Written Homework 5 Solutions

MA 4291 (Tilley)
C-Term 2022

Please turn in only the problems in **bold red font:**

pg. 170: **1a-e**, **2ab**,3,4,**5**,6,**7**,10
pg. 188: **2**,3,**4**,5,7,**8**
pg. 195: 1,**3**,6,**7**,**8**,11ab,12ab,**13**

Chapter 4 Section 52 : 1, 2, 3, 4, 5, 6, 7, 10

1. Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals:

(a) $\displaystyle\int_C \frac{e^{-z}\,dz}{z-(\pi i/2)}$; (b) $\displaystyle\int_C \frac{\cos z}{z(z^2+8)}\,dz$; (c) $\displaystyle\int_C \frac{z\,dz}{2z+1}$;

(d) $\displaystyle\int_C \frac{\cosh z}{z^4}\,dz$; (e) $\displaystyle\int_C \frac{\tan(z/2)}{(z-x_0)^2}\,dz \quad (-2 < x_0 < 2)$.

Ans. (a) 2π; (b) $\pi i/4$; (c) $-\pi i/2$; (d) 0; (e) $i\pi\sec^2(x_0/2)$.

Theorem. *Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then*

$$(1) \qquad f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-z_0}.$$

(a) $f(z) = e^{-z} \qquad z_0 = \pi i/2$

$$\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0) = 2\pi i e^{-i\pi/2} = -2\pi i^2 = \boxed{2\pi}$$

(b) $f(z) = \dfrac{\cos z}{z^2+8} \qquad z_0 = 0$

$$\int_C \frac{\cos z}{z(z^2+8)}\,dz = 2\pi i f(z_0) = 2\pi i/8 = \boxed{\pi i/4}$$

Note $0 = z^2+8 \Rightarrow z = \pm 2\sqrt{2}i$.
$f(z)$ analytic on and inside C

(c) $f(z) = z/2 \qquad z_0 = -1/2$

$$\int_C \frac{z}{2z+1}\,dz = \int_C \frac{z/2}{z-(-1/2)}\,dz = 2\pi i f(z_0)$$

$$= -2\pi i/4 = \boxed{-\pi i/2}$$

(d) $f(s) = \cosh(s) \qquad z = 0$

For f analytic on and within C, oriented positively,

$$(5) \qquad f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^{n+1}} \qquad (n = 1, 2, \ldots).$$ (pg 167)

$$\int_C \frac{\cosh z}{z^4}\,dz = \int_C \frac{\cosh s}{(s-0)^{3+1}}\,ds = \int_C \frac{f(s)}{(s-z)^{3+1}}\,ds$$

$$= 2\pi i\, f^{(3)}(z)/3! = \frac{\pi i}{3}\frac{d^3}{dz^3}\left[\cosh z\right]\Big|_{z=0}$$

$$= \frac{\pi i}{3}\sinh 0 = \frac{\pi i}{3}\cdot 0 = \boxed{0}$$

(e) $f(s) = \tan(s/2) \qquad z = x_0$

$$\int_C \frac{\tan(z/2)}{(z-x_0)^2}\,dz = \frac{2\pi i}{1!}\frac{d}{dz}\left[\tan(z/2)\right]\Big|_{z=x_0}$$

