

MA 4291 Written Homework 1

Chapter 1 Section 4 : 1, 3, 4, 6 (Page 12)

1. Locate the numbers $z_1 + z_2$ and $z_1 - z_2$ vectorially when

- (a) $z_1 = 2i$, $z_2 = \frac{2}{3} - i$; (b) $z_1 = (-\sqrt{3}, 1)$, $z_2 = (\sqrt{3}, 0)$;
 (c) $z_1 = (-3, 1)$, $z_2 = (1, 4)$; (d) $z_1 = x_1 + iy_1$, $z_2 = x_1 - iy_1$.

$$(a) z_1 + z_2 = (0, 2) + (2/3, -1) = (2/3, 1)$$

$$z_1 - z_2 = (0, 2) - (2/3, -1) = (-2/3, 3)$$

$$(b) z_1 + z_2 = (-\sqrt{3}, 1) + (\sqrt{3}, 0) = (0, 1)$$

$$z_1 - z_2 = (-\sqrt{3}, 1) - (\sqrt{3}, 0) = (-2\sqrt{3}, 1)$$

$$(c) z_1 + z_2 = (-3, 1) + (1, 4) = (-2, 5)$$

$$z_1 - z_2 = (-3, 1) - (1, 4) = (-4, -3)$$

$$(d) z_1 + z_2 = (x, y) + (x, -y) = (x, 0)$$

$$z_1 - z_2 = (x, y) - (x, -y) = (0, 2y)$$

3. Use established properties of moduli to show that when $|z_3| \neq |z_4|$,

$$\frac{\operatorname{Re}(z_1 + z_2)}{|z_3 + z_4|} \leq \frac{|z_1| + |z_2|}{||z_3| - |z_4||}.$$

$$\frac{\operatorname{Re}(z_1 + z_2)}{|z_3 + z_4|} \leq \frac{|z_1 + z_2|}{|z_3 + z_4|}$$

$$\leq \frac{|z_1| + |z_2|}{|z_3 + z_4|}$$

$$\operatorname{Re} z \leq |\operatorname{Re} z| \leq |z|$$

triangle inequality:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

$$\leq \frac{|z_1| + |z_2|}{||z_3| - |z_4||}$$

reverse triangle inequality:

$$0 < ||z_3| - |z_4|| \leq |z_3 + z_4|$$

4. Verify that $\sqrt{2}|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z|$.

Suggestion: Reduce this inequality to $(|x| - |y|)^2 \geq 0$.

$$\begin{aligned}
 0 &\leq (|\operatorname{Re} z| - |\operatorname{Im} z|)^2 = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 - 2|\operatorname{Re} z||\operatorname{Im} z| \\
 (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 &\leq 2[(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2] - 2|\operatorname{Re} z||\operatorname{Im} z| \\
 (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 + 2|\operatorname{Re} z||\operatorname{Im} z| &= 2[(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2] \\
 (|\operatorname{Re} z| + |\operatorname{Im} z|)^2 &\leq 2[(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2] \\
 (|\operatorname{Re} z| + |\operatorname{Im} z|)^2 &\leq 2|z|^2 \quad (|z|^2 = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2) \\
 |\operatorname{Re} z| + |\operatorname{Im} z| &\leq \sqrt{2}|z| \quad (a^2 + b^2 \rightarrow a \leq b \text{ for } a, b \geq 0)
 \end{aligned}$$

?

6. Using the fact that $|z_1 - z_2|$ is the distance between two points z_1 and z_2 , give a geometric argument that

- (a) $|z - 4i| + |z + 4i| = 10$ represents an ellipse whose foci are $(0, \pm 4)$;
- (b) $|z - 1| = |z + i|$ represents the line through the origin whose slope is -1 .

Write $z = x + iy$ where $x = \operatorname{Re} z \in \mathbb{R}$, $y = \operatorname{Im} z \in \mathbb{R}$

? (a) $10 = |z - 4i| + |z + 4i| = |x + i(y-4)| + |x + i(y+4)|$

$$10 = [x^2 + (y-4)^2]^{1/2} + [x^2 + (y+4)^2]^{1/2}$$

$$10^2 = x^2 + (y-4)^2 + x^2 + (y+4)^2 + 2[(x^2 + (y-4)^2)(x^2 + (y+4)^2)]^{1/2}$$

~~X~~ $10^2 = 2x^2 + 2y^2 + 32 + 2[\text{Not a perfect square}]^{1/2}$

Another try:

$$10^2 = |z - 4i|^2 + |z + 4i|^2 + 2|(z - 4i)(z + 4i)|$$

$$= 2x^2 + 2y^2 + 32 + 2|z^2 + 16|$$

~~X~~ $= 2x^2 + 2y^2 + 32 + \left\{ \begin{array}{l} x^4 + 2x^2y^2 + 32x^2 + y^4 - 32y^2 + 256 \\ x^2(x^2 + 2y^2 + 32) + y^2(y^2 - 32) + 256 \\ y^2(2x^2 + y^2 - 32) + (x^2 + 32)x^2 + 256 \end{array} \right\}^{1/2}$

$$|z^2 + 16| = |x^2 - y^2 + 16 + i \cdot 2xy|$$

$$= [(x^2 - y^2 + 16)^2 + (2xy)^2]^{1/2}$$

Need to get $1 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$, foci $(0, \pm c)$ $c = \sqrt{b^2 - a^2} = 4$

(b) $|z - 1| = |z + i| \iff |z - 1|^2 = |z + i|^2 \iff |x - 1 + iy|^2 = |x + i(y+1)|^2$
 $\iff x^2 - 2x + 1 + y^2 = x^2 + y^2 + 2y + 1 \iff \boxed{y = -x}$

Alternate forms
$(x^2 + y^2 - 4 \times 2y + 16)(x^2 + y^2 + 2 \times 4y + 16)$
$x^2(x^2 + 2y^2 + 32) + y^2(y^2 - 32) + 256$
$y^2(2x^2 + y^2 - 32) + (x^2 + 32)x^2 + 256$
Expanded form
$x^4 + 2x^2y^2 + 32x^2 + y^4 - 32y^2 + 256$

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Chapter 1 Section 5 : 2, 7, 9, 10, 11, 13 (pages 14, 15)

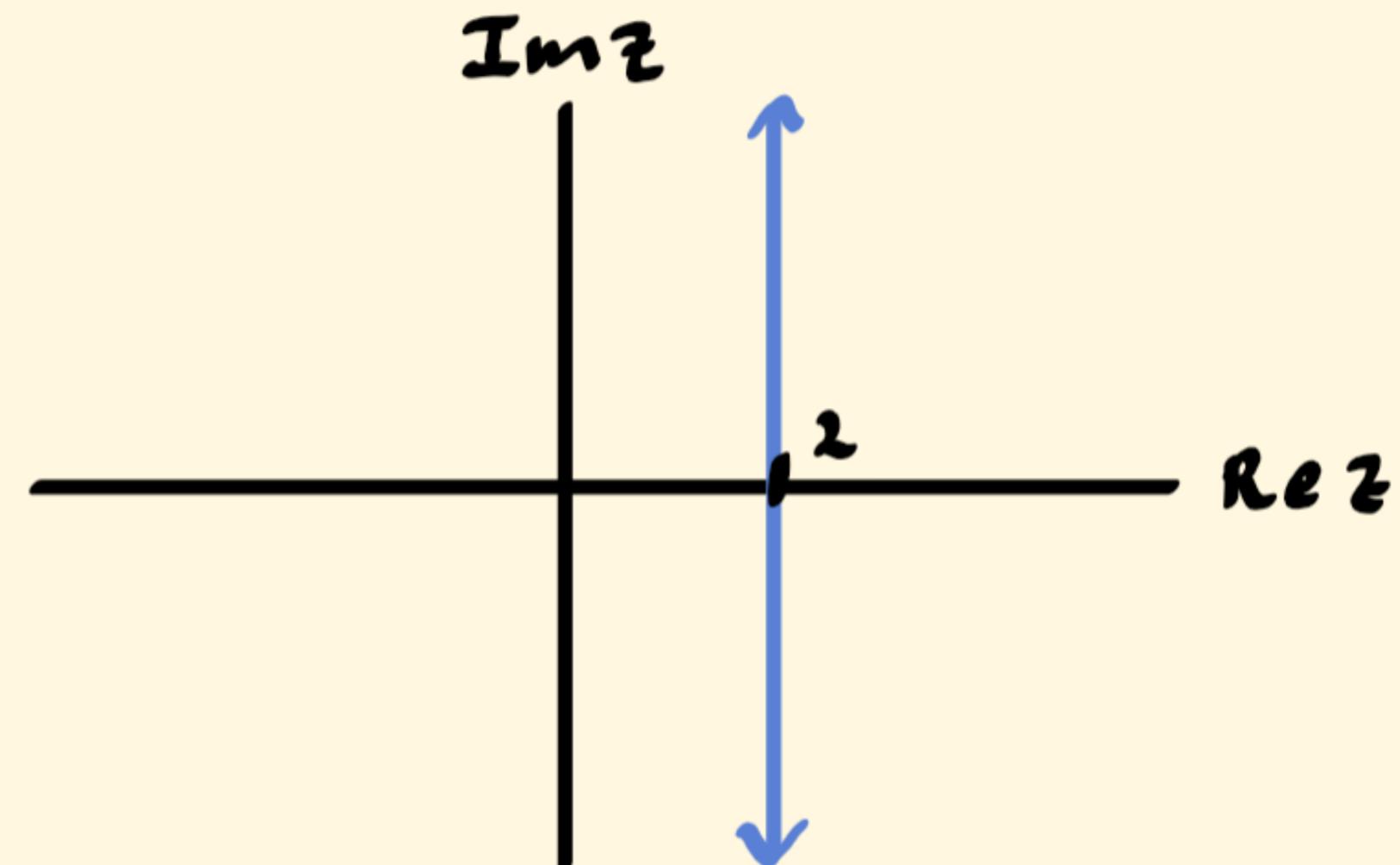
2. Sketch the set of points determined by the condition

