

Written Homework 3 Solutions

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Chapter 3 Section 31: 1b, 2b, 4, 6, 7, 9

1. Show that

$$(a) \operatorname{Log}(-ei) = 1 - \frac{\pi}{2}i; \quad (b) \operatorname{Log}(1-i) = \frac{1}{2}\ln 2 - \frac{\pi}{4}i.$$

Def: For $z=re^{i\theta}$, $\operatorname{Log} z = \ln r + i\theta$

$$(b) z = 1-i = \sqrt{2}e^{-i\pi/4}$$

$$\operatorname{Log}(1-i) = \ln\sqrt{2} + i(-\pi/4) = \frac{1}{2}\ln 2 - i\frac{\pi}{4}$$

2. Show that

$$(a) \log e = 1 + 2n\pi i \quad (n = 0, \pm 1, \pm 2, \dots);$$

$$(b) \log i = \left(2n + \frac{1}{2}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \dots);$$

$$(c) \log(-1 + \sqrt{3}i) = \ln 2 + 2\left(n + \frac{1}{3}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \dots).$$

Def: For $z=re^{i\theta}$, $\operatorname{Log} z = \ln r + i(\theta + 2n\pi)$ ($n = 0, \pm 1, \pm 2, \dots$)

$$(b) z = i = e^{i\pi/2}$$

$$\operatorname{Log} z = \ln 1 + i(\pi/2 + 2n\pi) = i(\pi/2 + 2n\pi), n \in \mathbb{Z}$$

4. Show that

$$(a) \log(i^2) = 2\log i \quad \text{when} \quad \operatorname{Log} z = \ln r + i\theta \left(r > 0, \frac{\pi}{4} < \theta < \frac{9\pi}{4}\right);$$

$$(b) \log(i^2) \neq 2\log i \quad \text{when} \quad \operatorname{Log} z = \ln r + i\theta \left(r > 0, \frac{3\pi}{4} < \theta < \frac{11\pi}{4}\right).$$

$$(a) \log i^2 = \log -1 = \ln 1 + i(\pi + 2n\pi) = i(\pi + 2n\pi), n \in \mathbb{Z}$$

For $\theta = \pi + 2n\pi$ to satisfy $\pi/4 < \theta < \pi/4 + 2\pi$, use $n=0$.

Then $\log i^2 = i\pi$

$$\log i = \ln 1 + i(\pi/2 + 2n\pi) = i(\pi/2 + 2n\pi), n \in \mathbb{Z}$$

For $\theta = \pi/2 + 2n\pi$ to satisfy $\pi/4 < \theta < \pi/4 + 2\pi$, use $n=0$.

Then $2\log i = 2i\pi/2 = i\pi$ as well.

$$(b) \log i^2 = i(\pi + 2n\pi). \text{ For } \theta = \pi + 2n\pi \text{ to satisfy } 3\pi/4 < \theta < 3\pi/4 + 2\pi, \text{ use } n=0. \text{ Then } \log i^2 = i\pi$$

$$\log i = i(\pi/2 + 2n\pi). \text{ For } \theta = \pi/2 + 2n\pi \text{ to satisfy } 3\pi/4 < \theta < 3\pi/4 + 2\pi, \text{ use } n=1. \text{ Then } 2\log i = 2i(\pi/2 + 2\pi) = 5i\pi \neq i\pi$$

6. Given that the branch $\log z = \ln r + i\theta$ ($r > 0, \alpha < \theta < \alpha + 2\pi$) of the logarithmic function is analytic at each point z in the stated domain, obtain its derivative by differentiating each side of the identity (Sec. 30)

$$e^{\log z} = z \quad (z \neq 0)$$

and using the chain rule.

$$\begin{aligned} 1 &= \frac{d}{dz} z = \frac{d}{dz} e^{\log z} = e^{\log z} \frac{d}{dz} \log z = z \frac{d}{dz} \log z \\ \therefore \frac{d}{dz} \log z &= \frac{1}{z}, \quad z \neq 0 \end{aligned}$$

7. Find all roots of the equation $\log z = i\pi/2$.

Ans. $z = i$.

$$\log z = i\pi/2 \Rightarrow z = e^{\log z} = e^{i\pi/2} = i$$

9. Show that

- (a) the function $f(z) = \text{Log}(z - i)$ is analytic everywhere except on the portion $x \leq 0$ of the line $y = 1$;
 (b) the function

$$f(z) = \frac{\text{Log}(z+4)}{z^2 + i}$$

is analytic everywhere except at the points $\pm(1-i)/\sqrt{2}$ and on the portion $x \leq -4$ of the real axis.

$$\begin{aligned} (a) \text{Log}(z-i) &= \text{Log}(x+iy-i) = \text{Log}(x+i(y-1)) \\ &= \ln\sqrt{x^2+(y-1)^2} + i\text{Arg}(x+i(y-1)) \end{aligned}$$

For $z = x+iy$ s.t. $y=1$, $\text{Arg}(x+i(y-1)) = \text{Arg } x$.

If $x \leq 0$ as well, $\text{Arg } x = \pi$. This is on the branch

cut for Log , so $\text{Log}(z-i)$ is discontinuous here

and therefore not analytic. Otherwise $-\pi < \text{Arg}(z-i) < \pi$

so $f(z)$ is analytic if it is not the case $y=1, x \leq 0$.