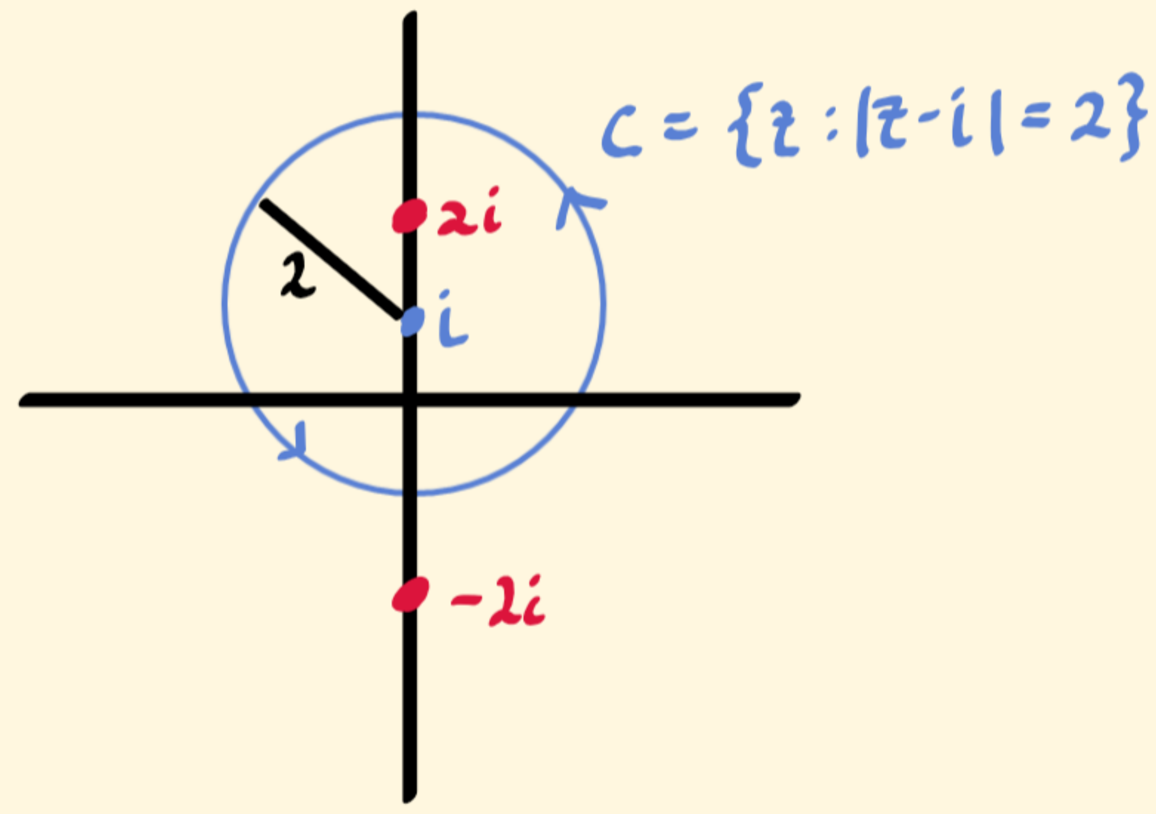
$$= 2\pi i/2\,\sec^2(x_0/2) = \boxed{i\pi\sec^2(x_0/2)}$$

2. Find the value of the integral of $g(z)$ around the circle $|z-i|=2$ in the positive sense when

(a) $g(z)=\dfrac{1}{z^2+4}$; (b) $g(z)=\dfrac{1}{(z^2+4)^2}$.

Ans. (a) $\pi/2$; (b) $\pi/16$.

(a) $$g(z)=\frac{1}{z^2+4}=\frac{1}{(z+2i)(z-2i)}=\frac{(z+2i)^{-1}}{z-2i}$$



Let $f(z)=(z+2i)^{-1}$. $f(z)$ is analytic on and within C.

$$\int_C g(z)dz=\int_C\frac{1}{z^2+4}dz=\int_C\frac{f(z)}{z-2i}dz$$

$$=2\pi i\, f(2i)=\frac{2\pi i}{4i}=\boxed{\frac{\pi}{2}}$$

(b) Let $f(z)=\dfrac{1}{(z+2i)^2}\longrightarrow f'(z)=-\dfrac{2}{(z+2i)^3}$

$$\int_C g(z)dz=\int_C\frac{1}{(z^2+4)^2}dz=\int_C\frac{(z+2i)^{-2}}{(z-2i)^2}dz$$

$$=\int_C\frac{f(z)}{(z-2i)^{1+1}}dz=2\pi i\, f'(2i)/1!$$

$$=2\pi i\cdot\frac{-2}{(4i)^3}=\boxed{\frac{\pi}{16}}$$

3. Let C be the circle $|z| = 3$, described in the positive sense. Show that if

$$g(z) = \int_C \frac{2s^2 - s - 2}{s - z}\, ds \qquad (|z| \neq 3),$$

then $g(2) = 8\pi i$. What is the value of $g(z)$ when $|z| > 3$?

