

Written Homework 5 Solutions

MA 4291 (Tilley)
C-Term 2022

Please turn in only the problems in **bold red font**:

pg. 170: **1a-e, 2ab, 3, 4, 5, 6, 7, 10**

pg. 188: **2, 3, 4, 5, 7, 8**

pg. 195: **1, 3, 6, 7, 8, 11ab, 12ab, 13**

Chapter 4 Section 52 : **1, 2, 3, 4, 5, 6, 7, 10**

1. Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals:

$$(a) \int_C \frac{e^{-z} dz}{z - (\pi i/2)}; \quad (b) \int_C \frac{\cos z}{z(z^2 + 8)} dz; \quad (c) \int_C \frac{z dz}{2z + 1};$$

$$(d) \int_C \frac{\cosh z}{z^4} dz; \quad (e) \int_C \frac{\tan(z/2)}{(z - x_0)^2} dz \quad (-2 < x_0 < 2).$$

Ans. (a) 2π ; (b) $\pi i/4$; (c) $-\pi i/2$; (d) 0; (e) $i\pi \sec^2(x_0/2)$.

Theorem. Let f be analytic everywhere inside and on a simple closed contour C , taken in the positive sense. If z_0 is any point interior to C , then

$$(1) \quad f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z) dz}{z - z_0}.$$

$$(a) \quad f(z) = e^{-z} \quad z_0 = \pi i/2$$

$$\int_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0) = 2\pi i e^{-i\pi/2} = -2\pi i^2 = 2\pi$$

$$(b) \quad f(z) = \frac{\cos z}{z^2 + 8} \quad z_0 = 0$$

$$\int_C \frac{\cos z}{z(z^2 + 8)} dz = 2\pi i f(z_0) = 2\pi i/8 = \boxed{\pi i/4}$$

Note $0 = z^2 + 8 \Rightarrow z = \pm 2\sqrt{2}i$.

$f(z)$ analytic on and inside C

$$(c) \quad f(z) = z/2 \quad z_0 = -1/2$$

$$\int_C \frac{z}{2z+1} dz = \int_C \frac{z/2}{z - (-1/2)} dz = 2\pi i f(z_0)$$

$$= -2\pi i/4 = \boxed{-\pi i/2}$$

$$(d) \quad f(s) = \cosh(s) \quad z = 0$$

For f analytic on and within C , oriented positively,

$$(5) \quad f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(s) ds}{(s-z)^{n+1}} \quad (n = 1, 2, \dots). \quad (\text{pg 167})$$

$$\int_C \frac{\cosh z}{z^4} dz = \int_C \frac{\cosh s}{(s-0)^{3+1}} ds = \int_C \frac{f(s)}{(s-z)^{3+1}} ds$$

$$= 2\pi i f^{(3)}(z)/3! = \frac{\pi i}{3} \frac{d^3}{dz^3} [\cosh z] \Big|_{z=0}$$

$$= \frac{\pi i}{3} \sinh 0 = \frac{\pi i}{3} \cdot 0 = \boxed{0}$$

$$(e) \quad f(s) = \tan(s/2) \quad z = x_0$$

$$\int_C \frac{\tan(z/2)}{(z-x_0)^2} dz = \frac{2\pi i}{1!} \frac{d}{dz} [\tan(z/2)] \Big|_{z=x_0}$$

