Written Homework 6 Solutions

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Chapter 5 Section 56: 4, 6, 7 (pg 188)

4. Write $z = re^{i\theta}$, where $0 < r < 1$, in the summation formula (10), Sec. 56. Then, with the aid of the theorem in Sec. 56, show that

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{r\cos\theta - r^2}{1 - 2r\cos\theta + r^2} \quad \text{and} \quad \sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1 - 2r\cos\theta + r^2}$$

when $0 < r < 1$. (Note that these formulas are also valid when $r = 0$.)

$$(10) \qquad \sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \text{whenever} \quad |z| < 1.$$

Notice that the summation in (10) starts at $n=0$

$$\sum_{n=1}^{\infty} z^n = -1 + \sum_{n=0}^{\infty} z^n = -1 + \frac{1}{1-z} = \frac{z}{1-z}$$

$$\sum_{n=1}^{\infty} (re^{i\theta})^n = \sum_{n=1}^{\infty} z^n = \frac{z}{1-z} = \frac{re^{i\theta}}{1-re^{i\theta}}$$

$$r^{-1}e^{-i\theta}\sum_{n=1}^{\infty} r^n e^{in\theta} = \frac{1}{1-re^{i\theta}} = \frac{1}{1-r\cos n\theta - ir\sin n\theta}$$

$$r^{-1}e^{-i\theta}\sum_{n=1}^{\infty} r^n e^{in\theta} = \frac{1-r\cos n\theta + ir\sin n\theta}{(1-r\cos n\theta)^2 + r^2\sin^2 n\theta}$$

$$r^{-1}e^{-i\theta}\sum_{n=1}^{\infty} r^n e^{in\theta} = \frac{1-r\cos n\theta + ir\sin n\theta}{1-2r\cos n\theta + r^2}$$

$$\sum_{n=1}^{\infty} r^n e^{in\theta} = \frac{r\cos n\theta - r^2(\cos^2 n\theta + \sin^2 n\theta)}{1-2r\cos n\theta + r^2} + i\,\frac{r\sin n\theta}{1-2r\cos n\theta + r^2}$$

$$\sum_{n=1}^{\infty} r^n e^{in\theta} = \frac{r\cos n\theta - r^2}{1-2r\cos n\theta + r^2} + i\,\frac{r\sin n\theta}{1-2r\cos n\theta + r^2}$$

6. Show that

$$\text{if} \quad \sum_{n=1}^{\infty} z_n = S, \quad \text{then} \quad \sum_{n=1}^{\infty} \overline{z_n} = \overline{S}.$$

Spse $\sum_{n=1}^{\infty} z_n = S$. There exist sequences (x_n) and (y_n) of real numbers s.t.

$$z_n = x_n + iy_n \quad , n = 1, 2, \ldots$$

$$\sum_{n=1}^{\infty} x_n + i \sum_{n=1}^{\infty} y_n = X + iY = S$$

Then $\overline{S} = X - iY = \sum_{n=1}^{\infty} x_n - i \sum_{n=1}^{\infty} y_n$

$$= \sum_{n=1}^{\infty} (x_n - iy_n)$$

$$= \sum_{n=1}^{\infty} \overline{z_n}$$

7. Let c denote any complex number and show that

$$\text{if } \sum_{n=1}^{\infty} z_n = S, \quad \text{then } \sum_{n=1}^{\infty} c z_n = cS.$$

Theorem. *Suppose that* $z_n = x_n + i y_n$ $(n = 1, 2, \ldots)$ *and* $S = X + iY$. *Then*

$$\sum_{n=1}^{\infty} z_n = S \tag{3}$$

if and only if

$$\sum_{n=1}^{\infty} x_n = X \quad \textit{and} \quad \sum_{n=1}^{\infty} y_n = Y. \tag{4}$$

We can write each z_n as $z_n = x_n + i y_n$

If $\sum_{n=1}^{\infty} z_n = S$, then $\sum_{n=1}^{\infty} x_n = X$ and $\sum_{n=1}^{\infty} y_n = Y$

Use the analagous result known for real seq's :

$$\sum_{n=1}^{\infty} c x_n = cX \quad \text{and} \quad \sum_{n=1}^{\infty} c y_n = cY$$

Since $c z_n = c x_n + i y_n$ and $cS = cX + icY$

$$\sum_{n=1}^{\infty} c z_n = cS$$

Chapter 5 Section 59: 3, 6, 7, 10 (pgs 196-197)

3. Find the Maclaurin series expansion of the function

$$f(z) = \frac{z}{z^4+9} = \frac{z}{9} \cdot \frac{1}{1+(z^4/9)}.$$

Ans. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+2}} z^{4n+1} \quad (|z| < \sqrt{3}).$

