Written Homework 4 Solutions

MA 4291 (Tilley)
C-Term 2022

Just turn in the problems in **bold red font**:

pg. 121: # **3**,4
pg. 125: # 5,**6**
pg. 135: # **1**,**3**,6,**7**,**10**
pg. 140: # 2,5,**6**,**8**
pg. 149: # 4,**5**
pg. 160: # 2,**4**

Chapter 4 Section 38: 3, 4 (page 121)

3. Show that if m and n are integers,

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta}\, d\theta = \begin{cases} 0 & \text{when } m \neq n, \\ 2\pi & \text{when } m = n. \end{cases}$$

For $m, n \in \mathbb{Z}$

If $m \neq n$,

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta}\, d\theta = \frac{1}{i(m-n)} e^{i(m-n)\theta}\Big|_0^{2\pi}$$

$$= \frac{1}{i(m-n)}\left[e^{i(m-n)2\pi} - e^0\right] = \frac{1}{i(m-n)}\left[1-1\right] = 0$$

If $m \neq n$,

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta}\, d\theta = \int_0^{2\pi} e^0\, d\theta = \int_0^{2\pi} d\theta = 2\pi$$

4. According to definition (2), Sec. 38, of definite integrals of complex-valued functions of a real variable,

$$\int_0^{\pi} e^{(1+i)x}\, dx = \int_0^{\pi} e^x \cos x\, dx + i \int_0^{\pi} e^x \sin x\, dx.$$

Evaluate the two integrals on the right here by evaluating the single integral on the left and then using the real and imaginary parts of the value found.

Ans. $-(1+e^{\pi})/2, \quad (1+e^{\pi})/2.$

$$\int_0^{\pi} e^{(1+i)x}\, dx = \int_0^{\pi} e^x e^{ix}\, dx = \int_0^{\pi} e^x \cos x\, dx + i\int_0^{\pi} e^x \sin x\, dx$$

$$\int_0^{\pi} e^{(1+i)x}\, dx = \frac{1}{1+i}\left[e^{(1+i)\pi} - 1\right] = \frac{1}{1+i}\left[e^{\pi}e^{i\pi} - 1\right] = -\frac{1}{1+i}\left[e^{\pi}+1\right]$$

$$= -\frac{1-i}{2}\left[e^{\pi}+1\right] = -\frac{1}{2}\left[e^{\pi}+1\right] + \frac{i}{2}\left[e^{\pi}+1\right]$$

$$\Rightarrow \int_0^{\pi} e^x \cos x\, dx = \mathrm{Re}\left[\int_0^{\pi} e^{(1+i)x}\, dx\right] = -\frac{1}{2}\left[e^{\pi}+1\right]$$

$$\int_0^{\pi} e^x \cos x\, dx = \mathrm{Im}\left[\int_0^{\pi} e^{(1+i)x}\, dx\right] = \frac{1}{2}\left[e^{\pi}+1\right]$$

$$* \quad * \quad * \quad * \quad *$$

Chapter 4 Section 39: 5, 6 (page 126)

5. Suppose that a function $f(z)$ is analytic at a point $z_0 = z(t_0)$ lying on a smooth arc $z = z(t)$ $(a \le t \le b)$. Show that if $w(t) = f[z(t)]$, then

$$w'(t) = f'[z(t)]z'(t)$$

when $t = t_0$.

Suggestion: Write $f(z) = u(x, y) + iv(x, y)$ and $z(t) = x(t) + iy(t)$, so that

$$w(t) = u[x(t), y(t)] + iv[x(t), y(t)].$$

Then apply the chain rule in calculus for functions of two real variables to write

$$w' = (u_x x' + u_y y') + i(v_x x' + v_y y'),$$

and use the Cauchy–Riemann equations.

$z : \mathbb{R} \to \mathbb{C}$, $z(t) = x(t) + iy(t)$

$f : \mathbb{C} \to \mathbb{C}$, $f(z) = f(x+iy) = u(x,y) + iv(x,y)$

$w : \mathbb{R} \to \mathbb{C}$, $w(t) = (f \circ z)(t) = f[z(t)] = u[x(t), y(t)] + iv[x(t), y(t)]$

$$w'(t_0) = \frac{d}{dt}\Big[u[x(t), y(t)] + iv[x(t), y(t)]\Big]\Big|_{t=t_0}$$

$$= \left[\frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + i\frac{\partial v}{\partial x}\frac{dx}{dt} + i\frac{\partial v}{\partial y}\frac{dy}{dt}\right]\Big|_{t=t_0}$$

$$= \left[u_x x' + u_y y' + i(v_x x' + v_y y')\right]\Big|_{t=t_0}$$

$$= \left[u_x x' - v_x y' + i(v_x x' + u_x y')\right]\Big|_{t=t_0}$$

$$\left(\begin{array}{l} \text{At } t = t_0, \\ u_x = v_y \\ u_y = -v_x \end{array}\right)$$

$$= \left[(u_x + iv_x)x' + (iu_x - v_x)y'\right]_{t=t_0}$$

$$= \left[(u_x + iv_x)x' + i(u_x + iv_x)y'\right]_{t=t_0}$$

$$= f'(z_0)x'(t_0) + f'(z_0)y'(t_0)$$

$$= f'(z(t_0))\, z'(t_0)$$

6. Let $y(x)$ be a real-valued function defined on the interval $0 \le x \le 1$ by means of the equations

$$y(x) = \begin{cases} x^3 \sin(\pi/x) & \text{when } 0 < x \le 1, \\ 0 & \text{when } x = 0. \end{cases}$$

(a) Show that the equation

$$z = x + iy(x) \qquad (0 \le x \le 1)$$

represents an arc C that intersects the real axis at the points $z = 1/n$ $(n = 1, 2, \ldots)$ and $z = 0$, as shown in Fig. 38.

(b) Verify that the arc C in part *(a)* is, in fact, a *smooth* arc.

Suggestion: To establish the continuity of $y(x)$ at $x = 0$, observe that

$$0 \le \left| x^3 \sin\left(\frac{\pi}{x}\right) \right| \le x^3$$

when $x > 0$. A similar remark applies in finding $y'(0)$ and showing that $y'(x)$ is continuous at $x = 0$.

<u>Def:</u> A set of points $z = (x, y)$ is said to be an arc if

$$x = x(t) \quad y = y(t) \qquad a \le t \le b$$

where $x(t), y(t) : [a,b] \subset \mathbb{R} \to \mathbb{R}$ are continuous functions.

(a) $z = x + iy(x)$ $(0 \le x \le 1)$. Parameterize $z = z(t) = t + it^3 \sin(\pi/t)$ using

$$x = x(t) = t \text{ for } 0 \le t \le 1$$

$$y = y(x(t)) = \begin{cases} x^3 \sin(\pi/x) = t^3 \sin(\pi/t), & 0 < t \le 1 \\ 0, & t = 0 \end{cases}$$

Then $x = x(t)$ and $y = y(t)$. $x(t) = t$ is continuous and $y(t)$ is continuous on $0 < t \le 1$ as the composition / product of continuous functions. Also $\lim_{t \to 0^+} t^3 \sin(\pi/t) = 0$ (part b)

So y is continuous at $t = 0$ as well.

