Written Homework 2 Solutions

Chapter 2 Section 18 : 1ac, 3ab, 4, 7, 10, 13 (page 55)

1. Use definition (2), Sec. 15, of limit to prove that

(a) $\lim_{z \to z_0} \operatorname{Re} z = \operatorname{Re} z_0$; (b) $\lim_{z \to z_0} \bar{z} = \overline{z_0}$; (c) $\lim_{z \to 0} \frac{\bar{z}^2}{z} = 0$.

Let a function f be defined at all points z in some deleted neighborhood (Sec. 11) of z_0. The statement that the *limit* of $f(z)$ as z approaches z_0 is a number w_0, or that

$$\lim_{z \to z_0} f(z) = w_0, \tag{1}$$

means that the point $w = f(z)$ can be made arbitrarily close to w_0 if we choose the point z close enough to z_0 but distinct from it. We now express the definition of limit in a precise and usable form.

Statement (1) means that for each positive number ε, there is a positive number δ such that

$$|f(z) - w_0| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta. \tag{2}$$

(a) $f(z) = \operatorname{Re} z$, $w_0 = \operatorname{Re} z_0$

Let $\varepsilon > 0$. Put $\delta = \varepsilon$. For z s.t. $0 < |z - z_0| < \delta$,

$$|f(z) - w_0| = |\operatorname{Re} z - \operatorname{Re} z_0| = |\operatorname{Re}(z - z_0)| \leq |z - z_0| < \delta = \varepsilon$$

(c) $f(z) = \bar{z}^2/z$, $w_0 = 0$, $z_0 = 0$

Let $\varepsilon > 0$. Put $\delta = \varepsilon$. For z s.t. $0 < |z - 0| < \delta$,

$$|f(z) - w_0| = \left|\frac{\bar{z}^2}{z} - 0\right| = \left|\frac{\bar{z}^2}{z}\right| = \frac{|\bar{z}^2|}{|z|} = \frac{|\bar{z}|^2}{|z|} = \frac{|z|^2}{|z|} = |z| < \delta = \varepsilon$$

3. Let n be a positive integer and let $P(z)$ and $Q(z)$ be polynomials, where $Q(z_0) \neq 0$. Use Theorem 2 in Sec. 16, as well as limits appearing in that section, to find

(a) $\lim_{z \to z_0} \frac{1}{z^n}$ $(z_0 \neq 0)$; *(b)* $\lim_{z \to i} \frac{iz^3 - 1}{z + i}$; *(c)* $\lim_{z \to z_0} \frac{P(z)}{Q(z)}$.

Ans. *(a)* $1/z_0^n$; *(b)* 0; *(c)* $P(z_0)/Q(z_0)$.

Theorem 2. *Suppose that*

(7) $$\lim_{z \to z_0} f(z) = w_0 \quad \textit{and} \quad \lim_{z \to z_0} F(z) = W_0.$$

Then

(8) $$\lim_{z \to z_0} [f(z) + F(z)] = w_0 + W_0,$$

(9) $$\lim_{z \to z_0} [f(z)F(z)] = w_0 W_0\,;$$

and, if $W_0 \neq 0$,

(10) $$\lim_{z \to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0}.$$

(a) $\lim_{z \to z_0} z^n = z_0^n$ is given on page 50 (follows from (9) of Thm 2).
A proof was not provided in the reading but will be provided in the following exercise.

By (10) of Thm 2, $\lim_{z \to z_0} 1/z^n = 1/z_0$ using $f(z) = 1$ and $F(z) = z^n$

(b) $\lim_{z \to i} z^3 i = i^4 = 1$ (9)

$\lim_{z \to i} (z^3 i - 1) = 1 - 1 = 0$ (8)

$\lim_{z \to i} (z + i) = i + i = 2i$ (8)

$\lim_{z \to i} \frac{iz^3 - 1}{z + i} = \frac{0}{2i} = 0$ (10)

4. Use mathematical induction and property (9), Sec. 16, of limits to show that

$$\lim_{z \to z_0} z^n = z_0^n$$

when n is a positive integer $(n = 1, 2, \ldots)$.

*Property (9)

Base Case $(n=1)$: $\lim_{z \to z_0} z^1 = \lim_{z \to z_0} z = z_0 = z_0^1$

Assuming $\lim_{z \to z_0} z^n = z_0^n$, $\lim_{z \to z_0} z^{n+1} = \lim_{z \to z_0} z^n z \overset{*}{=} \lim_{z \to z_0} z^n \lim_{z \to z_0} z = z_0^n z_0 = z_0^{n+1}$

7. Use definition (2), Sec. 15, of limit to prove that

$$\text{if} \quad \lim_{z \to z_0} f(z) = w_0, \quad \text{then} \quad \lim_{z \to z_0} |f(z)| = |w_0|.$$

Suggestion: Observe how the first of inequalities (9), Sec. 4, enables one to write

$$||f(z)| - |w_0|| \le |f(z) - w_0|.$$

Statement (1) means that for each positive number ε, there is a positive number δ such that

(2) $$|f(z) - w_0| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta.$$

(9) $$||z_1| - |z_2|| \le |z_1 \pm z_2| \le |z_1| + |z_2|.$$

Assume $\lim_{z \to z_0} f(z) = w_0$. Let $\varepsilon > 0$. There exists a $\delta > 0$ s.t.

$$|f(z) - w_0| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta$$

By (9) this means

$$||f(z)| - |w_0|| \le |f(z) - w_0| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta$$

Since $\varepsilon > 0$ was arbitrary, $\lim_{z \to z_0} |f(z_0)| = |w_0|$

10. Use the theorem in Sec. 17 to show that

(a) $\lim_{z\to\infty} \frac{4z^2}{(z-1)^2} = 4;$ *(b)* $\lim_{z\to 1} \frac{1}{(z-1)^3} = \infty;$ *(c)* $\lim_{z\to\infty} \frac{z^2+1}{z-1} = \infty.$

Theorem. *If z_0 and w_0 are points in the z and w planes, respectively, then*

(1) $$\lim_{z\to z_0} f(z) = \infty \quad \textit{if and only if} \quad \lim_{z\to z_0} \frac{1}{f(z)} = 0$$

and

(2) $$\lim_{z\to\infty} f(z) = w_0 \quad \textit{if and only if} \quad \lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0.$$

Moreover,

(3) $$\lim_{z\to\infty} f(z) = \infty \quad \textit{if and only if} \quad \lim_{z\to 0} \frac{1}{f(1/z)} = 0.$$