$$(a) \operatorname{Re}(\bar{z} - i) = 2; \quad (b) |2\bar{z} + i| = 4.$$

Let $z = x + iy \quad x, y \in \mathbb{R}$

$$(a) z = \operatorname{Re}(\bar{z} - i) = \operatorname{Re}(x + (-1-y)i) = x$$

All $z = x + iy$ with $x = 2$

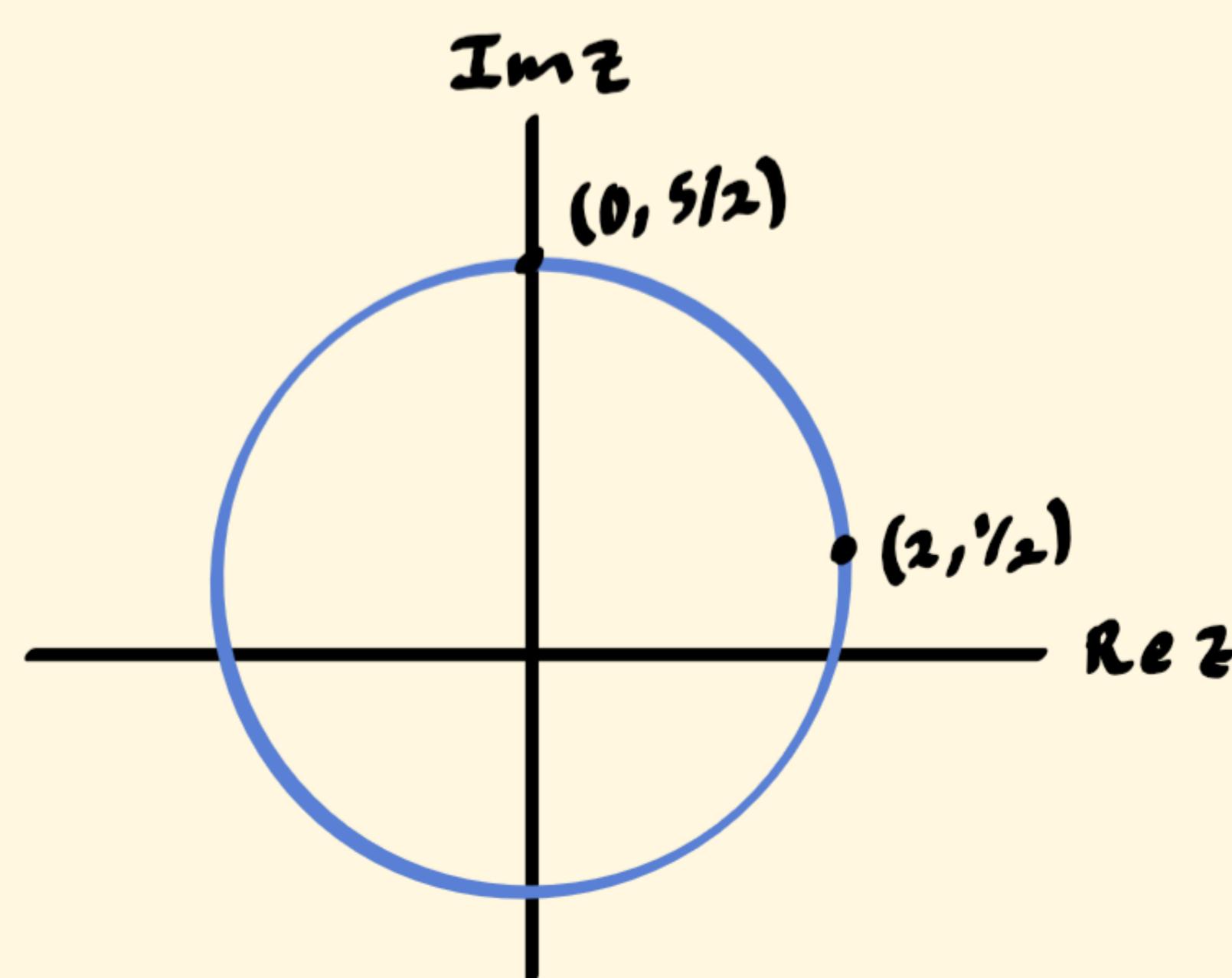


$$(b) 4 = |2\bar{z} + i| = |2x - 2yi + i|$$

$$4^2 = 4x^2 + 4(y - \frac{1}{2})^2$$

$$2^2 = x^2 + (y - \frac{1}{2})^2$$

All z on the circle of radius 2 centered at $(0, \frac{1}{2})$ in the complex plane



7. Show that

$$|\operatorname{Re}(2 + \bar{z} + z^3)| \leq 4 \quad \text{when } |z| \leq 1.$$

$$\begin{aligned} |\operatorname{Re}(2 + \bar{z} + z^3)| &\leq |2 + \bar{z} + z^3| \\ &\leq |2| + |\bar{z}| + |z^3| \\ &= 2 + |z| + |z|^3 \\ &\leq 2 + 1 + 1^3 = 4 \end{aligned}$$

$$\begin{aligned} |\operatorname{Re} w| &\leq |w| \quad \forall w \in \mathbb{C} \\ &\text{triangle inequality} \\ |\bar{z}| &= |z|, |z^3| = |z|^3 \\ |z| &\leq 1 \end{aligned}$$

10. Prove that

- (a) z is real if and only if $\bar{z} = z$;
- (b) z is either real or pure imaginary if and only if $\bar{z}^2 = z^2$.

Let $z = x + iy$ with $x, y \in \mathbb{R}$

$$(a) x + iy = z = \bar{z} = x - iy \rightarrow y = 0$$

$$\therefore z = x \in \mathbb{R}$$

$$(b) x^2 - y^2 + 2xyi = z^2 = \bar{z}^2 = x^2 - y^2 - 2xyi$$

$$\Rightarrow xy = 0 \Rightarrow x = 0 \text{ or } y = 0$$

If $x = 0$, z is pure imaginary. If $y = 0$, z is real.

11. Use mathematical induction to show that when $n = 2, 3, \dots$,

$$(a) \overline{z_1 + z_2 + \dots + z_n} = \overline{z_1} + \overline{z_2} + \dots + \overline{z_n}; \quad (b) \overline{z_1 z_2 \dots z_n} = \overline{z_1} \overline{z_2} \dots \overline{z_n}.$$

(a) Base case $n=2$: If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$,

$$\overline{z_1 + z_2} = \overline{(x_1 + x_2) + i(y_1 + y_2)} = (x_1 + x_2) - i(y_1 + y_2) = x_1 - iy_1 + x_2 - iy_2 = \overline{z}_1 + \overline{z}_2$$

Inductive step: Assume $\overline{z_1 + \dots + z_n} = \overline{z}_1 + \dots + \overline{z}_n$ for some $n \geq 2$

$$\overline{z_1 + \dots + z_n + z_{n+1}} = \overline{\overline{z}_1 + \dots + \overline{z}_n + \overline{z}_{n+1}} \quad (\text{by the base case})$$

$$= \overline{z}_1 + \dots + \overline{z}_n + \overline{z}_{n+1} \quad (\text{by the inductive hypothesis})$$

(b) Base case $n=2$: If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$,

$$\overline{z_1 z_2} = \overline{(x_1 + iy_1)(x_2 + iy_2)} = \overline{(x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)}$$

$$= (x_1 x_2 - y_1 y_2) - i(x_1 y_2 + x_2 y_1)$$

$$= (x_1 - iy_1)(x_2 - iy_2) = \overline{z}_1 \overline{z}_2$$

Inductive step: Assume $\overline{z_1 z_2 \dots z_n} = \overline{z}_1 \overline{z}_2 \dots \overline{z}_n$ for some $n \geq 2$

$$\overline{z_1 \dots z_n z_{n+1}} = \overline{\overline{z}_1 \dots \overline{z}_n \overline{z}_{n+1}} \quad (\text{by the base case})$$

$$= \overline{z}_1 \dots \overline{z}_n \overline{z}_{n+1} \quad (\text{by the inductive hypothesis})$$

13. Show that the equation $|z - z_0| = R$ of a circle, centered at z_0 with radius R , can be written

$$|z|^2 - 2 \operatorname{Re}(z\overline{z_0}) + |z_0|^2 = R^2.$$

$|z - z_0| = R$ iff $|z - z_0|^2 = R^2$ since $|z - z_0|, R \geq 0$

$$R^2 = |z - z_0|^2 = (z - z_0)(\overline{z - z_0})$$

$$= (z - z_0)(\bar{z} - \bar{z}_0)$$

$$= z\bar{z} + z_0\bar{z}_0 - z\bar{z}_0 - \bar{z}z_0$$

$$= |z|^2 + |z_0|^2 - (z\bar{z}_0 + \bar{z}\bar{z}_0)$$

$$= |z|^2 + |z_0|^2 - 2 \operatorname{Re}(z\bar{z}_0)$$

$$|w|^2 = w\bar{w} \quad \forall w \in \mathbb{C}$$

$$\overline{v-w} = \overline{v} - \overline{w} \quad \forall v, w \in \mathbb{C}$$

$$\overline{vw} = \bar{v}\bar{w} = \bar{v}w \quad \forall v, w \in \mathbb{C}$$

$$2 \operatorname{Re} w = w + \bar{w}$$

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Chapter 1 Section 8: 1, 3, 5, 9, 10 (pages 22, 23)

1. Find the principal argument Arg z when

$$(a) z = \frac{i}{-2-2i}; \quad (b) z = (\sqrt{3}-i)^6.$$

Ans. (a) $-3\pi/4$; (b) π .

$$(a) z_1 = i = e^{i(\pi/2 + 2k\pi)} \quad k \in \mathbb{Z}, \quad z_2 = -2-2i = 2\sqrt{2} e^{i(-3\pi/4 + 2j\pi)} \quad j \in \mathbb{Z}$$

$$\arg z_1 = \operatorname{Arg} z_1 + 2k\pi = \pi/2 + 2k\pi, \quad \arg z_2 = \operatorname{Arg} z_2 + 2j\pi = -3\pi/4 + 2j\pi$$

$$\arg z = \arg(z_1/z_2) = \arg[2\sqrt{2} e^{i(5\pi/4 + 2(k-j)\pi)}] = 5\pi/4 + 2n\pi, \quad n \in \mathbb{Z}$$

$$-\pi < \operatorname{Arg} z \leq \pi \rightarrow \operatorname{Arg} z = 5\pi/4 + 2 \cdot (-1)\pi = -3\pi/4$$

$$(b) w = \sqrt{3}-i = 2e^{-i\pi/6} \rightarrow z = w^6 = 2^6 e^{-i\pi}$$

$$\arg z = -\pi + 2n\pi \rightarrow \operatorname{Arg} z = -\pi + 2(1)\pi = \pi$$

3. Use mathematical induction to show that

$$e^{i\theta_1} e^{i\theta_2} \cdots e^{i\theta_n} = e^{i(\theta_1 + \theta_2 + \cdots + \theta_n)} \quad (n = 2, 3, \dots).$$

Base Case $n=2$:

$$\begin{aligned} e^{i\theta_1} e^{i\theta_2} &= (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \\ &= (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2, \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2) \\ &= \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) = e^{i(\theta_1 + \theta_2)} \end{aligned}$$

