MA 4291 Written Homework 1

Chapter 1 Section 4 : 1, 3, 4, 6 (page 12)

1. Locate the numbers $z_1 + z_2$ and $z_1 - z_2$ vectorially when

(a) $z_1 = 2i, \quad z_2 = \frac{2}{3} - i;$ *(b)* $z_1 = (-\sqrt{3}, 1), \quad z_2 = (\sqrt{3}, 0);$

(c) $z_1 = (-3, 1), \quad z_2 = (1, 4);$ *(d)* $z_1 = x_1 + iy_1, \quad z_2 = x_1 - iy_1.$

(a) $z_1+z_2 = (0,2)+(2/3,-1) = (2/3, 1)$

$z_1-z_2 = (0,2)-(2/3,-1) = (-2/3, 3)$

(b) $z_1+z_2 = (-\sqrt{3},1)+(\sqrt{3},0) = (0,1)$

$z_1-z_2 = (-\sqrt{3},1)-(\sqrt{3},0) = (-2\sqrt{3},1)$

(c) $z_1+z_2 = (-3,1)+(1,4) = (-2,5)$

$z_1-z_2 = (-3,1)-(1,4) = (-4,-3)$

(d) $z_1+z_2 = (x,y)+(x,-y) = (x,0)$

$z_1-z_2 = (x,y)-(x,-y) = (0,2y)$

3. Use established properties of moduli to show that when $|z_3| \neq |z_4|$,

$$\frac{\operatorname{Re}(z_1+z_2)}{|z_3+z_4|} \leq \frac{|z_1|+|z_2|}{||z_3|-|z_4||}.$$

$$\frac{\operatorname{Re}(z_1+z_2)}{|z_3+z_4|} \leq \frac{|z_1+z_2|}{|z_3+z_4|}$$

$\operatorname{Re} z \leq |\operatorname{Re} z| \leq |z|$

$$\leq \frac{|z_1|+|z_2|}{|z_3+z_4|}$$

triangle inequality:
$|z_1+z_2| \leq |z_1|+|z_2|$

$$\leq \frac{|z_1|+|z_2|}{||z_3|-|z_4||}$$

reverse triangle inequality:
$0 < ||z_3|-|z_4|| \leq |z_3+z_4|$

4. Verify that $\sqrt{2}\,|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z|$.

Suggestion: Reduce this inequality to $(|x| - |y|)^2 \geq 0$.

$$0 \leq (|\operatorname{Re} z| - |\operatorname{Im} z|)^2 = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 - 2|\operatorname{Re} z||\operatorname{Im} z|$$

$$(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 \leq 2[(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2] - 2|\operatorname{Re} z||\operatorname{Im} z|$$

$$(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 + 2|\operatorname{Re} z||\operatorname{Im} z| \leq 2[(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2]$$

$$(|\operatorname{Re} z| + |\operatorname{Im} z|)^2 \leq 2[(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2]$$

$$(|\operatorname{Re} z| + |\operatorname{Im} z|)^2 \leq 2|z|^2 \qquad (|z|^2 = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2)$$

$$|\operatorname{Re} z| + |\operatorname{Im} z| \leq \sqrt{2}\,|z| \qquad (a^2 \leq b^2 \rightarrow a \leq b \text{ for } a, b \geq 0)$$

6. Using the fact that $|z_1 - z_2|$ is the distance between two points z_1 and z_2, give a geometric argument that

(a) $|z - 4i| + |z + 4i| = 10$ represents an ellipse whose foci are $(0, \pm 4)$;

(b) $|z - 1| = |z + i|$ represents the line through the origin whose slope is -1.

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Chapter 1 Section 5 : 2, 7, 9, 10, 11, 13 (pages 14, 15)