(a) Use (2) with $f(z) = \frac{4z^2}{(z-1)^2}$ and $w_0 = 4$

$$\lim_{z\to 0} f(1/z) = \lim_{z\to 0} \frac{4/z^2}{(1/z-1)^2} = \lim_{z\to 0} \frac{4}{z^2(1/z-1)^2} = \lim_{z\to 0} \frac{4}{(1-z)^2} = 4$$

iff $\lim_{z\to\infty} f(z) = \lim_{z\to\infty} \frac{4z^2}{(z-1)^2} = 4$

(b) Use (1) with $f(z) = \frac{1}{(z-1)^3}$ and $z_0 = 1$

$$\lim_{z\to 1} 1/f(z) = \lim_{z\to 1} 1/[1/(z-1)^3] = \lim_{z\to 1} (z-1)^3 = (1-1)^3 = 0$$

iff $\lim_{z\to 1} f(z) = \lim_{z\to 1} \frac{1}{(z-1)^3} = \infty$

(c) Use (3) with $f(z) = \frac{z^2+1}{z-1}$

$$\lim_{z\to 0} 1/f(1/z) = \lim_{z\to 0} \frac{1/z - 1}{1/z^2 + 1} = \lim_{z\to 0} \frac{z - z^2}{1+z^2} = \frac{0-0}{1+0} = 0$$

iff $\lim_{z\to\infty} f(z) = \lim_{z\to\infty} \frac{z^2+1}{z-1} = \infty$

13. Show that a set S is unbounded (Sec. 11) if and only if every neighborhood of the point at infinity contains at least one point in S.

Suppose S is unbounded and consider the neighborhood of the point at infinity $N_\varepsilon = \{z : |z| > 1/\varepsilon\}$ for some small positive ε.

Since S is unbounded there exists, for any positive $R \in \mathbb{R}$, a point $z_0 \in S$ s.t. $|z_0| > R$. In particular, for $R = 1/\varepsilon$ there exists a $z_0 \in S$ s.t. $|z_0| > 1/\varepsilon$. Thus $z_0 \in N_\varepsilon$. Since N_ε was an arbitrary neighborhood of the point at, conclude that every neighborhood of the point at infinity contains a point in S.

Conversely, suppose that every neighborhood of the point at infinity contains a point of S. For S to be bounded there must exist some $R \in \mathbb{R}$ s.t. $|z| \leq R \; \forall z \in S$. But for any R we can make $1/\varepsilon > R$ by taking $0 < \varepsilon < 1/R$. There is a $z_0 \in S$ s.t. $|z_0| > 1/\varepsilon$ which means $|z_0| > R$. This shows that there is no R s.t. $|z| \leq R \; \forall z \in S$. Therefore S is unbounded.

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Chapter 2 Section 20: 4, 8 (pages 62, 63)

4. Suppose that $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$ and $g'(z_0)$ exist, where $g'(z_0) \neq 0$. Use definition (1), Sec. 19, of derivative to show that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

Let f be a function whose domain of definition contains a neighborhood $|z - z_0| < \varepsilon$ of a point z_0. The *derivative* of f at z_0 is the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \tag{1}$$

and the function f is said to be *differentiable* at z_0 when $f'(z_0)$ exists.

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{g(z) - g(z_0)} = \lim_{z \to z_0} \left[\frac{f(z) - f(z_0)}{z - z_0}\right]\left[\frac{g(z) - g(z_0)}{z - z_0}\right]^{-1}$$

$$= \lim_{z \to z_0} \left[\frac{f(z) - f(z_0)}{z - z_0}\right] \lim_{z \to z_0} \left[\frac{g(z) - g(z_0)}{z - z_0}\right]^{-1} = \frac{f'(z_0)}{g'(z_0)}$$

8. Use the method in Example 2, Sec. 19, to show that $f'(z)$ does not exist at any point z when
(a) $f(z) = \operatorname{Re} z$; *(b)* $f(z) = \operatorname{Im} z$.

(a) Let $\Delta z = \Delta x + i\Delta y$. The notation $\Delta z \to 0$ used in limits means $\Delta z \in \mathbb{C}$ is approaching the origin of the complex plane $(0,0)$.

$$\frac{\Delta w}{\Delta z} = \frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{\operatorname{Re}(z+\Delta z)-\operatorname{Re} z}{\Delta z}$$

$$= \frac{\operatorname{Re} z + \operatorname{Re}(\Delta z) - \operatorname{Re} z}{\Delta z} = \frac{\operatorname{Re}(\Delta z)}{\Delta z} = \frac{\Delta x}{\Delta x + i\Delta y}$$

If $\lim\limits_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$ exists this limit must be the same for any path through which Δz approaches $(0,0)$.

In particular consider as Δz approaches $(0,0)$ horizontally along the real axis $(\Delta y = 0)$. Then $\Delta z = (\Delta x, 0)$ on this path:

$$\frac{\Delta w}{\Delta z} = \frac{\Delta x}{\Delta x + i\Delta y} = \frac{\Delta x}{\Delta x + i\cdot 0} = 1 \Rightarrow \lim_{\substack{\Delta z \to 0 \\ \Delta y = 0}} \frac{\Delta w}{\Delta z} = \lim_{\substack{\Delta z \to 0 \\ \Delta y = 0}} 1 = 1$$

Also consider as Δz approaches $(0,0)$ vertically along the imaginary axis. Then $\Delta z = (0, \Delta y)$ along this path.

$$\frac{\Delta w}{\Delta z} = \frac{\Delta x}{\Delta x + i\Delta y} = \frac{0}{0 + i\Delta y} = 0 \Rightarrow \lim_{\substack{\Delta z \to 0 \\ \Delta x = 0}} \frac{\Delta w}{\Delta z} = \lim_{\substack{\Delta z \to 0 \\ \Delta x = 0}} 0 = 0$$

Since limits are unique conclude that $f'(z) := \lim\limits_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$ does not exist for any z.

(b) Let $\Delta z = \Delta x + i\Delta y$. The proof is similar to (a) so will be less detailed this time.

$$\frac{\Delta w}{\Delta z} = \frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{\operatorname{Im}(z+\Delta z)-\operatorname{Im} z}{\Delta z} = \frac{\Delta y}{\Delta x + i\Delta y}$$

$$\lim_{\substack{\Delta z \to 0 \\ \Delta y = 0}} \frac{\Delta w}{\Delta z} = \lim_{\substack{\Delta z \to 0 \\ \Delta y = 0}} \frac{0}{\Delta x + i\cdot 0} = \lim_{\substack{\Delta z \to 0 \\ \Delta y = 0}} 0 = 0$$

$$\lim_{\substack{\Delta z \to 0 \\ \Delta x = 0}} \frac{\Delta w}{\Delta z} = \lim_{\substack{\Delta z \to 0 \\ \Delta x = 0}} \frac{\Delta y}{0 + i\Delta y} = \lim_{\substack{\Delta z \to 0 \\ \Delta x = 0}} -i = -i$$

Conclude that $f'(z) := \lim\limits_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$ does not exist for any z

Chapter 2 Section 23: 1bc, 2ac, 5, 6 (pages 71, 72)

1. Use the theorem in Sec. 21 to show that $f'(z)$ does not exist at any point if

(a) $f(z) = \overline{z}$;　　(b) $f(z) = z - \overline{z}$;
(c) $f(z) = 2x + ixy^2$;　　(d) $f(z) = e^x e^{-iy}$.

