Written Homework 3 Solutions

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Chapter 3 Section 31: 1b, 2b, 4, 6, 7, 9

1. Show that

(a) $\text{Log}(-ei) = 1 - \frac{\pi}{2}i$; (b) $\text{Log}(1-i) = \frac{1}{2}\ln 2 - \frac{\pi}{4}i$.

$\boxed{\underline{\text{Def}}: \text{ For } z = re^{i\Theta},\ \text{Log}\, z = \ln r + i\Theta}$

(b) $z = 1 - i = \sqrt{2}\, e^{-i\pi/4}$

$\text{Log}(1-i) = \ln\sqrt{2} + i(-\pi/4) = \frac{1}{2}\ln 2 - i\frac{\pi}{4}$

2. Show that

(a) $\log e = 1 + 2n\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$;

(b) $\log i = \left(2n + \frac{1}{2}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$;

(c) $\log(-1 + \sqrt{3}i) = \ln 2 + 2\left(n + \frac{1}{3}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$.

$\boxed{\underline{\text{Def}}: \text{ For } z = re^{i\Theta},\ \log z = \ln r + i(\Theta + 2n\pi) \quad (n = 0, \pm 1, \pm 2, \ldots)}$

(b) $z = i = e^{i\pi/2}$

$\log z = \ln 1 + i(\pi/2 + 2n\pi) = \pi i(1/2 + 2n),\ n \in \mathbb{Z}$

4. Show that

(a) $\log(i^2) = 2\log i$ when $\log z = \ln r + i\theta \left(r > 0, \frac{\pi}{4} < \theta < \frac{9\pi}{4}\right)$;

(b) $\log(i^2) \neq 2\log i$ when $\log z = \ln r + i\theta \left(r > 0, \frac{3\pi}{4} < \theta < \frac{11\pi}{4}\right)$.

(a) $\log i^2 = \log -1 = \ln 1 + i(\pi + 2n\pi) = i(\pi + 2n\pi),\ n \in \mathbb{Z}$

For $\theta = \pi + 2n\pi$ to satisfy $\pi/4 < \theta < \pi/4 + 2\pi$, use $n = 0$.

Then $\log i^2 = i\pi$

$\log i = \ln 1 + i(\pi/2 + 2n\pi) = i(\pi/2 + 2n\pi),\ n \in \mathbb{Z}$

For $\theta = \pi/2 + 2n\pi$ to satisfy $\pi/4 < \theta < \pi/4 + 2\pi$, use $n = 0$.

Then $2\log i = 2i\pi/2 = i\pi$ as well.

(b) $\log i^2 = i(\pi + 2n\pi)$. For $\theta = \pi + 2n\pi$ to satisfy $3\pi/4 < \theta < 3\pi/4 + 2\pi$, use $n = 0$. Then $\log i^2 = i\pi$

$\log i = i(\pi/2 + 2n\pi)$. For $\theta = \pi/2 + 2n\pi$ to satisfy $3\pi/4 < \theta < 3\pi/4 + 2\pi$, use $n = 1$. Then $2\log i = 2i(\pi/2 + 2\pi) = 5i\pi \neq i\pi$

6. Given that the branch $\log z = \ln r + i\theta$ $(r > 0, \alpha < \theta < \alpha + 2\pi)$ of the logarithmic function is analytic at each point z in the stated domain, obtain its derivative by differentiating each side of the identity (Sec. 30)

$$e^{\log z} = z \qquad (z \neq 0)$$

and using the chain rule.

$$1 = \frac{d}{dz} z = \frac{d}{dz} e^{\log z} = e^{\log z} \frac{d}{dz} \log z = z \frac{d}{dz} \log z$$

$$\therefore \frac{d}{dz} \log z = \frac{1}{z}, \quad z \neq 0$$

7. Find all roots of the equation $\log z = i\pi/2$.

Ans. $z = i$.

$$\log z = i\pi/2 \Rightarrow z = e^{\log z} = e^{i\pi/2} = i$$

9. Show that

(a) the function $f(z) = \mathrm{Log}(z - i)$ is analytic everywhere except on the portion $x \leq 0$ of the line $y = 1$;

(b) the function

$$f(z) = \frac{\mathrm{Log}(z+4)}{z^2 + i}$$

is analytic everywhere except at the points $\pm(1 - i)/\sqrt{2}$ and on the portion $x \leq -4$ of the real axis.

(a) $\mathrm{Log}(z-i) = \mathrm{Log}(x+iy-i) = \mathrm{Log}(x+i(y-1))$
$= \ln\sqrt{x^2+(y-1)^2} + i\mathrm{Arg}(x+i(y-1))$

For $z = x+iy$ s.t. $y=1$, $\mathrm{Arg}(x+i(y-1)) = \mathrm{Arg}\, x$.
If $x \leq 0$ as well, $\mathrm{Arg}\, x = \pi$. This is on the branch cut for Log, so $\mathrm{Log}(z-i)$ is discontinuous here and therefore not analytic. Otherwise $-\pi < \mathrm{Arg}(z-i) < \pi$ so $f(z)$ is analytic if it is not the case $y=1$, $x \leq 0$.
The short answer is that $\mathrm{Log}\, z$ has a branch cut along $y=0$ and $x \leq 0$ so $\mathrm{Log}(z-i)$ has a branch cut shifted up to $y=1$, $x \leq 0$.