$\therefore$ $z = x + iy$ represents an arc. Call this arc C.

C intersects the real axis $\Leftrightarrow$ $y = 0$.

$y(x) = 0$ when $x = 0$ and also when $\sin(\pi/x) = 0$.

$\sin(\pi/x) = 0 \Leftrightarrow \pi/x = \pi n,\ n \in \mathbb{Z} \Leftrightarrow x = 1/n,\ n \in \mathbb{Z}$

Note that $x = 1/n \in [0,1]$ $\forall n$ (we assumed $x \in [0,1]$)

$\boxed{\begin{array}{l}\underline{\text{Def}}\text{: A smooth arc is an arc } z=z(t)\ (a\le t\le b) \text{ s.t. } z'(t) \text{ is} \\ \quad \text{continuous on } a\le t\le b \text{ (i.e. } z\in\mathbb{C}^1[a,b]) \text{ and } z'(t)\ne 0\ \forall t\in(a,b)\end{array}}$

(b) In order to be an arc (smooth or not), $z(t)$ must be continuous Equivalently, the components $x(t)$ and $y(t)$ must be continuous. The only point where continuity is questionable is $t=0$.

<u>Claim:</u> $\lim_{t\to 0^+} y(t) = 0$

Proof: With $y(t)$ parameterized as in part a,

$$|y(t)-0| = |t^3\sin(\pi/t)| \le |t^3| = t^3 \to 0 \text{ as } t\to 0^+ \quad \square$$

Since $\lim_{t\to 0^+} y(t) = y(0)$, $y(t)$ and thereby $z(t)$ is continuous at 0.

$\triangle$ Next show $z'(t) = x'(t) + iy'(t)$ is continuous on $[0,1]$

$z'(t) = 1 + i(3t^2\sin(\pi/t) - \pi t\cos(\pi/t))$ for $0<t\le 1$

$$z'(0) = \lim_{\Delta t\to 0} \frac{z(0+\Delta t) - z(0)}{\Delta t} = \lim_{\Delta t\to 0} \frac{\Delta t + i(\Delta t)^3\sin(\pi/\Delta t)}{\Delta t}$$

$$= \lim_{\Delta t\to 0} (1 + (\Delta t)^2\sin(\pi/\Delta t)) = 1 + 0 = 1$$ (by reasoning similar to that used to prove $y(t)$ is continuous at 0).

Since $z'(t)$ can be written using compositions, products, and difference of continuous functions on $0<t\le 1$, $z'(t)$ is also continuous on $0<t\le 1$. To show continuity at $t=0$,

$$|z'(t) - z'(0)| = |1 + i(3t^2\sin(\pi/t) - \pi t\cos(\pi/t)) - 1|$$
$$\le 3t^2|i||\sin(\pi/t)| + \pi t|i||\cos(\pi/t)|$$
$$\le 3t^2 + \pi t \to 0 \text{ as } t\to 0^+$$

Since $\lim_{t\to 0^+} z'(t) = z'(0)$, $z'(t)$ is continuous at $t=0$ as well.

$\triangle$ For $0<t<1$, $0 = z'(t) = 1 + i(3t^2\sin(\pi/t) - \pi t\cos(\pi/t))$ would require both $0=1$ and $0 = 3t^2\sin(\pi/t) - \pi t\cos(\pi/t)$ by matching real and imaginary components. Clearly $0\ne 1$ so $z'(t)\ne 0$ for $0<t<1$. Conclude $z(t)$ is a smooth arc.

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Chapter 4 Section 42 : 1, 3, 6, 7, 10 (pages 135, 136)

For the functions f and contours C in Exercises 1 through 7, use parametric representations for C, or legs of C, to evaluate

$$\int_C f(z)\,dz.$$

1. $f(z) = (z+2)/z$ and C is

(a) the semicircle $z = 2e^{i\theta}$ $(0 \le \theta \le \pi)$;

(b) the semicircle $z = 2e^{i\theta}$ $(\pi \le \theta \le 2\pi)$;

(c) the circle $z = 2e^{i\theta}$ $(0 \le \theta \le 2\pi)$.

Ans. *(a)* $-4 + 2\pi i$; *(b)* $4 + 2\pi i$; *(c)* $4\pi i$.

(a)

$$\int_0^{\pi} f[z(\theta)]\,z'(\theta)\,d\theta = \int_0^{\pi} \frac{2e^{i\theta}+2}{2e^{i\theta}}\,2ie^{i\theta}\,d\theta$$

$$= 2i\int_0^{\pi}\left(e^{i\theta}+1\right)d\theta$$

$$= 2i\left(\frac{1}{i}e^{i\theta}+\theta\right)\Big|_0^{\pi}$$

$$= 2i\left(-\frac{1}{i}+\pi-\frac{1}{i}\right) = \boxed{-4+2\pi i}$$

(b)

$$\int_{\pi}^{2\pi} f[z(\theta)]\,z'(\theta)\,d\theta = \int_{\pi}^{2\pi} \frac{2e^{i\theta}+2}{2e^{i\theta}}\,2ie^{i\theta}\,d\theta$$

$$= 2i\left(\frac{1}{i}e^{i\theta}+\theta\right)\Big|_{\pi}^{2\pi}$$

$$= 2i\left(\frac{1}{i}+2\pi+\frac{1}{i}-\pi\right) = \boxed{4+2\pi i}$$

(c)

$$\int_0^{2\pi} f[z(\theta)]\,z'(\theta)\,d\theta = \int_0^{2\pi} \frac{2e^{i\theta}+2}{2e^{i\theta}}\,2ie^{i\theta}\,d\theta$$

