

Written Homework 7 Solutions

MA 4291 (Tilley)
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Part 1 : Chapter 6 Exercises

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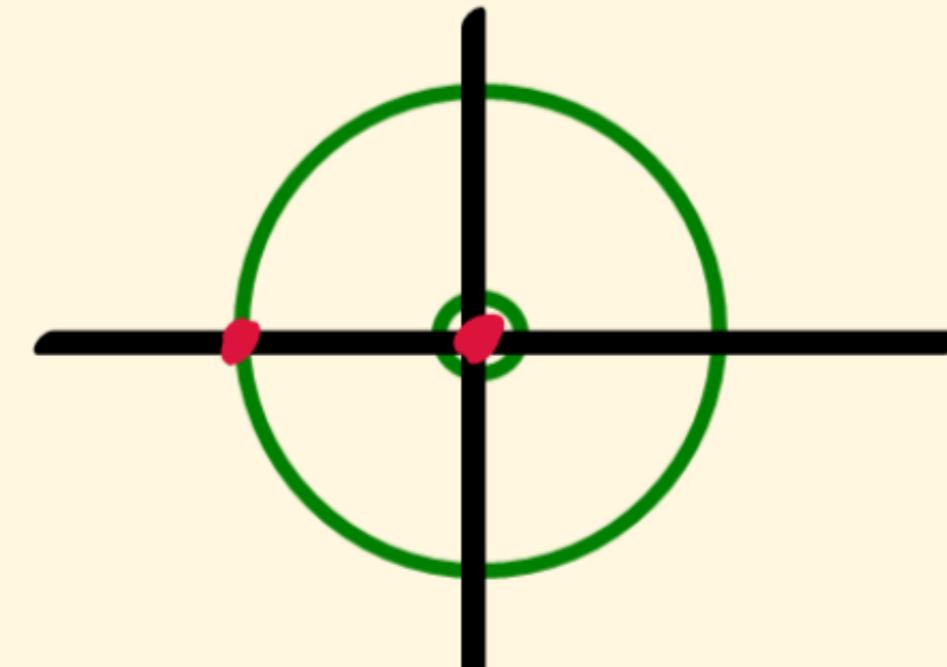
Chapter 6 Section 72: **1,2,3,4,6**

1. Find the residue at $z = 0$ of the function

$$(a) \frac{1}{z+z^2}; \quad (b) z \cos\left(\frac{1}{z}\right); \quad (c) \frac{z - \sin z}{z}; \quad (d) \frac{\cot z}{z^4}; \quad (e) \frac{\sinh z}{z^4(1-z^2)}.$$

Ans. (a) 1; (b) $-1/2$; (c) 0; (d) $-1/45$; (e) $7/6$.

$$(a) \frac{1}{z+z^2} = \frac{1}{z} \frac{1}{1+z}$$



$$= \sum_{n=0}^{\infty} (-1)^n z^{n-1} \quad (0 < |z| < 1)$$

$$= z^{-1} - 1 + z - z^2 + \dots$$

The residue is the coefficient of z^{-1} ,
which is 1

$$(b) \text{ Using } \cos z = 1 - \frac{z^2}{2} + \frac{z^4}{4!} - \dots \quad (|z| < \infty),$$

$$z \cos(1/z) = z - \frac{1}{2} \frac{1}{z} + \frac{1}{4!} \frac{1}{z^3} - \dots \quad (0 < |z| < \infty)$$

The residue is the coefficient of z^{-1} ,
which is -1/2

$$(c) \frac{z - \sin z}{z} = 1 - \frac{1}{2} \sin z$$

$$= 1 - \frac{1}{2} \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right) \quad (0 < |z| < \infty)$$

$$= \frac{z^2}{3!} - \frac{z^4}{5!} + \dots$$

The residue is the coefficient of z^{-1} ,
which is $\boxed{0}$

$$(d) \text{ Using } \cot z = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \frac{2}{945} z^5 - \theta(z^7)$$

for $0 < |z| < \infty$,

$$\frac{\cot z}{z^4} = \frac{1}{z^3} - \frac{1}{3} \frac{1}{z^2} - \frac{1}{45} \frac{1}{z} - \dots \quad (0 < |z| < \infty)$$

The residue is the coefficient of z^{-1} , $\boxed{-\frac{1}{45}}$

$$(e) \text{ Using } \sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots \quad (|z| < \infty)$$

$$\frac{\sinh z}{z^4(1-z^2)} = \left(\frac{1}{z^4} + \frac{1}{z^2} + 1 + z^2 + \dots \right) \left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots \right)$$

$$= \frac{1}{z^3} + \underbrace{\left(\frac{1}{3!} + 1 \right)}_{\frac{1}{6}} \frac{1}{z} + \left(\frac{1}{5!} + \frac{1}{3!} + 1 \right) z + \dots \quad (0 < |z| < 1)^*$$

The residue is the coefficient of z^{-1} , $\boxed{\frac{7}{6}}$

* Singular points : $z = 0, \pm 1$

2. Use Cauchy's residue theorem (Sec. 70) to evaluate the integral of each of these functions around the circle $|z| = 3$ in the positive sense:

$$(a) \frac{\exp(-z)}{z^2}; \quad (b) \frac{\exp(-z)}{(z-1)^2}; \quad (c) z^2 \exp\left(\frac{1}{z}\right); \quad (d) \frac{z+1}{z^2 - 2z}.$$

Ans. (a) $-2\pi i$; (b) $-2\pi i/e$; (c) $\pi i/3$; (d) $2\pi i$.

Theorem. Let C be a simple closed contour, described in the positive sense. If a function f is analytic inside and on C except for a finite number of singular points z_k ($k = 1, 2, \dots, n$) inside C (Fig. 87), then

$$(1) \quad \int_C f(z) dz = 2\pi i \sum_{k=1}^n \operatorname{Res}_{z=z_k} f(z).$$