Theorem. *Suppose that*

$$f(z) = u(x, y) + iv(x, y)$$

and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then the first-order partial derivatives of u and v must exist at (x_0, y_0), and they must satisfy the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x \tag{7}$$

there. Also, $f'(z_0)$ can be written

$$f'(z_0) = u_x + iv_x, \tag{8}$$

where these partial derivatives are to be evaluated at (x_0, y_0).

(b) $f(z) = z - \overline{z} = (x + iy) - (x - iy) = 0 + i2y = u(x,y) + iv(x,y)$

$$\forall x, y \in \mathbb{R} \begin{cases} u_x = 0 & v_y = 2 \\ u_y = 0 & v_x = 0 \end{cases}$$

Suppose $f'(z)$ exists at some point $z_0 = x_0 + iy_0$. Then the first-order partial derivatives of u and v must exist at (x_0, y_0) and satisfy <u>both</u> of the C-R equations. However,

$$u_x(x_0, y_0) = 0 \neq 2 = v_y(x_0, y_0)$$

Since the C-R equations for f are not satisfied at any z_0, $f'(z)$ must not exist for any z_0.

(c) $f(z) = 2x + ixy^2$
$\quad = u(x,y) + iv(x,y)$ $\longrightarrow$ $u_x = 2 \quad v_y = 2xy$, $\quad u_y = 0 \quad v_x = y^2$

If $f'(z)$ exists at a point $z_0 = x_0 + iy_0$ the C-R equations require:

$$\left.\begin{aligned} 2 = u_x(x_0, y_0) = v_y(x_0, y_0) = 2x_0 y_0 \\ 0 = u_y(x_0, y_0) = -v_x(x_0, y_0) = y_0^2 \end{aligned}\right\} \Rightarrow \begin{aligned} 2 &= 2x_0 \cdot 0 = 0 \\ 0 &= y_0 \end{aligned}$$

For any choice of z_0 the C-R equations require $2 = 0$. Since this isn't possible, $f'(z)$ must not exist at any point $z_0 = x_0 + iy_0$.

2. Use the theorem in Sec. 22 to show that $f'(z)$ and its derivative $f''(z)$ exist everywhere, and find $f''(z)$ when

(a) $f(z) = iz + 2$; *(b)* $f(z) = e^{-x}e^{-iy}$;
(c) $f(z) = z^3$; *(d)* $f(z) = \cos x \cosh y - i \sin x \sinh y$.

Ans. *(b)* $f''(z) = f(z)$; *(d)* $f''(z) = -f(z)$.

Satisfaction of the Cauchy–Riemann equations at a point $z_0 = (x_0, y_0)$ is not sufficient to ensure the existence of the derivative of a function $f(z)$ at that point. (See Exercise 6, Sec. 23.) But, with certain continuity conditions, we have the following useful theorem.

Theorem. *Let the function*

$$f(z) = u(x, y) + iv(x, y)$$

be defined throughout some ε *neighborhood of a point* $z_0 = x_0 + iy_0$*, and suppose that*

(a) the first-order partial derivatives of the functions u *and* v *with respect to* x *and* y *exist everywhere in the neighborhood;*

(b) those partial derivatives are continuous at (x_0, y_0) *and satisfy the Cauchy–Riemann equations*

$$u_x = v_y, \quad u_y = -v_x$$

at (x_0, y_0)*.*

Then $f'(z_0)$ *exists, its value being*

$$f'(z_0) = u_x + iv_x$$

where the right-hand side is to be evaluated at (x_0, y_0)*.*

(a) $f(z) = iz + 2 = i(x+iy) + 2 = 2 - y + ix = u(x,y) + iv(x,y)$

$$\forall x, y \in \mathbb{R} \begin{cases} u_x = 0 & v_y = 0 \\ u_y = -1 & -v_x = -1 \end{cases}$$

To show $f'(z)$ exists everywhere let $z_0 = x_0 + iy_0$ be arbitrary. $f(z)$ must be defined in some ε neighborhood of z_0. Since $f(z)$ is defined $\forall z \in \mathbb{C}$ any choice of $\varepsilon > 0$ will do; say $\varepsilon = 1$. This means the neighborhood is $N = \{z : |z - z_0| < 1\}$.

The first order partial derivatives exist everywhere in N and are continuous at (x_0, y_0). Also note that the C-R equations are satisfied at (x_0, y_0).

$\therefore f'(z_0)$ exists and $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = i$

Apply similar reasoning to show $f''(z)$ exists everywhere. We will just reassign $u(x,y)$ and $v(x,y)$ for notational convenience. Since z_0 was arbitrary, write $f'(z) = i$ for any $z \in \mathbb{C}$.

$f'(z) = i = 0 + i \cdot 1 = u(x,y) + iv(x,y)$

$$\forall x,y \in \mathbb{R} \begin{cases} u_x = 0 & v_y = 0 \\ u_y = 0 & v_x = 0 \end{cases}$$

Let $z_0 = x_0 + iy_0$. Since $f(z)$ is defined for all z, $f(z)$ is defined for all z in any ε-neighborhood of z_0. Pick $\varepsilon = 1$ and consider the corresponding ε-neighborhood $N = \{z : |z - z_0| < 1\}$. Since $u_x = u_y = v_x = v_y$, all the first order partial derivatives are defined on N, continuous at z_0, and satisfy the C-R equations at z_0. $\therefore f''(z_0)$ exists and $f''(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = 0$.

(c) $f(z) = z^3 = (x+iy)^3 = x^3 + 3x^2 iy - 3xy^2 - iy^3$

$= x^3 - 3xy^2 + i(3x^2 y - y^3) = u(x,y) + iv(x,y)$

$u_x = 3x^2 - 3y^2 \qquad v_y = 3x^2 - 3y^2$

$u_y = -6xy \qquad -v_x = -6xy$

Let $z_0 = x_0 + iy_0$. Verify:

1) $\exists$ an ε-neighborhood of z_0 s.t. $f(z)$ is defined everywhere throughout the neighborhood?

Yes since z^3 is defined $\forall z$, any choice of ε works.

2) u_x, u_y, v_x, v_y exist everywhere in that ε-neighborhood?

Yes since u and v are polynomials in x and y these partial derivates exist everywhere, in particular everywhere in the aforementioned ε-neighborhood.

3) u_x, u_y, v_x, v_y continuous at z_0 and satisfy the C-R equations at z_0?

