

# Written Homework 2 Solutions

## Chapter 2 Section 1B : 1ac, 3ab, 4, 7, 10, 13 (page 55)

1. Use definition (2), Sec. 15, of limit to prove that

$$(a) \lim_{z \rightarrow z_0} \operatorname{Re} z = \operatorname{Re} z_0; \quad (b) \lim_{z \rightarrow z_0} \bar{z} = \bar{z}_0; \quad (c) \lim_{z \rightarrow 0} \frac{\bar{z}^2}{z} = 0.$$

Let a function  $f$  be defined at all points  $z$  in some deleted neighborhood (Sec. 11) of  $z_0$ . The statement that the *limit* of  $f(z)$  as  $z$  approaches  $z_0$  is a number  $w_0$ , or that

$$(1) \quad \lim_{z \rightarrow z_0} f(z) = w_0,$$

means that the point  $w = f(z)$  can be made arbitrarily close to  $w_0$  if we choose the point  $z$  close enough to  $z_0$  but distinct from it. We now express the definition of limit in a precise and usable form.

Statement (1) means that for each positive number  $\varepsilon$ , there is a positive number  $\delta$  such that

$$(2) \quad |f(z) - w_0| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta.$$

$$(a) \quad f(z) = \operatorname{Re} z, \quad w_0 = \operatorname{Re} z_0$$

Let  $\varepsilon > 0$ . Put  $\delta = \varepsilon$ . For  $z$  s.t.  $0 < |z - z_0| < \delta$ ,

$$|f(z) - w_0| = |\operatorname{Re} z - \operatorname{Re} z_0| = |\operatorname{Re}(z - z_0)| \leq |z - z_0| < \delta = \varepsilon$$

$$(c) \quad f(z) = \frac{\bar{z}^2}{z}, \quad w_0 = 0, \quad z_0 = 0$$

Let  $\varepsilon > 0$ . Put  $\delta = \varepsilon$ . For  $z$  s.t.  $0 < |z - 0| < \delta$ ,

$$|f(z) - w_0| = \left| \frac{\bar{z}^2}{z} - 0 \right| = \left| \frac{\bar{z}^2}{z} \right| = \frac{|\bar{z}^2|}{|z|} = \frac{|\bar{z}|^2}{|z|} = \frac{|z|^2}{|z|} = |z| < \delta = \varepsilon$$

3. Let  $n$  be a positive integer and let  $P(z)$  and  $Q(z)$  be polynomials, where  $Q(z_0) \neq 0$ . Use Theorem 2 in Sec. 16, as well as limits appearing in that section, to find

$$(a) \lim_{z \rightarrow z_0} \frac{1}{z^n} \quad (z_0 \neq 0); \quad (b) \lim_{z \rightarrow i} \frac{iz^3 - 1}{z + i}; \quad (c) \lim_{z \rightarrow z_0} \frac{P(z)}{Q(z)}.$$

Ans. (a)  $1/z_0^n$ ; (b) 0; (c)  $P(z_0)/Q(z_0)$ .

**Theorem 2.** Suppose that

$$(7) \quad \lim_{z \rightarrow z_0} f(z) = w_0 \quad \text{and} \quad \lim_{z \rightarrow z_0} F(z) = W_0.$$

Then

$$(8) \quad \lim_{z \rightarrow z_0} [f(z) + F(z)] = w_0 + W_0,$$

$$(9) \quad \lim_{z \rightarrow z_0} [f(z)F(z)] = w_0 W_0;$$

and, if  $W_0 \neq 0$ ,

$$(10) \quad \lim_{z \rightarrow z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0}.$$

(a)  $\lim_{z \rightarrow z_0} z^n = z_0^n$  is given on page 50 (follows from (9) of Thm 2).

A proof was not provided in the reading but will be provided in the following exercise.

By (10) of Thm 2,  $\lim_{z \rightarrow z_0} 1/z^n = 1/z_0^n$  using  $f(z) = 1$  and  $F(z) = z^n$

$$(b) \lim_{z \rightarrow i} z^3 i = i^4 = 1 \quad (9)$$

$$\lim_{z \rightarrow i} (z^3 i - 1) = 1 - 1 = 0 \quad (8)$$

$$\lim_{z \rightarrow i} (z + i) = i + i = 2i \quad (8)$$

$$\lim_{z \rightarrow i} \frac{iz^3 - 1}{z + i} = \frac{0}{2i} = 0 \quad (10)$$

4. Use mathematical induction and property (9), Sec. 16, of limits to show that

$$\lim_{z \rightarrow z_0} z^n = z_0^n$$

when  $n$  is a positive integer ( $n = 1, 2, \dots$ ).

\*Property (9)

Base Case ( $n=1$ ):  $\lim_{z \rightarrow z_0} z^1 = \lim_{z \rightarrow z_0} z = z_0 = z_0^1$

Assuming  $\lim_{z \rightarrow z_0} z^n = z_0^n$ ,  $\lim_{z \rightarrow z_0} z^{n+1} = \lim_{z \rightarrow z_0} z^n z = \lim_{z \rightarrow z_0} z^n \lim_{z \rightarrow z_0} z = z_0^n z_0 = z_0^{n+1}$

7. Use definition (2), Sec. 15, of limit to prove that

$$\text{if } \lim_{z \rightarrow z_0} f(z) = w_0, \text{ then } \lim_{z \rightarrow z_0} |f(z)| = |w_0|.$$

Suggestion: Observe how the first of inequalities (9), Sec. 4, enables one to write

$$||f(z)| - |w_0|| \leq |f(z) - w_0|.$$

Statement (1) means that for each positive number  $\varepsilon$ , there is a positive number  $\delta$  such that

$$(2) \quad |f(z) - w_0| < \varepsilon \quad \text{whenever } 0 < |z - z_0| < \delta.$$

$$(9) \quad ||z_1| - |z_2|| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|.$$

Assume  $\lim_{z \rightarrow z_0} f(z) = w_0$ . Let  $\varepsilon > 0$ . There exists a  $\delta > 0$  s.t.

$$|f(z) - w_0| < \varepsilon \quad \text{whenever } 0 < |z - z_0| < \delta$$

By (9) this means

$$||f(z)| - |w_0|| \leq |f(z) - w_0| < \varepsilon \quad \text{whenever } 0 < |z - z_0| < \delta$$

Since  $\varepsilon > 0$  was arbitrary,  $\lim_{z \rightarrow z_0} |f(z)| = |w_0|$

10. Use the theorem in Sec. 17 to show that

$$(a) \lim_{z \rightarrow \infty} \frac{4z^2}{(z-1)^2} = 4; \quad (b) \lim_{z \rightarrow 1} \frac{1}{(z-1)^3} = \infty; \quad (c) \lim_{z \rightarrow \infty} \frac{z^2+1}{z-1} = \infty.$$

**Theorem.** If  $z_0$  and  $w_0$  are points in the  $z$  and  $w$  planes, respectively, then

$$(1) \quad \lim_{z \rightarrow z_0} f(z) = \infty \text{ if and only if } \lim_{z \rightarrow z_0} \frac{1}{f(z)} = 0$$

and

$$(2) \quad \lim_{z \rightarrow \infty} f(z) = w_0 \text{ if and only if } \lim_{z \rightarrow 0} f\left(\frac{1}{z}\right) = w_0.$$

Moreover,

$$(3) \quad \lim_{z \rightarrow \infty} f(z) = \infty \text{ if and only if } \lim_{z \rightarrow 0} \frac{1}{f(1/z)} = 0.$$