EXAMPLE 4. Another Maclaurin series representation is

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1), \tag{6}$$

From example 4,

$$\frac{1}{1+z^4/9} = \frac{1}{1-(-z^4/9)} = \sum_{n=0}^{\infty} (-z^4/9)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n z^{4n}/9^n \quad \text{for } |-z^4/9| < 1$$

$|-z^4/9| < 1$ iff $|z|^4 < 9$ iff $|z| < \sqrt{3}$

$$\therefore f(z) = \frac{z}{9} \frac{1}{1+z^4/9} = \frac{z}{9} \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n}}{9^n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+1}}{9^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+1}}{3^{2n+2}}$$

(for $|z| < \sqrt{3}$)

6. Use representation (2), Sec. 59, for $\sin z$ to write the Maclaurin series for the function

$$f(z) = \sin(z^2),$$

and point out how it follows that

$$f^{(4n)}(0) = 0 \quad \text{and} \quad f^{(2n+1)}(0) = 0 \qquad (n = 0, 1, 2, \ldots).$$

$$\text{(2)} \qquad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \qquad (|z| < \infty).$$

$$f(z) = \sin(z^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(z^2)^{2n+1}}{(2n+1)!} \qquad (|z^2| < \infty)$$

Since $|z^2| < \infty$ iff $|z| < \infty$,

$$f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{4n+2}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2(2n+1)}}{(2n+1)!}$$

$$= z^2 - \frac{1}{3!} z^6 + \frac{1}{5!} z^{10} - \ldots \quad (|z| < \infty)$$

Compare this with the general form for the MacLaurin series for $f(z)$:

$$f(z) = f(0) + f'(0) \cdot z + \frac{1}{2!} f''(0) \cdot z^2 + \frac{1}{3!} f'''(0) \cdot z^3 + \ldots$$

Since the series derived above has only terms s.t. the exponents K on z (i.e. z^K) satisfy $K \equiv 2 \pmod 4$, $K > 0$ we must have

$$f^{(4n)}(0) = 0 \quad \text{and} \quad f^{(2n+1)}(0) = 0$$

7. Derive the Taylor series representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}} \qquad (|z-i| < \sqrt{2}).$$

Suggestion: Start by writing

$$\frac{1}{1-z} = \frac{1}{(1-i)-(z-i)} = \frac{1}{1-i} \cdot \frac{1}{1-(z-i)/(1-i)}.$$

$$\frac{1}{1-z} = \frac{1}{1-i} \cdot \frac{1}{1-(z-i)/(1-i)}$$

$$= \frac{1}{1-i} \sum_{n=0}^{\infty} \left[\frac{z-i}{1-i}\right]^n \qquad \left|\frac{z-i}{1-i}\right| < 1$$

$$= \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}} \qquad \left|\frac{z-i}{1-i}\right| < 1$$

$$\left|\frac{z-i}{1-i}\right| < 1 \quad \text{iff} \quad |z-i| < |1-i| = \sqrt{2}$$

$$\therefore \frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}} \qquad (|z-i| < \sqrt{2})$$

10. What is the largest circle within which the Maclaurin series for the function $\tanh z$ converges to $\tanh z$? Write the first two nonzero terms of that series.

From section 57, pg 190, finding the largest circle within which the Maclaurin series for a function $f(z)$ converges to $f(z)$ is to find the largest circle within which $f(z)$ is analytic.

Since $\sinh z$ and $\cosh z$ are entire, $\tanh z$ is analytic wherever $\tanh z$ is defined:

