

# Written Homework 1 Solutions

MA 4291 (Tilley)  
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## Chapter 1 Section 4 : 1, 3, 4, 6 (Page 12)

1. Locate the numbers  $z_1 + z_2$  and  $z_1 - z_2$  vectorially when

$$(a) z_1 = 2i, \quad z_2 = \frac{2}{3} - i; \quad (b) z_1 = (-\sqrt{3}, 1), \quad z_2 = (\sqrt{3}, 0); \\ (c) z_1 = (-3, 1), \quad z_2 = (1, 4); \quad (d) z_1 = x_1 + iy_1, \quad z_2 = x_1 - iy_1.$$

$$(a) z_1 + z_2 = (0, 2) + \left(\frac{2}{3}, -1\right) = \left(\frac{2}{3}, 1\right)$$

$$z_1 - z_2 = (0, 2) - \left(\frac{2}{3}, -1\right) = \left(-\frac{2}{3}, 3\right)$$

$$(b) z_1 + z_2 = (-\sqrt{3}, 1) + (\sqrt{3}, 0) = (0, 1)$$

$$z_1 - z_2 = (-\sqrt{3}, 1) - (\sqrt{3}, 0) = (-2\sqrt{3}, 1)$$

$$(c) z_1 + z_2 = (-3, 1) + (1, 4) = (-2, 5)$$

$$z_1 - z_2 = (-3, 1) - (1, 4) = (-4, -3)$$

$$(d) z_1 + z_2 = (x, y) + (x, -y) = (x, 0)$$

$$z_1 - z_2 = (x, y) - (x, -y) = (0, 2y)$$

3. Use established properties of moduli to show that when  $|z_3| \neq |z_4|$ ,

$$\frac{\operatorname{Re}(z_1 + z_2)}{|z_3 + z_4|} \leq \frac{|z_1| + |z_2|}{||z_3| - |z_4||}.$$

$$\frac{\operatorname{Re}(z_1 + z_2)}{|z_3 + z_4|} \leq \frac{|z_1 + z_2|}{|z_3 + z_4|}$$

$$\leq \frac{|z_1| + |z_2|}{|z_3 + z_4|}$$

$$\leq \frac{|z_1| + |z_2|}{||z_3| - |z_4||}$$

$$\operatorname{Re} z \leq |\operatorname{Re} z| \leq |z|$$

triangle inequality:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

reverse triangle inequality:

$$0 < ||z_3| - |z_4|| \leq |z_3 + z_4|$$

4. Verify that  $\sqrt{2}|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z|$ .

Suggestion: Reduce this inequality to  $(|x| - |y|)^2 \geq 0$ .

$$\begin{aligned}
 0 &\leq (|\operatorname{Re} z| - |\operatorname{Im} z|)^2 = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 - 2|\operatorname{Re} z||\operatorname{Im} z| \\
 (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 &\leq 2[(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2] - 2|\operatorname{Re} z||\operatorname{Im} z| \\
 (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 + 2|\operatorname{Re} z||\operatorname{Im} z| &= 2[(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2] \\
 (|\operatorname{Re} z| + |\operatorname{Im} z|)^2 &\leq 2[(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2] \\
 (|\operatorname{Re} z| + |\operatorname{Im} z|)^2 &\leq 2|z|^2 \quad (|z|^2 = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2) \\
 |\operatorname{Re} z| + |\operatorname{Im} z| &\leq \sqrt{2}|z| \quad (a^2 + b^2 \rightarrow a \leq b \text{ for } a, b \geq 0)
 \end{aligned}$$

6. Using the fact that  $|z_1 - z_2|$  is the distance between two points  $z_1$  and  $z_2$ , give a geometric argument that

- (a)  $|z - 4i| + |z + 4i| = 10$  represents an ellipse whose foci are  $(0, \pm 4)$ ;
- (b)  $|z - 1| = |z + i|$  represents the line through the origin whose slope is  $-1$ .

(a) Consider this definition of an ellipse (source: Wikipedia)

Definition as locus of points [\[edit\]](#)

An ellipse can be defined geometrically as a set or [locus of points](#) in the Euclidean plane:

Given two fixed points  $F_1, F_2$  called the foci and a distance  $2a$  which is greater than the distance between the foci, the ellipse is the set of points  $P$  such that the sum of the distances  $|PF_1|, |PF_2|$  is equal to  $2a$ :  $E = \{P \in \mathbb{R}^2 \mid |PF_2| + |PF_1| = 2a\}$ .

While this definition refers to the Euclidean plane, we can still use it for geometric arguments since  $\mathbb{R}^2$  is isomorphic to  $\mathbb{C}$ .

$|z - 4i|$  is the distance between a point  $z$  and the point  $4i$ .

$|z + 4i|$  is the distance between a point  $z$  and the point  $-4i$ .

If we take  $a = 5$ , the distance between  $4i$  and  $-4i$  is less than  $2a$  ( $8 < 10$ ). Taking  $F_1 = 4i, F_2 = -4i$  conclude that  $|z - 4i| + |z + 4i| = 10$  is an ellipse in the complex plane.

(b) Write  $z = x + iy$  where  $x = \operatorname{Re} z \in \mathbb{R}, y = \operatorname{Im} z \in \mathbb{R}$

$$\begin{aligned}
 |z - 1| = |z + i| &\Leftrightarrow |z - 1|^2 = |z + i|^2 \Leftrightarrow |x - 1 + iy|^2 = |x + i(y+1)|^2 \\
 &\Leftrightarrow x^2 - 2x + 1 + y^2 = x^2 + y^2 + 2y + 1 \Leftrightarrow \boxed{y = -x}
 \end{aligned}$$

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## Chapter 1 Section 5 : 2, 7, 9, 10, 11, 13 (pages 14, 15)

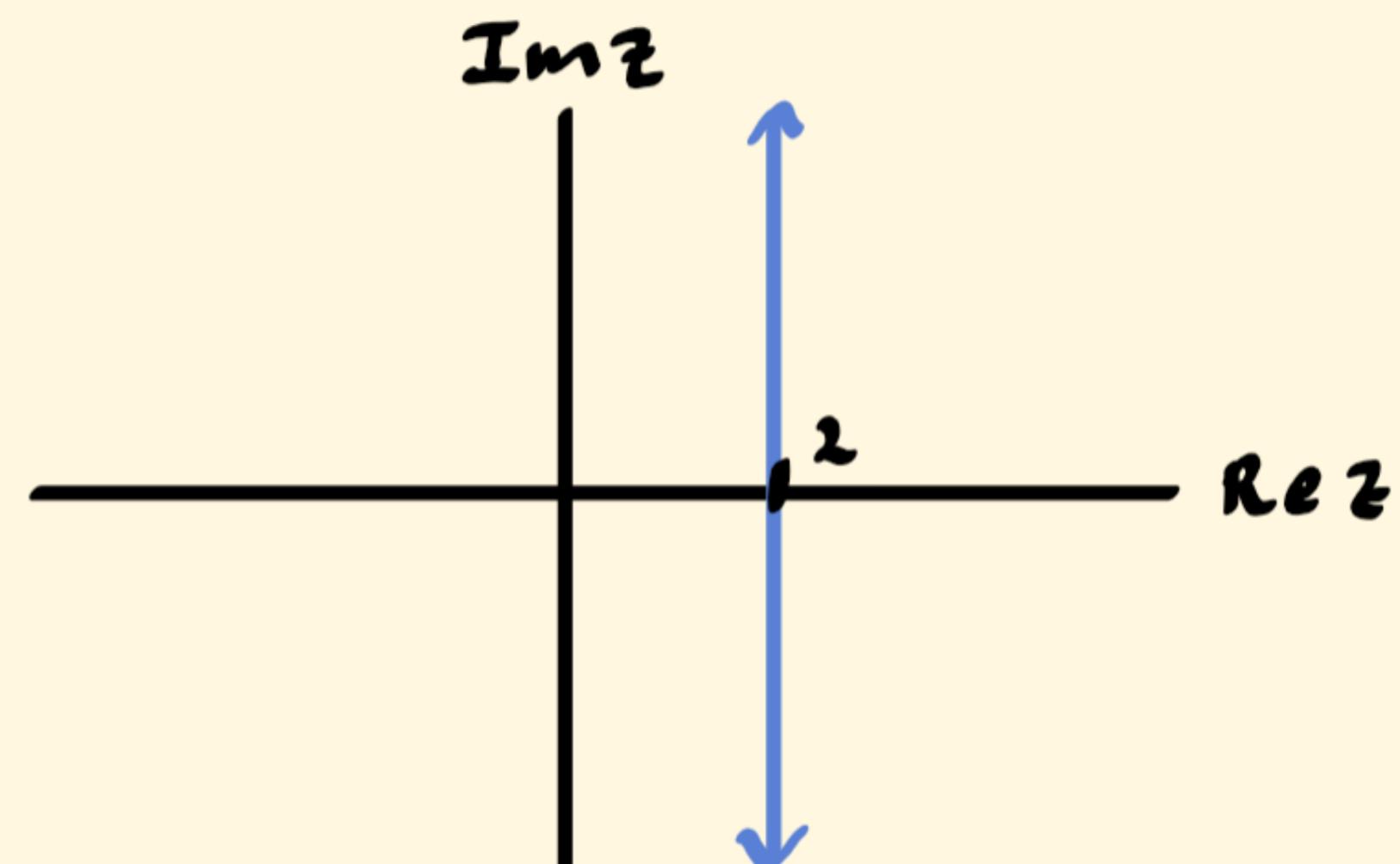
2. Sketch the set of points determined by the condition

