Written Homework 7 Solutions

pg. 267: # 1,3,5,7,9
pg. 275: # 1,3,5,7,9

Part 2: Chapter 7 Exercises

Chapter 7 Section 79: 1,3,5,7,9 (pg. 267)

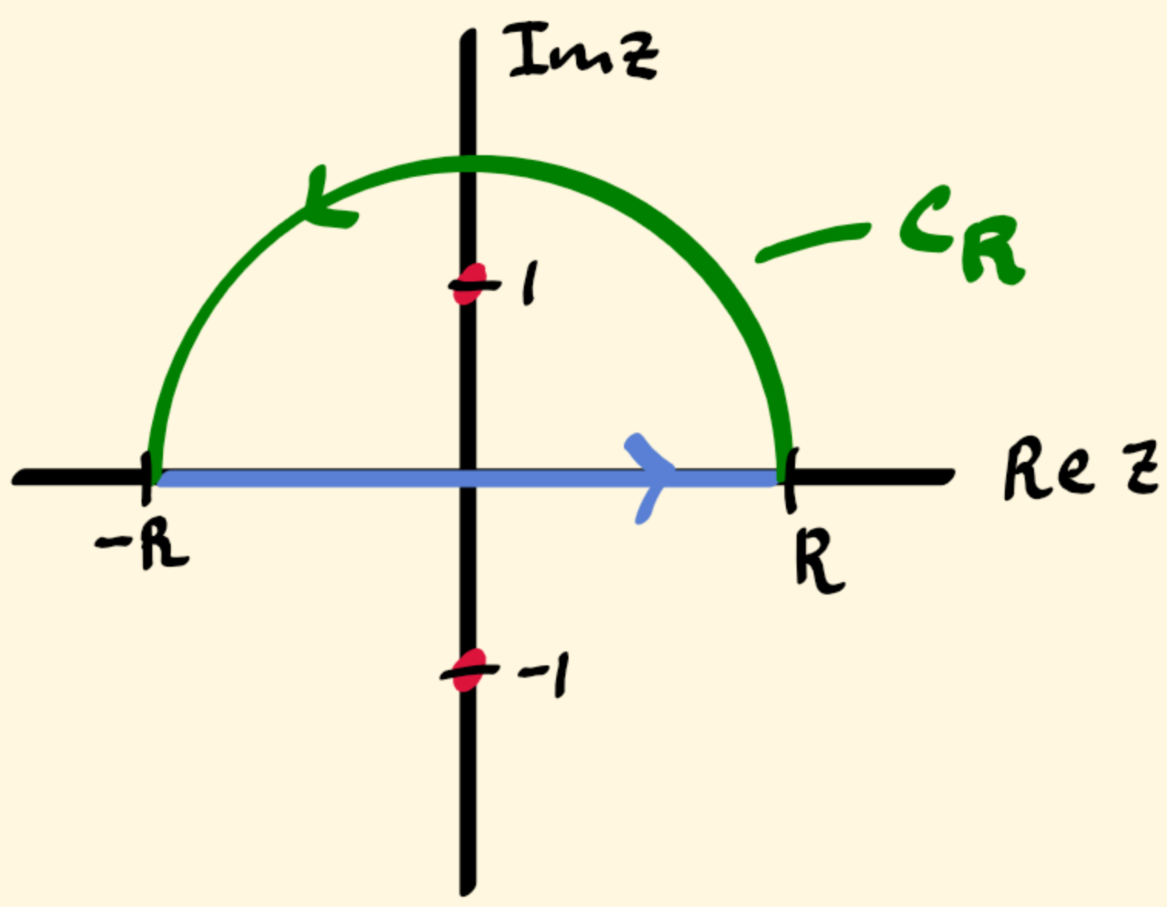
Use residues to evaluate the improper integrals in Exercises 1 through 5.

1. $\displaystyle\int_0^\infty \frac{dx}{x^2+1}$.

Ans. $\pi/2$.

$f(z) = \dfrac{1}{z^2+1}$ has isolated singularities at $z = \pm i$

when $R > 1$, the pt $z = i$ lies in the interior of the semicircular region bounded by the segment $z = x$ $(-R \leq x \leq R)$ of the real axis and the upper half C_R of the circle $|z| = R$.



Integrating ccw around the boundary:

$$\int_{-R}^{R} f(z)\,dz + \int_{C_R} f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz$$

$$= 2\pi i \cdot \operatorname*{Res}_{z=i} f(z) \quad \text{(Cauchy Residue Thm).}$$

Apply Thm 2 of Section 76 with $p = 1$ and $q = z^2+1$ ($\Rightarrow q' = 2z$) to get $\underset{z=i}{\text{Res}}\, f(z) = \frac{1}{2i}$

$$\Rightarrow \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \cdot \underset{z=i}{\text{Res}}\, f(z) = \pi$$

Next show $\int_{C_R} f(z)\,dz \longrightarrow 0$ as $R \to \infty$. May use:

Theorem. *Let C denote a contour of length L, and suppose that a function $f(z)$ is piecewise continuous on C. If M is a nonnegative constant such that*

(4) $$|f(z)| \le M$$

for all points z on C at which $f(z)$ is defined, then

(5) $$\left| \int_C f(z)\,dz \right| \le ML.$$

(pg 138)