- $\underline{z = 2}$

By the Cauchy-Integral Formula with $f(z) = 2z^2 - z - 2$,

$$g(2) = \int_C \frac{2s^2 - s - 2}{s - 2}\, ds = 2\pi i f(2) = 2\pi i \cdot 4 = \boxed{8\pi i}$$

- $\underline{|z| > 3}$

For $|z| > 3$, $s - z \neq 0$ and $h(s) = \dfrac{2s^2 - s - 2}{s - z}$ is analytic everywhere and interior to C. By Cauchy-Goursat Theorem,

$$\boxed{g(z) = \int_C h(s)\, ds = 0 \quad (|z| > 3)}$$

4. Let C be any simple closed contour, described in the positive sense in the z plane, and write

$$g(z) = \int_C \frac{s^3 + 2s}{(s-z)^3}\, ds.$$

Show that $g(z) = 6\pi i z$ when z is inside C and that $g(z) = 0$ when z is outside.

- z inside C

Let $f(s) = s^3 + 2s \rightarrow f''(s) = 6s$

$f(s)$ is analytic on and interior to C since f is entire. By the extension of the Cauchy Integral Formula (page 167):

$$g(z) = \int_C \frac{h(s)}{(s-z)^{2+1}}\, ds = 2\pi i f''(z)/2! = \boxed{6\pi i z}$$

- z is outside of C

If z is outside of C, $s - z \neq 0$ for any s on or interior to C.

For any z outside of C, $f(s) = \frac{s^3+2s}{(s-z)^3}$ is analytic on and interior to C. By Cauchy-Goursat,

$$\boxed{g(z) = \int_C f(s)\, ds = 0}$$

5. Show that if f is analytic within and on a simple closed contour C and z_0 is not on C, then

$$\int_C \frac{f'(z)\,dz}{z - z_0} = \int_C \frac{f(z)\,dz}{(z - z_0)^2}.$$

Theorem 1. *If a function f is analytic at a given point, then its derivatives of all orders are analytic there too.* (pg. 168)

By the theorem and the assumptions made about f, f' is analytic on and interior to the simple closed curve C. We do not know whether the curve is positively or negatively oriented but it won't matter.

Since z_0 is not on C, z_0 is either interior or exterior to C.

If z_0 is exterior to C, $z - z_0 \neq 0$ and both integrands are analytic everywhere on and exterior to C.

$$\int_C \frac{f'(z)}{z - z_0}\,dz = 0 = \int_C \frac{f(z)}{(z - z_0)^2}\,dz$$

If z_0 is interior to C, use C.I.F.* and its extension†:

$$\int_C \frac{f'(z)}{z - z_0}\,dz \overset{*}{=} 2\pi i f'(z_0)$$

$$= 2\pi i f'(z_0)/1! \overset{\dagger}{=} \int_C \frac{f(z)}{(z - z_0)^2}\,dz$$

6. Let f denote a function that is *continuous* on a simple closed contour C. Following a procedure used in Sec. 51, prove that the function

$$g(z) = \frac{1}{2\pi i} \int_C \frac{f(s)\,ds}{s - z}$$

is *analytic* at each point z interior to C and that

$$g'(z) = \frac{1}{2\pi i} \int_C \frac{f(s)\,ds}{(s - z)^2}$$

at such a point.

7. Let C be the unit circle $z = e^{i\theta}$ $(-\pi \le \theta \le \pi)$. First show that for any real constant a,

$$\int_C \frac{e^{az}}{z}\, dz = 2\pi i.$$

Then write this integral in terms of θ to derive the integration formula

$$\int_0^{\pi} e^{a\cos\theta} \cos(a \sin\theta)\, d\theta = \pi.$$

By the Cauchy Integral Formula with $f(z) = e^{az}$, $z_0 = 0$,

$$\int_C \frac{e^{az}}{z}\, dz = \int_C \frac{f(z)}{z}\, dz = 2\pi i f(0) = 2\pi i e^0 = 2\pi i$$

$$z = e^{i\theta} \rightarrow \frac{dz}{d\theta} = ie^{i\theta}$$

$$2\pi i = \int_C \frac{e^{az}}{z}\, dz = \int_{-\pi}^{\pi} \frac{e^{ae^{i\theta}}}{e^{i\theta}}\, ie^{i\theta}\, d\theta$$

$$= i\int_{-\pi}^{\pi} e^{a\cos\theta} e^{ia\sin\theta}\, d\theta$$

$$= i\int_{-\pi}^{\pi} e^{a\cos\theta}\left[\cos(a\sin\theta) + i\sin(a\sin\theta)\right] d\theta$$

$$= i\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\, d\theta - \int_{-\pi}^{\pi} e^{a\cos\theta}\sin(a\sin\theta)\, d\theta$$

$$\Rightarrow \pi = \frac{1}{2}\cdot\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\, d\theta \overset{*}{=} \int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\, d\theta$$

* Even integrand integrated over intvl symmetric wrt 0.

$$e^{a\cos(-\theta)}\cos(a\sin(-\theta)) = e^{a\cos\theta}\cos(a\sin\theta)$$

10. Let f be an entire function such that $|f(z)| \le A|z|$ for all z, where A is a fixed positive number. Show that $f(z) = a_1 z$, where a_1 is a complex constant.