The short answer is that $\text{Log } z$ has a branch cut along $y=0$ and $x \leq 0$ so $\text{Log}(z-i)$ has a branch cut shifted up to $y=1, x \leq 0$.

- (b) Since $z^2 + i = 0$ at $z = \pm(1-i)/\sqrt{2}$ ($z^2 = -i = e^{-i\pi/2} \Rightarrow z = \pm e^{-i\pi/4}$), $f(z)$ is not even defined at these points so cannot be analytic. Since the branch cut for $\text{Log } z$ is along $y=0$ and $x \leq 0$, the branch cut for $\text{Log}(z+4)$ is shifted left to $y=0$ and $x \leq -4$. Since $\text{Log}(z+4)$ is not analytic on the branch cut, neither is $f(z)$.

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Chapter 3 Section 32: 1, 2, 6 (page 100)

- Show that if $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$, then

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2.$$

Suggestion: Write $\Theta_1 = \operatorname{Arg} z_1$ and $\Theta_2 = \operatorname{Arg} z_2$. Then observe how it follows from the stated restrictions on z_1 and z_2 that $-\pi < \Theta_1 + \Theta_2 < \pi$.

$$z_1 = r_1 e^{i\Theta_1}, \quad z_2 = r_2 e^{i\Theta_2} \quad \text{where } \Theta_1 = \operatorname{Arg} z_1, \quad \Theta_2 = \operatorname{Arg} z_2$$

$$\operatorname{Re} z_1 > 0, \quad \operatorname{Re} z_2 > 0 \Rightarrow -\pi/2 < \operatorname{Arg} z_1 < \pi/2, \quad -\pi/2 < \operatorname{Arg} z_2 < \pi/2$$

$$\Rightarrow -\pi < \operatorname{Arg} z_1 + \operatorname{Arg} z_2 < \pi$$

$$\begin{aligned}\operatorname{Log}(z_1 z_2) &= \operatorname{Log}(r_1 e^{i\Theta_1} r_2 e^{i\Theta_2}) = \operatorname{Log}(r_1 r_2 e^{i(\Theta_1 + \Theta_2)}) \\ &= \ln(r_1 r_2) + i(\Theta_1 + \Theta_2) = \ln r_1 + i\Theta_1 + \ln r_2 + i\Theta_2 \\ &= \operatorname{Log} z_1 + \operatorname{Log} z_2\end{aligned}$$

- Show that for any two nonzero complex numbers z_1 and z_2 ,

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2 + 2N\pi i$$

where N has one of the values $0, \pm 1$. (Compare with Exercise 1.)

$$-\pi < \operatorname{Arg} z_1, \quad \operatorname{Arg} z_2 < \pi \Rightarrow -2\pi < \operatorname{Arg} z_1 + \operatorname{Arg} z_2 < 2\pi$$

Let $\Theta = \operatorname{Arg} z_1 z_2$, which must by definition satisfy $-\pi < \Theta < \pi$.

$$\Theta = \begin{cases} \Theta_1 + \Theta_2 - 2\pi, & \Theta_1 + \Theta_2 > \pi & (N = -1) \\ \Theta_1 + \Theta_2, & -\pi < \Theta_1 + \Theta_2 < \pi & (N = 0) \\ \Theta_1 + \Theta_2 + 2\pi, & \Theta_1 + \Theta_2 < -\pi & (N = 1) \end{cases}$$

$$\begin{aligned}\operatorname{Log}(z_1 z_2) &= \operatorname{Log}(r_1 e^{i\Theta_1} r_2 e^{i\Theta_2}) = \operatorname{Log}(r_1 r_2 e^{i(\Theta_1 + \Theta_2)}) \\ &= \ln(r_1 r_2) + i\Theta = \ln r_1 + i\Theta + \ln r_2 + i\Theta_2 + 2N\pi i\end{aligned}$$

Note that by the definition chosen in the textbook we cannot handle the case where $\Theta_1 + \Theta_2 = \pm\pi$ in this way since $N=0$, and $N=\pm 1$ all return us to $\Theta_1 + \Theta_2 = \pm\pi$ and we need $-\pi < \Theta < \pi$. In this case, add or subtract only π instead of 2π .

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Chapter 3 Section 33: 1, 2b, 4, 9 (page 104)

1. Show that

$$(a) (1+i)^i = \exp\left(-\frac{\pi}{4} + 2n\pi\right) \exp\left(i\frac{\ln 2}{2}\right) \quad (n = 0, \pm 1, \pm 2, \dots);$$

$$(b) (-1)^{1/\pi} = e^{(2n+1)i} \quad (n = 0, \pm 1, \pm 2, \dots).$$

$$(a) \log(1+i) = \ln\sqrt{2} + i(\pi/4 + 2m\pi), \quad m \in \mathbb{Z}$$

$$i \log(1+i) = i \ln\sqrt{2} - (\pi/4 + 2m\pi) = i \frac{\ln 2}{2} - \pi/4 + 2m\pi, \quad n := m$$

$$\begin{aligned} (1+i)^i &= z^c = \exp(c \log z) = \exp\left[i \frac{\ln 2}{2} - \pi/4 + 2m\pi\right] \\ &= \exp\left[i \frac{\ln 2}{2}\right] \exp[-\pi/4 + 2m\pi], \quad n \in \mathbb{Z} \end{aligned}$$

$$(b) \log(-1) = \ln 1 + i\pi(1+2n) = i\pi(1+2n)$$

$$\pi^{-1} \log(-1) = i(1+2n)$$

$$(-1)^{i/\pi} = \exp[i(1+2n)], \quad n \in \mathbb{Z}.$$

2. Find the principal value of

$$(a) i^i; \quad (b) \left[\frac{e}{2}(-1 - \sqrt{3}i)\right]^{3\pi i}; \quad (c) (1-i)^{4i}.$$

Ans. (a) $\exp(-\pi/2)$; (b) $-\exp(2\pi^2)$; (c) $e^\pi [\cos(2 \ln 2) + i \sin(2 \ln 2)]$.