$$(a) \operatorname{Re}(\bar{z} - i) = 2; \quad (b) |2\bar{z} + i| = 4.$$

Let  $z = x + iy \quad x, y \in \mathbb{R}$

$$(a) 2 = \operatorname{Re}(\bar{z} - i) = \operatorname{Re}(x + (-1-y)i) = x$$

All  $z = x + iy$  with  $x = 2$

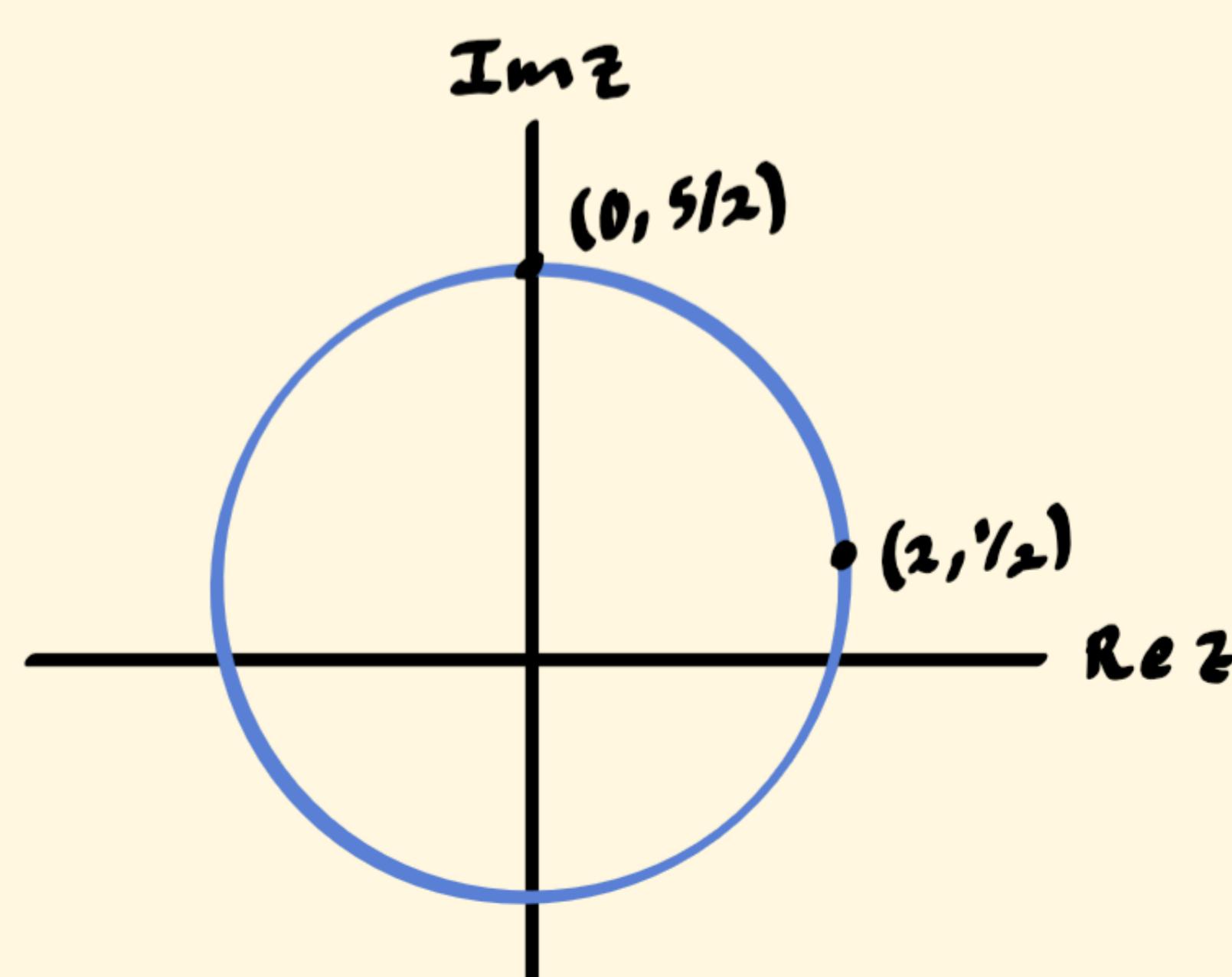


$$(b) 4 = |2\bar{z} + i| = |2x - 2yi + i|$$

$$4^2 = 4x^2 + 4(y - \frac{1}{2})^2$$

$$2^2 = x^2 + (y - \frac{1}{2})^2$$

All  $z$  on the circle of radius 2 centered at  $(0, \frac{1}{2})$  in the complex plane



7. Show that

$$|\operatorname{Re}(2 + \bar{z} + z^3)| \leq 4 \quad \text{when } |z| \leq 1.$$

$$\begin{aligned} |\operatorname{Re}(2 + \bar{z} + z^3)| &\leq |2 + \bar{z} + z^3| \\ &\leq |2| + |\bar{z}| + |z^3| \\ &= 2 + |z| + |z|^3 \\ &\leq 2 + 1 + 1^3 = 4 \end{aligned}$$

$$\begin{aligned} |\operatorname{Re} w| &\leq |w| \quad \forall w \in \mathbb{C} \\ &\text{triangle inequality} \\ |\bar{z}| &= |z|, |z^3| = |z|^3 \\ |z| &\leq 1 \end{aligned}$$

9. By factoring  $z^4 - 4z^2 + 3$  into two quadratic factors and using inequality (8), Sec. 4, show that if  $z$  lies on the circle  $|z| = 2$ , then

$$\left| \frac{1}{z^4 - 4z^2 + 3} \right| \leq \frac{1}{3}.$$

$$|z^4 - 4z^2 + 3| = |z^2 - 3||z^2 - 1| \geq ||z^2| - |3|| ||z^2| - |1|| \quad (\text{inequality 8})$$

$$= ||z|^2 - 3| |||z|^2 - 1| = |4 - 3| ||4 - 1| = 3$$

$$\text{Since } |z^4 - 4z^2 + 3| \geq 3, \quad \left| \frac{1}{z^4 - 4z^2 + 3} \right| = \frac{1}{|z^4 - 4z^2 + 3|} \leq \frac{1}{3}$$

10. Prove that

- (a)  $z$  is real if and only if  $\bar{z} = z$ ;
- (b)  $z$  is either real or pure imaginary if and only if  $\bar{z}^2 = z^2$ .

Let  $z = x + iy$  with  $x, y \in \mathbb{R}$

(a)  $x + iy = z = \bar{z} = x - iy \rightarrow y = 0$

$$\therefore z = x \in \mathbb{R}$$

(b)  $x^2 - y^2 + 2xyi = z^2 = \bar{z}^2 = x^2 - y^2 - 2xyi$

$$\Rightarrow xy = 0 \Rightarrow x = 0 \text{ or } y = 0$$

If  $x = 0$ ,  $z$  is pure imaginary. If  $y = 0$ ,  $z$  is real.

11. Use mathematical induction to show that when  $n = 2, 3, \dots$ ,

(a)  $\overline{z_1 + z_2 + \dots + z_n} = \overline{z_1} + \overline{z_2} + \dots + \overline{z_n}; \quad (b) \overline{z_1 z_2 \dots z_n} = \overline{z_1} \overline{z_2} \dots \overline{z_n}$

(a) Base case  $n=2$ : If  $z_1 = x_1 + iy_1$  and  $z_2 = x_2 + iy_2$ ,

$$\overline{z_1 + z_2} = \overline{(x_1 + x_2) + i(y_1 + y_2)} = (x_1 + x_2) - i(y_1 + y_2) = x_1 - iy_1 + x_2 - iy_2 = \overline{z}_1 + \overline{z}_2$$

Inductive step: Assume  $\overline{z_1 + \dots + z_n} = \overline{z}_1 + \dots + \overline{z}_n$  for some  $n \geq 2$

$$\begin{aligned} \overline{z_1 + \dots + z_n + z_{n+1}} &= \overline{\overline{z_1 + \dots + z_n} + \overline{z}_{n+1}} \quad (\text{by the base case}) \\ &= \overline{z}_1 + \dots + \overline{z}_n + \overline{z}_{n+1} \quad (\text{by the inductive hypothesis}) \end{aligned}$$

(b) Base case  $n=2$ : If  $z_1 = x_1 + iy_1$  and  $z_2 = x_2 + iy_2$ ,

$$\begin{aligned} \overline{z_1 z_2} &= \overline{(x_1 + iy_1)(x_2 + iy_2)} = \overline{(x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)} \\ &= (x_1 x_2 - y_1 y_2) - i(x_1 y_2 + x_2 y_1) \\ &= (x_1 - iy_1)(x_2 - iy_2) = \overline{z}_1 \overline{z}_2 \end{aligned}$$

Inductive step: Assume  $\overline{z_1 z_2 \dots z_n} = \overline{z}_1 \overline{z}_2 \dots \overline{z}_n$  for some  $n \geq 2$

$$\begin{aligned} \overline{z_1 \dots z_n z_{n+1}} &= \overline{\overline{z_1 \dots z_n} \overline{z}_{n+1}} \quad (\text{by the base case}) \\ &= \overline{z}_1 \dots \overline{z}_n \overline{z}_{n+1} \quad (\text{by the inductive hypothesis}) \end{aligned}$$

13. Show that the equation  $|z - z_0| = R$  of a circle, centered at  $z_0$  with radius  $R$ , can be written

$$|z|^2 - 2 \operatorname{Re}(z\bar{z}_0) + |z_0|^2 = R^2.$$

$|z - z_0| = R$  iff  $|z - z_0|^2 = R^2$  since  $|z - z_0|, R > 0$

$$\begin{aligned} R^2 &= |z - z_0|^2 = (z - z_0)(\bar{z} - \bar{z}_0) \\ &= (z - z_0)(\bar{z} - \bar{z}_0) \\ &= z\bar{z} + z_0\bar{z}_0 - z\bar{z}_0 - \bar{z}z_0 \\ &= |z|^2 + |z_0|^2 - (z\bar{z}_0 + \bar{z}\bar{z}_0) \\ &= |z|^2 + |z_0|^2 - 2\operatorname{Re}(z\bar{z}_0) \end{aligned}$$

$$\begin{aligned} |w|^2 &= w\bar{w} \quad \forall w \in \mathbb{C} \\ \overline{v-w} &= \overline{v} - \overline{w} \quad \forall v, w \in \mathbb{C} \\ \overline{vw} &= \overline{v}\bar{w} = \bar{v}w \quad \forall v, w \in \mathbb{C} \\ 2\operatorname{Re} w &= w + \bar{w} \end{aligned}$$

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## Chapter 1 Section B: 1, 3, 5, 9, 10 (pages 22, 23)

1. Find the principal argument  $\operatorname{Arg} z$  when

$$(a) z = \frac{i}{-2-2i}; \quad (b) z = (\sqrt{3}-i)^6.$$

Ans. (a)  $-3\pi/4$ ; (b)  $\pi$ .

$$(a) z_1 = i = e^{i(\pi/2 + 2k\pi)} \quad k \in \mathbb{Z}, \quad z_2 = -2-2i = 2\sqrt{2} e^{i(-3\pi/4 + 2j\pi)} \quad j \in \mathbb{Z}$$

$$\arg z_1 = \operatorname{Arg} z_1 + 2k\pi = \pi/2 + 2k\pi, \quad \arg z_2 = \operatorname{Arg} z_2 + 2j\pi = -3\pi/4 + 2j\pi$$

$$\arg z = \arg(z_1/z_2) = \arg\left[2\sqrt{2} e^{i(5\pi/4 + 2(k-j)\pi)}\right] = 5\pi/4 + 2n\pi, \quad n \in \mathbb{Z}$$

$$-\pi < \operatorname{Arg} z \leq \pi \rightarrow \operatorname{Arg} z = 5\pi/4 + 2 \cdot (-1)\pi = -3\pi/4$$

$$(b) w = \sqrt{3}-i = 2e^{-i\pi/6} \rightarrow z = w^6 = 2^6 e^{-i\pi}$$

$$\arg z = -\pi + 2n\pi \rightarrow \operatorname{Arg} z = -\pi + 2(1)\pi = \pi$$

3. Use mathematical induction to show that

$$e^{i\theta_1} e^{i\theta_2} \dots e^{i\theta_n} = e^{i(\theta_1 + \theta_2 + \dots + \theta_n)} \quad (n = 2, 3, \dots).$$