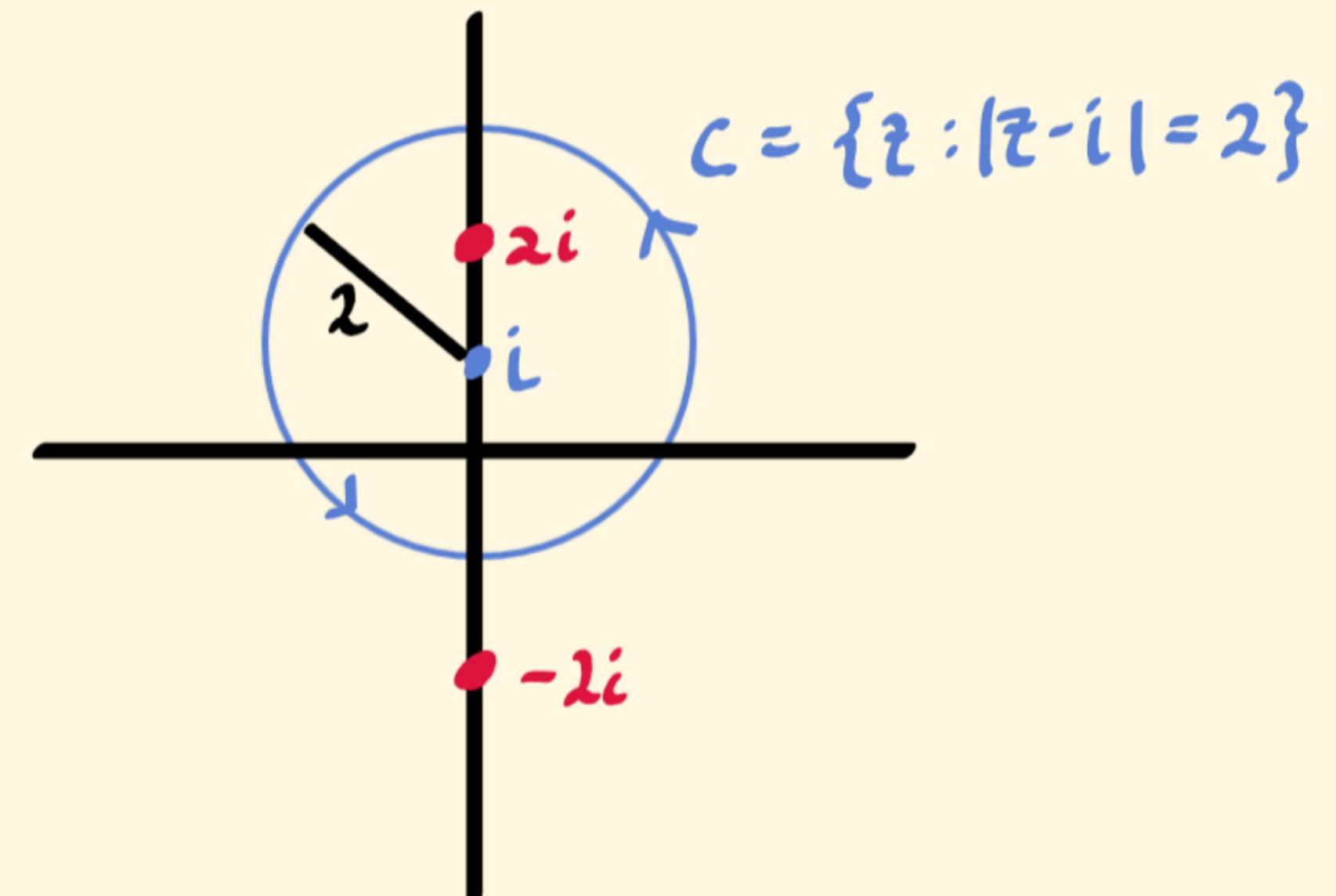
$$= 2\pi i/2 \sec^2(x_0/2) = \boxed{i\pi \sec^2(x_0/2)}$$

2. Find the value of the integral of $g(z)$ around the circle $|z - i| = 2$ in the positive sense when

$$(a) g(z) = \frac{1}{z^2 + 4}; \quad (b) g(z) = \frac{1}{(z^2 + 4)^2}.$$

Ans. (a) $\pi/2$; (b) $\pi/16$.

$$\begin{aligned}(a) \quad g(z) &= \frac{1}{z^2 + 4} \\ &= \frac{1}{(z+2i)(z-2i)} \\ &= \frac{(z+2i)^{-1}}{z-2i}\end{aligned}$$



Let $f(z) = (z+2i)^{-1}$. $f(z)$ is analytic on and within C .

$$\begin{aligned}\int_C g(z) dz &= \int_C \frac{1}{z^2 + 4} dz = \int_C \frac{f(z)}{z-2i} dz \\ &= 2\pi i f(2i) = \frac{2\pi i}{4i} = \boxed{\frac{\pi}{2}}\end{aligned}$$

$$(b) \text{ Let } f(z) = \frac{1}{(z+2i)^2} \rightarrow f'(z) = -\frac{2}{(z+2i)^3}$$

$$\begin{aligned}\int_C g(z) dz &= \int_C \frac{1}{(z^2 + 4)^2} dz = \int_C \frac{(z+2i)^{-2}}{(z-2i)^2} dz \\ &= \int_C \frac{f(z)}{(z-2i)^{1+1}} dz = 2\pi i f'(2i)/1! \\ &= 2\pi i \cdot \frac{-2}{(4i)^3} = \boxed{\frac{\pi}{16}}\end{aligned}$$

3. Let C be the circle $|z| = 3$, described in the positive sense. Show that if

$$g(z) = \int_C \frac{2s^2 - s - 2}{s - z} ds \quad (|z| \neq 3),$$

then $g(2) = 8\pi i$. What is the value of $g(z)$ when $|z| > 3$?