For C_R, $L = \pi R$. To find M,

$$|z^2+1| \geq \big|\,|z^2| - |1|\,\big| \quad \text{(reverse } \Delta\text{-inequality)}$$
$$= |z|^2 - 1$$
$$= R^2 - 1 \quad (|z| = R \text{ for } z \text{ on } C_R)$$

$$|f(z)| = \frac{1}{|z^2+1|} \le \frac{1}{R^2-1} =: M$$

$$\Rightarrow \left| \int_{C_R} f(z)\,dz \right| \le ML = \frac{\pi}{R^2-1}$$

$$\Rightarrow \left| \int_{C_R} f(z)\,dz \right| \longrightarrow 0 \text{ as } R \to \infty$$

$$\Rightarrow \int_{C_R} f(z)\,dz \longrightarrow 0 \text{ as } R \to \infty$$

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = \lim_{R\to\infty} \left[\pi - \int_{C_R} f(z)\,dz \right]$$

$$^{*}\,P.V. \int_{-\infty}^{\infty} f(x)\,dx = \pi - \lim_{R\to\infty} \int_{C_R} f(z)\,dz$$

$$P.V. \int_{-\infty}^{\infty} f(x)\,dx = \pi$$

*equation 3 pg. 262

Since $f(x)$ is an even function $(f(x) = f(-x))$,

$$\int_{0}^{\infty} \frac{1}{x^2+1}\,dx = \frac{1}{2}\left[P.V. \int_{-\infty}^{\infty} f(x)\,dx \right] \quad \left(\begin{matrix}\text{eq. 7}\\ \text{pg 263}\end{matrix}\right)$$

$$= \boxed{\frac{\pi}{2}}$$

3. $\displaystyle\int_0^\infty \frac{dx}{x^4+1}.$

Ans. $\pi/(2\sqrt{2})$.

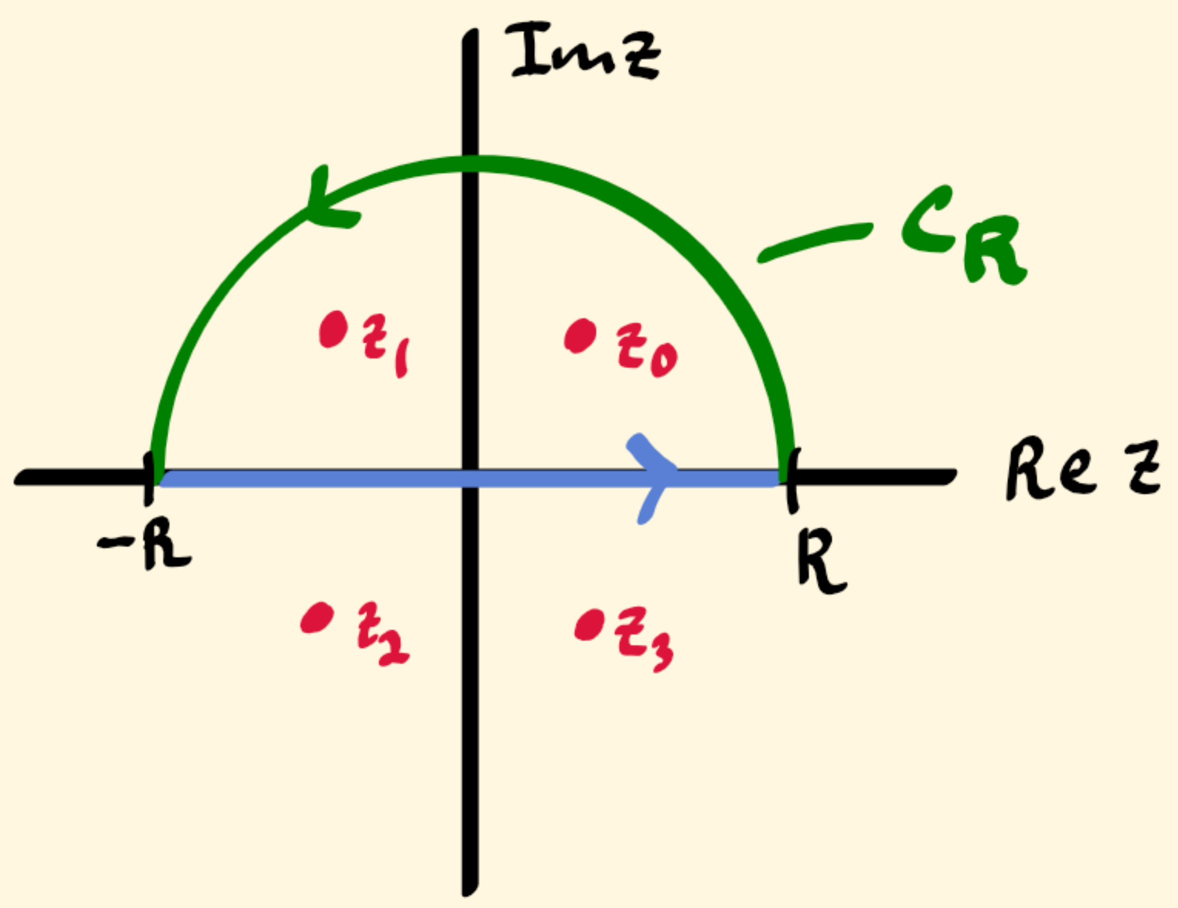
$f(z) = \dfrac{1}{z^4+1}$ has isolated singularities at the zeros of z^4+1.

$$0 = z^4 + 1$$

$$z^4 = -1 = e^{i(\pi + 2k\pi)}$$

$$z_k = e^{i(\pi/4 + n\pi/2)}$$

$(k = 0, 1, 2, 3)$



Using a process similar to Exercise 1:

$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \cdot \left[\operatorname*{Res}_{z=z_0} f(z) + \operatorname*{Res}_{z=z_1} f(z)\right]$$

$$= 2\pi i \cdot \left[\frac{1}{4z_0^3} + \frac{1}{4z_1^3}\right]$$

$$= \frac{\pi i}{2}\left[\left(e^{-i\pi/4}\right)^3 + \left(e^{-i3\pi/4}\right)^3\right]$$

$$= \frac{\pi i}{2}\left[e^{-i3\pi/4} + e^{-i9\pi/4}\right] = \frac{\pi i}{4}\left[-\sqrt{2} - \sqrt{2}i + \sqrt{2} - \sqrt{2}i\right]$$

$$= \frac{\pi}{\sqrt{2}}$$

Since $|z^4+1| \geq |z|^4 - 1 = R^4 - 1$ for z on C_R,
$|f(z)| \leq \dfrac{1}{R^4-1}$ for z on C_R.