Yes since u_x, u_y, v_x, and v_y are also polynomials they are continuous everywhere, including z_0. By the calculation above, the C-R equations are satisfied at all points z, in particular at $z_0 = (x_0, y_0) = x_0 + iy_0$

$\therefore f'(z_0)$ exists and $f'(z_0) = 3x_0^2 - 3y_0^2 + i6x_0y_0 = 3z^2$

Apply similar reasoning to show $f''(z)$ exists everywhere.
Reassign u and v for notational convenience.

$$f'(z) = 3x^2 - 3y^2 + i6xy = 3z^2$$

$$u_x = 6x \quad v_y = 6x$$
$$u_y = -6y \quad -v_x = -6y$$

$f'(z)$ defined on any choice of ε-neighborhood
u_x, u_y, v_x, v_y all exist on whatever ε-neighborhood was chosen
u_x, u_y, v_x, v_y all continuous and satisfy C-R at $z_0 = (x_0, y_0)$

$\therefore f''(z_0)$ exists and $f''(z_0) = 6x_0 + i6y_0 = 6z_0$

Since z_0 was arbitrary, $f''(z)$ exists $\forall z$ and $f''(z) = 6x + i6y = 6z$

5. Show that when $f(z) = x^3 + i(1-y)^3$, it is legitimate to write

$$f'(z) = u_x + iv_x = 3x^2$$

only when $z = i$.

Use the same theorem as in exercise 1 of this section: Suppose $f'(z)$ exists at some point $z_0 = (x_0, y_0)$. Then the first order partial derivatives must exist at z_0 and satisfy the C-R equations at z_0.

$$3x_0^2 = u_x(x_0, y_0) = v_y(x_0, y_0) = -3(1-y_0)^2$$

$$0 = u_y(x_0, y_0) = -v_x(x_0, y_0) = 0$$

Since $3x_0^2 \geq 0$ and $-3(1-y_0)^2$, $3x_0^2 = -3(1-y_0)^2$ can only be satisfied if $3x_0^2 = 0 = -3(1-y_0)^2 \Longrightarrow x_0 = 0,\ y_0 = 1$.

This shows the only point at which $f(z)$ may be differentiable is $z_0 = i$. Since satisfying C-R is only a necessary condition, use the theorem from exercise 2 to prove that $f'(z)$ does indeed exist at $z_0 = i$.

Since $f(z) = x^3 + i(1-y)^3$ is everywhere defined, has continuous partial derivatives everywhere and satisfies the C-R equations at $z_0 = i$ the conditions of the theorem are met.

$\therefore f'(z)$ exists at $z_0 = i$ and $f'(i) = u_x(0,1) + v_x(0,1) = 3 \cdot 0^2 + i \cdot 0 = 0$

6. Let u and v denote the real and imaginary components of the function f defined by means of the equations

$$f(z) = \begin{cases} \overline{z}^2/z & \text{when} \quad z \neq 0, \\ 0 & \text{when} \quad z = 0. \end{cases}$$

Verify that the Cauchy–Riemann equations $u_x = v_y$ and $u_y = -v_x$ are satisfied at the origin $z = (0, 0)$. [Compare with Exercise 9, Sec. 20, where it is shown that $f'(0)$ nevertheless fails to exist.]

For $z \neq 0$ $[(x,y) \neq (0,0)]$

$$f(z) = \bar{z}^2/z = \frac{(x-iy)^2}{x+iy} = \frac{(x-iy)^3}{x^2+y^2} = \frac{x^3-3xy^2}{x^2+y^2} + i\,\frac{y^3-3x^2y}{x^2+y^2}$$

$$u(x,y) = \frac{x^3-3xy^2}{x^2+y^2}, \quad v(x,y) = \frac{y^3-3x^2y}{x^2+y^2}$$

For $z=0$ $[(x,y)=(0,0)]$

$$f(0) = 0 = 0 + i\cdot 0, \quad u(0,0) = v(0,0) = 0$$

$$u_x(0,0) = \lim_{\Delta x \to 0} \frac{u(0+\Delta x, 0) - u(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{(\Delta x)^3 - 3(\Delta x)\cdot 0^2}{(\Delta x)^2 + 0^2} - 0}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (\Delta x)^3/(\Delta x)^3 = \lim_{\Delta x \to 0} 1 = 1$$

$$u_y(0,0) = \lim_{\Delta y \to 0} \frac{u(0, 0+\Delta y) - u(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\frac{0^3 - 3\cdot 0(\Delta y)^2}{0^2 + (\Delta y)^2} - 0}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{0}{\Delta y} = \lim_{\Delta y \to 0} 0 = 0$$

$$v_x(0,0) = \lim_{\Delta x \to 0} \frac{v(0+\Delta x, 0) - v(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{0^3 - 3(\Delta x)^2\cdot 0}{(\Delta x)^2 + 0^2} - 0}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{0}{\Delta x} = \lim_{\Delta x \to 0} 0 = 0$$

$$v_y(0,0) = \lim_{\Delta y \to 0} \frac{v(0, 0+\Delta y) - v(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\frac{(\Delta y)^3 - 0}{0^2 + (\Delta y)^2} - 0}{\Delta y} = 1$$

Since $u_x(0,0) = 1 = v_y(0,0)$ and $u_y(0,0) = 0 = -v_x(0,0)$, the C-R eqn's are satisfied at the origin.

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Chapter 2 Section 25: 1ac, 2b, 4ab, 6, 7 (pages 77, 78)

1. Apply the theorem in Sec. 22 to verify that each of these functions is entire:

(a) $f(z) = 3x + y + i(3y - x)$; *(b)* $f(z) = \sin x \cosh y + i \cos x \sinh y$;

(c) $f(z) = e^{-y} \sin x - ie^{-y} \cos x$; *(d)* $f(z) = (z^2 - 2)e^{-x}e^{-iy}$.

We are now ready to introduce the concept of an analytic function. A function f of the complex variable z is *analytic at a point* z_0 if it has a derivative at each point in some neighborhood of z_0.* It follows that if f is analytic at a point z_0, it must be analytic at each point in some neighborhood of z_0. A function f is *analytic in an open set* if it has a derivative everywhere in that set. If we should speak of a function f that is analytic in a set S which is not open, it is to be understood that f is analytic in an open set containing S.

An *entire* function is a function that is analytic at each point in the entire finite plane. Since the derivative of a polynomial exists everywhere, it follows that *every polynomial is an entire function.*

To show that f is entire, show that f is analytic at each point in the complex plane. To show that f is analytic at a point $z_0 \in \mathbb{C}$, show that there exists a neighborhood of z_0 s.t. $f'(z)$ exists throughout that neighborhood.

(a) Consider the theorem from sec. 22: $f(z)$ is defined everywhere and u_x, u_y, v_x, v_y exist everywhere, are continuous everywhere, and satisfy the C-R conditions everywhere:

$$u_x(x,y) = 3 = v_y(x,y) \qquad u_y(x,y) = 1 = v_y(x,y) \qquad (\forall z = x+iy \in \mathbb{C})$$

So for any choice of neighborhood as mentioned in the theorem the conditions of the theorem are satisfied so $f'(z)$ exists $\forall z \in \mathbb{C}$. But then for any choice of neighborhood about z_0 in the definition of analyticity, $f'(z)$ exists $\forall z$ in that neighborhood. $\therefore f$ is analytic everywhere in $\mathbb{C}$ and so f is entire.