Suggestion: Use Cauchy's inequality (Sec. 52) to show that the second derivative $f''(z)$ is zero everywhere in the plane. Note that the constant M_R in Cauchy's inequality is less than or equal to $A(|z_0| + R)$.

Cauchy's Inequality

Theorem 3. *Suppose that a function f is analytic inside and on a positively oriented circle C_R, centered at z_0 and with radius R (Fig. 69). If M_R denotes the maximum value of $|f(z)|$ on C_R, then*

$$|f^{(n)}(z_0)| \le \frac{n!M_R}{R^n} \quad (n = 1, 2, \ldots). \tag{2}$$

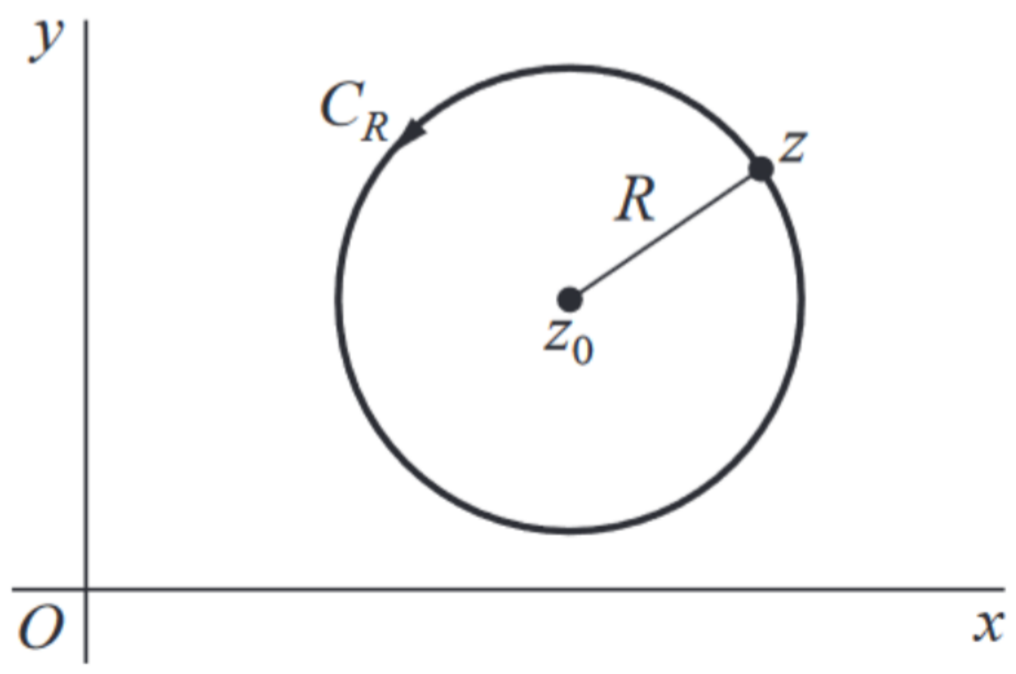


FIGURE 69

For each $z_0 \in \mathbb{C}$, Show $f''(z_0) = 0$.

For each $R \in \mathbb{N}^* = \{1, 2, 3, \ldots\}$

Consider the positive oriented circle C_R with center z_0 and radius R.

f is analytic on/inside C_R and $\forall z$ on C_R
$|f(z)| \le A|z| \le A(|z_0| + R)$ so $M_R \le A(|z_0| + R)$

$0 \le |f''(z_0)| \le 2M_R/R^2 \le 2A(|z_0|+R)/R^2 \le 2A/R$

$\therefore |f''(z_0)| = 0 \Rightarrow f''(z) = 0 \ \forall z \in \mathbb{C} \Rightarrow f(z) = az, \ a \in \mathbb{C}$

* * * * *

Chapter 5 Section 56 : 2, 3, 4, 5, 7, 8 (pg 188)

2. Let Θ_n $(n = 1, 2, \ldots)$ denote the principal arguments of the numbers

$$z_n = 2 + i\frac{(-1)^n}{n^2} \qquad (n = 1, 2, \ldots).$$

Point out why

$$\lim_{n\to\infty} \Theta_n = 0,$$

and compare with Example 2, Sec. 55.