Base Case  $n=2$ :

$$\begin{aligned} e^{i\theta_1} e^{i\theta_2} &= (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \\ &= (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2, \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2) \\ &= \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) = e^{i(\theta_1 + \theta_2)} \end{aligned}$$

Inductive Step: Assume  $e^{i\theta_1} e^{i\theta_2} \dots e^{i\theta_n} = e^{i(\theta_1 + \theta_2 + \dots + \theta_n)}$  for some  $n \geq 2$

$$\begin{aligned} e^{i\theta_1} e^{i\theta_2} \dots e^{i\theta_n} e^{i\theta_{n+1}} &= e^{i(\theta_1 + \dots + \theta_n)} e^{i\theta_{n+1}} \quad (\text{by the inductive hypothesis}) \\ &= e^{i((\theta_1 + \dots + \theta_n) + \theta_{n+1})} \quad (\text{by the base case}) \\ &= e^{i(\theta_1 + \dots + \theta_{n+1})} \end{aligned}$$

5. By writing the individual factors on the left in exponential form, performing the needed operations, and finally changing back to rectangular coordinates, show that

$$(a) i(1 - \sqrt{3}i)(\sqrt{3} + i) = 2(1 + \sqrt{3}i); \quad (b) 5i/(2+i) = 1+2i;$$

$$(c) (-1+i)^7 = -8(1+i); \quad (d) (1+\sqrt{3}i)^{-10} = 2^{-11}(-1+\sqrt{3}i).$$

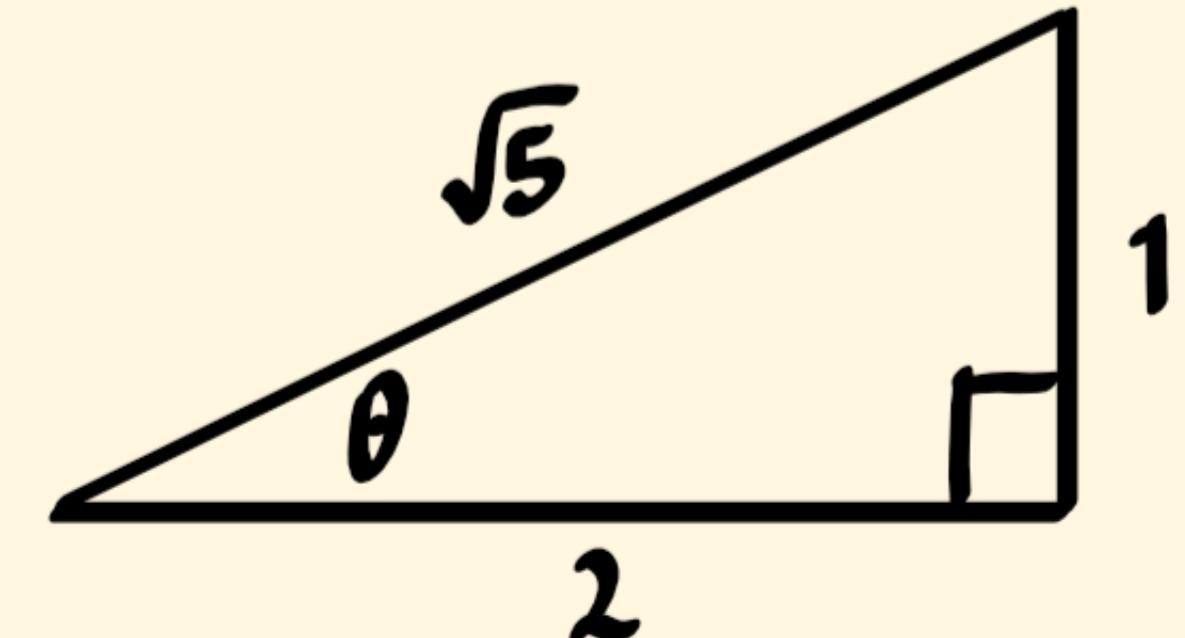
$$(a) i = e^{i\pi/2} \quad 1 - \sqrt{3}i = 2e^{-i\pi/3} \quad \sqrt{3} + i = 2e^{i\pi/6}$$

$$i(1 - \sqrt{3}i)(\sqrt{3} + i) = 4e^{i\pi(3-2+1)/6} = 4e^{i\pi/3} = 4(\cos \pi/3 + i \sin \pi/3) = 2(1 + \sqrt{3}i)$$

$$(b) 5i = 5e^{i\pi/2} \quad 2+i = \sqrt{5}e^{i\theta}, \quad \tan \theta = 1/2 \text{ with } \theta \text{ in quadrant 1}$$

$$\begin{aligned} 5i/(2+i) &= \sqrt{5}e^{i(\pi/2-\theta)} = \sqrt{5}(\cos(\pi/2-\theta) + i \sin(\pi/2-\theta)) \\ &= \sqrt{5}[(\cos \pi/2 \cos \theta + \sin \pi/2 \sin \theta) + i(\sin \pi/2 \cos \theta - \cos \pi/2 \sin \theta)] \\ &= \sqrt{5}(\sin \theta + i \cos \theta) \\ &= \sqrt{5}(1/\sqrt{5} + 2i/\sqrt{5}) \\ &= 1+2i \end{aligned}$$

$$\tan \theta = 1/2 \quad 0 < \theta < \pi/2$$



$$(c) -1+i = \sqrt{2}e^{3\pi i/4}$$

$$\begin{aligned} (-1+i)^7 &= \sqrt{2}^7 e^{21\pi i/4} = 8\sqrt{2} e^{16\pi i/4} e^{5\pi i/4} = 8\sqrt{2} e^{5\pi i/4} \\ &= 8\sqrt{2} (\cos 5\pi/4 + i \sin 5\pi/4) = 8\sqrt{2} (-\sqrt{2}/2 - \sqrt{2}i/2) \\ &= -8(1+i) \end{aligned}$$

$$(d) 1+\sqrt{3}i = 2e^{i\pi/3}$$

$$\begin{aligned} (1+\sqrt{3}i)^{-10} &= 2^{-10} e^{-10i\pi/3} = 2^{-10} e^{-12i\pi/3} e^{2\pi i/3} = 2^{-10} e^{2\pi i/3} \\ &= 2^{-10} (\cos 2\pi/3 + i \sin 2\pi/3) \\ &= 2^{-10} (-1/2 + \sqrt{3}i/2) \\ &= 2^{-11}(-1+\sqrt{3}i) \end{aligned}$$

9. Establish the identity

$$1 + z + z^2 + \dots + z^n = \frac{1 - z^{n+1}}{1 - z} \quad (z \neq 1)$$

and then use it to derive Lagrange's trigonometric identity:

$$1 + \cos \theta + \cos 2\theta + \dots + \cos n\theta = \frac{1}{2} + \frac{\sin[(2n+1)\theta/2]}{2 \sin(\theta/2)} \quad (0 < \theta < 2\pi).$$

Suggestion: As for the first identity, write  $S = 1 + z + z^2 + \dots + z^n$  and consider the difference  $S - zS$ . To derive the second identity, write  $z = e^{i\theta}$  in the first one.

## ○ Establish the first identity by telescoping

$$S = 1 + z + z^2 + \dots + z^n \rightarrow zS = z + z^2 + \dots + z^{n+1}$$

$$S(1-z) = S - zS = (1 + z + \dots + z^n) - (z + z^2 + \dots + z^{n+1}) = 1 - z^{n+1}$$

$$\therefore 1 + z + \dots + z^n = S = \frac{1 - z^{n+1}}{1 - z}$$

## △ Substitute $z = e^{i\theta}$ into this identity

$$1 + e^{i\theta} + e^{2i\theta} + \dots + e^{ni\theta} = \frac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}}$$

$$1 + \cos \theta + \dots + \cos n\theta + i(\sin \theta + \dots + \sin n\theta) = \frac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}}$$

## □ Simplify the right hand side

$$\begin{aligned} \frac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}} &= \frac{e^{(n+1)i\theta/2}}{e^{i\theta/2}} \frac{e^{-i(n+1)\theta/2} - e^{(n+1)i\theta/2}}{e^{-i\theta/2} - e^{i\theta/2}} \\ &= e^{ni\theta/2} \frac{e^{-i(n+1)\theta/2} - e^{(n+1)i\theta/2}}{e^{-i\theta/2} - e^{i\theta/2}} \\ &= e^{ni\theta/2} \frac{\cos((n+1)\theta/2) - i\sin((n+1)\theta/2) - (\cos((n+1)\theta/2) + i\sin((n+1)\theta/2))}{\cos(\theta/2) - i\sin(\theta/2) - (\cos(\theta/2) + i\sin(\theta/2))} \\ &= e^{ni\theta/2} \frac{\sin((n+1)\theta/2)}{\sin(\theta/2)} = (\cos(n\theta/2) + i\sin(n\theta/2)) \frac{\sin((n+1)\theta/2)}{\sin(\theta/2)} \end{aligned}$$

## ◊ Match real terms and simplify using $\sin \alpha \cos \beta = \frac{1}{2}(\sin(\alpha-\beta) + \sin(\alpha+\beta))$

$$1 + \cos \theta + \dots + \cos n\theta = \operatorname{Re} \left[ \frac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}} \right] = \frac{\cos(n\theta/2) \sin((n+1)\theta/2)}{\sin(\theta/2)}$$

$$1 + \cos \theta + \dots + \cos n\theta = \frac{1}{2} \frac{\sin(\theta/2) + \sin((2n+1)\theta/2)}{\sin(\theta/2)} = \frac{1}{2} + \frac{\sin((2n+1)\theta/2)}{2 \sin(\theta/2)}$$

◇ Alternatively, apply the first identity to both  $z = e^{i\theta}$  and  $z = e^{-i\theta}$ .

$$\begin{aligned}
 1 + \cos\theta + \dots + \cos n\theta &= \sum_{k=0}^n \cos k\theta = \frac{1}{2} \sum_{k=0}^n (e^{ik\theta} + e^{-ik\theta}) \\
 &= \frac{1}{2} \left( \frac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}} + \frac{1 - e^{-(n+1)i\theta}}{1 - e^{-i\theta}} \right) \\
 &= \frac{1}{2} \left( \frac{e^{-i\theta/2} - e^{(n+1/2)i\theta}}{e^{-i\theta/2} - e^{i\theta/2}} + \frac{-e^{i\theta/2} + e^{-(n+1/2)i\theta}}{-e^{i\theta/2} + e^{-i\theta/2}} \right) \\
 &= \frac{1}{2} \left( \frac{e^{-(n+1/2)i\theta} - e^{(n+1/2)i\theta}}{e^{-i\theta/2} - e^{i\theta/2}} + \frac{e^{-i\theta/2} - e^{i\theta/2}}{e^{-i\theta/2} - e^{i\theta/2}} \right) \\
 &= \frac{1}{2} \left( \frac{-2i \sin((n+1/2)\theta)}{-2i \sin(\theta/2)} + 1 \right) \\
 &= \frac{1}{2} + \frac{\sin((2n+1)\theta/2)}{2 \sin(\theta/2)}
 \end{aligned}$$