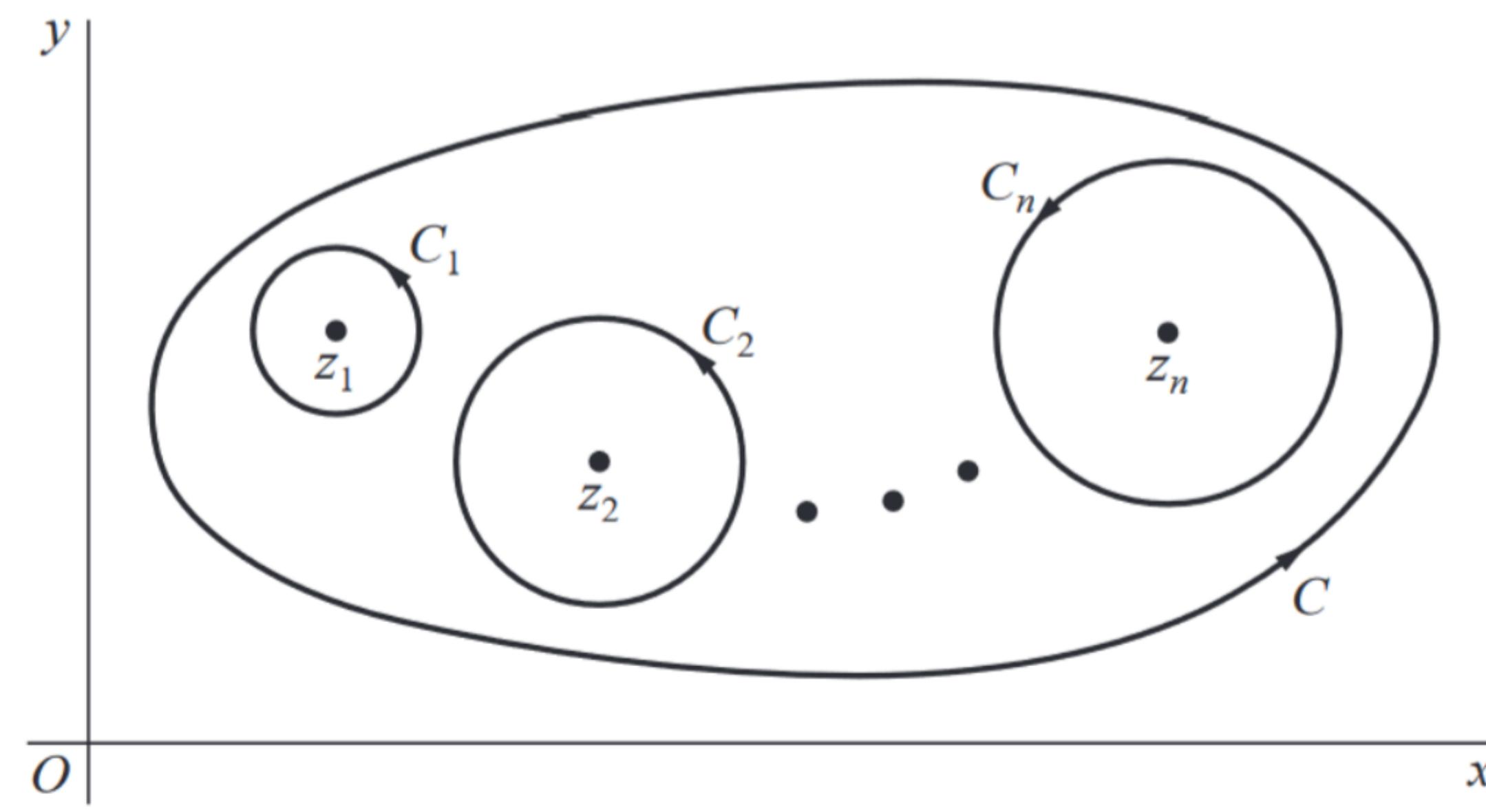


FIGURE 87

(a) The integrand has an isolated singularity at $z=0$, which is interior to C .

$$z^{-2} \exp(-z) = z^{-2} - z^{-1} + \frac{1}{2} - \frac{1}{3!} z + \frac{1}{4!} z^2 - \dots$$

$$\operatorname{Res}_{z=0} f(z) = -1 \Rightarrow \int_C z^{-2} \exp(-z) dz = \boxed{-2\pi i}$$

(b) The integrand has an isolated singularity at $z=1$, which is interior to C .

$$f(z) = \frac{e^{-z}}{(z-1)^2} = \frac{1}{e} \frac{1}{(z-1)^2} e^{1-z}$$

$$= \frac{1}{e} \frac{1}{(z-1)^2} \left(1 + (1-z) + \frac{(1-z)^2}{2} + \frac{(1-z)^3}{3!} + \dots \right)$$

$$= \frac{1}{e} \frac{1}{(z-1)^2} - \frac{1}{e} \frac{1}{z-1} + \frac{1}{2e} - \frac{1}{2e} (z-1) + \dots$$

for $0 < |z-1| < \infty$. The residue is the coefficient of $(z-1)^{-1}$.

$$\int_C (z-1)^{-2} e^{-z} dz = 2\pi i \operatorname{Res}_{z=1} f(z) = \boxed{-2\pi i/e}$$

(c) The integrand has an isolated singularity at $z=0$, which is interior to C .

$$\begin{aligned}f(z) &= z^2 \exp(1/z) \\&= z^2 \left(1 + \frac{1}{z} + \frac{1}{2} \frac{1}{z^2} + \frac{1}{3!} \frac{1}{z^3} + \frac{1}{4!} \frac{1}{z^4} + \dots \right) \\&= z^2 + z + \frac{1}{2} + \frac{1}{6} \frac{1}{z} + \frac{1}{24} \frac{1}{z^2} + \dots\end{aligned}$$

for $0 < |z| < \infty$. The residue is $\frac{1}{6}$.

$$\int_C z^2 \exp(1/z) dz = 2\pi i \operatorname{Res}_{z=0} f(z) = \boxed{\frac{\pi i}{3}}$$

(d) The integrand has isolated singularities at $z=0$ and $z=2$. By partial fractions:

$$f(z) = \frac{z+1}{z(z-2)} = -\frac{1/2}{z} + \frac{3/2}{z-2}$$

Since $-\frac{1}{2z}$ is a Laurent series when $0 < |z| < 1$

and $\frac{3/2}{z-2}$ is a Laurent series when $0 < |z-2| < 1$,

$$\int_C \frac{z+1}{z^2-2z} dz = 2\pi i \cdot (-\frac{1}{2}) + 2\pi i \cdot (3/2) = \boxed{2\pi i}$$

3. Use the theorem in Sec. 71, involving a single residue, to evaluate the integral of each of these functions around the circle $|z| = 2$ in the positive sense:

$$(a) \frac{z^5}{1-z^3}; \quad (b) \frac{1}{1+z^2}; \quad (c) \frac{1}{z}.$$

Ans. (a) $-2\pi i$; (b) 0; (c) $2\pi i$.

Theorem. If a function f is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C , then

$$(7) \quad \int_C f(z) dz = 2\pi i \operatorname{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right) \right].$$

$$\begin{aligned} (a) \quad \frac{1}{z^2} f\left(\frac{1}{z}\right) &= \frac{1}{z^2} \frac{\frac{1}{z^5}}{1 - \frac{1}{z^3}} = -\frac{1}{z^4} \frac{1}{1 - z^3} \\ &= -\frac{1}{z^4} (1 + z^3 + z^6 + \dots) \\ &= -\frac{1}{z^4} \underbrace{-\frac{1}{z}}_{z^2} - z^2 - \dots \quad (0 < |z| < 1) \end{aligned}$$

$$\int_C f(z) dz = 2\pi i \operatorname{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right) \right] = 2\pi i \cdot (-1) = \boxed{-2\pi i}$$

$$\begin{aligned} (b) \quad \frac{1}{z^2} f\left(\frac{1}{z}\right) &= \frac{1}{z^2} \frac{1}{1 + \frac{1}{z^2}} = \frac{1}{1 + z^2} = \frac{1}{1 - (-z^2)} \\ &= 1 - z^2 + z^4 - z^6 + \dots \quad (0 < |z| < 1) \end{aligned}$$

$$\int_C f(z) dz = 2\pi i \operatorname{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right) \right] = 2\pi i \cdot 0 = \boxed{0}$$

$$(c) \frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z} \quad (\text{Laurent series for } 0 < |z| < 1)$$

$$\int_C f(z) dz = 2\pi i \operatorname{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right) \right] = 2\pi i \cdot 1 = \boxed{2\pi i}$$

4. Let C denote the circle $|z| = 1$, taken counterclockwise, and use the following steps to show that

$$\int_C \exp\left(z + \frac{1}{z}\right) dz = 2\pi i \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}.$$

- (a) By using the Maclaurin series for e^z and referring to Theorem 1 in Sec. 65, which justifies the term by term integration that is to be used, write the above integral as

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_C z^n \exp\left(\frac{1}{z}\right) dz.$$

- (b) Apply the theorem in Sec. 70 to evaluate the integrals appearing in part (a) to arrive at the desired result.

$$(a) \quad \int_C \exp\left(z + \frac{1}{z}\right) dz = \int_C e^{1/z} e^z dz$$

$$= \int_C e^{1/z} \left(\sum_{n=0}^{\infty} \frac{z^n}{n!} \right) dz$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int_C z^n e^{1/z} dz$$

$$(b) \text{ For each } n, \text{ let } f_n(z) = z^n e^{1/z}$$

$$f_n(z) := z^n e^{1/z} = z^n \left(1 + \frac{1}{z} + \dots + \frac{1}{(n+1)!} \frac{1}{z^{n+1}} + \dots \right)$$

$$= z^n + z^{n-1} + \dots + \underbrace{\frac{1}{(n+1)!} \frac{1}{z}}_{\text{Residue}} + \dots$$

$$\int_C z^n e^{1/z} dz = 2\pi i \cdot \operatorname{Res}_{z=0} f_n(z) = 2\pi i \cdot \frac{1}{(n+1)!}$$

$$\therefore \int_C \exp\left(z + \frac{1}{z}\right) dz = \sum_{n=0}^{\infty} \frac{1}{n!} \int_C z^n e^{1/z} dz$$

$$= 2\pi i \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}$$

6. Let the degrees of the polynomials

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \quad (a_n \neq 0)$$

and

$$Q(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_m z^m \quad (b_m \neq 0)$$