(b) Since $z^2+i=0$ at $z = \pm(1-i)/\sqrt{2}$ $(z^2 = -i = e^{-i\pi/2} \Rightarrow z = \pm e^{-i\pi/4})$, $f(z)$ is not even defined at these points so cannot be analytic.
Since the branch cut for $\mathrm{Log}\, z$ is along $y=0$ and $x \leq 0$, the branch cut for $\mathrm{Log}(z+4)$ is shifted left to $y=0$ and $x \leq -4$.
Since $\mathrm{Log}(z+4)$ is not analytic on the branch cut, neither is $f(z)$.

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Chapter 3 Section 32: 1, 2, 6 (page 100)

1. Show that if $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$, then

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2.$$

Suggestion: Write $\Theta_1 = \operatorname{Arg} z_1$ and $\Theta_2 = \operatorname{Arg} z_2$. Then observe how it follows from the stated restrictions on z_1 and z_2 that $-\pi < \Theta_1 + \Theta_2 < \pi$.

$z_1 = r_1 e^{i\Theta_1}$, $z_2 = r_2 e^{i\Theta_2}$ where $\Theta_1 = \operatorname{Arg} z_1$, $\Theta_2 = \operatorname{Arg} z_2$

$\operatorname{Re} z_1 > 0$, $\operatorname{Re} z_2 > 0 \Rightarrow -\pi/2 < \operatorname{Arg} z_1 < \pi/2$, $-\pi/2 < \operatorname{Arg} z_2 < \pi/2$

$$\Rightarrow -\pi < \operatorname{Arg} z_1 + \operatorname{Arg} z_2 < \pi$$

$$\begin{aligned}\operatorname{Log}(z_1 z_2) &= \operatorname{Log}(r_1 e^{i\Theta_1} r_2 e^{i\Theta_2}) = \operatorname{Log}(r_1 r_2 e^{i(\Theta_1+\Theta_2)}) \\ &= \ln r_1 r_2 + i(\Theta_1 + \Theta_2) = \ln r_1 + i\Theta_1 + \ln r_2 + i\Theta_2 \\ &= \operatorname{Log} z_1 + \operatorname{Log} z_2\end{aligned}$$

2. Show that for any two nonzero complex numbers z_1 and z_2,

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2 + 2N\pi i$$

where N has one of the values $0, \pm 1$. (Compare with Exercise 1.)

$-\pi < \operatorname{Arg} z_1, \operatorname{Arg} z_2 < \pi \Rightarrow -2\pi < \operatorname{Arg} z_1 + \operatorname{Arg} z_2 < 2\pi$

Let $\Theta = \operatorname{Arg} z_1 z_2$, which must by definition satisfy $-\pi < \Theta < \pi$.

$$\Theta = \begin{cases} \Theta_1 + \Theta_2 - 2\pi, & \Theta_1 + \Theta_2 > \pi & (N = -1) \\ \Theta_1 + \Theta_2, & -\pi < \Theta_1 + \Theta_2 < \pi & (N = 0) \\ \Theta_1 + \Theta_2 + 2\pi, & \Theta_1 + \Theta_2 < -\pi & (N = 1) \end{cases}$$

$$\begin{aligned}\operatorname{Log}(z_1 z_2) &= \operatorname{Log}(r_1 e^{i\Theta_1} r_2 e^{i\Theta_2}) = \operatorname{Log}(r_1 r_2 e^{i(\Theta_1+\Theta_2)}) \\ &= \ln r_1 r_2 + i\Theta = \ln r_1 + i\Theta + \ln r_2 + i\Theta_2 + 2Ni\pi\end{aligned}$$

Note that by the definition chosen in the textbook we cannot handle the case where $\Theta_1 + \Theta_2 = \pm\pi$ in this way since $N=0$, and $N = \pm 1$ all return us to $\Theta_1 + \Theta_2 = \pm\pi$ and we need $-\pi < \Theta < \pi$. In this case, add or subtract only π instead of 2π.

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Chapter 3 Section 33: 1, 2b, 4, 9 (page 104)

1. Show that

(a) $(1+i)^i = \exp\left(-\frac{\pi}{4}+2n\pi\right)\exp\left(i\frac{\ln 2}{2}\right) \quad (n = 0, \pm 1, \pm 2, \ldots)$;

(b) $(-1)^{1/\pi} = e^{(2n+1)i} \quad (n = 0, \pm 1, \pm 2, \ldots)$.

(a) $\log(1+i) = \ln\sqrt{2} + i(\pi/4 + 2m\pi)$, $m \in \mathbb{Z}$

$i\log(1+i) = i\ln\sqrt{2} - (\pi/4 + 2m\pi) = i\frac{\ln 2}{2} - \pi/4 + 2n\pi$, $n := m$

$(1+i)^i = z^c = \exp(c\log z) = \exp\left[i\frac{\ln 2}{2} - \pi/4 + 2n\pi\right]$

$= \exp\left[i\frac{\ln 2}{2}\right]\exp\left[-\pi/4 + 2n\pi\right]$, $n \in \mathbb{Z}$

(b) $\log(-1) = \ln 1 + i\pi(1+2n) = i\pi(1+2n)$

$\pi^{-1}\log(-1) = i(1+2n)$

$(-1)^{1/\pi} = \exp[i(1+2n)]$, $n \in \mathbb{Z}$.