Inductive Step: Assume $e^{i\theta_1} e^{i\theta_2} \cdots e^{i\theta_n} = e^{i(\theta_1 + \theta_2 + \cdots + \theta_n)}$ for some $n \geq 2$

$$\begin{aligned} e^{i\theta_1} e^{i\theta_2} \cdots e^{i\theta_n} e^{i\theta_{n+1}} &= e^{i(\theta_1 + \cdots + \theta_n)} e^{i\theta_{n+1}} \quad (\text{by the inductive hypothesis}) \\ &= e^{i((\theta_1 + \cdots + \theta_n) + \theta_{n+1})} \quad (\text{by the base case}) \\ &= e^{i(\theta_1 + \cdots + \theta_{n+1})} \end{aligned}$$

5. By writing the individual factors on the left in exponential form, performing the needed operations, and finally changing back to rectangular coordinates, show that

$$(a) i(1 - \sqrt{3}i)(\sqrt{3} + i) = 2(1 + \sqrt{3}i); \quad (b) 5i/(2+i) = 1+2i;$$

$$(c) (-1+i)^7 = -8(1+i); \quad (d) (1+\sqrt{3}i)^{-10} = 2^{-11}(-1+\sqrt{3}i).$$

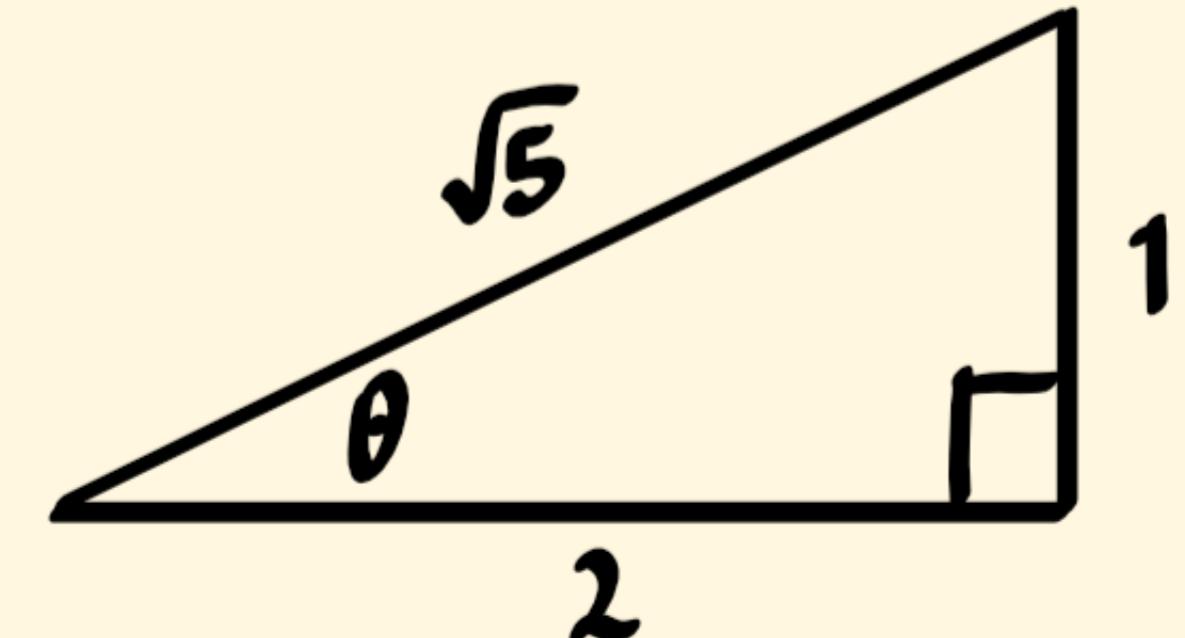
$$(a) i = e^{i\pi/2} \quad 1 - \sqrt{3}i = 2e^{-i\pi/3} \quad \sqrt{3} + i = 2e^{i\pi/6}$$

$$i(1 - \sqrt{3}i)(\sqrt{3} + i) = 4e^{i\pi(3-2+1)/6} = 4e^{i\pi/3} = 4(\cos \pi/3 + i \sin \pi/3) = 2(1 + \sqrt{3}i)$$

$$(b) 5i = 5e^{i\pi/2} \quad 2+i = \sqrt{5}e^{i\theta}, \quad \tan \theta = 1/2 \text{ with } \theta \text{ in quadrant 1}$$

$$\begin{aligned} 5i/(2+i) &= \sqrt{5}e^{i(\pi/2-\theta)} = \sqrt{5}(\cos(\pi/2-\theta) + i \sin(\pi/2-\theta)) \\ &= \sqrt{5}[(\cos \pi/2 \cos \theta + \sin \pi/2 \sin \theta) + i(\sin \pi/2 \cos \theta - \cos \pi/2 \sin \theta)] \\ &= \sqrt{5}(\sin \theta + i \cos \theta) \\ &= \sqrt{5}(1/\sqrt{5} + 2i/\sqrt{5}) \\ &= 1+2i \end{aligned}$$

$$\tan \theta = 1/2 \quad 0 < \theta < \pi/2$$



$$(c) -1+i = \sqrt{2}e^{3\pi i/4}$$

$$\begin{aligned} (-1+i)^7 &= \sqrt{2}^7 e^{21\pi i/4} = 8\sqrt{2} e^{16\pi i/4} e^{5\pi i/4} = 8\sqrt{2} e^{5\pi i/4} \\ &= 8\sqrt{2} (\cos 5\pi/4 + i \sin 5\pi/4) = 8\sqrt{2} (-\sqrt{2}/2 - \sqrt{2}i/2) \\ &= -8(1+i) \end{aligned}$$

$$(d) 1 + \sqrt{3}i = 2e^{i\pi/3}$$

$$\begin{aligned} (1 + \sqrt{3}i)^{-10} &= 2^{-10} e^{-10i\pi/3} = 2^{-10} e^{-12i\pi/3} e^{2\pi i/3} = 2^{-10} e^{2\pi i/3} \\ &= 2^{-10} (\cos 2\pi/3 + i \sin 2\pi/3) \\ &= 2^{-10} (-1/2 + \sqrt{3}i/2) \\ &= 2^{-11}(-1 + \sqrt{3}i) \end{aligned}$$

9. Establish the identity

$$1 + z + z^2 + \dots + z^n = \frac{1 - z^{n+1}}{1 - z} \quad (z \neq 1)$$

and then use it to derive Lagrange's trigonometric identity:

$$1 + \cos \theta + \cos 2\theta + \dots + \cos n\theta = \frac{1}{2} + \frac{\sin[(2n+1)\theta/2]}{2 \sin(\theta/2)} \quad (0 < \theta < 2\pi).$$

Suggestion: As for the first identity, write $S = 1 + z + z^2 + \dots + z^n$ and consider the difference $S - zS$. To derive the second identity, write $z = e^{i\theta}$ in the first one.

○ Establish the first identity by telescoping

$$S = 1 + z + z^2 + \dots + z^n \rightarrow zS = z + z^2 + \dots + z^{n+1}$$

$$S(1-z) = S - zS = (1 + z + \dots + z^n) - (z + z^2 + \dots + z^{n+1}) = 1 - z^{n+1}$$

$$\therefore 1 + z + \dots + z^n = S = \frac{1 - z^{n+1}}{1 - z}$$

△ Substitute $z = e^{i\theta}$ into this identity

$$1 + e^{i\theta} + e^{2i\theta} + \dots + e^{ni\theta} = \frac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}}$$

$$1 + \cos \theta + \dots + \cos n\theta + i(\sin \theta + \dots + \sin n\theta) = \frac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}}$$

□ Simplify the right hand side

$$\begin{aligned} \frac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}} &= \frac{e^{(n+1)i\theta/2}}{e^{i\theta/2}} \frac{e^{-i(n+1)\theta/2} - e^{(n+1)i\theta/2}}{e^{-i\theta/2} - e^{i\theta/2}} \\ &= e^{ni\theta/2} \frac{e^{-i(n+1)\theta/2} - e^{(n+1)i\theta/2}}{e^{-i\theta/2} - e^{i\theta/2}} \\ &= e^{ni\theta/2} \frac{\cos((n+1)\theta/2) - i\sin((n+1)\theta/2) - (\cos((n+1)\theta/2) + i\sin((n+1)\theta/2))}{\cos(\theta/2) - i\sin(\theta/2) - (\cos(\theta/2) + i\sin(\theta/2))} \\ &= e^{ni\theta/2} \frac{\sin((n+1)\theta/2)}{\sin(\theta/2)} = (\cos(n\theta/2) + i\sin(n\theta/2)) \frac{\sin((n+1)\theta/2)}{\sin(\theta/2)} \end{aligned}$$

◊ Match real terms and simplify using $\sin \alpha \cos \beta = \frac{1}{2}(\sin(\alpha-\beta) + \sin(\alpha+\beta))$

$$1 + \cos \theta + \dots + \cos n\theta = \operatorname{Re} \left[\frac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}} \right] = \frac{\cos(n\theta/2) \sin((n+1)\theta/2)}{\sin(\theta/2)}$$

$$1 + \cos \theta + \dots + \cos n\theta = \frac{1}{2} \frac{\sin(\theta/2) + \sin((2n+1)\theta/2)}{\sin(\theta/2)} = \frac{1}{2} + \frac{\sin((2n+1)\theta/2)}{2 \sin(\theta/2)}$$

◇ Alternatively, apply the first identity to both $z = e^{i\theta}$ and $z = e^{-i\theta}$.

$$\begin{aligned}
 1 + \cos\theta + \dots + \cos n\theta &= \sum_{k=0}^n \cos k\theta = \frac{1}{2} \sum_{k=0}^n (e^{ik\theta} + e^{-ik\theta}) \\
 &= \frac{1}{2} \left(\frac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}} + \frac{1 - e^{-(n+1)i\theta}}{1 - e^{-i\theta}} \right) \\
 &= \frac{1}{2} \left(\frac{e^{-i\theta/2} - e^{(n+1/2)i\theta}}{e^{-i\theta/2} - e^{i\theta/2}} + \frac{-e^{i\theta/2} + e^{-(n+1/2)i\theta}}{-e^{i\theta/2} + e^{-i\theta/2}} \right) \\
 &= \frac{1}{2} \left(\frac{e^{-(n+1/2)i\theta} - e^{(n+1/2)i\theta}}{e^{-i\theta/2} - e^{i\theta/2}} + \frac{e^{-i\theta/2} - e^{i\theta/2}}{e^{-i\theta/2} - e^{i\theta/2}} \right) \\
 &= \frac{1}{2} \left(\frac{-2i \sin((n+1/2)\theta)}{-2i \sin(\theta/2)} + 1 \right) \\
 &= \frac{1}{2} + \frac{\sin((2n+1)\theta/2)}{2 \sin(\theta/2)}
 \end{aligned}$$