$$\Rightarrow \left|\int_{C_R} f(z)\,dz\right| \leq ML = \frac{\pi R}{R^2-1} \longrightarrow 0 \text{ as } R \to \infty$$

$$\Rightarrow \text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{\sqrt{2}}$$

$$\Rightarrow \int_0^\infty \frac{1}{x^4+1}\,dx = \int_0^\infty f(x)\,dx = \boxed{\frac{\pi}{2\sqrt{2}}}$$

5. $\displaystyle\int_0^\infty \frac{x^2\,dx}{(x^2+9)(x^2+4)^2}$.

Ans. $\pi/200$.

Let $f(z) = \dfrac{z^2}{(z^2+9)(z^2+4)^2}$

$f(z)$ has isolated singularities at $z_k = \pm 3i, \pm 2i$. Of these, $z_k = 3i, 2i$ are enclosed by the semi circular region $z = x$ $(-R \le x \le R)$ and C_R $(R > 3)$.

Similar to Exercises 1, 3:

$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \cdot \left[\operatorname*{Res}_{z=2i} f(z) + \operatorname*{Res}_{z=3i} f(z)\right]$$

(pg 244)

Theorem. *An isolated singular point z_0 of a function f is a pole of order m if and only if $f(z)$ can be written in the form*

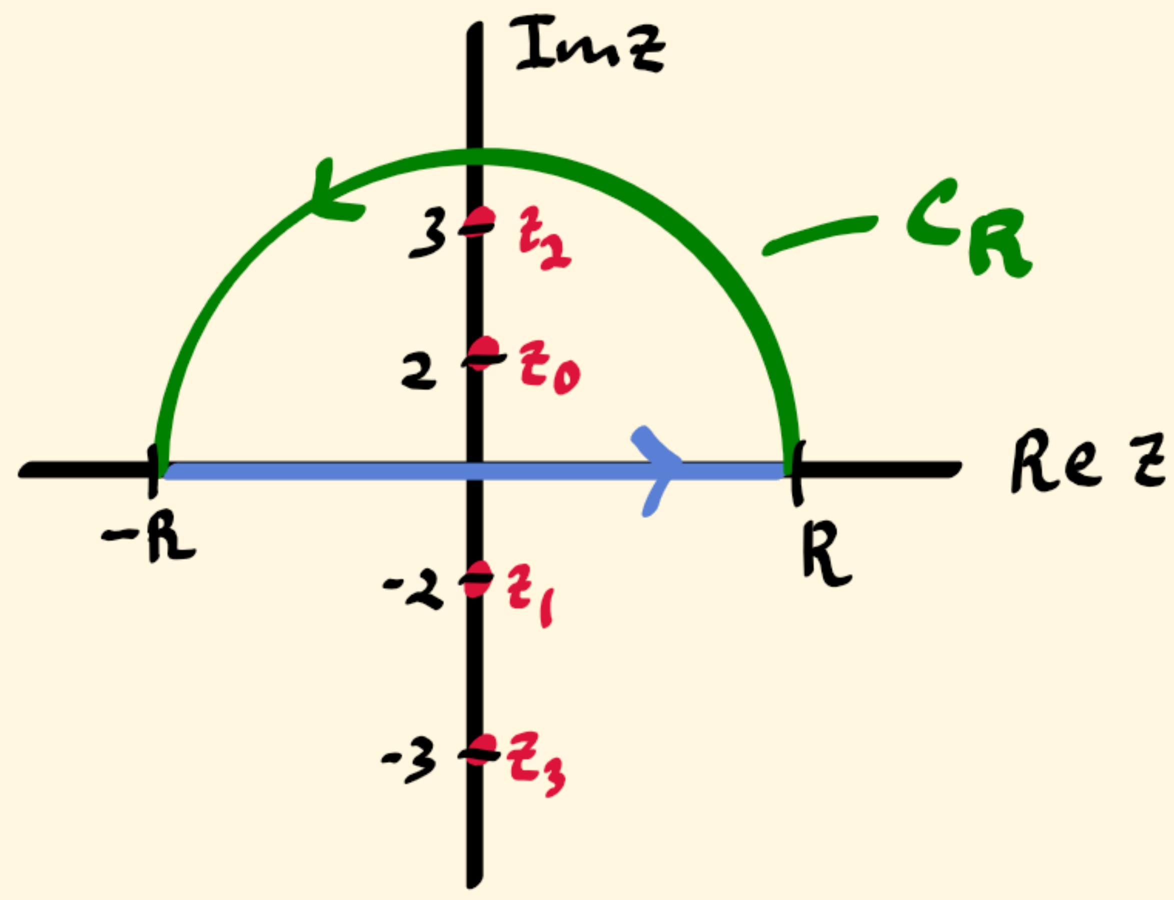
(1) $$f(z) = \frac{\phi(z)}{(z-z_0)^m},$$

where $\phi(z)$ is analytic and nonzero at z_0. Moreover,

(2) $$\operatorname*{Res}_{z=z_0} f(z) = \phi(z_0) \quad \text{if } m = 1$$

and

(3) $$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \quad \text{if } m \ge 2.$$



Let $\phi(z) = \dfrac{z^2}{(z+3i)(z^2+4)^2} \Rightarrow f(z) = \dfrac{\phi(z)}{z-3i}$

By the theorem, $\operatorname*{Res}_{z=3i} f(z) = \phi(3i) = \underline{\dfrac{3i}{50}}$

