MA 4291 Written Homework 1

Chapter 1 Section 4 : 1, 3, 4, 6 (page 12)

1. Locate the numbers $z_1 + z_2$ and $z_1 - z_2$ vectorially when

(a) $z_1 = 2i, \quad z_2 = \frac{2}{3} - i$; (b) $z_1 = (-\sqrt{3}, 1), \quad z_2 = (\sqrt{3}, 0)$;

(c) $z_1 = (-3, 1), \quad z_2 = (1, 4)$; (d) $z_1 = x_1 + iy_1, \quad z_2 = x_1 - iy_1$.

(a) $z_1 + z_2 = (0,2) + (2/3, -1) = (2/3, 1)$

$z_1 - z_2 = (0,2) - (2/3, -1) = (-2/3, 3)$

(b) $z_1 + z_2 = (-\sqrt{3}, 1) + (\sqrt{3}, 0) = (0, 1)$

$z_1 - z_2 = (-\sqrt{3}, 1) - (\sqrt{3}, 0) = (-2\sqrt{3}, 1)$

(c) $z_1 + z_2 = (-3, 1) + (1, 4) = (-2, 5)$

$z_1 - z_2 = (-3, 1) - (1, 4) = (-4, -3)$

(d) $z_1 + z_2 = (x, y) + (x, -y) = (x, 0)$

$z_1 - z_2 = (x, y) - (x, -y) = (0, 2y)$

3. Use established properties of moduli to show that when $|z_3| \neq |z_4|$,

$$\frac{\operatorname{Re}(z_1 + z_2)}{|z_3 + z_4|} \leq \frac{|z_1| + |z_2|}{||z_3| - |z_4||}.$$

$$\frac{\operatorname{Re}(z_1 + z_2)}{|z_3 + z_4|} \leq \frac{|z_1 + z_2|}{|z_3 + z_4|}$$

$\operatorname{Re} z \leq |\operatorname{Re} z| \leq |z|$

$$\leq \frac{|z_1| + |z_2|}{|z_3 + z_4|}$$

triangle inequality:
$|z_1 + z_2| \leq |z_1| + |z_2|$

$$\leq \frac{|z_1| + |z_2|}{||z_3| - |z_4||}$$

reverse triangle inequality:
$0 < ||z_3| - |z_4|| \leq |z_3 + z_4|$

4. Verify that $\sqrt{2}\,|z| \geq |\mathrm{Re}\, z| + |\mathrm{Im}\, z|$.

Suggestion: Reduce this inequality to $(|x| - |y|)^2 \geq 0$.

$$0 \leq (|\mathrm{Re}\,z| - |\mathrm{Im}\,z|)^2 = (\mathrm{Re}\,z)^2 + (\mathrm{Im}\,z)^2 - 2|\mathrm{Re}\,z||\mathrm{Im}\,z|$$

$$(\mathrm{Re}\,z)^2 + (\mathrm{Im}\,z)^2 \leq 2[(\mathrm{Re}\,z)^2 + (\mathrm{Im}\,z)^2] - 2|\mathrm{Re}\,z||\mathrm{Im}\,z|$$

$$(\mathrm{Re}\,z)^2 + (\mathrm{Im}\,z)^2 + 2|\mathrm{Re}\,z||\mathrm{Im}\,z| \leq 2[(\mathrm{Re}\,z)^2 + (\mathrm{Im}\,z)^2]$$

$$(|\mathrm{Re}\,z| + |\mathrm{Im}\,z|)^2 \leq 2[(\mathrm{Re}\,z)^2 + (\mathrm{Im}\,z)^2]$$

$$(|\mathrm{Re}\,z| + |\mathrm{Im}\,z|)^2 \leq 2|z|^2 \qquad (|z|^2 = (\mathrm{Re}\,z)^2 + (\mathrm{Im}\,z)^2)$$

$$|\mathrm{Re}\,z| + |\mathrm{Im}\,z| \leq \sqrt{2}\,|z| \qquad (a^2 \leq b^2 \to a \leq b \text{ for } a, b \geq 0)$$

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6. Using the fact that $|z_1 - z_2|$ is the distance between two points z_1 and z_2, give a geometric argument that

(a) $|z - 4i| + |z + 4i| = 10$ represents an ellipse whose foci are $(0, \pm 4)$;

(b) $|z - 1| = |z + i|$ represents the line through the origin whose slope is -1.

Write $z = x + iy$ where $x = \mathrm{Re}\,z \in \mathbb{R}$, $y = \mathrm{Im}\,z \in \mathbb{R}$

(a) $10 = |z - 4i| + |z + 4i| = |x + i(y-4)| + |x + i(y+4)|$

$$10 = [x^2 + (y-4)^2]^{1/2} + [x^2 + (y+4)^2]^{1/2}$$

$$10^2 = x^2 + (y-4)^2 + x^2 + (y+4)^2 + 2[(x^2 + (y-4)^2)(x^2 + (y+4)^2)]^{1/2}$$

✗ $10^2 = 2x^2 + 2y^2 + 32 + 2[\text{Not a perfect square}]^{1/2}$

Alternate forms

$(x^2 + y^2 - 4 \times 2\,y + 16)\,(x^2 + y^2 + 2 \times 4\,y + 16)$

$x^2\,(x^2 + 2\,y^2 + 32) + y^2\,(y^2 - 32) + 256$

$y^2\,(2\,x^2 + y^2 - 32) + (x^2 + 32)\,x^2 + 256$

Expanded form

$x^4 + 2\,x^2\,y^2 + 32\,x^2 + y^4 - 32\,y^2 + 256$

Another try:

$$10^2 = |z - 4i|^2 + |z + 4i|^2 + 2|(z - 4i)(z + 4i)|$$

$$= 2x^2 + 2y^2 + 32 + 2|z^2 + 16|$$

$$|z^2 + 16| = |x^2 - y^2 + 16 + i \cdot 2xy| = [(x^2 - y^2 + 16)^2 + (2xy)^2]^{1/2}$$

✗ $= 2x^2 + 2y^2 + 32 + \left\{ \begin{array}{l} x^4 + 2\,x^2\,y^2 + 32\,x^2 + y^4 - 32\,y^2 + 256 \\ x^2\,(x^2 + 2\,y^2 + 32) + y^2\,(y^2 - 32) + 256 \\ y^2\,(2\,x^2 + y^2 - 32) + (x^2 + 32)\,x^2 + 256 \end{array} \right\}^{1/2}$

Need to get $1 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$, foci $(0, \pm c)$ $c = \sqrt{b^2 - a^2} = 4$

(b) $|z - 1| = |z + i| \iff |z - 1|^2 = |z + i|^2 \iff |x - 1 + iy|^2 = |x + i(y+1)|^2$

$$\iff x^2 - 2x + 1 + y^2 = x^2 + y^2 + 2y + 1 \iff \boxed{y = -x}$$

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Chapter 1 Section 5 : 2, 7, 9, 10, 11, 13 (pages 14, 15)