2. Find the principal value of

(a) i^i; *(b)* $\left[\frac{e}{2}(-1-\sqrt{3}i)\right]^{3\pi i}$; *(c)* $(1-i)^{4i}$.

Ans. *(a)* $\exp(-\pi/2)$; *(b)* $-\exp(2\pi^2)$; *(c)* $e^{\pi}[\cos(2\ln 2) + i\sin(2\ln 2)]$.

$$\boxed{\underline{\text{Def}}: \text{P.V. } z^c = e^{c\,\text{Log}\,z}, \quad z \neq 0 \text{ and } c \in \mathbb{C}}$$

$z = e(-1-\sqrt{3}i)/2 = e\,e^{-i2\pi/3} = e^{1-i2\pi/3}$, $c = 3\pi i$

$c\,\text{Log}\,z = 3\pi i\,\text{Log}\,e^{1-i2\pi/3} = 3\pi i(1 - i2\pi/3) = 3\pi i + 2\pi^2$

$\therefore [e(-1-\sqrt{3}i)/2]^{3\pi i} = z^c = e^{c\,\text{Log}\,z} = e^{3\pi i + 2\pi^2} = (e^{i\pi})^3 e^{2\pi^2} = -e^{2\pi^2}$

9. Assuming that $f'(z)$ exists, state the formula for the derivative of $c^{f(z)}$.

$\frac{d}{dz}\left[c^{f(z)}\right] = \frac{d}{dz}\left[e^{f(z)\log c}\right] = \left[e^{f(z)\log c}\right]f'(z)\log c = c^{f(z)}f'(z)\log c$

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Chapter 3 Section 34: 2, 3, 7, 11, 16 (pages 108, 109)

2. *(a)* With the aid of expression (4), Sec. 34, show that

$$e^{iz_1}e^{iz_2} = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 + i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2).$$

Then use relations (3), Sec. 34, to show how it follows that

$$e^{-iz_1}e^{-iz_2} = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 - i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2).$$

(b) Use the results in part *(a)* and the fact that

$$\sin(z_1+z_2) = \frac{1}{2i}\left[e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}\right] = \frac{1}{2i}\left(e^{iz_1}e^{iz_2} - e^{-iz_1}e^{-iz_2}\right)$$

to obtain the identity

$$\sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$

in Sec. 34.

(a) $e^{iz_1}e^{iz_2} = (\cos z_1 + i\sin z_1)(\cos z_2 + i\sin z_2)$

$= \cos z_1\cos z_2 + i\cos z_1\sin z_2 + i\sin z_1\cos z_2 + i^2\sin z_1\sin z_2$

$= \cos z_1\cos z_2 - \sin z_1\sin z_2 + i(\sin z_1\cos z_2 + \cos z_1\sin z_2)$

$e^{-iz_1}e^{-iz_2} = \cos z_1\cos z_2 - (-1)^2\sin z_1\sin z_2 + i(-\sin z_1\cos z_2 - \cos z_1\sin z_2)$

$= \cos z_1\cos z_2 - \sin z_1\sin z_2 - i(\sin z_1\cos z_2 + \cos z_1\sin z_2)$

(b) $\sin(z_1+z_2) = \frac{1}{2i}\left(e^{iz_1}e^{iz_2} - e^{-iz_1}e^{-iz_2}\right)$

$= \frac{1}{2i}2i(\sin z_1\cos z_2 + \cos z_1\sin z_2) = \sin z_1\cos z_2 + \cos z_1\sin z_2$

3. According to the final result in Exercise 2*(b)*,

$$\sin(z+z_2) = \sin z \cos z_2 + \cos z \sin z_2.$$

By differentiating each side here with respect to z and then setting $z = z_1$, derive the expression

$$\cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

that was stated in Sec. 34.

By (2) on page 105 $\frac{d}{dz}\sin z = \cos z$ and $\frac{d}{dz}\cos z = -\sin z$, $z \in \mathbb{C}$ just like the derivatives of $\sin x$ and $\cos x$ for $x \in \mathbb{R}$.

$\frac{d}{dz}[\sin(z+z_2)]\Big|_{z=z_1} = \frac{d}{dz}[\sin z\cos z_2 + \cos z\sin z_2]\Big|_{z=z_1}$

$\cos(z+z_2)\Big|_{z=z_1} = [\cos z\cos z_2 - \sin z\sin z_2]\Big|_{z=z_1}$

$\cos(z_1+z_2) = \cos z_1\cos z_2 - \sin z_1\sin z_2$

7. In Sec. 34, use expressions (13) and (14) to derive expressions (15) and (16) for $|\sin z|^2$ and $|\cos z|^2$.

Suggestion: Recall the identities $\sin^2 x + \cos^2 x = 1$ and $\cosh^2 y - \sinh^2 y = 1$.

$$\sin z = \sin x \cosh y + i \cos x \sinh y, \tag{13}$$

$$\cos z = \cos x \cosh y - i \sin x \sinh y, \tag{14}$$

$$|\sin z|^2 = \sin^2 x + \sinh^2 y, \tag{15}$$

$$|\cos z|^2 = \cos^2 x + \sinh^2 y. \tag{16}$$

$$\begin{aligned} |\sin z|^2 &= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \\ &= \sin^2 x (1 + \sinh^2 y) + \cos^2 x \sinh^2 y \\ &= \sin^2 x + \sinh^2 y (\sin^2 x + \cos^2 x) \\ &= \sin^2 x + \sinh^2 y \end{aligned}$$

$$\begin{aligned} |\cos z|^2 &= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y \\ &= \cos^2 x (1 + \sinh^2 y) + \sin^2 x \sinh^2 y \\ &= \cos^2 x + \sinh^2 y (\cos^2 x + \sin^2 x) \\ &= \cos^2 x + \sinh^2 y \end{aligned}$$