(c) Apply the same analysis as in part (a). The argument is identical except that the C-R conditions take the form:

$$u_x = e^{-y}\cos x = v_y \qquad u_y = -e^{-y}\sin x = -v_x \qquad (\forall z = x+iy \in \mathbb{C})$$

<u>Note</u>: While the definitions and theorem used in this exercise require the existence of ε-neighborhoods wherein f satisfies certain conditions, the fact that $u(x,y)$ and $v(x,y)$ are "well-behaved" means that we don't really need to worry so much about stating explicit choices of neighborhoods. Focus on checking that all the first order partial derivatives are also well-behaved and check C-R. The neighborhoods usually only matter in pathological cases.

(a)

To show that f is entire, show that f is analytic at each point in the complex plane. To show that f is analytic at a point $z_0 \in \mathbb{C}$, show that there exists a neighborhood of z_0 s.t. $f'(z)$ exists througout that neighborhood.

(a) Consider the theorem from sec. 22: $f(z)$ is defined everywhere and u_x, u_y, v_x, v_y exist everywhere, are continuous everywhere, and satisfy the C-R conditions everywhere:

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(c) Apply the same analysis as in part (a). The argument is identical except that the C-R conditions take the form:

$$u_x = e^{-y}\cos x = v_y \qquad u_y = -e^{-y}\sin x = -v_x \qquad (\forall z = x+iy \in \mathbb{C})$$

2. With the aid of the theorem in Sec. 21, show that each of these functions is nowhere analytic:

(a) $f(z) = xy + iy$; *(b)* $f(z) = 2xy + i(x^2 - y^2)$; *(c)* $f(z) = e^y e^{ix}$.

(b) Suppose f is analytic at some point $z_0 \in \mathbb{C}$. This means there exists some neighborhood of z_0 s.t. $f'(z)$ exists for each z in that neighborhood. By the theorem in Sec. 21, this means u_x, u_y, v_x, and v_y must exist at each $z_0 = (x_0, y_0)$ and satisfy the C-R eqn's at $z_0 = (x_0, y_0)$. However,

$$\begin{aligned} u(x,y) &= 2xy \\ v(x,y) &= x^2 - y^2 \end{aligned} \longrightarrow \begin{aligned} 2y &= u_x = v_y = -2y \\ 2x &= u_y = -v_x = -2x \end{aligned} \longrightarrow (x,y) = (0,0)$$

The only point at which the C-R equations are satisfied is $z = 0$. No matter which z_0 was chosen (even $z_0 = 0$) it is impossible to find a neighborhood of z_0 s.t. $f'(z)$ exists throughout the neighborhood. This is because any neighborhood, however small, contains infinitely many points and we have shown that there exists at most one point at which $f'(z)$ may exist.

4. In each case, determine the singular points of the function and state why the function is analytic everywhere except at those points:

(a) $f(z) = \dfrac{2z+1}{z(z^2+1)}$; *(b)* $f(z) = \dfrac{z^3+i}{z^2-3z+2}$; *(c)* $f(z) = \dfrac{z^2+1}{(z+2)(z^2+2z+2)}$.

Ans. *(a)* $z = 0, \pm i$; *(b)* $z = 1, 2$; *(c)* $z = -2, -1 \pm i$.

(a) $f(z)$ is a rational function in z, that is, the quotient of polynomial expressions in z. So $f(z)$ is analytic at all points z at which $f(z)$ is defined, which is all z except:

$$\begin{aligned} 0 &= z(z^2+1) \\ 0 &= z(z+i)(z-i) \end{aligned} \longleftrightarrow z = 0, \pm i \text{ are the singular points}$$

(b) By the same reasoning as part a, $f(z)$ is analytic everywhere except where the denominator vanishes:

$$0 = z^2 - 3z + 2 = (z-2)(z-1) \longleftrightarrow z = 1, 2 \text{ are the singular points}$$

6. Use results in Sec. 23 to verify that the function

$$g(z) = \ln r + i\theta \qquad (r > 0, 0 < \theta < 2\pi)$$

is analytic in the indicated domain of definition, with derivative $g'(z) = 1/z$. Then show that the composite function $G(z) = g(z^2 + 1)$ is analytic in the quadrant $x > 0$, $y > 0$, with derivative

$$G'(z) = \frac{2z}{z^2 + 1}.$$

Suggestion: Observe that $\text{Im}(z^2 + 1) > 0$ when $x > 0$, $y > 0$.

Theorem. *Let the function*

$$f(z) = u(r, \theta) + iv(r, \theta)$$

be defined throughout some ε neighborhood of a nonzero point $z_0 = r_0 \exp(i\theta_0)$, and suppose that

(a) the first-order partial derivatives of the functions u and v with respect to r and θ exist everywhere in the neighborhood;

(b) those partial derivatives are continuous at (r_0, θ_0) and satisfy the polar form

$$ru_r = v_\theta, \quad u_\theta = -rv_r$$

of the Cauchy–Riemann equations at (r_0, θ_0).

Then $f'(z_0)$ exists, its value being

$$f'(z_0) = e^{-i\theta}(u_r + iv_r),$$

where the right-hand side is to be evaluated at (r_0, θ_0).

$g(z) = u(r,\theta) + iv(r,\theta)$ with $u = \ln r$ and $v = \theta$.

Pick any $z_0 = r_0 e^{i\theta_0}$ satisfying $0 < r_0$, $0 < \theta_0 < 2\pi$. The conditions of the theorem are met for any such z_0, including the C-R eqn's:

$$ru_r = r\,{}^1\!/_r = 1 = r \cdot 1 = rv_\theta$$
$$u_\theta = 0 = -r \cdot 0 = -rv_r$$

$\therefore g'(z_0)$ exists. Since z_0 was arbitrary, it follows that for any z in the specified region we can always find a neighborhood of z s.t. g is complex differentiable throughout the neighborhood. $\therefore g$ is analytic on $r > 0$, $0 < \theta < 2\pi$ and $g'(z)$ is given by

$$g'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}(1/r + i \cdot 0) = r^{-1}e^{-i\theta} = (re^{i\theta})^{-1} = z^{-1} = 1/z$$

Consider $G(z) = g(z^2 + 1)$ in the quadrant $x, y > 0$ $(r > 0, 0 < \theta < \pi/2)$

From section 20, page 60, we know that if $f(z)=z^2+1$ has a derivative at z_0 and if g has a derivative at $f(z_0)$ then the composite function $G(z)=g[f(z)]$ has a derivative at z_0 and $G'(z_0)=g'[f(z_0)]f'(z_0)$. From section 25, page 74, the composition of analytic functions is analytic.