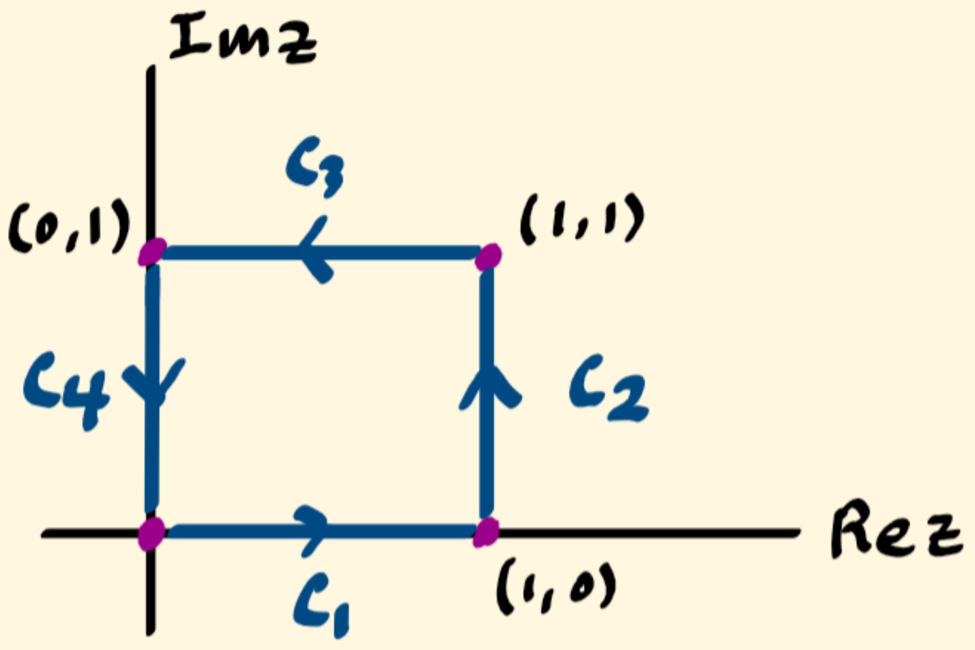
$$= \int_0^{\pi} \frac{2e^{i\theta}+2}{2e^{i\theta}}\,2ie^{i\theta}\,d\theta + \int_{\pi}^{2\pi} \frac{2e^{i\theta}+2}{2e^{i\theta}}\,2ie^{i\theta}\,d\theta$$

$$= -4+2\pi i + 4 + 2\pi i = \boxed{4\pi i}$$

3. $f(z) = \pi \exp(\pi \bar{z})$ and C is the boundary of the square with vertices at the points 0, 1, $1+i$, and i, the orientation of C being in the counterclockwise direction.

Ans. $4(e^{\pi} - 1)$.

$$z(t) = \begin{cases} t & 0 \le t \le 1 \\ 1 + (t-1)i & 1 \le t \le 2 \\ 1 - (t-2) + i & 2 \le t \le 3 \\ i - (t-3)i & 3 \le t \le 4 \end{cases}$$



$$z'(t) = \begin{cases} 1 & 0 \le t \le 1 \\ i & 1 \le t \le 2 \\ -1 & 2 \le t \le 3 \\ -i & 3 \le t \le 4 \end{cases}$$

$$\begin{aligned} \int_C f\,dz &= \int_{C_1} f\,dz + \int_{C_2} f\,dz + \int_{C_3} f\,dz + \int_{C_4} f\,dz \\ &= \int_0^1 f(z(t))z'(t)\,dt + \int_1^2 f(z(t))z'(t)\,dt \\ &\quad + \int_2^3 f(z(t))z'(t)\,dt + \int_3^4 f(z(t))z'(t)\,dt \\ &= \int_0^1 \pi \exp[\pi t]\,dt + \int_1^2 i\pi \exp[\pi(1-(t-1)i)]\,dt \\ &\quad - \int_2^3 \pi \exp[\pi(3-t-i)]\,dt - \int_3^4 i\pi \exp[\pi(t-4)i]\,dt \\ &= (e^{\pi}-1) + 2e^{\pi} + (e^{\pi}-1) - 2 = \boxed{4e^{\pi} - 4} \end{aligned}$$

Integrals evaluated by Wolfram Alpha.

6. $f(z)$ is the branch

$$z^{-1+i} = \exp[(-1+i)\log z] \qquad (|z|>0, 0<\arg z<2\pi)$$

of the indicated power function, and C is the unit circle $z = e^{i\theta}$ $(0 \le \theta \le 2\pi)$.

Ans. $i(1-e^{-2\pi})$.

$$\begin{aligned}\int_C f\,dz &= \int_0^{2\pi} \exp[(-1+i)\log\exp(i\theta)]\, i\exp[i\theta]\,d\theta \\ &= \int_0^{2\pi} i\exp[-i\theta+i^2\theta]\exp[i\theta]\,d\theta \qquad (*) \\ &= \int_0^{2\pi} ie^{-\theta}\,d\theta \\ &= -ie^{-\theta}\Big|_0^{2\pi} = -i(e^{-2\pi}-1) = \boxed{i(1-e^{-2\pi})}\end{aligned}$$

(*) $\log e^{i\theta} = i\theta$? See the end of page 101 for an explanation of why this is ok even though $0<\arg z<2\pi$ instead of $-\pi<\arg z<\pi$

7. $f(z)$ is the principal branch

$$z^i = \exp(i\,\mathrm{Log}\,z) \quad (|z|>0, -\pi < \mathrm{Arg}\,z < \pi)$$

of this power function, and C is the semicircle $z = e^{i\theta}$ $(0 \le \theta \le \pi)$.

Ans. $-\dfrac{1+e^{-\pi}}{2}(1-i)$.

$$\begin{aligned}\int_C f\,dz &= \int_0^{\pi} f[z(\theta)]z'(\theta)\,d\theta \\ &= \int_0^{\pi} \exp[i\,\mathrm{Log}\,e^{i\theta}]\, ie^{i\theta}\,d\theta \\ &= \int_0^{\pi} i\,e^{i^2\theta}e^{i\theta}\,d\theta \\ &= \int_0^{\pi} ie^{(i-1)\theta}\,d\theta \\ &= \frac{i}{i-1}\left[e^{(i-1)\theta}\right]\Big|_0^{\pi} \\ &= \frac{i}{i-1}\left[e^{i\pi}e^{-\pi}-1\right] \\ &= -\frac{i(-1-i)}{2}\left[e^{-\pi}+1\right] = \boxed{-\frac{1+e^{-\pi}}{2}(1-i)}\end{aligned}$$

10. Let C_0 and C denote the circles

$$z = z_0 + Re^{i\theta} \ (-\pi \le \theta \le \pi) \quad \text{and} \quad z = Re^{i\theta} \ (-\pi \le \theta \le \pi),$$

respectively.

(a) Use these parametric representations to show that

$$\int_{C_0} f(z - z_0)\, dz = \int_C f(z)\, dz$$

when f is piecewise continuous on C.

(b) Apply the result in part *(a)* to integrals (5) and (6) in Sec. 42 to show that

$$\int_{C_0} (z - z_0)^{n-1}\, dz = 0 \ (n = \pm 1, \pm 2, \ldots) \quad \text{and} \quad \int_{C_0} \frac{dz}{z - z_0} = 2\pi i.$$

(a)

$$\int_{C_0} f(z-z_0)\,dz = \int_{-\pi}^{\pi} f[z_0 + Re^{i\theta} - z_0](0 + iRe^{i\theta})\,dz = \int_{-\pi}^{\pi} f(Re^{i\theta})\,iRe^{i\theta}\,dz$$

$$\int_C f(z)\,dz = \int_{-\pi}^{\pi} f(Re^{i\theta})\,iRe^{i\theta}\,d\theta$$

Conclude $\int_{C_0} f(z-z_0)\,dz = \int_C f(z)\,dz$

(b) From Example 2 of Sec. 42 we have

$$(5) \qquad \int_C z^{n-1}\, dz = 0 \quad (n = \pm 1, \pm 2, \ldots).$$

If a is allowed to be zero, we have

$$(6) \qquad \int_C \frac{dz}{z} = \int_{-\pi}^{\pi} \frac{1}{Re^{i\theta}} iRe^{i\theta}\, d\theta = i \int_{-\pi}^{\pi} d\theta = 2\pi i.$$

for the same circle C parameterized by $z = Re^{i\theta}$, $-\pi \le \theta \le \pi$ given in this exercise.