* [Exercise 11 is on the next page] *

16. With the aid of expression (14), Sec. 34, show that the roots of the equaion $\cos z = 2$ are

$$z = 2n\pi + i \cosh^{-1} 2 \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Then express them in the form

$$z = 2n\pi \pm i \ln(2 + \sqrt{3}) \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

$2 = \cos z = \cos x \cosh y - i \sin x \sinh y \Rightarrow 2 = \cos x \cosh y$, $0 = \sin x \sinh y$

Since $0 = \sin x \sinh y$, $\sinh y = 0$ or $\sin x = 0$. $\sinh y = 0$ only if $y = 0$ and $\sin x = 0$ if $x = m\pi$, $m \in \mathbb{Z}$. If $y = 0$ then since $\cosh y = 1$, we must have $\cos x = 2$. But this isn't possible since $|\cos x| \leq 1$. So $y \neq 0$ and we must have $x = m\pi$. But $\cos m\pi = (-1)^m$ so we $2 = (-1)^m \cosh y$. Since $\cosh y > 0$ $\forall y$, only $2 = \cosh y$ can be satisfied $(m = 2n, n \in \mathbb{Z})$

$\therefore z = 2n\pi + i\cosh^{-1} 2 = 2n\pi \pm i \ln(2 + \sqrt{3})$ (see Sec. 36, Exercise 2)

11. Use the Cauchy–Riemann equations and the theorem in Sec. 21 to show that neither $\sin \bar{z}$ nor $\cos \bar{z}$ is an analytic function of z anywhere.

$\triangle$ $f(z) = \sin \bar{z} = \sin x \cosh(-y) + i \cos x \sinh(-y) = \sin x \cosh y - i \cos x \sinh y$

Suppose $f'(z)$ exists at some point $z = x + iy$. By the theorem:

$$\text{C-R} \begin{cases} \cos x \cosh y = u_x = v_y = -\cos x \cosh y \\ \sin x \sinh y = u_y = -v_x = -\sin x \sinh y \end{cases}$$

$u_x = v_y$ implies $\cos x \cosh y = 0$. Since $\cosh y > 0$ $\forall y \in \mathbb{R}$ we must have $\cos x = 0$ (iff $x = \pi/2 + m\pi$, $m \in \mathbb{Z}$).

$u_y = -v_x$ implies $\sin x \sinh y = 0$. Either $\sinh y = 0$ (iff $y = 0$) or $\sin x = 0$ (iff $x = n\pi$, $n \in \mathbb{Z}$). Both C-R equations must hold simultaneously. Since $\nexists\, x$ s.t. $\sin x = \cos x = 0$, the only possibility is $\cos x = 0$ and $\sinh y = 0$, which means $x = \pi/2 + m\pi$, $m \in \mathbb{Z}$ and $y = 0$. But this means the only points z at which $f'(z)$ may possibly exist are at these isolated points. Referring to the definition of analyticity, it follows that if $f'(z)$ only exists at isolated points or nowhere at all, $f(z)$ is nowhere analytic (think about the neighborhoods of these isolated pts).

$\triangle$ $g(z) = \cos \bar{z} = \cos x \cosh(-y) - i \sin x \sinh(-y) = \cos x \cosh y + i \sin x \sinh y$

Suppose $g'(z)$ exists at some point $z = x + iy$. By the theorem:

$$\text{C-R} \begin{cases} -\sin x \cosh y = u_x = v_y = \sin x \cosh y \\ \cos x \sinh y = u_y = -v_x = -\cos x \sinh y \end{cases}$$

$u_x = v_y \Rightarrow \sin x = 0 \Rightarrow x = n\pi$, $n \in \mathbb{Z}$

$u_y = -v_x \Rightarrow \cos x = 0$ or $\sinh y = 0 \Rightarrow x = \pi/2 + m\pi$, $m \in \mathbb{Z}$ or $y = 0$.

Since $u_x = v_y$ and $u_y = -v_x$ must both hold simultaneously, the only points z at which $g'(z)$ may possibly exist are the isolated pts $z = x + iy$ with $x = n\pi$, $n \in \mathbb{Z}$ and $y = 0$. It follows by the definition of analyticity that $g(z)$ is nowhere analytic.

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Chapter 3 Section 35: 4, 7, 16 (pages 111, 112)

7. Show that

(a) $\sinh(z+\pi i) = -\sinh z$; (b) $\cosh(z+\pi i) = \cosh z$;
(c) $\tanh(z+\pi i) = \tanh z$.