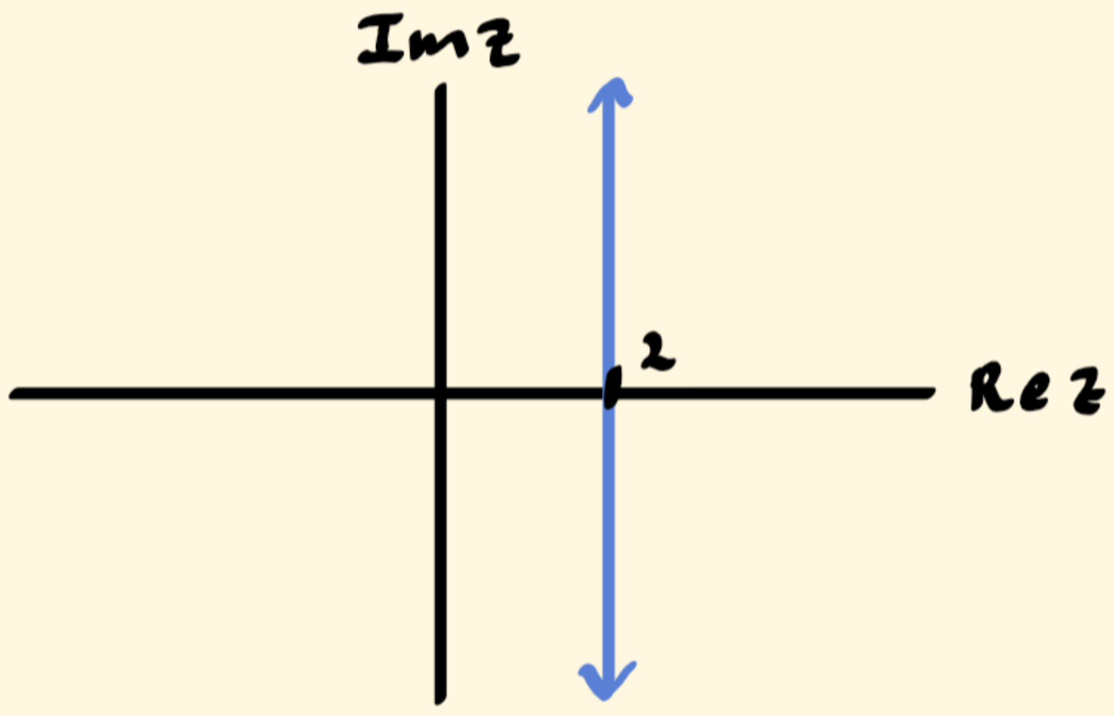
2. Sketch the set of points determined by the condition

(a) $\operatorname{Re}(\bar{z} - i) = 2$; (b) $|2\bar{z} + i| = 4$.

Let $z = x + iy$ $x, y \in \mathbb{R}$

(a) $2 = \operatorname{Re}(\bar{z} - i) = \operatorname{Re}(x + (-1-y)i) = x$

All $z = x + iy$ with $x = 2$

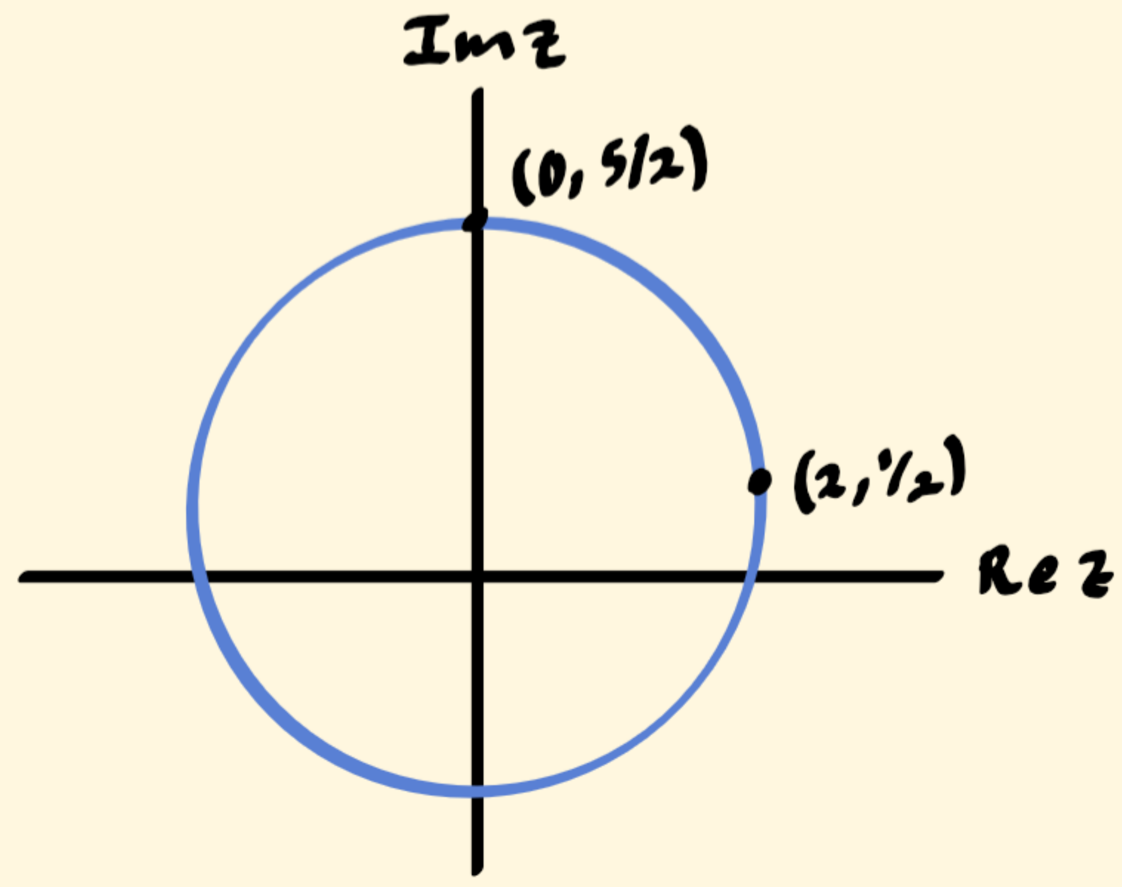


(b) $4 = |2\bar{z} + i| = |2x - 2yi + i|$

$4^2 = 4x^2 + 4(y - 1/2)^2$

$2^2 = x^2 + (y - 1/2)^2$

All z on the circle of radius 2 centered at $(0, 1/2)$ in the complex plane



7. Show that

$$|\operatorname{Re}(2 + \bar{z} + z^3)| \le 4 \qquad \text{when } |z| \le 1.$$

$\lvert\operatorname{Re}(2 + \bar{z} + z^3)\rvert \le \lvert 2 + \bar{z} + z^3\rvert$	$\lvert\operatorname{Re} w\rvert \le \lvert w\rvert \ \forall w \in \mathbb{C}$
$\le \lvert 2\rvert + \lvert\bar{z}\rvert + \lvert z^3\rvert$	triangle inequality
$= 2 + \lvert z\rvert + \lvert z\rvert^3$	$\lvert\bar{z}\rvert = \lvert z\rvert$, $\lvert z^3\rvert = \lvert z\rvert^3$
$\le 2 + 1 + 1^3 = 4$	$\lvert z\rvert \le 1$

10. Prove that

(a) z is real if and only if $\bar{z} = z$;

(b) z is either real or pure imaginary if and only if $\bar{z}^2 = z^2$.

Let $z = x + iy$ with $x, y \in \mathbb{R}$

(a) $x + iy = z = \bar{z} = x - iy \rightarrow y = 0$

$\therefore z = x \in \mathbb{R}$

(b) $x^2 - y^2 + 2xyi = z^2 = \bar{z}^2 = x^2 - y^2 - 2xyi$

$\Rightarrow xy = 0 \Rightarrow x = 0$ or $y = 0$

If $x = 0$, z is pure imaginary. If $y = 0$, z is real.

11. Use mathematical induction to show that when $n = 2, 3, \ldots,$

(a) $\overline{z_1 + z_2 + \cdots + z_n} = \overline{z_1} + \overline{z_2} + \cdots + \overline{z_n}$; (b) $\overline{z_1 z_2 \cdots z_n} = \overline{z_1}\,\overline{z_2} \cdots \overline{z_n}$.