10. Use de Moivre's formula (Sec. 7) to derive the following trigonometric identities:

$$(a) \cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta; \quad (b) \sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta.$$

de Moivre's formula:  $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta, n \in \mathbb{Z}$

Evaluate the formula with  $n=3$ .

$$\cos^3 \theta + 3i \cos^2 \theta \sin \theta - 3 \cos \theta \sin^2 \theta - i \sin^3 \theta = (\cos \theta + i \sin \theta)^3 = \cos 3\theta + i \sin 3\theta$$

$$(a) \text{Match real terms: } \cos^3 \theta - 3 \cos \theta \sin^2 \theta = \cos 3\theta$$

$$(b) \text{Match imaginary terms: } 3 \cos^2 \theta \sin \theta - \sin^3 \theta = \sin 3\theta$$

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## Chapter 1 Section 10: 1, 3, 6, 7 (pages 29, 30)

1. Find the square roots of (a)  $2i$ ; (b)  $1 - \sqrt{3}i$  and express them in rectangular coordinates.

$$\text{Ans. (a)} \pm (1+i); \quad \text{(b)} \pm \frac{\sqrt{3}-i}{\sqrt{2}}.$$

$$(a) z = 2i = 2e^{i(\pi/2 + 2k\pi)} \Rightarrow z^{1/2} = \sqrt{2} e^{i(\pi/4 + k\pi)} \quad (k \in \mathbb{Z})$$

Evaluate at  $k=0, 1$  to collect all distinct roots:

$$z^{1/2} = \sqrt{2} e^{i\pi/4}, \sqrt{2} e^{5\pi i/4} = \sqrt{2} (\sqrt{2}/2 + \sqrt{2}i/2), \sqrt{2} (-\sqrt{2}/2 - \sqrt{2}i/2) = \boxed{\pm(1+i)}$$

$$(b) z = 1 - \sqrt{3}i = 2e^{i(-\pi/3 + 2k\pi)} \Rightarrow z^{1/2} = \sqrt{2} e^{i(-\pi/6 + k\pi)} \quad (k \in \mathbb{Z})$$

Evaluate at  $k=0, 1$  to collect all distinct roots:

$$z^{1/2} = \sqrt{2} e^{-i\pi/6}, \sqrt{2} e^{5\pi i/6} = \sqrt{2} (\sqrt{3}/2 - i/2), \sqrt{2} (-\sqrt{3}/2 + i/2) = \boxed{\pm(\sqrt{3} - i)/\sqrt{2}}$$

3. In each case, find all the roots in rectangular coordinates, exhibit them as vertices of certain regular polygons, and identify the principal root:

$$(a) (-1)^{1/3}; \quad (b) 8^{1/6}.$$

$$\text{Ans. (b)} \pm \sqrt{2}, \pm \frac{1+\sqrt{3}i}{\sqrt{2}}, \pm \frac{1-\sqrt{3}i}{\sqrt{2}}.$$

$$(a) z = -1 = e^{i(\pi + 2k\pi)} \Rightarrow z^{1/3} = (-1)^{1/3} = e^{i(\pi/3 + 2k\pi/3)} \quad (k \in \mathbb{Z})$$

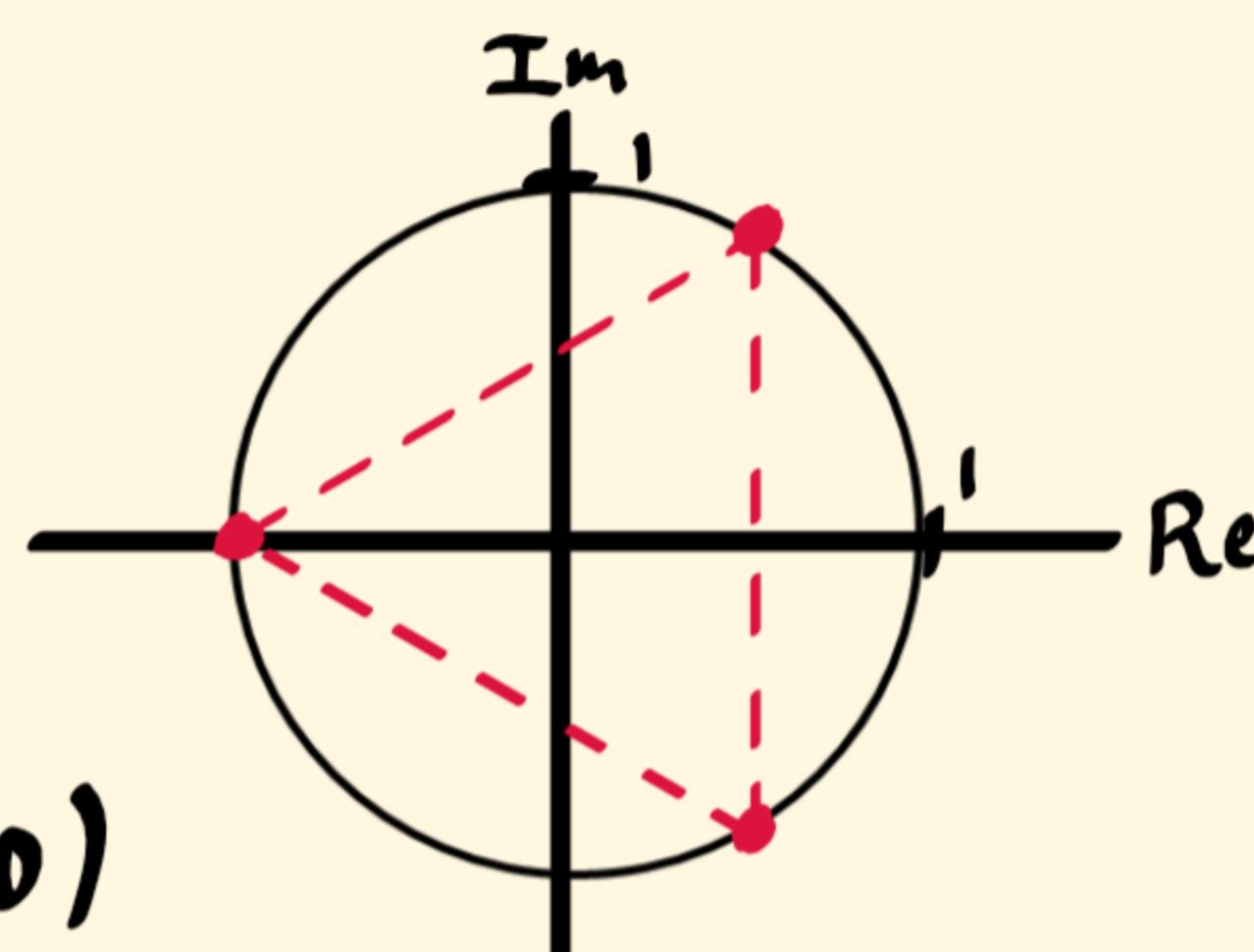
Evaluate at  $k=0, 1, 2$  to collect all distinct roots:

$$(-1)^{1/3} = e^{i\pi/3}, e^{i\pi}, e^{5\pi i/3}$$

$$= \frac{1}{2} + i\sqrt{3}/2, -1, \frac{1}{2} - i\sqrt{3}/2$$

The principal root is  $\frac{1}{2} + i\sqrt{3}/2$

(although the textbook defn requires  $z_0 > 0$ )



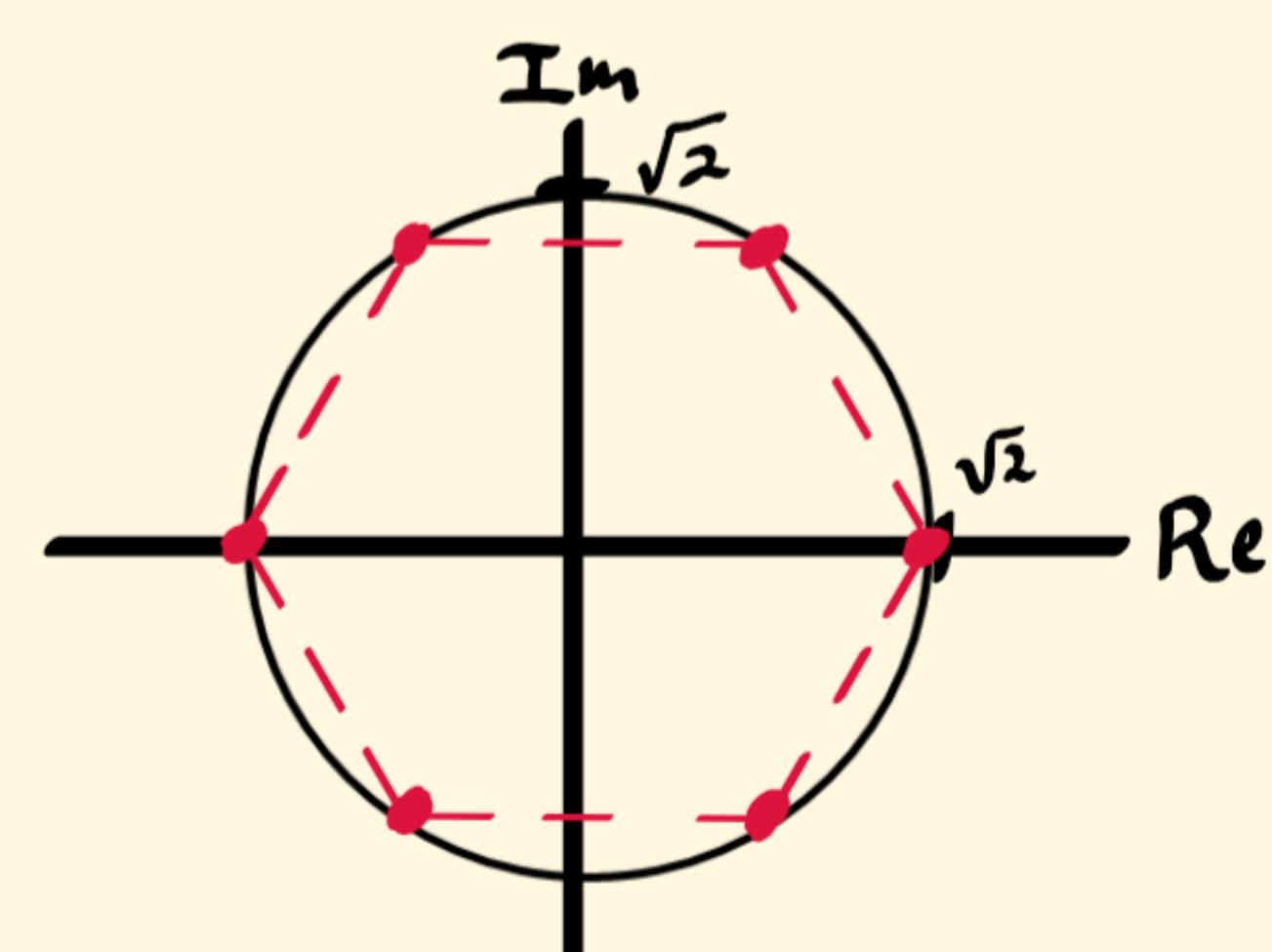
$$(b) z = 8 = 8e^{i(0 + 2k\pi)} = 8e^{2k\pi i} \Rightarrow z^{1/6} = 8^{1/6} = \sqrt{2} e^{k\pi i/3} \quad (k \in \mathbb{Z})$$