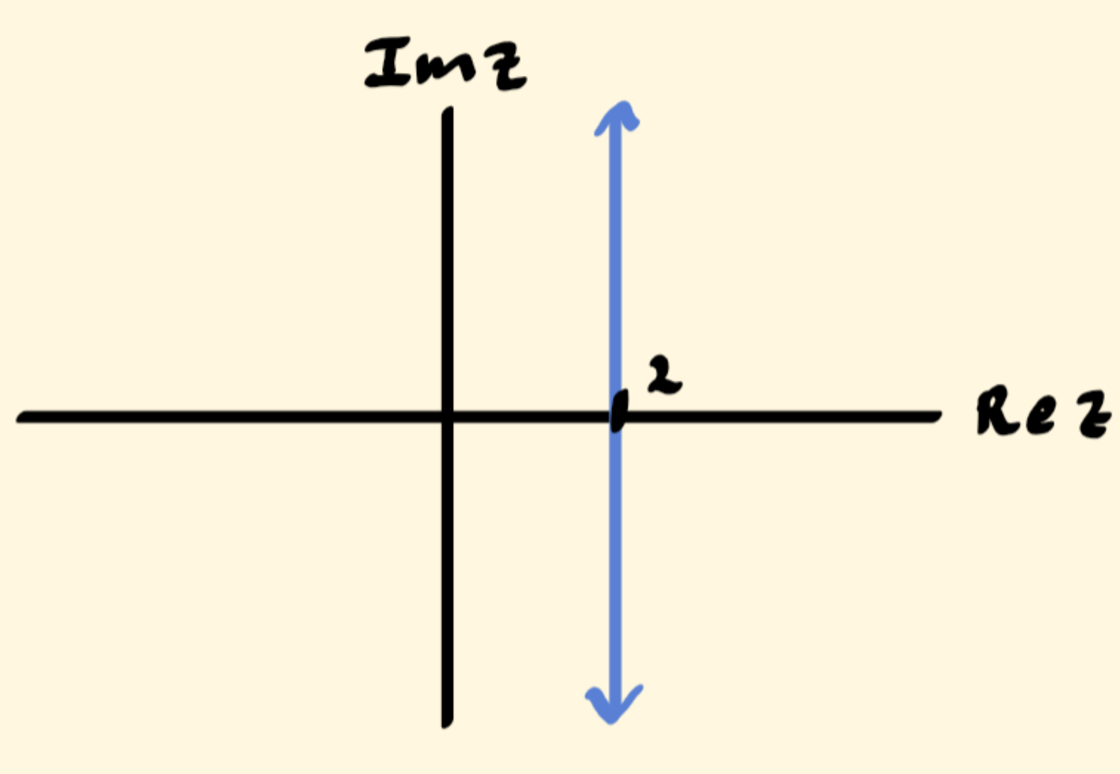
2. Sketch the set of points determined by the condition

(a) $\operatorname{Re}(\bar{z} - i) = 2$; *(b)* $|2\bar{z} + i| = 4$.