(a) Base case $n=2$: If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$,

$$\overline{z_1 + z_2} = \overline{(x_1 + x_2) + i(y_1 + y_2)} = (x_1 + x_2) - i(y_1 + y_2) = x_1 - iy_1 + x_2 - iy_2 = \overline{z_1} + \overline{z_2}$$

Inductive step: Assume $\overline{z_1 + \cdots + z_n} = \overline{z_1} + \cdots + \overline{z_n}$ for some $n \geq 2$

$$\overline{z_1 + \cdots + z_n + z_{n+1}} = \overline{z_1 + \cdots + z_n} + \overline{z_{n+1}} \quad \text{(by the base case)}$$

$$= \overline{z_1} + \cdots + \overline{z_n} + \overline{z_{n+1}} \quad \text{(by the inductive hypothesis)}$$

(b) Base case $n=2$: If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$,

$$\overline{z_1 z_2} = \overline{(x_1 + iy_1)(x_2 + iy_2)} = \overline{(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)}$$

$$= (x_1x_2 - y_1y_2) - i(x_1y_2 + x_2y_1)$$

$$= (x_1 - iy_1)(x_2 - iy_2) = \overline{z_1}\,\overline{z_2}$$

Inductive step: Assume $\overline{z_1 z_2 \cdots z_n} = \overline{z_1}\,\overline{z_2} \cdots \overline{z_n}$ for some $n \geq 2$

$$\overline{z_1 \cdots z_n z_{n+1}} = \overline{z_1 \cdots z_n}\,\overline{z_{n+1}} \quad \text{(by the base case)}$$

$$= \overline{z_1} \cdots \overline{z_n}\,\overline{z_{n+1}} \quad \text{(by the inductive hypothesis)}$$

13. Show that the equation $|z - z_0| = R$ of a circle, centered at z_0 with radius R, can be written

$$|z|^2 - 2\,\mathrm{Re}(z\overline{z_0}) + |z_0|^2 = R^2.$$

$|z - z_0| = R$ iff $|z - z_0|^2 = R^2$ since $|z - z_0|, R \geq 0$

$$R^2 = |z - z_0|^2 = (z - z_0)\overline{(z - z_0)} \qquad |w|^2 = w\overline{w} \quad \forall w \in \mathbb{C}$$

$$= (z - z_0)(\overline{z} - \overline{z_0}) \qquad \overline{v - w} = \overline{v} - \overline{w} \quad \forall v, w \in \mathbb{C}$$

$$= z\overline{z} + z_0\overline{z_0} - z\overline{z_0} - \overline{z}z_0$$

$$= |z|^2 + |z_0|^2 - (z\overline{z_0} + \overline{z\overline{z_0}}) \qquad \overline{v\overline{w}} = \overline{v}\,\overline{\overline{w}} = \overline{v}w \quad \forall v, w \in \mathbb{C}$$

$$= |z|^2 + |z_0|^2 - 2\,\mathrm{Re}(z\overline{z_0}) \qquad 2\,\mathrm{Re}\, w = w + \overline{w}$$

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Chapter 1 Section 8: 1, 3, 5, 9, 10 (pages 22, 23)

1. Find the principal argument Arg z when

(a) $z = \dfrac{i}{-2-2i}$; (b) $z = (\sqrt{3} - i)^6$.

Ans. (a) $-3\pi/4$; (b) π.

(a) $z_1 = i = e^{i(\pi/2 + 2k\pi)}$ $k \in \mathbb{Z}$, $z_2 = -2-2i = 2\sqrt{2}e^{i(-3\pi/4 + 2j\pi)}$ $j \in \mathbb{Z}$

$\arg z_1 = \operatorname{Arg} z_1 + 2k\pi = \pi/2 + 2k\pi$, $\arg z_2 = \operatorname{Arg} z_2 + 2j\pi = -3\pi/4 + 2j\pi$

$\arg z = \arg(z_1/z_2) = \arg\left[2\sqrt{2}e^{i(5\pi/4 + 2(k-j)\pi)}\right] = 5\pi/4 + 2n\pi$, $n \in \mathbb{Z}$

$-\pi < \operatorname{Arg} z \leq \pi \rightarrow \operatorname{Arg} z = 5\pi/4 + 2\cdot(-1)\pi = -3\pi/4$

(b) $w = \sqrt{3} - i = 2e^{-i\pi/6} \longrightarrow z = w^6 = 2^6 e^{-i\pi}$

$\arg z = -\pi + 2n\pi \longrightarrow \operatorname{Arg} z = -\pi + 2(1)\pi = \pi$

3. Use mathematical induction to show that

$$e^{i\theta_1} e^{i\theta_2} \cdots e^{i\theta_n} = e^{i(\theta_1 + \theta_2 + \cdots + \theta_n)} \qquad (n = 2, 3, \ldots).$$

Base Case $n=2$:

$e^{i\theta_1} e^{i\theta_2} = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1)$

$= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) = e^{i(\theta_1 + \theta_2)}$

Inductive Step: Assume $e^{i\theta_1} e^{i\theta_2} \cdots e^{i\theta_n} = e^{i(\theta_1 + \theta_2 + \cdots + \theta_n)}$ for some $n \geq 2$

$e^{i\theta_1} e^{i\theta_2} \cdots e^{i\theta_n} e^{i\theta_{n+1}} = e^{i(\theta_1 + \cdots + \theta_n)} e^{i\theta_{n+1}}$ (by the inductive hypothesis)

$= e^{i((\theta_1 + \cdots + \theta_n) + \theta_{n+1})}$ (by the base case)

$= e^{i(\theta_1 + \cdots + \theta_{n+1})}$

5. By writing the individual factors on the left in exponential form, performing the needed operations, and finally changing back to rectangular coordinates, show that

(a) $i(1-\sqrt{3}i)(\sqrt{3}+i) = 2(1+\sqrt{3}i)$; *(b)* $5i/(2+i) = 1+2i$;

(c) $(-1+i)^7 = -8(1+i)$; *(d)* $(1+\sqrt{3}i)^{-10} = 2^{-11}(-1+\sqrt{3}i)$.

(a) $i = e^{i\pi/2}$ $\quad 1-\sqrt{3}i = 2e^{-i\pi/3}$ $\quad \sqrt{3}+i = 2e^{i\pi/6}$

$$i(1-\sqrt{3}i)(\sqrt{3}+i) = 4e^{i\pi(3-2+1)/6} = 4e^{i\pi/3} = 4(\cos\pi/3 + i\sin\pi/3) = 2(1+\sqrt{3}i)$$

(b) $5i = 5e^{i\pi/2}$ $\quad 2+i = \sqrt{5}e^{i\theta}$, $\tan\theta = 1/2$ with θ in quadrant 1

$$5i/(2+i) = \sqrt{5}e^{i(\pi/2-\theta)} = \sqrt{5}\left(\cos(\pi/2-\theta) + i\sin(\pi/2-\theta)\right)$$

$$= \sqrt{5}\left[(\cos\pi/2\cos\theta + \sin\pi/2\sin\theta) + i(\sin\pi/2\cos\theta - \cos\pi/2\sin\theta)\right]$$

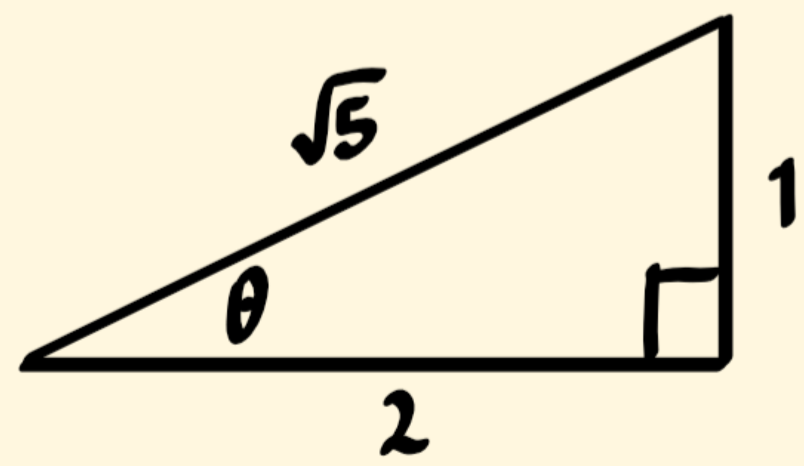
$$= \sqrt{5}(\sin\theta + i\cos\theta)$$