(a) $$\sinh(z+\pi i) = \frac{e^{z+\pi i} - e^{-z-\pi i}}{2} = \frac{e^z e^{i\pi} - e^{-z}e^{-i\pi}}{2}$$
$$= \frac{-e^z + e^{-z}}{2} = -\frac{e^z - e^{-z}}{2} = -\sinh z$$

(b) $$\cosh(z+\pi i) = \frac{e^{z+\pi i} + e^{-z-\pi i}}{2} = \frac{e^z e^{i\pi} + e^{-z}e^{-i\pi}}{2}$$
$$= \frac{-e^z - e^{-z}}{2} = -\frac{e^z + e^{-z}}{2} = -\cosh z$$

(c) $$\tanh(z+\pi i) = \frac{\sinh(z+\pi i)}{\cosh(z+\pi i)} = \frac{-\sinh z}{-\cosh z} = \frac{\sinh z}{\cosh z} = \tanh z$$

16. Find all roots of the equation $\cosh z = -2$. (Compare this exercise with Exercise 16, Sec. 34.)

Ans. $z = \pm\ln(2+\sqrt{3}) + (2n+1)\pi i \ (n = 0, \pm 1, \pm 2, \ldots)$.

$-2 = \cosh z = \cosh x \cos y + i \sinh x \sin y \Rightarrow -2 = \cosh x \cos y$, $0 = \sinh x \sin y$

$0 = \sinh x \sin y$ means $0 = \sinh x$ or $0 = \sin y$

If $0 = \sinh x$, $x = 0$ so $-2 = \cosh x \cos y = 1 \cdot \cos y = \cos y$.

Since $-2 \neq \cos y$ for any y, conclude $x \neq 0$. Then $0 = \sin y$.

If $0 = \sin y$, $y = m\pi$, $m \in \mathbb{Z}$. Since $\cos m\pi = (-1)^m$ we have $-2 = (-1)^m \cosh x$. Since $\cosh x > 0$ $\forall x$, this can only be satisfied when $(-1)^m = -1$. That is, $m = 2n+1$, $n \in \mathbb{Z}$ so $y = (2n+1)\pi$.

Finally, $-2 = -\cosh x \Rightarrow x = \cosh^{-1} 2 = \pm \ln(2+\sqrt{3})$ (this last equality is found in Exercise 2 of Section 36).

$\therefore z = \cosh^{-1} 2 + i(2n+1)\pi = \pm\ln(2+\sqrt{3}) + i(2n+1)\pi$, $n \in \mathbb{Z}$.

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Chapter 3 Section 36: 2, 4, 5, 7 (pages 114, 115)

2. Solve the equation $\sin z = 2$ for z by

(a) equating real parts and then imaginary parts in that equation;

(b) using expression (2), Sec. 36, for $\sin^{-1} z$.

Ans. $z = \left(2n + \dfrac{1}{2}\right)\pi \pm i \ln(2+\sqrt{3}) \quad (n = 0, \pm 1, \pm 2, \ldots)$.

(a) If $2 = \sin z = \sin x \cosh y + i \cos x \sinh y$

then $2 = \sin x \cosh y$ and $0 = \cos x \sinh y$.

Since $0 = \cos x \sinh y$ we must have $\sinh y = 0$ or $\cos x = 0$.

If $y = 0$, then $\sinh y = 0$ and $\cosh y = 1$ but then we'd need $2 = \sin x$. $|\sin x| \le 1$ so $y = 0$ doesn't work. Then it must be the case that $\cos x = 0$, which can occur if $x = \pi/2 + n\pi$, $n \in \mathbb{Z}$. Note that $\sin(\pi/2 + n\pi) = (-1)^n$ so from $2 = \sin x \cosh y$ we get $\pm 2 = \cosh y$. But since $\cosh y > 0 \ \forall y \in \mathbb{R}$, it must be the case that $\cos x = 0$ and $\sin x = 1$ to satisfy both equ's.

$\therefore 2 = \sin z \Rightarrow x = \pi/2 + 2n\pi, n \in \mathbb{Z}$ and $\cosh y = 2$.

Two ways to find y:

$$2 = \cosh y \qquad \left(\begin{matrix}\cosh y = 2 \text{ iff} \\ \cosh(-y) = 2\end{matrix}\right)$$

$$\begin{aligned} y &= \pm \cosh^{-1}(2) \\ &= \pm \log(2 + (2^2 - 1)^{1/2}) \\ &= \pm \log(2 + \sqrt{3}) \\ &= \pm \ln(2 + \sqrt{3}) \quad (\text{since } 2 + \sqrt{3} \in \mathbb{R}) \\ &\approx \pm 1.317 \end{aligned}$$

$$2 = \cosh y = (e^y + e^{-y})/2$$

$$4 = e^y + e^{-y}$$

$$4e^y = e^{2y} + 1$$

$$u^2 - 4u + 1 = 0 \quad (u := e^y)$$

$$u = \frac{4}{2} \pm \frac{\sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

$$e^y = 2 \pm \sqrt{3}$$

$$y = \ln(2 \pm \sqrt{3}) \approx \pm 1.317$$

$$\therefore z = x + iy = (\pi/2 + 2n\pi) + i \ln(2 \pm \sqrt{3}) = (2n + 1/2)\pi \pm i \ln(2 + \sqrt{3})$$

(b) (2) $$\sin^{-1} z = -i \log[iz + (1 - z^2)^{1/2}].$$

$$\sin z = 2 \to z = \sin^{-1} 2 = -i\log\left[2i + (1-2^2)^{1/2}\right] = -i\log\left[2i + \sqrt{3}\,i\right]$$
$$= -i\log\left[i(2+\sqrt{3})\right] = -i\log i - i\log(2+\sqrt{3})$$
$$= -i(\ln 1 + i(\pi/2 + 2n\pi)) - i\ln(2+\sqrt{3})$$
$$= \pi/2 + 2n\pi - i\ln(2+\sqrt{3})$$