Let $z = x + iy \quad x, y \in \mathbb{R}$

(a) $2 = \operatorname{Re}(\bar{z} - i) = \operatorname{Re}(x + (-1-y)i) = x$

All $z = x + iy$ with $x = 2$

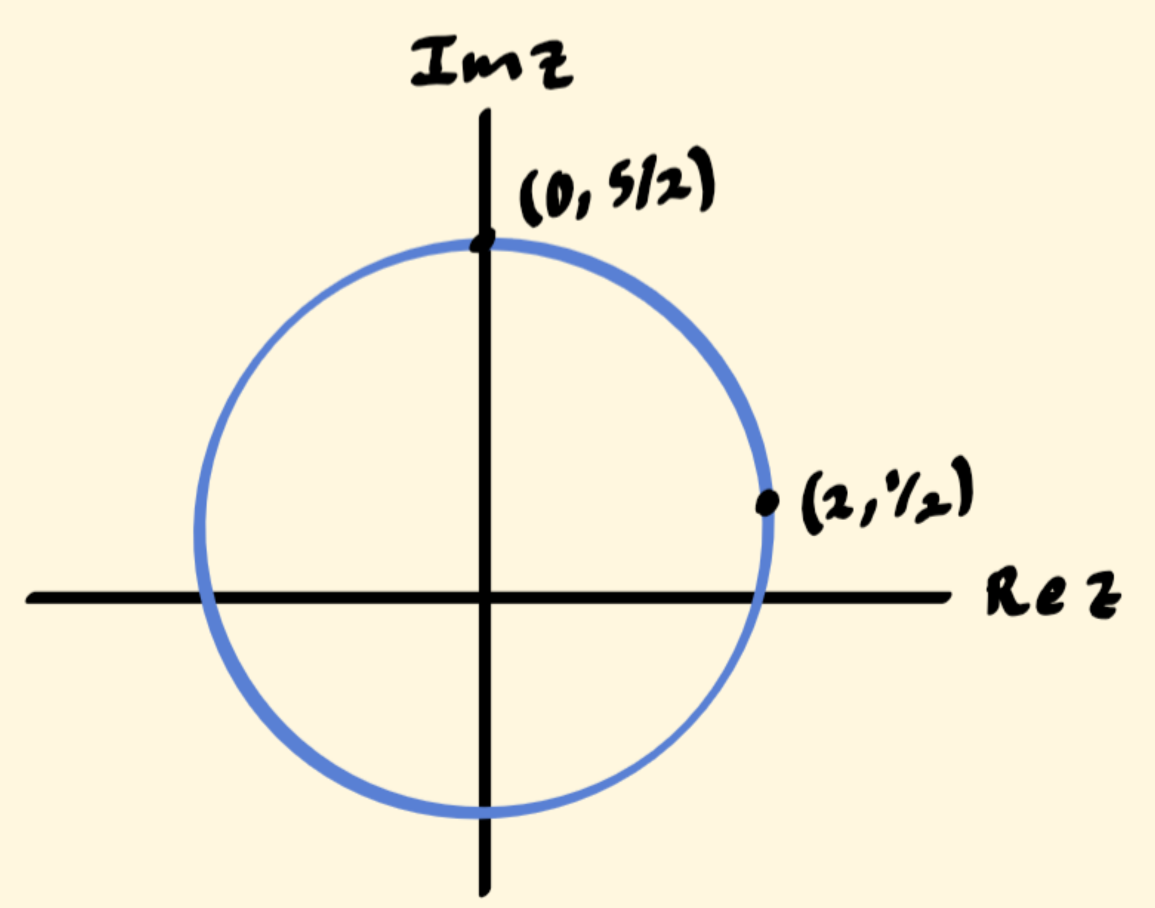


(b) $4 = |2\bar{z} + i| = |2x - 2yi + i|$

$4^2 = 4x^2 + 4(y - 1/2)^2$

$2^2 = x^2 + (y - 1/2)^2$

All z on the circle of radius 2 centered at $(0, 1/2)$ in the complex plane



7. Show that
$$|\mathrm{Re}(2+\overline{z}+z^3)| \leq 4 \qquad \text{when } |z| \leq 1.$$

$|\mathrm{Re}(2+\bar{z}+z^3)| \leq |2+\bar{z}+z^3|$ $\qquad$ $|\mathrm{Re}\, w| \leq |w| \ \forall w \in \mathbb{C}$

$\leq |2|+|\bar{z}|+|z^3|$ $\qquad$ triangle inequality

$= 2+|z|+|z|^3$ $\qquad$ $|\bar{z}|=|z|,\ |z^3|=|z|^3$

$\leq 2+1+1^3 = 4$ $\qquad$ $|z| \leq 1$

10. Prove that

(a) z is real if and only if $\overline{z}=z$;

(b) z is either real or pure imaginary if and only if $\overline{z}^2 = z^2$.

Let $z = x+iy$ with $x,y \in \mathbb{R}$

(a) $x+iy = z = \bar{z} = x-iy \rightarrow y=0$

$\therefore\ z = x \in \mathbb{R}$

(b) $x^2-y^2+2xyi = z^2 = \bar{z}^2 = x^2-y^2-2xyi$

$\Rightarrow xy=0 \Rightarrow x=0$ or $y=0$

If $x=0$, z is pure imaginary. If $y=0$, z is real.

11. Use mathematical induction to show that when $n = 2, 3, \ldots$,

(a) $\overline{z_1 + z_2 + \cdots + z_n} = \overline{z_1} + \overline{z_2} + \cdots + \overline{z_n}$; (b) $\overline{z_1 z_2 \cdots z_n} = \overline{z_1}\,\overline{z_2} \cdots \overline{z_n}$.

(a) Base case $n=2$: If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$,

$$\overline{z_1 + z_2} = \overline{(x_1 + x_2) + i(y_1 + y_2)} = (x_1 + x_2) - i(y_1 + y_2) = x_1 - iy_1 + x_2 - iy_2 = \overline{z_1} + \overline{z_2}$$

Inductive step: Assume $\overline{z_1 + \ldots + z_n} = \overline{z_1} + \ldots + \overline{z_n}$ for some $n \geq 2$

$$\begin{aligned} \overline{z_1 + \ldots + z_n + z_{n+1}} &= \overline{z_1 + \ldots + z_n} + \overline{z_{n+1}} && \text{(by the base case)} \\ &= \overline{z_1} + \ldots + \overline{z_n} + \overline{z_{n+1}} && \text{(by the inductive hypothesis)} \end{aligned}$$

(b) Base case $n=2$: If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$,

$$\begin{aligned} \overline{z_1 z_2} = \overline{(x_1 + iy_1)(x_2 + iy_2)} &= \overline{(x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)} \\ &= (x_1 x_2 - y_1 y_2) - i(x_1 y_2 + x_2 y_1) \\ &= (x_1 - iy_1)(x_2 - iy_2) = \overline{z_1}\,\overline{z_2} \end{aligned}$$

Inductive step: Assume $\overline{z_1 z_2 \ldots z_n} = \overline{z_1}\,\overline{z_2} \ldots \overline{z_n}$ for some $n \geq 2$

$$\begin{aligned} \overline{z_1 \ldots z_n z_{n+1}} &= \overline{z_1 \ldots z_n}\,\overline{z_{n+1}} && \text{(by the base case)} \\ &= \overline{z_1} \ldots \overline{z_n}\,\overline{z_{n+1}} && \text{(by the inductive hypothesis)} \end{aligned}$$