Evaluate at  $k=0, 1, \dots, 5$  to collect all distinct roots:

$$8^{1/6} = \sqrt{2}, \sqrt{2} e^{i\pi/3}, \sqrt{2} e^{2\pi i/3}, \dots, \sqrt{2} e^{5\pi i/3}$$

$$= \pm \sqrt{2}, (\pm 1 \pm \sqrt{3}i)/\sqrt{2}$$

The principle root is  $\sqrt{2}$



6. Find the four zeros of the polynomial  $z^4 + 4$ , one of them being

$$z_0 = \sqrt{2} e^{i\pi/4} = 1+i.$$

Then use those zeros to factor  $z^2 + 4$  into quadratic factors with real coefficients.

$$\text{Ans. } (z^2 + 2z + 2)(z^2 - 2z + 2).$$

$$z_0^4 + 4 = 0$$

$$z_0^4 = -4 = 4e^{i(\pi+2k\pi)}, \quad k \in \mathbb{Z}$$

$$z_0 = \sqrt[4]{4} e^{i(\pi+2k\pi)/4} = \sqrt{2} e^{i(\pi+2k\pi)/4}$$

$$z_0 = \sqrt{2} e^{i\pi/4}, \sqrt{2} e^{3\pi i/4}, \sqrt{2} e^{5\pi i/4}, \sqrt{2} e^{7\pi i/4} \quad (k=0,1,2,3)$$

$$z_0 = 1+i, -1+i, -1-i, 1-i$$

Note that  $(z-r)(z-\bar{r}) = z^2 - z(r+\bar{r}) + r\bar{r} = z^2 - 2z\operatorname{Re}(r) + |r|^2$  is a polynomial with real coefficients. This means we should multiply the linear terms corresponding to complex conjugate pairs

$$\begin{aligned}\therefore z^2 + 4 &= (z - (1+i))(z - (1-i))(z - (-1+i))(z - (-1-i)) \\ &= (z - 2z + 2)(z + 2z + 2)\end{aligned}$$

7. Show that if  $c$  is any  $n$ th root of unity other than unity itself, then

$$1 + c + c^2 + \dots + c^{n-1} = 0.$$

Suggestion: Use the first identity in Exercise 9, Sec. 8.

Suppose  $c \in \mathbb{C}$  is any root of unity other than unity itself, i.e.

$$c = 1^{\frac{m}{n}}, \quad c \neq 1$$

There are  $n-1$  such possible values of  $c$ , but all satisfy  $c^n = 1$  and  $1-c \neq 0$ . It follows that

$$1 + c + c^2 + \dots + c^{n-1} = \frac{1 - c^n}{1 - c} = \frac{1 - 1}{1 - c} = \frac{0}{1 - c} = 0$$

\* \* \* \*

## Chapter 1 Section 11: 1ae, 2, 3, 6, 7, 8 (page 33)

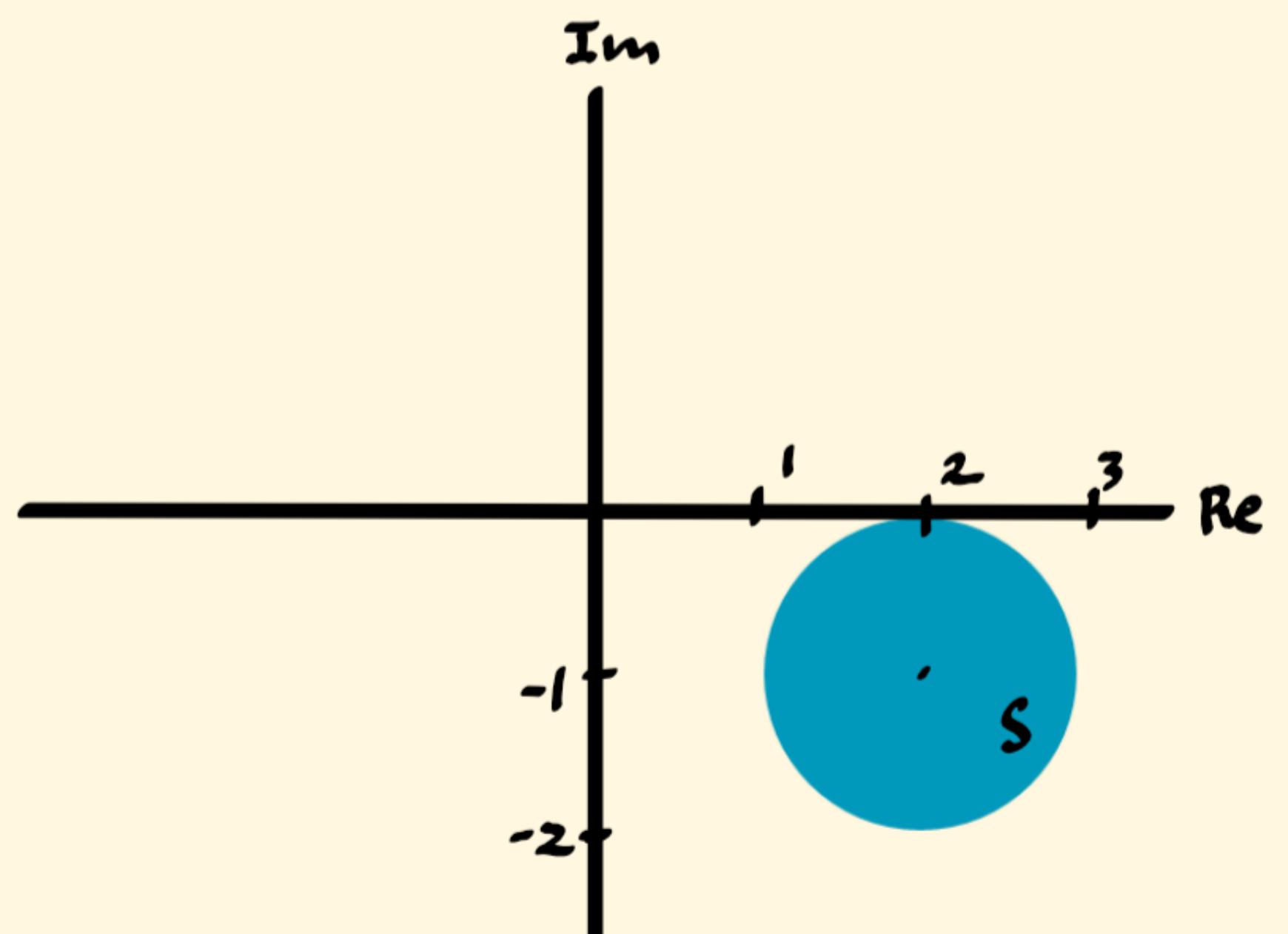
1. Sketch the following sets and determine which are domains:

- |  |                                 |
|--|---------------------------------|
| (a) $ z - 2 + i  \leq 1$ ;                     | (b) $ 2z + 3  > 4$ ;            |
| (c) $\operatorname{Im} z > 1$ ;                | (d) $\operatorname{Im} z = 1$ ; |
| (e) $0 \leq \arg z \leq \pi/4$ ( $z \neq 0$ ); | (f) $ z - 4  \geq  z $ .        |

Ans. (b), (c) are domains.

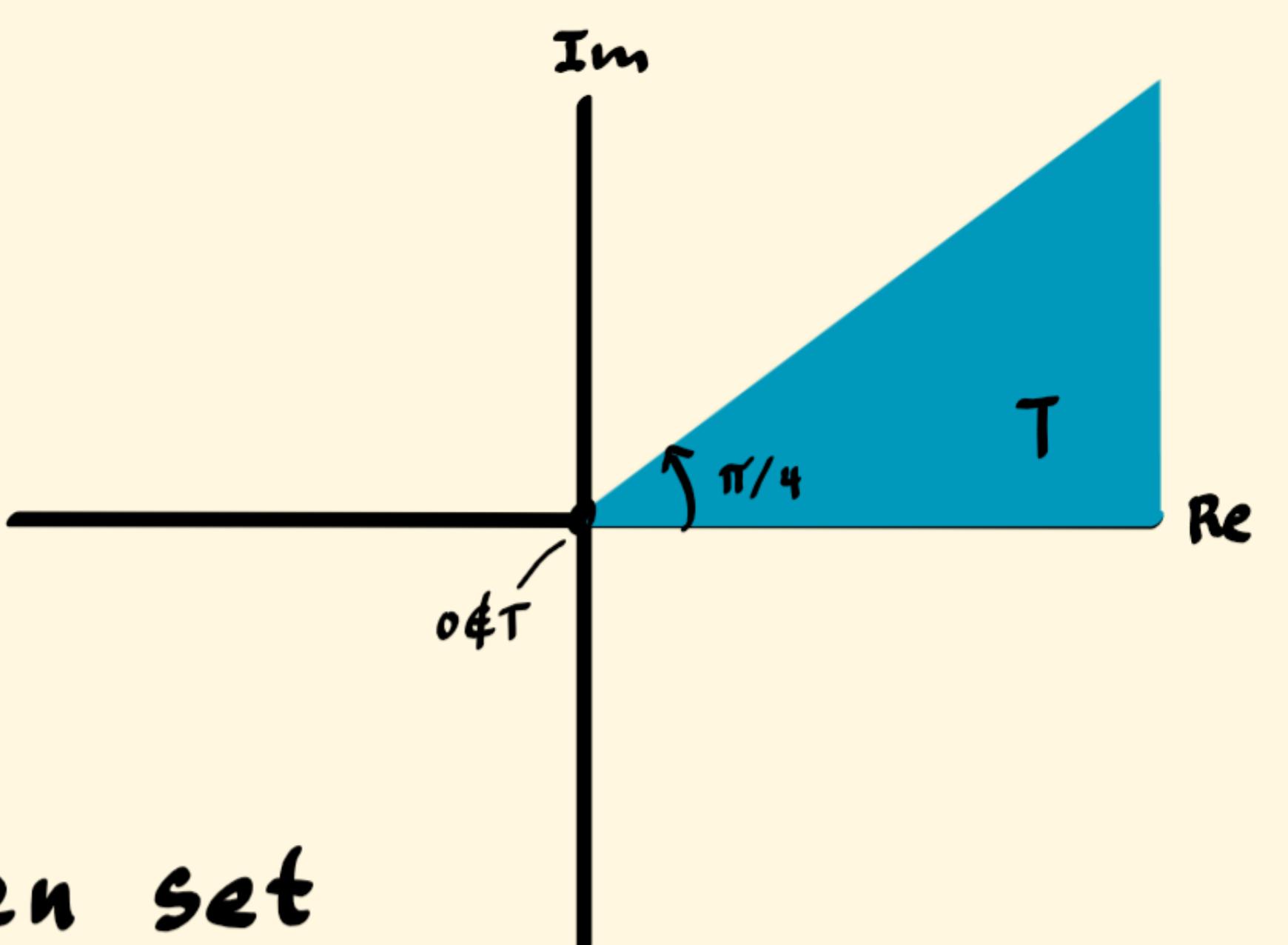
(a)  $S = \{z \in \mathbb{C} : |z - (2-i)| \leq 1\}$  is not a domain.