10. Use de Moivre's formula (Sec. 7) to derive the following trigonometric identities:

$$(a) \cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta; \quad (b) \sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta.$$

de Moivre's formula : $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$, $n \in \mathbb{Z}$

Evaluate the formula with $n=3$.

$$\cos^3 \theta + 3i \cos^2 \theta \sin \theta - 3 \cos \theta \sin^2 \theta - i \sin^3 \theta = (\cos \theta + i \sin \theta)^3 = \cos 3\theta + i \sin 3\theta$$

$$(a) \text{Match real terms: } \cos^3 \theta - 3 \cos \theta \sin^2 \theta = \cos 3\theta$$

$$(b) \text{Match imaginary terms: } 3 \cos^2 \theta \sin \theta - \sin^3 \theta = \sin 3\theta$$

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Chapter 1 Section 10: 1, 3, 6, 7 (pages 29, 30)

1. Find the square roots of (a) $2i$; (b) $1 - \sqrt{3}i$ and express them in rectangular coordinates.

$$\text{Ans. (a)} \pm (1+i); \quad \text{(b)} \pm \frac{\sqrt{3}-i}{\sqrt{2}}.$$

$$(a) z = 2i = 2e^{i(\pi/2 + 2k\pi)} \Rightarrow z^{1/2} = \sqrt{2} e^{i(\pi/4 + k\pi)} \quad (k \in \mathbb{Z})$$

Evaluate at $k=0, 1$ to collect all distinct roots:

$$z^{1/2} = \sqrt{2} e^{i\pi/4}, \sqrt{2} e^{5\pi i/4} = \sqrt{2} (\sqrt{2}/2 + \sqrt{2}i/2), \sqrt{2} (-\sqrt{2}/2 - \sqrt{2}i/2) = \boxed{\pm(1+i)}$$

$$(b) z = 1 - \sqrt{3}i = 2e^{i(-\pi/3 + 2k\pi)} \Rightarrow z^{1/2} = \sqrt{2} e^{i(-\pi/6 + k\pi)} \quad (k \in \mathbb{Z})$$

Evaluate at $k=0, 1$ to collect all distinct roots:

$$z^{1/2} = \sqrt{2} e^{-i\pi/6}, \sqrt{2} e^{5\pi i/6} = \sqrt{2} (\sqrt{3}/2 - i/2), \sqrt{2} (-\sqrt{3}/2 + i/2) = \boxed{\pm(\sqrt{3} - i)/\sqrt{2}}$$

3. In each case, find all the roots in rectangular coordinates, exhibit them as vertices of certain regular polygons, and identify the principal root:

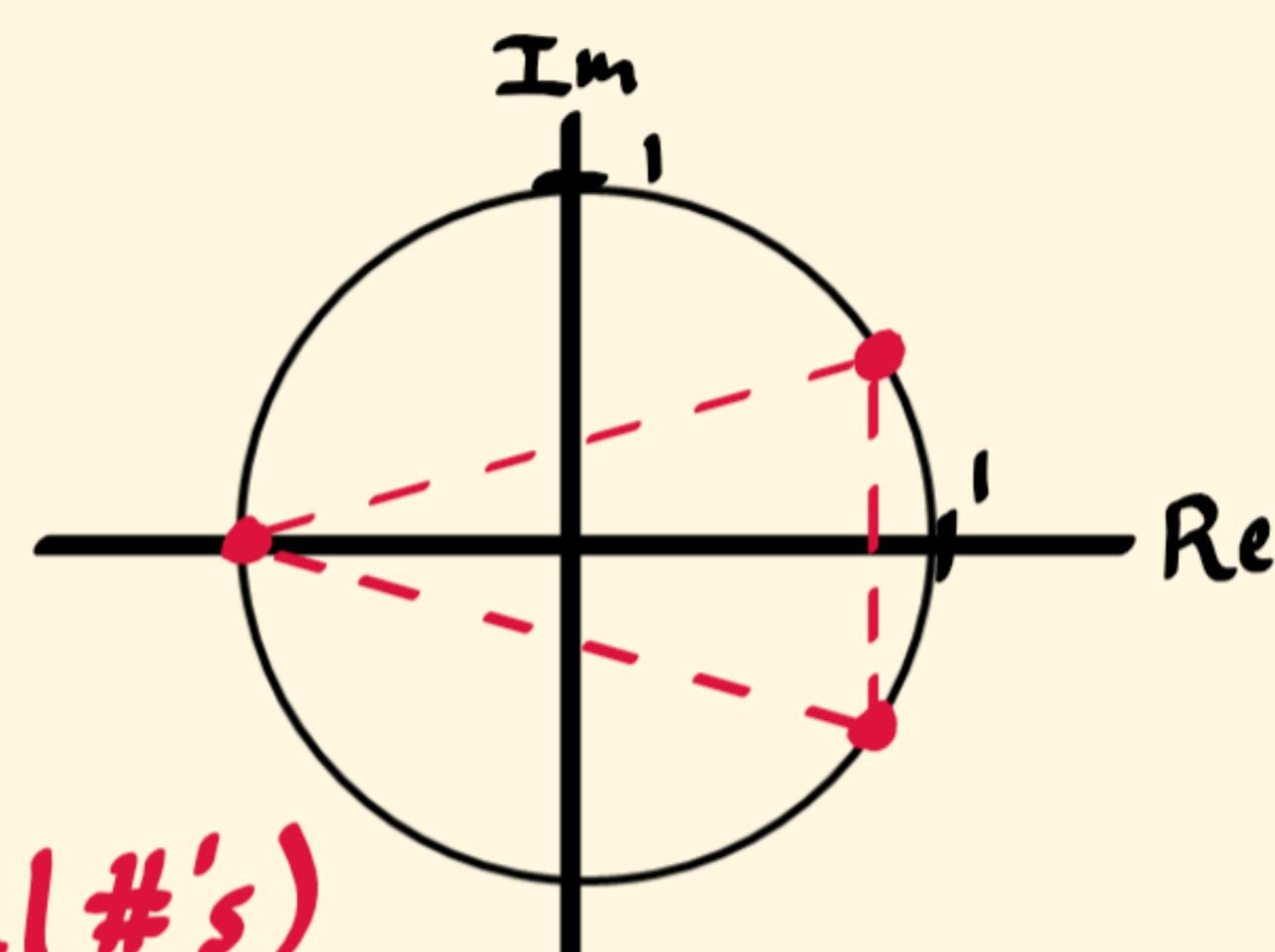
$$(a) (-1)^{1/3}; \quad (b) 8^{1/6}.$$

$$\text{Ans. (b)} \pm \sqrt{2}, \pm \frac{1+\sqrt{3}i}{\sqrt{2}}, \pm \frac{1-\sqrt{3}i}{\sqrt{2}}.$$

$$(a) z = -1 = e^{i(\pi + 2k\pi)} \Rightarrow z^{1/3} = (-1)^{1/3} = e^{i(\pi/3 + 2k\pi/3)} \quad (k \in \mathbb{Z})$$

Evaluate at $k=0, 1, 2$ to collect all distinct roots:

$$\begin{aligned} (-1)^{1/3} &= e^{i\pi/3}, e^{i\pi}, e^{5\pi i/3} \\ &= \sqrt{3}/2 + i/2, -1, \sqrt{3}/2 - i/2 \end{aligned}$$

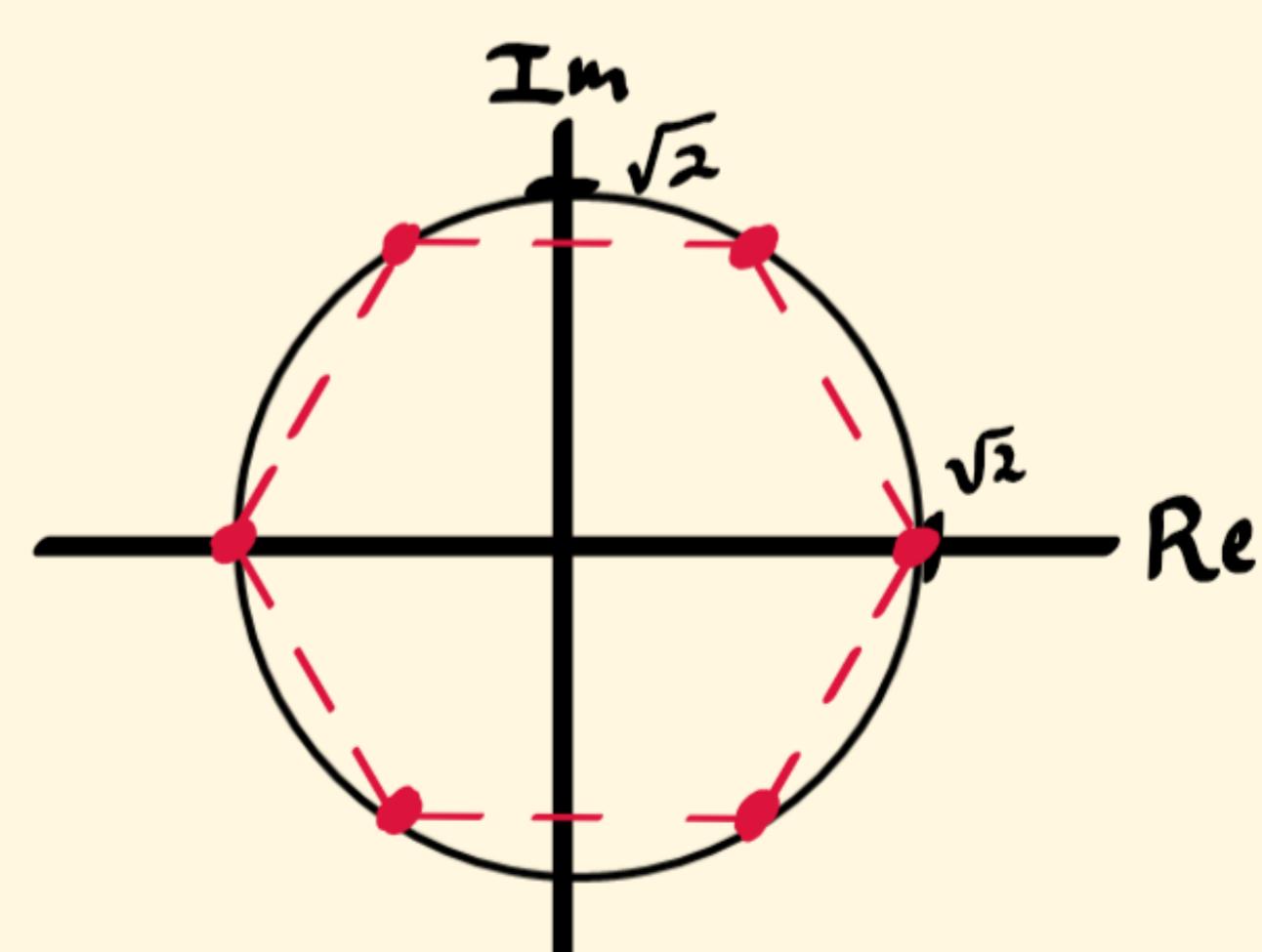


* The principle root is -1
(was not defined for roots of negative real #'s)

$$(b) z = 8 = 8e^{i(0 + 2k\pi)} = 8e^{2k\pi i} \Rightarrow z^{1/6} = 8^{1/6} = \sqrt{2} e^{k\pi i/3} \quad (k \in \mathbb{Z})$$