Since $f(z)=z^2+1$ is entire (polynomials in z are entire), $f'(z)$ certainly exists in the restricted region $x,y>0$ with $f'(z)=2z$. Since z lies in the first quadrant $x,y>0$ (or $r>0$, $0<\theta<\pi/2$), $2z$ lies in the first quadrant as well, which is contained in $r>0$, $0<\theta<2\pi$. We showed g is analytic in this region. It follows that G is analytic in the first quadrant and:

$$G'(z)=g'[f(z)]f'(z)=g'(z^2+1)\,2z=2z/(z^2+1)$$

7. Let a function f be analytic everywhere in a domain D. Prove that if $f(z)$ is real-valued for all z in D, then $f(z)$ must be constant throughtout D.

$\forall z\in D \quad f(z)=u(x,y)+iv(x,y)=u(x,y)$ as $v(x,y)\equiv 0$

Pick any $z_0\in D$. Since f is analytic at z_0, $f(z)$ has a derivative at z_0. This means the C-R equations must be satisfied at z_0 and $f'(z_0)=u_x(x_0,y_0)+iv_x(x_0,y_0)$.

$$\begin{aligned} u_x(x_0,y_0)&=v_y(x_0,y_0)=0 \\ u_y(x_0,y_0)&=-v_x(x_0,y_0)=0 \end{aligned} \quad \text{and} \quad f'(z_0)=0+i\cdot 0=0$$

Since $z_0\in D$ was arbitrary this shows $f'(z)=0 \quad \forall z\in D$.

Theorem. *If $f'(z)=0$ everywhere in a domain D, then $f(z)$ must be constant throughout D.*

(page 74)

By this theorem conclude that if $f(z)\in\mathbb{R}\ \forall z\in D$ then $f'(z)=0$ $\forall z\in D$ and therefore $f(z)$ must be constant throughout D.

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Chapter 2 Section 26: 1ad, 2, 3, 7, 9 (pages 81, 82)

1. Show that $u(x, y)$ is harmonic in some domain and find a harmonic conjugate $v(x, y)$ when

(a) $u(x, y) = 2x(1 - y)$; *(b)* $u(x, y) = 2x - x^3 + 3xy^2$;
(c) $u(x, y) = \sinh x \sin y$; *(d)* $u(x, y) = y/(x^2 + y^2)$.

Ans. *(a)* $v(x, y) = x^2 - y^2 + 2y$; *(b)* $v(x, y) = 2y - 3x^2y + y^3$;
(c) $v(x, y) = -\cosh x \cos y$; *(d)* $v(x, y) = x/(x^2 + y^2)$.

A real-valued function H of two real variables x and y is said to be *harmonic* in a given domain of the xy plane if, throughout that domain, it has continuous partial derivatives of the first and second order and satisfies the partial differential equation

$$H_{xx}(x, y) + H_{yy}(x, y) = 0, \tag{1}$$

known as *Laplace's equation.*

If two given functions u and v are harmonic in a domain D and their first-order partial derivatives satisfy the Cauchy–Riemann equations (2) throughout D, then v is said to be a *harmonic conjugate* of u. The meaning of the word conjugate here is, of course, different from that in Sec. 5, where $\bar{z}$ is defined.

(a) $\left.\begin{array}{ll} u_x = 2(1-y) & u_y = -2x \\ u_{xx} = 0 & u_{yy} = 0 \end{array}\right\} \rightarrow$ $u_{xx} + u_{yy} = 0 + 0 = 0$
$\therefore u$ is harmonic (any domain)

A harmonic conjugate $v(x,y)$ of $u(x,y)$ will satisfy

$$v_{xx} + v_{yy} = 0, \quad u_x = v_y, \quad u_y = -v_x$$

$$\left.\begin{array}{ll} v_y = u_x = 2 - 2y & \rightarrow v = 2y - y^2 + h(x) \\ v_x = -u_y = 2x & \rightarrow v = x^2 + g(y) \end{array}\right\} \rightarrow \boxed{v(x,y) = x^2 - y^2 + 2y}$$

Check that v is indeed harmonic: $v_{xx} + v_{yy} = 2 - 2 = 0$ ✓

(d) $\left.\begin{array}{ll} u_x = -\dfrac{2xy}{(x^2+y^2)^2} & u_y = \dfrac{x^2-y^2}{(x^2+y^2)^2} \\ u_{xx} = \dfrac{6x^2y - 2y^3}{(x^2+y^2)^3} & u_{yy} = \dfrac{2y^3 - 6x^2y}{(x^2+y^2)^3} \end{array}\right\} \rightarrow$ $u_{xx} + u_{yy} = 0$
$\therefore u$ is harmonic on any domain D that excludes the origin

$$\left.\begin{array}{l} v_y = u_x \rightarrow v = \int u_x\,dy + h(x) = \dfrac{x}{x^2+y^2} + h(x) \\ v_x = -u_y \rightarrow v = \int -u_y\,dx + g(y) = \dfrac{x}{x^2+y^2} + g(y) \end{array}\right\} \rightarrow \boxed{v(x,y) = \frac{x}{x^2+y^2}}$$

You can check that $v_{xx} + v_{yy} = 0$. I used Wolfram Alpha.

2. Show that if v and V are harmonic conjugates of $u(x, y)$ in a domain D, then $v(x, y)$ and $V(x, y)$ can differ at most by an additive constant.

If v and V are both harmonic conjugates of u in D,

$$u_{xx} + v_{yy} = 0 = u_{xx} + V_{yy}$$

$$V_{yy} - v_{yy} = 0$$

$$V_y - v_y = c \quad (c \in \mathbb{R} \text{ constant})$$

$$V - v = cy + d \quad (d \in \mathbb{R} \text{ constant})$$

Also $V_y = u_x = v_y$ from C-R so $c = V_y - v_y = 0$. Thus $V - v = d$ for some constant d, i.e. v and V differ at most by an additive constant ($V = v + d$).

3. Suppose that v is a harmonic conjugate of u in a domain D and also that u is a harmonic conjugate of v in D. Show how it follows that both $u(x, y)$ and $v(x, y)$ must be constant throughout D.

v is a harmonic conjugate of u in D $\Rightarrow$ $u_x = v_y$, $u_y = -v_x$

u is a harmonic conjugate of v in D $\Rightarrow$ $v_x = u_y$, $v_y = -u_x$

Since $u_x = v_y = -u_x$, $u_x = -u_x$. This means $u_x = 0$ so that $u(x,y) = g(y)$ by integration wrt x.

Since $u_y = -v_x = -u_y$, $u_y = -u_y$. This means $u_y = 0$ so that $u(x,y) = h(x)$ by integration wrt y.

The only way we can have $u = g(y)$ and $u = h(x)$ is if u does not vary wrt either of x or y. That is, u = constant in D.

Similarly, $v_x = -v_x$ implies $v_x = 0$ and $v_y = -v_y$ implies $v_y = 0$. By reasoning as we did for u, this implies that v doesn't vary wrt either of x or y. That is, v = constant in D.