Theorem. *Suppose that* $z_n = x_n + iy_n$ $(n = 1, 2, \ldots)$ *and* $z = x + iy$. *Then*

$$\lim_{n\to\infty} z_n = z \tag{4}$$

if and only if

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y. \tag{5}$$

Note how the theorem enables us to write

$$\lim_{n\to\infty} (x_n + iy_n) = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n$$

whenever we know that both limits on the right exist or that the one on the left exists.

Follow Example 2 with $z_n = 2 + i\,{(-1)^n}/{n^2}$ instead of the z_n from the example. The difference is that $-2 + i(-1)^n/n^2$ approaches -2 from the third quadrant if n is odd and from the second quadrant if n is even. In this case, $z_n = 2 + i(-1)^n/n^2$ approaches 2 which has $\Theta = 0$ from either direction (Quadrant 1 or quadrant 4).

3. Use the inequality (see Sec. 4) $||z_n| - |z|| \leq |z_n - z|$ to show that

$$\text{if} \quad \lim_{n\to\infty} z_n = z, \quad \text{then} \quad \lim_{n\to\infty} |z_n| = |z|.$$

Assume $\lim_{n\to\infty} z_n = z$

Equivalently $|z_n - z| \to 0$ as $n \to \infty$

Since $0 \leq ||z_n| - |z|| \leq |z_n - z| \quad \forall n \in \mathbb{N}$,

$$||z_n| - |z|| \to 0 \quad \text{as} \quad n \to \infty$$

(squeeze theorem)

$\therefore \lim_{n\to\infty} |z_n| = |z|$

4. Write $z = re^{i\theta}$, where $0 < r < 1$, in the summation formula (10), Sec. 56. Then, with the aid of the theorem in Sec. 56, show that

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{r\cos\theta - r^2}{1 - 2r\cos\theta + r^2} \quad \text{and} \quad \sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1 - 2r\cos\theta + r^2}$$

when $0 < r < 1$. (Note that these formulas are also valid when $r = 0$.)

$$(10) \qquad \sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \text{whenever} \quad |z| < 1.$$

Notice that the summation in (10) starts at $n=0$

$$\sum_{n=1}^{\infty} z^n = -1 + \sum_{n=0}^{\infty} z^n = -1 + \frac{1}{1-z} = \frac{z}{1-z}$$

$$\sum_{n=1}^{\infty} (re^{i\theta})^n = \sum_{n=1}^{\infty} z^n = \frac{z}{1-z} = \frac{re^{i\theta}}{1-re^{i\theta}}$$

$$r^{-1}e^{-i\theta}\sum_{n=1}^{\infty} r^n e^{in\theta} = \frac{1}{1-re^{i\theta}} = \frac{1}{1-r\cos n\theta - ir\sin n\theta}$$

$$r^{-1}e^{-i\theta}\sum_{n=1}^{\infty} r^n e^{in\theta} = \frac{1-r\cos n\theta + ir\sin n\theta}{(1-r\cos n\theta)^2 + r^2\sin^2 n\theta}$$

$$r^{-1}e^{-i\theta}\sum_{n=1}^{\infty} r^n e^{in\theta} = \frac{1-r\cos n\theta + ir\sin n\theta}{1-2r\cos n\theta + r^2}$$

$$\sum_{n=1}^{\infty} r^n e^{in\theta} = \frac{r\cos n\theta - r^2(\cos^2 n\theta + \sin^2 n\theta)}{1-2r\cos n\theta + r^2} + i\,\frac{r\sin n\theta}{1-2r\cos n\theta + r^2}$$

$$\sum_{n=1}^{\infty} r^n e^{in\theta} = \frac{r\cos n\theta - r^2}{1-2r\cos n\theta + r^2} + i\,\frac{r\sin n\theta}{1-2r\cos n\theta + r^2}$$

$$\sum_{n=1}^{\infty} r^n \cos n\theta + i \sum_{n=1}^{\infty} r^n \sin n\theta$$

$$= \frac{r\cos n\theta - r^2}{1 - 2r\cos n\theta + r^2} + i\,\frac{r\sin n\theta}{1 - 2r\cos n\theta + r^2}$$