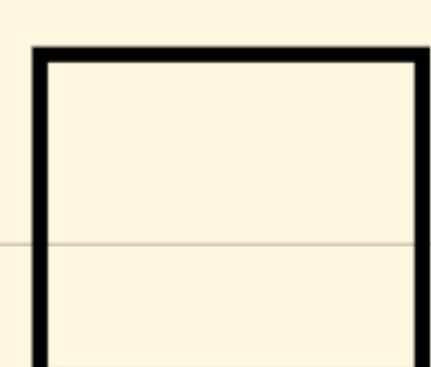
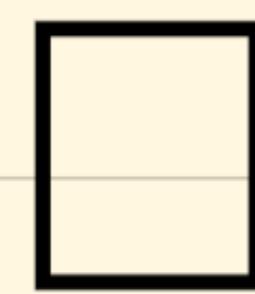
be such that $m \geq n + 2$. Use the theorem in Sec. 71 to show that if all of the zeros of $Q(z)$ are interior to a simple closed contour C , then

$$\int_C \frac{P(z)}{Q(z)} dz = 0.$$

[Compare with Exercise 3(b).]

Theorem. *If a function f is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C , then*

$$(7) \quad \int_C f(z) dz = 2\pi i \operatorname{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right) \right].$$



* * * *

Chapter 6 Section 72 : 1, 2, 3, 4 (pg 243)

1. In each case, write the principal part of the function at its isolated singular point and determine whether that point is a pole, a removable singular point, or an essential singular point:

$$(a) z \exp\left(\frac{1}{z}\right); \quad (b) \frac{z^2}{1+z}; \quad (c) \frac{\sin z}{z}; \quad (d) \frac{\cos z}{z}; \quad (e) \frac{1}{(2-z)^3}.$$

We saw in Sec. 69 that the theory of residues is based on the fact that if f has an isolated singular point at z_0 , then $f(z)$ has a Laurent series representation

$$(1) \quad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_n}{(z - z_0)^n} + \cdots$$

in a punctured disk $0 < |z - z_0| < R_2$. The portion

$$(2) \quad \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_n}{(z - z_0)^n} + \cdots$$

of the series, involving negative powers of $z - z_0$, is called the *principal part* of f at z_0 . We now use the principal part to identify the isolated singular point z_0 as one of three special types. This classification will aid us in the development of residue theory that appears in following sections.

(a) $f(z) = z \exp(1/z)$ has an isolated singularity at $z_0 = 0$.

$$\begin{aligned} f(z) &= z \left(1 + \frac{1}{z} + \frac{1}{2} \frac{1}{z^2} + \frac{1}{3!} \frac{1}{z^3} + \dots \right) \\ &= z + 1 + \frac{1}{2} \frac{1}{z} + \frac{1}{3!} \frac{1}{z^2} + \dots \quad |z| > 0 \end{aligned}$$

The principal part of $f(z)$ at $z_0 = 0$ is

$$\underline{\frac{1}{2} \frac{1}{z} + \frac{1}{3!} \frac{1}{z^2} + \dots}$$

$f(z)$ has an essential singular point at $z_0 = 0$, with the residue $b_1 = \frac{1}{2}$.

(b) $f(z) = \frac{z^2}{1+z}$ has an isolated singular point at $z_0 = -1$. $f(z)$ can be written in the form

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n \quad 0 < |z-z_0| < R_0$$

$$\begin{aligned} f(z) &= \frac{z^2 - 1}{1+z} + \frac{1}{1+z} = z-1 + \frac{1}{1+z} \\ &= -2 + z - (-1) + \frac{1}{1+z} \\ &= -2 \cdot (z-(-1))^0 + 1 \cdot (z-(-1)) + \frac{1}{z-(-1)} \quad 0 < |z+1| < \infty \end{aligned}$$

The principal part of $f(z)$ at $z_0 = -1$ is $\frac{1}{z+1}$.

This shows that $f(z)$ has a simple pole at $z_0 = -1$. Its residue there is $c_{-1} = 1$.

$$(c) f(z) = \frac{\sin z}{z} = 1 - \frac{1}{3!} z^2 + \frac{1}{5!} z^4 + \dots \quad (0 < |z| < \infty)$$

The principal part of $f(z)$ at $z_0 = 0$ is 0 and $\underset{z=0}{\text{Res}} f(z) = 0$.

$f(z)$ has a removable singularity at $z_0 = 0$.

$$(d) \quad f(z) = \frac{\cos z}{z} = \frac{1}{z} - \frac{1}{2}z + \frac{1}{4!}z^3 - \dots$$

The principal part of $f(z)$ at $z_0 = 0$ is $\frac{1}{z}$,
with residue $b_1 = 1$.

$f(z)$ has a simple pole at $z_0 = 0$.

$$(e) \quad f(z) = \frac{1}{(2-z)^3} = -\frac{1}{(z-2)^3}$$

$f(z)$ has a pole of order 3 at $z_0 = 2$
and $\underset{z=2}{\text{Res}} f(z) = b_1 = 0$.

The principal part of $f(z)$ at $z_0 = 2$ is
just $f(z)$ itself.

2. Show that the singular point of each of the following functions is a pole. Determine the order m of that pole and the corresponding residue B .

$$(a) \frac{1 - \cosh z}{z^3}; \quad (b) \frac{1 - \exp(2z)}{z^4}; \quad (c) \frac{\exp(2z)}{(z-1)^2}.$$

Ans. (a) $m = 1, B = -1/2$; (b) $m = 3, B = -4/3$; (c) $m = 2, B = 2e^2$.

(a) $f(z) = \frac{1 - \cosh z}{z^3}$ has a singular pt. $z_0 = 0$.

$$f(z) = \frac{1}{z^3} \left(1 - 1 - \frac{1}{2}z^2 - \frac{1}{4!}z^4 - \frac{1}{6!}z^6 - \dots \right) \quad (|z| > 0)$$

$$= -\frac{1}{2} \frac{1}{z} - \frac{1}{4!} z - \dots$$

$$\boxed{m=1} \\ \boxed{B=-1/2}$$

(b) $g(z) = \frac{1 - e^{2z}}{z^4}$ has a singular pt. $z_0 = 0$.

$$g(z) = z^{-4} \left(1 - 1 - 2z - \frac{1}{2}(2z)^2 - \frac{1}{3!}(2z)^3 - \dots \right)$$

$$= -\underline{2z^{-3}} - 2z^{-2} - \underline{\frac{4}{3}z^{-1}} - \frac{2}{3}z^0 - \dots \quad (|z| > 0)$$

$$\boxed{m=3}$$

$$\boxed{B=-4/3}$$

$$(c) h(z) = \frac{e^{2z}}{(z-1)^2} = \frac{e^2 e^{2(z-1)}}{(z-1)^2}$$

$$= \frac{e^2}{(z-1)^2} \left(1 + 2(z-1) + \frac{1}{2}2^2(z-1)^2 + \dots \right), \quad (0 < |z-1| < \infty)$$

$$= \underline{\frac{e^2}{(z-1)^2}} + \underline{\frac{2e^2}{(z-1)}} + 2e^2 + \dots$$

$$\boxed{m=2} \\ \boxed{B=2e^2}$$

3. Suppose that a function f is analytic at z_0 , and write $g(z) = f(z)/(z - z_0)$. Show that

- (a) if $f(z_0) \neq 0$, then z_0 is a simple pole of g , with residue $f(z_0)$;
- (b) if $f(z_0) = 0$, then z_0 is a removable singular point of g .