Evaluate at $k=0, 1, \dots, 5$ to collect all distinct roots:

$$\begin{aligned} 8^{1/6} &= \sqrt{2}, \sqrt{2} e^{i\pi/3}, \sqrt{2} e^{2\pi i/3}, \dots, \sqrt{2} e^{5\pi i/3} \\ &= \pm \sqrt{2}, (\pm 1 \pm \sqrt{3}i)/\sqrt{2} \end{aligned}$$



The principle root is $\sqrt{2}$

6. Find the four zeros of the polynomial $z^4 + 4$, one of them being

$$z_0 = \sqrt{2} e^{i\pi/4} = 1+i.$$

Then use those zeros to factor $z^2 + 4$ into quadratic factors with real coefficients.

$$\text{Ans. } (z^2 + 2z + 2)(z^2 - 2z + 2).$$

$$z_0^4 + 4 = 0$$

$$z_0^4 = -4 = 4e^{i(\pi + 2k\pi)}, \quad k \in \mathbb{Z}$$

$$z_0 = \sqrt[4]{4} e^{i(\pi + 2k\pi)/4} = \sqrt{2} e^{i(\pi + 2k\pi)/4}$$

$$z_0 = \sqrt{2} e^{i\pi/4}, \sqrt{2} e^{3\pi i/4}, \sqrt{2} e^{5\pi i/4}, \sqrt{2} e^{7\pi i/4} \quad (k=0,1,2,3)$$

$$z_0 = 1+i, -1+i, -1-i, 1-i$$

Note that $(z-r)(z-\bar{r}) = z^2 - z(r+\bar{r}) + r\bar{r} = z^2 - 2z\operatorname{Re}(r) + |r|^2$ is a polynomial with real coefficients. This means we should multiply the linear terms corresponding to complex conjugate pairs

$$\begin{aligned} \therefore z^2 + 4 &= (z - (1+i))(z - (1-i))(z - (-1+i))(z - (-1-i)) \\ &= (z - 2z + 2)(z + 2z + 2) \end{aligned}$$

7. Show that if c is any n th root of unity other than unity itself, then

$$1 + c + c^2 + \dots + c^{n-1} = 0.$$

Suggestion: Use the first identity in Exercise 9, Sec. 8.

Suppose $c \in \mathbb{C}$ is any root of unity other than unity itself, i.e.

$$c = 1^{\frac{m}{n}}, \quad c \neq 1$$

There are $n-1$ such possible values of c , but all satisfy $c^n = 1$ and $1-c \neq 0$. It follows that

$$1 + c + c^2 + \dots + c^{n-1} = \frac{1 - c^n}{1 - c} = \frac{1 - 1}{1 - c} = \frac{0}{1 - c} = 0$$

* * * *

Chapter 1 Section 11: 1ae, 2, 3, 6, 7, 8 (page 33)

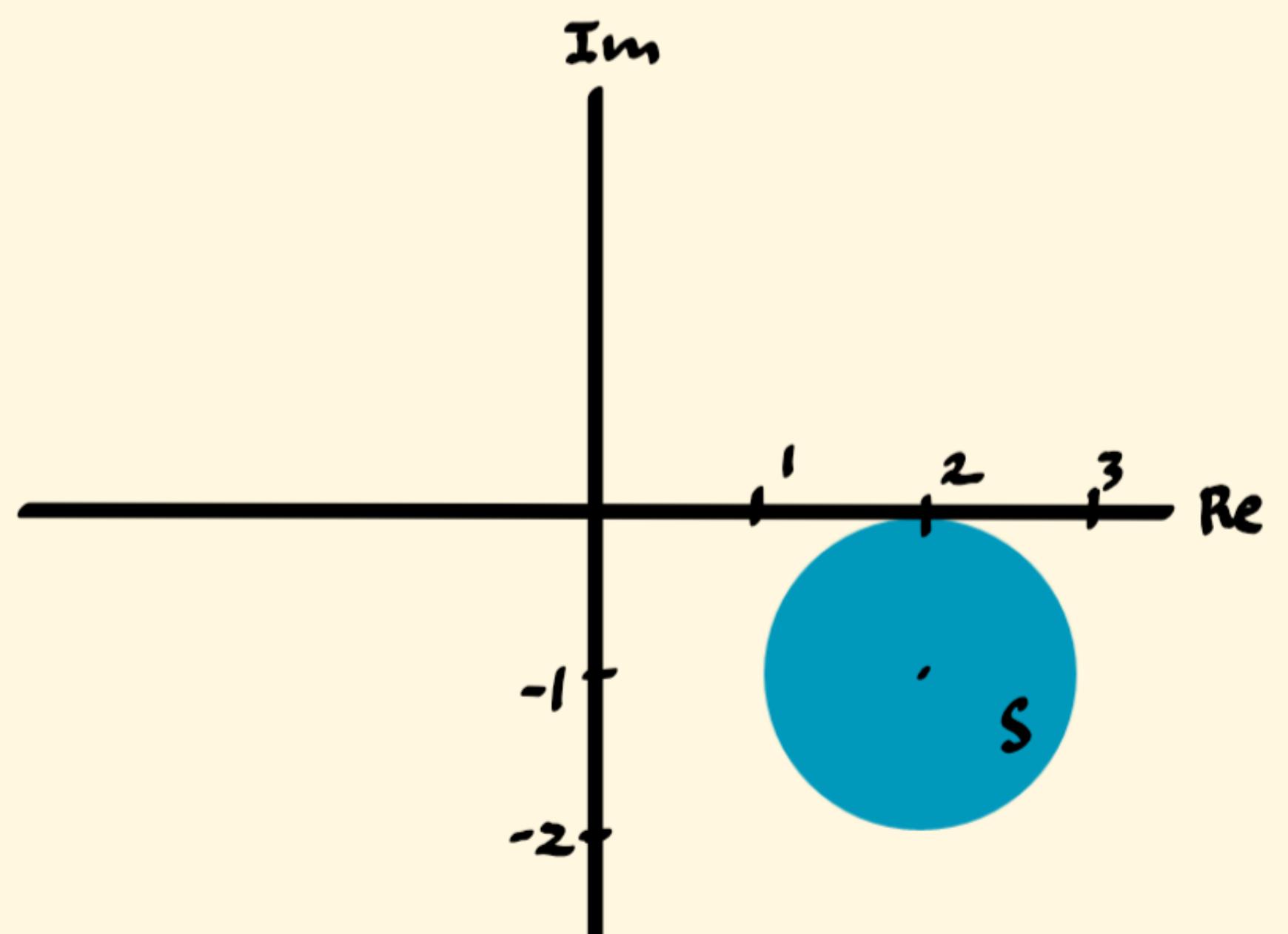
1. Sketch the following sets and determine which are domains:

- | | |
|--|---------------------------------|
| (a) $ z - 2 + i \leq 1$; | (b) $ 2z + 3 > 4$; |
| (c) $\operatorname{Im} z > 1$; | (d) $\operatorname{Im} z = 1$; |
| (e) $0 \leq \arg z \leq \pi/4$ ($z \neq 0$); | (f) $ z - 4 \geq z $. |

Ans. (b), (c) are domains.

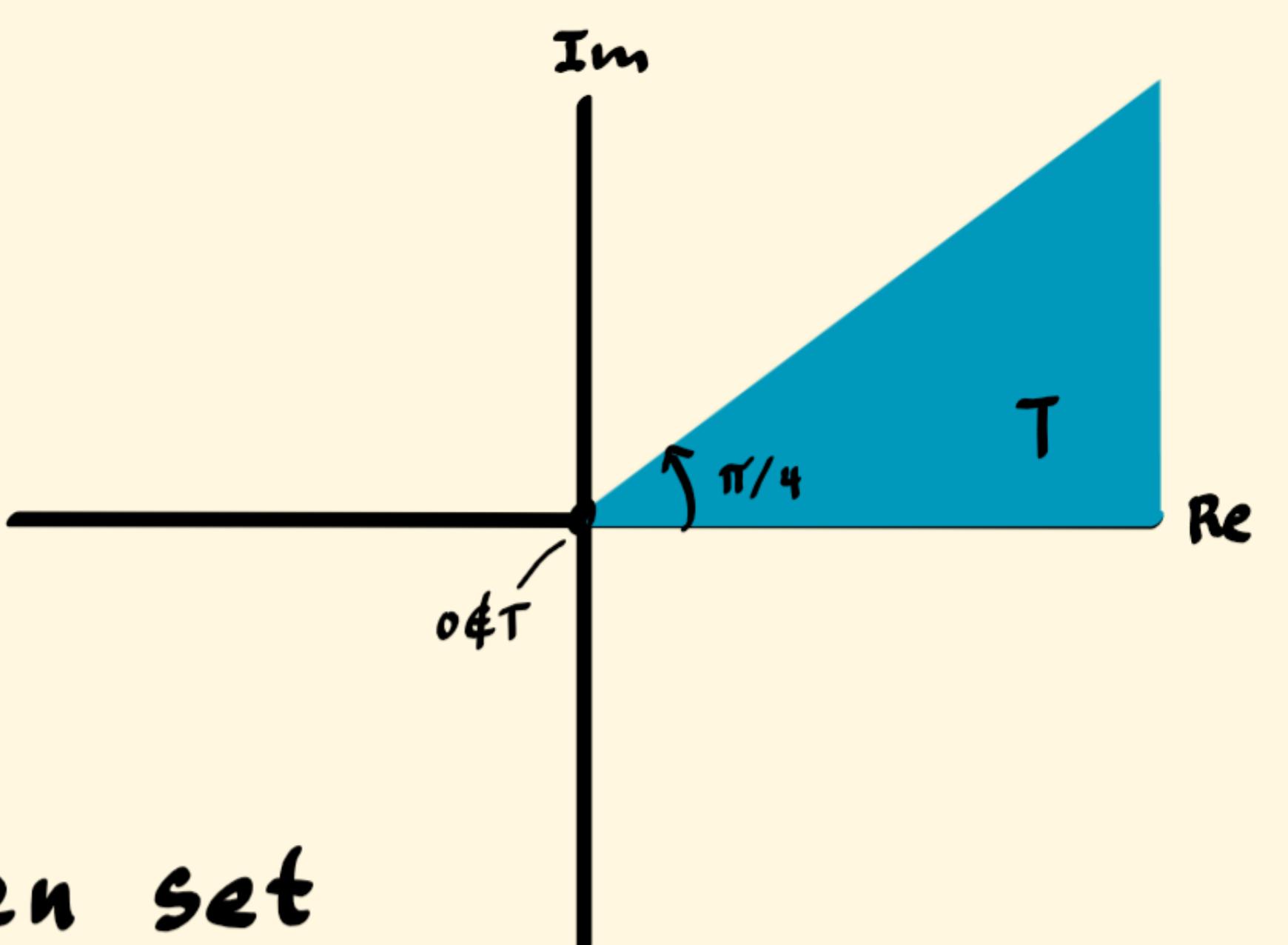
(a) $S = \{z \in \mathbb{C} : |z - (2-i)| \leq 1\}$ is not a domain.