13. Show that the equation $|z - z_0| = R$ of a circle, centered at z_0 with radius R, can be written

$$|z|^2 - 2\,\mathrm{Re}(z\overline{z_0}) + |z_0|^2 = R^2.$$

$|z - z_0| = R$ iff $|z - z_0|^2 = R^2$ since $|z - z_0|, R \geq 0$

$$\begin{aligned} R^2 = |z - z_0|^2 &= (z - z_0)\overline{(z - z_0)} && |w|^2 = w\overline{w} \ \forall w \in \mathbb{C} \\ &= (z - z_0)(\overline{z} - \overline{z_0}) && \overline{v - w} = \overline{v} - \overline{w} \ \forall v, w \in \mathbb{C} \\ &= z\overline{z} + z_0\overline{z_0} - z\overline{z_0} - \overline{z}z_0 \\ &= |z|^2 + |z_0|^2 - (z\overline{z_0} + \overline{z\overline{z_0}}) && \overline{v\overline{w}} = \overline{v}\,\overline{\overline{w}} = \overline{v}w \ \forall v, w \in \mathbb{C} \\ &= |z|^2 + |z_0|^2 - 2\,\mathrm{Re}(z\overline{z_0}) && 2\,\mathrm{Re}\, w = w + \overline{w} \end{aligned}$$

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Chapter 1 Section 8: 1, 3, 5, 9, 10 (pages 22, 23)

1. Find the principal argument Arg z when

(a) $z = \dfrac{i}{-2-2i}$; *(b)* $z = (\sqrt{3} - i)^6$.

Ans. *(a)* $-3\pi/4$; *(b)* π.

(a) $z_1 = i = e^{i(\pi/2 + 2k\pi)}$ $k \in \mathbb{Z}$, $z_2 = -2-2i = 2\sqrt{2}e^{i(-3\pi/4 + 2j\pi)}$ $j \in \mathbb{Z}$

$\arg z_1 = \text{Arg}\, z_1 + 2k\pi = \pi/2 + 2k\pi$, $\arg z_2 = \text{Arg}\, z_2 + 2j\pi = -3\pi/4 + 2j\pi$

$\arg z = \arg(z_1/z_2) = \arg\left[2\sqrt{2}e^{i(5\pi/4 + 2(k-j)\pi)}\right] = 5\pi/4 + 2n\pi$, $n \in \mathbb{Z}$

$-\pi < \text{Arg}\, z \leq \pi \rightarrow \text{Arg}\, z = 5\pi/4 + 2\cdot(-1)\pi = -3\pi/4$

(b) $w = \sqrt{3} - i = 2e^{-i\pi/6} \longrightarrow z = w^6 = 2^6 e^{-i\pi}$

$\arg z = -\pi + 2n\pi \longrightarrow \text{Arg}\, z = -\pi + 2(1)\pi = \pi$

3. Use mathematical induction to show that

$$e^{i\theta_1} e^{i\theta_2} \cdots e^{i\theta_n} = e^{i(\theta_1 + \theta_2 + \cdots + \theta_n)} \qquad (n = 2, 3, \ldots).$$

Base Case $n=2$:

$e^{i\theta_1}e^{i\theta_2} = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1)$

$= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) = e^{i(\theta_1 + \theta_2)}$

Inductive Step: Assume $e^{i\theta_1}e^{i\theta_2}\cdots e^{i\theta_n} = e^{i(\theta_1 + \theta_2 + \cdots + \theta_n)}$ for some $n \geq 2$

$e^{i\theta_1}e^{i\theta_2}\cdots e^{i\theta_n}e^{i\theta_{n+1}} = e^{i(\theta_1 + \cdots + \theta_n)}e^{i\theta_{n+1}}$ (by the inductive hypothesis)

$= e^{i((\theta_1 + \cdots + \theta_n) + \theta_{n+1})}$ (by the base case)

$= e^{i(\theta_1 + \cdots + \theta_{n+1})}$

5. By writing the individual factors on the left in exponential form, performing the needed operations, and finally changing back to rectangular coordinates, show that

(a) $i(1-\sqrt{3}i)(\sqrt{3}+i) = 2(1+\sqrt{3}i)$; *(b)* $5i/(2+i) = 1+2i$;

(c) $(-1+i)^7 = -8(1+i)$; *(d)* $(1+\sqrt{3}i)^{-10} = 2^{-11}(-1+\sqrt{3}i)$.