$$= \sqrt{5}(1/\sqrt{5} + 2i/\sqrt{5})$$

$$= 1+2i$$

$\tan\theta = 1/2$

$0<\theta<\pi/2$



(c) $-1+i = \sqrt{2}e^{3\pi i/4}$

$$(-1+i)^7 = \sqrt{2}^7 e^{21\pi i/4} = 8\sqrt{2}e^{16\pi i/4}e^{5\pi i/4} = 8\sqrt{2}e^{5\pi i/4}$$

$$= 8\sqrt{2}(\cos 5\pi/4 + i\sin 5\pi/4) = 8\sqrt{2}(-\sqrt{2}/2 - \sqrt{2}i/2)$$

$$= -8(1+i)$$

(d) $1+\sqrt{3}i = 2e^{i\pi/3}$

$$(1+\sqrt{3}i)^{-10} = 2^{-10}e^{-10i\pi/3} = 2^{-10}e^{-12i\pi/3}e^{2\pi i/3} = 2^{-10}e^{2\pi i/3}$$

$$= 2^{-10}(\cos 2\pi/3 + i\sin 2\pi/3)$$

$$= 2^{-10}(-1/2 + \sqrt{3}i/2)$$

$$= 2^{-11}(-1+\sqrt{3}i)$$

9. Establish the identity

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \qquad (z \neq 1)$$

and then use it to derive *Lagrange's trigonometric identity:*

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin[(2n+1)\theta/2]}{2\sin(\theta/2)} \qquad (0 < \theta < 2\pi).$$

Suggestion: As for the first identity, write $S = 1 + z + z^2 + \cdots + z^n$ and consider the difference $S - zS$. To derive the second identity, write $z = e^{i\theta}$ in the first one.

○ Establish the first identity by telescoping

$$S = 1 + z + z^2 + \cdots + z^n \rightarrow zS = z + z^2 + \cdots + z^{n+1}$$

$$S(1-z) = S - zS = (1 + z + \cdots + z^n) - (z + z^2 + \cdots + z^{n+1}) = 1 - z^{n+1}$$

$$\therefore 1 + z + \cdots + z^n = S = \frac{1 - z^n + 1}{1 - z}$$

△ Substitute $z = e^{i\theta}$ into this identity

$$1 + e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta} = \frac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}}$$

$$1 + \cos\theta + \cdots + \cos n\theta + i(\sin\theta + \cdots + \sin n\theta) = \frac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}}$$

□ Simplify the right hand side

$$\frac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}} = \frac{e^{(n+1)i\theta/2}}{e^{i\theta/2}} \frac{e^{-(n+1)i\theta/2} - e^{(n+1)i\theta/2}}{e^{-i\theta/2} - e^{i\theta/2}}$$

$$= e^{ni\theta/2} \frac{e^{-(n+1)i\theta/2} - e^{(n+1)i\theta/2}}{e^{-i\theta/2} - e^{i\theta/2}}$$

$$= e^{ni\theta/2} \frac{\cos((n+1)\theta/2) - i\sin((n+1)\theta/2) - \cos((n+1)\theta/2) - i\sin((n+1)\theta/2)}{\cos(\theta/2) - i\sin(\theta/2) - \cos(\theta/2) - i\sin(\theta/2)}$$

$$= e^{ni\theta/2} \frac{\sin((n+1)\theta/2)}{\sin(\theta/2)} = \left(\cos(n\theta/2) + i\sin(\theta/2)\right) \frac{\sin((n+1)\theta/2)}{\sin(\theta/2)}$$

◇ Match real terms and simplify using $\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha-\beta) + \sin(\alpha+\beta))$

$$1 + \cos\theta + \cdots + \cos n\theta = \mathrm{Re}\left[\frac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}}\right] = \frac{\cos(n\theta/2)\sin((n+1)\theta/2)}{\sin(\theta/2)}$$

$$1 + \cos\theta + \cdots + \cos n\theta = \frac{1}{2} \frac{\sin(\theta/2) + \sin((2n+1)\theta/2)}{\sin(\theta/2)} = \frac{1}{2} + \frac{\sin((2n+1)\theta/2)}{2\sin(\theta/2)}$$

⬠ Alternatively, apply the first identity to both $z = e^{i\theta}$ and $z = e^{-i\theta}$.

$$1 + \cos\theta + \dots + \cos n\theta = \sum_{k=0}^{n} \cos k\theta = \frac{1}{2}\sum_{k=0}^{n}\left(e^{ik\theta} + e^{-ik\theta}\right)$$

$$= \frac{1}{2}\left(\frac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}} + \frac{1 - e^{-(n+1)i\theta}}{1 - e^{-i\theta}}\right)$$

$$= \frac{1}{2}\left(\frac{e^{-i\theta/2} - e^{(n+1/2)i\theta}}{e^{-i\theta/2} - e^{i\theta/2}} + \frac{-e^{i\theta/2} + e^{-(n+1/2)i\theta}}{-e^{i\theta/2} + e^{-i\theta/2}}\right)$$

$$= \frac{1}{2}\left(\frac{e^{-(n+1/2)i\theta} - e^{(n+1/2)i\theta}}{e^{-i\theta/2} - e^{i\theta/2}} + \frac{e^{-i\theta/2} - e^{i\theta/2}}{e^{-i\theta/2} - e^{i\theta/2}}\right)$$

$$= \frac{1}{2}\left(\frac{-2i\sin((n+1/2)\theta)}{-2i\sin(\theta/2)} + 1\right)$$

$$= \frac{1}{2} + \frac{\sin((2n+1)\theta/2)}{2\sin(\theta/2)}$$

10. Use de Moivre's formula (Sec. 7) to derive the following trigonometric identities:

(a) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$; *(b)* $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

de Moivre's formula: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, $n \in \mathbb{Z}$

Evaluate the formula with $n=3$.

$$\cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta = (\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

(a) Match real terms: $\cos^3\theta - 3\cos\theta\sin^2\theta = \cos 3\theta$

(b) Match imaginary terms: $3\cos^2\theta\sin\theta - \sin^3\theta = \sin 3\theta$

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Chapter 1 Section 10: 1, 3, 6, 7 (pages 29, 30)

1. Find the square roots of *(a)* $2i$; *(b)* $1-\sqrt{3}i$ and express them in rectangular coordinates.

Ans. (a) $\pm(1+i)$; *(b)* $\pm\dfrac{\sqrt{3}-i}{\sqrt{2}}$.

(a) $z = 2i = 2e^{i(\pi/2+2k\pi)} \Rightarrow z^{1/2} = \sqrt{2}\, e^{i(\pi/4+k\pi)}$ $(k \in \mathbb{Z})$

Evaluate at $k = 0, 1$ to collect all distinct roots:

$z^{1/2} = \sqrt{2}e^{i\pi/4}, \sqrt{2}e^{5\pi i/4} = \sqrt{2}(\sqrt{2}/2 + \sqrt{2}i/2), \sqrt{2}(-\sqrt{2}/2 - \sqrt{2}i/2) = \boxed{\pm(1+i)}$

(b) $z = 1-\sqrt{3}i = 2e^{i(-\pi/3+2k\pi)} \Rightarrow z^{1/2} = \sqrt{2}\, e^{i(-\pi/6+k\pi)}$ $(k \in \mathbb{Z})$

Evaluate at $k = 0, 1$ to collect all distinct roots:

$z^{1/2} = \sqrt{2}e^{-i\pi/6}, \sqrt{2}e^{5\pi i/6} = \sqrt{2}(\sqrt{3}/2 - i/2), \sqrt{2}(-\sqrt{3}/2 + i/2) = \boxed{\pm(\sqrt{3}-i)/\sqrt{2}}$

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3. In each case, find all the roots in rectangular coordinates, exhibit them as vertices of certain regular polygons, and identify the principal root:

(a) $(-1)^{1/3}$; *(b)* $8^{1/6}$.