• $z=2$

By the Cauchy - Integral Formula with $f(z) = 2z^2 - z - 2$,

$$g(2) = \int_C \frac{2s^2 - s - 2}{s - 2} ds = 2\pi i f(2) = 2\pi i \cdot 4 = \boxed{8\pi i}$$

• $|z| > 3$

For $|z| > 3$, $s - z \neq 0$ and $h(s) = \frac{2s^2 - s - 2}{s - z}$ is analytic everywhere and interior to C . By Cauchy - Goursat Theorem,

$$\boxed{g(z) = \int_C h(s) ds = 0 \quad (|z| > 3)}$$

4. Let C be any simple closed contour, described in the positive sense in the z plane, and write

$$g(z) = \int_C \frac{s^3 + 2s}{(s - z)^3} ds.$$

Show that $g(z) = 6\pi iz$ when z is inside C and that $g(z) = 0$ when z is outside.

- z inside C

Let $f(s) = s^3 + 2s \rightarrow f''(s) = 6s$

$f(s)$ is analytic on and interior to C since f is entire. By the extension of the Cauchy Integral Formula (page 167) :

$$g(z) = \int_C \frac{h(s)}{(s - z)^{2+1}} ds = 2\pi i f''(z)/2! = \boxed{6\pi iz}$$

- z is outside of C

If z is outside of C , $s - z \neq 0$ for any s on or interior to C .

For any z outside of C , $f(s) = \frac{s^3 + 2s}{(s - z)^3}$ is analytic on and interior to C . By Cauchy-Goursat,

$$\boxed{g(z) = \int_C f(s) ds = 0}$$

5. Show that if f is analytic within and on a simple closed contour C and z_0 is not on C , then

$$\int_C \frac{f'(z) dz}{z - z_0} = \int_C \frac{f(z) dz}{(z - z_0)^2}.$$

Theorem 1. If a function f is analytic at a given point, then its derivatives of all orders are analytic there too.

(pg. 168)

By the theorem and the assumptions made about f , f' is analytic on and interior to the simple closed curve C . We do not know whether the curve is positively or negatively oriented but it won't matter.

Since z_0 is not on C , z_0 is either interior or exterior to C .

If z_0 is exterior to C , $z - z_0 \neq 0$ and both integrands are analytic everywhere on and exterior to C .

$$\int_C \frac{f'(z)}{z - z_0} dz = 0 = \int_C \frac{f(z)}{(z - z_0)^2} dz$$

If z_0 is interior to C , use C.I.F.* and its extension[†]:

$$\begin{aligned} \int_C \frac{f'(z)}{z - z_0} dz &\stackrel{*}{=} 2\pi i f'(z_0) \\ &= 2\pi i f'(z_0)/1! \stackrel{\dagger}{=} \int_C \frac{f(z)}{(z - z_0)^2} dz \end{aligned}$$

6. Let f denote a function that is *continuous* on a simple closed contour C . Following a procedure used in Sec. 51, prove that the function

$$g(z) = \frac{1}{2\pi i} \int_C \frac{f(s) ds}{s - z}$$

is *analytic* at each point z interior to C and that

$$g'(z) = \frac{1}{2\pi i} \int_C \frac{f(s) ds}{(s - z)^2}$$

at such a point.

7. Let C be the unit circle $z = e^{i\theta}$ ($-\pi \leq \theta \leq \pi$). First show that for any real constant a ,

$$\int_C \frac{e^{az}}{z} dz = 2\pi i.$$

Then write this integral in terms of θ to derive the integration formula

$$\int_0^\pi e^{a \cos \theta} \cos(a \sin \theta) d\theta = \pi.$$

By the Cauchy Integral Formula with $f(z) = e^{az}$, $z_0 = 0$,

$$\int_C \frac{e^{az}}{z} dz = \int_C \frac{f(z)}{z} dz = 2\pi i f(0) = 2\pi i e^0 = 2\pi i$$

$$z = e^{i\theta} \rightarrow \frac{dz}{d\theta} = ie^{i\theta}$$

$$\begin{aligned} 2\pi i &= \int_C \frac{e^{az}}{z} dz = \int_{-\pi}^{\pi} \frac{e^{ae^{i\theta}}}{e^{i\theta}} ie^{i\theta} d\theta \\ &= i \int_{-\pi}^{\pi} e^{a \cos \theta} e^{i a \sin \theta} d\theta \\ &= i \int_{-\pi}^{\pi} e^{a \cos \theta} [\cos(a \sin \theta) + i \sin(a \sin \theta)] d\theta \\ &= i \int_{-\pi}^{\pi} e^{a \cos \theta} \cos(a \sin \theta) d\theta - \int_{-\pi}^{\pi} e^{a \cos \theta} \sin(a \sin \theta) d\theta \end{aligned}$$

$$\Rightarrow \pi = \frac{1}{2} \cdot \int_{-\pi}^{\pi} e^{a \cos \theta} \cos(a \sin \theta) d\theta \stackrel{*}{=} \int_0^\pi e^{a \cos \theta} \cos(a \sin \theta) d\theta$$