Theorem. *Suppose that* $z_n = x_n + iy_n$ $(n = 1, 2, \ldots)$ *and* $S = X + iY$. *Then*

$$\sum_{n=1}^{\infty} z_n = S \tag{3}$$

if and only if

$$\sum_{n=1}^{\infty} x_n = X \quad \text{and} \quad \sum_{n=1}^{\infty} y_n = Y. \tag{4}$$

By the theorem,

$$\sum_{n=1}^{\infty} \overset{x_n}{r^n \cos n\theta} = \overset{X}{\frac{r\cos n\theta - r^2}{1 - 2r\cos n\theta + r^2}}$$

$$\sum_{n=1}^{\infty} \overset{y_n}{r^n \sin n\theta} = \overset{Y}{\frac{r\sin n\theta}{1 - 2r\cos n\theta + r^2}}$$

5. Show that a limit of a convergent sequence of complex numbers is unique by appealing to the corresponding result for a sequence of real numbers.

Let (z_n) be a convergent sequence of complex numbers with $z_n = x_n + iy_n$ for each n $(x_n, y_n \in \mathbb{R})$.

Spse (z_n) converges to $z = x + iy$ and $z' = x' + iy'$ with $z \neq z'$.

Theorem. *Suppose that* $z_n = x_n + iy_n$ $(n = 1, 2, \ldots)$ *and* $z = x + iy$. *Then*

$$\lim_{n\to\infty} z_n = z \tag{4}$$

if and only if

$$\lim_{n\to\infty} x_n = x \quad \textit{and} \quad \lim_{n\to\infty} y_n = y. \tag{5}$$

By the theorem,

$$x = \lim_{n\to\infty} x_n = x'$$

$$y = \lim_{n\to\infty} y_n = y'$$

But the limit of a convergent sequence of real numbers is unique. Therefore,

$$(x = x' \text{ and } y = y') \Leftrightarrow z = z'$$

7. Let c denote any complex number and show that

$$\text{if} \quad \sum_{n=1}^{\infty} z_n = S, \quad \text{then} \quad \sum_{n=1}^{\infty} cz_n = cS.$$

Theorem. *Suppose that $z_n = x_n + iy_n$ $(n = 1, 2, \ldots)$ and $S = X + iY$. Then*

$$\sum_{n=1}^{\infty} z_n = S \tag{3}$$

if and only if

$$\sum_{n=1}^{\infty} x_n = X \quad \text{and} \quad \sum_{n=1}^{\infty} y_n = Y. \tag{4}$$

We can write each z_n as $z_n = x_n + iy_n$

If $\sum_{n=1}^{\infty} z_n = S$, then $\sum_{n=1}^{\infty} x_n = X$ and $\sum_{n=1}^{\infty} y_n = Y$

Use the analagous result known for real seq's:

$$\sum_{n=1}^{\infty} cx_n = cX \quad \text{and} \quad \sum_{n=1}^{\infty} cy_n = cY$$

Since $cz_n = cx_n + icy_n$ and $cS = cX + icY$

$$\sum_{n=1}^{\infty} cz_n = cS$$

8. By recalling the corresponding result for series of real numbers and referring to the theorem in Sec. 56, show that

$$\text{if} \quad \sum_{n=1}^{\infty} z_n = S \quad \text{and} \quad \sum_{n=1}^{\infty} w_n = T, \qquad \text{then} \quad \sum_{n=1}^{\infty} (z_n + w_n) = S + T.$$

Write $z_n = a_n + ib_n$ and $w_n = c_n + id_n$ for each n, where a_n, b_n, c_n, and d_n are real sequences.

All summations are indexed from $n=1$ to ∞. The theorem mentioned is included with exercise 7.