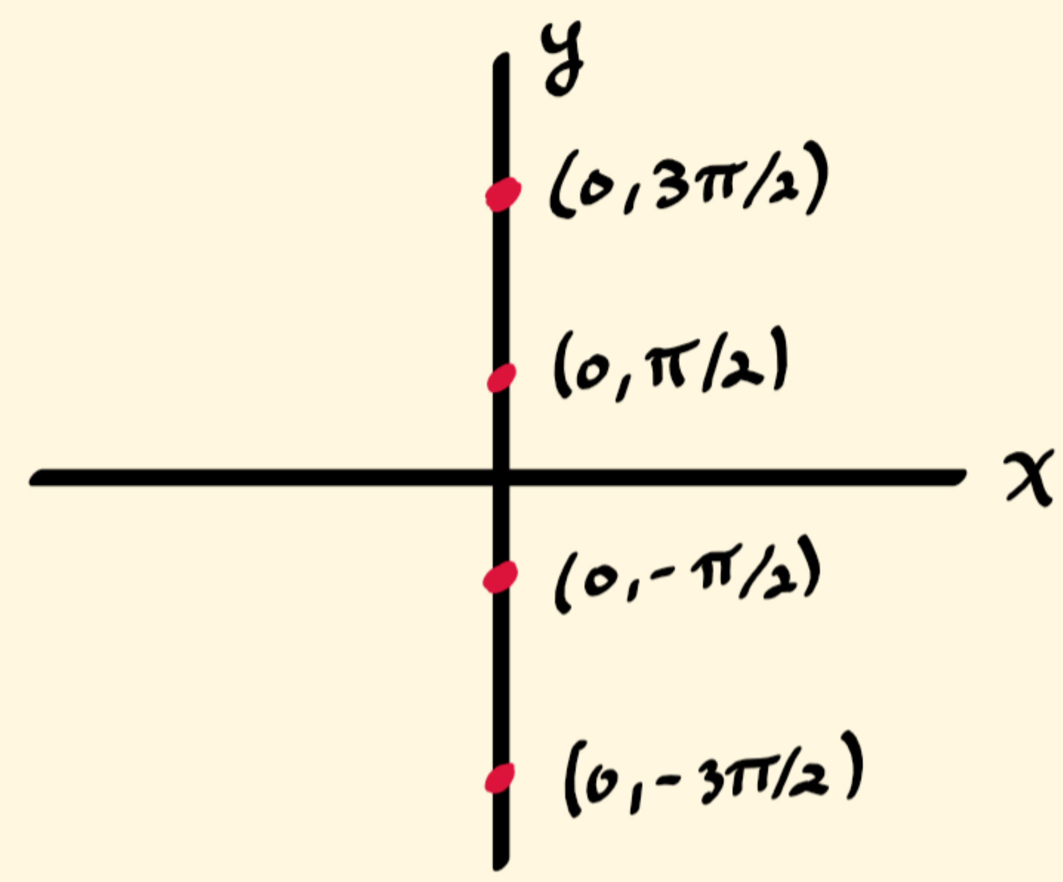
$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \cosh z \neq 0$$

Find the singularities $0 = \cosh z$

$$\begin{aligned} 0 &= \cosh z \\ &= \cosh x \cos y + i \sinh x \sin y \end{aligned}$$

$$\Rightarrow 0 = \cosh x \cos y, \quad 0 = \sinh x \sin y$$

$$(x, y) = (0, -\pi/2 + n\pi) \quad n \in \mathbb{Z}$$



The Maclaurin series for $\tanh z$ converges $\forall z$ within the circle $|z| = \pi/2$. The series does not, however, converge for all z on the circle. Specifically, convergence fails at $z = \pm i\pi/2$. →

Find the first 2 nonzero terms of the series

$$f(z) = \tanh z \qquad\qquad f(0) = 0$$
$$f'(z) = \operatorname{sech}^2 z \qquad \rightarrow \qquad f'(0) = 1$$
$$f''(z) = -2\operatorname{sech}^2 z \tanh z \qquad\qquad f''(0) = 0$$
$$f'''(z) = 4\operatorname{sech}^2 z \tanh^2 z - 2\operatorname{sech}^4 z \qquad\qquad f'''(0) = -2$$

$$f(z) = f(0) + z \cdot f'(0) + \frac{z^2}{2} \cdot f''(0) + \frac{z^3}{3!} \cdot f'''(0) + \dots$$

$$= 0 + z + 0 - 2 \cdot \frac{z^3}{6} + \dots = \boxed{z - \frac{z^3}{3} + \dots}$$

$$f^{(4K)}(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(4n+2)!}{(4n+2-4K)!} \frac{z^{4n-4K-1}}{(2n+1)!}$$

$$f^{(4K)}(0) = \text{division by } 0?$$

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Chapter 5 Section 52 : 2, 3, 4, 5, 8, 10, 11 (pg 205)

2. Derive the Laurent series representation

$$\frac{e^z}{(z+1)^2} = \frac{1}{e}\left[\sum_{n=0}^{\infty}\frac{(z+1)^n}{(n+2)!} + \frac{1}{z+1} + \frac{1}{(z+1)^2}\right] \qquad (0 < |z+1| < \infty).$$

$\dfrac{e^z}{(z+1)^2}$ is not defined if $(z+1)^2 = 0 \iff |z+1| = 0$.

So assume $|z+1| > 0$.

Replacing z with $z+1$ in $e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \qquad (|z| < \infty)$,

$$\frac{e^z}{(z+1)^2} = \frac{1}{e}\frac{e^{z+1}}{(z+1)^2} = \frac{1}{e}\sum_{n=0}^{\infty}\frac{(z+1)^{n-2}}{n!} \qquad (|z+1| < \infty)$$

$$= \frac{1}{e}\sum_{k=-2}^{\infty}\frac{(z+1)^k}{(k+2)!} \qquad (k := n-2)$$

$$= \frac{1}{e}\left[\frac{1}{(z+1)^2} + \frac{1}{z+1} + \sum_{n=0}^{\infty}\frac{(z+1)^n}{(n+2)!}\right] \qquad (n := k)$$

for $0 < |z+1| < \infty$

3. Find a representation for the function

$$f(z) = \frac{1}{1+z} = \frac{1}{z} \cdot \frac{1}{1+(1/z)}$$

in negative powers of z that is valid when $1 < |z| < \infty$.