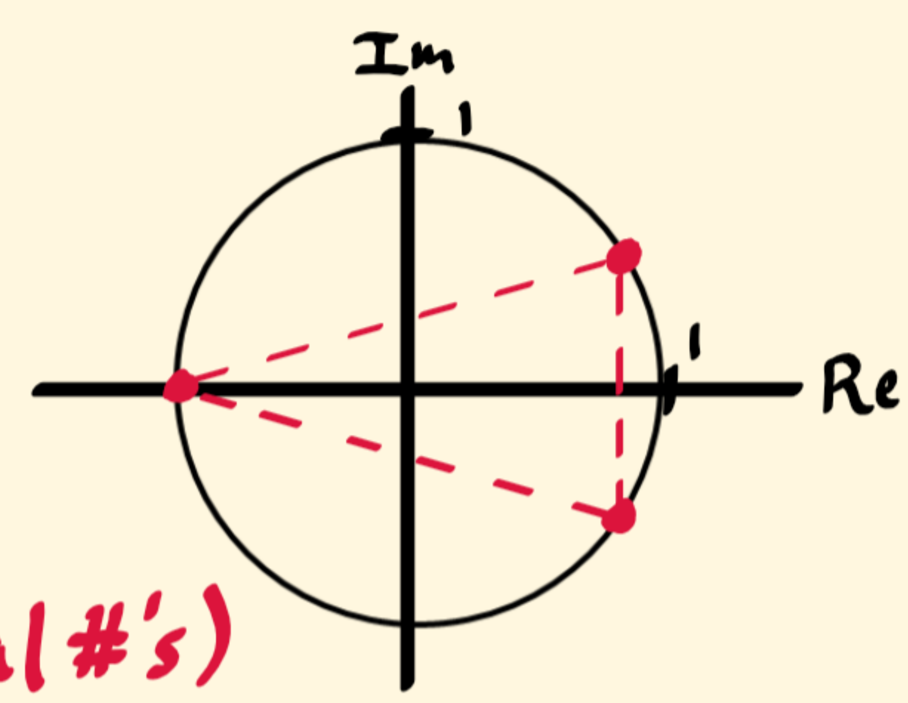
Ans. (b) $\pm\sqrt{2}$, $\pm\dfrac{1+\sqrt{3}i}{\sqrt{2}}$, $\pm\dfrac{1-\sqrt{3}i}{\sqrt{2}}$.

(a) $z = -1 = e^{i(\pi+2k\pi)} \Rightarrow z^{1/3} = (-1)^{1/3} = e^{i(\pi/3+2k\pi/3)}$ $(k \in \mathbb{Z})$

Evaluate at $k = 0, 1, 2$ to collect all distinct roots:

$(-1)^{1/3} = e^{i\pi/3}, e^{i\pi}, e^{5\pi i/3}$

$= \sqrt{3}/2 + i/2, -1, \sqrt{3}/2 - i/2$



* The principle root is -1
(was not defined for roots of negative real #'s)

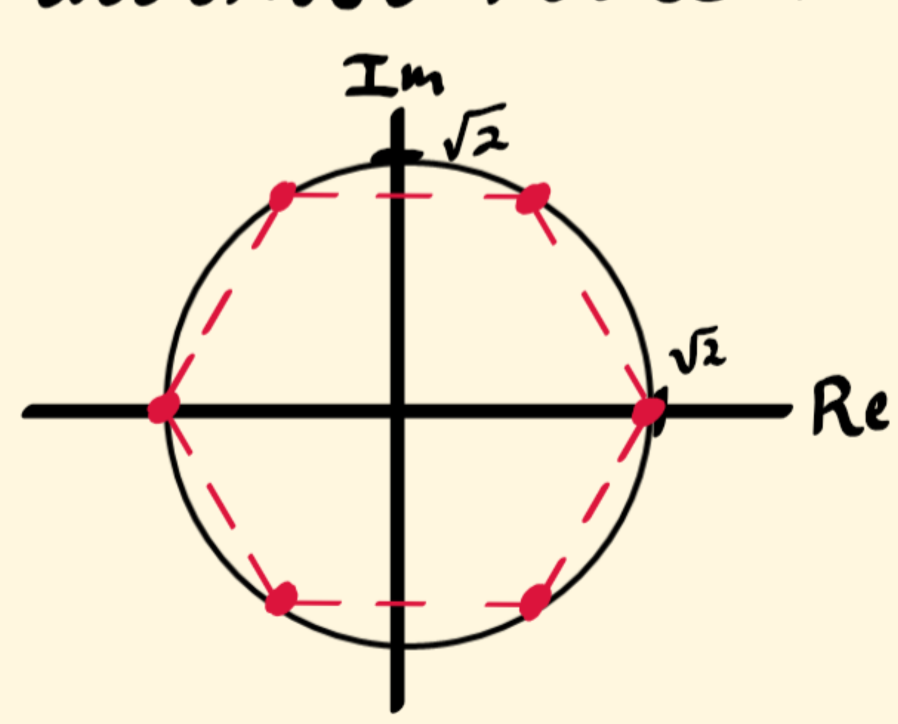
(b) $z = 8 = 8e^{i(0+2k\pi)} = 8e^{2k\pi i} \Rightarrow z^{1/6} = 8^{1/6} = \sqrt{2}e^{k\pi i/3}$ $(k \in \mathbb{Z})$

Evaluate at $k = 0, 1, \dots, 5$ to collect all distinct roots:

$8^{1/6} = \sqrt{2}, \sqrt{2}e^{i\pi/3}, \sqrt{2}e^{2\pi i/3}, \dots, \sqrt{2}e^{5\pi i/3}$

$= \pm\sqrt{2}, (\pm 1 \pm \sqrt{3}i)/\sqrt{2}$

The principle root is $\sqrt{2}$



6. Find the four zeros of the polynomial $z^4 + 4$, one of them being

$$z_0 = \sqrt{2}\, e^{i\pi/4} = 1 + i.$$

Then use those zeros to factor $z^2 + 4$ into quadratic factors with real coefficients.

Ans. $(z^2 + 2z + 2)(z^2 - 2z + 2)$.

$$z_0^4 + 4 = 0$$

$$z_0^4 = -4 = 4e^{i(\pi + 2k\pi)}, \quad k \in \mathbb{Z}$$

$$z_0 = \sqrt[4]{4}\, e^{i(\pi + 2k\pi)/4} = \sqrt{2}\, e^{i(\pi+2k\pi)/4}$$

$$z_0 = \sqrt{2}\, e^{i\pi/4},\ \sqrt{2}\, e^{3\pi i/4},\ \sqrt{2}\, e^{5\pi i/4},\ \sqrt{2}\, e^{7\pi i/4} \quad (k = 0, 1, 2, 3)$$

$$z_0 = 1+i,\ -1+i,\ -1-i,\ 1-i$$

Note that $(z-r)(z-\bar{r}) = z^2 - z(r+\bar{r}) + r\bar{r} = z^2 - 2z\mathrm{Re}(r) + |r|^2$ is a polynomial with real coefficients. This means we should multiply the linear terms corresponding to complex conjugate pairs

$$\therefore\ z^2 + 4 = (z-(1+i))(z-(1-i))(z-(-1+i))(z-(-1-i))$$

$$= (z - 2z + 2)(z + 2z + 2)$$

7. Show that if c is any nth root of unity other than unity itself, then

$$1 + c + c^2 + \cdots + c^{n-1} = 0.$$

Suggestion: Use the first identity in Exercise 9, Sec. 8.