A domain is a nonempty open set that is connected. The given set S is not open. An open set contains none of its boundary points. The boundary points of S are all z s.t. $|z - 2 + i| = 1$, and S contains all of these boundary points. Therefore S contains at least one boundary point



(e) $T = \{z \in \mathbb{C} : 0 \leq \arg z \leq \pi/4, z \neq 0\}$ is not a domain.

T is not open since it contains, for example, the boundary point $z = 1$ which has $\arg z = 0$.



2. Which sets in Exercise 1 are neither open nor closed?

Ans. (e).

For each of (a) - (f), let S denote the given set and ∂S the set of boundary points of S .

(a) S is closed since it contains all boundary points (Exercise 1a)

(b) $\partial S = \{z : |2z+3|=4\}$. For any $z \in \partial S$, $|2z+3| \neq 4$ so $z \notin S$. Since S contains no boundary points, S is open.

(c) $\partial S = \{z : \operatorname{Im} z = 1\}$. For any $z \in \partial S$, $\operatorname{Im} z \neq 1$ so $z \notin S$. S is open.

(d) $\partial S = \{z : \operatorname{Im} z = 1\} = S$. Since $\partial S = S$, $\partial S \subset S$. S is closed.

(e) S is not open (Exercise 1e). But also $0 \in \partial S$ with $0 \notin S$ so S is not closed. S is neither open nor closed.

(f) $S = \{z : |z-4| \geq |z|\} = \{z : \operatorname{Re} z \leq 2\} \Rightarrow \partial S = \{z : \operatorname{Re} z = 2\}$. Since $\partial S \subset S$, S is closed. The claim $|z-4| \geq |z|$ iff $\operatorname{Re} z \leq 2$ warrants proof.

*Proof that $S = \{z : |z-4| \geq |z|\} = \{z : \operatorname{Re} z \leq 2\}$:

$$0 \leq |z| \leq |z-4|^2$$

$$|z|^2 \leq |z-4|^2$$

$$(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 \leq (\operatorname{Re}(z-4))^2 + (\operatorname{Im}(z-4))^2$$

$$(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 \leq (\operatorname{Re} z - 4)^2 + (\operatorname{Im} z)^2$$

$$(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 \leq (\operatorname{Re} z)^2 - 8\operatorname{Re} z + 16 + (\operatorname{Im} z)^2$$

$$8\operatorname{Re} z \leq 16$$

$$\operatorname{Re} z \leq 2$$

$$\therefore |z| \leq |z-4| \text{ iff } \operatorname{Re} z \leq 2$$

3. Which sets in Exercise 1 are bounded?

Ans. (a).

A set S is bounded if every point of S lies in some circle $|z| \leq R$

(a) Since $|z| - \sqrt{5} = |z| - |-2+i| \leq |z| - |z+(-2+i)| \leq |z+(-2+i)| \leq 1$, all points of S lie in $|z| \leq 1 + \sqrt{5}$. S is bounded.

(b) Let $R > 2$ be arbitrary. Any point $z \in C$ s.t. $|z| > R$ satisfies $|z + 3/2| > R > 2$ and thus $|2z+3| > 4$. Since R was arbitrary this means that no matter how large we choose R to be there is always a point $z \in S$ such that $|z| \notin R$. S is not bounded.

(c) Let $R > 1$ be arbitrary. Then $z = (R+1)i$ satisfies $\operatorname{Im} z = R+1 > R > 1$ so $z \in S$ but $|z| = R+1 > R$. This shows that for arbitrarily large R we can always find a point $z \in S$ such that $|z| \notin R$. S is not bounded.

(d) Given $R > 0$ take $z = R+i$. Then $z \in S$ since $\operatorname{Im} z = 1$ but $|z| = (R^2+1)^{1/2} > R$. This shows that for arbitrarily large R we can always find a point $z \in S$ such that $|z| \notin R$. S is not bounded.

(e) Given $R > 0$ take $z = R+i$. Then since $\arg z = 0$, $0 \leq \arg z \leq \pi/4$ and $z \in S$ but $|z| > R$. S is not bounded.

(f) Given $R > 0$ take $z = -R-1$. Then since $\operatorname{Re} z = -R-1 < -1 \leq 2$, $z \in S$ but $|z| = (R^2+1)^{1/2} > R$. S is not bounded.

Definitions

A point z_0 is an interior point of S if there exists some neighborhood of z_0 that contains only points of S ; z_0 is an exterior point of S if there exists a neighborhood of z_0 containing no points of S . If z_0 is neither of these, z_0 is a boundary point of S : every neighborhood of z_0 contains at least one point of S and one point of S^c .

z_0 is an accumulation point of S if every deleted neighborhood of z_0 contains a point of S .

6. Show that a set S is open if and only if each point in S is an interior point.

Assume S is open and suppose $z_0 \in S$. By the definitions we know that if z_0 is not an interior point and z_0 is not an exterior point then z_0 is a boundary point. Since S is open, S does not contain any of its boundary points. In particular $z_0 \notin S$ means z_0 is not a boundary point of S . Thus z_0 must be either an interior point of S or an exterior point of S . Since every neighborhood of z_0 contains $z_0 \in S$, z_0 is not an exterior point. By exhaustion of all logical alternatives, z_0 must be an interior point.

Assume that each point of the set S is an interior point. Suppose z is a boundary point of S . If $z \in S$ then z is both an interior point and a boundary point of S . This is a contradiction since by the definitions a point cannot be both an interior point and a boundary point of S . Since z was arbitrary, conclude that no boundary point of S is contained in S . Therefore S is open.

7. Determine the accumulation points of each of the following sets:

$$(a) z_n = i^n \quad (n = 1, 2, \dots); \quad (b) z_n = i^n/n \quad (n = 1, 2, \dots); \\ (c) 0 \leq \arg z < \pi/2 \quad (z \neq 0); \quad (d) z_n = (-1)^n(1+i) \frac{n-1}{n} \quad (n = 1, 2, \dots).$$

Ans. (a) None; (b) 0; (d) $\pm(1+i)$.

(a) $S = \{i^n : n=1, 2, \dots\} = \{\pm i, \pm 1\}$

If $z \in S$, any deleted neighborhood of z with radius less than $\sqrt{2}$ contains no points of S . So $z \in S$ is not accumulation point of S .

If $z \in S^c$ then the fact that S contains only four isolated points means that a deleted neighborhood of sufficiently small radius (take e.g. $r = \inf \{ |z - z_0| : z_0 \in S \}$) will contain no points of S .

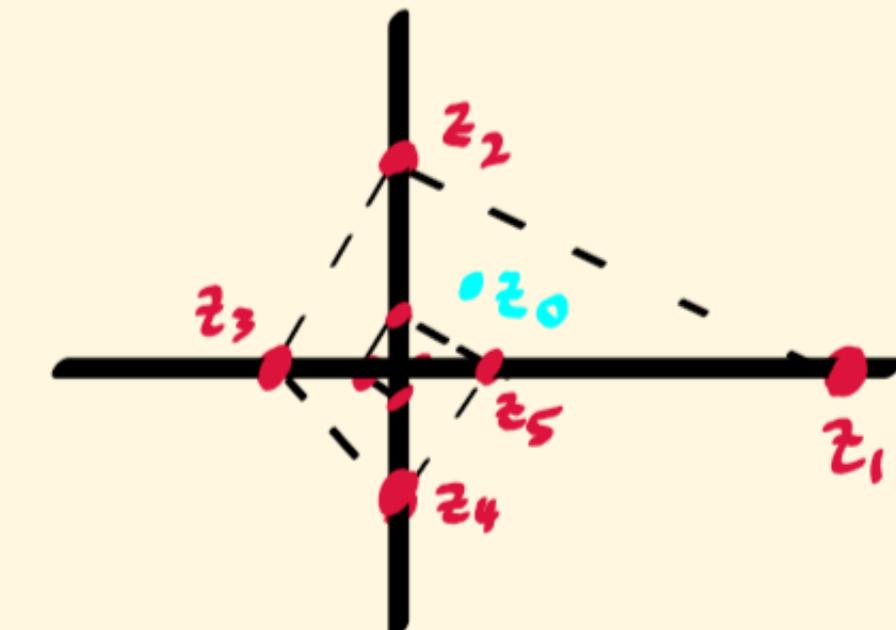
Since one of $z \in S$ or $z \in S^c$ must be true $\forall z \in \mathbb{C}$ and in either case z is not an accumulation point of S , S has no accumulation points.

$$(b) S = \{i^n/n : n = 1, 2, \dots\}$$

Since $i^n \in \{\pm 1, \pm i\}$ for each n , $|i^n/n| = 1/n$ for each n .

Given any ϵ neighborhood of $z_0 = 0$ there exists an N s.t. $1/n < \epsilon$ whenever $n \geq N$. Then $z_n = i^n/n$ belongs to this ϵ neighborhood for any $n \geq N$. This shows that there is a point of S (infinitely many in fact) in the ϵ neighborhood of 0. Since $0 \notin S$ the same is true of the deleted ϵ neighborhood as well and since ϵ was arbitrary this shows that $z_0 = 0$ is an accumulation point of S .

For $z_0 \neq 0$, let $D = \{|z_n - z_0| : z_n \neq z_0, n = 1, 2, \dots\}$, the set of distances between z_0 and any of the points z_n (the restriction $z_n \neq z_0$ handles the possible case that $z_0 = z_n$ for some n). D is a nonempty set of positive real numbers, so 0 is a lower bound of D .



$$\epsilon := \inf D \rightarrow |z_n - z_0| \geq \epsilon \quad \forall n$$

This choice of ϵ gives a deleted neighborhood of z_0 that contains no points of S . $z_0 \neq 0$ is not an accumulation point.