(a) $i = e^{i\pi/2}$ $\quad 1-\sqrt{3}i = 2e^{-i\pi/3}$ $\quad \sqrt{3}+i = 2e^{i\pi/6}$

$$i(1-\sqrt{3}i)(\sqrt{3}+i) = 4e^{i\pi(3-2+1)/6} = 4e^{i\pi/3} = 4(\cos\pi/3 + i\sin\pi/3) = 2(1+\sqrt{3}i)$$

(b) $5i = 5e^{i\pi/2}$ $\quad 2+i = \sqrt{5}e^{i\theta}$, $\tan\theta = 1/2$ with θ in quadrant 1

$$5i/(2+i) = \sqrt{5}e^{i(\pi/2-\theta)} = \sqrt{5}\,(\cos(\pi/2-\theta) + i\sin(\pi/2-\theta))$$

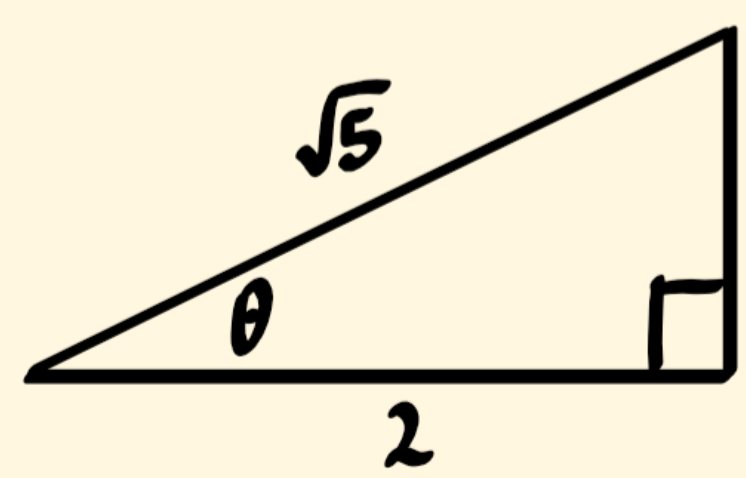
$$= \sqrt{5}\left[(\cos\pi/2\cos\theta + \sin\pi/2\sin\theta) + i(\sin\pi/2\cos\theta - \cos\pi/2\sin\theta)\right]$$

$$= \sqrt{5}(\sin\theta + i\cos\theta)$$

$$= \sqrt{5}(1/\sqrt{5} + 2i/\sqrt{5})$$

$$= 1+2i$$

$\tan\theta = 1/2$
$0<\theta<\pi/2$



(c) $-1+i = \sqrt{2}\,e^{3\pi i/4}$

$$(-1+i)^7 = \sqrt{2}^7 e^{21\pi i/4} = 8\sqrt{2}\,e^{16\pi i/4}e^{5\pi i/4} = 8\sqrt{2}\,e^{5\pi i/4}$$

$$= 8\sqrt{2}(\cos 5\pi/4 + i\sin 5\pi/4) = 8\sqrt{2}(-\sqrt{2}/2 - \sqrt{2}i/2)$$

$$= -8(1+i)$$

(d) $1+\sqrt{3}i = 2e^{i\pi/3}$

$$(1+\sqrt{3}i)^{-10} = 2^{-10}e^{-10i\pi/3} = 2^{-10}e^{-12i\pi/3}e^{2\pi i/3} = 2^{-10}e^{2\pi i/3}$$

$$= 2^{-10}(\cos 2\pi/3 + i\sin 2\pi/3)$$

$$= 2^{-10}(-1/2 + \sqrt{3}i/2)$$

$$= 2^{-11}(-1+\sqrt{3}i)$$

9. Establish the identity

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \qquad (z \neq 1)$$

and then use it to derive *Lagrange's trigonometric identity:*

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin[(2n+1)\theta/2]}{2\sin(\theta/2)} \qquad (0 < \theta < 2\pi).$$

Suggestion: As for the first identity, write $S = 1 + z + z^2 + \cdots + z^n$ and consider the difference $S - zS$. To derive the second identity, write $z = e^{i\theta}$ in the first one.