(a) Use (2) with  $f(z) = \frac{4z^2}{(z-1)^2}$  and  $w_0 = 4$

$$\lim_{z \rightarrow 0} f(1/z) = \lim_{z \rightarrow 0} \frac{4/z^2}{(1/z-1)^2} = \lim_{z \rightarrow 0} \frac{4}{z^2(1/z-1)^2} = \lim_{z \rightarrow 0} \frac{4}{(1-z)^2} = 4$$

$$\text{iff } \lim_{z \rightarrow \infty} f(z) = \lim_{z \rightarrow \infty} \frac{4z^2}{(z-1)^2} = 4$$

(b) Use (1) with  $f(z) = \frac{1}{(z-1)^3}$  and  $z_0 = 1$

$$\lim_{z \rightarrow 1} 1/f(z) = \lim_{z \rightarrow 1} 1/[1/(z-1)^3] = \lim_{z \rightarrow 1} (z-1)^3 = (1-1)^3 = 0$$

$$\text{iff } \lim_{z \rightarrow 1} f(z) = \lim_{z \rightarrow 1} \frac{1}{(z-1)^3} = \infty$$

(c) Use (3) with  $f(z) = \frac{z^2+1}{z-1}$

$$\lim_{z \rightarrow 0} 1/f(1/z) = \lim_{z \rightarrow 0} \frac{1/z-1}{1/z^2+1} = \lim_{z \rightarrow 0} \frac{z-1}{1+z^2} = \frac{0-0}{1+0} = 0$$

$$\text{iff } \lim_{z \rightarrow \infty} f(z) = \lim_{z \rightarrow \infty} \frac{z^2+1}{z-1} = \infty$$

13. Show that a set  $S$  is unbounded (Sec. 11) if and only if every neighborhood of the point at infinity contains at least one point in  $S$ .

Suppose  $S$  is unbounded and consider the neighborhood of the point at infinity  $N_\varepsilon = \{z : |z| > 1/\varepsilon\}$  for some small positive  $\varepsilon$ .

Since  $S$  is unbounded there exists, for any positive  $R \in \mathbb{R}$ , a point  $z_0 \in S$  s.t.  $|z_0| > R$ . In particular, for  $R = 1/\varepsilon$  there exists a  $z_0 \in S$  s.t.  $|z_0| > 1/\varepsilon$ . Thus  $z_0 \in N_\varepsilon$ . Since  $N_\varepsilon$  was an arbitrary neighborhood of the point at , conclude that every neighborhood of the point at infinity contains a point in  $S$ .

Conversely, suppose that every neighborhood of the point at infinity contains a point of  $S$ . For  $S$  to be bounded there must exist some  $R \in \mathbb{R}$  s.t.  $|z| \leq R \ \forall z \in S$ . But for any  $R$  we can make  $1/\varepsilon > R$  by taking  $0 < \varepsilon < 1/R$ . There is a  $z_0 \in S$  s.t.  $|z_0| > 1/\varepsilon$  which means  $|z_0| > R$ . This shows that there is no  $R$  s.t.  $|z| \leq R \ \forall z \in S$ . Therefore  $S$  is unbounded.

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## Chapter 2 Section 20 : 4, 8 (pages 62, 63)

4. Suppose that  $f(z_0) = g(z_0) = 0$  and that  $f'(z_0)$  and  $g'(z_0)$  exist, where  $g'(z_0) \neq 0$ . Use definition (1), Sec. 19, of derivative to show that

$$\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

Let  $f$  be a function whose domain of definition contains a neighborhood  $|z - z_0| < \varepsilon$  of a point  $z_0$ . The *derivative* of  $f$  at  $z_0$  is the limit

$$(1) \quad f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

and the function  $f$  is said to be *differentiable* at  $z_0$  when  $f'(z_0)$  exists.

$$\begin{aligned} \lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} &= \lim_{z \rightarrow z_0} \frac{\frac{f(z) - f(z_0)}{z - z_0}}{\frac{g(z) - g(z_0)}{z - z_0}} = \lim_{z \rightarrow z_0} \left[ \frac{f(z) - f(z_0)}{z - z_0} \right] \left[ \frac{g(z) - g(z_0)}{z - z_0} \right]^{-1} \\ &= \lim_{z \rightarrow z_0} \left[ \frac{f(z) - f(z_0)}{z - z_0} \right] \lim_{z \rightarrow z_0} \left[ \frac{g(z) - g(z_0)}{z - z_0} \right]^{-1} = \frac{f'(z_0)}{g'(z_0)} \end{aligned}$$

8. Use the method in Example 2, Sec. 19, to show that  $f'(z)$  does not exist at any point  $z$  when  
 (a)  $f(z) = \operatorname{Re} z$ ;      (b)  $f(z) = \operatorname{Im} z$ .

(a) Let  $\Delta z = \Delta x + i\Delta y$ . The notation  $\Delta z \rightarrow 0$  used in limits means  $\Delta z \in \mathbb{C}$  is approaching the origin of the complex plane  $(0,0)$ .

$$\begin{aligned}\frac{\Delta w}{\Delta z} &= \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\operatorname{Re}(z+\Delta z) - \operatorname{Re} z}{\Delta z} \\ &= \frac{\operatorname{Re} z + \operatorname{Re}(\Delta z) - \operatorname{Re} z}{\Delta z} = \frac{\operatorname{Re}(\Delta z)}{\Delta z} = \frac{\Delta x}{\Delta x + i\Delta y}\end{aligned}$$

If  $\lim_{\substack{\Delta z \rightarrow 0 \\ \Delta y=0}} \frac{\Delta w}{\Delta z}$  exists this limit must be the same for any path through which  $\Delta z$  approaches  $(0,0)$ .

In particular consider as  $\Delta z$  approaches  $(0,0)$  horizontally along the real axis ( $\Delta y=0$ ). Then  $\Delta z = (\Delta x, 0)$  on this path:

$$\frac{\Delta w}{\Delta z} = \frac{\Delta x}{\Delta x + i\Delta y} = \frac{\Delta x}{\Delta x + i \cdot 0} = 1 \Rightarrow \lim_{\substack{\Delta z \rightarrow 0 \\ \Delta y=0}} \frac{\Delta w}{\Delta z} = \lim_{\substack{\Delta z \rightarrow 0 \\ \Delta y=0}} 1 = 1$$

Also consider as  $\Delta z$  approaches  $(0,0)$  vertically along the imaginary axis. Then  $\Delta z = (0, \Delta y)$  along this path.

$$\frac{\Delta w}{\Delta z} = \frac{\Delta x}{\Delta x + i\Delta y} = \frac{0}{0 + i\Delta y} = 0 \Rightarrow \lim_{\substack{\Delta z \rightarrow 0 \\ \Delta x=0}} \frac{\Delta w}{\Delta z} = \lim_{\substack{\Delta z \rightarrow 0 \\ \Delta x=0}} 0 = 0$$

Since limits are unique conclude that  $f'(z) := \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z}$  does not exist for any  $z$ .