Since $\sin z = -\sin(-z)$, $\quad 2 = \sin z$ iff $-2 = \sin(-z)$.

$$\sin(-z) = -2 \to z = -\sin^{-1}(-2) = i\log\left[-2i + (1-(-2)^2)^{1/2}\right]$$
$$= i\log\left[-i(2-\sqrt{3})\right]$$
$$= i\log(-i) + i\log(2-\sqrt{3})$$
$$= i(\ln 1 + i(-\pi/2 + 2m\pi)) + i\ln(2-\sqrt{3})$$
$$= \pi/2 - 2m\pi + i\ln(2-\sqrt{3})$$
$$= \pi/2 + 2n\pi + i\ln(2-\sqrt{3}) \qquad n := -m \in \mathbb{Z}$$
$$= \pi/2 + 2n\pi - \ln(2+\sqrt{3})$$

$$\therefore z = \pi/2 + 2n\pi \pm i\ln(2+\sqrt{3})$$

5. Derive expression (4), Sec. 36, for $\tan^{-1} z$.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

$$w = \tan^{-1} z$$

$$z = \tan w = \frac{\sin w}{\cos w} = \frac{\frac{e^{iw} - e^{-iw}}{2i}}{\frac{e^{iw} + e^{-iw}}{2}} = \frac{1}{i}\,\frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} = -i\,\frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}}$$

$$iz(e^{iw} + e^{-iw}) = e^{iw} - e^{-iw}$$
$$iz(e^{2iw} + 1) = e^{2iw} - 1$$
$$(1 - iz)(e^{iw})^2 = 1 + iz$$
$$e^{iw} = \pm\left[\frac{1+iz}{1-iz}\right]^{1/2}$$
$$iw = \log \pm\left[\frac{1+iz}{1-iz}\right]^{1/2}$$

$$w = -i\log \pm\left[\frac{1+iz}{1-iz}\right]^{1/2}$$
$$w = -\frac{i}{2}\log \pm\left[\frac{i-z}{i+z}\right]$$
$$w = \frac{i}{2}\log \pm\left[\frac{i+z}{i-z}\right]$$
$$\tan^{-1}(z) = \frac{i}{2}\log\left[\frac{i+z}{i-z}\right]$$

$\Bigg\{$ see discussion following (4) about making $\tan^{-1} z$ single valued.

$* \quad * \quad * \quad * \quad *$

Chapter 8 Section 91 : 2, 3, 4 (page 313)

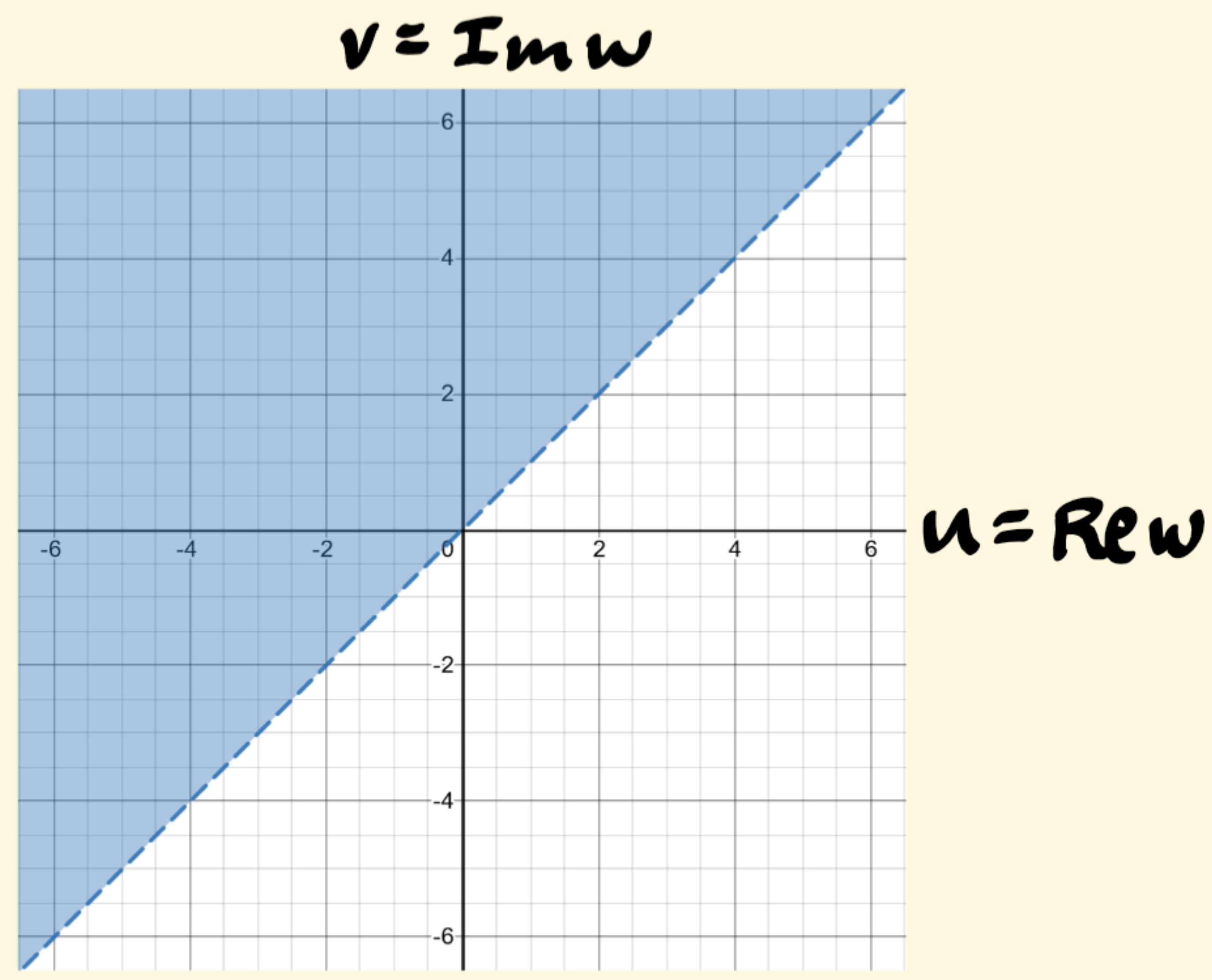
3. Find and sketch the region onto which the half plane $y > 0$ is mapped by the transformation $w = (1+i)z$.

