname*{Res}_{z=i\pi} \frac{\exp(zt)}{\sinh z} + \operatorname*{Res}_{z=-i\pi} \frac{\exp(zt)}{\sinh z} = \frac{p(i\pi)}{q'(i\pi)} + \frac{p(-i\pi)}{q'(-i\pi)}$$

$$= -e^{i\pi t} - e^{-i\pi t} = -(e^{i\pi t} + e^{-i\pi t}) = \underline{-2\cos(\pi t)}$$

4. Let C denote the positively oriented circle $|z| = 2$ and evaluate the integral

(a) $\int_C \tan z \, dz$; (b) $\int_C \frac{dz}{\sinh 2z}$.

Ans. (a) $-4\pi i$; (b) $-\pi i$.

(a) $\tan z = \frac{\sin z}{\cos z}$ has isolated singularities at $z = \pi/2 + n\pi$, $n \in \mathbb{Z}$. Of these, $z_0 = \pm\pi/2$ are enclosed by C.

$$p(z) = \sin z, \quad q(z) = \cos z \Rightarrow \begin{cases} p, q \text{ analytic at } \pm\pi/2 \\ p(\pm\pi/2) \neq 0 \\ q(\pm\pi/2) = 0 \\ q'(\pm\pi/2) = -\sin(\pm\pi/2) = \pm 1 \neq 0 \end{cases}$$

By Thm 2 (pg 253),

$$\operatorname*{Res}_{z=\pi/2} \tan z = \frac{p(\pi/2)}{q'(\pi/2)} = \frac{\sin(\pi/2)}{-\sin(\pi/2)} = -1$$

$$\operatorname*{Res}_{z=-\pi/2} \tan z = \frac{p(-\pi/2)}{q'(-\pi/2)} = \frac{\sin(-\pi/2)}{-\sin(-\pi/2)} = -1$$

$$\int_C \tan z = 2\pi i \cdot \left[\operatorname*{Res}_{z=\pi/2} \tan z + \operatorname*{Res}_{z=-\pi/2} \tan z\right]$$

$$= 2\pi i \cdot (-2) = \boxed{-4\pi i}$$

(b) $\frac{1}{\sinh 2z}$ has isolated singularities at $z_0 = \pm i\pi/2$:

For $z = x+iy$, $x, y \in \mathbb{R}$

$0 = \sinh 2z = \sinh 2x \cos 2y + i\cosh 2x \sin 2y$

$\Leftrightarrow$ $0 = \cosh 2x \sin 2y$ and $0 = \sinh 2x \cos 2y$

Since $\cosh 2x > 0 \ \forall x$, $0 = \sin 2y \Rightarrow y = m\pi/2$

Since $\cos(2m\pi/2) = \pm 1 \neq 0$, $0 = \sinh 2x \Rightarrow x = 0$

Then $\sinh z = 0$ for $z = im\pi/2$. Of these pts, $z_0 = 0, \pm i\pi/2$ are enclosed by C $(m = 0, \pm 1)$.

$$p(z) = 1, \quad q(z) = \sinh 2z \Rightarrow \begin{cases} p, q \text{ analytic at } \pm i\pi/2, 0 \\ p(\pm i\pi/2) = p(0) = 1 \neq 0 \\ q(\pm i\pi/2) = q(0) = 0 \\ q'(\pm i\pi/2) = 2\cosh(\pm i\pi) = -2 \neq 0 \\ q'(0) = 2\cosh(0) = 2 \neq 0 \end{cases}$$

$$\int_C \frac{1}{\sinh 2z}\,dz = 2\pi i \cdot \left[\frac{p(i\pi/2)}{q'(i\pi/2)} + \frac{p(-i\pi/2)}{q'(-i\pi/2)} + \frac{p(0)}{q'(0)}\right]$$

$$= 2\pi i \cdot \left[\frac{1}{-2} + \frac{1}{-2} + \frac{1}{2}\right] = \boxed{-\pi i}$$

6. Show that

$$\int_C \frac{dz}{(z^2-1)^2+3} = \frac{\pi}{2\sqrt{2}},$$

where C is the positively oriented boundary of the rectangle whose sides lie along the lines $x = \pm 2$, $y = 0$, and $y = 1$.

Suggestion: By observing that the four zeros of the polynomial $q(z) = (z^2-1)^2+3$ are the square roots of the numbers $1 \pm \sqrt{3}i$, show that the reciprocal $1/q(z)$ is analytic inside and on C except at the points

$$z_0 = \frac{\sqrt{3}+i}{\sqrt{2}} \quad \text{and} \quad -\overline{z_0} = \frac{-\sqrt{3}+i}{\sqrt{2}}.$$

Then apply Theorem 2 in Sec. 76.