Def: P.V. $z^c = e^{c \log z}, \quad z \neq 0 \text{ and } c \in \mathbb{C}$

$$z = e^{(-1-\sqrt{3}i)/2} = e e^{-i2\pi/3} = e^{1-i2\pi/3}, \quad c = 3\pi i$$

$$c \log z = 3\pi i \log e^{1-i2\pi/3} = 3\pi i (1-i2\pi/3) = 3\pi i + 2\pi^2$$

$$\therefore [e^{(-1-\sqrt{3}i)/2}]^{3\pi i} = z^c = e^{c \log z} = e^{3\pi i + 2\pi^2} = (e^{i\pi})^3 e^{2\pi^2} = -e^{2\pi^2}$$

9. Assuming that $f'(z)$ exists, state the formula for the derivative of $c^{f(z)}$.

$$\frac{d}{dz} [c^{f(z)}] = \frac{d}{dz} [e^{f(z) \log c}] = [e^{f(z) \log c}] f'(z) \log c = c^{f(z)} f'(z) \log c$$

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Chapter 3 Section 34: 2, 3, 7, 11, 16 (pages 108, 109)

2. (a) With the aid of expression (4), Sec. 34, show that

$$e^{iz_1} e^{iz_2} = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 + i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2).$$

Then use relations (3), Sec. 34, to show how it follows that

$$e^{-iz_1} e^{-iz_2} = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 - i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2).$$

- (b) Use the results in part (a) and the fact that

$$\sin(z_1 + z_2) = \frac{1}{2i} [e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}] = \frac{1}{2i} (e^{iz_1} e^{iz_2} - e^{-iz_1} e^{-iz_2})$$

to obtain the identity

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$

in Sec. 34.

$$\begin{aligned}
 (a) \quad & e^{iz_1} e^{iz_2} = (\cos z_1 + i \sin z_1)(\cos z_2 + i \sin z_2) \\
 &= \cos z_1 \cos z_2 + i \cos z_1 \sin z_2 + i \sin z_1 \cos z_2 + i^2 \sin z_1 \sin z_2 \\
 &= \cos z_1 \cos z_2 - \sin z_1 \sin z_2 + i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2) \\
 e^{-iz_1} e^{-iz_2} &= \cos z_1 \cos z_2 - (-1)^2 \sin z_1 \sin z_2 + i(-\sin z_1 \cos z_2 - \cos z_1 \sin z_2) \\
 &= \cos z_1 \cos z_2 - \sin z_1 \sin z_2 - i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2) \\
 (b) \quad & \sin(z_1 + z_2) = \frac{1}{2i} (e^{iz_1} e^{iz_2} - e^{-iz_1} e^{-iz_2}) \\
 &= \frac{1}{2i} 2i (\sin z_1 \cos z_2 + \cos z_1 \sin z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2
 \end{aligned}$$

3. According to the final result in Exercise 2(b),

$$\sin(z + z_2) = \sin z \cos z_2 + \cos z \sin z_2.$$

By differentiating each side here with respect to z and then setting $z = z_1$, derive the expression

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

that was stated in Sec. 34.

By (2) on page 105 $\frac{d}{dz} \sin z = \cos z$ and $\frac{d}{dz} \cos z = -\sin z$, $z \in \mathbb{C}$ just like the derivatives of $\sin x$ and $\cos x$ for $x \in \mathbb{R}$.

$$\frac{d}{dz} [\sin(z + z_2)]|_{z=z_1} = \frac{d}{dz} [\sin z \cos z_2 + \cos z \sin z_2]|_{z=z_1},$$

$$\cos(z + z_2)|_{z=z_1} = [\cos z \cos z_2 - \sin z \sin z_2]|_{z=z_1},$$

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

7. In Sec. 34, use expressions (13) and (14) to derive expressions (15) and (16) for $|\sin z|^2$ and $|\cos z|^2$.

Suggestion: Recall the identities $\sin^2 x + \cos^2 x = 1$ and $\cosh^2 y - \sinh^2 y = 1$.

$$(13) \quad \sin z = \sin x \cosh y + i \cos x \sinh y,$$

$$(14) \quad \cos z = \cos x \cosh y - i \sin x \sinh y,$$

$$(15) \quad |\sin z|^2 = \sin^2 x + \sinh^2 y,$$

$$(16) \quad |\cos z|^2 = \cos^2 x + \sinh^2 y.$$

$$\begin{aligned} |\sin z|^2 &= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \\ &= \sin^2 x (1 + \sinh^2 y) + \cos^2 x \sinh^2 y \\ &= \sin^2 x + \sinh^2 y (\sin^2 x + \cos^2 x) \\ &= \sin^2 x + \sinh^2 y \end{aligned}$$

$$\begin{aligned} |\cos z|^2 &= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y \\ &= \cos^2 x (1 + \sinh^2 y) + \sin^2 x \sinh^2 y \\ &= \cos^2 x + \sinh^2 y (\cos^2 x + \sin^2 x) \\ &= \cos^2 x + \sinh^2 y \end{aligned}$$