Suggestion: As pointed out in Sec. 57, there is a Taylor series for $f(z)$ about z_0 since f is analytic there. Start each part of this exercise by writing out a few terms of that series.

Theorem. Suppose that a function f is analytic throughout a disk $|z - z_0| < R_0$, centered at z_0 and with radius R_0 (Fig. 74). Then $f(z)$ has the power series representation

$$(1) \quad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (|z - z_0| < R_0),$$

where

$$(2) \quad a_n = \frac{f^{(n)}(z_0)}{n!} \quad (n = 0, 1, 2, \dots).$$

That is, series (1) converges to $f(z)$ when z lies in the stated open disk.

Since $f(z)$ is analytic at z_0 , \exists a disk $|z - z_0| < R_0$ s.t. $f(z)$ is analytic throughout the disk.

$f(z)$ then has a power series representation as written in the theorem.

$$\begin{aligned} g(z) &= (z - z_0)^{-1} f(z) \\ &= (z - z_0)^{-1} [f(z_0) + f'(z_0)(z - z_0) + \frac{1}{2} f''(z_0)(z - z_0)^2 + \dots] \\ &= \frac{f(z_0)}{z - z_0} + f'(z_0) + \frac{1}{2} f''(z_0)(z - z_0) + \dots \quad (|z - z_0| < R_0) \end{aligned}$$

(a) If $f(z_0) \neq 0$, the first term doesn't vanish
 $g(z)$ has a simple pole at z_0 and $\underset{z=z_0}{\text{Res}} g(z) = f(z_0)$

(b) If $f(z_0) = 0$, the first term vanishes so

$$g(z) = f'(z_0) + \frac{1}{2} f''(z_0)(z - z_0) + \dots$$

Then z_0 is a removable singularity of g

4. Use the fact (see Sec. 29) that $e^z = -1$ when

$$z = (2n+1)\pi i \quad (n = 0, \pm 1, \pm 2, \dots)$$

to show that $e^{1/z}$ assumes the value -1 an infinite number of times in each neighborhood of the origin. More precisely, show that $e^{1/z} = -1$ when

$$z = -\frac{i}{(2n+1)\pi} \quad (n = 0, \pm 1, \pm 2, \dots);$$

then note that if n is large enough, such points lie in any given ε neighborhood of the origin. Zero is evidently the exceptional value in Picard's theorem, stated in Example 5, Sec. 72.

Let $D_\varepsilon = \{z : |z| < \varepsilon\}$ be a neighborhood of the origin. We want to show there exist infinitely many $z \in D_\varepsilon$ s.t. $e^{1/z} = -1$. For $n \geq 0$,

$$\lim_{n \rightarrow \infty} \left| -\frac{i}{(2n+1)\pi} \right| = \lim_{n \rightarrow \infty} \frac{1}{(2n+1)\pi} = 0$$

$$\Rightarrow \exists N \text{ s.t. } \left| -\frac{i}{(2n+1)\pi} \right| < \varepsilon \quad \forall n \geq N.$$

(Similar for $n < 0$, $n \rightarrow -\infty$ although $n < 0$ not really needed to prove the statement)

$$\text{Let } z_n := -\frac{i}{(2n+1)\pi} \quad \text{for } n = 0, \pm 1, \pm 2, \dots$$

For $|n| \geq N$ we have $|z_n| < \varepsilon \Rightarrow z_n \in D_\varepsilon$ and

$$e^{1/z_n} = e^{-\frac{(2n+1)\pi i}{i}} = e^{i(2n+1)\pi} = -1$$

\therefore For any neighborhood of the origin, $e^{1/z} = -1$ for infinitely many z in the neighborhood.

* * * *

Chapter 6 Section 74: 1, 2, 3, 6, 7 (pg 248)

1. In each case, show that any singular point of the function is a pole. Determine the order m of each pole, and find the corresponding residue B .

$$(a) \frac{z^2+2}{z-1}; \quad (b) \left(\frac{z}{2z+1}\right)^3; \quad (c) \frac{\exp z}{z^2+\pi^2}.$$

Ans. (a) $m = 1, B = 3$; (b) $m = 3, B = -3/16$; (c) $m = 1, B = \pm i/2\pi$.

Theorem. An isolated singular point z_0 of a function f is a pole of order m if and only if $f(z)$ can be written in the form

$$(1) \quad f(z) = \frac{\phi(z)}{(z-z_0)^m},$$

where $\phi(z)$ is analytic and nonzero at z_0 . Moreover,

$$(2) \quad \underset{z=z_0}{\text{Res}} f(z) = \phi(z_0) \quad \text{if } m = 1$$

and

$$(3) \quad \underset{z=z_0}{\text{Res}} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \quad \text{if } m \geq 2.$$

$$(a) \quad f(z) = \frac{z^2+2}{z-1} = \frac{\phi(z)}{z-z_0}$$

$\phi(z) = z^2+2$ is analytic and nonzero at $z_0 = 1$

By the theorem, $m=1$ and $B=\phi(1)=3$

$$(b) \quad f(z) = \frac{z^3/8}{(z-(-1/2))^3} = \frac{\phi(z)}{(z-z_0)^3}$$

$\phi(z) = z^3/8$ is analytic and nonzero at $z_0 = -1/2$

By the theorem, $m=3$ and $B = \frac{1}{2!} \phi''(-\frac{1}{2})$

$$B = \underset{z=-1/2}{\text{Res}} f(z) = \frac{1}{2} \cdot 3 \cdot 2 \cdot (-1/2) \cdot \frac{1}{8} = -\frac{3}{16}$$

(c) $f(z) = \frac{e^z}{z^2 + \pi^2}$ has isolated singularities at $z_0 = \pm i\pi$.

• $z_0 = i\pi$:

$$f(z) = \frac{e^z}{(z - i\pi)(z + i\pi)} = \frac{e^z / (z + i\pi)}{z - i\pi} = \frac{\phi(z)}{z - z_0}$$

$\phi(z) = \frac{e^z}{z + i\pi}$ is analytic and nonzero at z_0

$$m = 1, \quad B = \phi(z_0) = \frac{e^{i\pi}}{2i\pi} = \frac{i}{2\pi}$$

• $z_0 = -i\pi$:

$$f(z) = \frac{e^z}{(z - i\pi)(z + i\pi)} = \frac{e^z / (z - i\pi)}{z - (-i\pi)} = \frac{\phi(z)}{z - z_0}$$

$\phi(z) = \frac{e^z}{z - i\pi}$ is analytic and nonzero at z_0

$$m = 1, \quad B = \phi(z_0) = \frac{e^{-i\pi}}{-2i\pi} = -\frac{i}{2\pi}$$

2. Show that

- (a) $\operatorname{Res}_{z=-1} \frac{z^{1/4}}{z+1} = \frac{1+i}{\sqrt{2}}$ ($|z| > 0, 0 < \arg z < 2\pi$);
- (b) $\operatorname{Res}_{z=i} \frac{\operatorname{Log} z}{(z^2+1)^2} = \frac{\pi+2i}{8}$;
- (c) $\operatorname{Res}_{z=i} \frac{z^{1/2}}{(z^2+1)^2} = \frac{1-i}{8\sqrt{2}}$ ($|z| > 0, 0 < \arg z < 2\pi$).