Let $f(z) = z^{n-1}$, $n \in \mathbb{Z} \setminus \{0\}$. By part (a) and equation (5):

$$\int_{C_0} (z-z_0)^{n-1}\,dz = \int_{C_0} f(z-z_0)\,dz = \int_C f(z)\,dz = \int_C z^{n-1} = 0 .$$

Let $f(z) = z^{-1}$. By part (a) and equation (6):

$$\int_{C_0} \frac{dz}{z-z_0} = \int_{C_0} f(z-z_0)\,dz = \int_C f(z)\,dz = \int_C \frac{dz}{z} = 2\pi i .$$

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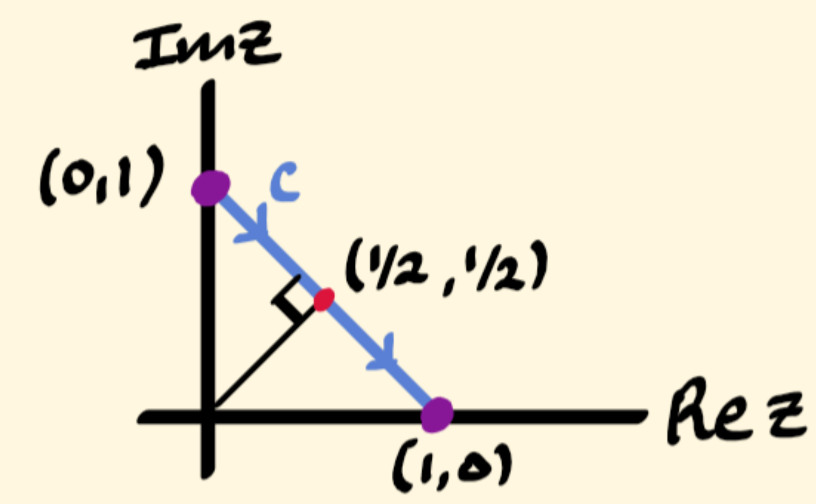
Chapter 4 Section 43 : 2, 5, 6, 8 (pages 140, 141)

2. Let C denote the line segment from $z = i$ to $z = 1$. By observing that of all the points on that line segment, the midpoint is the closest to the origin, show that

$$\left| \int_C \frac{dz}{z^4} \right| \leq 4\sqrt{2}$$

without evaluating the integral.



The shortest distance btwn the origin and C is along the line that passes through the origin and is perpendicular to C. This line intersects C at the midpoint $z_0 = \frac{1}{2} + \frac{1}{2} i$.

If z is a pt on C then $|z| \geq |z_0| = 1/\sqrt{2}$. Then $\forall z$ on C:

$$|f(z)| = |1/z^4| = 1/|z|^4 \leq \sqrt{2}^4 = 4 =: M$$

The length of C is $\sqrt{2} =: L$. By the Theorem on page 138,

$$\left|\int_C f\, dz\right| = \left|\int_C dz/z^4\right| \leq ML = 4\sqrt{2}$$

5. Let C_R be the circle $|z| = R\ (R > 1)$, described in the counterclockwise direction. Show that

$$\left| \int_{C_R} \frac{\operatorname{Log} z}{z^2}\, dz \right| < 2\pi \left(\frac{\pi + \ln R}{R} \right),$$

and then use l'Hospital's rule to show that the value of this integral tends to zero as R tends to infinity.

$\operatorname{Log} z = \ln R + i \operatorname{Arg} z, \; -\pi < \operatorname{Arg} z < \pi \quad \Rightarrow \quad |\operatorname{Log} z| < \ln R + \pi$

$|z| = R > 1 \Rightarrow |z|^2 > R^2 \Rightarrow 1/|z|^2 < 1/R^2$

$$|f(z)| = \left|\frac{\operatorname{Log} z}{z^2}\right| = \frac{|\operatorname{Log} z|}{|z|^2} < \frac{\ln R + \pi}{R^2} =: M$$

[$\exists \varepsilon > 0$ s.t. $|f(z)| \leq M - \varepsilon$ to handle the '$\leq$' in the theorem]

The length of the circle is $L = 2\pi R$

$$\therefore \left|\int_{C_R} f\, dz\right| = \left|\int_{C_R} \frac{\operatorname{Log} z}{z^2}\, dz\right| \leq L(M - \varepsilon) < LM = 2\pi \frac{\ln R + \pi}{R}$$

$$0 \leq \lim_{R\to\infty} \left|\int_{C_R} f\, dz\right| \leq \lim_{R\to\infty} 2\pi \frac{\ln R + \pi}{R} = \lim_{R\to\infty} 2\pi/R = 0$$

$$\Rightarrow \lim_{R\to\infty} \left|\int_{C_R} f\, dz\right| = 0 \Rightarrow \lim_{R\to\infty} \int_{C_R} f\, dz = 0$$

6. Let C_ρ denote a circle $|z| = \rho$ $(0 < \rho < 1)$, oriented in the counterclockwise direction, and suppose that $f(z)$ is analytic in the disk $|z| \le 1$. Show that if $z^{-1/2}$ represents any particular branch of that power of z, then there is a nonnegative constant M, *independent of* ρ, such that

$$\left| \int_{C_\rho} z^{-1/2} f(z)\, dz \right| \le 2\pi M \sqrt{\rho}.$$

Thus show that the value of the integral here approaches 0 as ρ tends to 0.

Suggestion: Note that since $f(z)$ is analytic, and therefore continuous, throughout the disk $|z| \le 1$, it is bounded there (Sec. 18).

- Since $f(z)$ is continuous on the compact set $\{z : |z| \le 1\}$, $f(z)$ is bounded on this set: $\exists M \ge 0$ s.t. $|f(z)| \le M$ $\forall z$ s.t. $|z| \le 1$

- $|z^{-1/2}| = |\exp[-\tfrac{1}{2} \log z]| = |\exp[-\tfrac{1}{2}(\ln\rho + i(\operatorname{Arg} z + 2n\pi))]|$

$$= |\exp[\ln \rho^{-1/2}]||\exp[i(\operatorname{Arg} z + 2n\pi)]| = \rho^{-1/2} \cdot 1 = 1/\sqrt{\rho}$$

- $g(z) := z^{-1/2} f(z)$

$$|g(z)| = |z^{-1/2} f(z)| = |z^{-1/2}||f(z)| \le M/\sqrt{\rho}$$

- The length of the circle C_ρ is the circumference: $L = 2\pi\rho$.