(b) Let  $\Delta z = \Delta x + i\Delta y$ . The proof is similar to (a) so will be less detailed this time.

$$\frac{\Delta w}{\Delta z} = \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\operatorname{Im}(z+\Delta z) - \operatorname{Im} z}{\Delta z} = \frac{\Delta y}{\Delta x + i\Delta y}$$

$$\lim_{\substack{\Delta z \rightarrow 0 \\ \Delta y=0}} \frac{\Delta w}{\Delta z} = \lim_{\substack{\Delta z \rightarrow 0 \\ \Delta y=0}} \frac{0}{\Delta x + i \cdot 0} = \lim_{\substack{\Delta z \rightarrow 0 \\ \Delta y=0}} 0 = 0$$

$$\lim_{\substack{\Delta z \rightarrow 0 \\ \Delta x=0}} \frac{\Delta w}{\Delta z} = \lim_{\substack{\Delta z \rightarrow 0 \\ \Delta x=0}} \frac{\Delta y}{0 + i\Delta y} = \lim_{\substack{\Delta z \rightarrow 0 \\ \Delta x=0}} -i = -i$$

Conclude that  $f'(z) := \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z}$  does not exist for any  $z$

## Chapter 2 Section 23 : 1bc, 2ac, 5, 6 (pages 71, 72)

1. Use the theorem in Sec. 21 to show that  $f'(z)$  does not exist at any point if

- (a)  $f(z) = \bar{z}$ ;
- (b)  $f(z) = z - \bar{z}$ ;
- (c)  $f(z) = 2x + ixy^2$ ;
- (d)  $f(z) = e^x e^{-iy}$ .

**Theorem.** Suppose that

$$f(z) = u(x, y) + iv(x, y)$$

and that  $f'(z)$  exists at a point  $z_0 = x_0 + iy_0$ . Then the first-order partial derivatives of  $u$  and  $v$  must exist at  $(x_0, y_0)$ , and they must satisfy the Cauchy-Riemann equations

$$(7) \quad u_x = v_y, \quad u_y = -v_x$$

there. Also,  $f'(z_0)$  can be written

$$(8) \quad f'(z_0) = u_x + iv_x,$$

where these partial derivatives are to be evaluated at  $(x_0, y_0)$ .

$$(b) \quad f(z) = z - \bar{z} = (x + iy) - (x - iy) = 0 + i2y = u(x, y) + iv(x, y)$$

$$\forall x, y \in \mathbb{R} \quad \begin{cases} u_x = 0 & v_y = 2 \\ u_y = 0 & v_x = 0 \end{cases}$$

Suppose  $f'(z)$  exists at some point  $z_0 = x_0 + iy_0$ . Then the first-order partial derivatives of  $u$  and  $v$  must exist at  $(x_0, y_0)$  and satisfy both of the C-R equations. However,

$$u_x(x_0, y_0) = 0 \neq 2 = v_y(x_0, y_0)$$

Since the C-R equations for  $f$  are not satisfied at any  $z_0$ ,  $f'(z)$  must not exist for any  $z_0$ .

$$(c) \quad \begin{aligned} f(z) &= 2x + ixy^2 \\ &= u(x, y) + iv(x, y) \end{aligned} \quad \rightarrow \quad \begin{aligned} u_x &= 2 & v_y &= 2xy \\ u_y &= 0 & v_x &= y^2 \end{aligned}$$

If  $f'(z)$  exists at a point  $z_0 = x_0 + iy_0$  the C-R equations require:

$$\left. \begin{aligned} 2 &= u_x(x_0, y_0) = v_y(x_0, y_0) = 2x_0 y_0 \\ 0 &= u_y(x_0, y_0) = -v_x(x_0, y_0) = y_0^2 \end{aligned} \right\} \Rightarrow \begin{aligned} 2 &= 2x_0 \cdot 0 = 0 \\ 0 &= y_0 \end{aligned}$$

For any choice of  $z_0$  the C-R equations require  $2=0$ . Since this isn't possible,  $f'(z)$  must not exist at any point  $z_0 = x_0 + iy_0$ .

2. Use the theorem in Sec. 22 to show that  $f'(z)$  and its derivative  $f''(z)$  exist everywhere, and find  $f''(z)$  when

- (a)  $f(z) = iz + 2$ ;      (b)  $f(z) = e^{-x}e^{-iy}$ ;  
 (c)  $f(z) = z^3$ ;      (d)  $f(z) = \cos x \cosh y - i \sin x \sinh y$ .

Ans. (b)  $f''(z) = f(z)$ ; (d)  $f''(z) = -f(z)$ .

Satisfaction of the Cauchy–Riemann equations at a point  $z_0 = (x_0, y_0)$  is not sufficient to ensure the existence of the derivative of a function  $f(z)$  at that point. (See Exercise 6, Sec. 23.) But, with certain continuity conditions, we have the following useful theorem.

**Theorem.** Let the function

$$f(z) = u(x, y) + iv(x, y)$$

be defined throughout some  $\varepsilon$  neighborhood of a point  $z_0 = x_0 + iy_0$ , and suppose that

- (a) the first-order partial derivatives of the functions  $u$  and  $v$  with respect to  $x$  and  $y$  exist everywhere in the neighborhood;  
 (b) those partial derivatives are continuous at  $(x_0, y_0)$  and satisfy the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x$$

at  $(x_0, y_0)$ .

Then  $f'(z_0)$  exists, its value being

$$f'(z_0) = u_x + iv_x$$

where the right-hand side is to be evaluated at  $(x_0, y_0)$ .

$$(a) f(z) = iz + 2 = i(x+iy) + 2 = 2-y + ix = u(x, y) + iv(x, y)$$

$$\forall x, y \in \mathbb{R} \quad \begin{cases} u_x = 0 & v_y = 0 \\ u_y = -1 & -v_x = -1 \end{cases}$$

To show  $f'(z)$  exists everywhere let  $z_0 = x_0 + iy_0$  be arbitrary.  $f(z)$  must be defined in some  $\varepsilon$  neighborhood of  $z_0$ . Since  $f(z)$  is defined  $\forall z \in \mathbb{C}$  any choice of  $\varepsilon > 0$  will do; say  $\varepsilon = 1$ . This means the neighborhood is  $N = \{z : |z - z_0| < 1\}$ .

The first order partial derivatives exist everywhere in  $N$  and are continuous at  $(x_0, y_0)$ . Also note that the C-R equations are satisfied at  $(x_0, y_0)$ .

$$\therefore f'(z_0) \text{ exists and } f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = i$$

Apply similar reasoning to show  $f''(z)$  exists everywhere. We will just reassign  $u(x, y)$  and  $v(x, y)$  for notational convenience. Since  $z_0$  was arbitrary, write  $f'(z) = i$  for any  $z \in \mathbb{C}$ .

$$f'(z) = i = 0 + i \cdot 1 = u(x, y) + i v(x, y)$$

$$\forall x, y \in \mathbb{R} \begin{cases} u_x = 0 & v_y = 0 \\ u_y = 0 & v_x = 0 \end{cases}$$

Let  $z_0 = x_0 + iy_0$ . Since  $f(z)$  is defined for all  $z$ ,  $f(z)$  is defined for all  $z$  in any  $\varepsilon$ -neighborhood of  $z_0$ . Pick  $\varepsilon = 1$  and consider the corresponding  $\varepsilon$ -neighborhood

$N = \{z : |z - z_0| < 1\}$ . Since  $u_x = u_y = v_x = v_y$ , all the first order partial derivatives are defined on  $N$ , continuous at  $z_0$ , and satisfy the C-R equations at  $z_0$ .  $\therefore f''(z_0)$  exists and  $f''(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0) = 0$ .

$$(c) f(z) = z^3 = (x+iy)^3 = x^3 + 3x^2iy - 3xy^2 - iy^3 \\ = x^3 - 3xy^2 + i(3x^2y - y^3) = u(x, y) + i v(x, y)$$

$$u_x = 3x^2 - 3y^2 \quad v_y = 3x^2 - 3y^2$$

$$u_y = -6xy \quad -v_x = -6xy$$

Let  $z_0 = x_0 + iy_0$ . Verify:

1)  $\exists$  an  $\varepsilon$ -neighborhood of  $z_0$  s.t.  $f(z)$  is defined everywhere throughout the neighborhood?