7. Let the function $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain D, and consider the families of *level curves* $u(x, y) = c_1$ and $v(x, y) = c_2$, where c_1 and c_2 are arbitrary real constants. Prove that these families are orthogonal. More precisely, show that if $z_0 = (x_0, y_0)$ is a point in D which is common to two particular curves $u(x, y) = c_1$ and $v(x, y) = c_2$ and if $f'(z_0) \neq 0$, then the lines tangent to those curves at (x_0, y_0) are perpendicular.

Suggestion: Note how it follows from the pair of equations $u(x, y) = c_1$ and $v(x, y) = c_2$ that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0 \quad \text{and} \quad \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy}{dx} = 0.$$

Consider a point z common to both curves s.t. $f'(z)$ exists and $f'(z) \neq 0$

$0 \neq f(z) = u_x(x,y) + iv_x(x,y)$ implies:

$0 \neq |f'(z)|^2 = u_x^2 + v_x^2 = u_x^2 + u_y^2 = |\nabla u|^2$ since $v_x = -u_y$

$0 \neq |f'(z)|^2 = u_x^2 + v_x^2 = v_y^2 + v_x^2 = |\nabla v|^2$ since $u_x = v_y$

Hence the 2-d vector gradients $\nabla u = (u_x, u_y) \in \mathbb{R}^2$ and $\nabla v = (v_x, v_y) \in \mathbb{R}^2$ are nonzero. From vector calculus we know the gradient is orthogonal to its level curve: $du = \nabla u \cdot d\vec{s} = 0$ where $d\vec{s}$ points in the direction of the tangent to the level curve and similar for v. These gradients are orthogonal since their dot product is 0:

$$\nabla u \cdot \nabla v = u_x v_x + u_y v_y = -u_x u_y + u_y u_x = 0$$

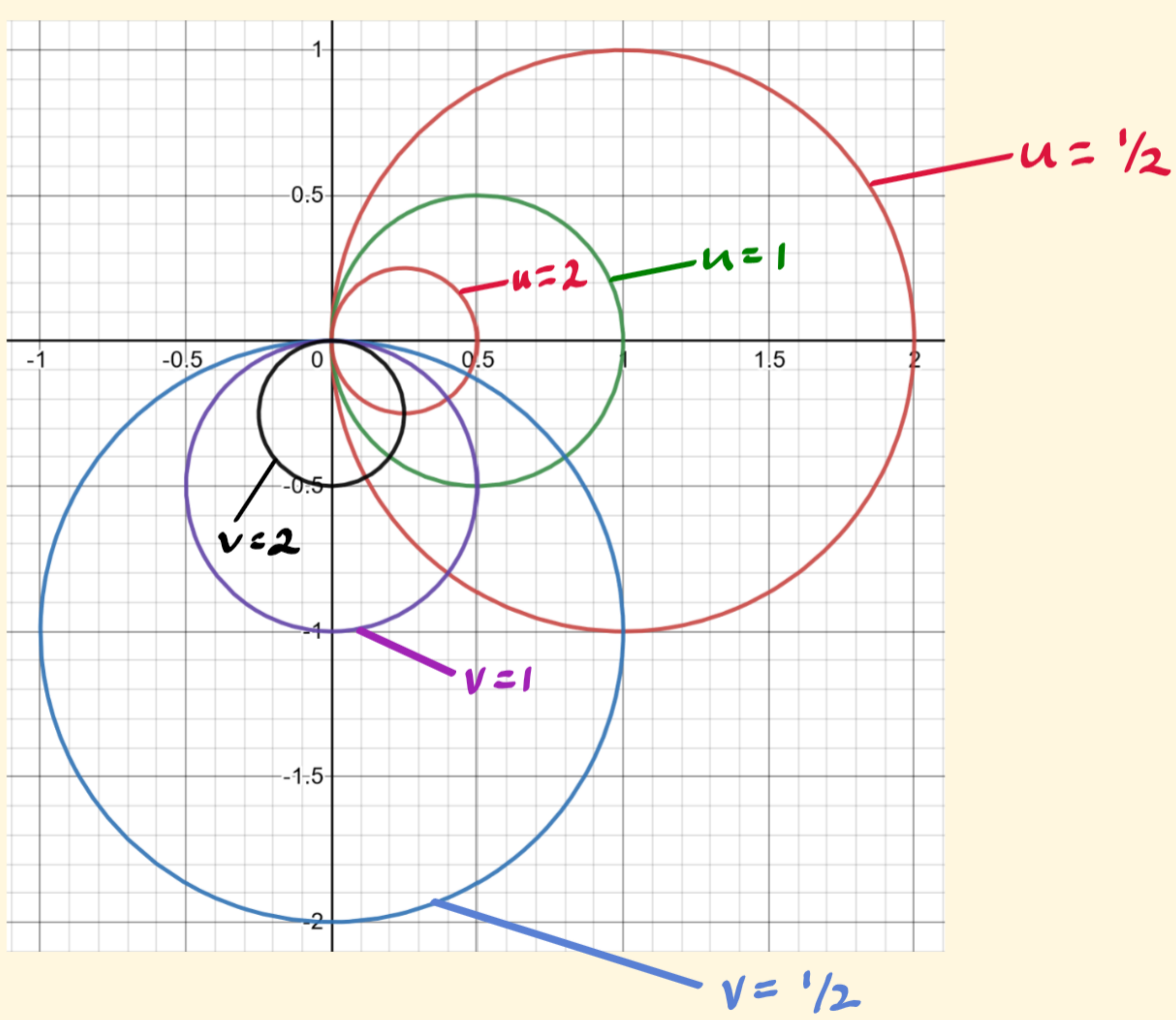
Since orthogonality is a transitive relationship: $(u=c_1) \perp \nabla u$, $\nabla u \perp \nabla v$, and $\nabla v \perp (v=c_2)$ together imply $(u=c_1) \perp (v=c_2)$, i.e. the level curves $u=c_1$ and $v=c_2$ are orthogonal.

9. Sketch the families of level curves of the component functions u and v when $f(z) = 1/z$, and note the orthogonality described in Exercise 7.

$$f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = u(x,y) + iv(x,y)$$

A level curve $u(x,y) = c$ is a circle in the complex plane with radius $1/2c$ and center $(1/2c, 0)$.

A level curve $v(x,y) = d$ is a circle in the complex plane with radius $1/2d$ and center $(0, -1/2d)$.



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Chapter 3 Section 29 : 3, 5, 11, 13 (page 92)

3. Use the Cauchy–Riemann equations and the theorem in Sec. 21 to show that the function $f(z) = \exp \bar{z}$ is not analytic anywhere.