(a) $z^{1/4} = \sqrt[4]{|z|} \exp\left[\frac{i}{4}\operatorname{Arg} z + k\pi/2\right], k=0,1,2,3$

For a single-valued fn, let

$$\phi(z) = \sqrt[4]{|z|} \exp\left[\frac{i}{4}\operatorname{Arg} z\right] \quad (k=0)$$

$$\operatorname{Res}_{z=-1} \frac{z^{1/4}}{z+1} = \phi(-1) = e^{i\pi/4} = \boxed{\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}}$$

(b) $\operatorname{Log} z = \ln|z| + i\operatorname{Arg} z, z \neq 0$

$$f(z) = \frac{\operatorname{Log} z}{(z^2+1)^2} = \frac{(\operatorname{Log} z)/(z+i)^2}{(z-i)^2} = \frac{\phi(z)}{(z-i)^2}$$

$\phi(z) = (\operatorname{Log} z)/(z+i)^2$ is analytic and non-zero at the order $m=2$ pole at $z_0 = i$.

We will need $\phi'(z)$:

$$\phi'(z) = \frac{(z+i)^2/z - 2(z+i)\operatorname{Log} z}{(z+i)^4}$$

$$\begin{aligned}\operatorname{Res}_{z=i} f(z) &= \phi'(z_0) = \frac{(2i)^2/i - 4i \log i}{(2i)^4} \\ &= \frac{4i - 4i^2\pi/2}{16} = \boxed{\frac{2i + \pi}{16}}\end{aligned}$$

$$(c) f(z) = \frac{z^{1/2}/(z+i)^2}{(z-i)^2} = \frac{\phi(z)}{(z-i)^2}$$

where we take the principal root of
 $z^{1/2} = \sqrt{|z|} \exp(i/2 \cdot \operatorname{Arg} z + k\pi)$ $k=0, 1$.
 That is, use $k=0$ so that $\phi(z)$ is
 single valued on $|z|>0$, $0<\arg z<2\pi$.

$\phi(z)$ is analytic and nonzero at $z_0=i$.

$$\phi'(z) = \frac{1/2 z^{-1/2} (z+i)^2 - 2z^{1/2}(z+i)}{(z+i)^4}$$

$$\begin{aligned}\operatorname{Res}_{z=i} f(z) &= \phi'(i) = \frac{\frac{(2i)^2\sqrt{2}}{2+2i} - \frac{(2+2i)}{\sqrt{2}}(2i)}{16} \\ &= \frac{1}{16} \left[\frac{-4 \cdot 2 \cdot (2-2i)}{8\sqrt{2}} - \frac{(2+2i) \cdot (16i)}{8\sqrt{2}} \right]\end{aligned}$$

$$\begin{aligned}&= \frac{1}{16} \left[\frac{-16 + 16i - 32i + 32}{8\sqrt{2}} \right] = \boxed{\frac{1-i}{8\sqrt{2}}}\end{aligned}$$

3. Find the value of the integral

$$\int_C \frac{3z^3 + 2}{(z-1)(z^2+9)} dz,$$

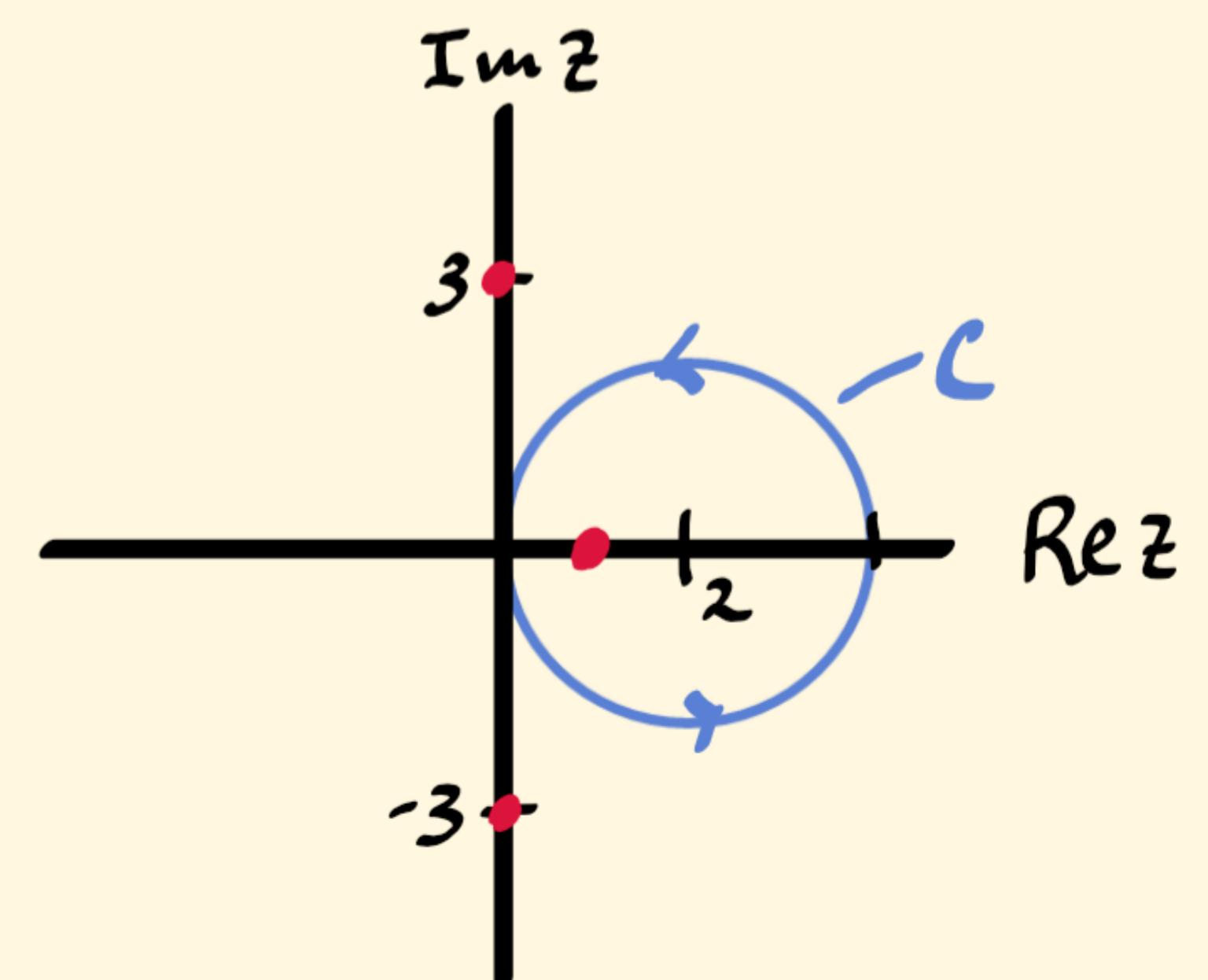
taken counterclockwise around the circle (a) $|z-2|=2$; (b) $|z|=4$.

Ans. (a) πi ; (b) $6\pi i$.

The integrand $f(z) = \frac{3z^3 + 2}{(z-1)(z^2+9)}$ has isolated singularities at $z_0 = 1, \pm 3i$ (all simple poles)

(a) The singular pt. $z_0 = 1$ is enclosed by C

$\phi(z) = (3z^3 + 2)/(z^2 + 9)$ is nonzero and analytic at $z_0 = 1$.