* [Exercise 11 is on the next page] *

16. With the aid of expression (14), Sec. 34, show that the roots of the equation $\cos z = 2$ are

$$z = 2n\pi + i \cosh^{-1} 2 \quad (n = 0, \pm 1, \pm 2, \dots).$$

Then express them in the form

$$z = 2n\pi \pm i \ln(2 + \sqrt{3}) \quad (n = 0, \pm 1, \pm 2, \dots).$$

$2 = \cos z = \cos x \cosh y - i \sin x \sinh y \Rightarrow 2 = \cos x \cosh y, 0 = \sin x \sinh y$
 Since $0 = \sin x \sinh y$, $\sinh y = 0$ or $\sin x = 0$. $\sinh y = 0$ only if $y = 0$ and $\sin x = 0$ if $x = m\pi$, $m \in \mathbb{Z}$. If $y = 0$ then since $\cosh y = 1$, we must have $\cos x = 2$. But this isn't possible since $|\cos x| \leq 1$. So $y \neq 0$ and we must have $x = m\pi$. But $\cos m\pi = (-1)^m$ so we $2 = (-1)^m \cosh y$. Since $\cosh y > 0 \forall y$, only $2 = \cosh y$ can be satisfied ($m = 2n$, $n \in \mathbb{Z}$)
 $\therefore z = 2n\pi + i \cosh^{-1} 2 = 2n\pi \pm i \ln(2 + \sqrt{3})$ (see Sec. 36, Exercise 2)

11. Use the Cauchy-Riemann equations and the theorem in Sec. 21 to show that neither $\sin \bar{z}$ nor $\cos \bar{z}$ is an analytic function of z anywhere.

$$\triangle f(z) = \sin \bar{z} = \sin x \cosh(-y) + i \cos x \sinh(-y) = \sin x \cosh y - i \cos x \sinh y$$

Suppose $f'(z)$ exists at some point $z = x+iy$. By the theorem:

$$C-R \quad \begin{cases} \cos x \cosh y = u_x = v_y = -\cos x \cosh y \\ \sin x \sinh y = u_y = -v_x = -\sin x \sinh y \end{cases}$$

$u_x = v_y$ implies $\cos x \cosh y = 0$. Since $\cosh y > 0 \quad \forall y \in \mathbb{R}$ we must have $\cos x = 0$ (iff $x = \pi/2 + m\pi, m \in \mathbb{Z}$).

$u_y = -v_x$ implies $\sin x \sinh y = 0$. Either $\sinh y = 0$ (iff $y = 0$) or $\sin x = 0$ (iff $x = n\pi, n \in \mathbb{Z}$). Both C-R equations must hold simultaneously.

Since $\nexists x$ s.t. $\sin x = \cos x = 0$, the only possibility is $\cos x = 0$ and $\sinh y = 0$, which means $x = \pi/2 + m\pi, m \in \mathbb{Z}$ and $y = 0$. But this means the only points z at which $f'(z)$ may possibly exist are at these isolated points. Referring to the definition of analyticity, it follows that if $f'(z)$ only exists at isolated points or nowhere at all, $f(z)$ is nowhere analytic (think about the neighborhoods of these isolated pts).

$$\triangle g(z) = \cos \bar{z} = \cos x \cosh(-y) - i \sin x \sinh(-y) = \cos x \cosh y + i \sin x \sinh y$$

Suppose $g'(z)$ exists at some point $z = x+iy$. By the theorem:

$$C-R \quad \begin{cases} -\sin x \cosh y = u_x = v_y = \sin x \cosh y \\ \cos x \sinh y = u_y = -v_x = -\cos x \sinh y \end{cases}$$

$$u_x = v_y \Rightarrow \sin x = 0 \Rightarrow x = n\pi, n \in \mathbb{Z}$$

$$u_y = -v_x \Rightarrow \cos x = 0 \text{ or } \sinh y = 0 \Rightarrow x = \pi/2 + m\pi, m \in \mathbb{Z} \text{ or } y = 0.$$

Since $u_x = v_y$ and $u_y = -v_x$ must both hold simultaneously, the only points z at which $g'(z)$ may possibly exist are the isolated pts $z = x+iy$ with $x = n\pi, n \in \mathbb{Z}$ and $y = 0$. It follows by the definition of analyticity that $g(z)$ is nowhere analytic.

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Chapter 3 Section 35: 4, 7, 16 (pages 111, 112)

7. Show that

- (a) $\sinh(z + \pi i) = -\sinh z$;
- (b) $\cosh(z + \pi i) = \cosh z$;
- (c) $\tanh(z + \pi i) = \tanh z$.

$$(a) \sinh(z + \pi i) = \frac{e^{z+\pi i} - e^{-z-\pi i}}{2} = \frac{e^z e^{i\pi} - e^{-z} e^{-i\pi}}{2}$$

$$= \frac{-e^z + e^{-z}}{2} = -\frac{e^z - e^{-z}}{2} = -\sinh z$$

$$(b) \cosh(z + \pi i) = \frac{e^{z+\pi i} + e^{-z-\pi i}}{2} = \frac{e^z e^{i\pi} + e^{-z} e^{-i\pi}}{2}$$

$$= \frac{-e^z - e^{-z}}{2} = -\frac{e^z + e^{-z}}{2} = -\cosh z$$

$$(c) \tanh(z + \pi i) = \frac{\sinh(z + \pi i)}{\cosh(z + \pi i)} = \frac{-\sinh z}{-\cosh z} = \frac{\sinh z}{\cosh z} = \tanh z$$