○ Establish the first identity by telescoping

$$S = 1 + z + z^2 + \dots + z^n \rightarrow zS = z + z^2 + \dots + z^{n+1}$$

$$S(1-z) = S - zS = (1 + z + \dots + z^n) - (z + z^2 + \dots + z^{n+1}) = 1 - z^{n+1}$$

$$\therefore 1 + z + \dots + z^n = S = \frac{1 - z^{n+1}}{1 - z}$$

△ Substitute $z = e^{i\theta}$ into this identity

$$1 + e^{i\theta} + e^{2i\theta} + \dots + e^{ni\theta} = \frac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}}$$

$$1 + \cos\theta + \dots + \cos n\theta + i(\sin\theta + \dots + \sin n\theta) = \frac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}}$$

□ Simplify the right hand side

$$\frac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}} = \frac{e^{(n+1)i\theta/2}}{e^{i\theta/2}} \, \frac{e^{-(n+1)i\theta/2} - e^{(n+1)i\theta/2}}{e^{-i\theta/2} - e^{i\theta/2}}$$

$$= e^{ni\theta/2} \, \frac{e^{-(n+1)i\theta/2} - e^{(n+1)i\theta/2}}{e^{-i\theta/2} - e^{i\theta/2}}$$

$$= e^{ni\theta/2} \, \frac{\cos((n+1)\theta/2) - i\sin((n+1)\theta/2) - \cos((n+1)\theta/2) - i\sin((n+1)\theta/2)}{\cos(\theta/2) - i\sin(\theta/2) - \cos(\theta/2) - i\sin(\theta/2)}$$

$$= e^{ni\theta/2} \, \frac{\sin((n+1)\theta/2)}{\sin(\theta/2)} = (\cos(n\theta/2) + i\sin(\theta/2)) \frac{\sin((n+1)\theta/2)}{\sin(\theta/2)}$$

◊ Match real terms and simplify using $\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha-\beta) + \sin(\alpha+\beta))$

$$1 + \cos\theta + \dots + \cos n\theta = \operatorname{Re}\left[\frac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}}\right] = \frac{\cos(n\theta/2)\sin((n+1)\theta/2)}{\sin(\theta/2)}$$

$$1 + \cos\theta + \dots + \cos n\theta = \frac{1}{2} \, \frac{\sin(\theta/2) + \sin((2n+1)\theta/2)}{\sin(\theta/2)} = \frac{1}{2} + \frac{\sin((2n+1)\theta/2)}{2\sin(\theta/2)}$$

⬠ Alternatively, apply the first identity to both $z = e^{i\theta}$ and $z = e^{-i\theta}$.

$$1 + \cos\theta + \dots + \cos n\theta = \sum_{k=0}^{n} \cos k\theta = \frac{1}{2}\sum_{k=0}^{n} \left(e^{ik\theta} + e^{-ik\theta}\right)$$

$$= \frac{1}{2}\left(\frac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}} + \frac{1 - e^{-(n+1)i\theta}}{1 - e^{-i\theta}}\right)$$

$$= \frac{1}{2}\left(\frac{e^{-i\theta/2} - e^{(n+1/2)i\theta}}{e^{-i\theta/2} - e^{i\theta/2}} + \frac{-e^{i\theta/2} + e^{-(n+1/2)i\theta}}{-e^{i\theta/2} + e^{-i\theta/2}}\right)$$

$$= \frac{1}{2}\left(\frac{e^{-(n+1/2)i\theta} - e^{(n+1/2)i\theta}}{e^{-i\theta/2} - e^{i\theta/2}} + \frac{e^{-i\theta/2} - e^{i\theta/2}}{e^{-i\theta/2} - e^{i\theta/2}}\right)$$

$$= \frac{1}{2}\left(\frac{-2i\sin((n+1/2)\theta)}{-2i\sin(\theta/2)} + 1\right)$$

$$= \frac{1}{2} + \frac{\sin((2n+1)\theta/2)}{2\sin(\theta/2)}$$

10. Use de Moivre's formula (Sec. 7) to derive the following trigonometric identities:

(a) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$; *(b)* $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

de Moivre's formula: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, $n \in \mathbb{Z}$

Evaluate the formula with $n=3$.

$$\cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta = (\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

(a) Match real terms: $\cos^3\theta - 3\cos\theta\sin^2\theta = \cos 3\theta$

(b) Match imaginary terms: $3\cos^2\theta\sin\theta - \sin^3\theta = \sin 3\theta$

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Chapter 1 Section 10: 1, 3, 6, 7 (pages 29, 30)

1. Find the square roots of *(a)* $2i$; *(b)* $1-\sqrt{3}i$ and express them in rectangular coordinates.

Ans. (a) $\pm(1+i)$; *(b)* $\pm\dfrac{\sqrt{3}-i}{\sqrt{2}}$.