Conclude $z_0 = 0$ is the only accumulation point of S .

$$(c) S = \{z \in \mathbb{C} : z \neq 0, 0 \leq \arg z < \pi/2\}$$

$z_0 = 0$ is an accumulation point of S since every deleted neighborhood of 0 contains (infinitely many) points of S .

Any $z_0 \in S$ is an accumulation point of S since if $0 < \arg z_0 < \pi/2$ we can find a deleted neighborhood entirely contained in S and if $\arg z_0 = 0$ then any deleted neighborhood of z_0 contains points z with $0 < \arg z < \pi/2$.

Any point $z_0 \neq 0$ with $\arg z_0 = \pi/2$ is an accumulation point since these points are boundary points of S , just as the points along $\arg z_0 = 0$ are.

The set of accumulation points of S is the set

$$\bar{S} = \{z_0 : z_0 = 0 \text{ or } 0 \leq \arg z_0 \leq \pi/2\}$$

$$(d) S = \left\{ (-1)^n (1+i) \frac{n-1}{n} : n=1, 2, \dots \right\}$$

Given $\epsilon > 0$, $\exists N$ s.t. $\sqrt{2} \left| \frac{n-1}{n} - 1 \right| < \epsilon \quad \forall n \geq N$ since $\frac{n-1}{n} \rightarrow 1$ as $n \rightarrow \infty$.

For any even $n \geq N$

$$0 < \left| (-1)^n (1+i) \frac{n-1}{n} - (1+i) \right| = |1+i| \left| \frac{n-1}{n} - 1 \right| = \sqrt{2} \left| \frac{n-1}{n} - 1 \right| < \epsilon$$

For any odd $n \geq N$

$$0 < \left| (-1)^n (1+i) \frac{n-1}{n} - (-1-i) \right| = |1+i| \left| 1 - \frac{n-1}{n} \right| = \sqrt{2} \left| \frac{n-1}{n} - 1 \right| < \epsilon$$

This shows that every deleted neighborhood of $1+i$ contains a point of S and every deleted neighborhood of $-1-i$ contains a point of S . Therefore $z_0 = \pm(1+i)$ are accumulation points of S .

Since $z_{2n} \rightarrow 1+i$ and $z_{2n+1} \rightarrow -1-i$, $z_0 \neq \pm(1+i)$ cannot be an accumulation point of S since if z_0 was some third accumulation point of S we would be able to find a subsequence of (z_n) converging to z_0 .

8. Prove that if a set contains each of its accumulation points, then it must be a closed set.

Assume the set S contains each of its accumulation points. To show that S is closed, show that S contains all of its boundary points.

Suppose $z_0 \in \partial S$. Either $z_0 \in S$ or $z_0 \notin S$. If $z_0 \in S$ we are done. Otherwise if $z_0 \notin S$ then since every neighborhood of z_0 contains a point of S (by the definition of a boundary point) and $z_0 \notin S$ itself, every neighborhood of z_0 contains a point of S other than z_0 , i.e. every deleted neighborhood of z_0 contains a point of S . Thus z_0 is also an accumulation point of S so that $z_0 \in S$ by the initial assumption. This contradicts $z_0 \notin S$.

Since $z_0 \in S$ or $z_0 \notin S$ and the latter case produces a contradiction, conclude $z_0 \in S$. Since $z_0 \in \partial S$ was arbitrary, this proves S is closed.

* * * *

Chapter 2 Section 12 : 2, 3, 4 (pages 37, 38)

2. Write the function $f(z) = z^3 + z + 1$ in the form $f(z) = u(x, y) + iv(x, y)$.

$$Ans. f(z) = (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y).$$

$$z = x + iy \quad x, y \in \mathbb{R}$$

$$\begin{aligned} f(z) &= z^3 + z + 1 = (x + iy)^3 + x + iy + 1 = x^3 + 3ix^2y - 3xy^2 - iy^3 + x + iy + 1 \\ &= (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y) = u(x, y) + iv(x, y) \end{aligned}$$

3. Suppose that $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$, where $z = x + iy$. Use the expressions (see Sec. 5)

$$x = \frac{z + \bar{z}}{2} \quad \text{and} \quad y = \frac{z - \bar{z}}{2i}$$

to write $f(z)$ in terms of z , and simplify the result.

$$Ans. f(z) = \bar{z}^2 + 2iz.$$

$$\begin{aligned} z &= x + iy \rightarrow \bar{z} = x - iy \\ x &= \frac{z + \bar{z}}{2} \rightarrow x^2 = \frac{1}{4}(z^2 + 2z\bar{z} + \bar{z}^2) \\ y &= \frac{z - \bar{z}}{2i} \rightarrow -y^2 = \frac{1}{4}(z^2 - 2z\bar{z} + \bar{z}^2) \quad \left. \begin{array}{l} x^2 - y^2 = \frac{1}{2}z^2 + \frac{1}{2}\bar{z}^2 \\ xy = \frac{1}{4i}(z^2 - \bar{z}^2) \rightarrow -2ixy = -\frac{1}{2}(z^2 - \bar{z}^2) = \frac{1}{2}\bar{z}^2 - \frac{1}{2}z^2 \end{array} \right\} \\ f(z) &= x^2 - y^2 - 2y + i(2x - 2xy) \\ &= \frac{1}{2}z^2 + \frac{1}{2}\bar{z}^2 - \frac{1}{2}(z - \bar{z}) + i(z + \bar{z}) - \frac{1}{2}(z^2 - \bar{z}^2) \\ &= \frac{1}{2}z^2 + \frac{1}{2}\bar{z}^2 + i(z - \bar{z}) + i(z + \bar{z}) + \frac{1}{2}\bar{z}^2 - \frac{1}{2}z^2 \\ &= \bar{z}^2 + 2iz \end{aligned}$$

4. Write the function

$$f(z) = z + \frac{1}{z} \quad (z \neq 0)$$

in the form $f(z) = u(r, \theta) + iv(r, \theta)$.

$$Ans. f(z) = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta.$$

$$z = re^{i\theta}$$

$$\begin{aligned} f(z) &= z + z^{-1} = re^{i\theta} + r^{-1}e^{-i\theta} = r(\cos\theta + i\sin\theta) + r^{-1}(\cos\theta - i\sin\theta) \\ &= (r + r^{-1})\cos\theta + i(r - r^{-1})\sin\theta = u(r, \theta) + iv(r, \theta) \end{aligned}$$

* * * *

Chapter 2 Section 14 : 2, 3c, 4, 5 (page 44)

- ? 2. Find and sketch, showing corresponding orientations, the images of the hyperbolas

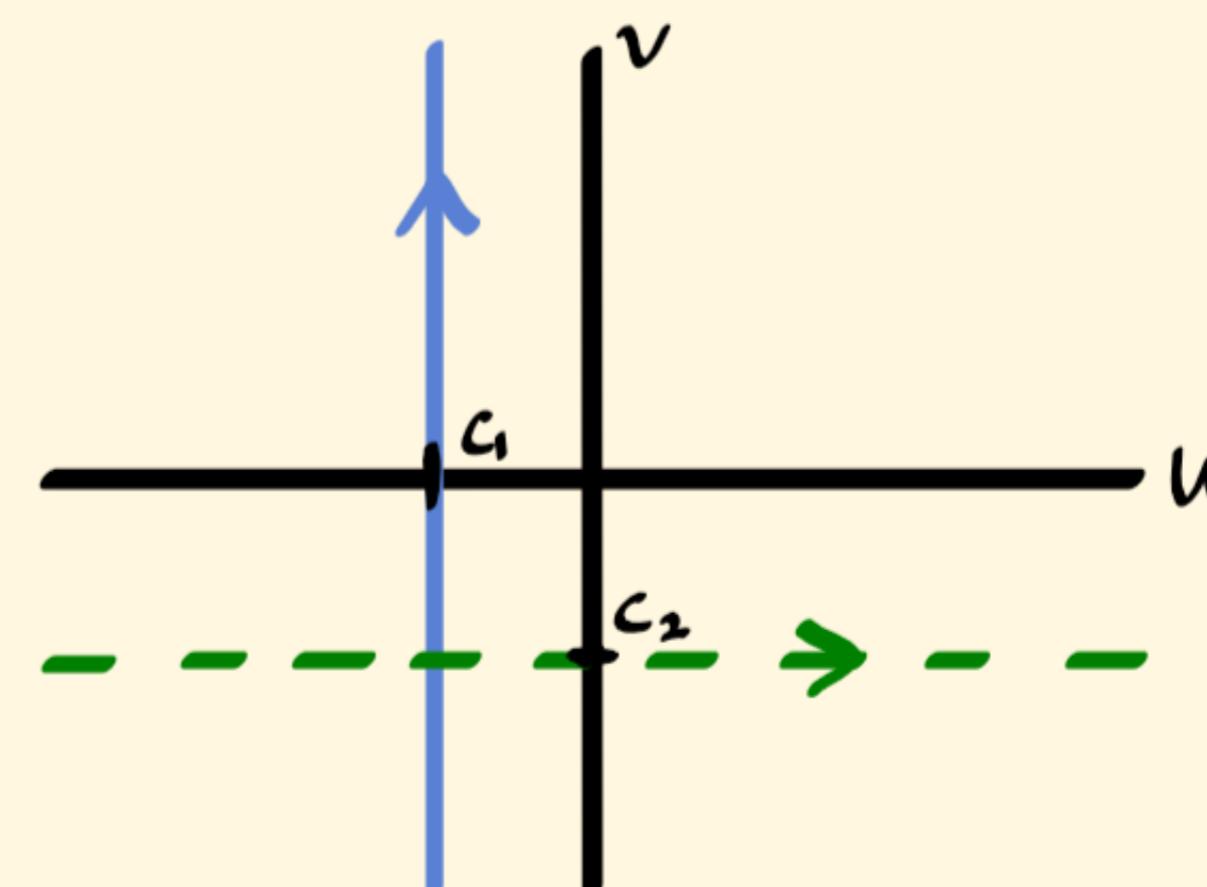
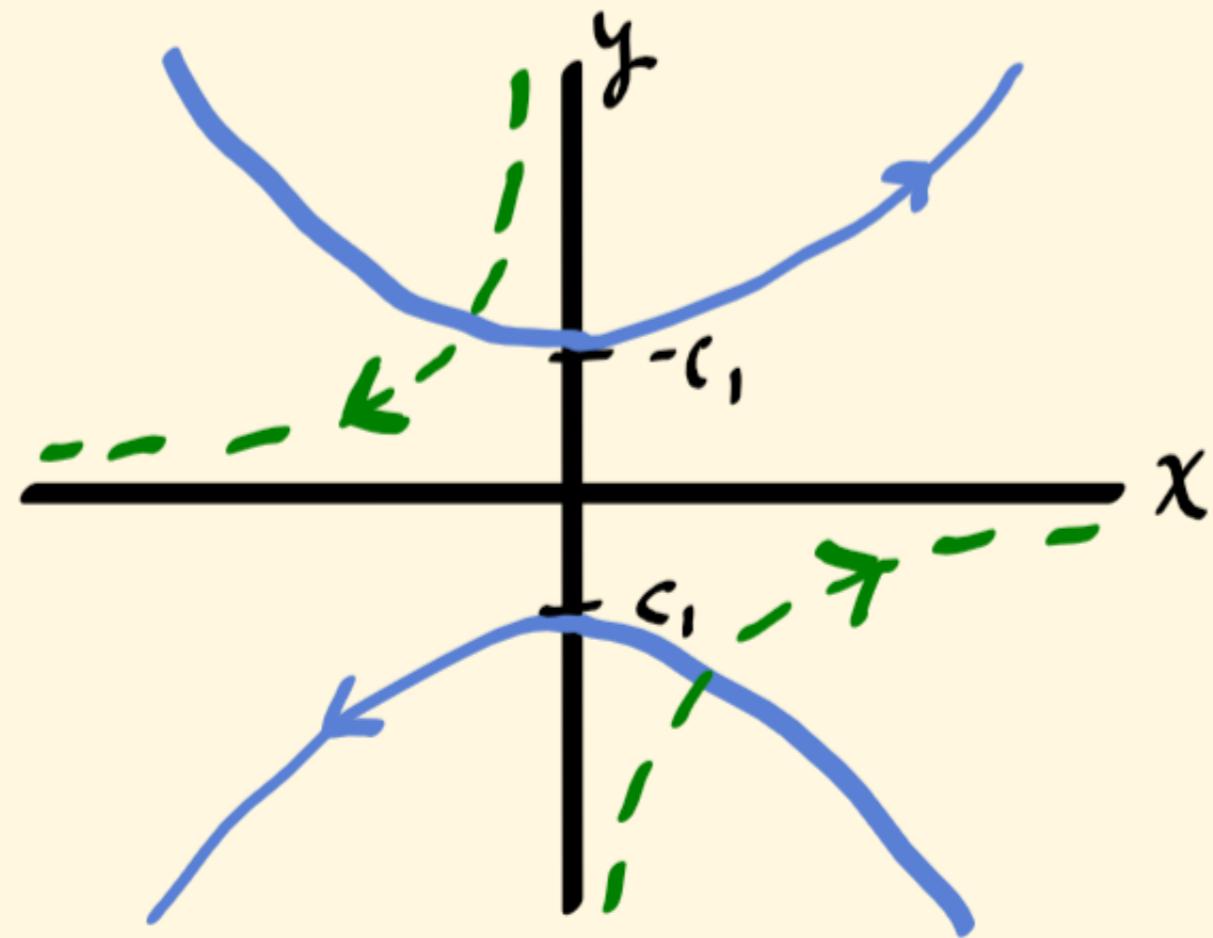
$$x^2 - y^2 = c_1 \quad (c_1 < 0) \quad \text{and} \quad 2xy = c_2 \quad (c_2 < 0)$$