If $\sum z_n = S$ and $\sum w_n = T$, then $\exists\, A, B, C, D \in \mathbb{R}$ s.t.

$$\sum a_n = A \qquad \sum b_n = B \qquad S = A + Bi$$

$$\sum c_n = C \qquad \sum d_n = D \qquad T = C + Di$$

Using the corresponding result for real series':

$$\sum (a_n + c_n) = \sum a_n + \sum c_n = A + C$$

$$\sum (b_n + d_n) = \sum b_n + \sum d_n = B + D$$

Since $z_n + w_n = (a_n + c_n) + i(b_n + d_n)$ and $S + T = (A+C) + i(B+D)$

$$\sum (z_n + w_n) = A + C + i(B + D) = S + T$$

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Chapter 5 Section 59: 1, 3, 6, 7, 8, 11, 12, 13 (pg 195-197)

1. Obtain the Maclaurin series representation

$$z\cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!} \qquad (|z| < \infty).$$

From page 194:

$$(5) \qquad \cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \qquad (|z| < \infty).$$

Substituting z^2 for z in (5) ($|z|<\infty$ iff $|z^2|<\infty$),

$$z\cosh z^2 = z\sum_{n=0}^{\infty} \frac{(z^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!}$$

3. Find the Maclaurin series expansion of the function

$$f(z) = \frac{z}{z^4+9} = \frac{z}{9} \cdot \frac{1}{1+(z^4/9)}.$$

Ans. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+2}} z^{4n+1} \quad (|z| < \sqrt{3}).$

EXAMPLE 4. Another Maclaurin series representation is

(6) $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1),$$

From example 4,

$$\frac{1}{1+z^4/9} = \frac{1}{1-(-z^4/9)} = \sum_{n=0}^{\infty} (-z^4/9)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n z^{4n} / 9^n \quad \text{for } |-z^4/9| < 1$$

$$|-z^4/9| < 1 \quad \text{iff} \quad |z|^4 < 9 \quad \text{iff} \quad |z| < \sqrt{3}$$

$$\therefore f(z) = \frac{z}{9} \frac{1}{1+z^4/9} = \frac{z}{9} \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n}}{9^n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+1}}{9^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+1}}{3^{2n+2}}$$

$$(\text{for } |z| < \sqrt{3})$$

6. Use representation (2), Sec. 59, for $\sin z$ to write the Maclaurin series for the function

$$f(z) = \sin(z^2),$$

and point out how it follows that

$$f^{(4n)}(0) = 0 \quad \text{and} \quad f^{(2n+1)}(0) = 0 \qquad (n = 0, 1, 2, \ldots).$$

$$(2) \qquad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \qquad (|z| < \infty).$$

$$f(z) = \sin(z^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(z^2)^{2n+1}}{(2n+1)!} \qquad (|z^2| < \infty)$$

Since $|z^2| < \infty$ iff $|z| < \infty$,

$$f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{4n+2}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2(2n+1)}}{(2n+1)!}$$

$$= z^2 - \frac{1}{3!} z^6 + \frac{1}{5!} z^{10} - \ldots \quad (|z| < \infty)$$

Compare this with the general form for the Maclaurin series for $f(z)$:

$$f(z) = f(0) + f'(0) \cdot z + \frac{1}{2!} f''(0) \cdot z^2 + \frac{1}{3!} f'''(0) \cdot z^3 + \ldots$$

Since the series derived above has only terms s.t. the exponents K on z (i.e. z^K) satisfy $K \equiv 2 \pmod 4$, $K > 0$ we must have

$$f^{(4n)}(0) = 0 \quad \text{and} \quad f^{(2n+1)}(0) = 0$$

7. Derive the Taylor series representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}} \qquad (|z-i| < \sqrt{2}).$$

Suggestion: Start by writing

$$\frac{1}{1-z} = \frac{1}{(1-i)-(z-i)} = \frac{1}{1-i} \cdot \frac{1}{1-(z-i)/(1-i)}.$$

$$\frac{1}{1-z} = \frac{1}{1-i} \cdot \frac{1}{1-(z-i)/(1-i)}$$

$$= \frac{1}{1-i} \sum_{n=0}^{\infty} \left[\frac{z-i}{1-i}\right]^n \qquad \left|\frac{z-i}{1-i}\right| < 1$$

$$= \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}} \qquad \left|\frac{z-i}{1-i}\right| < 1$$

$$\left|\frac{z-i}{1-i}\right| < 1 \quad \text{iff} \quad |z-i| < |1-i| = \sqrt{2}$$

$$\therefore \frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}} \qquad (|z-i| < \sqrt{2})$$

8. With the aid of the identity (see Sec. 34)

$$\cos z = -\sin\left(z - \frac{\pi}{2}\right),$$

expand $\cos z$ into a Taylor series about the point $z_0 = \pi/2$.