A domain is a nonempty open set that is connected. The given set  $S$  is not open. An open set contains none of its boundary points. The boundary points of  $S$  are all  $z$  s.t.  $|z - 2 + i| = 1$ , and  $S$  contains all of these boundary points. Therefore  $S$  contains at least one boundary point



(e)  $T = \{z \in \mathbb{C} : 0 \leq \arg z \leq \pi/4, z \neq 0\}$  is not a domain.

$T$  is not open since it contains, for example, the boundary point  $z = 1$  which has  $\arg z = 0$ .



2. Which sets in Exercise 1 are neither open nor closed?

Ans. (e).

For each of (a) - (f), let  $S$  denote the given set and  $\partial S$  the set of boundary points of  $S$ .

(a)  $S$  is closed since it contains all boundary points (Exercise 1a)

(b)  $\partial S = \{z : |2z+3|=4\}$ . For any  $z \in \partial S$ ,  $|2z+3| \neq 4$  so  $z \notin S$ . Since  $S$  contains no boundary points,  $S$  is open.

(c)  $\partial S = \{z : \operatorname{Im} z = 1\}$ . For any  $z \in \partial S$ ,  $\operatorname{Im} z \neq 1$  so  $z \notin S$ .  $S$  is open.

(d)  $\partial S = \{z : \operatorname{Im} z = 1\} = S$ . Since  $\partial S = S$ ,  $\partial S \subset S$ .  $S$  is closed.

(e)  $S$  is not open (Exercise 1e). But also  $0 \in \partial S$  with  $0 \notin S$  so  $S$  is not closed.  $S$  is neither open nor closed.

(f)  $S = \{z : |z-4| \geq |z|\} = \{z : \operatorname{Re} z \leq 2\} \Rightarrow \partial S = \{z : \operatorname{Re} z = 2\}$ . Since  $\partial S \subset S$ ,  $S$  is closed. The claim  $|z-4| \geq |z|$  iff  $\operatorname{Re} z \leq 2$  warrants proof.

\*Proof that  $S = \{z : |z-4| \geq |z|\} = \{z : \operatorname{Re} z \leq 2\}$ :

$$0 \leq |z| \leq |z-4|^2$$

$$|z|^2 \leq |z-4|^2$$

$$(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 \leq (\operatorname{Re}(z-4))^2 + (\operatorname{Im}(z-4))^2$$

$$(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 \leq (\operatorname{Re} z - 4)^2 + (\operatorname{Im} z)^2$$

$$(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 \leq (\operatorname{Re} z)^2 - 8\operatorname{Re} z + 16 + (\operatorname{Im} z)^2$$

$$8\operatorname{Re} z \leq 16$$

$$\operatorname{Re} z \leq 2$$

$$\therefore |z| \leq |z-4| \text{ iff } \operatorname{Re} z \leq 2$$

3. Which sets in Exercise 1 are bounded?

Ans. (a).

A set  $S$  is bounded if every point of  $S$  lies in some circle  $|z| \leq R$

(a) Since  $|z| - \sqrt{5} = |z| - |-2+i| \leq |z| - |z+(-2+i)| \leq |z+(-2+i)| \leq 1$ , all points of  $S$  lie in  $|z| \leq 1 + \sqrt{5}$ .  $S$  is bounded.

(b) Let  $R > 2$  be arbitrary. Any point  $z \in C$  s.t.  $|z| > R$  satisfies  $|z + 3/2| > R > 2$  and thus  $|2z+3| > 4$ . Since  $R$  was arbitrary this means that no matter how large we choose  $R$  to be there is always a point  $z \in S$  such that  $|z| \notin R$ .  $S$  is not bounded.

(c) Let  $R > 1$  be arbitrary. Then  $z = (R+1)i$  satisfies  $\operatorname{Im} z = R+1 > R > 1$  so  $z \in S$  but  $|z| = R+1 > R$ . This shows that for arbitrarily large  $R$  we can always find a point  $z \in S$  such that  $|z| \notin R$ .  $S$  is not bounded.

(d) Given  $R > 0$  take  $z = R+i$ . Then  $z \in S$  since  $\operatorname{Im} z = 1$  but  $|z| = (R^2+1)^{1/2} > R$ . This shows that for arbitrarily large  $R$  we can always find a point  $z \in S$  such that  $|z| \notin R$ .  $S$  is not bounded.

(e) Given  $R > 0$  take  $z = R+i$ . Then since  $\arg z = 0$ ,  $0 \leq \arg z \leq \pi/4$  and  $z \in S$  but  $|z| > R$ .  $S$  is not bounded.

(f) Given  $R > 0$  take  $z = -R-1$ . Then since  $\operatorname{Re} z = -R-1 < -1 \leq 2$ ,  $z \in S$  but  $|z| = (R^2+1)^{1/2} > R$ .  $S$  is not bounded.

## Definitions

A point  $z_0$  is an interior point of  $S$  if there exists some neighborhood of  $z_0$  that contains only points of  $S$ ;  $z_0$  is an exterior point of  $S$  if there exists a neighborhood of  $z_0$  containing no points of  $S$ . If  $z_0$  is neither of these,  $z_0$  is a boundary point of  $S$ : every neighborhood of  $z_0$  contains at least one point of  $S$  and one point of  $S^c$ .

$z_0$  is an accumulation point of  $S$  if every deleted neighborhood of  $z_0$  contains a point of  $S$ .

6. Show that a set  $S$  is open if and only if each point in  $S$  is an interior point.

Assume  $S$  is open and suppose  $z_0 \in S$ . By the definitions we know that if  $z_0$  is not an interior point and  $z_0$  is not an exterior point then  $z_0$  is a boundary point. Since  $S$  is open,  $S$  does not contain any of its boundary points. In particular  $z_0 \notin S$  means  $z_0$  is not a boundary point of  $S$ . Thus  $z_0$  must be either an interior point of  $S$  or an exterior point of  $S$ . Since every neighborhood of  $z_0$  contains  $z_0 \in S$ ,  $z_0$  is not an exterior point. By exhaustion of all logical alternatives,  $z_0$  must be an interior point.

Assume that each point of the set  $S$  is an interior point. Suppose  $z$  is a boundary point of  $S$ . If  $z \in S$  then  $z$  is both an interior point and a boundary point of  $S$ . This is a contradiction since by the definitions a point cannot be both an interior point and a boundary point of  $S$ . Since  $z$  was arbitrary, conclude that no boundary point of  $S$  is contained in  $S$ . Therefore  $S$  is open.

7. Determine the accumulation points of each of the following sets:

$$(a) z_n = i^n \quad (n = 1, 2, \dots); \quad (b) z_n = i^n/n \quad (n = 1, 2, \dots); \\ (c) 0 \leq \arg z < \pi/2 \quad (z \neq 0); \quad (d) z_n = (-1)^n(1+i) \frac{n-1}{n} \quad (n = 1, 2, \dots).$$

Ans. (a) None; (b) 0; (d)  $\pm(1+i)$ .

(a)  $S = \{i^n : n=1, 2, \dots\} = \{\pm i, \pm 1\}$

If  $z \in S$ , any deleted neighborhood of  $z$  with radius less than  $\sqrt{2}$  contains no points of  $S$ . So  $z \in S$  is not accumulation point of  $S$ .

If  $z \in S^c$  then the fact that  $S$  contains only four isolated points means that a deleted neighborhood of sufficiently small radius (take e.g.  $r = \inf \{|z - z_0| : z_0 \in S\}$ ) will contain no points of  $S$ .

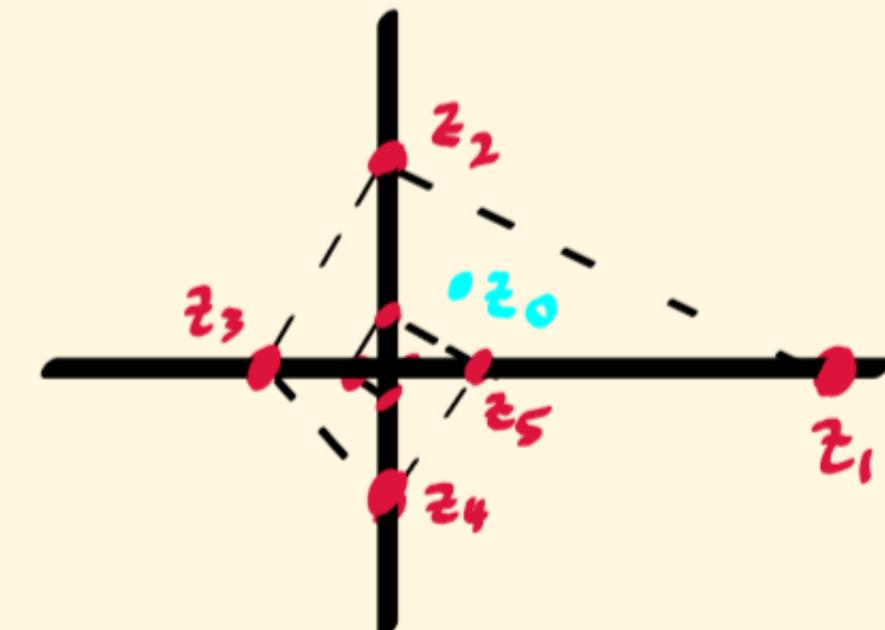
Since one of  $z \in S$  or  $z \in S^c$  must be true  $\forall z \in \mathbb{C}$  and in either case  $z$  is not an accumulation point of  $S$ ,  $S$  has no accumulation points.

$$(b) S = \{i^n/n : n = 1, 2, \dots\}$$

Since  $i^n \in \{\pm 1, \pm i\}$  for each  $n$ ,  $|i^n/n| = 1/n$  for each  $n$ .

Given any  $\epsilon$  neighborhood of  $z_0 = 0$  there exists an  $N$  s.t.  $1/n < \epsilon$  whenever  $n \geq N$ . Then  $z_n = i^n/n$  belongs to this  $\epsilon$  neighborhood for any  $n \geq N$ . This shows that there is a point of  $S$  (infinitely many in fact) in the  $\epsilon$  neighborhood of  $0$ . Since  $0 \notin S$  the same is true of the deleted  $\epsilon$  neighborhood as well and since  $\epsilon$  was arbitrary this shows that  $z_0 = 0$  is an accumulation point of  $S$ .