Ans. $v > u$.

$z = re^{i\theta}$, $0 < r < \infty$, $0 < \theta < \pi$ for z in the half plane $y > 0$.

$$\begin{aligned} w = (1+i)z &= \sqrt{2}\, e^{i\pi/4}\, r e^{i\theta} \\ &= \sqrt{2}\, r e^{i(\theta + \pi/4)}, \quad 0 < \sqrt{2}\, r < \infty, \quad \pi/4 < \theta + \pi/4 < 5\pi/4 \\ &= \rho e^{i\varphi}, \quad 0 < \rho < \infty, \quad \pi/4 < \varphi < 5\pi/4 \quad (\rho := \sqrt{2}\, r, \ \varphi := \theta + \pi/4) \\ &= u + iv \end{aligned}$$

The polar conditions $0 < \rho < \infty$ and $\pi/4 < \varphi < 5\pi/4$ include all pts $w = u + iv$ satisfying $v > u$.



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Chapter 8 Section 92: 2, 3, 4, 9, 11 (page 318)

2. Show that when $c_1 < 0$, the image of the half plane $x < c_1$ under the transformation $w = 1/z$ is the interior of a circle. What is the image when $c_1 = 0$?

$A(x^2+y^2) + Bx + Cy + D = 0 \longrightarrow$ lines $x = K$, $K < c_1 < 0$

$(A = C = 0,\ B = 1,\ D = -K)$

$D(u^2+v^2) + Bu - Cv + A = 0$

$-K(u^2+v^2) + u = 0$

$K(u^2 - u/K) + Kv^2 = 0$

$(u^2 - u/K + 1/4K^2 - 1/4K^2) + v^2 = 0$

$(u - 1/2K)^2 + v^2 = (1/2K)^2$ for each line $x = K$ in the half plane $x < c_1$

For each K, this is a circle with radius $-1/2K$ and center $(1/2K, 0)$ in the complex plane $(w = u + iv)$. Since $K < c_1 < 0$, the center is on the negative real axis.

As K decreases ($K<0$ with increasing magnitude), the center $(1/2K, 0)$ approaches the origin $(0,0)$ and the radius tends toward 0, i.e. circles of smaller and smaller radii shifting right along the negative real axis.

As K increases ($K<0$ but K becoming less negative) toward $c_1 < 0$ we have circles of larger and larger radii shifting left along the negative real axis. The condition $x < c_1$ means that these circles approach $(u - 1/2c_1)^2 + v^2 = (1/2c_1)^2$ in this manner but are all strictly contained within this circle.

$\therefore$ The image of the half plane $x < c_1$ under the transformation $w = 1/z$ is the interior of the circle $(u - 1/2c_1)^2 + v^2 = (1/2c_1)^2$.

9. Find the image of the semi-infinite strip $x > 0,\ 0 < y < 1$ when $w = i/z$. Sketch the strip and its image.

Ans. $\left(u - \frac{1}{2}\right)^2 + v^2 > \left(\frac{1}{2}\right)^2,\ u > 0,\ v > 0.$

The transformation $w = i/z = i^2/iz = -1/(xi - y) = 1/(y + i(-x))$ performs same transformation as $w = 1/z$ if we replace x with y and replace y with $-x$ in the latter.

$$Bx + Cy + D = 0 \rightarrow By - Cx + D = 0.$$

Transforming a horizontal semi-infinite line segment $x > 0$, $y = K \in (0,1)$ we have (modifying the calculation from exercise 2):