$$\cos z = -\sin(z - \pi/2)$$

$$= -\sum_{n=0}^{\infty} (-1)^n \frac{(z-\pi/2)^{2n+1}}{(2n+1)!} \qquad (|z-\pi/2| < \infty)$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(z-\pi/2)^{2n+1}}{(2n+1)!} \qquad (|z| < \infty)$$

11. Show that when $z \neq 0$,

(a) $\dfrac{e^z}{z^2} = \dfrac{1}{z^2} + \dfrac{1}{z} + \dfrac{1}{2!} + \dfrac{z}{3!} + \dfrac{z^2}{4!} + \cdots;$

(b) $\dfrac{\sin(z^2)}{z^4} = \dfrac{1}{z^2} - \dfrac{z^2}{3!} + \dfrac{z^6}{5!} - \dfrac{z^{10}}{7!} + \cdots.$

(a) $$\frac{e^z}{z^2} = \frac{1}{z^2}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\right)$$

$$= \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \ldots$$

(b) $$\frac{\sin(z^2)}{z^4} = \frac{1}{z^4}\left(z^2 - \frac{(z^2)^3}{3!} + \frac{(z^2)^5}{5!} - \frac{(z^2)^7}{7!} + \cdots\right)$$

$$= \frac{1}{z^2} - \frac{z^2}{3!} + \frac{z^6}{5!} - \frac{z^{10}}{7!} + \cdots$$

12. Derive the expansions

(a) $\dfrac{\sinh z}{z^2} = \dfrac{1}{z} + \displaystyle\sum_{n=0}^{\infty} \dfrac{z^{2n+1}}{(2n+3)!} \quad (0 < |z| < \infty);$

(b) $z^3 \cosh\left(\dfrac{1}{z}\right) = \dfrac{z}{2} + z^3 + \displaystyle\sum_{n=1}^{\infty} \dfrac{1}{(2n+2)!} \cdot \dfrac{1}{z^{2n-1}} \quad (0 < |z| < \infty).$

(a) $$\frac{\sinh z}{z^2} = \frac{1}{z^2}\sum_{K=0}^{\infty}\frac{z^{2K+1}}{(2K+1)!} = \frac{1}{z^2}z + \frac{1}{z^2}\sum_{K=1}^{\infty}\frac{z^{2K+1}}{(2K+1)!}$$

$$= \frac{1}{z} + \sum_{K=1}^{\infty}\frac{z^{2K-1}}{(2K+1)!} = \frac{1}{z} - \sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+3)!} \quad (0<|z|<\infty)$$

(b) $$z^3\cosh(1/z) = z^3\sum_{K=0}^{\infty}\frac{(1/z)^{2K}}{(2K)!} = \sum_{K=0}^{\infty}\frac{z^{3-2K}}{(2K)!} \quad (|1/z|<\infty)$$

$$= z^3 + z/2 + \sum_{K=2}^{\infty}\frac{z^{3-2K}}{(2K)!} = z/2 + z^3 + \sum_{n=1}^{\infty}\frac{z^{1-2n}}{(2n+2)!}$$

$$= z/2 + z^3 + \sum_{n=1}^{\infty}\frac{1}{(2n+2)!}\cdot\frac{1}{z^{2n-1}} \quad (0<|z|<\infty)$$

13. Show that when $0 < |z| < 4$,

$$\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}.$$

Partial Fractions:

$$\frac{1}{4z-z^2} = \frac{1}{z(4-z)} = \frac{A}{z} + \frac{B}{4-z}$$

$$1 = A(4-z) + Bz$$

$$\left.\begin{array}{ll} z=0: & 1 = 4A \\ z=4: & 1 = 4B \end{array}\right\} \longrightarrow A = B = 1/4$$

$$\frac{1}{4z-z^2} = \frac{1}{z(4-z)} = \frac{1}{4z} + \frac{1/4}{4(1-z/4)}$$

$$= \frac{1}{4z} + \frac{1}{16}\frac{1}{1-z/4}$$

for series convergence

$$= \frac{1}{4z} + \frac{1}{16}\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n \quad (|z/4| < 1)$$

$$= \frac{1}{4z} + \sum_{n=0}^{\infty}\frac{z^n}{4^{n+2}} \quad (0 < |z| < 4)$$

The additional constraint $|z| > 0$ is necessary to avoid division by 0.