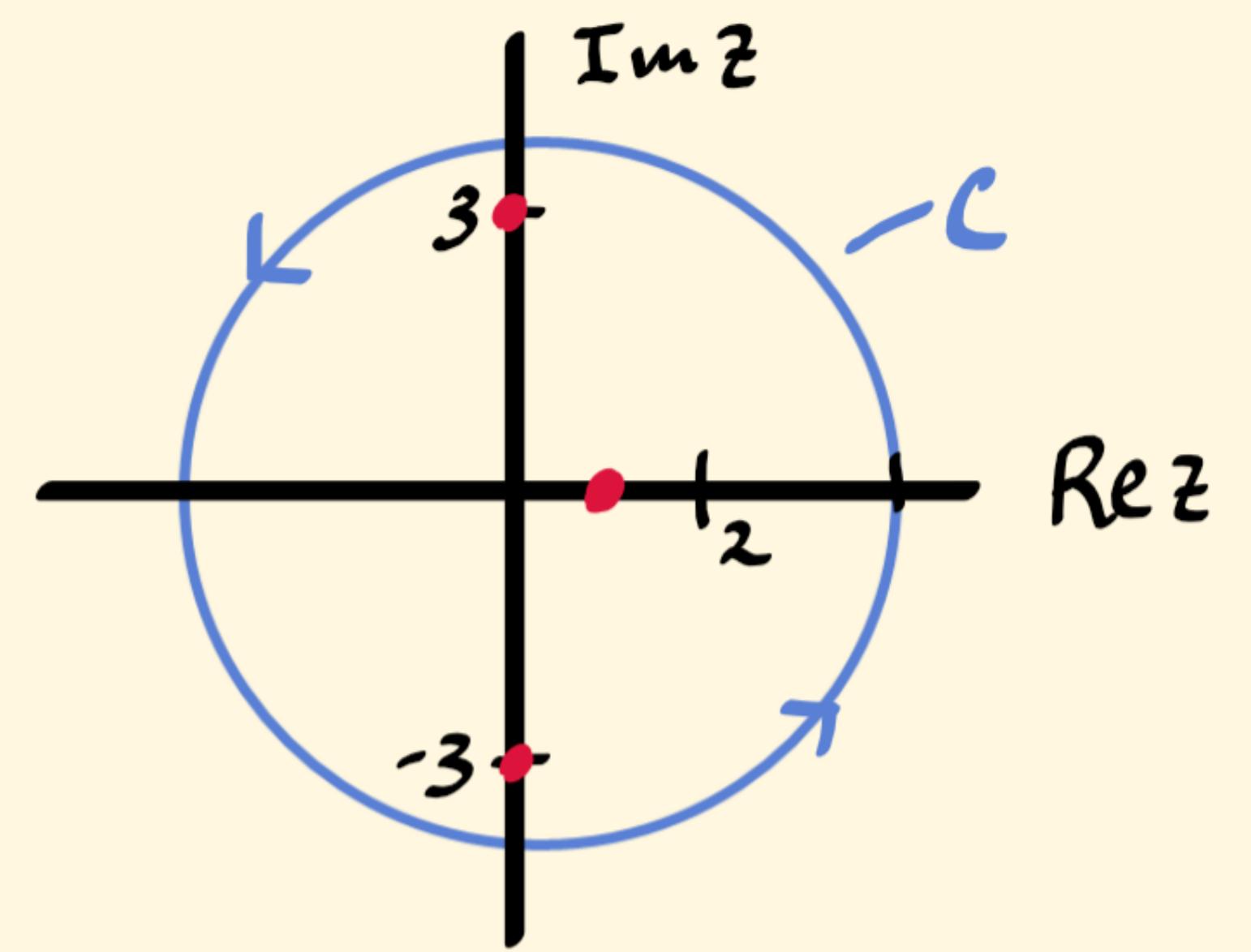
Use Cauchy's Residue Theorem :

Theorem. Let C be a simple closed contour, described in the positive sense. If a function f is analytic inside and on C except for a finite number of singular points z_k ($k = 1, 2, \dots, n$) inside C (Fig. 87), then

$$(1) \quad \int_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}_{z=z_k} f(z).$$

$$\int_C f(z) dz = 2\pi i \text{Res}_{z=1} f(z) = 2\pi i \phi(1) = 2\pi i \cdot \frac{5}{10} = \pi i$$

(b) All three singular points are enclosed by C .



To calculate $\int_C f(z) dz$, find the residues at each singular pt. and apply Cauchy's Residue Theorem

- $\underset{z=-1}{\text{Res}} f(z) = \frac{1}{2}$ by part a.

- For $z_0 = 3i$, let $\phi(z) = \frac{3z^3 + 2}{(z-1)(z+3i)}$

Since $\phi(z)$ is nonzero and analytic at $3i$

$$\underset{z=3i}{\text{Res}} f(z) = \phi(3i) = \frac{3(3i)^3 + 2}{(3i-1)(6i)} = \frac{1}{12}(15 + 49i)$$

- For $z_0 = -3i$, let $\phi(z) = \frac{3z^3 + 2}{(z-1)(z-3i)}$

Since $\phi(z)$ is nonzero and analytic at $-3i$

$$\underset{z=-3i}{\text{Res}} f(z) = \phi(-3i) = \frac{3(-3i)^3 + 2}{(-3i-1)(-6i)} = -\frac{1}{12}(-15 + 49i)$$

$$\begin{aligned} \int_C f(z) dz &= 2\pi i \left[\frac{1}{2} + \frac{1}{12}(15 + 49i) - \frac{1}{12}(-15 + 49i) \right] \\ &= 2\pi i \left[\frac{1}{2} + \frac{5}{2} \right] = 6\pi i \end{aligned}$$

6. Use the theorem in Sec. 71, involving a single residue, to evaluate the integral of $f(z)$ around the positively oriented circle $|z| = 3$ when

$$(a) f(z) = \frac{(3z+2)^2}{z(z-1)(2z+5)}; \quad (b) f(z) = \frac{z^3(1-3z)}{(1+z)(1+2z^4)}; \quad (c) f(z) = \frac{z^3 e^{1/z}}{1+z^3}.$$

Ans. (a) $9\pi i$; (b) $-3\pi i$; (c) $2\pi i$.

Theorem. If a function f is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C , then

$$(7) \quad \int_C f(z) dz = 2\pi i \operatorname{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right) \right].$$

$$(a) \quad \frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} \cdot \frac{(3/z+2)^2}{1/z(1/z-1)(2/z+5)} \cdot \left(\frac{z^2}{z^2}\right)$$

$$= \frac{(3+2z)^2}{z(1-z)(2+5z)}$$

$$\phi(z) = \frac{(3+2z)^2}{(1-z)(2+5z)} \Rightarrow \frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{\phi(z)}{z}$$

Since $\phi(z)$ is analytic and nonzero at $z_0=0$,

$$\operatorname{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right) \right] = \phi(0) = \frac{3^2}{1 \cdot 2} = \frac{9}{2}$$

$$\int_C f(z) dz = 2\pi i \cdot \operatorname{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right) \right] = \boxed{9\pi i}$$

$$(b) \quad \frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} \cdot \frac{\frac{1}{z^3}(1-3/z)}{(1+\frac{1}{z})(1+2/z^4)} \cdot \frac{z^4}{z^4}$$

$$= \frac{z^3 - 3}{z(z+1)(z^4+2)}$$

$$\phi(z) = \frac{z^3 - 3}{(z+1)(z^4+2)} \Rightarrow \frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{\phi(z)}{z}$$

Since $\phi(z)$ is analytic and nonzero at $z_0=0$,

$$\int_C f(z) dz = 2\pi i \cdot \operatorname{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right) \right] = 2\pi i \phi(0)$$

$$= 2\pi i \cdot \frac{-3}{1 \cdot 2} = \boxed{-3\pi i}$$

$$(c) \quad \frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} \cdot \frac{z^{-3} e^z}{1+z^{-3}} \cdot \frac{z^3}{z^3} = \frac{e^z}{z^2(z^3+1)}$$

$$\phi(z) = \frac{e^z}{z^3+1} \Rightarrow \frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{\phi(z)}{z^2}$$

$$\phi'(z) = \frac{(z^3+1)e^z - 3z^2e^z}{(z^3+1)^2} \quad \begin{cases} \text{Needed since } 0 \\ \text{is an order 2 pole} \end{cases}$$

$$\int_C f(z) dz = 2\pi i \cdot \phi'(0) = 2\pi i \cdot \frac{1-0}{1^2} = \boxed{2\pi i}$$

7. Let z_0 be an isolated singular point of a function f and suppose that

$$f(z) = \frac{\phi(z)}{(z - z_0)^m},$$

where m is a positive integer and $\phi(z)$ is analytic and nonzero at z_0 . By applying the extended form (6), Sec. 51, of the Cauchy integral formula to the function $\phi(z)$,

show that

$$\operatorname{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!},$$

as stated in the theorem of Sec. 73.