Suppose $c \in \mathbb{C}$ is any root of unity other than unity itself, i.e.

$$c = 1^{1/n}, \quad c \neq 1$$

There are $n-1$ such possible values of c, but all satisfy $c^n = 1$ and $1 - c \neq 0$. It follows that

$$1 + c + c^2 + \ldots + c^{n-1} = \frac{1 - c^n}{1 - c} = \frac{1-1}{1-c} = \frac{0}{1-c} = 0$$

* * * * *

Chapter 1 Section 11: 1ae, 2, 3, 6, 7, 8 (page 33)

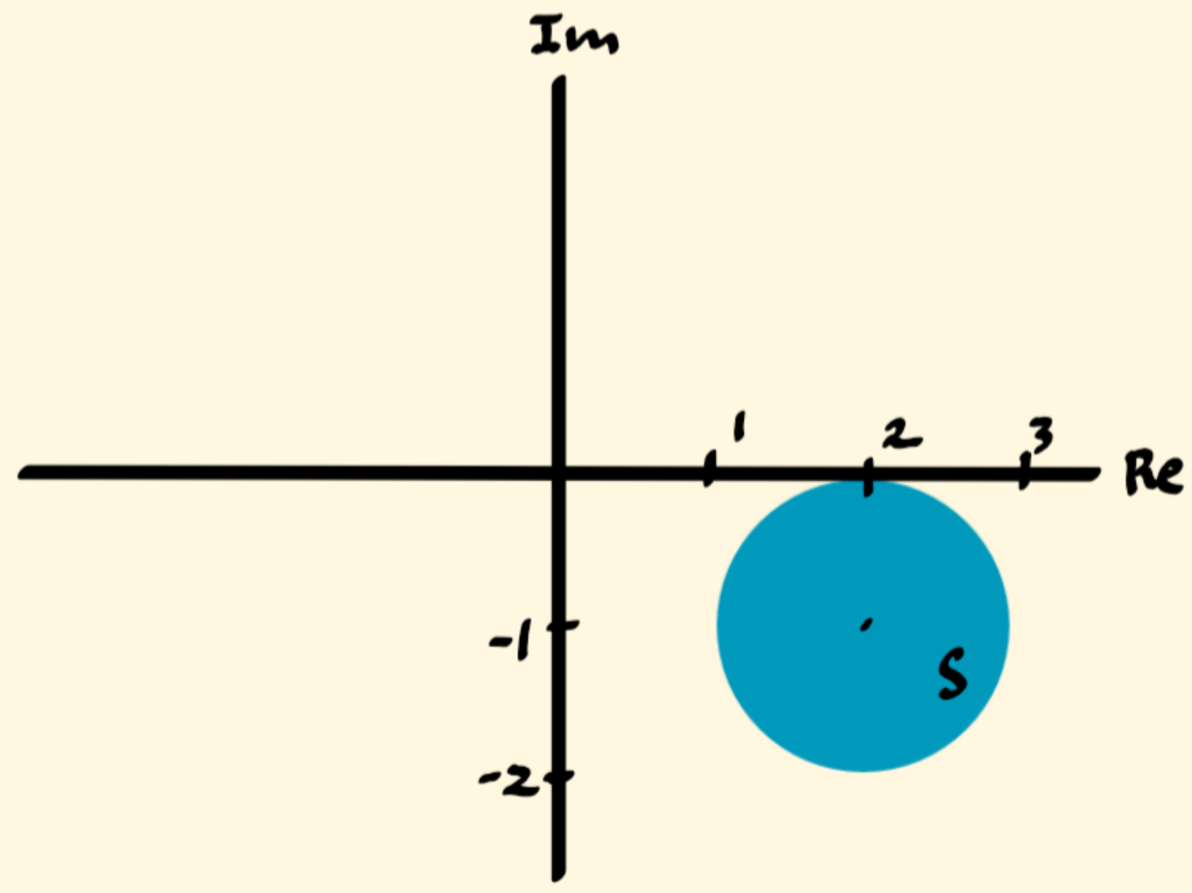
1. Sketch the following sets and determine which are domains:

(a) $|z-2+i| \le 1$; *(b)* $|2z+3| > 4$;
(c) $\operatorname{Im} z > 1$; *(d)* $\operatorname{Im} z = 1$;
(e) $0 \le \arg z \le \pi/4$ $(z \ne 0)$; *(f)* $|z-4| \ge |z|$.

Ans. *(b)*, *(c)* are domains.

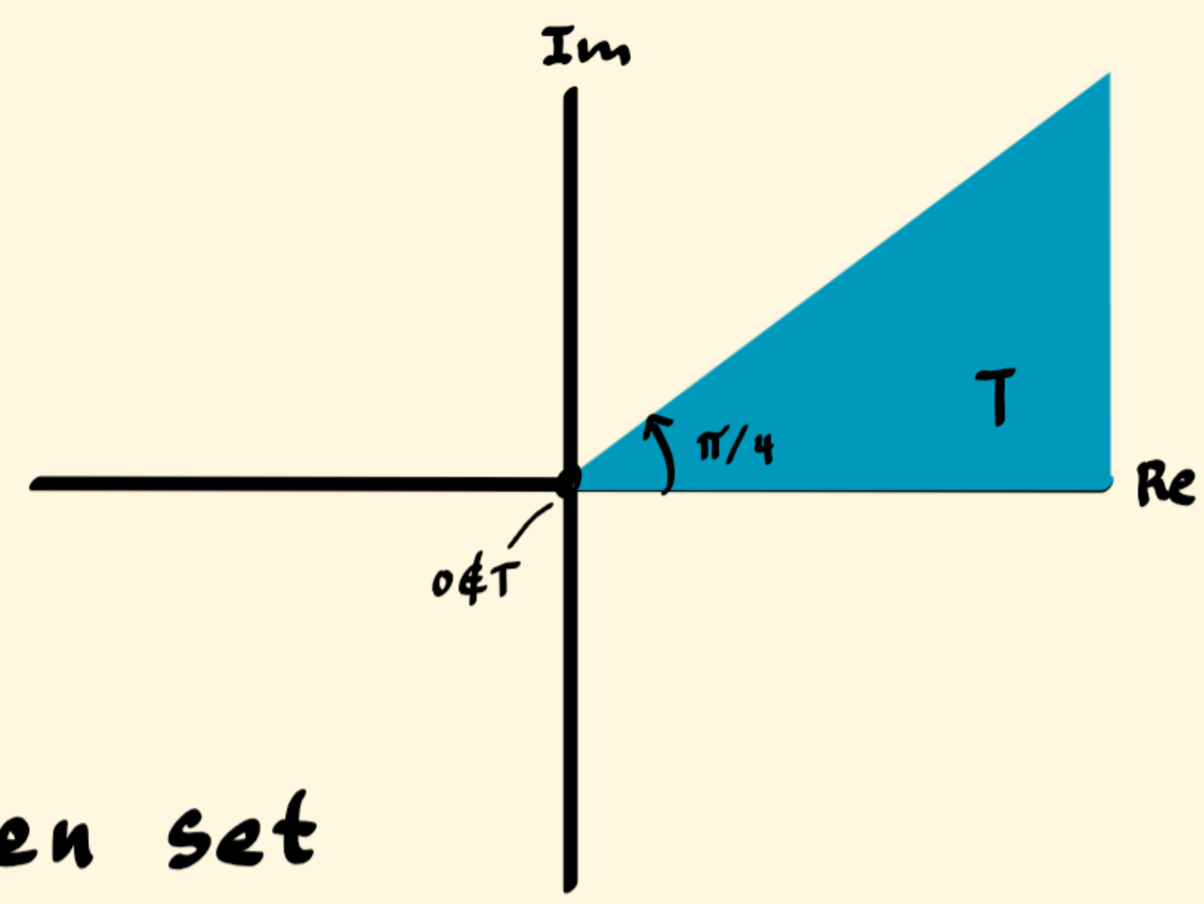
(a) $S = \{z \in \mathbb{C} : |z-(2-i)| \le 1\}$ is <u>not</u> a domain.

A domain is a nonempty open set that is connected. The given set S is not open. An open set contains none of its boundary points. The boundary points of S are all z s.t. $|z-2+i| = 1$, and S contains all of these boundary points. Therefore S contains at least one boundary point



(e) $T = \{z \in \mathbb{C} : 0 \le \arg z \le \pi/4,\ z \ne 0\}$ is <u>not</u> a domain.

T is not open since it contains, for example, the boundary point $z = 1$ which has $\arg z = 0$.



2. Which sets in Exercise 1 are neither open nor closed?

Ans. *(e)*.