(a) $z = 2i = 2e^{i(\pi/2+2k\pi)} \Rightarrow z^{1/2} = \sqrt{2}\,e^{i(\pi/4+k\pi)}$ $(k \in \mathbb{Z})$

Evaluate at $k=0,1$ to collect all distinct roots:

$z^{1/2} = \sqrt{2}e^{i\pi/4}, \sqrt{2}e^{5\pi i/4} = \sqrt{2}(\sqrt{2}/2+\sqrt{2}i/2), \sqrt{2}(-\sqrt{2}/2-\sqrt{2}i/2) = \boxed{\pm(1+i)}$

(b) $z = 1-\sqrt{3}i = 2e^{i(-\pi/3+2k\pi)} \Rightarrow z^{1/2} = \sqrt{2}\,e^{i(-\pi/6+k\pi)}$ $(k \in \mathbb{Z})$

Evaluate at $k=0,1$ to collect all distinct roots:

$z^{1/2} = \sqrt{2}e^{-i\pi/6}, \sqrt{2}e^{5\pi i/6} = \sqrt{2}(\sqrt{3}/2-i/2), \sqrt{2}(-\sqrt{3}/2+i/2) = \boxed{\pm(\sqrt{3}-i)/\sqrt{2}}$

???

3. In each case, find all the roots in rectangular coordinates, exhibit them as vertices of certain regular polygons, and identify the principal root:

(a) $(-1)^{1/3}$; *(b)* $8^{1/6}$.

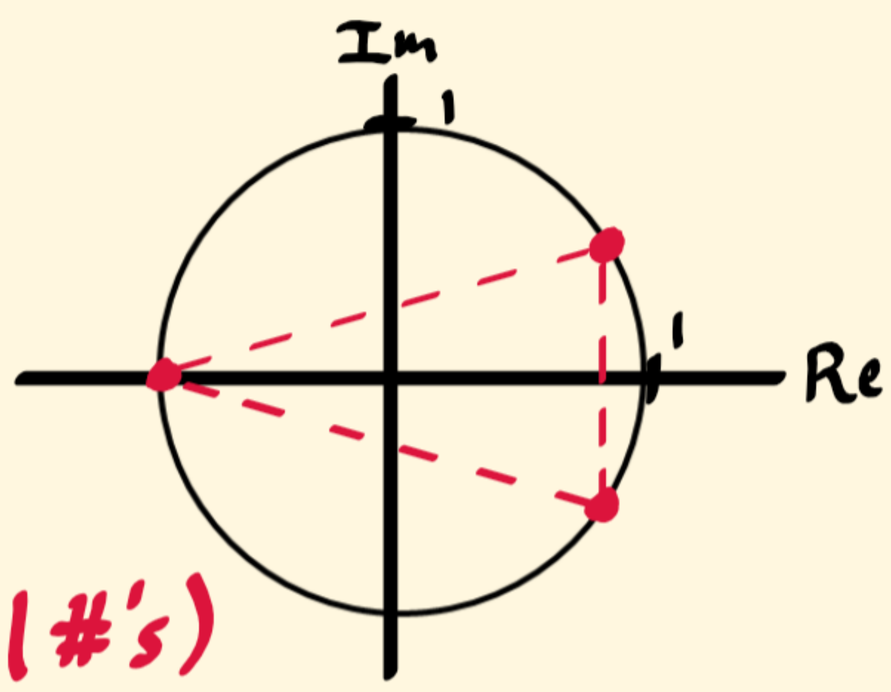
Ans. (b) $\pm\sqrt{2}$, $\pm\dfrac{1+\sqrt{3}i}{\sqrt{2}}$, $\pm\dfrac{1-\sqrt{3}i}{\sqrt{2}}$.

(a) $z = -1 = e^{i(\pi+2k\pi)} \Rightarrow z^{1/3} = (-1)^{1/3} = e^{i(\pi/3+2k\pi/3)}$ $(k \in \mathbb{Z})$

Evaluate at $k=0,1,2$ to collect all distinct roots:

$(-1)^{1/3} = e^{i\pi/3}, e^{i\pi}, e^{5\pi i/3}$

$= \sqrt{3}/2+i/2, -1, \sqrt{3}/2-i/2$



* The principle root is -1
(was not defined for roots of negative real #'s)