* Even integrand integrated over interval symmetric wrt 0.

$$e^{a \cos(-\theta)} \cos(a \sin(-\theta)) = e^{a \cos \theta} \cos(a \sin \theta)$$

10. Let f be an entire function such that $|f(z)| \leq A|z|$ for all z , where A is a fixed positive number. Show that $f(z) = a_1 z$, where a_1 is a complex constant.

Suggestion: Use Cauchy's inequality (Sec. 52) to show that the second derivative $f''(z)$ is zero everywhere in the plane. Note that the constant M_R in Cauchy's inequality is less than or equal to $A(|z_0| + R)$.

Cauchy's Inequality

Theorem 3. Suppose that a function f is analytic inside and on a positively oriented circle C_R , centered at z_0 and with radius R (Fig. 69). If M_R denotes the maximum value of $|f(z)|$ on C_R , then

$$(2) \quad |f^{(n)}(z_0)| \leq \frac{n! M_R}{R^n} \quad (n = 1, 2, \dots).$$

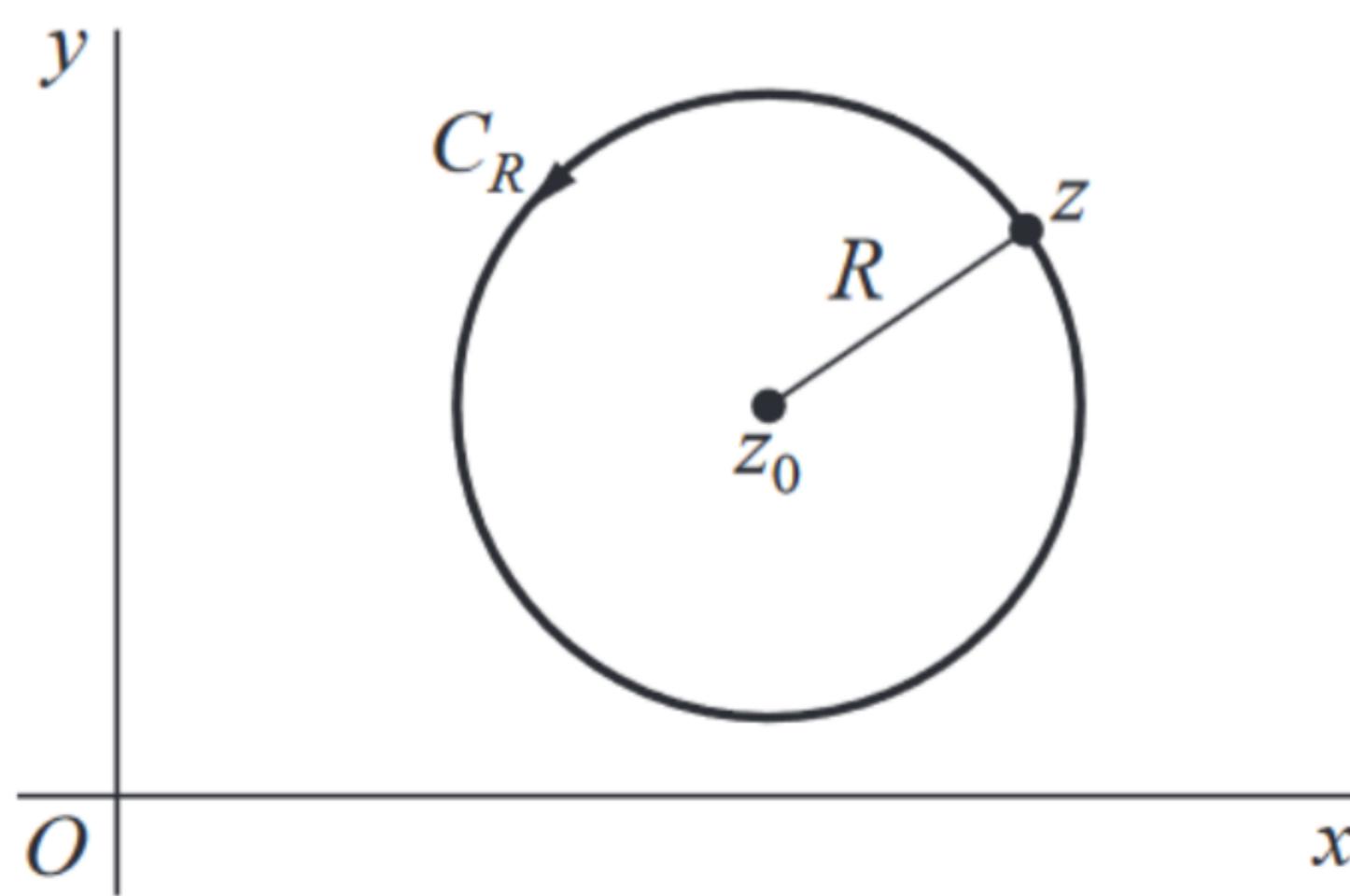


FIGURE 69

For each $z_0 \in \mathbb{C}$, Show $f''(z_0) = 0$.

For each $R \in \mathbb{N}^* = \{1, 2, 3, \dots\}$

Consider the positive oriented circle C_R with center z_0 and radius R .

f is analytic on/inside C_R and $\forall z$ on C_R
 $|f(z)| \leq A|z| \leq A(|z_0| + R)$ so $M_R \leq A(|z_0| + R)$

$$0 \leq |f''(z_0)| \leq 2M_R/R^2 \leq 2A(|z_0| + R)/R^2 \leq 2A/R$$

$$\therefore |f''(z_0)| = 0 \Rightarrow f''(z) = 0 \quad \forall z \in \mathbb{C} \Rightarrow f(z) = az, \quad a \in \mathbb{C}$$

* * * *

Chapter 5 Section 56 : 2, 3, 4, 5, 7, 8 (pg 188)

2. Let Θ_n ($n = 1, 2, \dots$) denote the principal arguments of the numbers

$$z_n = 2 + i \frac{(-1)^n}{n^2} \quad (n = 1, 2, \dots).$$

Point out why

$$\lim_{n \rightarrow \infty} \Theta_n = 0,$$

and compare with Example 2, Sec. 55.

Theorem. Suppose that $z_n = x_n + iy_n$ ($n = 1, 2, \dots$) and $z = x + iy$. Then

$$(4) \quad \lim_{n \rightarrow \infty} z_n = z$$