Ans. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^n}$.

Replacing z with $-1/z$ in $\displaystyle\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1)$,

$$\frac{1}{1+1/z} = \sum_{n=0}^{\infty} \left[-\frac{1}{z}\right]^n \quad (|-1/z| < 1)$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{z^n} \quad (|z| > 1)$$

Then $\displaystyle f(z) = \frac{1}{1+z} = \frac{1}{z} \frac{1}{1+1/z}$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{z^{n+1}}$$

$$= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{z^k} \quad (k := n+1)$$

$$= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{z^k}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{z^n} \quad (n := k)$$

4. Give two Laurent series expansions in powers of z for the function

$$f(z) = \frac{1}{z^2(1-z)},$$

and specify the regions in which those expansions are valid.

Ans. $\sum_{n=0}^{\infty} z^n + \frac{1}{z} + \frac{1}{z^2} \quad (0 < |z| < 1); \quad -\sum_{n=3}^{\infty} \frac{1}{z^n} \quad (1 < |z| < \infty).$

(a) $\underbrace{\frac{1}{z^2}}_{(|z|>0)} \frac{1}{1-z} = \frac{1}{z^2} \sum_{n=0}^{\infty} z^n \qquad (|z|<1)$

$$= \sum_{n=0}^{\infty} z^{n-2}$$

$$= \sum_{n=-2}^{\infty} z^n \qquad (n := n-2)$$

$$= \frac{1}{z^2} + \frac{1}{z} + \sum_{n=0}^{\infty} z^n,$$

where we've collected the restrictions $0 < |z| < 1$.

(b) $\underbrace{\frac{1}{z^2}}_{|z|>0} \frac{1}{1-z} = -\frac{1}{z^3} \frac{1}{1 - 1/z}$

$$= -\frac{1}{z^3} \sum_{n=0}^{\infty} (1/z)^n \qquad |1/z| < 1 \iff |z| > 1$$

$$= -\sum_{n=0}^{\infty} \frac{1}{z^{n+3}}$$

$$= -\sum_{n=3}^{\infty} \frac{1}{z^n} \qquad (n := n+3)$$

5. Represent the function

$$f(z) = \frac{z+1}{z-1}$$

(a) by its Maclaurin series, and state where the representation is valid;
(b) by its Laurent series in the domain $1 < |z| < \infty$.

Ans. *(a)* $-1 - 2\sum_{n=1}^{\infty} z^n \quad (|z| < 1)$; *(b)* $1 + 2\sum_{n=1}^{\infty} \frac{1}{z^n}$.

(a)
$$f(z) = \frac{z}{z-1} + \frac{1}{z-1}$$
$$= -z\frac{1}{1-z} - \frac{1}{1-z}$$
$$= -\sum_{n=0}^{\infty} z^{n+1} - \sum_{n=0}^{\infty} z^n \qquad (|z|<1)$$
$$= -\sum_{n=1}^{\infty} z^n - \left(1 + \sum_{n=1}^{\infty} z^n\right)$$
$$= -1 - 2\sum_{n=1}^{\infty} z^n$$

(b)
$$f(z) = \frac{1 + 1/z}{1 - 1/z} = \frac{1}{z}\frac{1}{1-1/z} + \frac{1}{1-1/z} \qquad (|z|>0)$$
$$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n \qquad (|1/z| < 1 \Leftrightarrow |z| > 1)$$
$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{1}{z^n}$$
$$= \sum_{n=1}^{\infty} \frac{1}{z^n} + \left(1 + \sum_{n=1}^{\infty}\frac{1}{z^n}\right) = 1 + 2\sum_{n=1}^{\infty}\frac{1}{z^n}$$

8. *(a)* Let a denote a real number, where $-1 < a < 1$, and derive the Laurent series representation

$$\frac{a}{z-a} = \sum_{n=1}^{\infty} \frac{a^n}{z^n} \qquad (|a| < |z| < \infty).$$

(b) After writing $z = e^{i\theta}$ in the equation obtained in part *(a)*, equate real parts and then imaginary parts on each side of the result to derive the summation formulas

$$\sum_{n=1}^{\infty} a^n \cos n\theta = \frac{a\cos\theta - a^2}{1 - 2a\cos\theta + a^2} \quad \text{and} \quad \sum_{n=1}^{\infty} a^n \sin n\theta = \frac{a\sin\theta}{1 - 2a\cos\theta + a^2},$$

where $-1 < a < 1$. (Compare with Exercise 4, Sec. 56.)