Let $\phi(z) = \dfrac{z^2}{(z^2+9)(z+2i)^2} \Rightarrow f(z) = \dfrac{\phi(z)}{(z-2i)^2}$

$$\phi'(z) = \frac{2z(z^3-18i)}{(z+2i)^3(z^2+9)^2}$$

By the Theorem, $\operatorname*{Res}_{z=2i} f(z) = \phi'(2i) = -\dfrac{13i}{200}$

Now that we know the residues:

$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \cdot \left[\frac{3i}{50} - \frac{13i}{200}\right] = \frac{\pi}{100}$$

Next show that $\int_{C_R} f(z)\,dz \longrightarrow 0$ as $R \to \infty$

For z on C_R,

$$|f(z)| = \frac{|z^2|}{|z^2+9|\,|z^2+4|^2} \leq \frac{R^2}{(R^2-9)(R^2-4)^2}$$

$$\left|\int_{C_R} f(z)\,dz\right| \leq \pi R \frac{R^2}{(R^2-9)(R^2-4)^2} \to 0 \quad \text{as } R \to \infty$$

$$\Rightarrow \int_{C_R} f(z)\,dz \longrightarrow 0 \quad \text{as } R \to \infty$$

$$\therefore \int_0^\infty \frac{x^2}{(x^2+9)(x^2+4)^2}\,dx = \frac{1}{2}\,\text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx = \boxed{\frac{\pi}{200}}$$

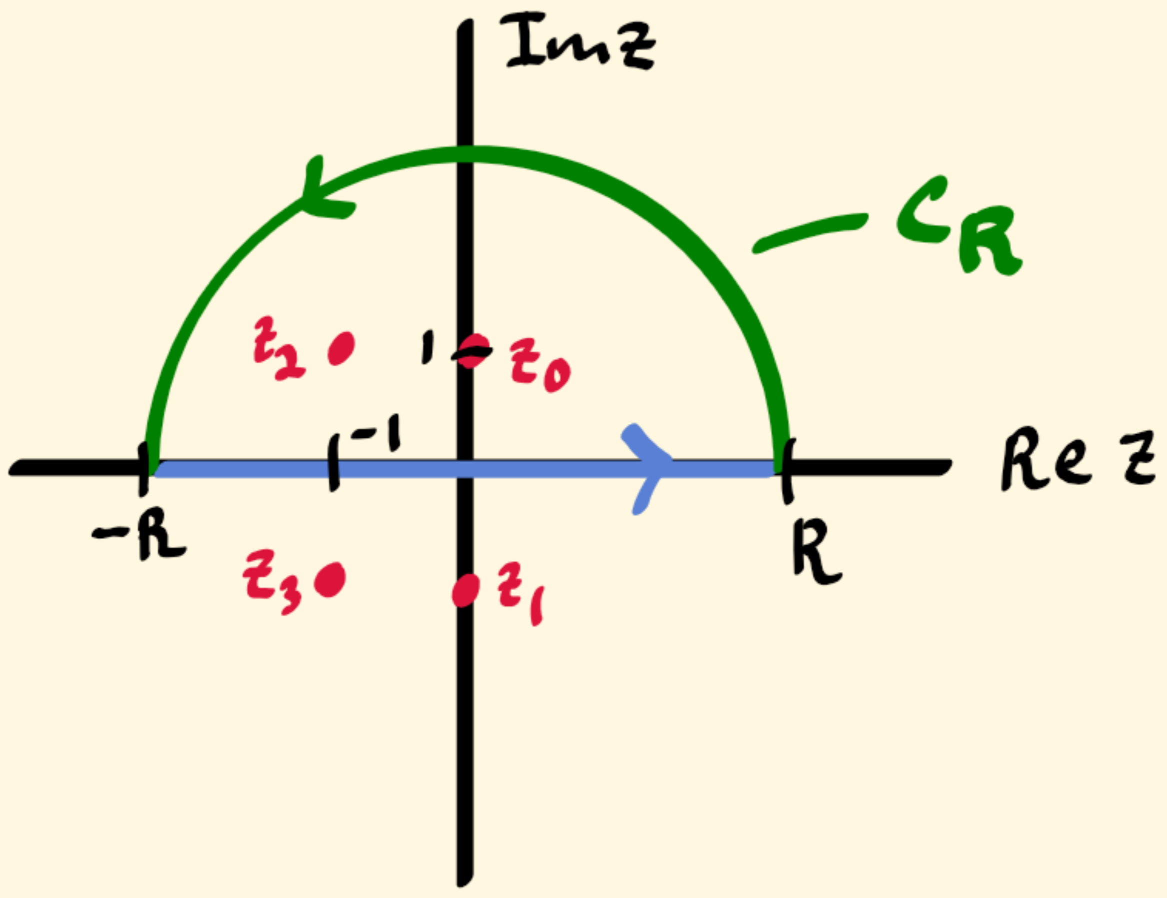
Use residues to find the Cauchy principal values of the integrals in Exercises 6 and 7.

6. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$.

7. $\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+1)(x^2+2x+2)}$.