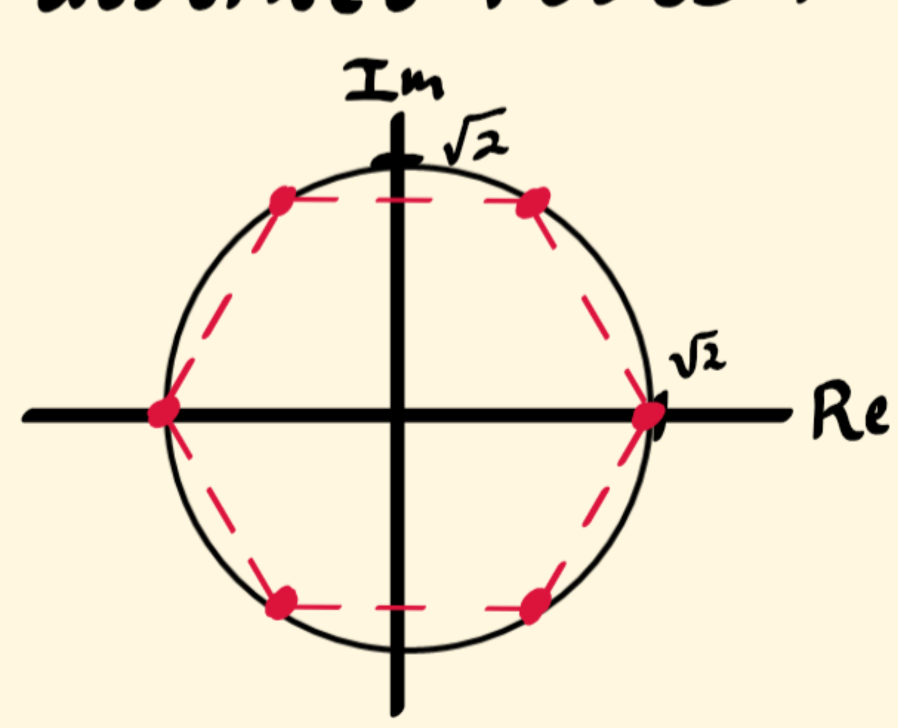
(b) $z = 8 = 8e^{i(0+2k\pi)} = 8e^{2k\pi i} \Rightarrow z^{1/6} = 8^{1/6} = \sqrt{2}\,e^{k\pi i/3}$ $(k \in \mathbb{Z})$

Evaluate at $k=0,1,\ldots,5$ to collect all distinct roots:

$8^{1/6} = \sqrt{2}, \sqrt{2}e^{i\pi/3}, \sqrt{2}e^{2\pi i/3}, \ldots, \sqrt{2}e^{5\pi i/3}$

$= \pm\sqrt{2}, (\pm1\pm\sqrt{3}i)/\sqrt{2}$

The principle root is $\sqrt{2}$



6. Find the four zeros of the polynomial $z^4 + 4$, one of them being

$$z_0 = \sqrt{2}\, e^{i\pi/4} = 1 + i.$$

Then use those zeros to factor $z^2 + 4$ into quadratic factors with real coefficients.

Ans. $(z^2 + 2z + 2)(z^2 - 2z + 2)$.

$z_0^4 + 4 = 0$

$z_0^4 = -4 = 4e^{i(\pi + 2k\pi)}, \quad k \in \mathbb{Z}$

$z_0 = \sqrt[4]{4}\, e^{i(\pi + 2k\pi)/4} = \sqrt{2} e^{i(\pi+2k\pi)/4}$

$z_0 = \sqrt{2} e^{i\pi/4}, \sqrt{2} e^{3\pi i/4}, \sqrt{2} e^{5\pi i/4}, \sqrt{2} e^{7\pi i/4} \quad (k = 0, 1, 2, 3)$

$z_0 = 1+i, -1+i, -1-i, 1-i$

Note that $(z-r)(z-\bar{r}) = z^2 - z(r+\bar{r}) + r\bar{r} = z^2 - 2z\mathrm{Re}(r) + |r|^2$ is a polynomial with real coefficients. This means we should multiply the linear terms corresponding to complex conjugate pairs

$\therefore\ z^2 + 4 = (z-(1+i))(z-(1-i))(z-(-1+i))(z-(-1-i))$

$= (z - 2z + 2)(z + 2z + 2)$

7. Show that if c is any nth root of unity other than unity itself, then

$$1 + c + c^2 + \cdots + c^{n-1} = 0.$$

Suggestion: Use the first identity in Exercise 9, Sec. 8.

Suppose $c \in \mathbb{C}$ is any root of unity other than unity itself, i.e.

$$c = 1^{1/n}, \quad c \neq 1$$

There are $n-1$ such possible values of c, but all satisfy $c^n = 1$ and $1 - c \neq 0$. It follows that

$$1 + c + c^2 + \ldots + c^{n-1} = \frac{1-c^n}{1-c} = \frac{1-1}{1-c} = \frac{0}{1-c} = 0$$

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Chapter 1 Section 11: 1ae, 2, 3, 6, 7, 8 (page 33)

1. Sketch the following sets and determine which are domains:

(a) $|z - 2 + i| \leq 1$;

(b) $|2z + 3| > 4$;

(c) $\operatorname{Im} z > 1$;

(d) $\operatorname{Im} z = 1$;

(e) $0 \leq \arg z \leq \pi/4 \ (z \neq 0)$;

(f) $|z - 4| \geq |z|$.

Ans. *(b)*, *(c)* are domains.