Yes since  $z^3$  is defined  $\forall z$ , any choice of  $\varepsilon$  works.

2)  $u_x, u_y, v_x, v_y$  exist everywhere in that  $\varepsilon$ -neighborhood?

Yes since  $u$  and  $v$  are polynomials in  $x$  and  $y$  these partial derivatives exist everywhere, in particular everywhere in the aforementioned  $\varepsilon$ -neighborhood.

3)  $u_x, u_y, v_x, v_y$  continuous at  $z_0$  and satisfy the C-R equations at  $z_0$ ?

Yes since  $u_x, u_y, v_x$ , and  $v_y$  are also polynomials they are continuous everywhere, including  $z_0$ . By the calculation above, the C-R equations are satisfied at all points  $z$ , in particular at  $z_0 = (x_0, y_0) = x_0 + iy_0$

$\therefore f'(z_0)$  exists and  $f'(z_0) = 3x_0^2 - 3y_0^2 + i6x_0y_0 = 3z^2$

Apply similar reasoning to show  $f''(z)$  exists everywhere.  
Reassign  $u$  and  $v$  for notational convenience.

$$f'(z) = 3x^2 - 3y^2 + i6xy = 3z^2$$

$$\begin{aligned} u_x &= 6x & v_y &= 6x \\ u_y &= -6y & -v_x &= -6y \end{aligned}$$

$f'(z)$  defined on any choice of  $\epsilon$ -neighborhood

$u_x, u_y, v_x, v_y$  all exist on whatever  $\epsilon$ -neighborhood was chosen  
 $u_x, u_y, v_x, v_y$  all continuous and satisfy C-R at  $z_0 = (x_0, y_0)$

$$\therefore f''(z_0) \text{ exists and } f''(z_0) = 6x_0 + i6y_0 = 6z_0$$

Since  $z_0$  was arbitrary,  $f''(z)$  exists  $\forall z$  and  $f''(z) = 6x + i6y = 6z$

5. Show that when  $f(z) = x^3 + i(1-y)^3$ , it is legitimate to write

$$f'(z) = u_x + iv_x = 3x^2$$

only when  $z = i$ .

Use the same theorem as in exercise 1 of this section: Suppose  $f'(z)$  exists at some point  $z_0 = (x_0, y_0)$ . Then the first order partial derivatives must exist at  $z_0$  and satisfy the C-R equations at  $z_0$ .

$$3x_0^2 = u_x(x_0, y_0) = v_y(x_0, y_0) = -3(1-y_0)^2$$

$$0 = u_y(x_0, y_0) = -v_x(x_0, y_0) = 0$$

Since  $3x_0^2 \geq 0$  and  $-3(1-y_0)^2 \leq 0$ ,  $3x_0^2 = -3(1-y_0)^2$  can only be satisfied if  $3x_0^2 = 0 = -3(1-y_0)^2 \Rightarrow x_0 = 0, y_0 = 1$ .

This shows the only point at which  $f(z)$  may be differentiable is  $z_0 = i$ . Since satisfying C-R is only a necessary condition, use the theorem from exercise 2 to prove that  $f'(z)$  does indeed exist at  $z_0 = i$ .

Since  $f(z) = x^3 + i(1-y)^3$  is everywhere defined, has continuous partial derivatives everywhere and satisfies the C-R equations at  $z_0 = i$  the conditions of the theorem are met.

$$\therefore f'(z) \text{ exists at } z_0 = i \text{ and } f'(i) = u_x(0, 1) + v_x(0, 1) = 3 \cdot 0^2 + i \cdot 0 = 0$$

6. Let  $u$  and  $v$  denote the real and imaginary components of the function  $f$  defined by means of the equations

$$f(z) = \begin{cases} \bar{z}^2/z & \text{when } z \neq 0, \\ 0 & \text{when } z = 0. \end{cases}$$

Verify that the Cauchy-Riemann equations  $u_x = v_y$  and  $u_y = -v_x$  are satisfied at the origin  $z = (0, 0)$ . [Compare with Exercise 9, Sec. 20, where it is shown that  $f'(0)$  nevertheless fails to exist.]

For  $z \neq 0$   $[(x, y) \neq (0, 0)]$

$$f(z) = \bar{z}^2/z = \frac{(x-iy)^2}{x+iy} = \frac{(x-iy)^3}{x^2+y^2} = \frac{x^3-3xy^2}{x^2+y^2} + i \frac{y^3-3x^2y}{x^2+y^2}$$

$$u(x, y) = \frac{x^3-3xy^2}{x^2+y^2}, \quad v(x, y) = \frac{y^3-3x^2y}{x^2+y^2}$$

For  $z=0$   $[(x, y) = (0, 0)]$

$$f(0) = 0 = 0 + i \cdot 0, \quad u(0, 0) = v(0, 0) = 0$$

$$u_x(0, 0) = \lim_{\Delta x \rightarrow 0} \frac{u(0+\Delta x, 0) - u(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{(\Delta x)^3 - 3(\Delta x) \cdot 0^2}{(\Delta x)^2 + 0^2} - 0}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{(\Delta x)^3 / (\Delta x)^3}{(\Delta x)^3} = \lim_{\Delta x \rightarrow 0} 1 = 1$$

$$u_y(0, 0) = \lim_{\Delta y \rightarrow 0} \frac{u(0, 0+\Delta y) - u(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{\frac{0^3 - 3 \cdot 0 (\Delta y)^2}{0^2 + (\Delta y)^2} - 0}{\Delta y}$$

$$= \lim_{\Delta y \rightarrow 0} \frac{0}{\Delta y} = \lim_{\Delta y \rightarrow 0} 0 = 0$$

$$v_x(0, 0) = \lim_{\Delta x \rightarrow 0} \frac{v(0+\Delta x, 0) - v(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{0^3 - 3(\Delta x)^2 \cdot 0}{(\Delta x)^2 + 0^2} - 0}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{0}{\Delta x} = \lim_{\Delta x \rightarrow 0} 0 = 0$$

$$v_y(0, 0) = \lim_{\Delta y \rightarrow 0} \frac{v(0, 0+\Delta y) - v(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{\frac{(\Delta y)^3 - 0}{0^2 + (\Delta y)^2} - 0}{\Delta y} = 1$$

Since  $u_x(0, 0) = 1 = v_y(0, 0)$  and  $u_y(0, 0) = 0 = -v_x(0, 0)$ , the C-R eqn's are satisfied at the origin.

\* \* \* \*

## Chapter 2 Section 25: 1ac, 2b, 4ab, 6, 7 (pages 77, 78)

1. Apply the theorem in Sec. 22 to verify that each of these functions is entire:

$$(a) f(z) = 3x + y + i(3y - x); \quad (b) f(z) = \sin x \cosh y + i \cos x \sinh y;$$

$$(c) f(z) = e^{-y} \sin x - ie^{-y} \cos x; \quad (d) f(z) = (z^2 - 2)e^{-x}e^{-iy}.$$

We are now ready to introduce the concept of an analytic function. A function  $f$  of the complex variable  $z$  is *analytic at a point*  $z_0$  if it has a derivative at each point in some neighborhood of  $z_0$ .\* It follows that if  $f$  is analytic at a point  $z_0$ , it must be analytic at each point in some neighborhood of  $z_0$ . A function  $f$  is *analytic in an open set* if it has a derivative everywhere in that set. If we should speak of a function  $f$  that is analytic in a set  $S$  which is not open, it is to be understood that  $f$  is analytic in an open set containing  $S$ .

An *entire* function is a function that is analytic at each point in the entire finite plane. Since the derivative of a polynomial exists everywhere, it follows that *every polynomial is an entire function*.