Ans. $-\pi/5$.

$$f(z) = \frac{z}{(z^2+1)(z^2+2z+2)}$$



isolated singularities:

$z_K = \pm i,\ -1 \pm i$

$z_K = i, -1+i$ enclosed by the upper semi-circle

May apply Thm 2 of Section 76 with
$$\begin{cases} p = z \\ q = z^4 + 2z^3 + 3z^2 + 2z + 2 \\ q' = 4z^3 + 6z^2 + 6z + 2 \end{cases}$$

$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \cdot \left[\frac{p(i)}{q'(i)} + \frac{p(-1+i)}{q'(-1+i)}\right]$$
$$= 2\pi i \cdot \frac{i}{10} = -\frac{\pi}{5}$$

For z on C_R,
$$|f(z)| \leq \frac{|z|}{(|z|^2-1)(|z|^2-2|z|-2)} = \frac{R}{(R^2-1)(R^2-2R-2)}$$

$$\Rightarrow \left|\int_{C_R} f(z)\,dz\right| \leq \frac{\pi R^2}{(R^2-1)(R^2-2R-2)} \to 0 \text{ as } R \to \infty$$

$$\therefore \text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = \boxed{-\frac{\pi}{5}}$$

9. Let m and n be integers, where $0 \le m < n$. Follow the steps below to derive the integration formula

$$\int_0^\infty \frac{x^{2m}}{x^{2n}+1}\,dx = \frac{\pi}{2n}\csc\left(\frac{2m+1}{2n}\pi\right).$$

(a) Show that the zeros of the polynomial $z^{2n}+1$ lying above the real axis are

$$c_k = \exp\left[i\frac{(2k+1)\pi}{2n}\right] \qquad (k = 0, 1, 2, \ldots, n-1)$$

and that there are none on that axis.

(a) $0 = c_k^{2n} + 1$

$c_k^{2n} = e^{i\pi(1+2k)}$

$c_k = e^{i\pi(1+2k)/2n}$, $k = 0, 1, \ldots, n-1$

$c_k = \cos\left[\frac{(2k+1)\pi}{2n}\right] + i\sin\left[\frac{(2k+1)\pi}{2n}\right]$

c_k on the real axis iff $\mathrm{Im}\, c_k = 0$

Since $0 < \frac{\pi}{2n} \le \frac{(2k+1)\pi}{2n} \le \frac{2n-1}{2n}\pi < \pi$

for $k = 0, 1, \ldots, n-1$ and $\sin z > 0$ for $0 < z < \pi$,
$\mathrm{Im}\, c_k \neq 0$ $\forall k$. No c_k lies on the real axis.

(b) With the aid of Theorem 2 in Sec. 76, show that

$$\operatorname*{Res}_{z=c_k} \frac{z^{2m}}{z^{2n}+1} = -\frac{1}{2n} e^{i(2k+1)\alpha} \qquad (k = 0, 1, 2, \ldots, n-1)$$

where c_k are the zeros found in part *(a)* and

$$\alpha = \frac{2m+1}{2n}\pi.$$

Then use the summation formula

$$\sum_{k=0}^{n-1} z^k = \frac{1-z^n}{1-z} \qquad (z \neq 1)$$

(see Exercise 9, Sec. 8) to obtain the expression

$$2\pi i \sum_{k=0}^{n-1} \operatorname*{Res}_{z=c_k} \frac{z^{2m}}{z^{2n}+1} = \frac{\pi}{n \sin\alpha}.$$

Theorem 2. *Let two functions p and q be analytic at a point z_0. If*

$$p(z_0) \neq 0, \quad q(z_0) = 0, \quad \text{and} \quad q'(z_0) \neq 0,$$

then z_0 is a simple pole of the quotient $p(z)/q(z)$ and

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}. \tag{2}$$

Let $p(z) = z^{2m}$, $q(z) = z^{2n}+1$

For $k = 0, \ldots, n-1$

$$\operatorname*{Res}_{z=c_k} \frac{z^{2m}}{z^{2n}+1} = \frac{p(c_k)}{q'(c_k)} = \frac{(c_k)^{2m}}{2n(c_k)^{2n-1}}$$

$$= \frac{\left(e^{i\pi(2k+1)/2n}\right)^{2m}}{2n\left(e^{i\pi(2k+1)/2n}\right)^{2n-1}}$$

$$= \frac{1}{2n}\left[e^{i\pi(2k+1)}\right]^{\frac{2m-2n+1}{2n}}$$

$$= \frac{1}{2n}\left[e^{i\pi(2k+1)}\right]^{\frac{2m+1}{2n}}\left[e^{i\pi(2k+1)}\right]^{-1}$$

$$= \frac{1}{2n} e^{i(2k+1)\alpha}\; e^{-i\pi(2k+1)}$$

$$= \frac{1}{2n} e^{i(2k+1)\alpha} \qquad \alpha = \frac{2m+1}{2n}\pi$$

$$\sum_{k=0}^{n-1} \operatorname*{Res}_{z=c_k} \frac{z^{2m}}{z^{2n}+1} = \sum_{k=0}^{n-1} \frac{1}{2n} e^{i(2k+1)\alpha}$$

$$= \frac{1}{2n} e^{i\alpha} \sum_{k=0}^{n-1} \left(e^{i2\alpha}\right)^k = \frac{1}{2n} e^{i\alpha}\, \frac{1-e^{i2n\alpha}}{1-e^{i2\alpha}}$$

$$= \frac{1}{2n} e^{i\alpha}\, \frac{1-\cos[(2m+1)\pi]-i\sin[(2m+1)\pi]}{1-e^{i2\alpha}}$$

$$= \frac{1}{2n} e^{i\alpha}\, \frac{1-(-1)-0}{1-e^{i2\alpha}} = \frac{1}{n}\, \frac{e^{i\alpha}}{1-e^{i2\alpha}}$$

$$= \frac{1}{n} \frac{1}{e^{-i\alpha} - e^{i\alpha}} = \frac{1}{n} \frac{1}{-2i\sin\alpha}$$

$$\boxed{\therefore 2\pi i \sum_{k=0}^{n-1} \operatorname*{Res}_{z=c_k} \frac{z^{2m}}{z^{2n}+1} = \frac{2\pi i}{-2ni\sin\alpha} = \frac{\pi}{n\sin\alpha}}$$

(c) Use the final result in part *(b)* to complete the derivation of the integration formula.

$$\int_{-R}^{R} \frac{x^{2m}}{x^{2n}+1}\,dx = \frac{\pi}{n\sin\alpha} - \int_{C_R} \frac{z^{2m}}{z^{2n}+1}\,dz$$