For  $z_0 \neq 0$ , let  $D = \{|z_n - z_0| : z_n \neq z_0, n = 1, 2, \dots\}$ , the set of distances between  $z_0$  and any of the points  $z_n$  (the restriction  $z_n \neq z_0$  handles the possible case that  $z_0 = z_n$  for some  $n$ ).  $D$  is a nonempty set of positive real numbers, so  $0$  is a lower bound of  $D$ .



$$\epsilon := \inf D \rightarrow |z_n - z_0| \geq \epsilon \quad \forall n$$

This choice of  $\epsilon$  gives a deleted neighborhood of  $z_0$  that contains no points of  $S$ .  $z_0 \neq 0$  is not an accumulation point.

Conclude  $z_0 = 0$  is the only accumulation point of  $S$ .

$$(c) S = \{z \in \mathbb{C} : z \neq 0, 0 \leq \arg z < \pi/2\}$$

$z_0 = 0$  is an accumulation point of  $S$  since every deleted neighborhood of  $0$  contains (infinitely many) points of  $S$ .

Any  $z_0 \in S$  is an accumulation point of  $S$  since if  $0 < \arg z_0 < \pi/2$  we can find a deleted neighborhood entirely contained in  $S$  and if  $\arg z_0 = 0$  then any deleted neighborhood of  $z_0$  contains points  $z$  with  $0 < \arg z < \pi/2$ .

Any point  $z_0 \neq 0$  with  $\arg z_0 = \pi/2$  is an accumulation point since these points are boundary points of  $S$ , just as the points along  $\arg z_0 = 0$  are.

The set of accumulation points of  $S$  is the set

$$\bar{S} = \{z_0 : z_0 = 0 \text{ or } 0 \leq \arg z_0 \leq \pi/2\}$$

$$(d) S = \left\{ (-1)^n (1+i) \frac{n-1}{n} : n=1, 2, \dots \right\}$$

Given  $\epsilon > 0$ ,  $\exists N$  s.t.  $\sqrt{2} \left| \frac{n-1}{n} - 1 \right| < \epsilon \quad \forall n \geq N$  since  $\frac{n-1}{n} \rightarrow 1$  as  $n \rightarrow \infty$ .

For any even  $n \geq N$

$$0 < \left| (-1)^n (1+i) \frac{n-1}{n} - (1+i) \right| = |1+i| \left| \frac{n-1}{n} - 1 \right| = \sqrt{2} \left| \frac{n-1}{n} - 1 \right| < \epsilon$$

For any odd  $n \geq N$

$$0 < \left| (-1)^n (1+i) \frac{n-1}{n} - (-1-i) \right| = |1+i| \left| 1 - \frac{n-1}{n} \right| = \sqrt{2} \left| \frac{n-1}{n} - 1 \right| < \epsilon$$

This shows that every deleted neighborhood of  $1+i$  contains a point of  $S$  and every deleted neighborhood of  $-1-i$  contains a point of  $S$ . Therefore  $z_0 = \pm(1+i)$  are accumulation points of  $S$ .

Since  $z_{2n} \rightarrow 1+i$  and  $z_{2n+1} \rightarrow -1-i$ ,  $z_0 \neq \pm(1+i)$  cannot be an accumulation point of  $S$  since if  $z_0$  was some third accumulation point of  $S$  we would be able to find a subsequence of  $(z_n)$  converging to  $z_0$ .

8. Prove that if a set contains each of its accumulation points, then it must be a closed set.

Assume the set  $S$  contains each of its accumulation points. To show that  $S$  is closed, show that  $S$  contains all of its boundary points.

Suppose  $z_0 \in \partial S$ . Either  $z_0 \in S$  or  $z_0 \notin S$ . If  $z_0 \in S$  we are done. Otherwise if  $z_0 \notin S$  then since every neighborhood of  $z_0$  contains a point of  $S$  (by the definition of a boundary point) and  $z_0 \notin S$  itself, every neighborhood of  $z_0$  contains a point of  $S$  other than  $z_0$ , i.e. every deleted neighborhood of  $z_0$  contains a point of  $S$ . Thus  $z_0$  is also an accumulation point of  $S$  so that  $z_0 \in S$  by the initial assumption. This contradicts  $z_0 \notin S$ .

Since  $z_0 \in S$  or  $z_0 \notin S$  and the latter case produces a contradiction, conclude  $z_0 \in S$ . Since  $z_0 \in \partial S$  was arbitrary, this proves  $S$  is closed.

\* \* \* \*

## Chapter 2 Section 12 : 2, 3, 4 (pages 37, 38)

2. Write the function  $f(z) = z^3 + z + 1$  in the form  $f(z) = u(x, y) + iv(x, y)$ .

$$Ans. f(z) = (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y).$$

$$z = x + iy \quad x, y \in \mathbb{R}$$

$$\begin{aligned} f(z) &= z^3 + z + 1 = (x + iy)^3 + x + iy + 1 = x^3 + 3ix^2y - 3xy^2 - iy^3 + x + iy + 1 \\ &= (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y) = u(x, y) + iv(x, y) \end{aligned}$$

3. Suppose that  $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$ , where  $z = x + iy$ . Use the expressions (see Sec. 5)

$$x = \frac{z + \bar{z}}{2} \quad \text{and} \quad y = \frac{z - \bar{z}}{2i}$$

to write  $f(z)$  in terms of  $z$ , and simplify the result.

$$Ans. f(z) = \bar{z}^2 + 2iz.$$

$$\begin{aligned} z &= x + iy \rightarrow \bar{z} = x - iy \\ x &= \frac{z + \bar{z}}{2} \rightarrow x^2 = \frac{1}{4}(z^2 + 2z\bar{z} + \bar{z}^2) \\ y &= \frac{z - \bar{z}}{2i} \rightarrow -y^2 = \frac{1}{4}(z^2 - 2z\bar{z} + \bar{z}^2) \quad \left. \begin{array}{l} x^2 - y^2 = \frac{1}{2}z^2 + \frac{1}{2}\bar{z}^2 \\ xy = \frac{1}{4i}(z^2 - \bar{z}^2) \rightarrow -2ixy = -\frac{1}{2}(z^2 - \bar{z}^2) = \frac{1}{2}\bar{z}^2 - \frac{1}{2}z^2 \end{array} \right\} \\ f(z) &= x^2 - y^2 - 2y + i(2x - 2xy) \\ &= \frac{1}{2}z^2 + \frac{1}{2}\bar{z}^2 - \frac{1}{2}(z - \bar{z}) + i(z + \bar{z}) - \frac{1}{2}(z^2 - \bar{z}^2) \\ &= \frac{1}{2}z^2 + \frac{1}{2}\bar{z}^2 + i(z - \bar{z}) + i(z + \bar{z}) + \frac{1}{2}\bar{z}^2 - \frac{1}{2}z^2 \\ &= \bar{z}^2 + 2iz \end{aligned}$$

4. Write the function

$$f(z) = z + \frac{1}{z} \quad (z \neq 0)$$

in the form  $f(z) = u(r, \theta) + iv(r, \theta)$ .

$$Ans. f(z) = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta.$$

$$z = re^{i\theta}$$

$$\begin{aligned} f(z) &= z + z^{-1} = re^{i\theta} + r^{-1}e^{-i\theta} = r(\cos\theta + i\sin\theta) + r^{-1}(\cos\theta - i\sin\theta) \\ &= (r + r^{-1})\cos\theta + i(r - r^{-1})\sin\theta = u(r, \theta) + iv(r, \theta) \end{aligned}$$