- By the theorem on page 138 (using $L := 2\pi\rho$, $M := M/\sqrt{\rho}$),

$$\left|\int_{C_\rho} w(z)\,dz\right| = \left|\int_{C_\rho} z^{-1/2} f(z)\,dz\right| \le (2\pi\rho)(M/\sqrt{\rho}) = 2\pi M\sqrt{\rho}$$

- Since $0 \le \left|\int_{C_\rho} z^{-1/2} f(z)\,dz\right| \le 2\pi M\sqrt{\rho}$ for each $\rho \in (0,1)$,

$$0 \le \lim_{\rho\to 0^+} \left|\int_{C_\rho} z^{-1/2} f(z)\,dz\right| \le \lim_{\rho\to 0^+} 2\pi M\sqrt{\rho} = 0$$

$$\Rightarrow \lim_{\rho\to 0^+} \left|\int_{C_\rho} z^{-1/2} f(z)\,dz\right| = 0$$

$$\Rightarrow \lim_{\rho\to 0^+} \int_{C_\rho} z^{-1/2} f(z)\,dz = 0$$

8. Let C_N denote the boundary of the square formed by the lines

$$x = \pm\left(N + \frac{1}{2}\right)\pi \quad \text{and} \quad y = \pm\left(N + \frac{1}{2}\right)\pi,$$

where N is a positive integer and the orientation of C_N is counterclockwise.

(a) With the aid of the inequalities

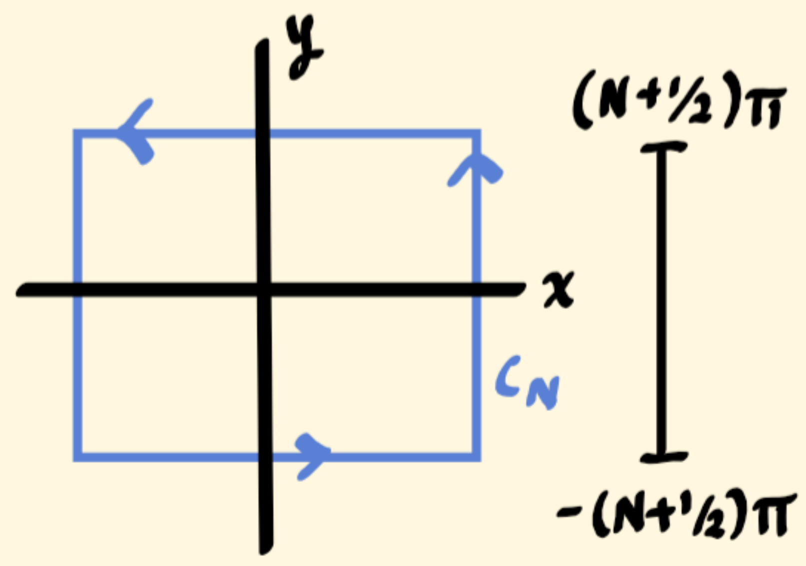
$$|\sin z| \geq |\sin x| \quad \text{and} \quad |\sin z| \geq |\sinh y|,$$

obtained in Exercises 8*(a)* and 9*(a)* of Sec. 34, show that $|\sin z| \geq 1$ on the vertical sides of the square and that $|\sin z| > \sinh(\pi/2)$ on the horizontal sides. Thus show that there is a positive constant A, *independent of* N, such that $|\sin z| \geq A$ for all points z lying on the contour C_N.

(b) Using the final result in part *(a)*, show that

$$\left|\int_{C_N} \frac{dz}{z^2 \sin z}\right| \leq \frac{16}{(2N+1)\pi A}$$

and hence that the value of this integral tends to zero as N tends to infinity.



(a)

△ For $N \in \{1, 2, 3, \dots\}$

$$\sin[(N+\tfrac{1}{2})\pi] = (-1)^N$$

$$\sin[-(N+\tfrac{1}{2})\pi] = -\sin[(N+\tfrac{1}{2})\pi] = (-1)^{N+1}$$

$$\Rightarrow |\sin[(N+\tfrac{1}{2})\pi]| = |\sin[-(N+\tfrac{1}{2})\pi]| = 1$$

On the vertical sides of the square $x = \pm(N+\frac{1}{2})\pi$:

$$|\sin z| \geq |\sin x| = 1$$

△ Since $\sinh y$ is strictly increasing and $N+\frac{1}{2} > \frac{1}{2}$,

$$\sinh[(N+\tfrac{1}{2})\pi] > \sinh[\pi/2]$$

$$\Rightarrow |\sinh[(N+\tfrac{1}{2})\pi]| > |\sinh[\pi/2]| = \sinh[\pi/2]$$

$$\sinh[-(N+\tfrac{1}{2})\pi] = -\sinh[(N+\tfrac{1}{2})\pi] < -\sinh[\pi/2]$$

$$\Rightarrow |\sinh[-(N+\tfrac{1}{2})\pi]| > \sinh[\pi/2]$$

On the horizontal sides of the square $x = \pm(N+\frac{1}{2})\pi$:

$$|\sin z| \geq |\sinh y| > \sinh[\pi/2]$$

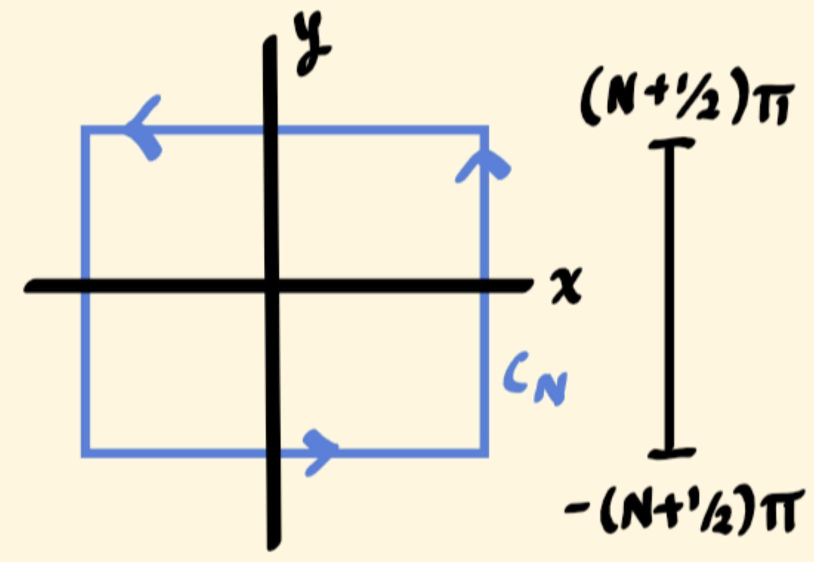
$\triangle$ For any $N \in \{1, 2, \ldots\}$, if z is on the contour C_N:

$$2|\sin z| > 1 + \sinh[\pi/2]$$

$$|\sin z| > A := (1 + \sinh[\pi/2])/2$$

$$|\sin z| > A \Rightarrow |\sin z| \geq A$$

(b) The length of the square C_N is the perimeter $4(2N+1)\pi$



For any z on C_N, $|z| \geq (N+1/2)\pi$. Using this and (a),

$$|z^2 \sin z| = |z|^2 |\sin z| \geq [(N+1/2)\pi]^2 A$$

$$\Rightarrow \left| \frac{1}{z^2 \sin z} \right| = \frac{1}{|z^2 \sin z|} \leq \frac{1}{[(N+1/2)\pi]^2 A}$$