Since $\left|\frac{z^{2m}}{z^{2n}+1}\right| \leq \frac{R^{2m}}{R^{2n}-1}$ for z on C_R,

$$\left|\int_{C_R} \frac{z^{2m}}{z^{2n}+1}\,dz\right| \leq \frac{\pi R^{2m+1}}{R^{2n}-1} \longrightarrow 0 \text{ as } R \longrightarrow \infty$$

$$(0 < m < n \Rightarrow n \geq m+1 \Rightarrow 2n \geq 2m+2 > 2m+1)$$

$$\therefore \int_0^\infty \frac{x^{2m}}{x^{2n}+1}\,dx = \frac{1}{2}\frac{\pi}{n\sin\alpha} = \frac{\pi}{2n}\csc\alpha$$

$$\boxed{\int_0^\infty \frac{x^{2m}}{x^{2n}+1}\,dx = \frac{\pi}{2n}\csc\left[\frac{2m+1}{2n}\pi\right]}$$

*   *   *   *   *

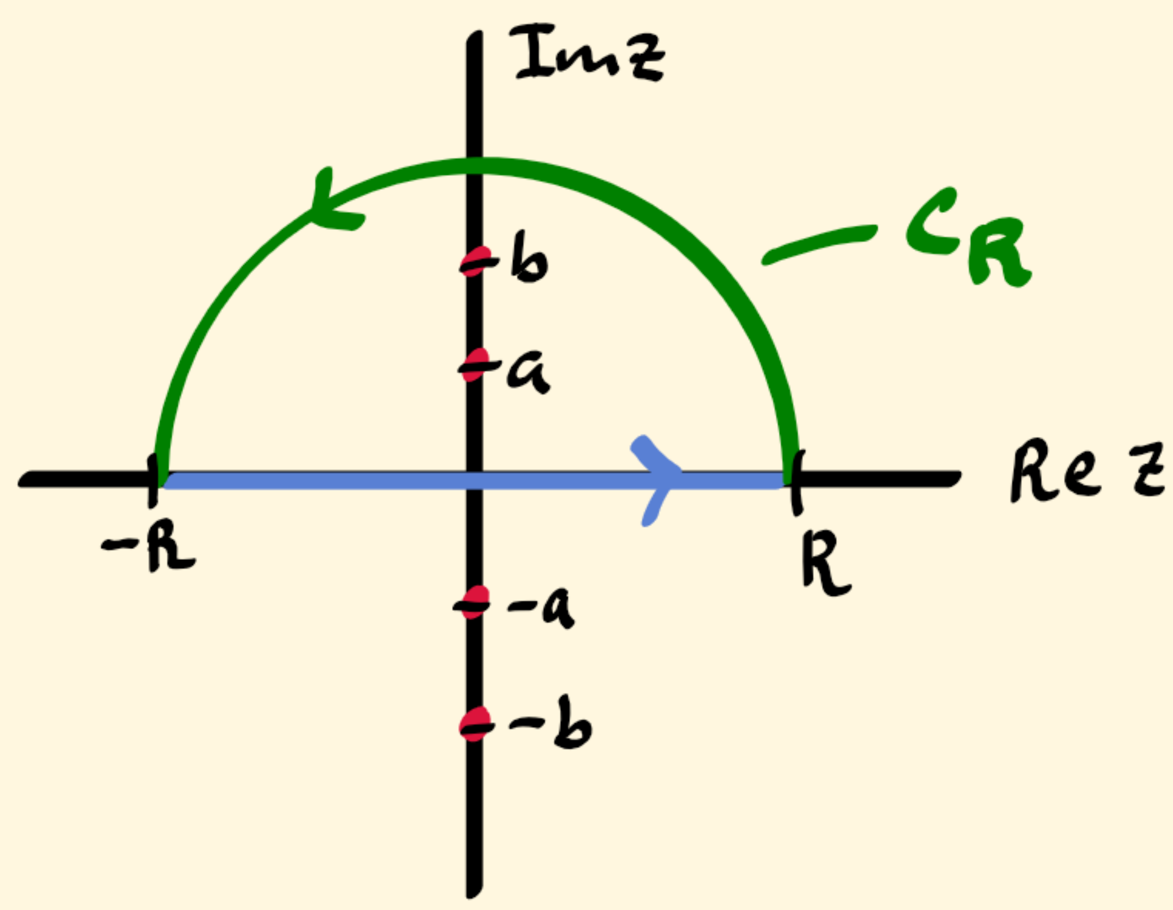
Chapter 7 Section 81 : 1, 3, 5, 7, 9 (pg. 275)

Use residues to evaluate the improper integrals in Exercises 1 through 8.

1. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x \, dx}{(x^2+a^2)(x^2+b^2)} \quad (a > b > 0).$

Ans. $\displaystyle\frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right).$

$$f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}$$



singularities at $z = \pm ai, \pm bi$

$$\int_{-R}^{R} f(x)e^{ix}\,dx + \int_{C_R} f(z)e^{iz}\,dz = 2\pi i\left[A_1 + B_1\right]$$

where $A_1 = \operatorname*{Res}_{z=ai} f(z)e^{iz}$, $B_1 = \operatorname*{Res}_{z=bi} f(z)e^{iz}$