(a) $$\frac{a}{z-a} = \frac{a/z}{1-a/z} = \frac{a}{z}\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n = \sum_{n=1}^{\infty}\frac{a^n}{z^n}$$

$$(|a/z| < 1 \Leftrightarrow |a| < |z|)$$

(b) By part (a), with $z = e^{i\theta}$:

$$\frac{a}{e^{i\theta}-a} = \sum_{n=1}^{\infty} a^n e^{-in\theta} = \sum_{n=1}^{\infty} a^n \cos n\theta - i\sum_{n=1}^{\infty} a^n \sin n\theta$$

By algebraic manipulation:

$$\frac{a}{e^{i\theta}-a} = \frac{a}{\cos\theta + i\sin\theta - a} \cdot \frac{-a+\cos\theta - i\sin\theta}{-a+\cos\theta - i\sin\theta}$$

$$= \frac{-a^2 + a\cos\theta}{a^2 - 2a\cos\theta + 1} - i\,\frac{a\sin\theta}{a^2 - 2a\cos\theta + 1}$$

Equating real and imaginary components:

$$\sum_{n=1}^{\infty} a^n \cos n\theta = \frac{-a^2 + a\cos\theta}{a^2 - 2a\cos\theta + 1}$$

$$\sum_{n=1}^{\infty} a^n \sin n\theta = \frac{a\sin\theta}{a^2 - 2a\cos\theta + 1}$$

10. *(a)* Let z be any complex number, and let C denote the unit circle

$$w = e^{i\phi} \qquad (-\pi \le \phi \le \pi)$$

in the w plane. Then use that contour in expression (5), Sec. 60, for the coefficients in a Laurent series, adapted to such series about the origin in the w plane, to show that

$$\exp\left[\frac{z}{2}\left(w - \frac{1}{w}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z) w^n \qquad (0 < |w| < \infty)$$

where

$$J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[-i(n\phi - z\sin\phi)]\, d\phi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

(b) With the aid of Exercise 5, Sec. 38, regarding certain definite integrals of even and odd complex-valued functions of a real variable, show that the coefficients in part *(a)* here can be written[†]

$$J_n(z) = \frac{1}{\pi} \int_0^{\pi} \cos(n\phi - z\sin\phi)\, d\phi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

11. *(a)* Let $f(z)$ denote a function which is analytic in some annular domain about the origin that includes the unit circle $z = e^{i\phi}$ $(-\pi \le \phi \le \pi)$. By taking that circle as the path of integration in expressions (2) and (3), Sec. 60, for the coefficients a_n and b_n in a Laurent series in powers of z, show that

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})\,d\phi + \frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(e^{i\phi})\left[\left(\frac{z}{e^{i\phi}}\right)^n + \left(\frac{e^{i\phi}}{z}\right)^n\right] d\phi$$

when z is any point in the annular domain.

(b) Write $u(\theta) = \text{Re}[f(e^{i\theta})]$ and show how it follows from the expansion in part *(a)* that

$$u(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\phi)\,d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} u(\phi)\cos[n(\theta-\phi)]\,d\phi.$$

This is one form of the *Fourier series* expansion of the real-valued function $u(\theta)$ on the interval $-\pi \le \theta \le \pi$. The restriction on $u(\theta)$ is more severe than is necessary in order for it to be represented by a Fourier series.*

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Chapter 5 Section 66 : 2, 3, 4, 6, 7, 9 (pg 219)

2. By substituting $1/(1-z)$ for z in the expansion

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty}(n+1)\,z^n \qquad (|z|<1),$$

found in Exercise 1, derive the Laurent series representation

$$\frac{1}{z^2} = \sum_{n=2}^{\infty}\frac{(-1)^n(n-1)}{(z-1)^n} \qquad (1<|z-1|<\infty).$$

(Compare with Example 2, Sec. 65.)

For $|z|>0$,

$$\frac{1}{\left(1-\frac{1}{1-z}\right)^2} = \frac{1}{\left(\frac{1-z-1}{1-z}\right)^2} = \frac{(1-z)^2}{z^2}$$

For $\frac{1}{|1-z|}<1 \iff 1<|z-1|$

$$\frac{1}{\left(1-\frac{1}{1-z}\right)^2} = \sum_{n=0}^{\infty}(n+1)\left(\frac{1}{1-z}\right)^2 = \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{(z-1)^n}$$

If $|z-1|>1$, then $|z|>0$. So for $1<|z-1|<\infty$,

$$\frac{(z-1)^2}{z^2} = \frac{(1-z)^2}{z^2} = \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{(z-1)^n}$$

$$\frac{1}{z^2} = \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{(z-1)^{n+2}} = \sum_{n=2}^{\infty}\frac{(-1)^n(n-1)}{(z-1)^{n+2}}$$

3. Find the Taylor series for the function

$$\frac{1}{z} = \frac{1}{2+(z-2)} = \frac{1}{2} \cdot \frac{1}{1+(z-2)/2}$$

about the point $z_0 = 2$. Then, by differentiating that series term by term, show that

$$\frac{1}{z^2} = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n \qquad (|z-2|<2).$$

$$\frac{1}{z} = \frac{1}{2} \cdot \frac{1}{1+(z-2)/2}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(z-2)^n}{2^n}(-1)^n \quad \left(\left|-\frac{z-2}{2}\right| < 1 \iff |z-2|<2\right)$$

$$\longrightarrow \quad -\frac{1}{z} = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{(z-2)^n}{2^n}(-1)^n$$