By the theorem from section 21, if $f'(z)$ exists at a point z, then the C-R equations must be satisfied at z. However,

$$f(z) = \exp \bar{z} = \exp(x - iy) = e^x \cos y + i(-e^x \sin y) = u(x,y) + iv(x,y)$$

$$e^x \cos y = u_x = v_y = -e^x \cos y \Rightarrow 0 = e^x \cos y \Rightarrow \cos y = 0$$

$$-e^x \sin y = u_y = -v_x = e^x \sin y \Rightarrow 0 = e^x \sin y \Rightarrow \sin y = 0$$

This means $f'(z)$ may possibly exist only at points where both $\cos y = 0$ and $\sin y = 0$, which are the points where $y = \pi/2 + m\pi$ and $y = n\pi$ respectively. These points never coincide so $f'(z)$ does not exist anywhere. Since f is nowhere differentiable, f is nowhere analytic. We have used the fact that:

the statement : f analytic at $z \Rightarrow f$ differentiable at z

is equivalent to: f is not differentiable at $z \Rightarrow f$ is not analytic at z

5. Write $|\exp(2z + i)|$ and $|\exp(iz^2)|$ in terms of x and y. Then show that

$$|\exp(2z + i) + \exp(iz^2)| \le e^{2x} + e^{-2xy}.$$

$$\exp(2z+i) = \exp(2x + 2yi + i) = e^{2x}\cos(2y+1) + ie^{2x}\sin(2y+1)$$

$$|\exp(2z+i)| = [e^{4x}\cos^2(2y+1) + e^{4x}\sin^2(2y+1)]^{1/2} = [e^{4x}]^{1/2} = e^{2x}$$

$$\exp(iz^2) = \exp[i(x^2 - y^2 + i2xy)] = e^{-2xy}\cos(x^2-y^2) + ie^{-2xy}\sin(x^2-y^2)$$

$$|\exp(iz^2)| = [e^{-4xy}\cos^2(x^2-y^2) + e^{-4xy}\sin^2(x^2-y^2)]^{1/2} = [e^{-4xy}]^{1/2} = e^{-2xy}$$

$$\therefore |\exp(2z+i) + \exp(iz^2)| \le |\exp(2z+i)| + |\exp(iz^2)| = e^{2x} + e^{-2xy}$$

11. Describe the behavior of $e^z = e^x e^{iy}$ as *(a)* x tends to $-\infty$; *(b)* y tends to ∞.

(a) $\lim\limits_{x \to -\infty} e^x e^{iy} = 0$

(b) $\lim\limits_{y \to \infty} e^x e^{iy}$ does not exist since $e^{iy} = \cos y + i \sin y$ oscillates.

More specifically, for x held constant:

$$\left.\begin{aligned} \mathrm{Re}(e^z) &= e^x \cos y \text{ oscillates between } e^x \text{ and } -e^x \\ \mathrm{Im}(e^z) &= e^x \sin y \text{ oscillates between } e^x \text{ and } -e^x \end{aligned}\right\} \begin{array}{l} e^z \text{ revolves} \\ \text{about the} \\ \text{circle } |z| = e^x \end{array}$$

13. Let the function $f(z) = u(x, y) + iv(x, y)$ be analytic in some domain D. State why the functions

$$U(x, y) = e^{u(x,y)} \cos v(x, y), \quad V(x, y) = e^{u(x,y)} \sin v(x, y)$$

are harmonic in D and why $V(x, y)$ is, in fact, a harmonic conjugate of $U(x, y)$.

Theorem 1. *If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then its component functions u and v are harmonic in D.*

$$\Rightarrow \begin{aligned} u_{xx} + u_{yy} &= 0 \\ v_{xx} + v_{yy} &= 0 \end{aligned}$$

If two given functions u and v are harmonic in a domain D and their first-order partial derivatives satisfy the Cauchy–Riemann equations (2) throughout D, then v is said to be a *harmonic conjugate* of u. The meaning of the word conjugate here is, of course, different from that in Sec. 5, where $\bar{z}$ is defined.

Theorem 2. *A function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D if and only if v is a harmonic conjugate of u.*

$$\Rightarrow \begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

$$U_x = u_x e^u \cos v - v_x e^u \sin v = e^u (u_x \cos v - v_x \sin v)$$

$$U_{xx} = e^u [u_{xx} \cos v + u_x^2 \cos v - u_x v_x \sin v - v_{xx} \sin v - u_x v_x \sin v - v_x^2 \cos v]$$

$$U_y = u_y e^u \cos v - v_y e^u \sin v = e^u (u_y \cos v - v_y \sin v)$$

$$U_{yy} = e^u [u_{yy} \cos v + u_y^2 \cos v - u_y v_y \sin v - v_{yy} \sin v - u_y v_y \sin v - v_y^2 \cos v]$$

$$\begin{aligned} U_{xx} + U_{yy} &= e^u [\cos v \overbrace{(u_{xx} + u_{yy})}^{=0} + \cos v (u_x^2 - v_x^2 + u_y^2 - v_y^2) \\ &\qquad - \sin v \underbrace{(v_{xx} + v_{yy})}_{=0} - 2 \sin v (u_x v_x + u_y v_y)] \\ &= e^u [\cos v (u_x^2 - u_y^2 + u_y^2 - u_x^2) - 2 \sin v (-u_x u_y + u_y u_x)] \\ &= 0 \end{aligned}$$

$\therefore U$ is harmonic in D.

$$V = e^u \sin v$$

$$V_x = u_x e^u \sin v + v_x e^u \cos v = e^u (u_x \sin v + v_x \cos v)$$

$$V_{xx} = u_x e^u (u_x \sin v + v_x \cos v) + e^u (u_{xx} \sin v + u_x v_x \cos v + v_{xx} \cos v - v_x^2 \sin v)$$

$$= e^u [(u_x^2 + u_{xx} - v_x^2) \sin v + (2 u_x v_x + v_{xx}) \cos v]$$

$$V_y = u_y e^u \sin v + v_y e^u \cos v = e^u (u_y \sin v + v_y \cos v)$$

$$V_{yy} = u_y e^u (u_y \sin v + v_y \cos v) + e^u (u_{yy} \sin v + u_y v_y \cos v + v_{yy} \cos v - v_y^2 \sin v)$$

$$= e^u [(u_y^2 + u_{yy} - v_y^2) \sin v + (2 u_y v_y + v_{yy}) \cos v]$$

$$V_{xx} + V_{yy} = e^u [\overbrace{(u_{xx} + u_{yy})}^{=0} \sin v + (u_x^2 - v_x^2 + u_y^2 - v_y^2) \sin v$$
$$+ (v_{xx} + v_{yy}) \cos v + 2 (u_x v_x + u_y v_y) \cos v]$$

$$= e^u (u_x^2 - u_y^2 + u_y^2 - u_x^2) \sin v + e^u (-u_x u_y + u_y u_x) \cos v$$

$$= 0$$

$\therefore$ V is harmonic in D.

To conclude V is a harmonic conjugate of U, check C-R eqn's:

$$U_x = e^u (u_x \cos v - v_x \sin v) = e^u (v_y \cos v + u_y \sin v) = V_y \quad \checkmark$$

$$U_y = e^u (u_y \cos v - v_y \sin v) = e^u (-v_x \cos v - u_x \sin v) = -V_x \quad \checkmark$$