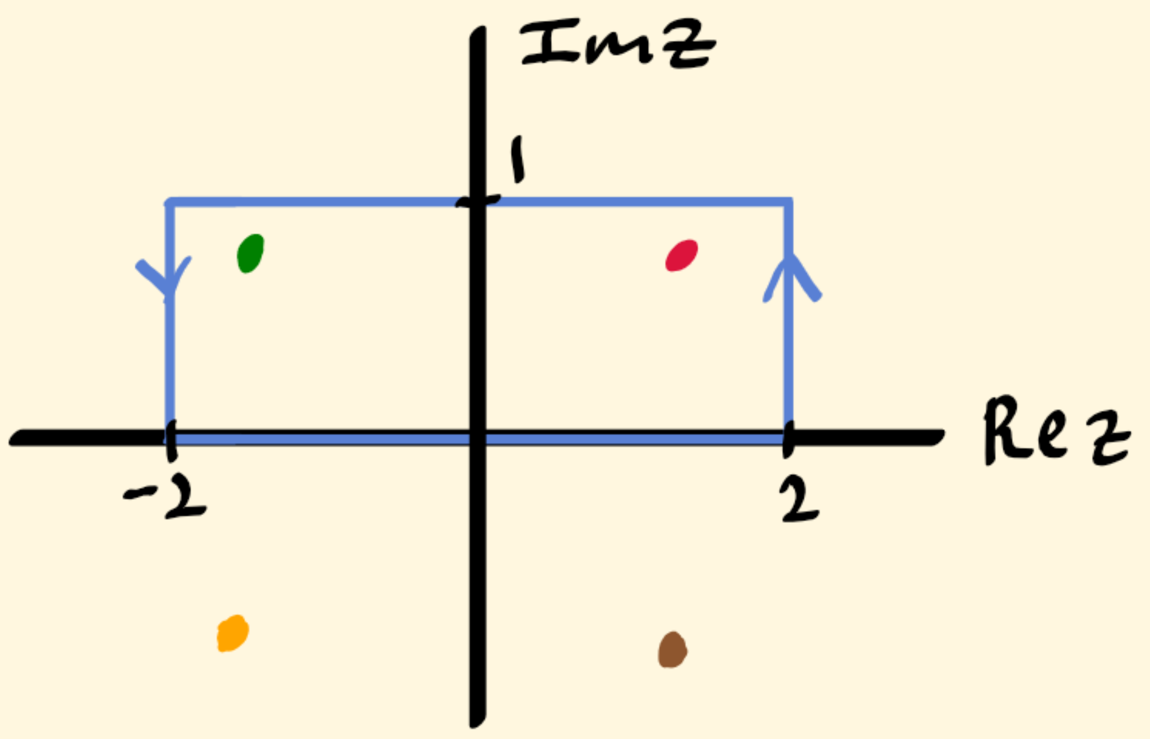
$$0 = (z^2-1)^2+3$$

$$z^2 = 1 \pm \sqrt{3}i = 2e^{i(\pm\pi/3 + 2n\pi)}$$

$$z = \sqrt{2}\, e^{i(\pm\pi/6 + n\pi)} \qquad (n = 0, 1)$$

This provides the four zeroes of $q(z)$:

- $z_0 = \sqrt{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \frac{\sqrt{3}+i}{\sqrt{2}}$
- $z_1 = \sqrt{2}\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = -\frac{\sqrt{3}+i}{\sqrt{2}}$
- $z_2 = \sqrt{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \frac{\sqrt{3}-i}{\sqrt{2}}$
- $z_4 = \sqrt{2}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \frac{-\sqrt{3}+i}{\sqrt{2}}$



Out of these four zeroes, the first and fourth are enclosed by C (z_0, z_3).

Note that $-\overline{z}_0 = -\left(\frac{\sqrt{3}-i}{\sqrt{2}}\right) = \frac{-\sqrt{3}+i}{\sqrt{2}} = z_3$

Therefore the rational fn $1/q(z)$ is analytic at all pts on or enclosed by C except:

$$z_0 = \frac{\sqrt{3}+i}{\sqrt{2}} \quad \text{and} \quad -\overline{z}_0 = \frac{-\sqrt{3}+i}{\sqrt{2}}$$

$$\begin{aligned} p(z) &= 1 \\ q(z) &= (z^2-1)^2+3 \end{aligned} \quad \Rightarrow \quad \begin{cases} p, q \text{ analytic at } z_0, -\overline{z}_0 \\ p(z_0) = p(-\overline{z}_0) = 1 \neq 0 \\ q(z_0) = q(-\overline{z}_0) = 0 \\ q'(z_0) \neq 0,\ q'(-\overline{z}_0) \neq 0 \quad * \end{cases}$$

$$q'(z) = 4z(z^2-1)$$

$$q'(z_0) = 4 \cdot \frac{\sqrt{3}+i}{\sqrt{2}}\left[\left(\frac{\sqrt{3}+i}{\sqrt{2}}\right)^2 - 1\right] = -2\sqrt{6} + 6\sqrt{2}\,i \quad *$$

$$q'(-\overline{z}_0) = 4 \cdot \frac{-\sqrt{3}+i}{\sqrt{2}}\left[\left(\frac{-\sqrt{3}+i}{\sqrt{2}}\right)^2 - 1\right] = 2\sqrt{6} + 6\sqrt{2}\,i \quad *$$

$$\int_C \frac{dz}{(z^2-1)^2+3} = 2\pi i \cdot \left[\frac{p(z_0)}{q'(z_0)} + \frac{p(-\overline{z}_0)}{q'(-\overline{z}_0)}\right]$$

$$= 2\pi i\left[\frac{1}{-2\sqrt{6}+6\sqrt{2}\,i} + \frac{1}{2\sqrt{6}+6\sqrt{2}\,i}\right] = \boxed{\frac{\pi}{2\sqrt{2}}}$$

(Several of these computations by Wolfram Alpha)