16. Find all roots of the equation $\cosh z = -2$. (Compare this exercise with Exercise 16, Sec. 34.)

$$\text{Ans. } z = \pm \ln(2 + \sqrt{3}) + (2n + 1)\pi i \quad (n = 0, \pm 1, \pm 2, \dots).$$

$$-2 = \cosh z = \cosh x \cos y + i \sinh x \sin y \Rightarrow -2 = \cosh x \cos y, 0 = \sinh x \sin y$$

$$0 = \sinh x \sin y \text{ means } 0 = \sinh x \text{ or } 0 = \sin y$$

$$\text{If } 0 = \sinh x, x = 0 \text{ so } -2 = \cosh x \cos y = 1 \cdot \cos y = \cos y.$$

Since $-2 \neq \cos y$ for any y , conclude $x \neq 0$. Then $0 = \sin y$.

If $0 = \sin y, y = m\pi, m \in \mathbb{Z}$. Since $\cos m\pi = (-1)^m$ we have

$-2 = (-1)^m \cosh x$. Since $\cosh x > 0 \forall x$, this can only be satisfied when $(-1)^m = -1$. That is, $m = 2n+1, n \in \mathbb{Z}$ so $y = (2n+1)\pi$.

Finally, $-2 = -\cosh x \Rightarrow x = \cosh^{-1} 2 = \pm \ln(2 + \sqrt{3})$ (this last equality is found in Exercise 2 of Section 36).

$$\therefore z = \cosh^{-1} 2 + i(2n+1)\pi = \pm \ln(2 + \sqrt{3}) + i(2n+1)\pi, n \in \mathbb{Z}.$$

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Chapter 3 Section 36: 2, 4, 5, 7 (pages 114, 115)

2. Solve the equation $\sin z = 2$ for z by

- (a) equating real parts and then imaginary parts in that equation;
- (b) using expression (2), Sec. 36, for $\sin^{-1} z$.

$$\text{Ans. } z = \left(2n + \frac{1}{2}\right)\pi \pm i \ln(2 + \sqrt{3}) \quad (n = 0, \pm 1, \pm 2, \dots).$$

(a) If $2 = \sin z = \sin x \cosh y + i \cos x \sinh y$

then $2 = \sin x \cosh y$ and $0 = \cos x \sinh y$.

Since $0 = \cos x \sinh y$ we must have $\sinh y = 0$ or $\cos x = 0$.

If $y=0$, then $\sinh y = 0$ and $\cosh y = 1$ but then we'd need

$2 = \sin x$. $|\sin x| \leq 1$ so $y=0$ doesn't work. Then it must be the case that $\cos x = 0$, which can occur if $x = \pi/2 + n\pi$,

$n \in \mathbb{Z}$. Note that $\sin(\pi/2 + n\pi) = (-1)^n$ so from $2 = \sin x \cosh y$

we get $\pm 2 = \cosh y$. But since $\cosh y > 0 \quad \forall y \in \mathbb{R}$, it must

be the case that $\cos x = 0$ and $\sin x = 1$ to satisfy both eqn's.

$\therefore 2 = \sin z \Rightarrow x = \pi/2 + 2n\pi, n \in \mathbb{Z}$ and $\cosh y = 2$.

Two ways to find y :

$$2 = \cosh y$$

$$y = \pm \cosh^{-1}(2) \quad (\cosh y = 2 \text{ iff } \cosh(-y) = 2)$$

$$= \pm \log(2 + (2^2 - 1)^{1/2})$$

$$= \pm \log(2 + \sqrt{3})$$

$$= \pm \ln(2 + \sqrt{3}) \quad (\text{since } 2 + \sqrt{3} \in \mathbb{R})$$

$$\approx \pm 1.317$$

$$2 = \cosh y = (e^y + e^{-y})/2$$

$$4 = e^y + e^{-y}$$

$$4e^y = e^{2y} + 1$$

$$u^2 - 4u + 1 = 0 \quad (u := e^y)$$

$$u = \frac{4}{2} \pm \frac{\sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

$$e^y = 2 \pm \sqrt{3}$$

$$y = \ln(2 \pm \sqrt{3}) \approx \pm 1.317$$

$$\therefore z = x + iy = (\pi/2 + 2n\pi) + i \ln(2 \pm \sqrt{3}) = (2n + 1/2)\pi \pm i \ln(2 + \sqrt{3})$$

(b) (2)

$$\sin^{-1} z = -i \log [iz + (1 - z^2)^{1/2}].$$

$$\begin{aligned}\sin z = 2 \rightarrow z &= \sin^{-1} 2 = -i \log [2i + (1 - 2^2)^{1/2}] = -i \log [2i + \sqrt{3}i] \\ &= -i \log [i(2 + \sqrt{3})] = -i \log i - i \log (2 + \sqrt{3}) \\ &= -i(\ln 1 + i(\pi/2 + 2n\pi)) - i \ln (2 + \sqrt{3}) \\ &= \pi/2 + 2n\pi - i \ln (2 + \sqrt{3})\end{aligned}$$