For each of (a) – (f), let S denote the given set and ∂S the set of boundary points of S.

(a) S is <u>closed</u> since it contains all boundary points (Exercise 1a)

(b) $\partial S = \{z : |2z+3| = 4\}$. For any $z \in \partial S$, $|2z+3| \not> 4$ so $z \notin S$. Since S contains no boundary points, S is <u>open</u>.

(c) $\partial S = \{z : \operatorname{Im} z = 1\}$. For any $z \in \partial S$, $\operatorname{Im} z \not> 1$ so $z \notin S$. S is <u>open</u>.

(d) $\partial S = \{z : \operatorname{Im} z = 1\} = S$. Since $\partial S = S$, $\partial S \subset S$. S is <u>closed</u>.

(e) S is <u>not open</u> (Exercise 1e). But also $0 \in \partial S$ with $0 \notin S$ so S is <u>not closed</u>. S is neither open nor closed.

(f) $S = \{z : |z-4| \ge |z|\} \overset{*}{=} \{z : \operatorname{Re} z \le 2\} \Rightarrow \partial S = \{z : \operatorname{Re} z = 2\}$. Since $\partial S \subset S$, S is <u>closed</u>. The claim $|z-4| \ge |z|$ iff $\operatorname{Re} z \le 2$ warrants proof.

*Proof that $S = \{z : |z-4| \geq |z|\} = \{z : \operatorname{Re} z \leq 2\}$:

$0 \leq |z| \leq |z-4|^2$

$|z|^2 \leq |z-4|^2$

$(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 \leq (\operatorname{Re}(z-4))^2 + (\operatorname{Im}(z-4))^2$

$(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 \leq (\operatorname{Re} z - 4)^2 + (\operatorname{Im} z)^2$

$(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 \leq (\operatorname{Re} z)^2 - 8\operatorname{Re} z + 16 + (\operatorname{Im} z)^2$

$8 \operatorname{Re} z \leq 16$

$\operatorname{Re} z \leq 2$ $\qquad\qquad \therefore\ |z| \leq |z-4|$ iff $\operatorname{Re} z \leq 2$

3. Which sets in Exercise 1 are bounded?

Ans. (a).

A set S is bounded if every point of S lies in some circle $|z| \leq R$

(a) Since $|z| - \sqrt{5} = |z| - |-2+i| \leq ||z| - |-2+i|| \leq |z + (-2+i)| \leq 1$, all points of S lie in $|z| \leq 1 + \sqrt{5}$. S is <u>bounded</u>.

(b) Let $R > 2$ be arbitrary. Any point $z \in \mathbb{C}$ s.t. $|z| > R$ satisfies $|z + 3/2| > R > 2$ and thus $|2z+3| > 4$. Since R was arbitrary this means that no matter how large we choose R to be there is always a point $z \in S$ such that $|z| \not\leq R$. S is <u>not bounded</u>.

(c) Let $R > 1$ be arbitrary. Then $z = (R+1)i$ satisfies $\operatorname{Im} z = R+1 > R > 1$ so $z \in S$ but $|z| = R+1 > R$. This shows that for arbitrarily large R we can always find a point $z \in S$ such that $|z| \not\leq R$. S is <u>not bounded</u>.

(d) Given $R > 0$ take $z = R + i$. Then $z \in S$ since $\operatorname{Im} z = 1$ but $|z| = (R^2+1)^2 > R$. This shows that for arbitrarily large R we can always find a point $z \in S$ such that $|z| \not\leq R$. S is <u>not bounded</u>.

(e) Given $R > 0$ take $z = R+1$. Then since $\arg z = 0$, $0 \leq \arg z \leq \pi/4$ and $z \in S$ but $|z| > R$. S is <u>not bounded</u>.

(f) Given $R > 0$ take $z = -R-1$. Then since $\operatorname{Re} z = -R-1 < -1 \leq 2$, $z \in S$ but $|z| = (R^2+1)^{1/2} > R$. S is <u>not bounded</u>.

Definitions

A point z_0 is an interior point of S if there exists some neighborhood of z_0 that contains only points of S; z_0 is an exterior point of S if there exists a neighborhood of z_0 containing no points of S. If z_0 is neither of these, z_0 is a boundary point of S: every neighborhood of z_0 contains at least one point of S and one point of S^c.

z_0 is an accumulation point of S if every deleted neighborhood of z_0 contains a point of S.

6. Show that a set S is open if and only if each point in S is an interior point.

Assume S is open and suppose $z_0 \in S$. By the definitions we know that if z_0 is not an interior point and z_0 is not an exterior point then z_0 is a boundary point. Since S is open, S does not contain any of its boundary points. In particular $z_0 \in S$ means z_0 is not a boundary point of S. Thus z_0 must be either an interior point of S or an exterior point of S. Since every neighborhood of z_0 contains $z_0 \in S$, z_0 is not an exterior point. By exhaustion of all logical alternatives, z_0 must be an interior point.

Assume that each point of the set S is an interior point. Suppose z is a boundary point of S. If $z \in S$ then z is both an interior point and a boundary point of S. This is a contradiction since by the definitions a point cannot be both an interior point and a boundary point of S. Since z was arbitrary, conclude that no boundary point of S is contained in S. Therefore S is open.

7. Determine the accumulation points of each of the following sets:

(a) $z_n = i^n \ (n = 1, 2, \ldots)$; *(b)* $z_n = i^n/n \ (n = 1, 2, \ldots)$;

(c) $0 \le \arg z < \pi/2 \ (z \ne 0)$; *(d)* $z_n = (-1)^n(1+i)\dfrac{n-1}{n} \ (n = 1, 2, \ldots)$.

Ans. (a) None; *(b)* 0; *(d)* $\pm(1+i)$.

(a) $S = \{ i^n : n = 1, 2, \ldots \} = \{ \pm i, \pm 1 \}$

If $z \in S$, any deleted neighborhood of z with radius less than $\sqrt{2}$ contains no points of S. So $z \in S$ is not accumulation point of S.

If $z \in S^c$ then the fact that S contains only four isolated points means that a deleted neighborhood of sufficiently small radius (take e.g. $r = \inf\{ |z - z_0| : z_0 \in S \}$) will contain no points of S.

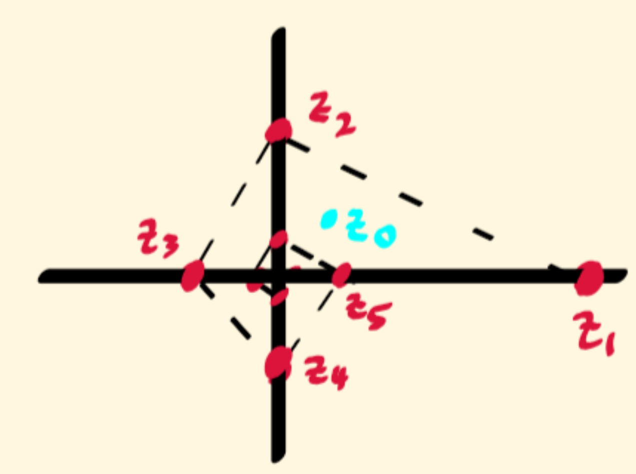
Since one of $z \in S$ or $z \in S^c$ must be true $\forall z \in \mathbb{C}$ and in either case z is not an accumulation point of S, S has no accumulation points.

(b) $S = \{i^n/n : n = 1, 2, \ldots\}$

Since $i^n \in \{\pm 1, \pm i\}$ for each n, $|i^n/n| = 1/n$ for each n.

Given any ε neighborhood of $z_0 = 0$ there exists an N s.t. $1/n < \varepsilon$ whenever $n \geq N$. Then $z_n = i^n/n$ belongs to this ε neighborhood for any $n \geq N$. This shows that there is a point of S (infinitely many in fact) in the ε neighborhood of 0. Since $0 \notin S$ the same is true of the deleted ε neighborhood as well and since ε was arbitrary this shows that $z_0 = 0$ is an accumulation point of S.