By the theorem on page 138,

$$\left| \int_{C_N} \frac{dz}{z^2 \sin z} \right| \leq 4(2N+1)\pi \frac{1}{[(N+1/2)\pi]^2 A}$$

$$= \frac{4(2N+1)\pi}{1/4(2N+1)^2\pi^2 A} = \frac{16}{(2N+1)\pi A}$$

Since $0 \leq \left| \int_{C_N} \frac{dz}{z^2 \sin z} \right| \leq \frac{16}{(2N+1)\pi A}$, $N = 1, 2, 3, \ldots$

and $\frac{16}{(2N+1)\pi A} \longrightarrow 0$ as $N \longrightarrow \infty$

$$\lim_{N \to \infty} \left| \int_{C_N} \frac{dz}{z^2 \sin z} \right| = 0 \Rightarrow \int_{C_N} \frac{dz}{z^2 \sin z} = 0$$

* * * * *

Chapter 4 Section 45 : 4, 5 (page 149)

4. Find an antiderivative $F_2(z)$ of the branch $f_2(z)$ of $z^{1/2}$ in Example 4, Sec. 44, to show that integral (6) there has value $2\sqrt{3}(-1+i)$. Note that the value of the integral of the function (5) around the closed contour $C_2 - C_1$ in that example is, therefore, $-4\sqrt{3}$.

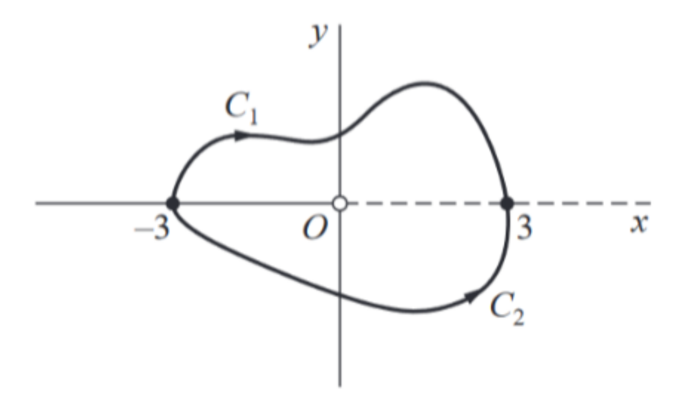


FIGURE 52

$$f_2(z) = z^{1/2} = \sqrt{r}\, e^{i\theta/2}$$

$$F_2(z) = \tfrac{2}{3} z^{3/2} = \tfrac{2}{3} r\sqrt{r}\, e^{3i\theta/2}$$

with $z = re^{i\theta}$, $r > 0$, $\pi/2 < \theta < 5\pi/2$

$$\int_{C_2} z^{1/2}\,dz = \int_{-3}^{3} f_2(z)\,dz = F_2(z)\Big|_{-3}^{3}$$

$$= \tfrac{2}{3}\cdot 3\cdot\sqrt{3}\, e^{3\pi i} - \tfrac{2}{3}\cdot 3\cdot \sqrt{3}\, e^{3i\pi/2} \qquad \left\{\begin{matrix} 3 = 3e^{2\pi i} \\ -3 = 3e^{\pi i} \end{matrix}\right\}$$

$$= -2\sqrt{3} - 2\sqrt{3}\, e^{i3\pi/2} = 2\sqrt{3}(-1+i)$$

$C = C_2 - C_1$, $f(z) = z^{1/2} = \sqrt{r}\, e^{i\theta/2}$, $r > 0$, $0 < \theta < 2\pi$

$$\Rightarrow \int_C f(z)\,dz = \int_C z^{1/2}\,dz = \int_{C_2} z^{1/2}\,dz - \int_{C_1} z^{1/2}\,dz$$

$$= \int_{C_2} f_2(z)\,dz - \int_{C_1} f_1(z)\,dz$$

$$= 2\sqrt{3}(-1+i) - 2\sqrt{3}(1+i) = -4\sqrt{3}$$

5. Show that

$$\int_{-1}^{1} z^i \, dz = \frac{1+e^{-\pi}}{2}(1-i),$$

where the integrand denotes the principal branch

$$z^i = \exp(i \operatorname{Log} z) \qquad (|z| > 0, -\pi < \operatorname{Arg} z < \pi)$$

of z^i and where the path of integration is any contour from $z = -1$ to $z = 1$ that, except for its end points, lies above the real axis. (Compare with Exercise 7, Sec. 42.)

Suggestion: Use an antiderivative of the branch

$$z^i = \exp(i \log z) \qquad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

of the same power function.

Let C be any contour from -1 to 1 that lies above the real axis except at its endpoints.

Although the integrand is piecewise continuous on C, and the integral therefore exists, the given branch is not defined on the ray $\theta = \pi$ (including $z=-1$).

The branch $f(z) = z^i = e^{i \log z}$ $(|z|>0, -\pi/2 < \arg z < 3\pi/2)$ is defined an continuous everywhere on C and is equal to the given branch of z^i at all points except $z=-1$. Then the integral will be the same if we replace the given integrand with this other branch $f(z)$.

$$F(z) = \frac{1}{1+i} z^{i+1} = \frac{1-i}{2} z^{i+1} = \frac{1-i}{2} e^{(1+i)\log z}$$
$$(|z|>0, -\pi/2 < \arg z < 3\pi/2)$$

$F(z)$ is an antiderivative of $f(z)$.

$$\int_{-1}^{1} z^i \, dz = \int_C f(z)\,dz = F(z)\Big|_{-1}^{1} \qquad \left\{ \begin{array}{l} 1 = e^{i\cdot 0} = e^0 \\ -1 = e^{i\pi} \end{array} \right\}$$

$$= \frac{1-i}{2}\left[e^{(1+i)\log e^0} - e^{(1+i)\log e^{i\pi}}\right]$$

$$= \frac{1-i}{2}\left[e^{(1+i)\cdot 0} - e^{(1+i)i\pi}\right]$$

$$= \frac{1-i}{2}\left[e^0 - e^{i\pi}e^{-\pi}\right]$$

$$= \frac{1-i}{2}\left[1 + e^{-\pi}\right] = \frac{1+e^{-\pi}}{2}(1-i)$$

In Exercise 7 of Section 42 we also integrated z^i but over a contour C that went from 1 to -1 so it makes sense that our result here has the opposite sign since the endpoints are reversed.

* * * * *

Chapter 4 Section 49 : 2, 4 (pages 161, 162)

2. Let C_1 denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 1$, $y = \pm 1$ and let C_2 be the positively oriented circle $|z| = 4$ (Fig. 63). With the aid of the corollary in Sec. 49, point out why

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

when

(a) $f(z) = \dfrac{1}{3z^2+1}$; *(b)* $f(z) = \dfrac{z+2}{\sin(z/2)}$; *(c)* $f(z) = \dfrac{z}{1-e^z}$.