Since $\sin z = -\sin(-z)$, $z = \sin z$ iff $-z = \sin(-z)$.

$$\begin{aligned}\sin(-z) = -2 \rightarrow z &= -\sin^{-1}(-2) = i \log [-2i + (1 - (-2)^2)^{1/2}] \\ &= i \log [-i(2 - \sqrt{3})] \\ &= i \log(-i) + i \log(2 - \sqrt{3}) \\ &= i(\ln 1 + i(-\pi/2 + 2m\pi)) + i \ln (2 - \sqrt{3}) \\ &= \pi/2 - 2m\pi + i \ln (2 - \sqrt{3}) \\ &= \pi/2 + 2n\pi + i \ln (2 - \sqrt{3}) \quad n := -m \in \mathbb{Z} \\ &= \pi/2 + 2n\pi - i \ln (2 + \sqrt{3})\end{aligned}$$

$$\therefore z = \pi/2 + 2n\pi \pm i \ln (2 + \sqrt{3})$$

5. Derive expression (4), Sec. 36, for $\tan^{-1} z$.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

$$w = \tan^{-1} z$$

$$z = \tan w = \frac{\sin w}{\cos w} = \frac{\frac{e^{iw} - e^{-iw}}{2i}}{\frac{e^{iw} + e^{-iw}}{2}} = \frac{1}{i} \frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} = -i \frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}}$$

$$iz(e^{iw} + e^{-iw}) = e^{iw} - e^{-iw}$$

$$(iz(e^{2iw} + 1) = e^{2iw} - 1$$

$$(1 - iz)(e^{iw})^2 = 1 + iz$$

$$e^{iw} = \pm \left[\frac{1 + iz}{1 - iz} \right]^{1/2}$$

$$iw = \log \pm \left[\frac{1 + iz}{1 - iz} \right]^{1/2}$$

$$w = -i \log \pm \left[\frac{1 + iz}{1 - iz} \right]^{1/2}$$

$$w = -\frac{i}{2} \log \pm \left[\frac{i - z}{i + z} \right]$$

$$w = \frac{i}{2} \log \pm \left[\frac{i + z}{i - z} \right]$$

$$\tan^{-1}(z) = \frac{i}{2} \log \left[\frac{i + z}{i - z} \right]$$

see discussion
following (4)
about making
 $\tan^{-1} z$ single
valued.

* * * *

Chapter 8 Section 91 : 2, 3, 4 (page 313)

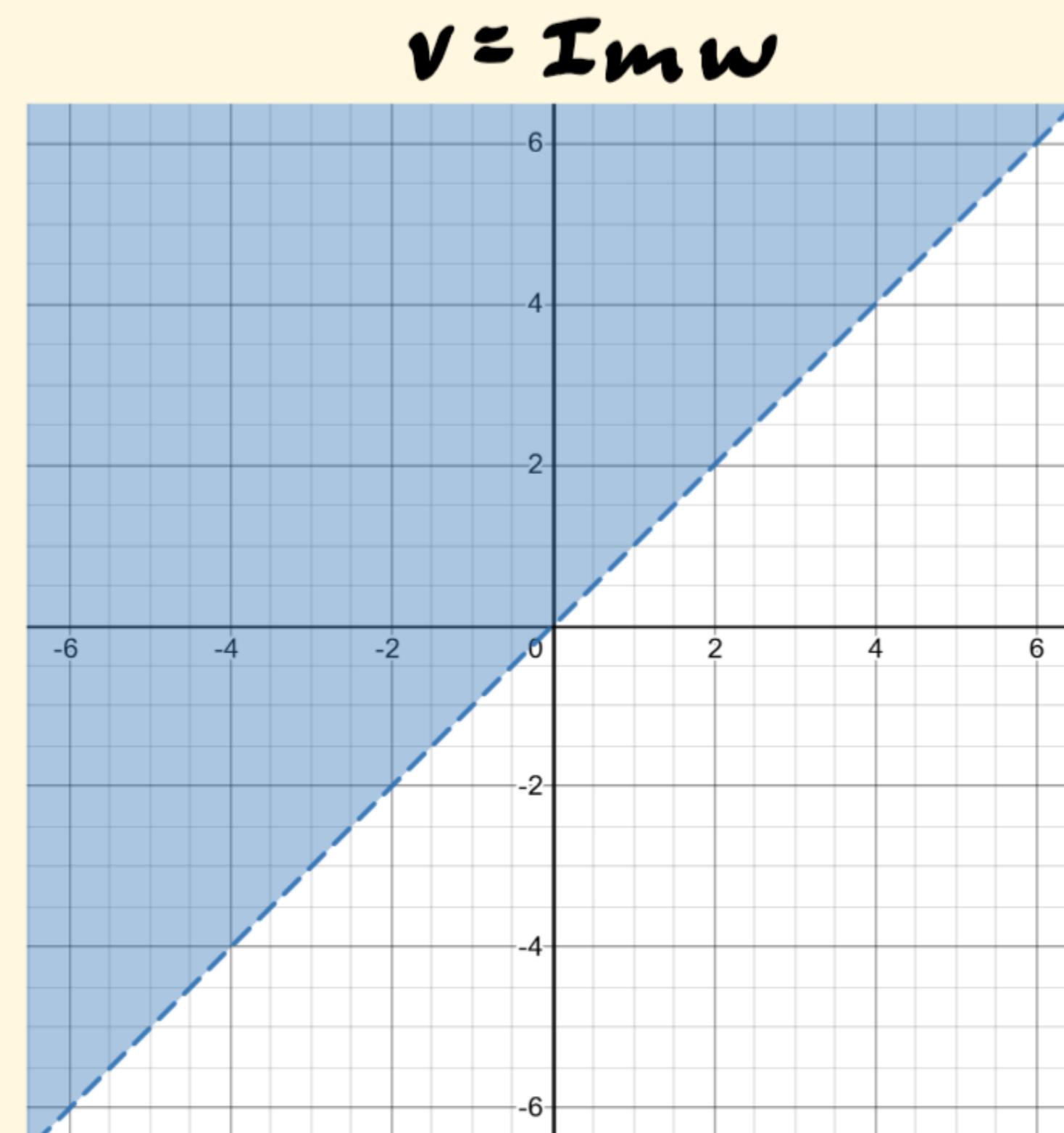
3. Find and sketch the region onto which the half plane $y > 0$ is mapped by the transformation $w = (1+i)z$.