\* \* \* \*

## Chapter 2 Section 14 : 2, 3c, 4, 5 (page 44)

2. Find and sketch, showing corresponding orientations, the images of the hyperbolas

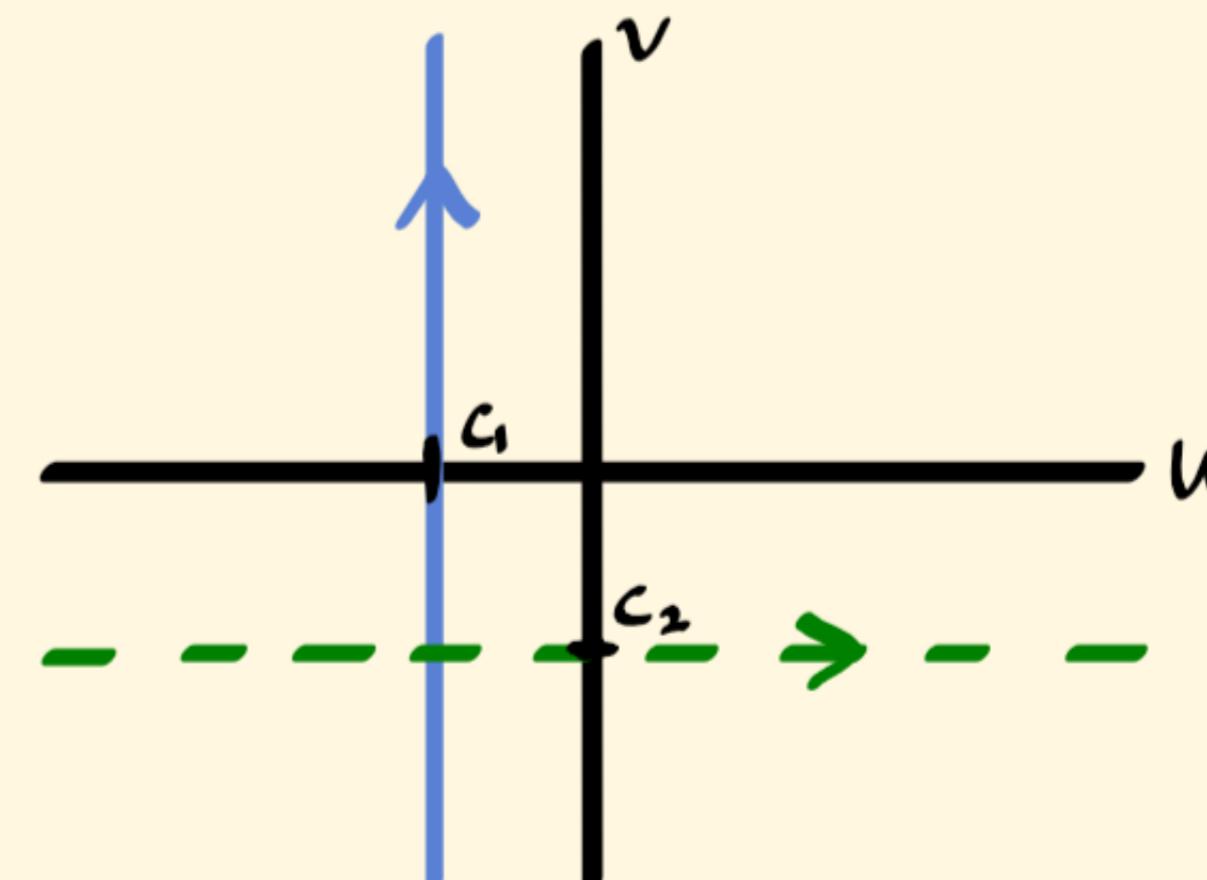
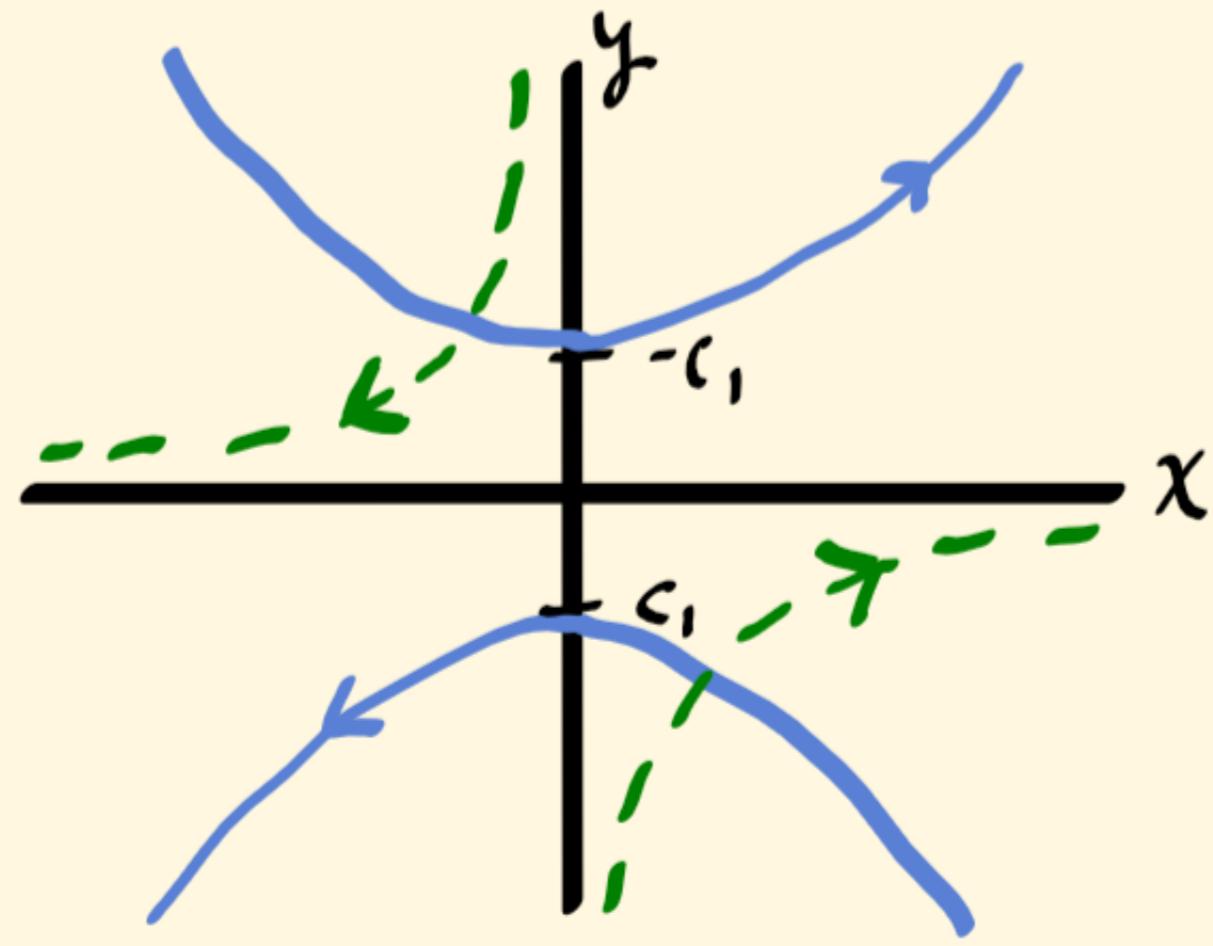
$$x^2 - y^2 = c_1 \quad (c_1 < 0) \quad \text{and} \quad 2xy = c_2 \quad (c_2 < 0)$$

under the transformation  $w = z^2$ .

$$z = x + iy$$

$$w = z^2 = x^2 - y^2 + i2xy = u + iv$$

$$\begin{array}{ll} \text{---} & x^2 - y^2 = c_1, \quad u = c_1, \\ \text{---} & 2xy = c_2, \quad v = c_2 \end{array}$$



$$\left. \begin{array}{l} x^2 - y^2 = c_1, \\ y^2 = x^2 - c_1 > 0 \end{array} \right\} \rightarrow u = c_1, \quad v = 2x\sqrt{x^2 - c_1} \quad (y > 0 \text{ branch}) \quad x \in \mathbb{R}$$

$$v = -2x\sqrt{x^2 - c_1} \quad (y < 0 \text{ branch})$$

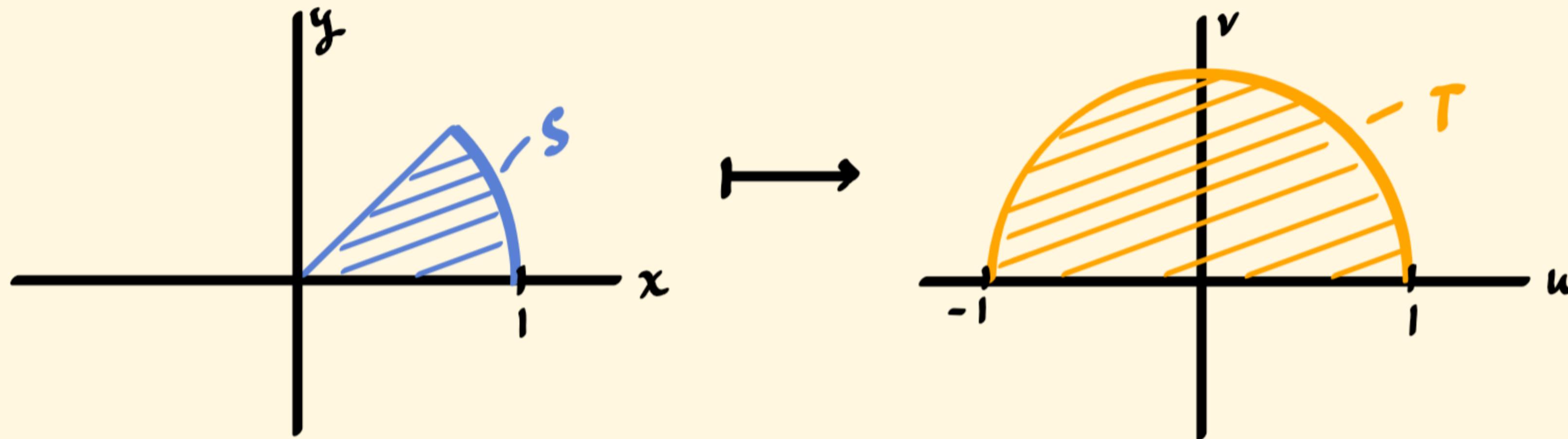
The image of the hyperbola  $x^2 - y^2 = c_1$  is the vertical line  $u = c_1$ .  $\operatorname{Im} w = v$  can take on any real numbered value. To provide orientation, use the parameterization  $v = \pm 2x\sqrt{x^2 - c_1}$ . Increasing  $x$  along the upper branch corresponds to increasing  $v$  along the line  $u = c_1$ , while decreasing  $x$  along the lower branch corresponds to increasing  $v$  along the line  $u = c_1$ .

$$\left. \begin{array}{l} 2xy = c_2 \\ y = c_2/2x \\ x = c_2/2y \end{array} \right\} \rightarrow v = c_2, \quad \begin{array}{ll} u = x^2 - c_2^2/4x^2 & \lim_{x \rightarrow \infty} u = +\infty \\ u = c_2^2/4y^2 - y^2 & \lim_{y \rightarrow \infty} u = -\infty \end{array} \quad \begin{array}{ll} \lim_{x \rightarrow 0^+} u = -\infty & \lim_{x \rightarrow 0^-} u = +\infty \\ \lim_{y \rightarrow 0^+} u = +\infty & \lim_{y \rightarrow 0^-} u = -\infty \end{array}$$

The image of the hyperbola  $2xy = c_2$  is the horizontal line  $v = c_2$ .  $\operatorname{Re} w = u$  can take on any real numbered value. Parameterize  $u$  along the branch in quadrant 4 of the  $xy$  plane as  $u = x^2 - c_2^2/4x^2$  ( $0 < x < \infty$ ) to see that increasing  $x$  along this branch corresponds to increasing  $u$  (and vice versa). Parameterize  $u$  along the branch in quadrant 2 as  $u = c_2^2/4y^2 - y^2$  ( $0 < y < \infty$ ) to see that increasing  $y$  along this branch corresponds to decreasing  $u$  (and vice versa with  $y$  decreasing toward 0). These observations provide the orientation in the graphs.

3. Sketch the region onto which the sector  $r \leq 1, 0 \leq \theta \leq \pi/4$  is mapped by the transformation (a)  $w = z^2$ ; (b)  $w = z^3$ ; (c)  $w = z^4$ .

$$(c) z = re^{i\theta}, w = z^4, re^{i\theta} \mapsto r^4 e^{4i\theta}$$



The region  $S = \{(r, \theta) : r \leq 1, 0 \leq \theta \leq \pi/4\}$  is mapped to the region  $T = \{(r, \theta) : r \leq 1, 0 \leq \theta \leq \pi\}$  by the transformation  $w = z^4$ .

4. Show that the lines  $ay = x$  ( $a \neq 0$ ) are mapped onto the spirals  $\rho = \exp(a\phi)$  under the transformation  $w = \exp z$ , where  $w = \rho \exp(i\phi)$ .

$$z = x + iy, ay = x$$

$$w = e^z = e^x e^{iy} = e^{ay} e^{iy} = e^{a\phi} e^{i\phi} = \rho e^{i\phi} \quad (\rho := e^{a\phi} \quad \phi := y)$$

5. By considering the images of horizontal line segments, verify that the image of the rectangular region  $a \leq x \leq b, c \leq y \leq d$  under the transformation  $w = \exp z$  is the region  $e^a \leq \rho \leq e^b, c \leq \phi \leq d$ , as shown in Fig. 21 (Sec. 14).

Consider the horizontal line segment  $\ell = \{(x, y_0) : a \leq x \leq b\}$  for some fixed  $y_0 \in [c, d]$ . The image of  $\ell$  under this transformation is:

$$\{\rho e^{i\phi} : e^a \leq \rho \leq e^b, \phi = y_0\}$$

This is a segment oriented at the angle  $\phi = y_0$  extending from a distance  $\rho = e^a$  from the origin to a distance  $\rho = e^b$  from the origin. Repeat this transformation for every  $y_0 \in [c, d]$ . The result is

$$\{w = \exp(x + iy) : a \leq x \leq b, c \leq y \leq d\} = \{\rho e^{i\phi} : e^a \leq \rho \leq e^b, c \leq \phi \leq d\}$$