To show that  $f$  is entire, show that  $f$  is analytic at each point in the complex plane. To show that  $f$  is analytic at a point  $z_0 \in \mathbb{C}$ , show that there exists a neighborhood of  $z_0$  s.t.  $f'(z)$  exists throughout that neighborhood.

(a) Consider the theorem from sec. 22:  $f(z)$  is defined everywhere and  $u_x, u_y, v_x, v_y$  exist everywhere, are continuous everywhere, and satisfy the C-R conditions everywhere:

$$u_x(x, y) = 3 = v_y(x, y) \quad u_y(x, y) = 1 = v_y(x, y) \quad (\forall z = x+iy \in \mathbb{C})$$

So for any choice of neighborhood as mentioned in the theorem the conditions of the theorem are satisfied so  $f'(z)$  exists  $\forall z \in \mathbb{C}$ . But then for any choice of neighborhood about  $z_0$  in the definition of analyticity,  $f'(z)$  exists  $\forall z$  in that neighborhood.  
 $\therefore f$  is analytic everywhere in  $\mathbb{C}$  and so  $f$  is entire.

(c) Apply the same analysis as in part (a). The argument is identical except that the C-R conditions take the form:

$$u_x = e^{-y} \cos x = v_y \quad u_y = -e^{-y} \sin x = -v_x \quad (\forall z = x+iy \in \mathbb{C})$$

Note: While the definitions and theorem used in this exercise require the existence of  $\epsilon$ -neighborhoods wherein  $f$  satisfies certain conditions, the fact that  $u(x, y)$  and  $v(x, y)$  are "well-behaved" means that we don't really need to worry so much about stating explicit choices of neighborhoods. Focus on checking that all the first order partial derivatives are also well-behaved and check C-R. The neighborhoods usually only matter in pathological cases.

2. With the aid of the theorem in Sec. 21, show that each of these functions is nowhere analytic:

$$(a) f(z) = xy + iy; \quad (b) f(z) = 2xy + i(x^2 - y^2); \quad (c) f(z) = e^y e^{ix}.$$

(b) Suppose  $f$  is analytic at some point  $z_0 \in \mathbb{C}$ . This means there exists some neighborhood of  $z_0$  s.t.  $f'(z)$  exists for each  $z$  in that neighborhood. By the theorem in Sec. 21, this means  $u_x, u_y, v_x$ , and  $v_y$  must exist at each  $z_0 = (x_0, y_0)$  and satisfy the C-R eqn's at  $z_0 = (x_0, y_0)$ . However,

$$\begin{aligned} u(x, y) &= 2xy & 2y &= u_x = v_y = -2y \\ v(x, y) &= x^2 - y^2 & 2x &= u_y = -v_x = -2x \end{aligned} \rightarrow (x, y) = (0, 0)$$

The only point at which the C-R equations are satisfied is  $z = 0$ . No matter which  $z_0$  was chosen (even  $z_0 = 0$ ) it is impossible to find a neighborhood of  $z_0$  s.t.  $f'(z)$  exists throughout the neighborhood. This is because any neighborhood, however small, contains infinitely many points and we have shown that there exists at most one point at which  $f'(z)$  may exist.

4. In each case, determine the singular points of the function and state why the function is analytic everywhere except at those points:

$$(a) f(z) = \frac{2z+1}{z(z^2+1)}; \quad (b) f(z) = \frac{z^3+i}{z^2-3z+2}; \quad (c) f(z) = \frac{z^2+1}{(z+2)(z^2+2z+2)}.$$

$$\text{Ans. (a)} z = 0, \pm i; \quad \text{(b)} z = 1, 2; \quad \text{(c)} z = -2, -1 \pm i.$$

(a)  $f(z)$  is a rational function in  $z$ , that is, the quotient of polynomial expressions in  $z$ . So  $f(z)$  is analytic at all points  $z$  at which  $f(z)$  is defined, which is all  $z$  except:

$$\begin{aligned} 0 &= z(z^2+1) \\ 0 &= z(z+i)(z-i) \end{aligned} \leftrightarrow z = 0, \pm i \text{ are the singular points}$$

(b) By the same reasoning as part a,  $f(z)$  is analytic everywhere except where the denominator vanishes:

$$0 = z^2 - 3z + 2 = (z-2)(z-1) \leftrightarrow z = 1, 2 \text{ are the singular points}$$

6. Use results in Sec. 23 to verify that the function

$$g(z) = \ln r + i\theta \quad (r > 0, 0 < \theta < 2\pi)$$

is analytic in the indicated domain of definition, with derivative  $g'(z) = 1/z$ . Then show that the composite function  $G(z) = g(z^2 + 1)$  is analytic in the quadrant  $x > 0, y > 0$ , with derivative

$$G'(z) = \frac{2z}{z^2 + 1}.$$

*Suggestion:* Observe that  $\operatorname{Im}(z^2 + 1) > 0$  when  $x > 0, y > 0$ .

**Theorem.** Let the function

$$f(z) = u(r, \theta) + iv(r, \theta)$$

be defined throughout some  $\varepsilon$  neighborhood of a nonzero point  $z_0 = r_0 \exp(i\theta_0)$ , and suppose that

- (a) the first-order partial derivatives of the functions  $u$  and  $v$  with respect to  $r$  and  $\theta$  exist everywhere in the neighborhood;
- (b) those partial derivatives are continuous at  $(r_0, \theta_0)$  and satisfy the polar form

$$ru_r = v_\theta, \quad u_\theta = -rv_r$$

of the Cauchy-Riemann equations at  $(r_0, \theta_0)$ .

Then  $f'(z_0)$  exists, its value being

$$f'(z_0) = e^{-i\theta} (u_r + iv_r),$$

where the right-hand side is to be evaluated at  $(r_0, \theta_0)$ .

$$g(z) = u(r, \theta) + iv(r, \theta) \quad \text{with } u = \ln r \quad \text{and} \quad v = \theta.$$

Pick any  $z_0 = r_0 e^{i\theta_0}$  satisfying  $0 < r_0, 0 < \theta_0 < 2\pi$ . The conditions of the theorem are met for any such  $z_0$ , including the C-R eqn's:

$$\begin{aligned} ru_r &= r'/r = 1 = r \cdot 1 = rv_\theta \\ u_\theta &= 0 = -r \cdot 0 = -rv_r \end{aligned}$$

$\therefore g'(z_0)$  exists. Since  $z_0$  was arbitrary, it follows that for any  $z$  in the specified region we can always find a neighborhood of  $z$  s.t.  $g$  is complex differentiable throughout the neighborhood.  
 $\therefore g$  is analytic on  $r > 0, 0 < \theta < 2\pi$  and  $g'(z)$  is given by

$$g'(z) = e^{-i\theta} (u_r + iv_r) = e^{-i\theta} (1/r + i \cdot 0) = r^{-1} e^{-i\theta} = (re^{i\theta})^{-1} = z^{-1} = 1/z$$

Consider  $G(z) = g(z^2 + 1)$  in the quadrant  $x, y > 0$  ( $r > 0, 0 < \theta < \pi/2$ )

From section 20, page 60, we know that if  $f(z) = z^2 + 1$  has a derivative at  $z_0$  and if  $g$  has a derivative at  $f(z_0)$  then the composite function  $G(z) = g[f(z)]$  has a derivative at  $z_0$  and  $G'(z_0) = g'[f(z_0)] f'(z_0)$ . From section 25, page 74, the composition of analytic functions is analytic.