if and only if

$$(5) \quad \lim_{n \rightarrow \infty} x_n = x \quad \text{and} \quad \lim_{n \rightarrow \infty} y_n = y.$$

Note how the theorem enables us to write

$$\lim_{n \rightarrow \infty} (x_n + iy_n) = \lim_{n \rightarrow \infty} x_n + i \lim_{n \rightarrow \infty} y_n$$

whenever we know that both limits on the right exist or that the one on the left exists.

Follow Example 2 with $z_n = 2 + i \frac{(-1)^n}{n^2}$ instead of the z_n from the example. The difference is that $-2 + i \frac{(-1)^n}{n^2}$ approaches -2 from the third quadrant if n is odd and from the second quadrant if n is even. In this case, $z_n = 2 + i \frac{(-1)^n}{n^2}$ approaches 2 which has $\Theta = 0$ from either direction (quadrant 1 or quadrant 4).

3. Use the inequality (see Sec. 4) $||z_n| - |z|| \leq |z_n - z|$ to show that

$$\text{if } \lim_{n \rightarrow \infty} z_n = z, \text{ then } \lim_{n \rightarrow \infty} |z_n| = |z|.$$

Assume $\lim_{n \rightarrow \infty} z_n = z$

Equivalently $|z_n - z| \rightarrow 0$ as $n \rightarrow \infty$

Since $0 \leq ||z_n| - |z|| \leq |z_n - z| \quad \forall n \in \mathbb{N},$

$||z_n| - |z|| \rightarrow 0$ as $n \rightarrow \infty$
(squeeze theorem)

$\therefore \lim_{n \rightarrow \infty} |z_n| = |z|$

4. Write $z = re^{i\theta}$, where $0 < r < 1$, in the summation formula (10), Sec. 56. Then, with the aid of the theorem in Sec. 56, show that

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{r \cos \theta - r^2}{1 - 2r \cos \theta + r^2} \quad \text{and} \quad \sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r \sin \theta}{1 - 2r \cos \theta + r^2}$$

when $0 < r < 1$. (Note that these formulas are also valid when $r = 0$.)

$$(10) \quad \sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \text{whenever } |z| < 1.$$