Then for $|z-2|<2$,

$$\frac{1}{z^2} = \frac{d}{dz}\left[-\frac{1}{z}\right] = -\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n n\frac{(z-2)^{n-1}}{2^n}$$

$$= \frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{z-2}{2}\right)^{n-1}$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-2}{2}\right)^n$$

4. With the aid of series, show that the function f defined by means of the equations

$$f(z) = \begin{cases} (\sin z)/z & \text{when } z \neq 0, \\ 1 & \text{when } z = 0 \end{cases}$$

is entire. Use that result to establish the limit

$$\lim_{z \to 0} \frac{\sin z}{z} = 1.$$

(See Example 1, Sec. 65.)

6. In the w plane, integrate the Taylor series expansion (see Example 4, Sec. 59)

$$\frac{1}{w} = \sum_{n=0}^{\infty} (-1)^n (w-1)^n \qquad (|w-1| < 1)$$

along a contour interior to the circle of convergence from $w = 1$ to $w = z$ to obtain the representation

$$\operatorname{Log} z = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (z-1)^n \qquad (|z-1| < 1).$$

Let C be a contour in the w-plane that goes from $w=1$ to $w=z$ within the circle $|w-1|<1$

$$\operatorname{Log} z = \int_1^z \frac{1}{w}\,dw \quad \left(= \int_C \frac{1}{w}\,dw\right)$$

$$= \int_1^z \sum_{n=0}^{\infty} (-1)^n (w-1)^n \, dw \qquad (|w-1|<1)$$

$$= \sum_{n=0}^{\infty} (-1)^n \int_1^z (w-1)^n \, dw$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} (w-1)^{n+1} \Big|_{w=1}^{w=z}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} (z-1)^{n+1} \qquad \left(\begin{array}{c} |z-1|<1 \\ \text{by assumption} \end{array}\right)$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} (z-1)^n$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (z-1)^n$$

7. Use the result in Exercise 6 to show that if

$$f(z) = \frac{\text{Log } z}{z - 1} \qquad \text{when } z \neq 1$$

?

and $f(1) = 1$, then f is analytic throughout the domain

$$0 < |z| < \infty, \ -\pi < \text{Arg } z < \pi.$$

9. Suppose that a function $f(z)$ has a power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

inside some circle $|z - z_0| = R$. Use Theorem 2 in Sec. 65, regarding term by term differentiation of such a series, and mathematical induction to show that

$$f^{(n)}(z) = \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} a_{n+k} (z - z_0)^k \qquad (n = 0, 1, 2, \ldots)$$

when $|z - z_0| < R$. Then, by setting $z = z_0$, show that the coefficients a_n $(n = 0, 1, 2, \ldots)$ are the coefficients in the Taylor series for f about z_0. Thus give an alternative proof of Theorem 1 in Sec. 66.

Theorem 2. *The power series (1) can be differentiated term by term. That is, at each point z interior to the circle of convergence of that series,*

$$S'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}. \tag{6}$$

Prove the identity holds in the base case $(n=0)$:

$$f^{(0)}(z) = f(z) = \sum_{K=0}^{\infty} a_n (z - z_0)^K$$

$$= \sum_{K=0}^{\infty} \frac{(0+K)!}{K!} a_{0+K} (z - z_0)^K$$

Assume the identity holds for some $n \geq 0$,

$$f^{(n+1)}(z) = \frac{d}{dz}\left[f^{(n)}(z)\right]$$

$$= \sum_{K=0}^{\infty} K \frac{(n+K)!}{K} a_{n+K} (z - z_0)^{K-1}$$

$(K := K-1)$

$$= \sum_{K=0}^{\infty} \frac{(n+1+K)!}{K!} a_{n+1+K} (z - z_0)^K$$

* * * * *

Chapter 5 Section 67 : 1, 3, 4, 5, 7 (pg 225)