Since  $f(z) = z^2 + 1$  is entire (polynomials in  $z$  are entire),  $f'(z)$  certainly exists in the restricted region  $x, y > 0$  with  $f'(z) = 2z$ . Since  $z$  lies in the first quadrant  $x, y > 0$  (or  $r > 0, 0 < \theta < \pi/2$ ),  $2z$  lies in the first quadrant as well, which is contained in  $r > 0, 0 < \theta < 2\pi$ . We showed  $g$  is analytic in this region. It follows that  $G$  is analytic in the first quadrant and:

$$G'(z) = g'[f(z)] f'(z) = g'(z^2 + 1) 2z = 2z/(z^2 + 1)$$

7. Let a function  $f$  be analytic everywhere in a domain  $D$ . Prove that if  $f(z)$  is real-valued for all  $z$  in  $D$ , then  $f(z)$  must be constant throughout  $D$ .

$$\forall z \in D \quad f(z) = u(x, y) + i v(x, y) = u(x, y) \quad \text{as} \quad v(x, y) \equiv 0$$

Pick any  $z_0 \in D$ . Since  $f$  is analytic at  $z_0$ ,  $f(z)$  has a derivative at  $z_0$ . This means the C-R equations must be satisfied at  $z_0$  and  $f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0)$ .

$$\begin{aligned} u_x(x_0, y_0) &= v_y(x_0, y_0) = 0 \\ u_y(x_0, y_0) &= -v_x(x_0, y_0) = 0 \end{aligned} \quad \text{and} \quad f'(z_0) = 0 + i \cdot 0 = 0$$

Since  $z_0 \in D$  was arbitrary this shows  $f'(z) = 0 \quad \forall z \in D$ .

*Theorem.* If  $f'(z) = 0$  everywhere in a domain  $D$ , then  $f(z)$  must be constant throughout  $D$ .

(page 74)

By this theorem conclude that if  $f(z) \in \mathbb{R} \quad \forall z \in D$  then  $f'(z) = 0 \quad \forall z \in D$  and therefore  $f(z)$  must be constant throughout  $D$ .

\* \* \* \*

## Chapter 2 Section 26: 1ad, 2, 3, 7, 9 (pages 81, 82)

1. Show that  $u(x, y)$  is harmonic in some domain and find a harmonic conjugate  $v(x, y)$  when

$$(a) u(x, y) = 2x(1 - y); \quad (b) u(x, y) = 2x - x^3 + 3xy^2; \\ (c) u(x, y) = \sinh x \sin y; \quad (d) u(x, y) = y/(x^2 + y^2).$$

$$\text{Ans. } (a) v(x, y) = x^2 - y^2 + 2y; \quad (b) v(x, y) = 2y - 3x^2y + y^3; \\ (c) v(x, y) = -\cosh x \cos y; \quad (d) v(x, y) = x/(x^2 + y^2).$$

A real-valued function  $H$  of two real variables  $x$  and  $y$  is said to be *harmonic* in a given domain of the  $xy$  plane if, throughout that domain, it has continuous partial derivatives of the first and second order and satisfies the partial differential equation

$$(1) \quad H_{xx}(x, y) + H_{yy}(x, y) = 0,$$

known as *Laplace's equation*.

If two given functions  $u$  and  $v$  are harmonic in a domain  $D$  and their first-order partial derivatives satisfy the Cauchy–Riemann equations (2) throughout  $D$ , then  $v$  is said to be a *harmonic conjugate* of  $u$ . The meaning of the word conjugate here is, of course, different from that in Sec. 5, where  $\bar{z}$  is defined.

$$(a) \left. \begin{array}{l} u_x = 2(1-y) \\ u_{xx} = 0 \end{array} \right. \quad \left. \begin{array}{l} u_y = -2x \\ u_{yy} = 0 \end{array} \right\} \rightarrow u_{xx} + u_{yy} = 0 + 0 = 0 \\ \therefore u \text{ is harmonic (any domain)}$$

A harmonic conjugate  $v(x, y)$  of  $u(x, y)$  will satisfy

$$\left. \begin{array}{l} v_{xx} + v_{yy} = 0, \quad u_x = v_y, \quad u_y = -v_x \\ v_y = u_x = 2 - 2y \rightarrow v = 2y - y^2 + h(x) \\ v_x = -u_y = 2x \rightarrow v = x^2 + g(y) \end{array} \right\} \rightarrow v(x, y) = x^2 - y^2 + 2y$$

Check that  $v$  is indeed harmonic:  $v_{xx} + v_{yy} = 2 - 2 = 0 \checkmark$

$$(d) \left. \begin{array}{l} u_x = -\frac{2xy}{(x^2+y^2)^2} \quad u_y = \frac{x^2-y^2}{(x^2+y^2)^2} \\ u_{xx} = \frac{6x^2y-2y^3}{(x^2+y^2)^3} \quad u_{yy} = \frac{2y^3-6x^2y}{(x^2+y^2)^3} \end{array} \right\} \rightarrow \left. \begin{array}{l} u_{xx} + u_{yy} = 0 \\ \therefore u \text{ is harmonic on any domain } D \text{ that excludes the origin} \end{array} \right.$$

$$\left. \begin{array}{l} v_y = u_x \rightarrow v = \int u_x dy + h(x) = \frac{x}{x^2+y^2} + h(x) \\ v_x = -u_y \rightarrow v = \int -u_y dx + g(y) = \frac{x}{x^2+y^2} + g(y) \end{array} \right\} \rightarrow v(x, y) = \frac{x}{x^2+y^2}$$

You can check that  $v_{xx} + v_{yy} = 0$ . I used Wolfram Alpha.

2. Show that if  $v$  and  $V$  are harmonic conjugates of  $u(x, y)$  in a domain  $D$ , then  $v(x, y)$  and  $V(x, y)$  can differ at most by an additive constant.

If  $v$  and  $V$  are both harmonic conjugates of  $u$  in  $D$ ,

$$u_{xx} + v_{yy} = 0 = u_{xx} + V_{yy}$$

$$V_{yy} - v_{yy} = 0$$

$$V_y - v_y = c \quad (c \in \mathbb{R} \text{ constant})$$

$$V - v = cy + d \quad (d \in \mathbb{R} \text{ constant})$$

Also  $V_y = u_x = v_y$  from C-R so  $c = V_y - v_y = 0$ . Thus  $V - v = d$  for some constant  $d$ , i.e.  $v$  and  $V$  differ at most by an additive constant ( $V = v + d$ ).

3. Suppose that  $v$  is a harmonic conjugate of  $u$  in a domain  $D$  and also that  $u$  is a harmonic conjugate of  $v$  in  $D$ . Show how it follows that both  $u(x, y)$  and  $v(x, y)$  must be constant throughout  $D$ .

$$v \text{ is a harmonic conjugate of } u \text{ in } D \Rightarrow \begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

$$u \text{ is a harmonic conjugate of } v \text{ in } D \Rightarrow \begin{aligned} v_x &= u_y \\ v_y &= -u_x \end{aligned}$$

Since  $u_x = v_y = -u_x$ ,  $u_x = -u_x$ . This means  $u_x = 0$  so that  $u(x, y) = g(y)$  by integration wrt  $x$ .

Since  $u_y = -v_x = -u_y$ ,  $u_y = -u_y$ . This means  $u_y = 0$  so that  $u(x, y) = h(x)$  by integration wrt  $y$ .

The only way we can have  $u = g(y)$  and  $u = h(x)$  is if  $u$  does not vary wrt either of  $x$  or  $y$ . That is,  $u = \text{constant}$  in  $D$ .