Notice that the summation in (10) starts at $n=0$

$$\sum_{n=1}^{\infty} z^n = -1 + \sum_{n=0}^{\infty} z^n = -1 + \frac{1}{1-z} = \frac{z}{1-z}$$

$$\sum_{n=1}^{\infty} (re^{i\theta})^n = \sum_{n=1}^{\infty} z^n = \frac{z}{1-z} = \frac{re^{i\theta}}{1-re^{i\theta}}$$

$$r^{-1}e^{-i\theta} \sum_{n=1}^{\infty} r^n e^{in\theta} = \frac{1}{1-re^{i\theta}} = \frac{1}{1-r\cos\theta - ir\sin\theta}$$

$$r^{-1}e^{-i\theta} \sum_{n=1}^{\infty} r^n e^{in\theta} = \frac{1-r\cos\theta + ir\sin\theta}{(1-r\cos\theta)^2 + r^2\sin^2\theta}$$

$$r^{-1}e^{-i\theta} \sum_{n=1}^{\infty} r^n e^{in\theta} = \frac{1-r\cos\theta + ir\sin\theta}{1-2r\cos\theta + r^2}$$

$$\sum_{n=1}^{\infty} r^n e^{in\theta} = \frac{r\cos\theta - r^2(\cos^2\theta + \sin^2\theta)}{1-2r\cos\theta + r^2} + i \frac{r\sin\theta}{1-2r\cos\theta + r^2}$$

$$\sum_{n=1}^{\infty} r^n e^{in\theta} = \frac{r\cos\theta - r^2}{1-2r\cos\theta + r^2} + i \frac{r\sin\theta}{1-2r\cos\theta + r^2}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} r^n \cos n\theta + i \sum_{n=1}^{\infty} r^n \sin n\theta \\ &= \frac{r \cos n\theta - r^2}{1 - 2r \cos n\theta + r^2} + i \frac{r \sin n\theta}{1 - 2r \cos n\theta + r^2} \end{aligned}$$

Theorem. Suppose that $z_n = x_n + iy_n$ ($n = 1, 2, \dots$) and $S = X + iY$. Then

$$(3) \quad \sum_{n=1}^{\infty} z_n = S$$

if and only if

$$(4) \quad \sum_{n=1}^{\infty} x_n = X \quad \text{and} \quad \sum_{n=1}^{\infty} y_n = Y.$$

By the theorem,

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{x}{1 - 2r \cos n\theta + r^2}$$

$$\sum_{n=1}^{\infty} r^n \sin n\theta = \frac{y}{1 - 2r \cos n\theta + r^2}$$

5. Show that a limit of a convergent sequence of complex numbers is unique by appealing to the corresponding result for a sequence of real numbers.

Let (z_n) be a convergent sequence of complex numbers with $z_n = x_n + iy_n$ for each n ($x_n, y_n \in \mathbb{R}$).

Spse (z_n) converges to $z = x + iy$ and $z' = x' + iy'$ with $z \neq z'$.

Theorem. Suppose that $z_n = x_n + iy_n$ ($n = 1, 2, \dots$) and $z = x + iy$. Then

$$(4) \quad \lim_{n \rightarrow \infty} z_n = z$$

if and only if

$$(5) \quad \lim_{n \rightarrow \infty} x_n = x \quad \text{and} \quad \lim_{n \rightarrow \infty} y_n = y.$$

By the theorem,

$$x = \lim_{n \rightarrow \infty} x_n = x'$$

$$y = \lim_{n \rightarrow \infty} y_n = y'$$

But the limit of a convergent sequence of real numbers is unique. Therefore,

$$(x = x' \text{ and } y = y') \Leftrightarrow z = z'$$

7. Let c denote any complex number and show that

$$\text{if } \sum_{n=1}^{\infty} z_n = S, \text{ then } \sum_{n=1}^{\infty} cz_n = cS.$$

Theorem. Suppose that $z_n = x_n + iy_n$ ($n = 1, 2, \dots$) and $S = X + iY$. Then

$$(3) \quad \sum_{n=1}^{\infty} z_n = S$$

if and only if

$$(4) \quad \sum_{n=1}^{\infty} x_n = X \text{ and } \sum_{n=1}^{\infty} y_n = Y.$$

We can write each z_n as $z_n = x_n + iy_n$

If $\sum_{n=1}^{\infty} z_n = S$, then $\sum_{n=1}^{\infty} x_n = X$ and $\sum_{n=1}^{\infty} y_n = Y$

Use the analogous result known for real seq's :

$$\sum_{n=1}^{\infty} cx_n = cX \text{ and } \sum_{n=1}^{\infty} cy_n = cY$$

Since $cz_n = cx_n + iy_n$ and $cS = cX + icY$

$$\sum_{n=1}^{\infty} cz_n = cS$$

8. By recalling the corresponding result for series of real numbers and referring to the theorem in Sec. 56, show that

$$\text{if } \sum_{n=1}^{\infty} z_n = S \text{ and } \sum_{n=1}^{\infty} w_n = T, \text{ then } \sum_{n=1}^{\infty} (z_n + w_n) = S + T.$$

Write $z_n = a_n + i b_n$ and $w_n = c_n + i d_n$ for each n , where a_n, b_n, c_n , and d_n are real sequences.