under the transformation $w = z^2$.

$$z = x + iy$$

$$w = z^2 = x^2 - y^2 + i2xy = u + iv$$

$$\begin{array}{l} \text{--- } x^2 - y^2 = c_1, \quad u = c_1, \\ \text{--- } 2xy = c_2, \quad v = c_2 \end{array}$$



$$\left. \begin{array}{l} x^2 - y^2 = c_1, \\ y^2 = x^2 - c_1 > 0 \end{array} \right\} \rightarrow u = c_1, \quad v = 2x\sqrt{x^2 - c_1} \quad (y > 0 \text{ branch}) \quad x \in \mathbb{R}$$

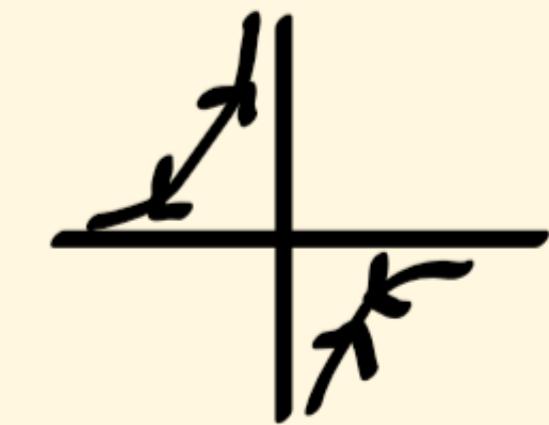
$$v = -2x\sqrt{x^2 - c_1} \quad (y < 0 \text{ branch})$$

The image of the hyperbola $x^2 - y^2 = c_1$ is the vertical line $u = c_1$. $\operatorname{Im} w = v$ can take on any real numbered value. To provide orientation, use the parameterization $v = \pm 2x\sqrt{x^2 - c_1}$. Increasing x along the upper branch corresponds to increasing v along the line $u = c_1$, while decreasing x along the lower branch corresponds to increasing v along the line $u = c_1$.

$$\left. \begin{array}{l} 2xy = c_2 \\ y = c_2/2x \\ x = c_2/2y \end{array} \right\} \rightarrow v = c_2, \quad \begin{array}{ll} u = x^2 - c_2^2/4x^2 & \lim_{x \rightarrow \infty} u = +\infty \\ u = c_2^2/4y^2 - y^2 & \lim_{y \rightarrow \infty} u = -\infty \end{array} \quad \begin{array}{ll} \lim_{x \rightarrow 0} u = -\infty & \lim_{x \rightarrow 0} u = +\infty \\ \lim_{y \rightarrow 0} u = +\infty & \lim_{y \rightarrow 0} u = -\infty \end{array}$$

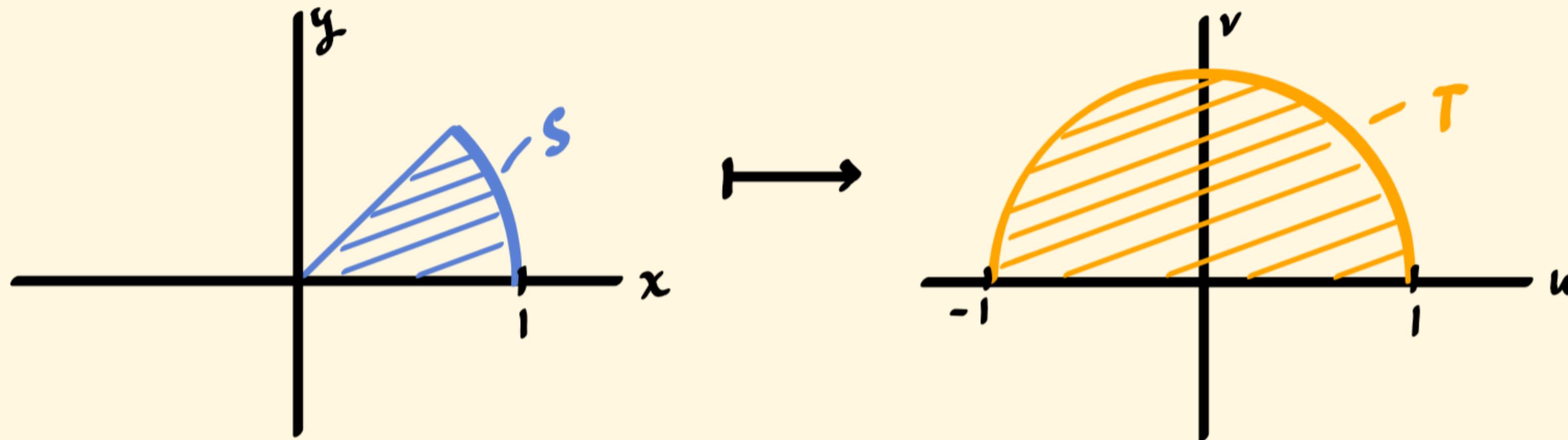
The image of the hyperbola $2xy = c_2$ is the horizontal line $v = c_2$. $\operatorname{Re} w = u$ can take on any real numbered value. Parameterize u along the branch in quadrant 4 of the xy plane as $u = x^2 - c_2^2/4x^2$ ($0 < x < \infty$) to see that increasing x along this branch corresponds to increasing u (and vice versa). Parameterize u along the branch in quadrant 2 as $u = c_2^2/4y^2 - y^2$ ($0 < y < \infty$) to see that increasing y along this branch corresponds to decreasing u (and vice versa with y decreasing toward 0). These observations provide the orientation in the graphs.

? $\lim_{x \rightarrow -\infty} u = +\infty \quad \lim_{x \rightarrow 0} u = +\infty \quad \lim_{y \rightarrow -\infty} u = -\infty \quad \lim_{y \rightarrow 0} u = -\infty \rightarrow$



3. Sketch the region onto which the sector $r \leq 1, 0 \leq \theta \leq \pi/4$ is mapped by the transformation (a) $w = z^2$; (b) $w = z^3$; (c) $w = z^4$.

$$(c) z = re^{i\theta}, w = z^4, re^{i\theta} \mapsto r^4 e^{4i\theta}$$



The region $S = \{(r, \theta) : r \leq 1, 0 \leq \theta \leq \pi/4\}$ is mapped to the region $T = \{(r, \theta) : r \leq 1, 0 \leq \theta \leq \pi\}$ by the transformation $w = z^4$.

4. Show that the lines $ay = x$ ($a \neq 0$) are mapped onto the spirals $\rho = \exp(a\phi)$ under the transformation $w = \exp z$, where $w = \rho \exp(i\phi)$.

$$z = x + iy, \quad w = e^z = e^x e^{iy} = e^{ay} e^{iy} = e^{a\phi} e^{i\phi} = \rho e^{i\phi} \quad (\rho := e^{a\phi}, \phi := y)$$

Is this all that is necessary to show?

5. By considering the images of *horizontal* line segments, verify that the image of the rectangular region $a \leq x \leq b, c \leq y \leq d$ under the transformation $w = \exp z$ is the region $e^a \leq \rho \leq e^b, c \leq \phi \leq d$, as shown in Fig. 21 (Sec. 14).

Consider the horizontal line segment $\ell = \{(x, y_0) : a \leq x \leq b\}$ for some fixed $y_0 \in [c, d]$. The image of ℓ under this transformation is:

$$\{\rho e^{i\phi} : e^a \leq \rho \leq e^b, \phi = y_0\}$$

This is a segment oriented at the angle $\phi = y_0$ extending from a distance $\rho = e^a$ from the origin to a distance $\rho = e^b$ from the origin. Repeat this transformation for every $y_0 \in [c, d]$. The result is

$$\{w = \exp(x+iy) : a \leq x \leq b, c \leq y \leq d\} = \{\rho e^{i\phi} : e^a \leq \rho \leq e^b, c \leq \phi \leq d\}$$