1. Use multiplication of series to show that

$$\frac{e^z}{z(z^2+1)} = \frac{1}{z} + 1 - \frac{1}{2}z - \frac{5}{6}z^2 + \cdots \qquad (0 < |z| < 1).$$

$$\frac{e^z}{z(z^2+1)} = \frac{1}{1-(-z^2)}\frac{1}{z}e^z$$

$|z|>0$

$|-z^2|<1 \iff |z|<1$

$$= \left(1 - z^2 + z^4 - z^6 + \dots\right)\frac{1}{z}\left(1 + z + \frac{z^2}{2} + \dots\right)$$

$$= \left(\frac{1}{z} - z + z^3 - z^5 + \dots\right)\left(1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \dots\right)$$

$$= \frac{1}{z} + 1 + \left(-1 + \frac{1}{2}\right)z + \left(\frac{1}{3!} - 1\right)z^2 + \dots$$

$$= \frac{1}{z} + 1 - \frac{1}{2}z - \frac{5}{6}z^2 + \dots \qquad (0<|z|<1)$$

3. Use division to obtain the Laurent series representation

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \frac{1}{12}z - \frac{1}{720}z^3 + \cdots \qquad (0 < |z| < 2\pi).$$

$$\frac{1}{e^z - 1} = \frac{1}{(1 + z + z^2/2 + z^3/3! + \ldots) - 1}$$

$$= \frac{1}{z} \frac{1}{1 + z/2 + z^2/3! + \ldots}$$

$$\overset{*}{=} \frac{1}{z}\left(1 - \frac{1}{2}z + \frac{1}{12}z^2 - \frac{1}{720}z^4 + \ldots\right)$$

$$= \frac{1}{z} - \frac{1}{2} + \frac{1}{12}z - \frac{1}{720}z^3 + \ldots$$

Since $e^z - 1 = 0$ at $z = 0 + 2n\pi i$, this representation can be defined on the annular region $\underline{0 < |z| < 2\pi}$ between the singularities $z = 0$ and $z = 2\pi$.

*

$$1 - \frac{1}{2}z + \frac{1}{12}z^2 + \frac{1}{720}z^4 + \ldots$$

$$1 + \frac{1}{2}z + \frac{1}{3!}z^2 + \frac{1}{4!}z^3 + \ldots \;\Big|\; 1$$

$$-\left(1 + \frac{1}{2}z + \frac{1}{3!}z^2 + \frac{1}{4!}z^3 + \ldots\right)$$

$$-\frac{1}{2}z - \frac{1}{3!}z^2 - \frac{1}{4!}z^3 - \frac{1}{5!}z^5 - \ldots$$

$$-\left(-\frac{1}{2}z - \frac{1}{4}z^2 - \frac{1}{12}z^3 - \frac{1}{48}z^4 - \ldots\right)$$

$$\frac{1}{12}z^2 + \frac{1}{24}z^3 + \frac{9}{720}z^4 + \ldots$$

$$-\left(\frac{1}{12}z^2 + \frac{1}{24}z^3 + \frac{10}{720}z^4 + \ldots\right)$$

$$-\frac{1}{720}z^4 - \ldots$$

4. Use the expansion

$$\frac{1}{z^2 \sinh z} = \frac{1}{z^3} - \frac{1}{6} \cdot \frac{1}{z} + \frac{7}{360} z + \cdots \qquad (0 < |z| < \pi)$$

in Example 2, Sec. 67, and the method illustrated in Example 1, Sec. 62, to show that

$$\int_C \frac{dz}{z^2 \sinh z} = -\frac{\pi i}{3},$$

when C is the positively oriented unit circle $|z| = 1$.

5. Follow these steps, which illustrate an alternative to straightforward division, to obtain representation (8) in Example 2, Sec. 67.

(a) Write

$$\frac{1}{1 + z^2/3! + z^4/5! + \cdots} = d_0 + d_1 z + d_2 z^2 + d_3 z^3 + d_4 z^4 + \cdots,$$

where the coefficients in the power series on the right are to be determined by multiplying the two series in the equation

$$1 = \left(1 + \frac{1}{3!}z^2 + \frac{1}{5!}z^4 + \cdots\right)(d_0 + d_1 z + d_2 z^2 + d_3 z^3 + d_4 z^4 + \cdots).$$

Perform this multiplication to show that

$$(d_0 - 1) + d_1 z + \left(d_2 + \frac{1}{3!}d_0\right)z^2 + \left(d_3 + \frac{1}{3!}d_1\right)z^3 + \left(d_4 + \frac{1}{3!}d_2 + \frac{1}{5!}d_0\right)z^4 + \cdots = 0$$

when $|z| < \pi$.

(b) By setting the coefficients in the last series in part *(a)* equal to zero, find the values of d_0, d_1, d_2, d_3, and d_4. With these values, the first equation in part *(a)* becomes equation (8), Sec. 67.