All summations are indexed from $n=1$ to ∞ . The theorem mentioned is included with exercise 7.

If $\sum z_n = S$ and $\sum w_n = T$, then $\exists A, B, C, D \in \mathbb{R}$ s.t.

$$\sum a_n = A \quad \sum b_n = B \quad S = A + Bi$$

$$\sum c_n = C \quad \sum d_n = D \quad T = C + Di$$

Using the corresponding result for real series':

$$\sum (a_n + c_n) = \sum a_n + \sum c_n = A + C$$

$$\sum (b_n + d_n) = \sum b_n + \sum d_n = B + D$$

Since $z_n + w_n = (a_n + c_n) + i(b_n + d_n)$ and $S + T = (A + C) + i(B + D)$

$$\sum (z_n + w_n) = A + C + i(B + D) = S + T$$

* * * *

Chapter 5 Section 59: 1, 3, 6, 7, 8, 11, 12, 13 (pg 195-197)

- Obtain the Maclaurin series representation

$$z \cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!} \quad (|z| < \infty).$$

From page 194:

$$(5) \quad \cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \quad (|z| < \infty).$$

Substituting z^2 for z in (5) ($|z| < \infty$ iff $|z^2| < \infty$),

$$z \cosh z^2 = z \sum_{n=0}^{\infty} \frac{(z^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!}$$

3. Find the Maclaurin series expansion of the function

$$f(z) = \frac{z}{z^4 + 9} = \frac{z}{9} \cdot \frac{1}{1 + (z^4/9)}.$$

$$\text{Ans. } \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+2}} z^{4n+1} \quad (|z| < \sqrt{3}).$$

EXAMPLE 4. Another Maclaurin series representation is

$$(6) \quad \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1),$$

From example 4,

$$\begin{aligned} \frac{1}{1+z^4/q} &= \frac{1}{1-(-z^4/q)} = \sum_{n=0}^{\infty} (-z^4/q)^n \\ &= \sum_{n=0}^{\infty} (-1)^n z^{4n}/q^n \quad \text{for } |-z^4/q| < 1 \end{aligned}$$

$$|-z^4/q| < 1 \quad \text{iff} \quad |z|^4 < q \quad \text{iff} \quad |z| < \sqrt[4]{q}$$

$$\therefore f(z) = \frac{z}{q} \frac{1}{1+z^4/q} = \frac{z}{q} \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n}}{q^n}$$

$$\begin{aligned} &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+1}}{q^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+1}}{3^{2n+2}} \\ &\quad (\text{for } |z| < \sqrt[4]{q}) \end{aligned}$$

6. Use representation (2), Sec. 59, for $\sin z$ to write the Maclaurin series for the function

$$f(z) = \sin(z^2),$$

and point out how it follows that

$$f^{(4n)}(0) = 0 \quad \text{and} \quad f^{(2n+1)}(0) = 0 \quad (n = 0, 1, 2, \dots).$$

$$(2) \quad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \quad (|z| < \infty).$$

$$f(z) = \sin(z^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(z^2)^{2n+1}}{(2n+1)!} \quad (|z^2| < \infty)$$

Since $|z^2| < \infty$ iff $|z| < \infty$,

$$f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{4n+2}}{(2n+1)!} \quad (|z| < \infty)$$

$$f^{(4k)}(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(4n+2)!}{(4n+2-4k)!} \frac{z^{4n-4k-1}}{(2n+1)!}$$

$$f^{(4k)}(0) = \text{division by 0?}$$

7. Derive the Taylor series representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}} \quad (|z-i| < \sqrt{2}).$$

Suggestion: Start by writing

$$\frac{1}{1-z} = \frac{1}{(1-i) - (z-i)} = \frac{1}{1-i} \cdot \frac{1}{1 - (z-i)/(1-i)}.$$

$$\frac{1}{1-z} = \frac{1}{1-i} \cdot \frac{1}{1 - (z-i)/(1-i)}$$

$$= \frac{1}{1-i} \sum_{n=0}^{\infty} \left[\frac{z-i}{1-i} \right]^n \quad \left| \frac{z-i}{1-i} \right| < 1$$

$$= \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}} \quad \left| \frac{z-i}{1-i} \right| < 1$$

$$\left| \frac{z-i}{1-i} \right| < 1 \quad \text{iff} \quad |z-i| < |1-i| = \sqrt{2}$$

$$\therefore \frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}} \quad (|z-i| < \sqrt{2})$$