$$(u - 1/2K)^2 + v^2 = (1/2K)^2 \quad , \quad K \in (0,1)$$

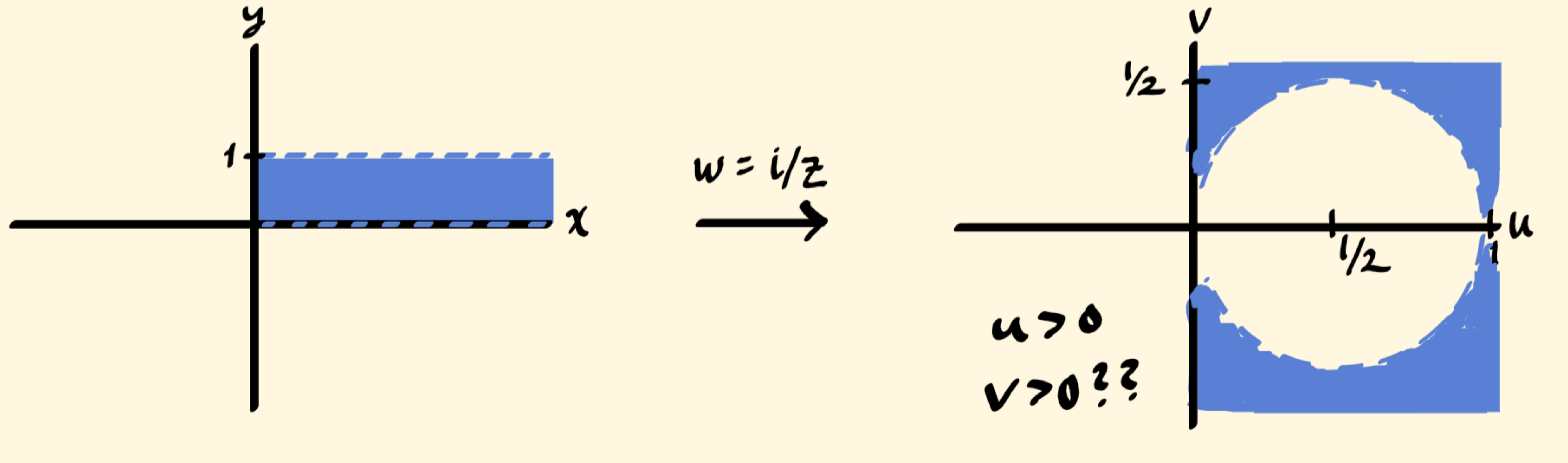
This is circle in the complex (u,v) plane centered at $(1/2K, 0)$ with radius $1/2K$. Note that $1/2K > 1/2$.

Apply this transformation to all line segments $x > 0$, $y = K \in (0,1)$ to produce infinitely many such circles.

For $K \rightarrow 0^+$ we have circles of increasingly large radii and centers shifting rightward. Each circle still passes through $(0,0)$.

For $K \rightarrow 1^-$ we have circles of smaller and smaller radii (with radius length bounded below by $1/2$) and centers shifting left toward $(1/2, 0)$ (but never including this circle).

$\therefore$ The image of $x > 0$, $0 < y < 1$ under $w = i/z$ is $(u - 1/2)^2 + v^2 > (1/2)^2$.



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Chapter 8 Section 94: 1, 4, 6, 7 (page 324)

1. Find the linear fractional transformation that maps the points $z_1 = 2$, $z_2 = i$, $z_3 = -2$ onto the points $w_1 = 1$, $w_2 = i$, $w_3 = -1$.

Ans. $w = \dfrac{3z + 2i}{iz + 6}$.

$$w = \frac{az+b}{cz+d}, \quad ad-bc \neq 0 \rightarrow \begin{cases} -c+di = ia+b \\ 2c+d = 2a+b \\ 2c-d = -2a+b \end{cases}$$

$2c+d+2c-d = 2a+b-2a+b \rightarrow 2c = b$

$2c+d-2c+d = 2a+b+2a-b \rightarrow 2a = d$

Set $b = 2i$. Then $c = i$ and $-c+di = ia+b$ becomes

$-i+2ai = ia+2i \rightarrow ai = 3i \rightarrow a = 3$

Then $a = 3$, $b = 2i$, $c = i$, $d = 6$ and $ad-bc = 20 \neq 0$.

$$\therefore w = \frac{3z+2i}{iz+6}$$

4. Find the bilinear transformation that maps distinct points z_1, z_2, z_3 onto the points $w_1 = 0$, $w_2 = 1$, $w_3 = \infty$.

Ans. $w = \dfrac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$.

First consider eqn (1) of section 94 (page 322):

$$(1) \qquad \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Following Example 2 and the discussion preceding Example 2 on page 323, delete all factors involving $w_3 = \infty$ from (1):

$$\frac{w-0}{1-0} = \frac{w-w_1}{w_2-w_1} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$w = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

6. A *fixed point* of a transformation $w = f(z)$ is a point z_0 such that $f(z_0) = z_0$. Show that every linear fractional transformation, with the exception of the identity transformation $w = z$, has at most two fixed points in the extended plane.

$$z_0 = \frac{az_0 + b}{cz_0 + d}$$

$$cz_0^2 + dz_0 = az_0 + b$$

$$cz_0^2 + (d-a)z_0 - b = 0$$

$f(z_0) = z_0$ produces a quadratic equation in z_0. By the Fundamental Theorem of Algebra this has 2 soln's counting multiplicity: so either 2 distinct solutions for z_0 (see 7a) or 1 solution for z_0 of multiplicity 2 (see 7b).

7. Find the fixed points (see Exercise 6) of the transformation

(a) $w = \dfrac{z-1}{z+1}$; *(b)* $w = \dfrac{6z-9}{z}$.

Ans. *(a)* $z = \pm i$; *(b)* $z = 3$.

(a)

$$z_0(z_0+1) = z_0 - 1$$

$$z_0^2 + z_0 = z_0 - 1$$

$$z_0^2 = -1$$

$$\boxed{z_0 = \pm i}$$

(b)

$$z_0^2 = 6z_0 - 9$$

$$z_0^2 - 6z_0 + 9 = 0$$

$$(z_0 - 3)^2 = 0$$

$$\boxed{z_0 = 3}$$