Similarly,  $v_x = -v_x$  implies  $v_x = 0$  and  $v_y = -v_y$  implies  $v_y = 0$ . By reasoning as we did for  $u$ , this implies that  $v$  doesn't vary wrt either of  $x$  or  $y$ . That is,  $v = \text{constant}$  in  $D$ .

7. Let the function  $f(z) = u(x, y) + iv(x, y)$  be analytic in a domain  $D$ , and consider the families of level curves  $u(x, y) = c_1$  and  $v(x, y) = c_2$ , where  $c_1$  and  $c_2$  are arbitrary real constants. Prove that these families are orthogonal. More precisely, show that if  $z_0 = (x_0, y_0)$  is a point in  $D$  which is common to two particular curves  $u(x, y) = c_1$  and  $v(x, y) = c_2$  and if  $f'(z_0) \neq 0$ , then the lines tangent to those curves at  $(x_0, y_0)$  are perpendicular.

*Suggestion:* Note how it follows from the pair of equations  $u(x, y) = c_1$  and  $v(x, y) = c_2$  that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{dy}{dx} = 0 \quad \text{and} \quad \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \frac{dy}{dx} = 0.$$

Consider a point  $z$  common to both curves s.t.  $f'(z)$  exists and  $f'(z) \neq 0$ .  $0 \neq f'(z) = u_x(x, y) + iv_x(x, y)$  implies (evaluate all terms using this fixed  $z$ ):

$$0 \neq |f'(z)|^2 = u_x^2 + v_x^2 = u_x^2 + u_y^2 = |\nabla u|^2 \text{ since } v_x = -u_y$$

$$0 \neq |f'(z)|^2 = u_x^2 + v_x^2 = v_y^2 + v_x^2 = |\nabla v|^2 \text{ since } u_x = v_y$$

Hence the 2-d vector gradients  $\nabla u = (u_x, u_y) \in \mathbb{R}^2$  and  $\nabla v = (v_x, v_y) \in \mathbb{R}^2$  are nonzero. From vector calculus we know the gradient is orthogonal to its level curve:  $d\mathbf{u} = \nabla u \cdot d\vec{s} = 0$  where  $d\vec{s}$  points in the direction of the tangent to the level curve and similar for  $v$ . These gradients are orthogonal since their dot product is 0:

$$\nabla u \cdot \nabla v = u_x v_x + u_y v_y = -u_x u_y + u_y u_x = 0$$

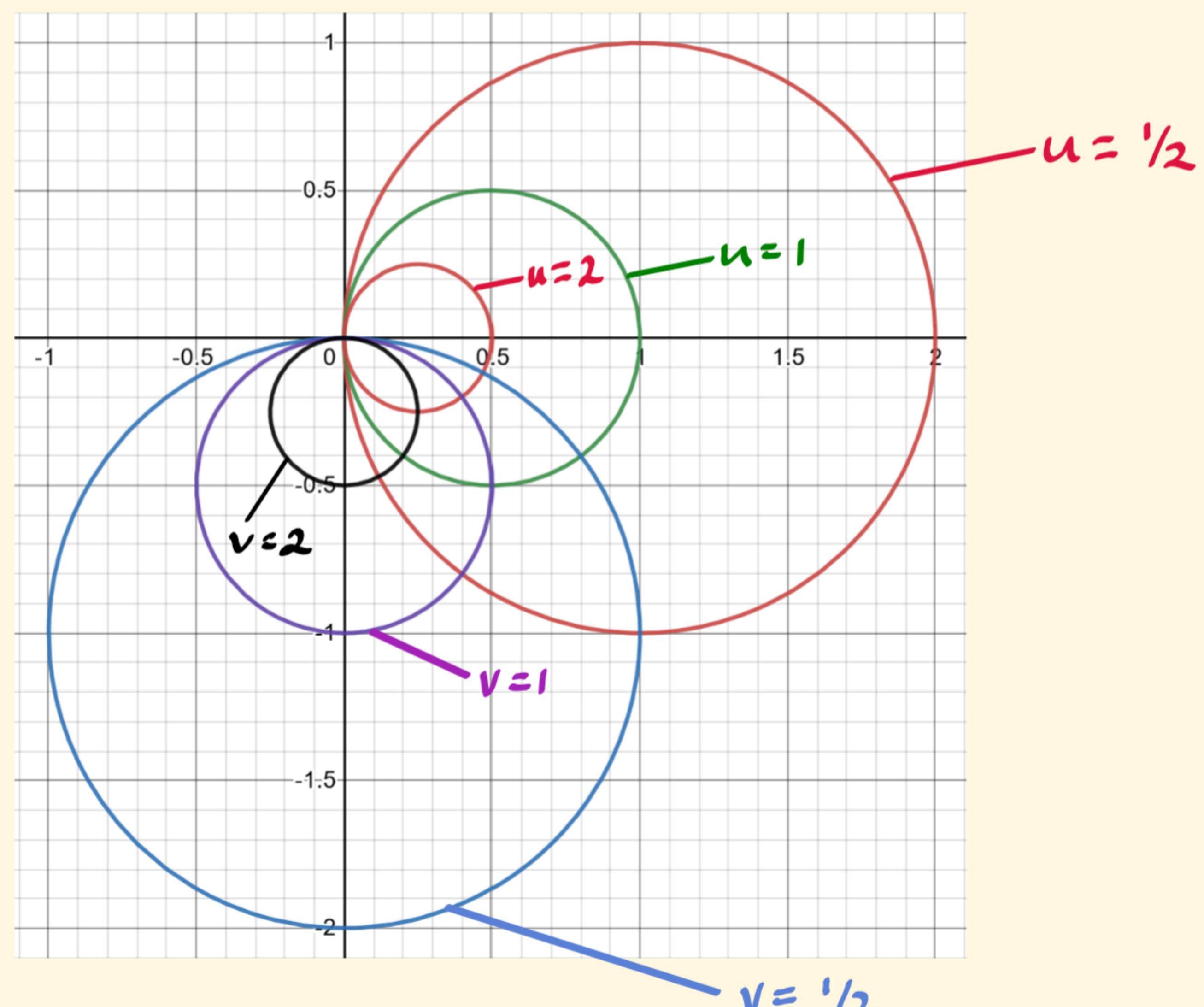
Since orthogonality is a transitive relationship:  $(u=c_1) \perp \nabla u$ ,  $\nabla u \perp \nabla v$ , and  $\nabla v \perp (v=c_2)$  together imply  $(u=c_1) \perp (v=c_2)$ , i.e. the level curves  $u=c_1$  and  $v=c_2$  are orthogonal.

9. Sketch the families of level curves of the component functions  $u$  and  $v$  when  $f(z) = 1/z$ , and note the orthogonality described in Exercise 7.

$$\begin{aligned} f(z) &= \frac{1}{z} = \frac{1}{x+iy} \\ &= \frac{x}{x^2+y^2} - i \frac{y}{x^2+y^2} \\ &= u(x, y) + iv(x, y) \end{aligned}$$

A level curve  $u(x, y) = c$  is a circle in the complex plane with radius  $1/2|c|$  and center  $(1/2|c|, 0)$ .

A level curve  $v(x, y) = d$  is a circle in the complex plane with radius  $1/2|d|$  and center  $(0, -1/2|d|)$ .



\* \* \* \*

## Chapter 3 Section 29 : 3, 5, 11, 13 (page 92)

3. Use the Cauchy-Riemann equations and the theorem in Sec. 21 to show that the function  $f(z) = \exp \bar{z}$  is not analytic anywhere.