8. With the aid of the identity (see Sec. 34)

$$\cos z = -\sin\left(z - \frac{\pi}{2}\right),$$

expand $\cos z$ into a Taylor series about the point $z_0 = \pi/2$.

$$\cos z = -\sin(z - \pi/2)$$

$$= -\sum_{n=0}^{\infty} (-1)^n \frac{(z - \pi/2)^{2n+1}}{(2n+1)!} \quad (|z - \pi/2| < \infty)$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(z - \pi/2)^{2n+1}}{(2n+1)!} \quad (|z| < \infty)$$

11. Show that when $z \neq 0$,

$$(a) \frac{e^z}{z^2} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \dots;$$

$$(b) \frac{\sin(z^2)}{z^4} = \frac{1}{z^2} - \frac{z^2}{3!} + \frac{z^6}{5!} - \frac{z^{10}}{7!} + \dots.$$

$$\begin{aligned} (a) \frac{e^z}{z^2} &= \frac{1}{z^2} \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \right) \\ &= \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \dots \end{aligned}$$

$$\begin{aligned} (b) \frac{\sin(z^2)}{z^4} &= \frac{1}{z^4} \left(z^2 - \frac{(z^2)^3}{3!} + \frac{(z^2)^5}{5!} - \frac{(z^2)^7}{7!} + \dots \right) \\ &= \frac{1}{z^2} - \frac{z^2}{3!} + \frac{z^6}{5!} - \frac{z^{10}}{7!} + \dots \end{aligned}$$

12. Derive the expansions

$$(a) \frac{\sinh z}{z^2} = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+3)!} \quad (0 < |z| < \infty);$$

$$(b) z^3 \cosh\left(\frac{1}{z}\right) = \frac{z}{2} + z^3 + \sum_{n=1}^{\infty} \frac{1}{(2n+2)!} \cdot \frac{1}{z^{2n-1}} \quad (0 < |z| < \infty).$$

$$(a) \frac{\sinh z}{z^2} = \frac{1}{z^2} \sum_{K=0}^{\infty} \frac{z^{2K+1}}{(2K+1)!} = \frac{1}{z^2} z + \frac{1}{z^2} \sum_{K=1}^{\infty} \frac{z^{2K+1}}{(2K+1)!}$$

$$= \frac{1}{z} + \sum_{K=1}^{\infty} \frac{z^{2K-1}}{(2K+1)!} = \frac{1}{z} - \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+3)!} \quad (0 < |z| < \infty)$$

$$\begin{aligned} (b) z^3 \cosh\left(\frac{1}{z}\right) &= z^3 \sum_{K=0}^{\infty} \frac{(1/z)^{2K}}{(2K)!} = \sum_{K=0}^{\infty} \frac{z^{3-2K}}{(2K)!} \quad (|1/z| < \infty) \\ &= z^3 + z/2 + \sum_{K=2}^{\infty} \frac{z^{3-2K}}{(2K)!} = z/2 + z^3 + \sum_{n=1}^{\infty} \frac{z^{1-2n}}{(2n+2)!} \end{aligned}$$

$$= z/2 + z^3 + \sum_{n=1}^{\infty} \frac{1}{(2n+2)!} \cdot \frac{1}{z^{2n-1}} \quad (0 < |z| < \infty)$$

13. Show that when $0 < |z| < 4$,

$$\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}.$$

Partial Fractions:

$$\frac{1}{4z - z^2} = \frac{1}{z(4-z)} = \frac{A}{z} + \frac{B}{4-z}$$

$$1 = A(4-z) + Bz$$

$$\left. \begin{array}{l} z=0 : 1 = 4A \\ z=4 : 1 = 4B \end{array} \right\} \rightarrow A = B = 1/4$$

$$\frac{1}{4z - z^2} = \frac{1}{z(4-z)} = \frac{1}{4z} + \frac{1/4}{4(1-z/4)}$$

$$= \frac{1}{4z} + \frac{1}{16} \frac{1}{1-z/4}$$

$$= \frac{1}{4z} + \frac{1}{16} \sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n \quad (|z/4| < 1) \quad \text{for series convergence}$$

$$= \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}} \quad (0 < |z| < 4)$$

The additional constraint $|z| > 0$ is necessary to avoid division by 0.