$\underline{A_1}$

Let $\phi(z) = \dfrac{e^{iz}}{(z+ai)(z^2+b^2)} \longrightarrow f(z)e^{iz} = \dfrac{\phi(z)}{z-ai}$

$$A_1 = \phi(ai) = \frac{e^{-a}}{2ai\,(b^2-a^2)}$$

$\underline{B_1}$

$$\text{Let } \phi(z) = \frac{e^{iz}}{(z+bi)(z^2+a^2)} \rightarrow f(z)e^{iz} = \frac{\phi(z)}{z-bi}$$

$$B_1 = \phi(bi) = \frac{e^{-b}}{2bi(a^2-b^2)}$$

$$\int_{-R}^{R} f(x)e^{ix}\,dx + \int_{C_R} f(z)e^{iz}\,dz = 2\pi i\left[A_1 + B_1\right]$$

$$= \frac{2\pi i}{2i}\frac{1}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right) = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

$$\mathrm{Re}\int_{-R}^{R} f(x)e^{ix}\,dx = \mathrm{Re}\left\{2\pi i\left[A_1+B_1\right] - \int_{C_R} f(z)e^{iz}\,dz\right\}$$

$$\int_{-R}^{R}\frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right) - \mathrm{Re}\int_{C_R} f(z)e^{iz}\,dz$$

$$\left|f(z)e^{iz}\right| = \left|f(z)\right| \le \frac{1}{(R^2-a^2)(R^2-b^2)} \quad \text{for } z \text{ on } C_R$$

$$\left|\mathrm{Re}\int_{C_R} f(z)\,dz\right| \le \left|\int_{C_R} f(z)\,dz\right| \le \frac{\pi R}{(R^2-a^2)(R^2-b^2)} \xrightarrow{(R\to\infty)} 0$$

$$\therefore \int_{-\infty}^{\infty}\frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

3. $\displaystyle\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2}\,dx \quad (a>0, b>0).$

Ans. $\dfrac{\pi}{4b^3}(1+ab)e^{-ab}$.

Let $f(z) = \dfrac{1}{(z^2+b^2)^2}$

$$\int_{-R}^{R} f(x)e^{iax}\,dx = 2\pi i \cdot B_1 - \int_{C_R} f(z)e^{iaz}\,dz$$

$$\left(B_1 = \operatorname*{Res}_{z=bi} f(z)e^{iaz}\right)$$

$$\phi(z) := \frac{e^{iaz}}{(z+bi)^2} \rightarrow \phi'(z) = -\frac{2e^{iaz}}{(z+bi)^3} + i\,\frac{ae^{iaz}}{(z+bi)^2}$$

$$B_1 = \phi'(bi) = -\frac{2e^{-ab}}{(2bi)^3} + \frac{iae^{-ab}}{(2bi)^2} = \frac{\pi e^{-ab}(1+ab)}{4b^3 i}$$

$$\int_{-R}^{R} f(x)e^{iax}\,dx = \frac{\pi e^{-ab}(1+ab)}{2b^3} - \int_{C_R} f(z)e^{iaz}\,dz$$

$$\int_{-R}^{R} f(x)\cos ax\,dx = \frac{\pi e^{-ab}(1+ab)}{2b^3} - \operatorname{Re}\int_{C_R} f(z)e^{iaz}\,dz$$

$$|f(z)e^{iaz}| = |f(z)| \le (R^2-b^2)^{-2} \quad \text{for } z \text{ on } C_R$$

$$\left|\operatorname{Re}\int_{C_R} f(z)e^{iaz}\,dz\right| \le \left|\int_{C_R} f(z)e^{iaz}\,dz\right| \le \frac{\pi R}{(R^2-b^2)^2} \xrightarrow{(R\to\infty)} 0$$

$\therefore$

$$\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos ax}{(x^2+b^2)^2}\,dx = \frac{\pi e^{-ab}(1+ab)}{4b^3}$$

5. $\int_{-\infty}^{\infty} \frac{x \sin ax}{x^4+4}\,dx \quad (a>0).$

Ans. $\frac{\pi}{2} e^{-a} \sin a.$

Let $f(z) = \dfrac{z}{z^4+4}$

$0 = z^4 + 4 \longrightarrow z^4 = -4 = 4e^{i(\pi + 2n\pi)} \longrightarrow f(z)e^{iaz}$ has

singularities at $z_k = \sqrt{2}\, e^{i(\pi/4 + k\pi/2)}$ $(k = 0, 1, 2, 3)$

$$z_0 = 1+i \qquad z_2 = -1-i$$
$$z_1 = -1+i \qquad z_3 = 1-i$$

$$\int_{-R}^{R} f(x)e^{iax}\,dx = 2\pi i \cdot [A+B] - \int_{C_R} f(z)e^{iaz}\,dz$$

$$A = \operatorname*{Res}_{z=z_0} f(z)e^{iaz}, \quad B = \operatorname*{Res}_{z=z_1} f(z)e^{iaz}$$

(z_2, z_3 not enclosed by the upper semicircular region)

$$\phi_A(z) := \frac{z e^{iaz}}{(z-z_1)(z-z_2)(z-z_3)}$$

$$A = \phi_A(z_0) = \frac{(1+i)e^{ia(1+i)}}{(z_0-z_1)(z_0-z_2)(z_0-z_3)}$$

$$= \frac{(1+i)e^{ia}e^{-a}}{2\cdot(2+2i)(2i)} = \frac{1+i}{-8+8i} e^{ia}e^{-a} = \underline{-\frac{i}{8} e^{ia}e^{-a}}$$

$$\phi_B(z) := \frac{z e^{iaz}}{(z-z_0)(z-z_2)(z-z_3)}$$

$$B = \phi_B(z_1) = \frac{(-1+i)e^{ia(-1+i)}}{(z_1-z_0)(z_1-z_2)(z_1-z_3)}$$

$$= \frac{(-1+i)e^{-ia}e^{-a}}{-2\cdot 2i\cdot(-2+2i)} = \underline{\frac{i}{8}e^{-ia}e^{-a}}$$

$$2\pi i\cdot[A+B] = 2\pi i\cdot\frac{ie^{-a}}{8}\left[e^{-ia}-e^{ia}\right]$$

$$= -\frac{\pi e^{-a}}{4}\left[-2i\sin a\right] = i\frac{\pi}{2}e^{-a}\sin a$$

$$\int_{-R}^{R} f(x)e^{iax}\,dx = i\frac{\pi}{2}e^{-a}\sin a - \int_{C_R} f(z)e^{iaz}\,dz$$

$$\int_{-R}^{R} f(x)\sin ax\,dx = \frac{\pi}{2}e^{-a}\sin a - \operatorname{Im}\int_{C_R} f(z)e^{iaz}\,dz$$

$$\left|\operatorname{Im}\int_{C_R} f(z)e^{iz}\,dz\right| \le \left|\int_{C_R} f(z)e^{iz}\,dz\right|$$

$$\le \frac{\pi R^2}{R^4-4} \longrightarrow 0 \text{ as } R\to\infty$$

$$\int_{-\infty}^{\infty}\frac{x\sin ax}{x^4+4}\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\sin ax\,dx = \frac{\pi}{2}e^{-a}\sin a$$