By the theorem from section 21, if  $f'(z)$  exists at a point  $z$ , then the C-R equations must be satisfied at  $z$ . However,

$$f(z) = \exp \bar{z} = \exp(x - iy) = e^x \cos y + i(-e^x \sin y) = u(x, y) + i v(x, y)$$

$$e^x \cos y = u_x = v_y = -e^x \cos y \Rightarrow 0 = e^x \cos y \Rightarrow \cos y = 0$$

$$-e^x \sin y = u_y = -v_x = e^x \sin y \Rightarrow 0 = e^x \sin y \Rightarrow \sin y = 0$$

This means  $f'(z)$  may possibly exist only at points where both  $\cos y = 0$  and  $\sin y = 0$ , which are the points where  $y = \pi/2 + m\pi$  and  $y = n\pi$  respectively. These points never coincide so  $f'(z)$  does not exist anywhere. Since  $f$  is nowhere differentiable,  $f$  is nowhere analytic. We have used the fact that:

the statement:  $f$  analytic at  $z \Rightarrow f$  differentiable at  $z$

is equivalent to:  $f$  is not differentiable at  $z \Rightarrow f$  is not analytic at  $z$

5. Write  $|\exp(2z+i)|$  and  $|\exp(i z^2)|$  in terms of  $x$  and  $y$ . Then show that

$$|\exp(2z+i) + \exp(i z^2)| \leq e^{2x} + e^{-2xy}.$$

$$\exp(2z+i) = \exp(2x + 2yi + i) = e^{2x} \cos(2y+1) + ie^{2x} \sin(2y+1)$$

$$|\exp(2z+i)| = [e^{4x} \cos^2(2y+1) + e^{4x} \sin^2(2y+1)]^{1/2} = [e^{4x}]^{1/2} = e^{2x}$$

$$\exp(i z^2) = \exp[i(x^2 - y^2 + i2xy)] = e^{-2xy} \cos(x^2 - y^2) + ie^{-2xy} \sin(x^2 - y^2)$$

$$|\exp(i z^2)| = [e^{-4xy} \cos^2(x^2 - y^2) + e^{-4xy} \sin^2(x^2 - y^2)]^{1/2} = [e^{-4xy}]^{1/2} = e^{-2xy}$$

$$\therefore |\exp(2z+i) + \exp(i z^2)| \leq |\exp(2z+i)| + |\exp(i z^2)| = e^{2x} + e^{-2xy}$$

11. Describe the behavior of  $e^z = e^x e^{iy}$  as (a)  $x$  tends to  $-\infty$ ; (b)  $y$  tends to  $\infty$ .

$$(a) \lim_{x \rightarrow -\infty} e^x e^{iy} = 0$$

(b)  $\lim_{y \rightarrow \infty} e^x e^{iy}$  does not exist since  $e^{iy} = \cos y + i \sin y$  oscillates.

More specifically, for  $x$  held constant:

$\operatorname{Re}(e^z) = e^x \cos y$  oscillates between  $e^x$  and  $-e^x$  }  $e^z$  revolves  
 $\operatorname{Im}(e^z) = e^x \sin y$  oscillates between  $e^x$  and  $-e^x$  } about the circle  $|z| = e^x$

13. Let the function  $f(z) = u(x, y) + iv(x, y)$  be analytic in some domain  $D$ . State why the functions

$$U(x, y) = e^{u(x, y)} \cos v(x, y), \quad V(x, y) = e^{u(x, y)} \sin v(x, y)$$

are harmonic in  $D$  and why  $V(x, y)$  is, in fact, a harmonic conjugate of  $U(x, y)$ .

**Theorem 1.** If a function  $f(z) = u(x, y) + iv(x, y)$  is analytic in a domain  $D$ , then its component functions  $u$  and  $v$  are harmonic in  $D$ .

$$\Rightarrow \begin{cases} u_{xx} + u_{yy} = 0 \\ v_{xx} + v_{yy} = 0 \end{cases}$$

If two given functions  $u$  and  $v$  are harmonic in a domain  $D$  and their first-order partial derivatives satisfy the Cauchy–Riemann equations (2) throughout  $D$ , then  $v$  is said to be a *harmonic conjugate* of  $u$ . The meaning of the word conjugate here is, of course, different from that in Sec. 5, where  $\bar{z}$  is defined.

**Theorem 2.** A function  $f(z) = u(x, y) + iv(x, y)$  is analytic in a domain  $D$  if and only if  $v$  is a harmonic conjugate of  $u$ .

$$\Rightarrow \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

$$U_x = u_x e^u \cos v - v_x e^u \sin v = e^u (u_x \cos v - v_x \sin v)$$

$$U_{xx} = e^u [u_{xx} \cos v + u_x^2 \cos v - u_x v_x \sin v - v_{xx} \sin v - u_x v_x \sin v - v_x^2 \cos v]$$

$$U_y = u_y e^u \cos v - v_y e^u \sin v = e^u (u_y \cos v - v_y \sin v)$$

$$U_{yy} = e^u [u_{yy} \cos v + u_y^2 \cos v - u_y v_y \sin v - v_{yy} \sin v - u_y v_y \sin v - v_y^2 \cos v]$$

$$\begin{aligned} U_{xx} + U_{yy} &= e^u [\cos v (u_{xx} + u_{yy}) + \cos v (u_x^2 - v_x^2 + u_y^2 - v_y^2) \\ &\quad - \sin v (v_{xx} + v_{yy}) - 2 \sin v (u_x v_x + u_y v_y)] \\ &\stackrel{=} {=} e^u [\cos v (u_x^2 - u_y^2 + u_y^2 - u_x^2) - 2 \sin v (-u_x u_y + u_y u_x)] \\ &= 0 \end{aligned}$$

$\therefore U$  is harmonic in  $D$ .

$$V = e^u \sin v$$

$$V_x = u_x e^u \sin v + v_x e^u \cos v = e^u (u_x \sin v + v_x \cos v)$$

$$\begin{aligned} V_{xx} &= u_x e^u (u_x \sin v + v_x \cos v) + e^u (u_{xx} \sin v + u_x v_x \cos v + v_{xx} \cos v - v_x^2 \sin v) \\ &= e^u [(u_x^2 + u_{xx} - v_x^2) \sin v + (2u_x v_x + v_{xx}) \cos v] \end{aligned}$$

$$V_y = u_y e^u \sin v + v_y e^u \cos v = e^u (u_y \sin v + v_y \cos v)$$

$$\begin{aligned} V_{yy} &= u_y e^u (u_y \sin v + v_y \cos v) + e^u (u_{yy} \sin v + u_y v_y \cos v + v_{yy} \cos v - v_y^2 \sin v) \\ &= e^u [(u_y^2 + u_{yy} - v_y^2) \sin v + (2u_y v_y + v_{yy}) \cos v] \end{aligned}$$

$$\begin{aligned} V_{xx} + V_{yy} &= e^u [(u_{xx} + u_{yy}) \overset{=0}{\sin v} + (u_x^2 - v_x^2 + u_y^2 - v_y^2) \sin v \\ &\quad + (v_{xx} + v_{yy}) \cos v + 2(u_x v_x + u_y v_y) \cos v] \\ &= e^u (u_x^2 - u_y^2 + u_y^2 - u_x^2) \sin v + e^u (-u_x u_y + u_y u_x) \cos v \\ &= 0 \end{aligned}$$

$\therefore V$  is harmonic in D.

To conclude  $V$  is a harmonic conjugate of  $U$ , check C-R eqn's:

$$U_x = e^u (u_x \cos v - v_x \sin v) = e^u (v_y \cos v + u_y \sin v) = V_y \quad \checkmark$$

$$U_y = e^u (u_y \cos v - v_y \sin v) = e^u (-v_x \cos v - u_x \sin v) = -V_x \quad \checkmark$$