Suggestion: Since there is a neighborhood $|z - z_0| < \varepsilon$ throughout which $\phi(z)$ is analytic (see Sec. 24), the contour used in the extended Cauchy integral formula can be the positively oriented circle $|z - z_0| = \varepsilon/2$.

* * * *

Chapter 6 Section 77: 1, 2, 4, 6 (pg 255)

1. Show that the point $z = 0$ is a simple pole of the function

$$f(z) = \csc z = \frac{1}{\sin z}$$

and that the residue there is unity by appealing to

- (a) Theorem 2 in Sec. 76;
- (b) the Laurent series for $\csc z$ that was found in Exercise 2, Sec. 67.

(a)

$$\begin{aligned} p(z) &= 1 \\ q(z) &= \sin z \end{aligned} \Rightarrow \begin{cases} P, q \text{ analytic at } 0 \\ P(0) \neq 0 \\ q(0) = 0 \\ q'(0) = \cos 0 = 1 \neq 0 \end{cases}$$

Theorem 2. Let two functions p and q be analytic at a point z_0 . If

$$p(z_0) \neq 0, \quad q(z_0) = 0, \quad \text{and} \quad q'(z_0) \neq 0,$$

then z_0 is a simple pole of the quotient $p(z)/q(z)$ and

$$(2) \quad \operatorname{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.$$

By Thm 2, 0 is a simple pole of $f(z) = \frac{p(z)}{q(z)}$ and

$$\operatorname{Res}_{z=0} f(z) = \operatorname{Res}_{z=0} \frac{p(z)}{q(z)} = \frac{p(0)}{q'(0)} = \frac{1}{1} = 1$$

(b)

2. By writing $\csc z = 1/\sin z$ and then using division, show that

$$\csc z = \frac{1}{z} + \frac{1}{3!}z + \left[\frac{1}{(3!)^2} - \frac{1}{5!} \right] z^3 + \dots \quad (0 < |z| < \pi).$$

(Section 67)
(pg 225)

Since $f(z)$ has the form $f(z) = \frac{b_1}{z} + \sum_{n=0}^{\infty} a_n z^n$,

$z_0 = 0$ is a pole of order 1 (simple). The residue is $b_1 = 1$

2. Show that

$$(a) \operatorname{Res}_{z=\pi i} \frac{z - \sinh z}{z^2 \sinh z} = \frac{i}{\pi};$$

$$(b) \operatorname{Res}_{z=\pi i} \frac{\exp(zt)}{\sinh z} + \operatorname{Res}_{z=-\pi i} \frac{\exp(zt)}{\sinh z} = -2 \cos(\pi t).$$

$$(a) \quad p(z) = \frac{z - \sinh z}{z^2} \quad q(z) = \sinh z \quad \Rightarrow \quad \begin{cases} P, q \text{ analytic at } i\pi \\ P(i\pi) \neq 0 \\ q(i\pi) = 0 \\ q'(i\pi) = \cosh i\pi = -1 \neq 0 \end{cases}$$

By Thm 2 (Pg 253),

$$\begin{aligned} \operatorname{Res}_{z=i\pi} \frac{(z - \sinh z)/z^2}{\sinh z} &= \frac{P(i\pi)}{q'(i\pi)} \\ &= \frac{i\pi/(i\pi)^2}{-1} = \frac{i}{\pi} \end{aligned}$$

$$(b) \quad p(z) = \exp(zt) \quad q(z) = \sinh z \quad \Rightarrow \quad \begin{cases} P, q \text{ analytic at } \pm i\pi \\ P(\pm i\pi) \neq 0 \\ q(\pm i\pi) = 0 \\ q'(\pm i\pi) = \cosh[\pm i\pi] = -1 \neq 0 \end{cases}$$

By Thm 2 (Pg 253),

$$\operatorname{Res}_{z=i\pi} \frac{\exp(zt)}{\sinh z} + \operatorname{Res}_{z=-i\pi} \frac{\exp(zt)}{\sinh z} = \frac{P(i\pi)}{q'(i\pi)} + \frac{P(-i\pi)}{q'(-i\pi)}$$

$$= -e^{i\pi t} - e^{-i\pi t} = -(e^{i\pi t} + e^{-i\pi t}) = \underline{-2 \cos(\pi t)}$$

4. Let C denote the positively oriented circle $|z| = 2$ and evaluate the integral

$$(a) \int_C \tan z \, dz; \quad (b) \int_C \frac{dz}{\sinh 2z}.$$

Ans. (a) $-4\pi i$; (b) $-\pi i$.

?

(a) $\tan z = \frac{\sin z}{\cos z}$ has isolated singularities at $z = \pi/2 + n\pi, n \in \mathbb{Z}$. Of these, $z_0 = \pm\pi/2$ are enclosed by C .

$$P(z) = \sin z \quad q(z) = \cos z \Rightarrow \begin{cases} P, q \text{ analytic at } \pm\pi/2 \\ P(\pm\pi/2) \neq 0 \\ q(\pm\pi/2) = 0 \\ q'(\pm\pi/2) = -\sin(\pm\pi/2) = \pm 1 \neq 0 \end{cases}$$

$$\underset{z=\pi/2}{\operatorname{Res}} \tan z = \frac{P(\pi/2)}{q'(\pi/2)} = \frac{\sin(\pi/2)}{-\sin(\pi/2)} = -1$$

$$\underset{z=-\pi/2}{\operatorname{Res}} \tan z = \frac{P(-\pi/2)}{q'(-\pi/2)} = \frac{\sin(-\pi/2)}{-\sin(-\pi/2)} = -1$$

$$\int_C \tan z \, dz = 2\pi i \cdot \left[\underset{z=\pi/2}{\operatorname{Res}} \tan z + \underset{z=-\pi/2}{\operatorname{Res}} \tan z \right]$$

$$= 2\pi i \cdot (-2) = \boxed{-4\pi i}$$

? (b) $\frac{1}{\sinh 2z}$ has isolated singularities at $z_0 = \pm i\pi/2$:

For $z = x+iy$, $x, y \in \mathbb{R}$

$$0 = \sinh 2z = \sinh 2x \cos 2y + i \cosh 2x \sin 2y$$

$$\Leftrightarrow 0 = \cosh 2x \sin 2y \quad \text{and} \quad 0 = \sinh 2x \cos 2y$$

$$\text{Since } \cosh 2x > 0 \quad \forall x, \quad 0 = \sin 2y \Rightarrow y = m\pi/2$$

$$\text{Since } \cos(2m\pi/2) = \pm 1 \neq 0, \quad 0 = \sinh 2x \Rightarrow x = 0$$

Then $\sinh z = 0$ for $z = im\pi/2$. Of these pts, $z_0 = \pm i\pi/2$ are enclosed by C ($m = \pm 1$).

$$P(z) = 1 \quad q(z) = \sinh 2z \quad \Rightarrow \quad \begin{cases} P, q \text{ analytic at } \pm i\pi/2 \\ P(\pm i\pi/2) = 1 \neq 0 \\ q(\pm i\pi/2) = 0 \\ q'(\pm i\pi/2) = 2\cosh(\pm i\pi/2) = -2 \neq 0 \end{cases}$$

$$\int_C \frac{1}{\sinh 2z} dz = 2\pi i \cdot \left[\frac{P(i\pi/2)}{q'(i\pi/2)} + \frac{P(-i\pi/2)}{q'(-i\pi/2)} \right]$$

$$= 2\pi i \cdot \left[\frac{1}{-2} + \frac{1}{-2} \right] = \boxed{-2\pi i}$$

6. Show that

$$\int_C \frac{dz}{(z^2 - 1)^2 + 3} = \frac{\pi}{2\sqrt{2}},$$

where C is the positively oriented boundary of the rectangle whose sides lie along the lines $x = \pm 2$, $y = 0$, and $y = 1$.