Ans. $v > u$.

$z = re^{i\theta}$, $0 < r < \infty$, $0 < \theta < \pi$ for z in the half plane $y > 0$.

$$\begin{aligned} w &= (1+i)z = \sqrt{2}e^{i\pi/4}re^{i\theta} \\ &= \sqrt{2}re^{i(\theta+\pi/4)}, \quad 0 < \sqrt{2}r < \infty, \quad \pi/4 < \theta + \pi/4 < 5\pi/4 \\ &= \rho e^{i\varphi}, \quad 0 < \rho < \infty, \quad \pi/4 < \varphi < 5\pi/4 \quad (\rho := \sqrt{2}r, \varphi := \theta + \pi/4) \\ &= u + iv \end{aligned}$$

The polar conditions $0 < \rho < \infty$ and $\pi/4 < \varphi < 5\pi/4$ include all pts $w = u + iv$ satisfying $v > u$.



$u = \text{Re } w$

* * * *

Chapter 8 Section 92: 2, 3, 4, 9, 11 (page 318)

2. Show that when $c_1 < 0$, the image of the half plane $x < c_1$ under the transformation $w = 1/z$ is the interior of a circle. What is the image when $c_1 = 0$?

$$A(x^2 + y^2) + Bx + Cy + D = 0 \rightarrow \text{lines } x = K, \quad K < c_1 < 0 \\ (A = C = 0, B = 1, D = -K)$$

$$D(u^2 + v^2) + Bu - Cv + A = 0$$

$$-K(u^2 + v^2) + u = 0$$

$$K(u^2 - u/K) + Kv^2 = 0$$

$$(u^2 - u/K + 1/4K^2 - 1/4K^2) + v^2 = 0$$

$$(u - 1/2K)^2 + v^2 = (1/2K)^2 \text{ for each line } x = K \text{ in the half plane } x < c_1,$$

For each K , this is a circle with radius $-1/2K$ and center $(1/2K, 0)$ in the complex plane ($w = u + iv$). Since $K < c_1 < 0$, the center is on the negative real axis.

As K decreases ($K < 0$ with increasing magnitude), the center $(1/2K, 0)$ approaches the origin $(0, 0)$ and the radius tends toward 0, i.e. circles of smaller and smaller radii shifting right along the negative real axis.

As K increases ($K < 0$ but K becoming less negative) toward $c_1 < 0$ we have circles of larger and larger radii shifting left along the negative real axis. The condition $x < c_1$ means that these circles approach $(u - 1/2c_1)^2 + v^2 = (1/2c_1)^2$ in this manner but are all strictly contained within this circle.

\therefore The image of the half plane $x < c_1$ under the transformation $w = 1/z$ is the interior of the circle $(u - 1/2c_1)^2 + v^2 = (1/2c_1)^2$.

9. Find the image of the semi-infinite strip $x > 0, 0 < y < 1$ when $w = i/z$. Sketch the strip and its image.

$$\text{Ans. } \left(u - \frac{1}{2}\right)^2 + v^2 > \left(\frac{1}{2}\right)^2, \quad u > 0, \quad v > 0.$$

The transformation $w = i/z = i^2/iz = -1/(xi-y) = 1/(y+i(-x))$ performs same transformation as $w = 1/z$ if we replace x with y and replace y with $-x$ in the latter.

$$Bx + Cy + D = 0 \rightarrow By - Cx + D = 0.$$

Transforming a horizontal semi-infinite line segment $x > 0, y = k \in (0, 1)$ we have (modifying the calculation from exercise 2):