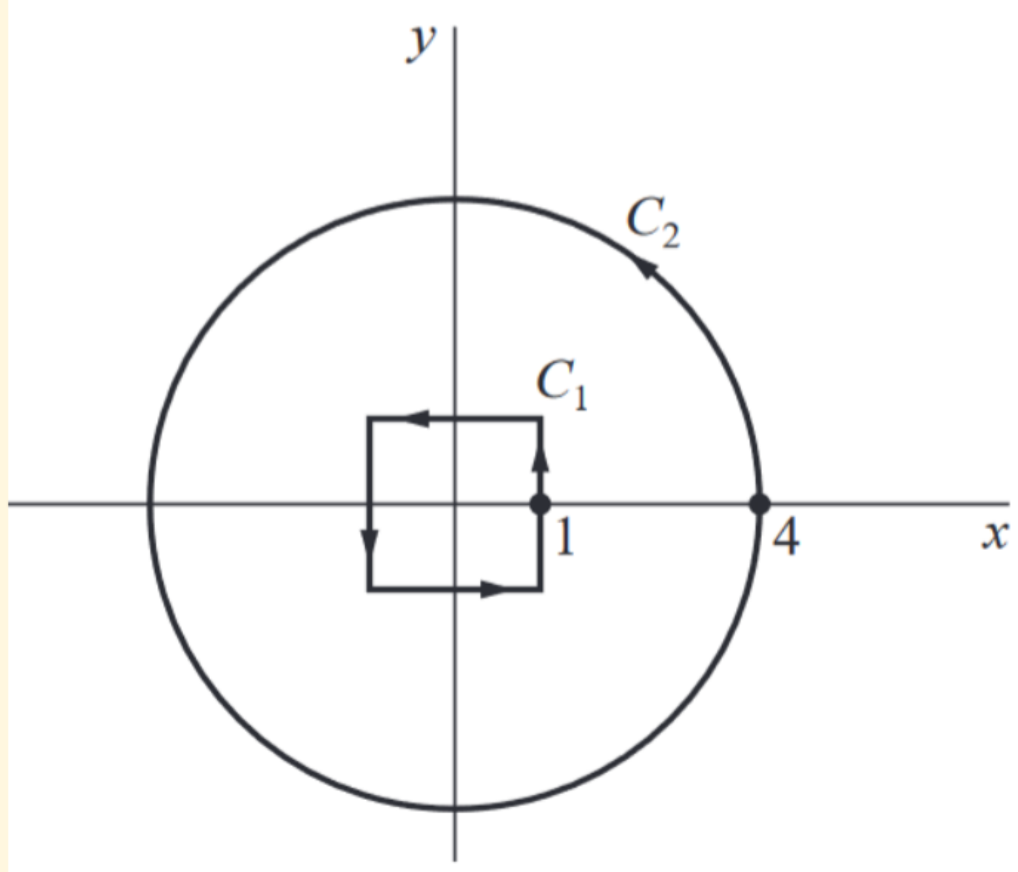


FIGURE 63

Corollary. *Let C_1 and C_2 denote positively oriented simple closed contours, where C_1 is interior to C_2 (Fig. 61). If a function f is analytic in the closed region consisting of those contours and all points between them, then*

$$\int_{C_2} f(z)\,dz = \int_{C_1} f(z)\,dz. \tag{2}$$

(a) Since $f(z)$ is a rational function of z, $f(z)$ is analytic at all points z where $f(z)$ is defined. $f(z)$ is defined $\forall z$ except $z = \pm 1/\sqrt{3}\, i$ $(3z^2+1=0)$. Note that the points $\pm 1/\sqrt{3}\, i \approx \pm 0.577i$ are enclosed by the contour C_1. Let S be the region on between the two squares with sides $x = \pm 0.6$, $y = \pm 0.6$ and $x = \pm 4$, $y = \pm 4$. Let $\bar{S}$ include S and all points lying on these two squares. $\bar{S}$ is closed, contains C_1, C_2 and all pts between C_1, C_2 and $f(z)$ is analytic on $\bar{S}$. By the corollary, $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$.

(b) $f(z)$ is analytic except where $\sin(z/2) = 0$.

$0 = \sin(z/2) = \sin x/2 \cosh y/2 + i \cos x/2 \sinh y/2$

$\Rightarrow \begin{cases} 0 = \sin x/2 \cosh y/2 \\ 0 = \cos x/2 \sinh y/2 \end{cases}$

$\Rightarrow$ $x/2 = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$
$y = 0$

$f(z)$ is analytic except at $(x,y) = (2n\pi, 0)$, $n = 0, \pm 1, \pm 2, \ldots$
Note that only $(x,y) = (0,0)$ is enclosed by C_1, C_2 $(n=0)$.
For $n = \pm 1, \pm 2, \ldots$, (x,y) it outside of C_2. We can take $\bar{S}$ to be the same region as part (a) (not the only option of course) and the conditions of the corollary are met on $\bar{S}$. $\therefore \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$.

(c) $f(z)$ is analytic except where $1 - e^z = 0$.

$0 = 1 - e^z \rightarrow 1 = e^z = e^x \cos y + i e^x \sin y \Rightarrow \begin{cases} 1 = e^x \cos y \\ 0 = e^x \sin y \end{cases}$

$0 = e^x \sin y \Rightarrow y = n\pi$, $n = 0, \pm 1, \ldots$

$1 = e^x \cos y$ and $e^x > 0 \Rightarrow x = 0$, $y = 2n\pi$, $n = 0, \pm 1, \ldots$

$f(z)$ is analytic everywhere except $(x,y) = (0, 2n\pi)$.
Note that $(0,0)$ is enclosed by C_1, C_2 but $(0, 2n\pi)$ is outside of C_2 for $n = \pm 1, \pm 2, \ldots$ The conditions of the corollary are met by $\bar{S}$ from parts (a) and (b).
$\therefore \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$

4. Use the following method to derive the integration formula

$$\int_0^\infty e^{-x^2}\cos 2bx\,dx = \frac{\sqrt{\pi}}{2}e^{-b^2} \qquad (b>0).$$

(a) Show that the sum of the integrals of e^{-z^2} along the lower and upper horizontal legs of the rectangular path in Fig. 64 can be written