(a)

$$1 = \left(1 + \frac{1}{3!}z^2 + \frac{1}{5!}z^4 + \ldots\right)\left(d_0 + d_1 z + d_2 z^2 + d_3 z^3 + \ldots\right)$$

$$= d_0 + d_1 z + \left(\frac{1}{3!} + d_2\right)z^2 + \left(d_3 + \frac{1}{3!}d_1\right)z^3 + \left(d_4 + \frac{1}{3!}d_2 + \frac{1}{5!}d_0\right)z^4 + \ldots$$

$$0 = (d_0 - 1) + d_1 z + \left(\frac{1}{3!} + d_2\right)z^2 + \left(d_3 + \frac{1}{3!}d_1\right)z^3 + \left(d_4 + \frac{1}{3!}d_2 + \frac{1}{5!}d_0\right)z^4 + \ldots$$

(b) $d_0 = 1$ $\quad d_1 = 0$ $\quad d_2 = -\frac{1}{6}$ $\quad d_3 = 0$ $\quad d_4 = \frac{7}{360}$

$$\frac{1}{1 + z^2/3! + z^4/5! + \ldots} = 1 - \frac{1}{6}z^2 + \frac{7}{360}z^4 + \ldots$$

7. Let $f(z)$ be an entire function that is represented by a series of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \qquad (|z| < \infty).$$

(a) By differentiating the composite function $g(z) = f[f(z)]$ successively, find the first three nonzero terms in the Maclaurin series for $g(z)$ and thus show that

$$f[f(z)] = z + 2a_2 z^2 + 2(a_2^2 + a_3)z^3 + \cdots \qquad (|z| < \infty).$$

(b) Obtain the result in part *(a)* in a *formal* manner by writing

$$f[f(z)] = f(z) + a_2[f(z)]^2 + a_3[f(z)]^3 + \cdots,$$

replacing $f(z)$ on the right-hand side here by its series representation, and then collecting terms in like powers of z.

(c) By applying the result in part *(a)* to the function $f(z) = \sin z$, show that

$$\sin(\sin z) = z - \frac{1}{3}z^3 + \cdots \qquad (|z| < \infty).$$

(a) $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots \qquad (|z|<\infty)$

$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \ldots$

$f''(z) = 2a_2 + 6a_3 z + 12a_4 z^2 + \ldots$

$f'''(z) = 6a_3 + 24a_4 z + \ldots$

$g(z) = f[f(z)]$

$g'(z) = f'[f(z)]f'(z)$

$g''(z) = f''[f(z)](f'(z))^2 + f'[f(z)]f''(z)$

$g'''(z) = f'''[f(z)](f'(z))^3 + 2f''[f(z)]f'(z)f''(z)$
$\quad + f''[f(z)]f'(z)f''(z) + f'[f(z)]f'''(z)$

$$g(0) = f[f(0)] = f[0] = 0$$

$$g'(0) = f'[f(0)]f'(0) = (f'(0))^2 = 1^2 = 1$$

$$g''(0) = f''[f(0)](f'(0))^2 + f'[f(0)]f''(0)$$

$$= 2f''(0) = 4a_2$$

$$g'''(0) = f'''[f(0)](f'(0))^3 + 2f''[f(0)]f'(0)f''(0)$$

$$+ f''[f(0)]f'(0)f''(0) + f'[f(0)]f'''(0)$$

$$= f'''(0) + 2\cdot(f''(0))^2 + (f''(0))^2 + f'''(0)$$

$$= 12a_3 + 3\cdot(2a_2)^2 = 12a_3 + 12a_2^2$$

$$f[f(z)] = g(z) = g(0) + zg'(0) + \frac{1}{2}z^2 g''(0) + \frac{1}{3!}z^3 g'''(0) + \ldots$$

$$= 0 + z + 2a_2 z^2 + \frac{12}{6}(a_2^2 + a_3)z^3 + \ldots$$

$$= \boxed{z + 2a_2 z^2 + 2(a_2^2 + a_3)z^3 + \ldots}$$

(b) $f[f(z)] = f(z) + a_2[f(z)]^2 + a_3[f(z)]^3 + \dots$

$= (z + a_2 z^2 + a_3 z^3 + \dots) + a_2(z + a_2 z^2 + a_3 z^3 + \dots)^2$

$+ a_3(z + a_2 z^2 + a_3 z^3 + \dots)^3 + \mathcal{O}(z^4) \leftarrow \begin{pmatrix}\text{all exponents} \\ \text{4 or higher}\end{pmatrix}$

$= (z + a_2 z^2 + a_3 z^3 + \dots) + (a_2 z^2 + 2a_2^2 z^3 + \dots)$

$+ (a_3 z^3 + \dots) + \mathcal{O}(z^4)$

$= z + 2a_2 z^2 + 2(a_2^2 + a_3) z^3 + \dots$

(c) $z + a_2 z^2 + a_3 z^3 + \dots = f(z) = \sin z = z - \frac{1}{3!} z^3 + \dots$

$\rightarrow a_2 = 0, \quad a_3 = -\frac{1}{3!} = -\frac{1}{6}$

$\sin[\sin z] = f[f(z)]$

$= z + 2a_2 z^2 + 2(a_2^2 + a_3) z^3 + \dots$

$= z + 2 \cdot 0 \cdot z^2 + 2(0^2 - \frac{1}{6}) z^3 + \dots$

$= \boxed{z - \frac{1}{3} z^3 + \dots \quad (|z| < \infty)}$