Suggestion: By observing that the four zeros of the polynomial $q(z) = (z^2 - 1)^2 + 3$ are the square roots of the numbers $1 \pm \sqrt{3}i$, show that the reciprocal $1/q(z)$ is analytic inside and on C except at the points

$$z_0 = \frac{\sqrt{3} + i}{\sqrt{2}} \quad \text{and} \quad -\bar{z}_0 = \frac{-\sqrt{3} + i}{\sqrt{2}}.$$

Then apply Theorem 2 in Sec. 76.

$$0 = (z^2 - 1)^2 + 3$$

$$z^2 = 1 \pm \sqrt{3}i = 2e^{i(\pm \pi/3 + 2n\pi)}$$

$$z = \sqrt{2} e^{i(\pm \pi/6 + n\pi)} \quad (n = 0, 1)$$

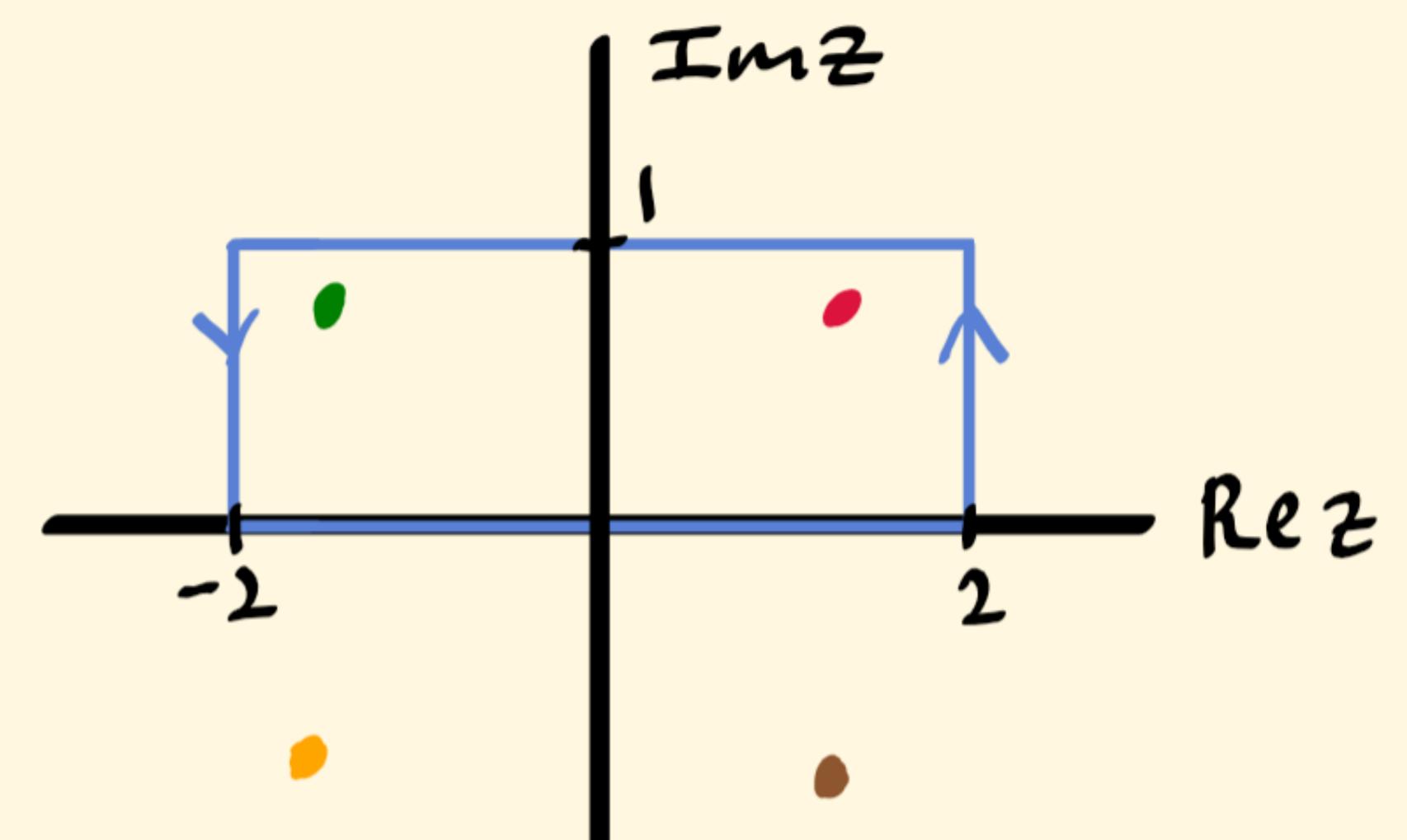
This provides the four zeroes of $q(z)$:

$$\bullet z_0 = \sqrt{2} \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i \right) = \frac{\sqrt{3} + i}{\sqrt{2}}$$

$$\bullet z_1 = \sqrt{2} \left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i \right) = -\frac{\sqrt{3} + i}{\sqrt{2}}$$

$$\bullet z_2 = \sqrt{2} \left(\frac{\sqrt{3}}{2} - \frac{1}{2}i \right) = \frac{\sqrt{3} - i}{\sqrt{2}}$$

$$\bullet z_3 = \sqrt{2} \left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i \right) = \frac{-\sqrt{3} + i}{\sqrt{2}}$$



Out of these four zeroes, the first and fourth are enclosed by C (z_0, z_3).

Note that $-\bar{z}_0 = -\left(\frac{\sqrt{3}-i}{\sqrt{2}}\right) = \frac{-\sqrt{3}+i}{\sqrt{2}} = z_3$

Therefore the rational fn $\frac{1}{q(z)}$ is analytic at all pts on or enclosed by C except:

$$z_0 = \frac{\sqrt{3}+i}{\sqrt{2}} \quad \text{and} \quad -\bar{z}_0 = \frac{-\sqrt{3}+i}{\sqrt{2}}$$

$$\begin{aligned} p(z) &= 1 \\ q(z) &= (z^2-1)^2 + 3 \end{aligned} \Rightarrow \begin{cases} p, q \text{ analytic at } z_0, -\bar{z}_0 \\ p(z_0) = p(-\bar{z}_0) = 1 \neq 0 \\ q(z_0) = q(-\bar{z}_0) = 0 \\ q'(z_0) \neq 0, q'(-\bar{z}_0) \neq 0 \end{cases} *$$

$$q'(z) = 4z(z^2-1)$$

$$q'(z_0) = 4 \cdot \frac{\sqrt{3}+i}{\sqrt{2}} \left[\left(\frac{\sqrt{3}+i}{\sqrt{2}} \right)^2 - 1 \right] = -2\sqrt{6} + 6\sqrt{2}i *$$

$$q'(-\bar{z}_0) = 4 \cdot \frac{-\sqrt{3}+i}{\sqrt{2}} \left[\left(\frac{-\sqrt{3}+i}{\sqrt{2}} \right)^2 - 1 \right] = 2\sqrt{6} + 6\sqrt{2}i *$$

$$\int_C \frac{dz}{(z^2-1)^2 + 3} = 2\pi i \cdot \left[\frac{p(z_0)}{q'(z_0)} + \frac{p(-\bar{z}_0)}{q'(-\bar{z}_0)} \right]$$

$$= 2\pi i \left[\frac{1}{-2\sqrt{6} + 6\sqrt{2}i} + \frac{1}{2\sqrt{6} + 6\sqrt{2}i} \right] = \boxed{\frac{\pi}{2\sqrt{2}}}$$

(Several of these computations by Wolfram Alpha)