$$(u - \frac{1}{2k})^2 + v^2 = (\frac{1}{2k})^2, \quad k \in (0, 1)$$

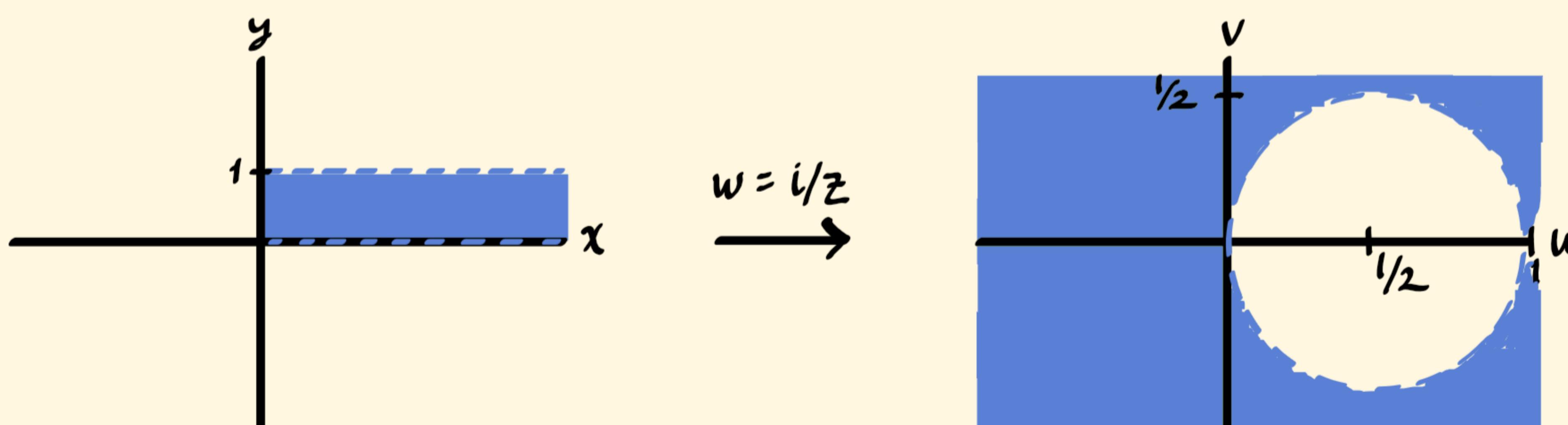
This is circle in the complex (u, v) plane centered at $(\frac{1}{2k}, 0)$ with radius $\frac{1}{2k}$. Note that $\frac{1}{2k} > \frac{1}{2}$.

Apply this transformation to all line segments $x > 0, y = k \in (0, 1)$ to produce infinitely many such circles.

For $k \rightarrow 0^+$ we have circles of increasingly large radii and centers shifting rightward. Each circle still passes through $(0, 0)$.

For $k \rightarrow 1^-$ we have circles of smaller and smaller radii (with radius length bounded below by $\frac{1}{2}$) and centers shifting left toward $(\frac{1}{2}, 0)$ (but never including this circle).

\therefore The image of $x > 0, 0 < y < 1$ under $w = i/z$ is $(u - \frac{1}{2})^2 + v^2 > (\frac{1}{2})^2$.



* * * *

Chapter 8 Section 94: 1, 4, 6, 7 (page 324)

1. Find the linear fractional transformation that maps the points $z_1 = 2, z_2 = i, z_3 = -2$ onto the points $w_1 = 1, w_2 = i, w_3 = -1$.

$$\text{Ans. } w = \frac{3z + 2i}{iz + 6}.$$

$$w = \frac{az + b}{cz + d}, \quad ad - bc \neq 0 \rightarrow \begin{cases} -c + di = ia + b \\ 2c + d = 2a + b \\ 2c - d = -2a + b \end{cases}$$

$$2c + d + 2c - d = 2a + b - 2a + b \rightarrow 2c = b$$

$$2c + d - 2c + d = 2a + b + 2a - b \rightarrow 2a = d$$

Set $b = 2i$. Then $c = i$ and $-c + di = ia + b$ becomes

$$-i + 2ai = ia + 2i \rightarrow ai = 3i \rightarrow a = 3$$

Then $a = 3, b = 2i, c = i, d = 6$ and $ad - bc = 20 \neq 0$.

$$\therefore w = \frac{3z + 2i}{iz + 6}$$

4. Find the bilinear transformation that maps distinct points z_1, z_2, z_3 onto the points $w_1 = 0, w_2 = 1, w_3 = \infty$.

$$\text{Ans. } w = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}.$$

First consider eqn (1) of section 94 (page 322) :

$$(1) \quad \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

Following Example 2 and the discussion preceding Example 2 on page 323, delete all factors involving $w_3 = \infty$ from (1) :

$$\frac{w - 0}{1 - 0} = \frac{w - w_1}{w_2 - w_1} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

$$w = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

6. A *fixed point* of a transformation $w = f(z)$ is a point z_0 such that $f(z_0) = z_0$. Show that every linear fractional transformation, with the exception of the identity transformation $w = z$, has at most two fixed points in the extended plane.

$$z_0 = \frac{az_0 + b}{cz_0 + d}$$

$$cz_0^2 + dz_0 = az_0 + b$$

$$cz_0^2 + (d-a)z_0 - b = 0$$

$f(z_0) = z_0$ produces a quadratic equation in z_0 . By the Fundamental Theorem of Algebra this has 2 soln's counting multiplicity: so either 2 distinct solutions for z_0 (see 7a) or 1 solution for z_0 of multiplicity 2 (see 7b).

7. Find the fixed points (see Exercise 6) of the transformation

$$(a) w = \frac{z-1}{z+1}; \quad (b) w = \frac{6z-9}{z}.$$

Ans. (a) $z = \pm i$; (b) $z = 3$.

$$(a) z_0(z_0+1) = z_0 - 1$$

$$z_0^2 + z_0 = z_0 - 1$$

$$z_0^2 = -1$$

$$\boxed{z_0 = \pm i}$$

$$(b) z_0^2 = 6z_0 - 9$$

$$z_0^2 - 6z_0 + 9 = 0$$

$$(z_0 - 3)^2 = 0$$

$$\boxed{z_0 = 3}$$