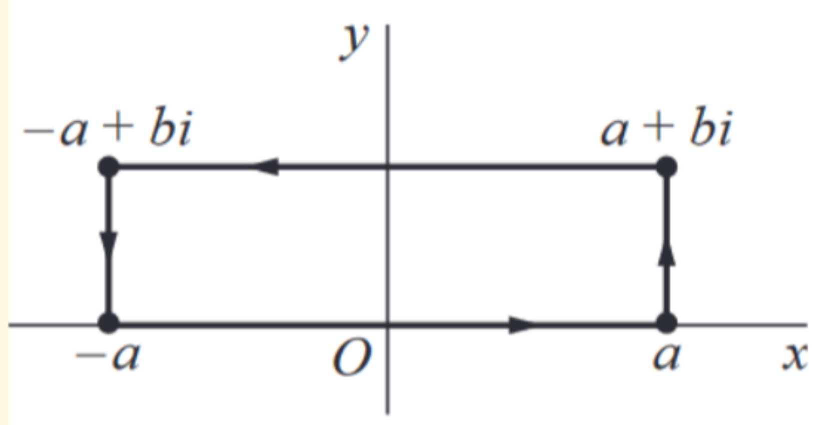


FIGURE 64

$$2\int_0^a e^{-x^2}dx - 2e^{b^2}\int_0^a e^{-x^2}\cos 2bx\,dx$$

and that the sum of the integrals along the vertical legs on the right and left can be written

$$ie^{-a^2}\int_0^b e^{y^2}e^{-i2ay}dy - ie^{-a^2}\int_0^b e^{y^2}e^{i2ay}dy.$$

Thus, with the aid of the Cauchy–Goursat theorem, show that

$$\int_0^a e^{-x^2}\cos 2bx\,dx = e^{-b^2}\int_0^a e^{-x^2}dx + e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin 2ay\,dy.$$

(b) By accepting the fact that*

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

and observing that

$$\left|\int_0^b e^{y^2}\sin 2ay\,dy\right| \le \int_0^b e^{y^2}dy,$$

obtain the desired integration formula by letting a tend to infinity in the equation at the end of part *(a)*.

Let C denote the rectangular path (Fig. 64).

$$f(z) = e^{-z^2} = e^{y^2-x^2}e^{-2xyi}$$

$$\int_C f(z)\,dz = \sum_{k=1}^{4}\int_{C_k} f(z)\,dz$$

Along C_1 ($y=0$, $x=-a$ to a), $f(z) = e^{-x^2}$

Along C_3 ($y=b$, $x=a$ to $-a$), $f(z) = e^{b^2-x^2}e^{-2bxi}$

Along C_2 ($x=a$, $y=0$ to b), $f(z) = e^{y^2-a^2}e^{-2ayi}$

Along C_4 ($x=-a$, $y=b$ to 0), $f(z) = e^{y^2-a^2}e^{2ayi}$

$$\int_{C_1} f(z)\,dz = \int_{-a}^{a} e^{-x^2}dx = 2\int_0^a e^{-x^2}dx$$

$$\int_{C_3} f(z)\,dz = \int_a^{-a} e^{b^2-x^2}\cos 2bx\,dx - i\int_a^{-a} e^{b^2-x^2}\sin 2bx\,dx$$

$$= 2\int_0^a e^{b^2-x^2}\cos 2bx\,dx - 0$$

$$= 2e^{b^2}\int_0^a e^{-x^2}\cos 2bx\,dx$$

$$\int_{C_2} f(z)\,dz = \int_0^b e^{y^2-a^2}e^{-2ayi}\,i\,dy$$

$$= ie^{-a^2}\int_0^b e^{y^2}e^{-2ayi}\,dy$$

$$\int_{C_4} f(z)\,dz = \int_b^0 e^{y^2-a^2}e^{2ayi}\,i\,dy$$

$$= -ie^{-a^2}\int_0^b e^{y^2}e^{2ayi}\,dy$$

<u>Horizontal:</u>

$$\int_{C_1} f\,dz + \int_{C_3} f\,dz = 2\int_0^a e^{-x^2}dx - 2e^{b^2}\int_0^a e^{-x^2}\cos 2bx\,dx$$

<u>Vertical:</u>

$$\int_{C_2} f\,dz + \int_{C_4} f\,dz = ie^{-a^2}\int_0^b e^{y^2}e^{-2ayi}\,dy$$
$$-ie^{-a^2}\int_0^b e^{y^2}e^{2ayi}\,dy$$

Cauchy-Goursat Theorem:

Theorem. *If a function f is analytic at all points interior to and on a simple closed contour C, then*

$$\int_C f(z)\, dz = 0.$$

$g(z) = -z^2 = y^2 - x^2 - 2xyi$ is analytic everywhere

$$u_x = -2x = v_y$$
$$u_y = 2y = -v_x$$

$h(w) = e^w$ is also analytic everywhere

For $w = x+iy$, $e^w = e^x \cos y + i e^x \sin y$

$$u_x = e^x \cos y = v_y$$
$$u_y = -e^x \sin y = -v_x$$

Thus $f(z) = e^{-z^2}$ is analytic everywhere (by composition $f = h \circ g$), in particular on and inside C.

By Cauchy-Goursat,

$$0 = \int_C f(z)\, dz = \sum_{k=1}^{4} \int_{C_k} f(z)\, dz$$

$$0 = 2\int_0^a e^{-x^2}dx - 2e^{b^2}\int_0^a e^{-x^2}\cos 2bx\,dx$$

$$+ ie^{-a^2}\int_0^b e^{y^2}e^{-2ayi}\,dy - ie^{-a^2}\int_0^b e^{y^2}e^{2ayi}\,dy$$

$$2e^{b^2}\int_0^a e^{-x^2}\cos 2bx\,dx = 2\int_0^a e^{-x^2}dx$$

$$+ ie^{-a^2}\left[\int_0^b e^{y^2}(\cos 2ay - i\sin 2ay - \cos 2ay - i\sin 2ay)\,dy\right]$$

$$e^{b^2}\int_0^a e^{-x^2}\cos 2bx\,dx = \int_0^a e^{-x^2}dx + e^{-a^2}\int_0^b e^{y^2}\sin 2ay\,dy$$

$$\boxed{\int_0^a e^{-x^2}\cos 2bx\,dx = e^{-b^2}\int_0^a e^{-x^2}dx + e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin 2ay\,dy}$$

(b) We are given that $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$. Also,

$$\left|\int_0^b e^{y^2}\sin 2ay\,dy\right| \le \int_0^b |e^{y^2}\sin 2ay|\,dy \le \int_0^b |e^{y^2}|\,dy = \int_0^b e^{y^2}dy$$

$$\Rightarrow 0 \le \lim_{a\to\infty}\left|e^{-a^2}\int_0^b e^{y^2}\sin 2ay\,dy\right| \le \lim_{a\to\infty} e^{-a^2}\int_0^b e^{y^2}dy = 0$$

$$\Rightarrow \lim_{a\to\infty} e^{-a^2}e^{-b^2}\int_0^b e^{y^2}\sin 2ay\,dy = 0$$

$$\therefore \int_0^\infty e^{-x^2}\cos 2bx = e^{-b^2}\frac{\sqrt{\pi}}{2} + \lim_{a\to\infty} e^{-a^2}e^{-b^2}\int_0^b e^{y^2}\sin 2ay\,dy$$

$$\int_0^\infty e^{-x^2}\cos 2bx = e^{-b^2}\frac{\sqrt{\pi}}{2}, \quad b>0$$