For $z_0 \neq 0$, let $D = \{|z_n - z_0| : z_n \neq z_0, n = 1, 2, \ldots\}$, the set of distances between z_0 and any of the points z_n (the restriction $z_n \neq z_0$ handles the possible case that $z_0 = z_n$ for some n). D is a nonempty set of positive real numbers, so 0 is a lower bound of D.



$\varepsilon := \inf D \longrightarrow |z_n - z_0| \geq \varepsilon \ \forall n$

This choice of ε gives a deleted neighborhood of z_0 that contains no points of S. $z_0 \neq 0$ is not an accumulation point.

Conclude $z_0 = 0$ is the only accumulation point of S.

(c) $S = \{z \in \mathbb{C} : z \neq 0, \ 0 \leq \arg z < \pi/2\}$

$z_0 = 0$ is an accumulation point of S since every deleted neighborhood of 0 contains (infinitely many) points of S.

Any $z_0 \in S$ is an accumulation point of S since if $0 < \arg z_0 < \pi/2$ we can find a deleted neighborhood entirely contained in S and if $\arg z_0 = 0$ then any deleted neighborhood of z_0 contains points z with $0 < \arg z < \pi/2$.

Any point $z_0 \neq 0$ with $\arg z_0 = \pi/2$ is an accumulation point since these points are boundary points of S, just as the points along $\arg z_0 = 0$ are.

The set of accumulation points of S is the set

$$\bar{S} = \{z_0 : z_0 = 0 \text{ or } 0 \leq \arg z_0 \leq \pi/2\}$$

(d) $S = \left\{ (-1)^n (1+i) \frac{n-1}{n} : n = 1, 2, \dots \right\}$

Given $\varepsilon > 0$, $\exists N$ s.t. $\sqrt{2} \left| \frac{n-1}{n} - 1 \right| < \varepsilon$ $\forall n \geq N$ since $\frac{n-1}{n} \to 1$ as $n \to \infty$.

For any even $n \geq N$

$$0 < \left| (-1)^n (1+i) \frac{n-1}{n} - (1+i) \right| = |1+i| \left| \frac{n-1}{n} - 1 \right| = \sqrt{2} \left| \frac{n-1}{n} - 1 \right| < \varepsilon$$

For any odd $n \geq N$

$$0 < \left| (-1)^n (1+i) \frac{n-1}{n} - (-1-i) \right| = |1+i| \left| 1 - \frac{n-1}{n} \right| = \sqrt{2} \left| \frac{n-1}{n} - 1 \right| < \varepsilon$$

This shows that every deleted neighborhood of $1+i$ contains a point of S and every deleted neighborhood of $-1-i$ contains a point of S. Therefore $z_0 = \pm(1+i)$ are accumulation points of S.

Since $z_{2n} \to 1+i$ and $z_{2n-1} \to -1-i$, $z_0 \neq \pm(1+i)$ cannot be an accumulation point of S since if if z_0 was some third accumulation point of S we would be able to find a subsequence of (z_n) converging to z_0.

8. Prove that if a set contains each of its accumulation points, then it must be a closed set.

Assume the set S contains each of its accumulation points. To show that S is closed, show that S contains all of its boundary points.

Suppose $z_0 \in \partial S$. Either $z_0 \in S$ or $z_0 \notin S$. If $z_0 \in S$ we are done. Otherwise if $z_0 \notin S$ then since every neighborhood of z_0 contains a point of S (by the definition of a boundary point) and $z_0 \notin S$ itself, every neighborhood of z_0 contains a point of S other than z_0, i.e. every deleted neighborhood of z_0 contains a point of S. Thus z_0 is also an accumulation point of S so that $z_0 \in S$ by the initial assumption. This contradicts $z_0 \notin S$.

Since $z_0 \in S$ or $z_0 \notin S$ and the latter case produces a contradiction, conclude $z_0 \in S$. Since $z_0 \in \partial S$ was arbitrary, this proves S is closed.

*   *   *   *   *

Chapter 2 Section 12 : 2, 3, 4 (pages 37, 38)

2. Write the function $f(z) = z^3 + z + 1$ in the form $f(z) = u(x, y) + iv(x, y)$.

Ans. $f(z) = (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y)$.

$z = x + iy \quad x, y \in \mathbb{R}$

$$f(z) = z^3 + z + 1 = (x+iy)^3 + x + iy + 1 = x^3 + 3ix^2y - 3xy^2 - iy^3 + x + iy + 1$$

$$= (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y) = u(x,y) + iv(x,y)$$

3. Suppose that $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$, where $z = x + iy$. Use the expressions (see Sec. 5)

$$x = \frac{z + \bar{z}}{2} \quad \text{and} \quad y = \frac{z - \bar{z}}{2i}$$

to write $f(z)$ in terms of z, and simplify the result.

Ans. $f(z) = \bar{z}^2 + 2iz$.

$z = x + iy \longrightarrow \bar{z} = x - iy$

$$\left.\begin{aligned} x = \frac{z+\bar{z}}{2} &\longrightarrow x^2 = \tfrac{1}{4}(z^2 + 2z\bar{z} + \bar{z}^2) \\ y = \frac{z-\bar{z}}{2i} &\longrightarrow -y^2 = \tfrac{1}{4}(z^2 - 2z\bar{z} + \bar{z}^2) \end{aligned}\right\} x^2 - y^2 = \tfrac{1}{2}z^2 + \tfrac{1}{2}\bar{z}^2$$

$$xy = \tfrac{1}{4i}(z^2 - \bar{z}^2) \rightarrow -2ixy = -\tfrac{1}{2}(z^2 - \bar{z}^2) = \tfrac{1}{2}\bar{z}^2 - \tfrac{1}{2}z^2$$

$$f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$$

$$= \tfrac{1}{2}z^2 + \tfrac{1}{2}\bar{z}^2 - \tfrac{1}{i}(z - \bar{z}) + i(z + \bar{z}) - \tfrac{1}{2}(z^2 - \bar{z}^2)$$

$$= \tfrac{1}{2}z^2 + \tfrac{1}{2}\bar{z}^2 + i(z - \bar{z}) + i(z + \bar{z}) + \tfrac{1}{2}\bar{z}^2 - \tfrac{1}{2}z^2$$

$$= \bar{z}^2 + 2iz$$

4. Write the function

$$f(z) = z + \frac{1}{z} \qquad (z \neq 0)$$

in the form $f(z) = u(r, \theta) + iv(r, \theta)$.

Ans. $f(z) = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$.

$z = re^{i\theta}$

$$f(z) = z + z^{-1} = re^{i\theta} + r^{-1}e^{-i\theta} = r(\cos\theta + i\sin\theta) + r^{-1}(\cos\theta - i\sin\theta)$$

$$= (r + r^{-1})\cos\theta + i(r - r^{-1})\sin\theta = u(r,\theta) + iv(r,\theta)$$

* * * * *

Chapter 2 Section 14 : 2, 3c, 4, 5 (page 44)

2. Find and sketch, showing corresponding orientations, the images of the hyperbolas

$$x^2 - y^2 = c_1 \ (c_1 < 0) \quad \text{and} \quad 2xy = c_2 \ (c_2 < 0)$$

under the transformation $w = z^2$.

3. Sketch the region onto which the sector $r \leq 1, 0 \leq \theta \leq \pi/4$ is mapped by the transformation *(a)* $w = z^2$; *(b)* $w = z^3$; *(c)* $w = z^4$.

4. Show that the lines $ay = x$ $(a \neq 0)$ are mapped onto the spirals $\rho = \exp(a\phi)$ under the transformation $w = \exp z$, where $w = \rho \exp(i\phi)$.

5. By considering the images of *horizontal* line segments, verify that the image of the rectangular region $a \leq x \leq b, c \leq y \leq d$ under the transformation $w = \exp z$ is the region $e^a \leq \rho \leq e^b, c \leq \phi \leq d$, as shown in Fig. 21 (Sec. 14).