7. $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin x \, dx}{(x^2+1)(x^2+4)}$. Let $f(z) = \dfrac{z}{(z^2+1)(z^2+4)}$

$f(z)e^{iz}$ has isolated singularities at $z = \pm i, \pm 2i$

$$\int_{-R}^{R} f(x)e^{ix}\,dx = 2\pi i \cdot [A+B] - \int_{C_R} f(z)e^{iz}\,dz$$

$$\phi_A(z) = \frac{ze^{iz}}{(z+i)(z^2+4)}$$

$$A := \operatorname*{Res}_{z=i} f(z)e^{iz} = \phi_A(i) = \frac{ie^{i^2}}{2i \cdot 3} = \frac{1}{6e}$$

$$\phi_B(z) = \frac{ze^{iz}}{(z^2+1)(z+2i)}$$

$$B := \operatorname*{Res}_{z=2i} f(z)e^{iz} = \phi_B(2i) = \frac{2ie^{2i^2}}{(-3)(4i)} = -\frac{1}{6e^2}$$

$$\int_{-R}^{R} f(x)e^{ix}\,dx = \frac{\pi i}{3}\left(\frac{1}{e} - \frac{1}{e^2}\right) - \int_{C_R} f(z)e^{iz}\,dz$$

$$\int_{-R}^{R} f(x)\sin x\,dx = \frac{\pi}{3}\left(\frac{1}{e} - \frac{1}{e^2}\right) - \operatorname{Im}\int_{C_R} f(z)e^{iz}\,dz$$

Show $\operatorname{Im}\int_{C_R} f(z)e^{iz}\,dz \to 0$ as $R \to \infty$

$$|f(z)e^{iz}| = |f(z)| \le \frac{R}{(R^2-1)(R^2-4)} \quad \text{for } z \text{ on } C_R$$

$$\left| \operatorname{Im} \int_{C_R} f(z)e^{iz}\,dz \right| \le \left| \int_{C_R} f(z)e^{iz}\,dz \right|$$

$$\le \frac{\pi R^2}{(R^2-1)(R^2-4)} \longrightarrow 0 \text{ as } R \to \infty$$

$\therefore$

$$\int_{-\infty}^{\infty} \frac{x\sin x}{(x^2+1)(x^2+4)}\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\sin x\,dx$$

$$= \boxed{\frac{\pi}{3}\left(\frac{1}{e} - \frac{1}{e^2}\right)}$$

Use residues to find the Cauchy principal values of the improper integrals in Exercises 9 through 11.

9. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x\, dx}{x^2+4x+5}$.

Ans. $-\dfrac{\pi}{e}\sin 2$.

$$f(z) = \frac{1}{z^2+4z+5}$$

$f(z)e^{iz}$ has isolated singularities at

$$0 = z^2+4z+5 \longrightarrow z = -2 \pm \sqrt{-4}/2 = -2 \pm i$$

$z_0 = -2+i$ enclosed by C_R

$$\int_{-R}^{R} f(x)e^{ix}\,dx + \int_{C_R} f(z)e^{iz}\,dz = 2\pi i \cdot \operatorname*{Res}_{z=z_0} f(z)e^{iz}$$

$$\phi(z) = \frac{e^{iz}}{z-(-2-i)} \longrightarrow \frac{\phi(z)}{z-z_0} = f(z)e^{iz}$$

$$\operatorname*{Res}_{z=z_0} f(z) = \phi(z_0) = \frac{e^{i(-2+i)}}{2i} = \frac{e^{-1-2i}}{2i}$$

$$\int_{-R}^{R} f(x)e^{ix}\,dx + \int_{C_R} f(z)e^{iz}\,dz = 2\pi i \cdot \operatorname*{Res}_{z=z_0} f(z)e^{iz}$$

$$= \pi e^{-1-2i} = \pi e^{-1}(\cos 2 - i\sin 2)$$

$$\int_{-R}^{R} f(x)e^{ix}\,dx = \pi e^{-1}(\cos 2 - i\sin 2) - \int_{C_R} f(z)e^{iz}\,dz$$

Show $\int_{C_R} f(z)e^{iz}\,dz$ decays

$$|z^2+4z+5| \geq |z^2+4z| - |5| \geq |z|^2 - 4|z| - 5$$

$$|f(z)e^{iz}| \leq |f(z)| \leq \frac{1}{|z|^2 - 4|z| - 5} = \frac{1}{R^2-4R-5}$$

$$\left| \int_{C_R} f(z)e^{iz}\,dz \right| \leq \frac{\pi R}{R^2-4R-5} \longrightarrow 0 \quad \text{as } R \to \infty$$

$$\text{P.V. } \int_{-\infty}^{\infty} f(x)e^{ix}\,dx = \pi e^{-1}(\cos 2 - i\sin 2)$$

$$\text{P.V. } \int_{-\infty}^{\infty} f(x)\sin x\,dx = \operatorname{Im}\left[\pi e^{-1}(\cos 2 - i\sin 2)\right]$$

$$= -\pi e^{-1}\sin 2$$